

ASPECTS OF LIMIT DESIGN OF
REINFORCED CONCRETE STRUCTURES

by

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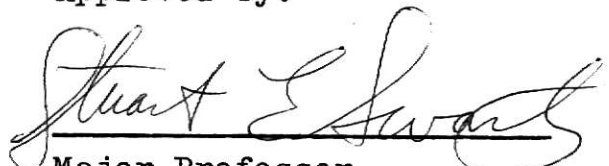
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NOTATION

- a = the depth of neutral axis at ultimate.
 a' = the depth of neutral axis at elastic stage.
 A_s = area of reinforcement.
 b = the width of a cross section.
 C = the total compression force in concrete.
 d = the depth of a cross section.
 E = modulus of elasticity.
 E_c = modulus of elasticity of concrete.
 E_s = modulus of elasticity of steel.
 f_c = concrete stress.
 f'_c = specified compressive strength of concrete.
 f_{sy} = specified yield strength of steel.
 f'_{sy} = specified yield strength of stirrup steel.
 f_r = modulus of rupture of concrete, psi.
 f_s = steel stress.
 I = moment of inertia of a section.
 I_{cr} = moment of inertia of cracked section transformed to concrete.
 I_e = effective moment of inertia for computation of deflection.
 I_g = moment of inertia of gross section, neglecting the reinforcement.
 $K_1 K_2 K_3$ = parameters used to describe the shape of the stress block in the concrete.
 $k = a'/d$, $k_u = a/d$.

- L_p = the length of plastic hinge (over which plastic rotation occurs as a constant).
 L'_p = the length of plastic hinge.
 M = moment.
 M_a = maximum moment in member at stage for which deflection is being computed.
 M_{cr} = cracking moment.
 M^F = the fixed end moment.
 M_{io} = the bending moment at section i due to external load acting on the statically determinate structure.
 M_p = the plastic moment or the ultimate moment of a section. (M_u)
 M_y = moment at yield of tension steel.
 m_{ij} = bending moment at section i due to unit load acting at assumed hinge j .
 N = the degree of statically indeterminacy.
 n = ratio of E_s/E_c .
 p = ratio of tension reinforcement.
 p'' , p''' = ratio of the volume of the binding reinforcement to the volume of the concrete bound.
 p_b = reinforcement ratio to produce balanced condition.
 q_y = a parameter describing the effective percentage of steel.
 R_1 = a parameter for the influence of tension steel.
 R_2 = a parameter for the influence of axial load.

- R_3 = a parameter for the influence of grade of concrete.
 T = the total tension force in steel.
 u_a = the depth to the center of compression force.
 Y_t = distance from centroidal axis of gross section, neglecting the reinforcement, to extreme fiber in tension.
 z = the distance from the critical section to the point of contraflexure.
 θ_{Ri}, θ_i = rotation required at a plastic hinge i .
 θ_T = the total available rotation on one side of a hinge.
 ϕ = curvature
 ϕ_a = capacity reduction.
 ϕ_y = the curvature at elastic stage.
 ϕ_u = the curvature at ultimate.
 ϕ_i = $\phi_u - \phi_y$ = the increasing curvature at a hinge .
 ϵ_c = compression strain in concrete.
 ϵ_o = compression strain in concrete corresponding to maximum stress.
 ϵ_{cu} = the ultimate concrete strain.
 ϵ_{cy} = the strain in the concrete at the extreme fiber at start of steel yielding.
 $\epsilon_p = \epsilon_{cu} - \epsilon_{cy}$ = the increment of strain in the concrete at the compression edge.
 ϵ_{su} = strain in the steel at the failure of the beam.

ϵ_{sy} = the strain in steel at yield.

ϵ_y = the increment of strain in steel due to yield

s_{io} = relative rotation at hinge i due to external load.

s_{ij} = relative rotation at hinge i due to unit moment
acting at hinge j.

ρ = the radius of curvature of the neutral axis.

$\bar{\sigma}_c$ = the average compression stress in concrete distributed
over the compression zone.

Δ = deflection.

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1. INTRODUCTION

A principal responsibility of a structural engineering designer is to calculate the dimensions of a structure, in order that it is safe to carry the service loads. In the past, most reinforced concrete structures carried the load safely, but the strength of reinforced concrete was not fully utilized in those designs. Most of the buildings over the world have been designed using the elastic theory. The exact value of their safety factor was not obtained since the inelastic behavior of the materials, used in the members in those structures, was not considered. In other words these structures may be safe enough against failure, but material may have been wasted. Therefore, engineers have attempted to develop more effective design methods such as ultimate strength design, prestressed concrete design, yielding line design of slabs, plastic design of steel structures and limit design of reinforced concrete structures.

Plastic design of steel structures is more effective than the elastic design method since the strength of materials is almost completely developed (neglecting strain hardening range) in this design method. In the elastic design method, there is a considerable reserve of load-carrying capacity beyond the yield load P_y . By using plastic design this reserve of load-carrying capacity may

be utilized, as indicated in Fig.1-1.

The basic idea of limit design of reinforced concrete structures is the same as the plastic design of steel structures. The main difference is that steel is a ductile material and concrete is a brittle material. Comparison of the two materials is shown in Fig.1-2. The structural steel has the rotation capacity to permit other plastic hinges to form after the first one, until a failure mechanism is formed. Reinforced concrete generally doesn't have this ability. If we want a reinforced concrete member to work like a steel member, we have to improve its ductility; that is its rotation capacity should be increased.

Although it is not very clear how much plasticity and rotation capacity at ultimate load are required to insure satisfactory hinge action in an indeterminate structure, the possibility of increasing the ductility of reinforced concrete members appears to be an important consideration. After many tests, trials and studies, it is believed that one can improve concrete ductility and increase its rotation capacity in order to permit the structure to have the ability of moment redistribution.

In this report, the basic idea of plastic design and limit design; the relationship between moment and rotation angle in simple, rectangular, reinforced concrete beams; ways to increase rotation capacity; the Cambridge method^{1*}

* Superscripts denote references listed in References.

of limit design; and Baker's method¹ of limit design will be reviewed. In addition, experimental results will be described and two numerical examples will be given.

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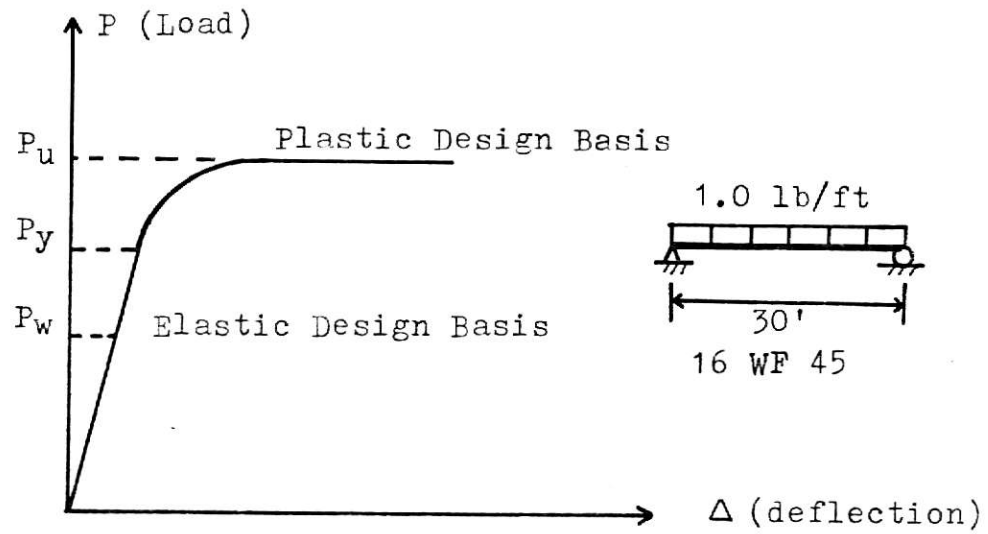


Fig.1-1 Load-Deflection Curve for a Steel Beam

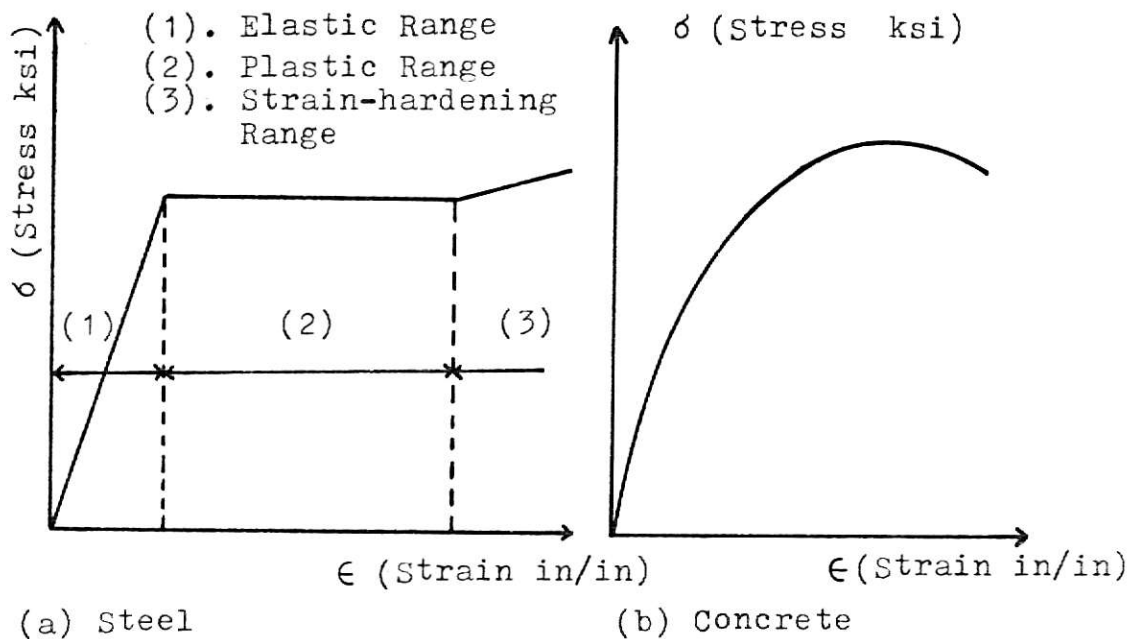


Fig.1-2 Stress-Strain Curves

2. CONCEPT OF PLASTIC DESIGN AND LIMIT DESIGN

a). Plastic Design of Steel Structures

The traditional theory of structural design is based on the assumption of elastic behavior of materials. This also implies that under the maximum service loading condition the structure has no permanent deformation occurring during its working life. The largest stress which a material can withstand without being permanently deformed is called the elastic limit. Using elastic design, one may ignore the plastic behavior of structural materials. However, any advantage of the plastic plateau is not considered. In plastic design, it is assumed that a ductile material can continue to deform after yielding without failure.

The basic theory of plastic analysis indicates that a major change in the distribution of stress occurs at critical points in a structure after yielding has been reached. The idea is that those parts of the structure which have yielded may not resist further increase in stress. Instead of any additional stress, they will deform the amount which is required to allow the loads to be transferred to other parts of the structure which are still in the elastic range and able to resist increased stress. In other words, the distribution of stresses or moments throughout the structure at failure is different from those distributions in the working range. For this discussion the stress-strain diagram is

assumed to have the idealized shape as shown in Fig.1-2a. Beyond the plastic range is the range of strain hardening. This range can theoretically permit the steel member to resist more loads after yielding has been reached, but from a practical standpoint the strains occurring are so large that they can not be considered.

The yield moment of a cross section is defined as the moment which will produce yielding in the outer fibers of the section. As the moment increases beyond the yield moment, the outer fibers will continue to deform without any increase in stress. The duty of resisting the increasing moment will fall on the fibers which are nearer the neutral axis and have not yet yielded. This process will continue with more and more fibers of the cross-section being stressed to the yield point, until finally a hinge is fully formed (neglecting the influence of strain hardening). No more additional moment can be taken then by this section. The maximum moment that can be resisted by a plastic hinge is the plastic moment, M_u .

The bending moment diagram of an indeterminate structure, such as a beam with fixed ends, will have several peak points where the plastic hinges will form. The outer fiber of the cross section with greatest stress will reach the yield point first. As more loads are added, this section will gradually yield to form a plastic hinge, but other parts of this structure may still be elastic. This plastic hinge will not carry any more moment and other parts are forced to resist

any further increased in load. Then the next plastic hinge will form at one of the remaining peak-moment sections and when a sufficient number of plastic hinges has been developed to produce a failure mechanism, the structure will collapse.

A simply supported beam under a given set of loads has one point where the moment is a maximum. As the loading increases, this moment increases proportionately until the extreme fiber stress equals the yield point of the steel. If the loading is further increased, the material will deform more rapidly. Although there will be an increase in load above the elastic limit, this increase is very small and accompanied by rapidly increasing deflection. Therefore, plastic design of steel structures has little advantage over elastic design for steel structures with statically determinate beams and simple structures with effectively pin-connected members, but it may have a great advantage for indeterminate structures.

b). Limit Design of Reinforced Concrete Structures

The limit design concept for reinforced concrete structures was developed after the plastic design concept for steel structures. The classical, elastic theory for the behavior of reinforced concrete, in which it was considered as an elastic material is not necessarily valid for reinforced concrete structures under working load. Although, when a reinforced concrete structure is lightly loaded it behaves almost elastically, concrete is not really an elastic material; on the contrary it is a rather brittle material. Actually the behavior of a reinforced concrete structure is very complicated during its working life. In spite of these complexities and the brittle behavior of concrete, many limit or plastic design methods have been developed. Two of them, the Cambridge method¹ and Baker's method¹, are reviewed here. By using these two methods, a more satisfactory building can be designed and established.

The basis of the Cambridge method is completely the same as for plastic design. The method of plastic design has to satisfy the following three conditions:

1). Mechanism Condition:

A sufficient number of plastic hinges has to be formed in order to develop a failure mechanism.

2). Equilibrium Condition:

The sum of forces, and moments has to be zero.

$$(\Sigma F_x = 0, \quad \Sigma F_y = 0, \quad \Sigma M = 0)$$

3). Plastic Moment Condition:

The moments in the structure have to be nowhere greater than the plastic moment, ie.

$$|M| \leq |M_p| \quad .$$

These three conditions are also required in the Cambridge method, but are not sufficient. Since concrete is a brittle material, one more condition is needed; that is to check whether the rotation capacities of the hinges formed, before the last one, are sufficient to allow the failure mechanism to develop or not. With these four requirements, we may say that the moment redistribution will occur before failure and the structure is safe.

For an under-reinforced concrete beam, the moment-curvature relationship is shown as in Fig.2-1a. As concrete is a brittle material, it is usually considered that there will be a limit corresponding to the point A in Fig.2-1b at which the member will fail completely, the available rotation capacity then is θ_a . It is also necessary to determine whether θ_a is large enough to meet the fourth requirement in the Cambridge method. However, recent research² has shown that the normally designed reinforced concrete beam usually has enough rotation capacity to permit a failure mechanism to form and point A will not be reached at any point in a structure before failure occurs. If the rotation capacity is not enough, this can be improved by using helical binders,

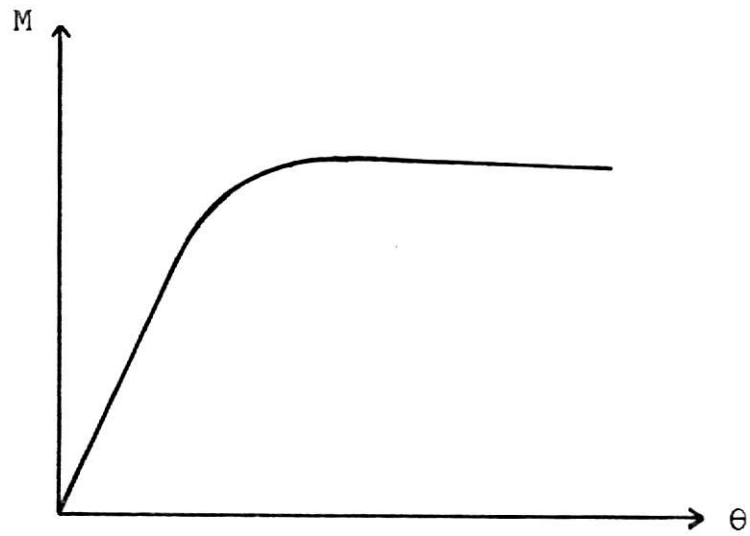


Fig.2-1a Typical M- θ Curve for An Under-Reinforced Concrete Section

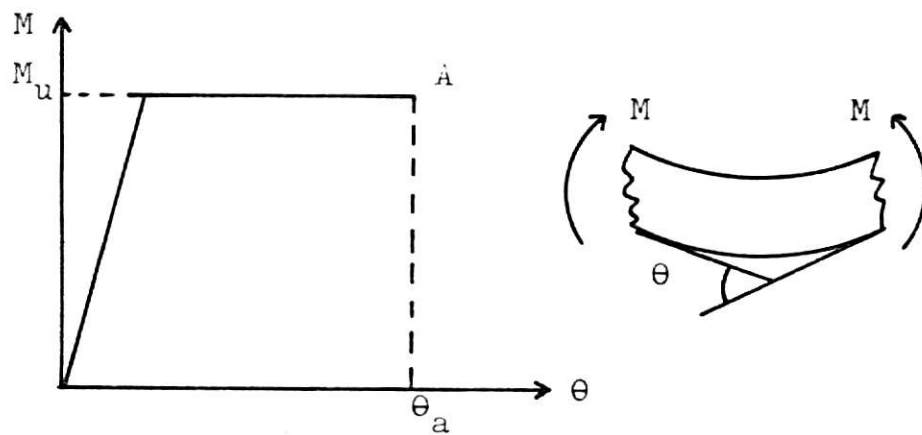


Fig.2-1b Idealized M- θ Curve for Reinforced Concrete

small-space stirrups, rectangular spiral binders or increasing cross section depth.

The Cambridge method was originally developed for steel structures, and then extended to reinforced concrete structures. In utilizing this method it is necessary to improve the behavior of concrete to satisfy the requirements of the plastic design theory. However a method presented by Baker¹ is primarily for reinforced concrete structures. It is based on the observed inelastic behavior of reinforced concrete structures. So Baker's method is the more complete one and also has the advantage of being more general and simpler to use.

In the Cambridge method, it is assumed that the plastic hinges will maintain their plastic moment while developing sufficient rotation to enable other plastic hinges to form and full moment redistribution to occur. This assumption is true for steel structures, but is not true for concrete structures. Baker realized this difference and developed his own method for limit analysis of reinforced concrete structures. The Cambridge method calculates the ultimate load of a structure where $(N+1)^*$ hinges have formed a failure mechanism. But in Baker's method, the ultimate load is limited to that at which N hinges have formed to transform the structure into one which is statically determinate, and the structure is analyzed on this basis. At this stage the

* N = The degree of statical indeterminacy of a structure.

structure is stable and will collapse with the formation of one more plastic hinge.

This method is based on the following concept³:

" when a frame has developed sufficient plastic hinges to become statically determinate, it can be treated as if it has been made statically determinate in the conventional way by the insertion, at the plastic hinge sections, of frictionless hinges, with equal and opposite couples acting at each hinge on the adjacent members, the couples being equal in value to the plastic moment of resistance of the section."

Insert enough hinges to transform the structure into a statically determinate one as shown in Fig.2-2. At each hinge i , apply a pair of couples $\bar{X}_i$, to the adjacent members. According to elastic structural analysis, the relative rotation angle θ_i , at hinge i is:

$$\epsilon_{i0} + \bar{X}_1 \epsilon_{i1} + \bar{X}_2 \epsilon_{i2} + \dots + \bar{X}_j \epsilon_{ij} = -\theta_i \dots (2-1)$$

For instance the simultaneous equations for Fig.2-2 are:

$$\epsilon_{20} + \bar{X}_2 \epsilon_{22} + \bar{X}_3 \epsilon_{23} = -\theta_2$$

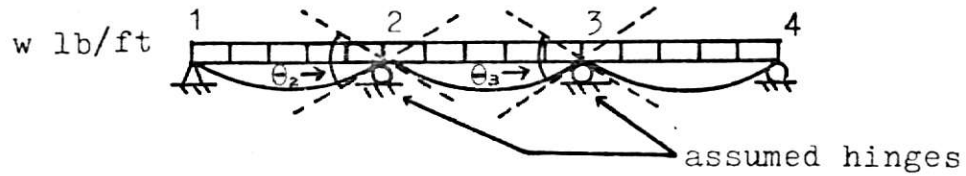
$$\epsilon_{30} + \bar{X}_2 \epsilon_{32} + \bar{X}_3 \epsilon_{33} = -\theta_3$$

$$i = 1, 2, 3, \dots, n$$

$$j = 1, 2, 3, \dots, n$$

$\bar{X}_i$ = the magnitude of assumed moment at hinge i .

$$\epsilon_{i0} = \sum \int \frac{M_{i0} m_{ii}}{EI} ds \dots \dots \dots (2-2)$$

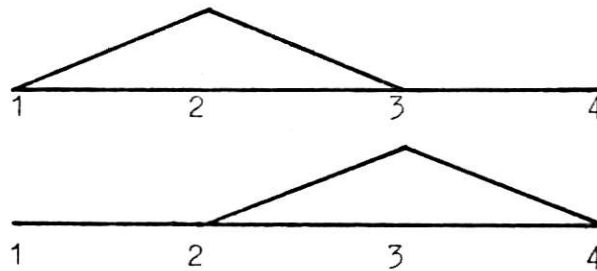


a). Statically Indeterminate to 2nd Degree



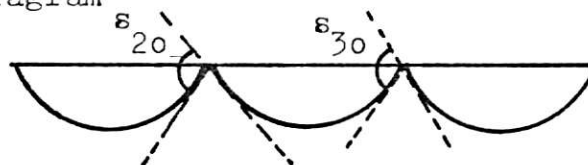
moment due to external load on beam made statically determinate by assumed hinges at 2 and 3

b). M_0 Diagram

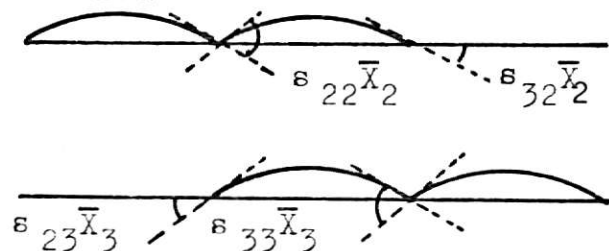


moment due to assumed unit moment at hinge 2 and 3

c). m_i Diagram



d). Relative Rotation at Hinge 2 and 3 Due to External load



e). Relative Rotation at Hinge 2 and 3 Due to Moment $\bar{X}_2$ at Hinge 2 and Moment $\bar{X}_3$ at Hinge 3

Fig.2-2

$$\delta_{ij} = \sum \int \frac{m_{ij}m_{ii}}{EI} ds \dots\dots\dots(2-3)$$

m_{ii} = the bending moment when $\bar{X}_i = 1$ while all other redundants vanish.

M_{i0} = the bending moment produced by the external load acting on the statically determinate structure.

δ_{i0} = relative rotation at hinge i due to external loads.

δ_{ij} = relative rotation at hinge i due to unit moment acting at hinge j.

θ_i = the plastic rotations necessary for the full redistribution of moments.

The location of plastic hinges and the value of the plastic moment may be arbitrarily chosen, because the reinforced concrete structures can always be designed to ensure the desired failure mechanism.

3. MOMENT-CURVATURE RELATIONSHIPS

a). The Rotation Capacity Available in a Section

Steel structural components generally can take more load after the first yielding. This extra capacity can come from two sources:

(1). full plastification of individual cross sections, and

(2). redistribution of moment in an indeterminate structure to give a failure mechanism.

The increase in loading capacity due to moment redistribution requires that the rotation capacity of the hinges formed before the last one are sufficient to allow the failure mechanism to develop.

The solution of the limit design method should be checked for the satisfaction of rotation compatibility. This condition is satisfied and the design is assumed correct if the inelastic rotation θ_{Ri} at a critical section, i , of the structure under the design loading condition does not exceed the rotation capacity θ_{Ti} of this section;

$$\theta_{Ri} \leq \theta_{Ti} .$$

The assumptions for the analysis of moment and rotation capacity for a reinforced concrete section are as follows:

(1). The reinforced concrete section is a cracked section.

- (2). The concrete stress distribution across a section may be described by parameters, as shown in Fig.3-1a.
- (3). The strain varies linearly across a section.
- (4). The steel reinforcement is perfectly bonded to the concrete.
- (5). The stress-strain relationship of steel is known as an idealized type. (see Fig.1-2a).
- (6). The concrete will crush when the ultimate strain ϵ_{cu} is reached.
- (7). The discussion is limited to an under-reinforced section, where the failure starts with the yielding of the steel and ends with the crushing of the concrete.

As mentioned before, most reinforced concrete structures behave very differently from the way a steel structure does. In the region subjected to a sufficiently great bending moment, the concrete deforms plastically in compression and the reinforcement yields in tension. This narrow region may be regarded as the plastic hinge and its length is called the length of plastic hinge L_p . At fracture, the maximum compressive strain of the concrete in the fibers of a beam varies from 0.003 in/in to 0.005 in/in. Because this value is small, the rotation capacity of the plastic hinges is decided by this compressive strain.

The available rotation capacity of a reinforced concrete

section with tensile reinforcement only is now determined.

From elastic theory, the moment M_y in Fig.3-1b is:

$$M_y = A_s f_{sy} d \left(1 - \frac{a'}{3d}\right) = p b d^2 f_{sy} \left(1 - \frac{a'}{3d}\right) = q_y b d^2 f'_c \left(1 - \frac{a'}{3d}\right). \quad \dots\dots\dots(3-1)$$

M_y = moment at yield of tension steel.

$p = \frac{A_s}{bd}$ = ratio of tension reinforcement.

$q_y = \frac{p f_{sy}}{f'_c}$ = a parameter describing the effective percentage of steel.

The curvature at the elastic limit is:

$$\phi_y = \frac{M_y}{EI} \quad \dots\dots\dots(3-2)$$

E = modulus of elasticity of a reinforced concrete section.

I = moment of inertia of a reinforced concrete section.

f'_c = specified compressive strength of concrete.

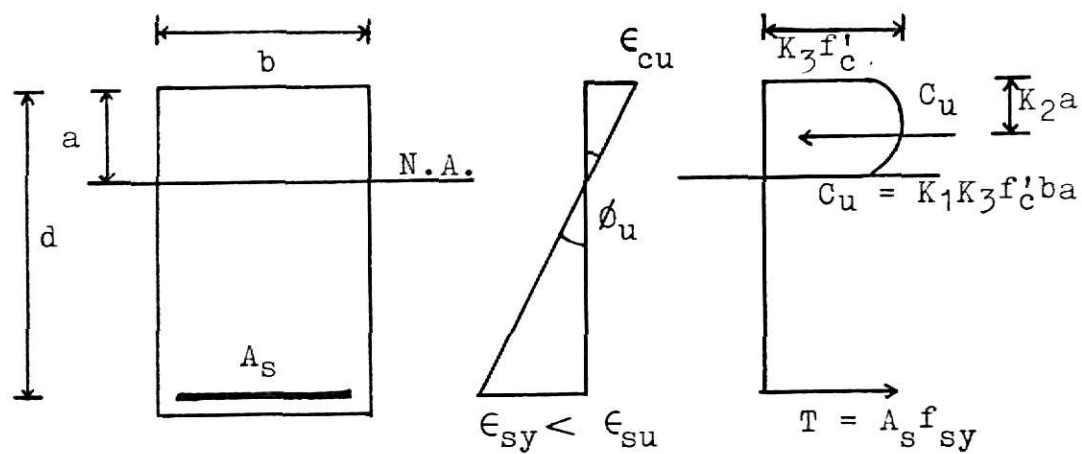
f_{sy} = specified yield strength of steel.

From the analysis of the forces shown in Fig.3-1a, the ultimate moment at this section is derived as:

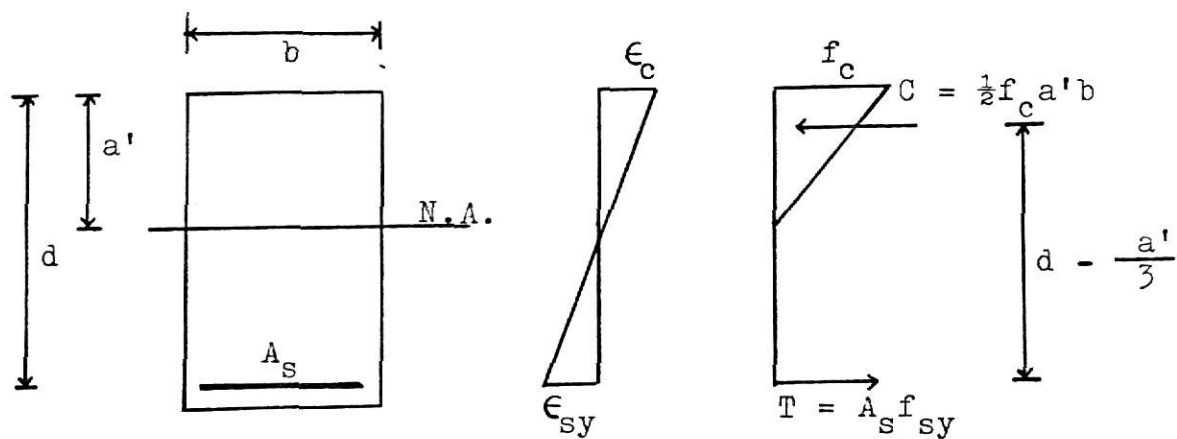
$$M_u = p f_{sy} b d^2 \left(1 - \frac{K_2}{K_1 K_3} \frac{p f_{sy}}{f'_c}\right) \quad \dots\dots\dots(3-3)$$

K_1, K_2, K_3 = parameters used to describe the shape of the stress-block in the concrete and are given¹ as:

$$K_1 = 0.94 - \frac{f'_c}{2600} ;$$



(a). At Ultimate



(b). At First Yield

Fig.3-1 Stress-Block in Reinforced Concrete Beam

$$K_2 = 0.50 - \frac{f'_c}{80000} ;$$

$$K_1 K_3 = \frac{3900 + 0.35f'_c}{3200 + f'_c} , \quad (K_1 K_3 = 0.8 \text{ at ultimate});$$

$$\frac{K_2}{K_1 K_3} = \frac{(1600 + 0.46f'_c - \frac{f'^2_c}{80000})}{3900 + 0.35f'_c} ,$$

$$\frac{K_2}{K_1 K_3} = 0.59 \text{ at ultimate};$$

$$\epsilon_{cu} = 0.004 - \frac{f'_c}{6500000} \quad \nless 0.003.$$

Introducing these values into Eq.3-3,

$$M_u = pf_{sy} bd^2 (1 - 0.59 \frac{pf_{sy}}{f'_c}) \dots \dots \dots (3-4)$$

The curvature at ultimate is

$$\phi_{max} = \phi_u = \frac{\epsilon_{su} + \epsilon_{cu}}{d}$$

$$\frac{a}{d} = \frac{\epsilon_{cu}}{\epsilon_{su} + \epsilon_{cu}} .$$

From Fig.3-1a, $C_u = T$,

$$\frac{a}{d} = \frac{A_s f_{sy}}{b d f'_c K_1 K_3} = \frac{pf_{sy}}{f'_c K_1 K_3} ,$$

$$\text{So,} \quad \epsilon_{su} = \epsilon_{cu} (\frac{d}{a} - 1) = \epsilon_{cu} (\frac{K_1 K_3 f'_c}{pf_{sy}} - 1).$$

Substituting this into the expression for $\phi_{\max}$

$$\begin{aligned}\phi_u = \phi_{\max} &= \frac{\epsilon_{cu} \left(\frac{K_1 K_3 f'_c}{p f_{sy}} - 1 \right) + \epsilon_{cu}}{d} \\ &= \frac{\epsilon_{cu} K_1 K_3 f'_c}{p d f_{sy}} \dots\dots\dots(3-5)\end{aligned}$$

From Fig.3-2⁴

$$\phi_u = \phi_y + \phi_i$$

$$\phi_x = \frac{d\theta}{dx} \quad , \quad d\theta = \phi_x \, dx \quad , \quad \theta = \int \phi_x \, dx \quad .$$

Therefore,

$$\begin{aligned}\theta_T &= \int_0^{L_p'} (\phi_{ux} - \phi_y) \, dx \div (\phi_u - \phi_y) L_p \\ &= \phi_i L_p \quad ,\end{aligned}$$

where $\phi_i = \phi_u - \phi_y \quad .$

Therefore,

$$\frac{\theta_T}{L_p} = \phi_u - \phi_y \quad ,$$

or, $\theta_T = L_p (\phi_u - \phi_y) \dots\dots\dots(3-6)$

and,

ϕ_y = the curvature at elastic limit (at first yielding).

$\phi_i = \frac{\theta_T}{L_p}$ = the increasing curvature at the plastic hinge and is equal to the total angle of rotation or the angle of discontinuity between the two elastic parts of the beam

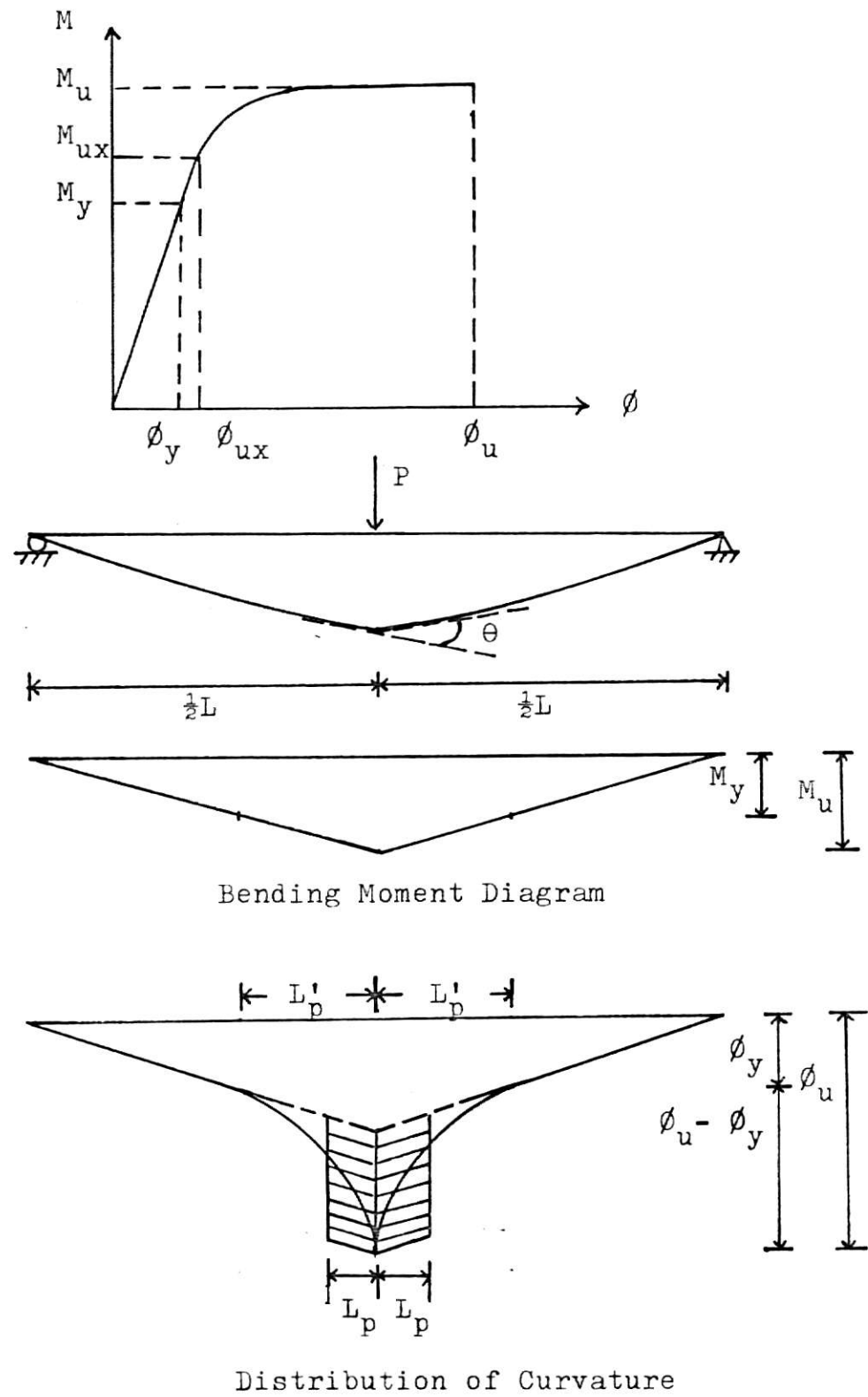


Fig.3-2 Bending Moment and Curvature

spread over the hinge length.

θ_T = the total available rotation on one side of a hinge.

L_p = the length of plastic hinge (over which plastic rotation occurs as a constant).

From Fig.3-3,

$$\phi_y = \frac{\epsilon_{cy}}{a'} ; \quad \phi_u = \frac{\epsilon_{cu} K_1 K_3 f'_c}{pdf_{sy}} ;$$

and,

$$\phi_i = \frac{\epsilon_{cu} K_1 K_3 f'_c}{pdf_{sy}} - \frac{\epsilon_{cy}}{a'} ,$$

$$\begin{aligned} \theta_T &= L_p \phi_i \\ &= \left(\frac{\epsilon_{cu} K_1 K_3 f'_c}{pdf_{sy}} - \frac{\epsilon_{cy}}{a'} \right) L_p . \end{aligned}$$

Therefore, as a' is increasing or decreasing, θ_T will behave in the same manner respectively.

$$a < a'$$

If $a' = a = k_u d$ (on safe side),

then

$$\frac{a'}{d} = \frac{a}{d} = \frac{pf_{sy}}{f'_c K_1 K_3} = \frac{\epsilon_{cy}}{\epsilon_{cy} + \epsilon_{sy}} ;$$

and,

$$\epsilon_{sy} = \epsilon_{cy} \left(\frac{K_1 K_3 f'_c}{pf_{sy}} - 1 \right) \dots \dots \dots (3-7a)$$

Therefore,

$$\phi_y = \frac{\epsilon_{cy}}{a'} = \frac{\epsilon_{sy} + \epsilon_{cy}}{d} = \frac{\epsilon_{cy} K_1 K_3 f'_c}{pdf_{sy}} \dots \dots (3-7)$$

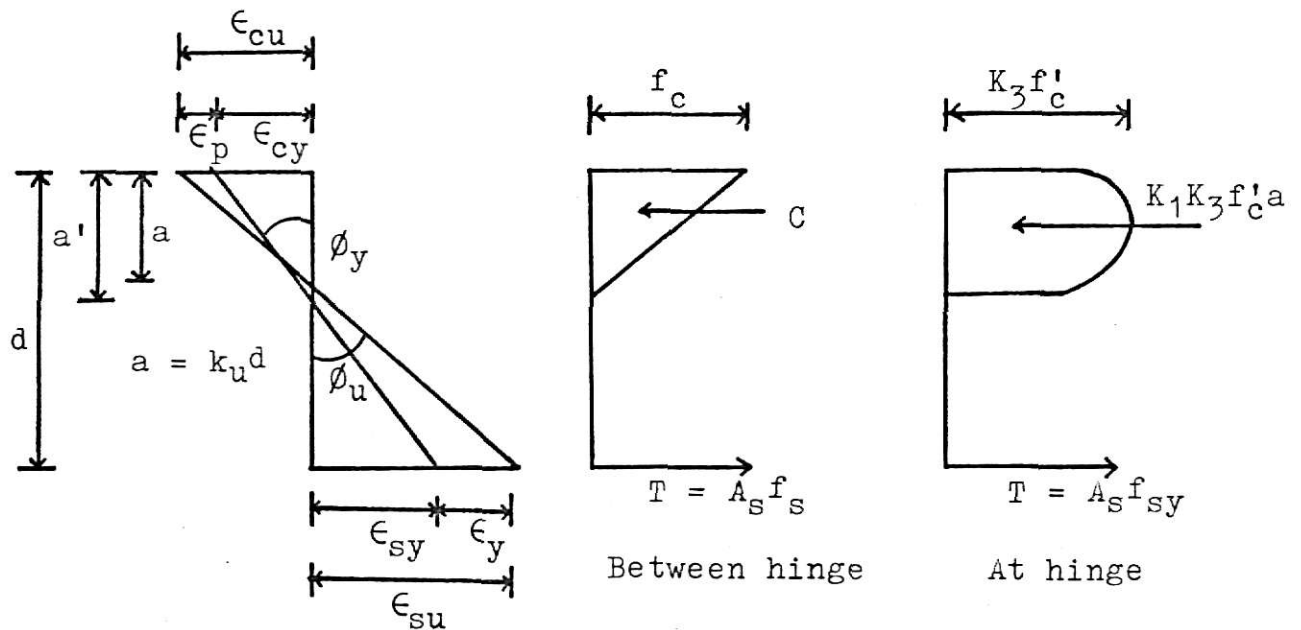


Fig.3-3 Stress-Strain Distribution at a Reinforced Concrete Section

So

$$\begin{aligned} \phi_i &= \frac{\epsilon_{cu} K_1 K_3 f'_c}{pdf_{sy}} - \frac{\epsilon_{cy} K_1 K_3 f'_c}{pdf_{sy}} \\ &= \frac{(\epsilon_{cu} - \epsilon_{cy}) K_1 K_3 f'_c}{pdf_{sy}} \dots\dots\dots(a) \end{aligned}$$

Substituting Eq.a into Eq.3-6

$$\begin{aligned} \theta_T &= L_p \phi_i \\ &= \left[\frac{(\epsilon_{cu} - \epsilon_{cy}) K_1 K_3 f'_c}{pdf_{sy}} \right] L_p \dots\dots\dots(3-8) \end{aligned}$$

This result is the same as the one used in Baker's method¹. Since in this method, the total rotation θ_T is

defined as the angle of discontinuity between the elastic parts of the member on either side of the hinge. So the total available rotation at a plastic hinge zone is $2\theta_T$.

ϵ_{cu} = the ultimate concrete strain.

ϵ_{cy} = the strain in the concrete at the extreme fiber at start of steel yielding.

ϵ_{sy} = the strain of steel at yield.

ϵ_{su} = the strain in the steel at the failure of the beam.

ϵ_y = the increment of strain in steel due to yield.

$\epsilon_p = (\epsilon_{cu} - \epsilon_{cy})$ = the increment of strain in the concrete at the compression edge which occurs due to increasing load between yielding of steel and crushing of concrete.

$L_p = R_1 R_2 R_3 (z/d)^{\frac{1}{4}} d$. (Reference 1)(3-9)

R_1 = a parameter for the influence of tensile steel.

R_2 = a parameter for the influence of axial load and equal to $(1 + (0.5)(P/P_w))$, where P is the ultimate axial load for the member (allowing for the bending moment when present) and P_w is the ultimate capacity of the member for axial load when no bending moment acts.

R_3 = a parameter for the influence of grade of concrete.

z = the distance from the critical section to the point of contraflexure.

Based on test results¹ and A. C. I. Research Committee⁵,
 R_1 , R_2 , and R_3 can be obtained as:

	A.C.I.	Test Results
R_1	0.80	0.7 for mild steel. 0.9 for cold worked steel.
R_2	1.25	
R_3		0.6 for concrete with a cylinder strength equal to 4700 psi. 0.9 for concrete with a cylinder strength equal to 1600 psi.

b). Calculating The Rotation Required at a Plastic Hinge θ_R .

θ_R is the angle of discontinuity between the elastic parts of the members spread over the hinge length, and it can be found by slope deflection. Since it is assumed that the inelastic behavior only occurs over the hinge, the change of slope is equal to the actual change of slope due to inelastic behavior. In the following analysis, the secondary stresses and the influence of axial and shear force are neglected, and it is assumed that the plastic hinge occurs at a point and that the segments between hinge-sections remain elastic. It is also assumed that the rotation at a hinge never stops once it has developed.

Fig.3-4a shows a uniform member AB of length, L, and elastic flexural rigidity, EI, which is subjected to an arbitrary loading and also to moment M_{AB} and M_{BA} at its ends, the sign conventions are shown in Fig.3-4b.

The rotation at the hinges can be given by the slope-deflection equations⁶.

$$\left. \begin{aligned} \theta_{AB} &= \frac{\Delta}{L} + \frac{L}{6EI} \left[2(M_{AB} - M_{AB}^F) - (M_{BA} - M_{BA}^F) \right] \\ \theta_{BA} &= \frac{\Delta}{L} + \frac{L}{6EI} \left[2(M_{BA} - M_{BA}^F) - (M_{AB} - M_{AB}^F) \right] \end{aligned} \right\} \dots (3-10)$$

M_{AB} , M_{BA} = the ultimate moment at ends A and B respectively.

M_{AB}^F, M_{BA}^F = the fixed end moments which would be produced at ends A and B respectively if the member was subjected to the same load but both ends were held clamped in position and direction.

An example will explain this equation more clearly. The example to be considered is the fixed-end beam of span $3L$, which is subjected to a load, w , at C, as shown in Fig.3-5. The beam is supposed to be of uniform section, with a fully plastic moment, M_p , and elastic flexural rigidity, EI . The collapse mechanism and the deflected form of the beam at collapse are shown in Fig.3-5, also. For no load acting on segments AC and BC, the fixed-end moments, M^F , are zero. From Eq.3-10, the rotations at sections A, B, and C can be obtained as:

$$\theta_{AC} = \theta_{CA} = \frac{\Delta}{2L} - \frac{M_p L}{3EI}$$

$$\theta_{CB} = \theta_{BC} = -\frac{\Delta}{L} + \frac{M_p L}{6EI}$$

In Fig.3-5, the required hinge rotation, θ_R , at each hinge is given in magnitude and sign, a positive hinge rotation being defined as one which causes tension in the bottom fiber of the section.

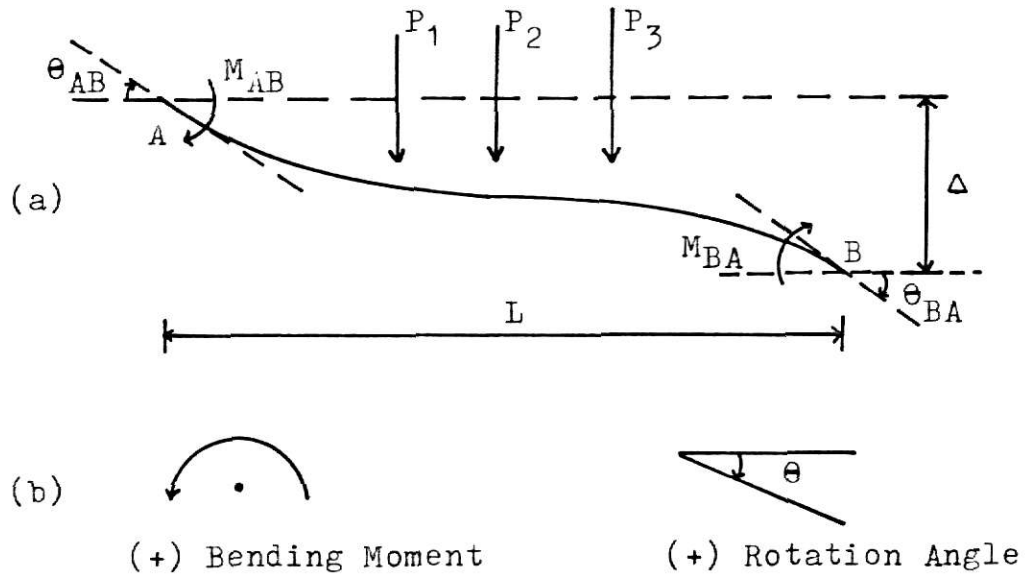


Fig.3-4 Definition of Terms and Sign Convention in Slope-Deflection Equation

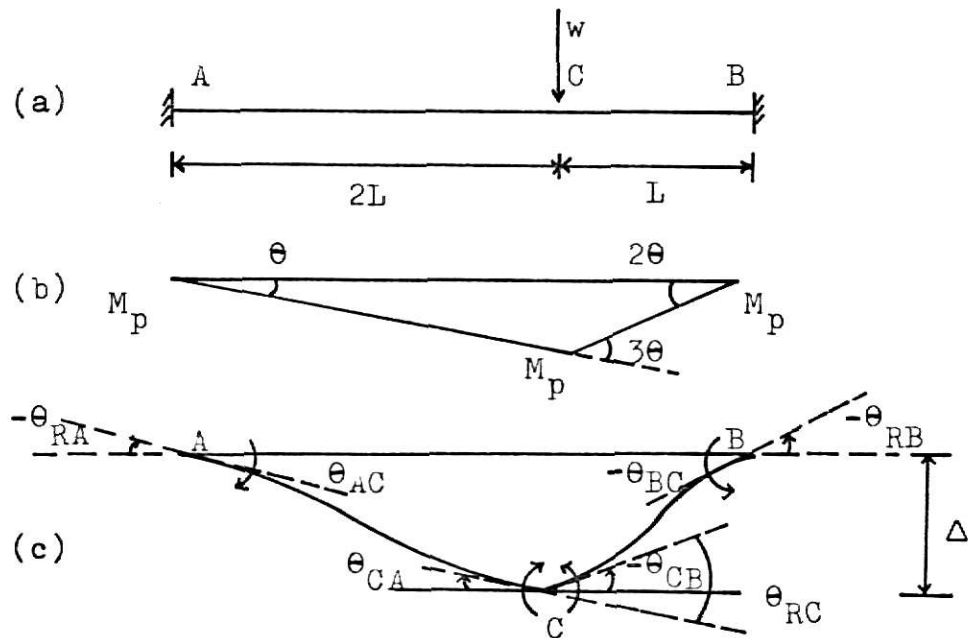


Fig.3-5 Fixed-Ended Beam with Concentrated Load

$$\begin{array}{lcl}
 \theta_{RA} : & - & \\
 \theta_{RA} = -\theta_{AC} = -\left(\frac{\Delta}{2L}\right) + \frac{M_p L}{3EI} & & \\
 \theta_{RB} : & - & \\
 \theta_{RB} = \theta_{BC} = -\left(\frac{\Delta}{L}\right) + \frac{M_p L}{6EI} & & \\
 \theta_{RC} : & + & \\
 \theta_{RC} = \theta_{CA} + (-\theta_{CB}) = \frac{3\Delta}{2L} - \frac{M_p L}{2EI} & &
 \end{array}
 \left. \vphantom{\begin{array}{l} \theta_{RA} \\ \theta_{RB} \\ \theta_{RC} \end{array}} \right\} \dots\dots\dots(a)$$

When the fully plastic moment has just been attained at the position where the last hinge forms, there will be no change in rotation occurring at this hinge just prior to collapse. It will be seen from Eq.a that if any one of the hinge rotations is assumed to be zero, as it would be at the last hinge, the value of Δ is determined and thus the other two hinge rotations can be found. In order to decide whether this assumption is correct or not, it is necessary to make use of the fact that the rotation at a hinge prior to collapse must be in the same sense as its rotation at collapse, since it is assumed that once a hinge has formed it will continue to rotate. And the deflection Δ will be larger than the deflection obtained from a wrong assumption. Referring to Fig.3-5 and Eq.a,

$$\theta_{RA} : - , \quad \theta_{RB} : - , \quad \theta_{RC} : +$$

For instance:

(1) Assume $\theta_{RC} = 0$

then $\Delta_1 = \frac{M_p L^2}{3EI}$

$$\theta_{RA} = -\frac{M_p L}{6EI} + \frac{M_p L}{3EI} = \frac{M_p L}{6EI} \quad (+)$$

$$\theta_{RB} = -\frac{M_p L}{3EI} + \frac{M_p L}{6EI} = -\frac{M_p L}{6EI} \quad (-)$$

$$\theta_{RC} = 0$$

NO GOOD

(2) Assume $\theta_{RB} = 0$

then $\Delta_2 = \frac{M_p L^2}{6EI}$

$$\theta_{RA} = \frac{-M_p L}{12EI} + \frac{M_p L}{3EI} = \frac{M_p L}{4EI} \quad (+)$$

$$\theta_{RB} = 0$$

$$\theta_{RC} = \frac{M_p L}{4EI} - \frac{M_p L}{2EI} = -\frac{M_p L}{4EI} \quad (-)$$

NO GOOD

(3) Assume $\theta_{RA} = 0$

$$\Delta_3 = \frac{2M_p L^2}{3EI}$$

$$\theta_{RA} = 0$$

$$\theta_{RB} = - \frac{2M_p L}{3EI} + \frac{M_p L}{6EI} = - \frac{M_p L}{2EI} \quad (-)$$

$$\theta_{RC} = \frac{M_p L}{EI} - \frac{M_p L}{2EI} = \frac{M_p L}{2EI} \quad (+)$$

ALL RIGHT

And $\Delta_1 < \Delta_2 < \Delta_3$

Since both θ_{RB} and θ_{RC} are of the correct sign, it confirms that this assumption, $\theta_{RA} = 0$, is a correct one.

In reinforced concrete an additional problem arises: the question of the value of E , the modulus of elasticity, and I , the moment of inertia of the section, to be used in Eq.3-10. At failure a reinforced concrete beam is usually not in a uniform condition along its length. Part of it is plastic and badly cracked, and so it is impossible to get a single accurate value of EI which would apply to the whole length of the member. If it is assumed that the whole beam is elastic, but in the cracked condition, then

$$I = (1/3)ba^3 + nA_s(d - a)^2 \quad \dots\dots\dots(3-11)$$

n = the elastic modular ratio E_s/E_c .

a = the neutral axis depth for the elastic condition, as shown in Fig.3-1a.

An approximate value of E_c^6 is given by

$$E_c = (57000) \sqrt{f'_c} \quad \dots\dots\dots(3-12)$$

and then the value of EI to be used in Eq.3-10 is

$$EI = 57000 \sqrt{f'_c} \left[(1/3)ba^3 + nA_s(d - a)^2 \right] \dots\dots\dots(3-13)$$

A numerical example of calculating θ_R is now given.

The problem is shown in Fig.3-6.

$$- M_{AB}^F = M_{CB}^F = \frac{wL^2}{12} + \frac{P(80)(40)^2}{L^2} = 584000 \text{ lb-in}$$

$$- M_{BC}^F = M_{BA}^F = \frac{wL^2}{12} + \frac{P(80)^2(40)}{L^2} = 1015000 \text{ lb-in}$$

$$M_{AB} = M_{BA} = - M_p = - 2350000 \text{ lb-in}$$

$$M_{BC} = M_{CB} = M_p = 2350000 \text{ lb-in}$$

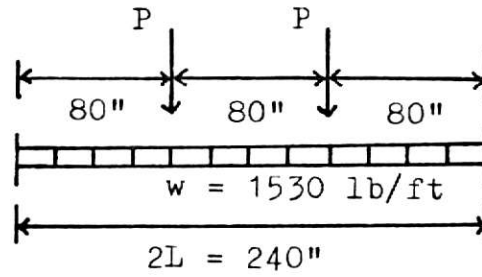
then

$$\begin{aligned} \theta_{AB} &= \frac{\Delta}{120} + \frac{20}{EI} \left[2(-2350000 + 584000) - (-2350000 - 1015000) \right] \\ &= \frac{\Delta}{120} + \frac{20}{EI} (-167000) \end{aligned}$$

$$\begin{aligned} \theta_{BA} &= \frac{\Delta}{120} + \frac{20}{EI} \left[2(-2350000 - 1015000) - (-2350000 + 584000) \right] \\ &= \frac{\Delta}{120} + \frac{20}{EI} (-4964000) \end{aligned}$$

$$\begin{aligned} \theta_{BC} &= \frac{-\Delta}{120} + \frac{20}{EI} \left[2(2350000 + 1015000) - (2350000 - 584000) \right] \\ &= \frac{-\Delta}{120} + \frac{20}{EI} (4964000) \end{aligned}$$

$$\begin{aligned} \theta_{CB} &= \frac{-\Delta}{120} + \frac{20}{EI} \left[2(2350000 - 584000) - (2350000 + 1015000) \right] \\ &= \frac{-\Delta}{120} + \frac{20}{EI} (167000) \end{aligned}$$



$EI = 27000000000 \text{ lb-in}^2$
 uniform section: $14'' \times 26''$
 $P = 45800 \text{ lb}$
 $M_p = 2350000 \text{ lb-in}$

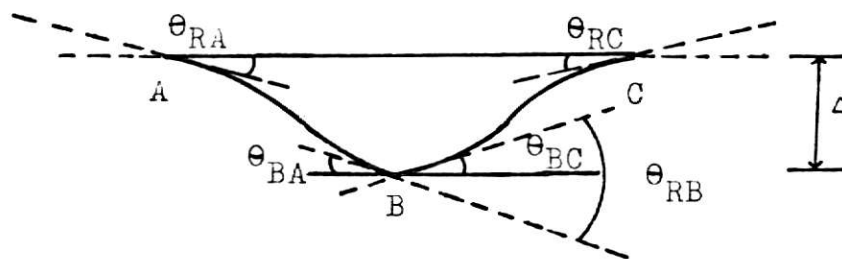
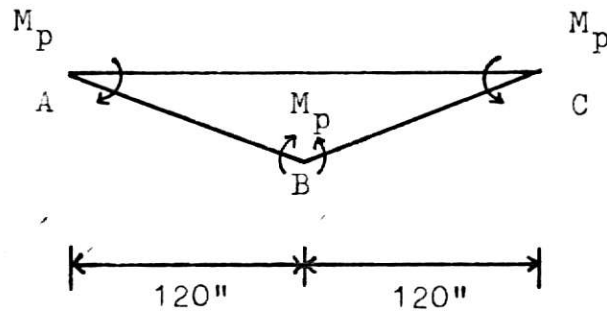


Fig.3-6 Fixed-Ended Beam with Uniform and Concentrated Load

$$\theta_{RA} = -\theta_{AB} = \frac{-\Delta}{120} + \frac{20}{EI} (167000) \quad (-)$$

$$\theta_{RB} = \theta_{BA} - \theta_{BC} = \frac{\Delta}{60} - \frac{20}{EI} (9928000) \quad (+)$$

$$\theta_{RC} = \theta_{CB} = \frac{-\Delta}{120} + \frac{20}{EI} (167000) \quad (-)$$

(1). Assume $\theta_{RA} = 0$ ($\theta_{RC} = 0$)

$$\frac{\Delta}{120} = \frac{20}{EI} (167000)$$

$$\theta_{RA} = 0$$

$$\theta_{RB} = \frac{20}{EI} (334000) - \frac{20}{EI} (9928000) < 0 \quad (-)$$

$$\theta_{RC} = 0$$

NO GOOD

(2). Assume $\theta_{RB} = 0$

$$\frac{\Delta}{60} = \frac{20}{EI} (9928000)$$

$$\theta_{RA} = -\frac{20}{EI} (4964000) + \frac{20}{EI} (167000) < 0 \quad (-)$$

$$\theta_{RB} = 0$$

$$\theta_{RC} = -\frac{20}{EI} (4964000) + \frac{20}{EI} (167000) < 0 \quad (-)$$

ALL RIGHT

$$\begin{aligned}\theta_{RA} = \theta_{RC} &= - \frac{20}{EI} (4964000) + \frac{20}{EI} (167000) \\ &= \frac{-(20)(4797000)}{27000000000} = - 0.00355 \text{ rad. } \# \end{aligned}$$

4. WAYS TO INCREASE THE ROTATION CAPACITY OF A SECTION

In limit design of reinforced concrete structures, the basic requirement is rotation compatibility of the plastic hinges. Improved rotation capacity of the hinges permits better redistribution of moment. In order to reach this goal, many methods have been developed. From these, equations for rotation capacity θ_T , as follows:

$$\begin{aligned}\theta_T &= \frac{(\epsilon_{cu} - \epsilon_{cy})K_1K_3f'_c}{\rho d f_{sy}} \cdot L_p \\ &= \frac{(\epsilon_{cu} - \epsilon_{cy})K_1K_3f'_c b}{A_s f_{sy}} \cdot L_p \dots\dots\dots(3-8)\end{aligned}$$

$$L_p = R_1 R_2 R_3 (z/d)^{\frac{1}{4}} d \dots\dots\dots(3-9)$$

and the equation presented by W.G. Corly⁸,

$$\epsilon_{cu} = 0.003 + 0.02(b/z) + \left(\frac{P'' f'_{sy}}{20}\right)^2 \dots\dots\dots(4-1)$$

b = the width of the cross section. (unit: in.)

f'_{sy} = the yield point strength of stirrup steel.

(usually same as main reinforcement) (unit: ksi.)

P'' = the ratio of the volume of the binding reinforcement (one stirrup plus compression steel) to the volume of the concrete bound (area enclosed by one stirrup multiplied by the stirrup spacing).

It is shown that the total rotation capacity, θ_T , is proportional to the concrete strength, effective beam depth, amount of binding steel and the ductility of concrete.

a). Test Results from N.H. Burn and C.P. Siess⁹

In tests on beams as shown in Fig.4-1 and Table 4-1, the effect of the depth of a section and the effect of the compression steel are indicated as:

(1). Effect of Depth --- The effect of the depth may be shown by the load-deflection curve in Fig.4-2. The beam in each group has the same amount of tension and compression steel and substantially the same concrete strength and steel yield strength. The only variable is the depth of the beam and the ratio of reinforcement ($p = A_s/bd$, $p' = A'_s/bd$). For each group of curves the yield load increases with depth and the increase in load after yield is a result of strain hardening of the tension steel. The increase in depth results in greater differences between yield and crushing load as well as greater differences between yield and ultimate load. The deflection at crushing is practically unaffected by depth, but the deflection at ultimate load shows a slight variation with depth. The beam with equal amounts of tension and compression steel ($p/p' = 1.0$), shows a slight decrease in ultimate deflection as the depth increased, in this case undoubtedly reducing the ductility. The reason for this decrease in ductility

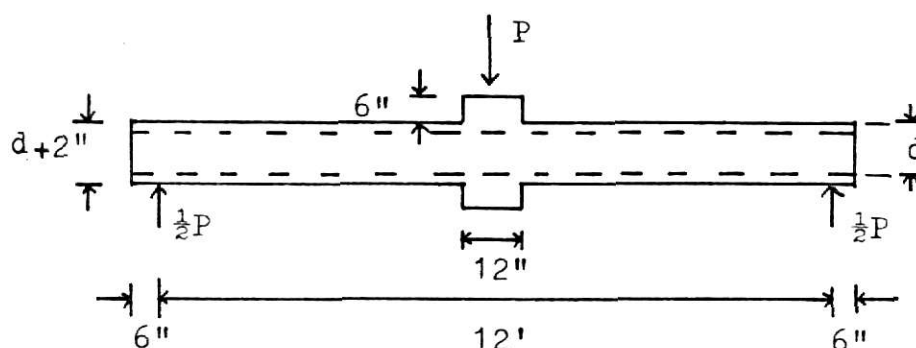


Fig.4-1 Test Specimens
(For the Test of V.H. Burn and C.P. Siess)

TABLE 4-1. Properties of Beams
(for the test of N. H. Burn and C. P. Siess)

Beam	f'_c (psi)	A_s		f_{sy} (ksi)		d, in inches	Stirrup size and spacing, in inches
		Tension	Compression	Tension	Compression		
J-1	4930	2-#8	--	47.6	--	10.0	#3 at 6
J-11	4110	2-#8	--	46.9	--	10.0	#3 at 6
J-2	4080	2-#8	2-#6	48.0	48.6	10.0	#3 at 6
J-8	4680	2-#8	2-#8	45.4	45.5	10.0	#3 at 6
J-17	3900	2-#8	2-#8	46.9	46.8	10.0	#3 at 6
J-10	3590	2-#8	--	45.1	--	14.0	#3 at 6
J-14	4500	2-#8	2-#6	47.1	50.0	14.0	#3 at 6
J-13	4800	2-#8	2-#8	45.6	46.0	14.0	#3 at 6
J-4	4820	2-#8	--	44.9	--	18.0	#3 at 6
J-9	4190	2-#8	--	47.0	--	18.0	#3 at 6
J-5	5000	2-#8	2-#6	45.1	48.9	18.0	#3 at 6
J-6	5160	2-#8	2-#8	46.2	46.4	18.0	#3 at 6

b = 8 inches for all beams except as noted.

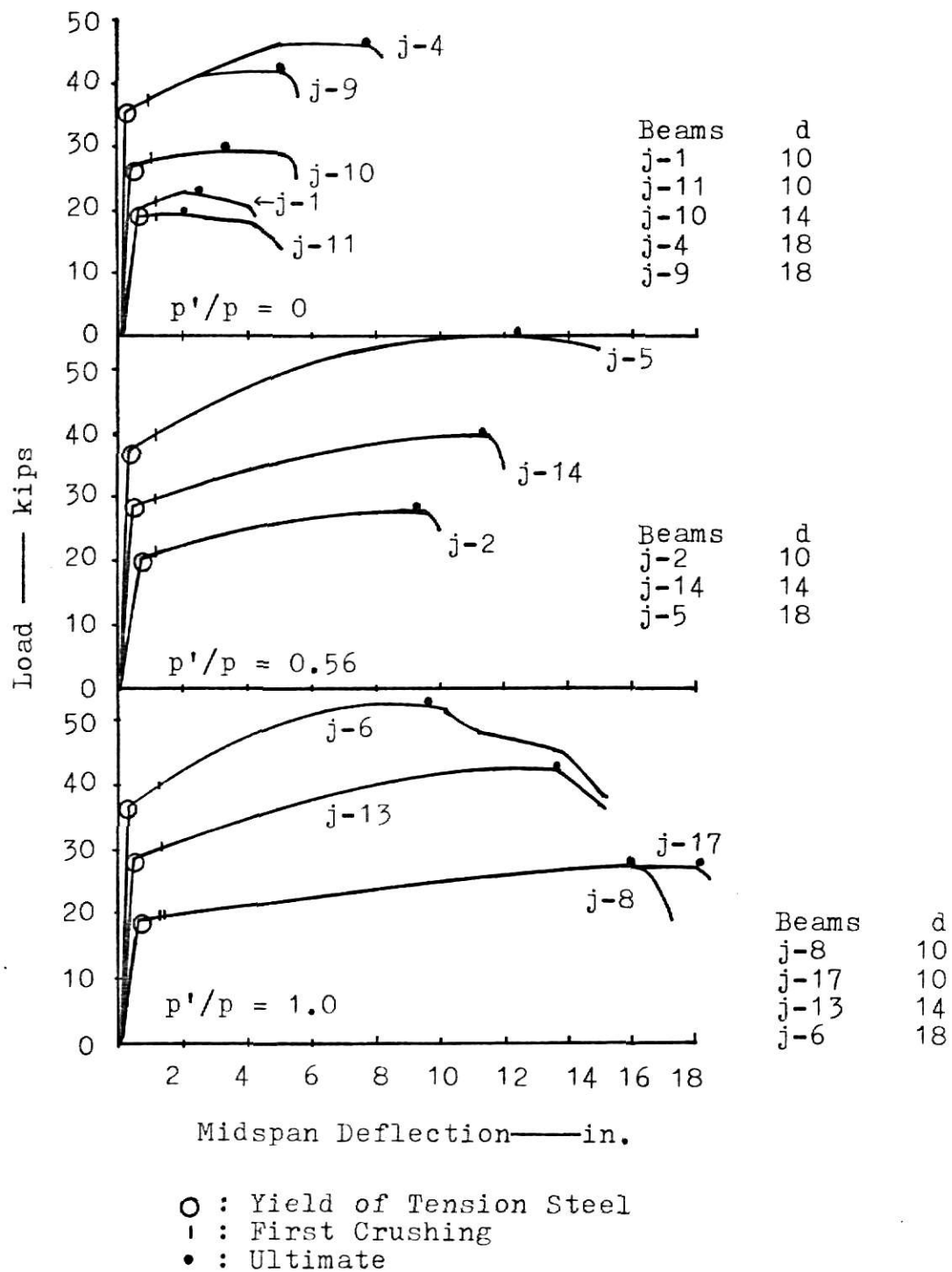


Fig.4-2 Effect of Depth for Beams with Various p'/p
(For the Test of V.H. Burn and C.P. Siess)

is the shearing movement associated with the flexural failure of the beam.

(2). Effect of Compression Steel --- The effect of compression steel may be shown by the load-deflection curve in Fig.4-3. The most obvious trend is the increase in ductility that results from the addition of compression steel. The deflection at crushing is still practically unaffected by using additional compression steel. But the allowable deformation capacity before failure shows a marked increase with the addition of compression steel.

For a beam without compression steel, the internal compressive force is carried by the confined concrete core. When this confined core can no longer carry the compressive force, then failure occurs and results in a rapid decrease in load-carrying capacity as deflection increases. The crushing of the concrete between stirrups shows the confining effect of the closed stirrups. A closer spacing of the closed stirrups in the region of greatest curvature will have resulted in greater deformation capacity before failure due to the increased confinement of the concrete core. On the other hand, for a beam with compression steel, the internal compressive force is carried by both the concrete and compression steel. As crushing of concrete progressed, the reinforcement is forced to take a large proportion of the total compressive force until the steel yielded. At ultimate,

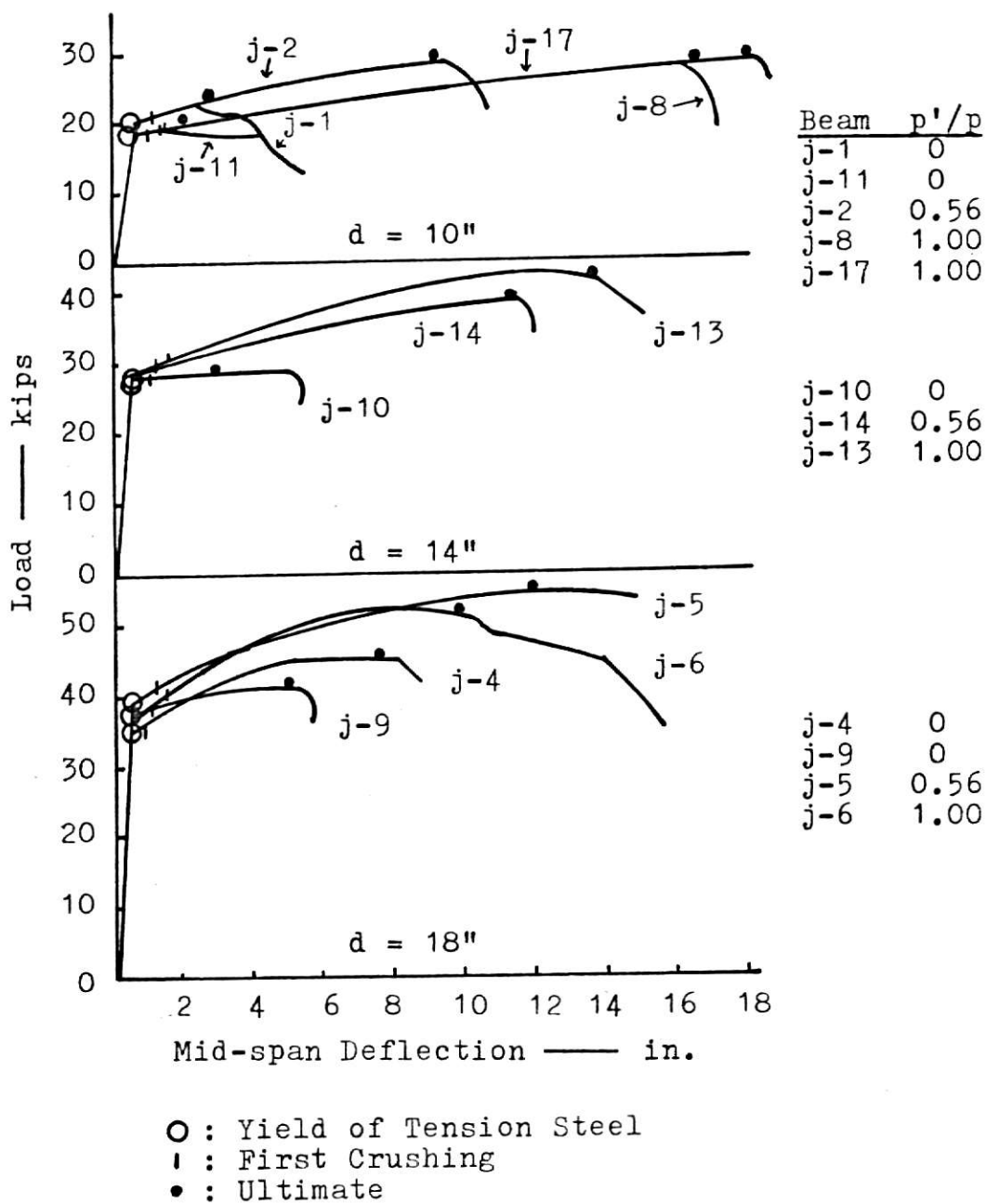


Fig.4-3 Effect of Compression Steel
(For the Test of V.H. Burn and C.P. Siess)

the strain of the compression steel is in the yield range, and only the confined concrete core is left to act in carrying additional compressive force. In some cases, the failure mode involves buckling of the compression steel. The buckling of the compression steel transfers more load to the concrete core than it can carry, resulting in crushing of the concrete and a rapid decrease in load as buckling continued.

b). Test Results from G.D. Base and J.B. Read²

In their test, they showed the effectiveness of helical binding in the compression zone of concrete beams. Thirteen of the sixteen beams tested were reinforced concrete beams and the other three were prestressed concrete beams*. All of these beams had a rectangular section (6" x 11") and were 120in. long. They were tested by single point loading at mid-span on a 110in. span. Table 4-2 and Fig.4-4 give the details of the beams. The results of these tests can be described in three parts.

(1). Beams 1---3 (under-reinforced section)

Each of these beams failed primarily because of yielding of the steel reinforcement. In Fig.4-5 the moment-rotation curves for these beams are given with applied moment plotted as a proportion of M_u . It can be seen that all three beams

* These 3 prestressed concrete beams aren't discussed here.

TABLE 4-2. Properties of Beams
(for the test of G. D. Base and J. B. Read)

Beam No.	Cube strength, psi	Type of helix	Weight of helices, lb per ft	Weight of stirrups, lb per ft	Total weight of secondary reinforcement, lb per ft	
(U)	1	5140	Light	0.814	0.624 *	1.438
	2	5060	Heavy	2.9	0.624 *	3.524
	3	4110	---	---	0.624 *	0.624
(B)	4	4650	Light	0.814	0.624 *	1.438
	5	6000	Heavy	2.9	0.624 *	3.524
	6	4960	---	---	0.624 *	0.624
	7	4300	---	---	0.624 *	0.624
	8	4200	---	---	2.49 +	2.49
(O)	9	3500	Light	0.814	0.624 *	1.438
	10	4300	---	---	0.624 *	0.624
	11	4560	---	---	2.49 +	2.49
	16	5300	Light	0.814	0.624 *	1.438
	17	6500	---	---	2.49 +	2.49

All stirrups 1/4 in. diameter mild steel, rectangular in shape.

Light helix, 3/16 in. diameter mild steel, 2 in. pitch, 3-1/4 in. over-all diameter, 4 ft long.

Heavy helix, 1/4 in. diameter mild steel, 1 in. pitch, 3-1/4 in. over-all diameter, 4 ft long.

U = under-reinforced section

B = balanced Section

O = over-reinforced section

* The spacing of stirrups is 8" center to center.

+ The spacing of stirrups is 2" center to center.

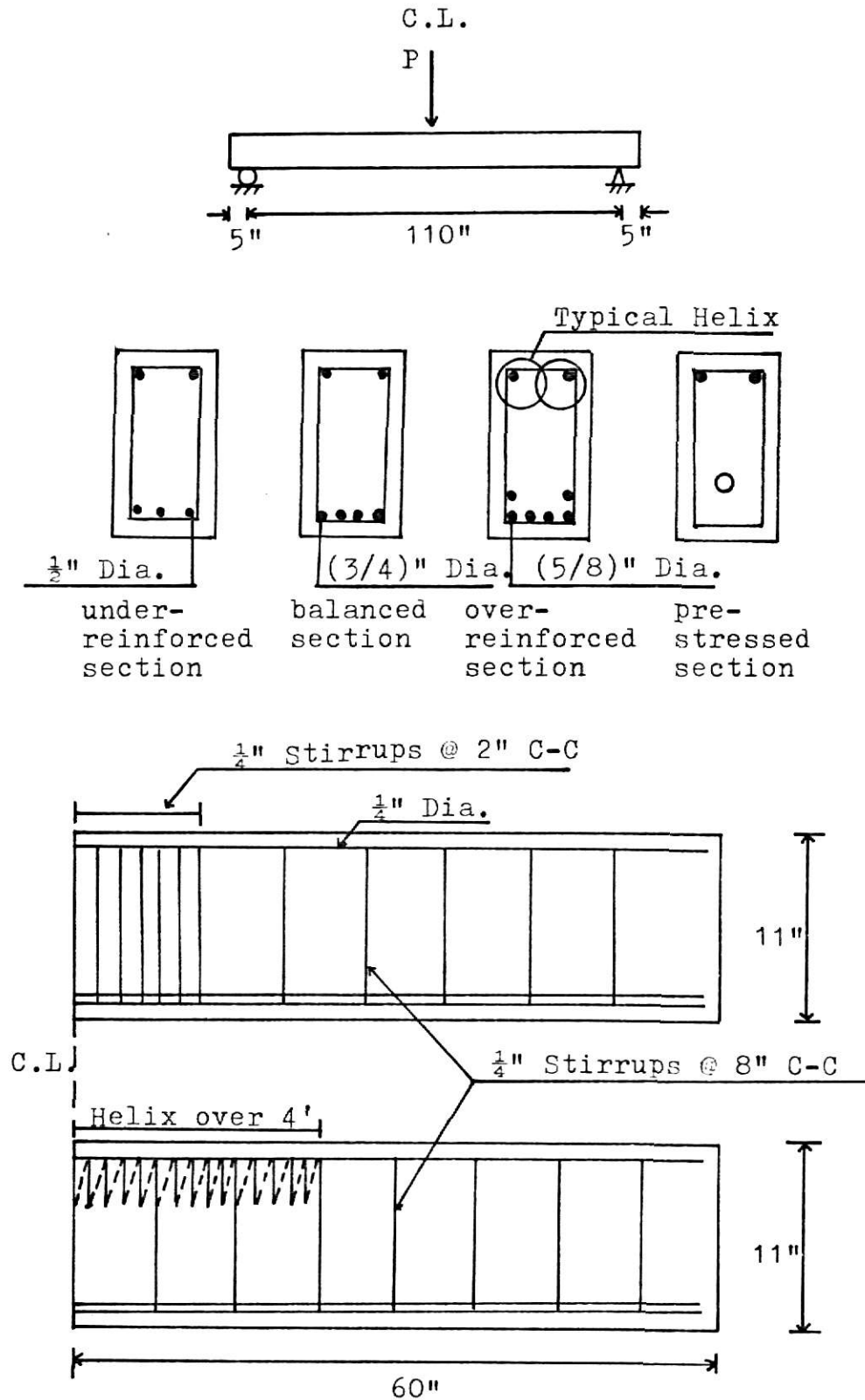


Fig.4-4 Typical Details of the Test Beam and Loading
(For the Test of G.D. Base and J.B. Read)

had a large rotation capacity without significant decrease in the resisting moment of a plastic hinge. At 0.2 radians total rotation (change of slope from end to end of beam), the resisting moment of the beam was still not less than 95% of M_u . The low cube strength, 4110 psi., of beam 3 does not affect the validity of the results because it was high enough to insure yielding of the reinforcement. (Table 4-2 and Fig.4-5).

(2) Beams 4---8 (balanced section)

In this group, the beams failed when the main reinforcement reached the yield point. Almost immediately after the steel yielded, the outer fiber of the compression side of the concrete started to crush. In beams without binding in the compression zone, the cracks formed right across the width of the beams, but in beams with considerable binding (a helix or closely spaced stirrups) the cracks were only in the concrete outside the binding. From Fig.4-6, it is found that the crushing of the concrete in the two beams with stirrups at 8" center to center only (Beams 6 and 7) was accompanied by a rapid decrease in the resisting moment of the section. In the beam with stirrups at 2" centers (Beam 8), the binding effect of the stirrups was completely effective until the total rotation was approximately 0.08 radians. The stirrups then deformed outwards and allowed

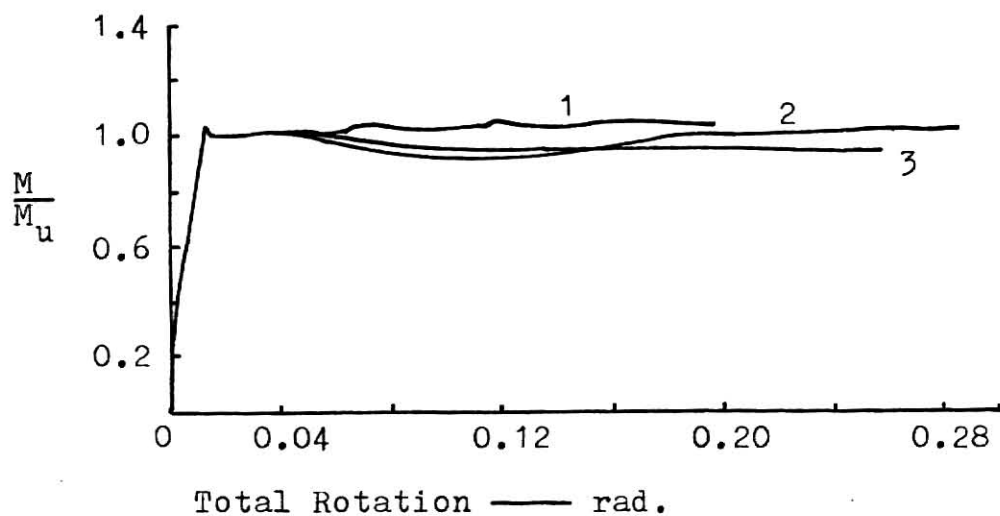


Fig.4-5 Moment Rotation Curves for Under-Reinforced Section
(For the Test of G.D. Base and J.B. Read)

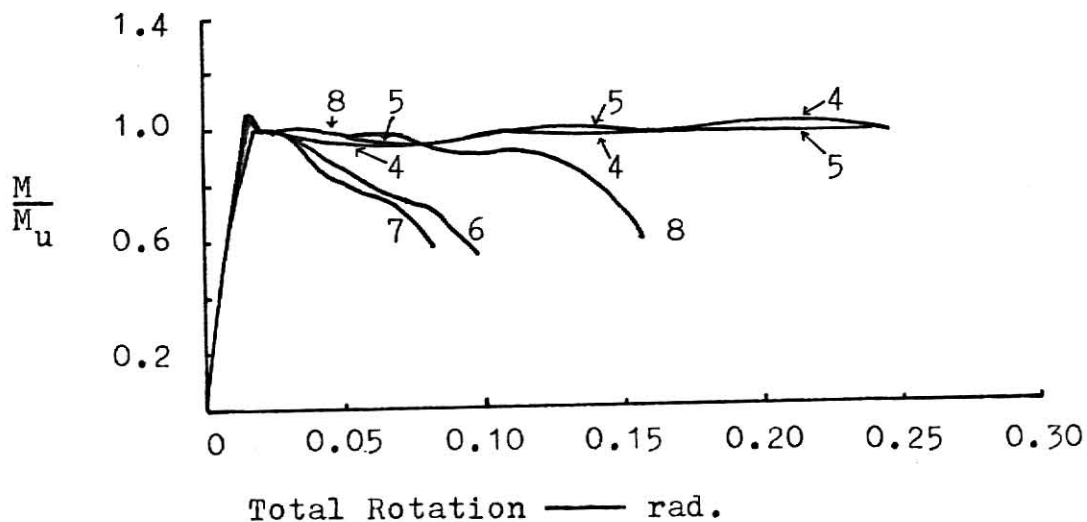


Fig.4-6 Moment Rotation Curves for Balanced Section
(For the Test of G.D. Base and J.B. Read)

the concrete within them to crush and the moment of resistance to fall quite rapidly. The two beams with helical binding in the compression zone (Beams 4 and 5) had the same type of moment-rotation relation that the under-reinforced concrete beams had. When total rotation was 0.24 radians the resisting moment had not decreased significantly from M_u . (Fig.4-6).

The light helices worked just as well as the heavy helices despite the fact that the beam with light helices had lower cube strength. The closely spaced stirrups produced a good binding effect, but the weight of this secondary reinforcement per foot was considerably more than in the beam with light helices.

(3). Beams 9, 10, 11, 16, and 17 (over-reinforced section)

In this group the stronger reinforcement caused the concrete to crush before the elastic range of the steel had been exceeded. The test results for this group are shown in Fig.4-7. The cube strength of Beam 9 was significantly less than the cube strength of Beams 10 and 11, but the comparison between Beams 9 and 10 gives a clear indication of its improved rotation capacity. The results from Beam 11 were indecisive since the closely spaced stirrups did not extend as far as the helices in Beam 9. Beam 16 with helical binding provided a considerable increase in moment after the cover

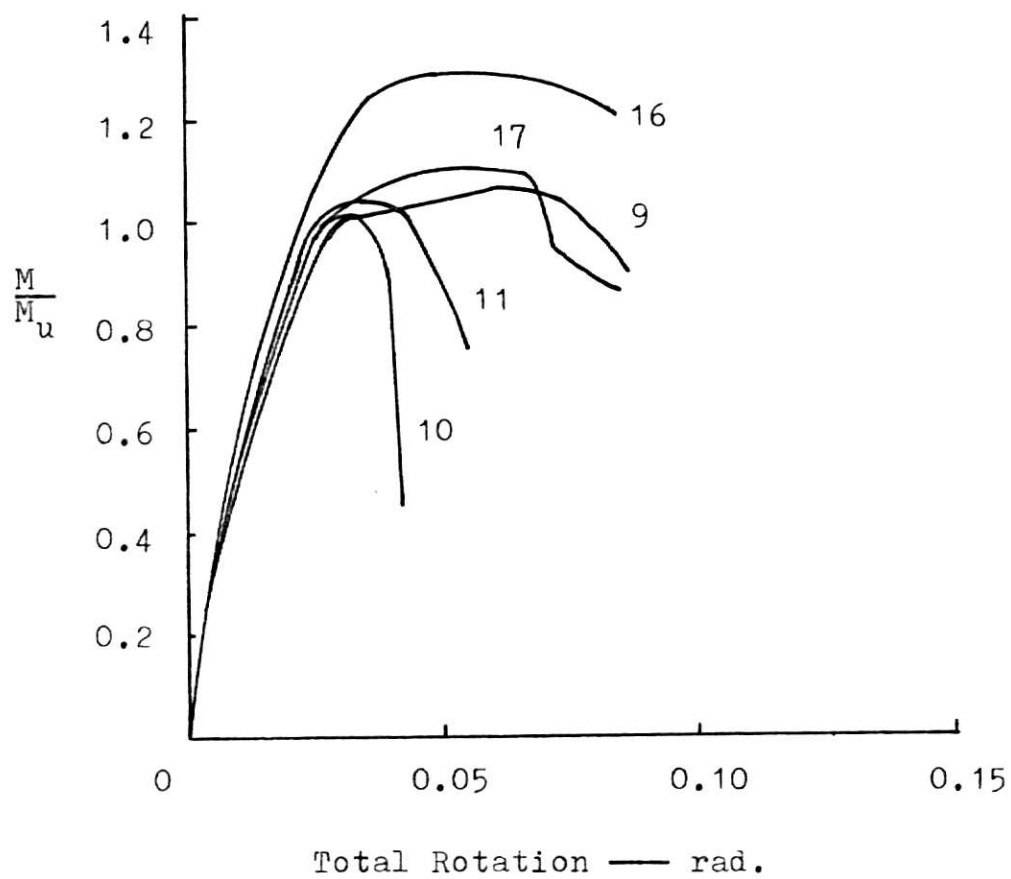


Fig.4-7 Moment Rotation Curves for Over-Reinforced
Section
(For the Test of G.D. Base and J.B. Read)

to the helix started to spall, while Beam 17, with stirrups at 2" centers, gave a small increase. Both increases were due to the ability of the bound concrete in the compression zone to carry stresses greater than unbound concrete.

In the elastic range, the different form of secondary reinforcement caused little or no difference in behavior. The under-reinforced concrete beams probably have adequate plasticity prior to failure and should not need any special secondary reinforcement. The balanced section reinforced concrete beams will fail in a rather brittle manner unless a suitable secondary reinforcement is used in the section. Helical binding appeared to be more economical than closely spaced stirrups but both methods can improve the moment-rotation relationship greatly. The over-reinforced concrete beams without special secondary reinforcement failed in a very brittle manner and the failure was terminated by shear collapse. Helical binding can delay the shear collapse to some extent. Closely spaced stirrups can completely prevent the shear collapse and allow crushing to occur in the compression zone. A combined reinforcement would appear to be necessary to produce an ideal moment-rotation relationship.

c). Test Results from E.G. Nawy, R.F. Danesi and
J.J. Grosko¹⁰

In their tests, they showed the effect of rectangular

spiral binders in reinforced concrete beams. Two series were tested: (A). tension hinges in beams with 84" span subjected to transverse load only, and (B). compression hinges in beams with 80" span subjected to both transverse and axial load. Only beams with tension hinges in series (A) are reviewed here. It is found that rectangular spiral binders are effective in increasing the rotation capacity of a tension plastic hinge.

Eight simply-supported beams having a 7' span were loaded transversely at midspan. The cross section of each beam was 8" x 12", having an effective depth of 11". All the beams were under-reinforced and had the same amounts of tension and compression steel. The only variable parameter was the percentage of rectangular binding spirals p'' . This is the ratio of the volume of spiral steel to the volume of the bound concrete (similar to p''). The details of each beam are shown in Fig.4-8 and Table 4-3. It is necessary to expand the confining spirals area beyond the expected plastic hinge zone.

The values of the elastic total rotation, θ_y , denoting the rotation at yield of the tension steel seem to be the same in all the beams, as shown in Table 4-4. This indicates that there is no significant effect of the spiral binders on the concrete beam in the elastic range. As the compression steel at the top starts to yield, there is a significant

TABLE 4-3. Properties of Beams
(for the test of E. G. Nawy, R. F. Danesi and J. J. Grosko)

Beam No.	Tension steel		Compression steel		f'_c , psi	Spiral p''' , percent	Spiral pitch, in.
	Bar No.	f_y , psi	Bar No.	f_y , psi			
P9G1	2#8	47,500	2#5	65,500	4870	0	0
P10G2	2#8	65,500	2#5	65,500	5150	1.0	2.0
P11G3	2#8	47,500	2#5	65,500	5090	1.0	2.0
P3G4	2#8	65,500	2#5	65,500	5440	2.0	1.0
P4G5	2#8	65,500	2#5	65,500	5670	2.0	1.0
P14G6	2#8	47,500	2#5	65,500	5640	3.0	0.67
P5G7	2#8	65,500	2#5	65,500	5010	4.0	0.5
P6G8	2#8	65,500	2#5	65,500	4710	4.0	0.5

Note: Spiral reinforcement $f_y = 47,800$ psi at 0.2 percent offset

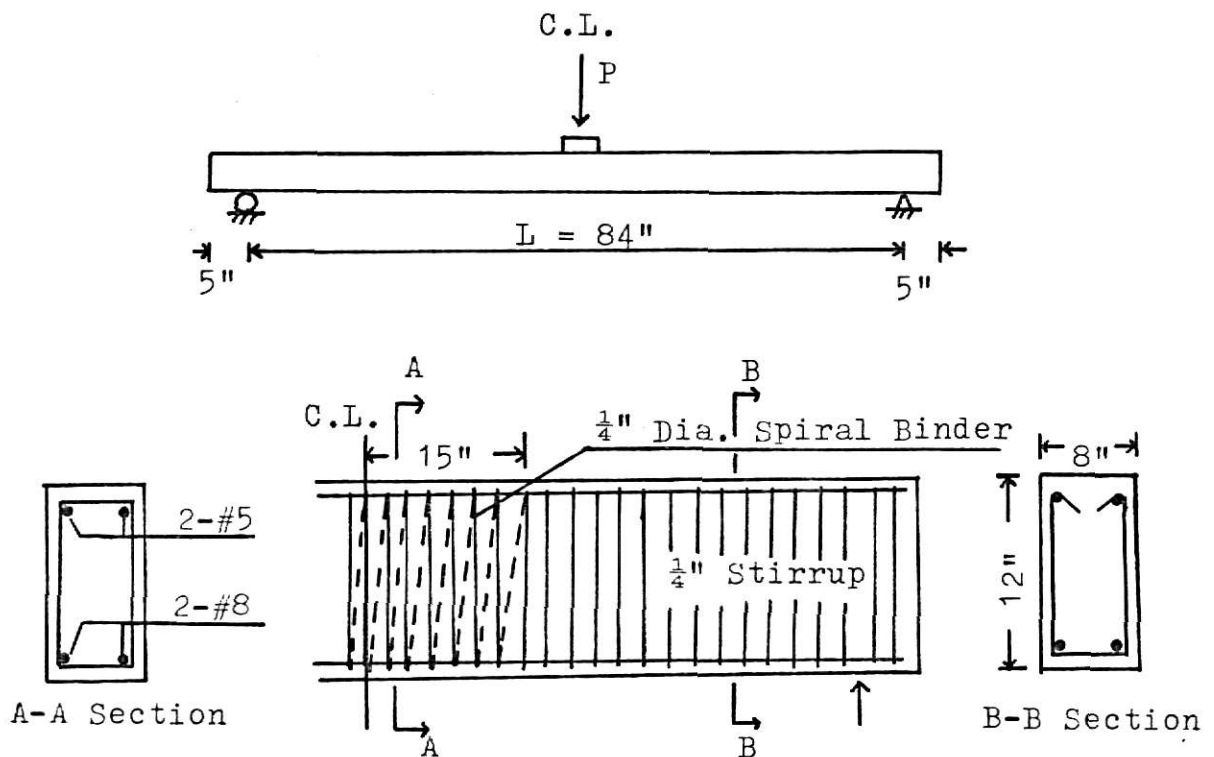


Fig.4-8 Details of the Test Beams
(For the Test of E.G.Nawy and R.F. Danesi
and J.J. Grosko)

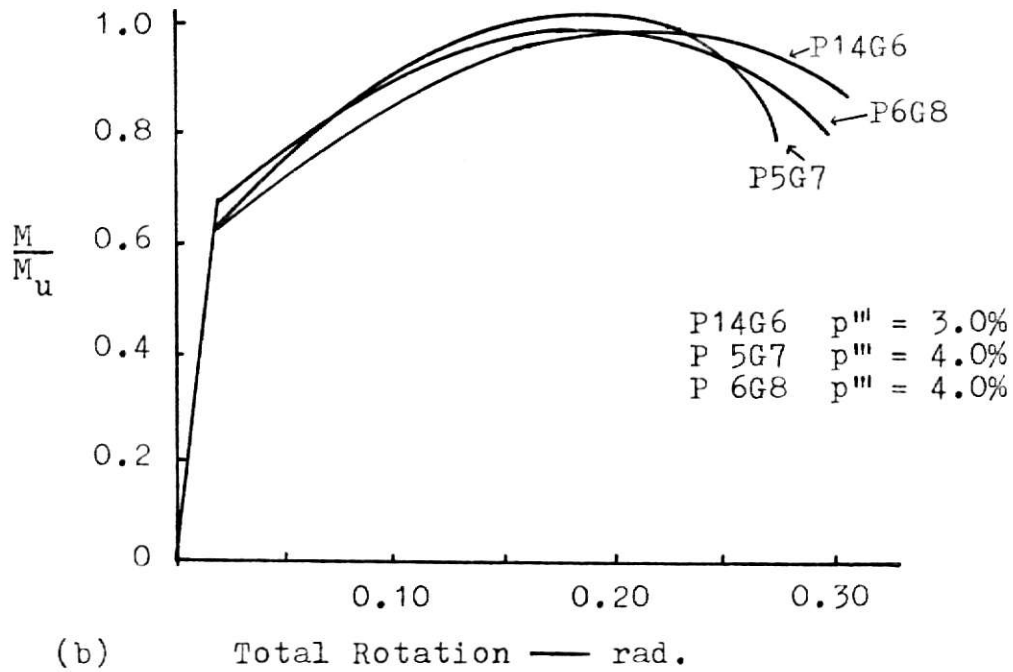
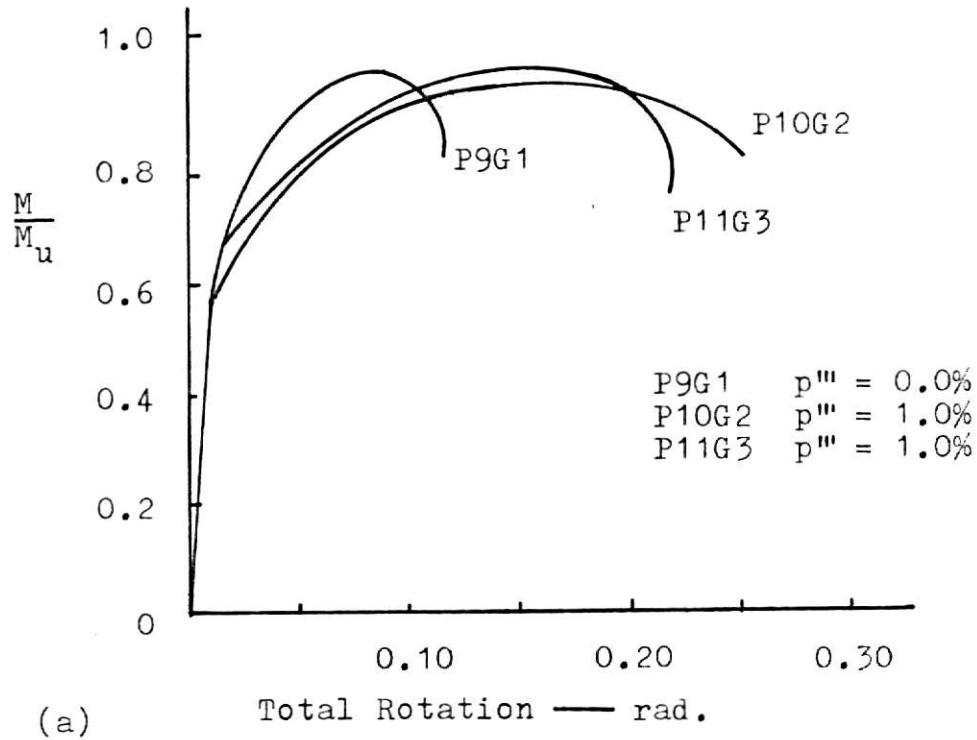


Fig.4-9 Moment-Rotation Relationship
(For the Test of E.G. Nawy and R.F. Danesi
and J.J. Grosko)

TABLE 4-4. Elastic and Plastic Rotations of Beams Subjected to Transverse Load Only

Beam No.	Spiral p''' , percent	M_y/M_u	θ_y , radians	θ_{yp} , radians	θ_p , radians
P9G1	0	0.803	0.018	0.083	0.065
P10G2	1.0	0.700	0.016	0.141	0.125
P11G3	1.0	0.685	0.020	0.131	0.111
P3G4	2.0	0.682	0.021	0.145	0.134
P4G5	2.0	0.725	0.019	0.155	0.136
P14G6	3.0	0.630	0.020	0.185	0.165
P5G7	4.0	0.648	0.018	0.174	0.156
P6G8	4.0	0.660	0.021	0.180	0.159

θ = total rotation in radians over whole span at linear limit

θ_y = total rotation at plastic limit

θ_p = magnitude of plastic rotation = $\theta_{yp} - \theta_y$

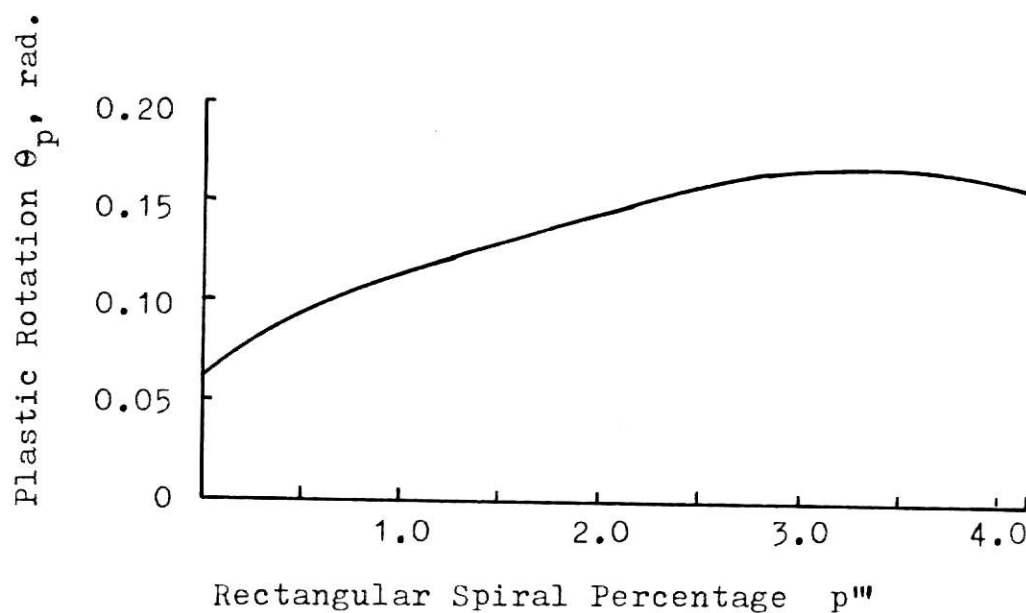


Fig.4-10 Plastic Rotation Versus p'''
(For the Test of E.G. Nawy and R.F. Danesi
and J.J. Grosko)

difference in the total rotation. For example, when p''' is 0%, θ_{yp} is 0.083 radians, but when p''' is 4%, θ_{yp} is 0.180 radians. This increase in ductility can be explained by the use of spirals. Plots of moment ratio versus steel ratio over the whole span are given in Fig.4-9a and b. A distinct trend in the effect of the p''' percentage on the degree of confinement is shown in Fig.4-10 and Table 4-4. It also shows that a maximum effective degree of confinement can be expected at an approximate value of $p''' = 3\%$. Beyond this value, θ_p , can not be increased. Reduction in the effectiveness of confinement for very high p''' values can be explained by stress concentration, which induces cracking.

d). Test Results from S.P. Shah and B.V. Rangan¹

In their tests, they showed that the ductility of concrete can be improved by using closely spaced stirrups. To explain this effect, beams with three different stirrup spacings, 1", 2", and 4", all having the same volume of steel, 0.5%, were tested. In order to keep the volume of steel constant, the stirrups were tied together at higher spacing. The specimens used for this study were 10" long, 2" deep, 2" wide, and had no longitudinal reinforcement. The results are shown in Fig.4-11 and Table 4-5. They made the following conclusions:

- (1). Decreasing the spacing of stirrups does not influence the ultimate strength but considerably increases the ductility of concrete.
- (2). Reducing the spacing of stirrups increases the relative values of critical stress ($\frac{\sigma_{cr}}{\sigma_u}$), indicating that decreased spacing retards crack propagation.

TABLE 4-5. Effect of Spacing of Reinforcement
(for the test of S. P. Shah and B. V. Rangan)

Specimen group	Reinforcement		σ_u	Peak strain (10^{-6} in/in)	$\frac{\sigma_{cr}}{\sigma_u}$
	Type	$p = A_s/bd$			
(1)	(2)	(3)	(4)	(5)	(6)
P03	None	0	2,500	2,800	0.84
PS4	Stirrups at 4 in.	0.50	2,620	3,000	0.87
PS5	Stirrups at 2 in.	0.50	2,610	3,000	0.88
PS6	Stirrups at 1 in.	0.50	2,630	3,300	0.92
PF4	Random fibers	0.50	2,530	3,000	0.86

σ_u = the ultimate strength of concrete.

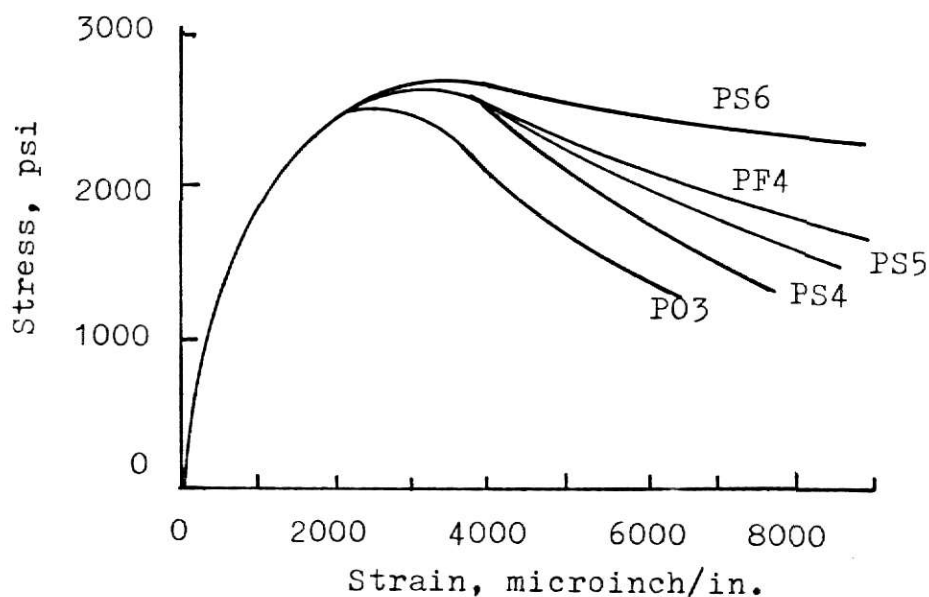


Fig.4-11 Effect of Spacing of Reinforcement
(For the Test of S.F. Shah and B.V. Rangan)

5. DESIGN PROCEDURES AND NUMERICAL EXAMPLES

The design procedures given by the Cambridge method and Baker's method and two numerical design examples are introduced here. From these design procedures, we can see the advantages and the differences between limit design and elastic design.

(1). Cambridge Method.

Limit Analysis: For the analysis under the condition of collapse, two methods are available.

(a). Virtual Work Method: In this method, the work done at failure by the external load is equal to the internal work done by the corresponding rotation at the hinges. This method can be explained by using a simple case of a propped cantilever, as shown in Fig.5-1.

$$\text{External Work} = \sum w \Delta$$

$$= w(1/x)(x)(\frac{1}{2}L) = \frac{1}{2}wL$$

$$\text{Internal Work} = \sum M\theta$$

$$= M'_u(1/x) + M_u(\frac{1}{x} + \frac{1}{L-x})$$

$$\frac{1}{2}wL = M'_u(\frac{1}{x}) + M_u(\frac{1}{x} + \frac{1}{L-x})$$

$$w = \frac{2}{L} \left[M'_u(\frac{1}{x}) + M_u(\frac{1}{x} + \frac{1}{L-x}) \right] \dots\dots\dots(5-1)$$

To determine the location of hinge C, the value of x, which

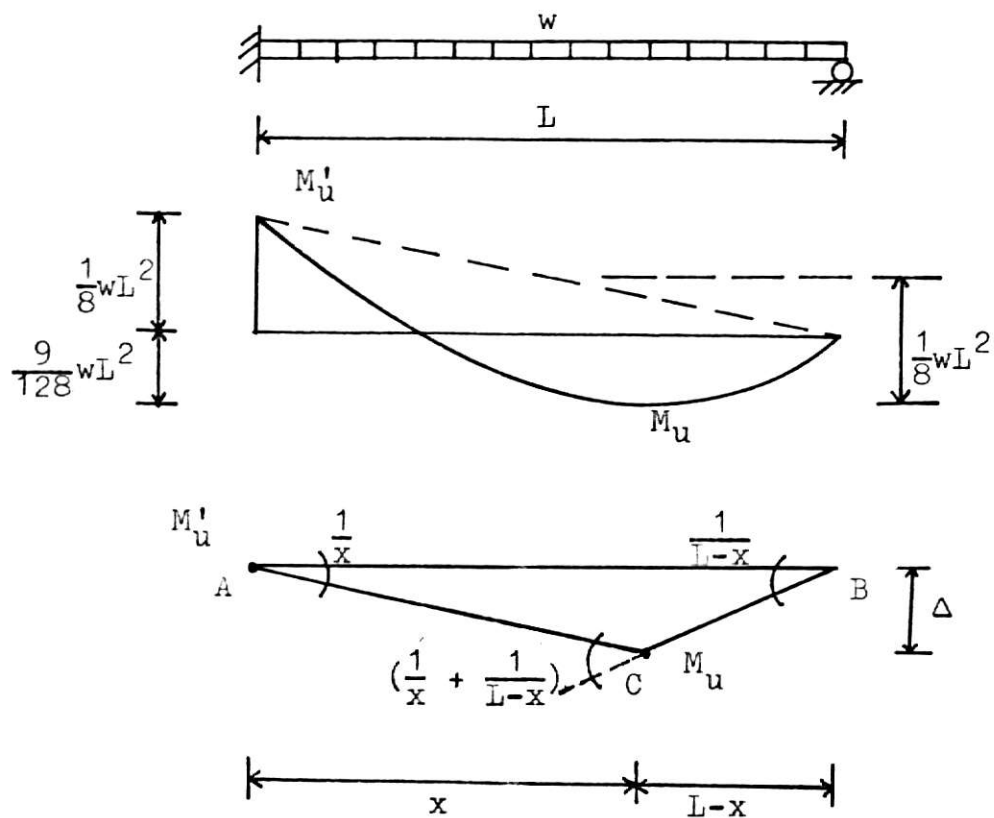


Fig.5-1 Collapse Mechanism and Rotation of a Propped Cantilever

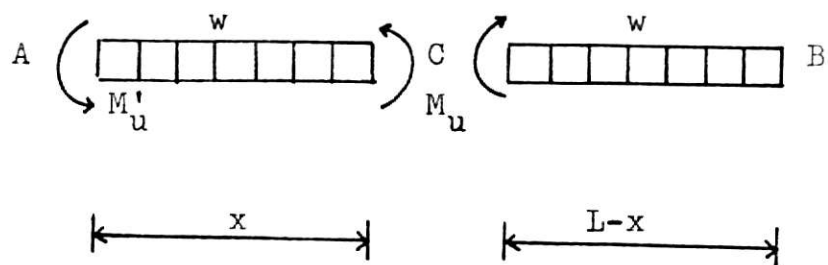


Fig.5-2 Forces on Two Segments of Beam at Collapse

will produce the minimum collapse load, we can differentiate Eq.5-1 with respect to x and set the first derivative equation to zero.

$$\begin{aligned}\frac{dw}{dx} &= \frac{d}{dx} \left[\frac{2}{L} \left(M'_u \left(\frac{1}{x} \right) + M_u \left(\frac{1}{x} + \frac{1}{L-x} \right) \right) \right] \\ &= \frac{2}{L} \left(-M'_u \left(\frac{1}{x^2} \right) - M_u \left(\frac{1}{x^2} - \frac{1}{(L-x)^2} \right) \right) \dots\dots\dots(5-2)\end{aligned}$$

Let $\frac{dw}{dx} = 0$

$$-\frac{M'_u}{x^2} - \frac{M_u}{x^2} + \frac{M_u}{(L-x)^2} = 0$$

$$x = L \frac{\sqrt{1 + \frac{M'_u}{M_u}}}{1 + \sqrt{1 + \frac{M'_u}{M_u}}} \dots\dots\dots(5-3)$$

and $x < L$

Substituting the value of x found from Eq.5-3 into Eq.5-1 gives the collapse load

$$w_u = \frac{2M_u}{L^2} \left(1 + \sqrt{1 + \frac{M'_u}{M_u}} \right)^2 \dots\dots\dots(5-4)$$

(b). Equilibrium Method: In this method, we can find the minimum collapse load by analyzing the equilibrium of the separate segments of the beam at collapse, as shown in Fig.5-2.

Taking the AC Free Body:

$$\Sigma M_A = 0$$

$$M_u - \frac{1}{2}wx^2 + M'_u = 0$$

$$M_u + M'_u = \frac{1}{2}wx^2 \quad \dots\dots\dots(5-5)$$

Taking the CB Free Body:

$$\Sigma M_B = 0$$

$$M_u - \frac{1}{2}w(L-x)^2 = 0$$

$$M_u = \frac{1}{2}w(L-x)^2 \quad \dots\dots\dots(5-6)$$

From Eq.5-5 and Eq.5-6.

$$w_u = \frac{2(M_u + M'_u)}{x^2} = \frac{2M_u}{(L-x)^2}$$

$$\frac{M_u + M'_u}{M_u} = \frac{x^2}{(L-x)^2}$$

So

$$x = L \frac{\sqrt{1 + (M'_u/M_u)}}{1 + \sqrt{1 + (M'_u/M_u)}} \quad \dots\dots\dots(5-3)$$

Substituting Eq.5-3 into Eq.5-5 gives the minimum collapse load, w_u , as before

$$w_u = \frac{2M_u}{L^2} (1 + \sqrt{1 + (M'_u/M_u)})^2 \quad \dots\dots\dots(5-4)$$

M_u and M'_u are the plastic moments at section A and C respectively.

Check the Compatibility of Rotation:

Due to the basic idea of limit design of reinforced concrete structures, it is necessary to check the compatibility of rotation at hinges. After the bending moment diagrams in the elastic range and at collapse have been obtained, the rotations required at a hinge, θ_R , can be found by using the method discussed in Chapter 3 and Eq.3-10.

$$\theta_{AB} = \frac{\Delta}{L} + \frac{L}{6EI} \left[2(M_{AB} - M_{AB}^F) - (M_{BA} - M_{BA}^F) \right] \quad \dots(3-10)$$

$$\theta_{BA} = \frac{\Delta}{L} + \frac{L}{6EI} \left[2(M_{BA} - M_{BA}^F) - (M_{AB} - M_{AB}^F) \right]$$

$$EI = 57000 \sqrt{f'_c} \left[\frac{ba^3}{3} + nA_s(d-a)^2 \right] \quad \dots\dots\dots(3-13)$$

and the available rotation capacity can be obtained from:

$$\theta_T = \left[\frac{(\epsilon_{cu} - \epsilon_{cy})K_1K_3f'_c}{\rho d f_{sy}} \right] L_p \quad \dots\dots\dots(3-8)$$

$$L_p = R_1R_2R_3(z/d)^{\frac{1}{4}}d \quad \dots\dots\dots(3-9)$$

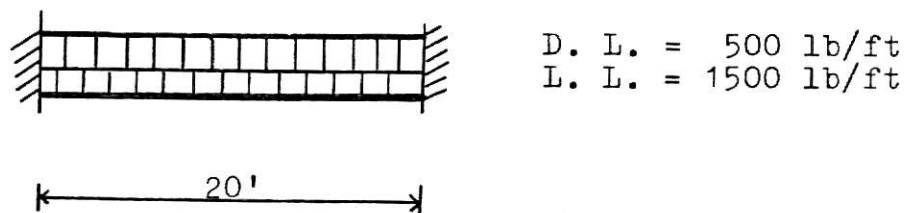
Other Factors for Consideration:

After the structure has been analyzed and designed for bending by satisfying the results of the limit analysis and the compatibility of rotation at hinges, it is necessary to consider other factors such as: shear stress, bond slip and failure due to instability. To prevent premature failures by shear stress, bond slip and instability, the requirements and design methods given by the A.C.I. Building Code (ACI 318-71) should be followed.

In order to make sure of the serviceability of the structure at working loads, a rough elastic analysis must be carried out. The maximum deflection of the structure at working loads also has to be estimated and checked to determine whether this value is smaller than the allowed maximum deflection or not.

(2). Numerical Example⁺

A fixed-end beam with a 20FT. span and the loads acting as shown in Fig.5-3 is to be designed.



The yield strength of the steel $f_{sy} = 40000$ psi.

The strength of the concrete (28 days, cylinder)

$f'_c = 3000$ psi.

Fig.5-3 Beam with Loads

+ According to ACI Building Code (ACI 318-71)

Loading Analysis: According to the ACI Building Code⁷, the load factors are 1.7 for live load and 1.4 for dead load.

$$w = 1.7(\text{L.L.}) + 1.4(\text{D.L.})$$

$$w = 1.7(1500) + 1.4(500) = 3250 \text{ lb/ft}$$

Limit Analysis: Using the virtual work method and referring to Fig.5-4, the analysis can be carried out as follows:

$$\text{External Work} = W_E = \sum w \Delta = \frac{1}{2} w L x \theta$$

$$\text{Internal Work} = W_I = \sum M \theta = M_A \theta + M_B \frac{x}{L-x} \theta + M_C \frac{L}{L-x} \theta$$

$$W_E = W_I$$

$$\frac{1}{2} w L x \theta = (M_A + M_B \frac{x}{L-x} + M_C \frac{L}{L-x}) \theta \quad \dots\dots\dots(1)$$

Let $M_A = M_B = M_C = M_p$

$$\frac{1}{2} w L x \theta = M_p \frac{2L}{L-x} \theta \quad (\theta \neq 0 , \quad L \neq 0)$$

$$\frac{1}{2} w x = M_p \frac{2}{L-x}$$

$$w = M_p \frac{4}{(L-x)x} \quad \dots\dots\dots(2)$$

$$\frac{dw}{dx} = 0$$

$$4M_p \frac{2x-L}{(Lx-x^2)^2} = 0$$

$$M_p \neq 0 , \quad (Lx-x^2)^2 \neq 0$$

So $2x-L = 0 , \quad x = \frac{1}{2}L = 10' .$

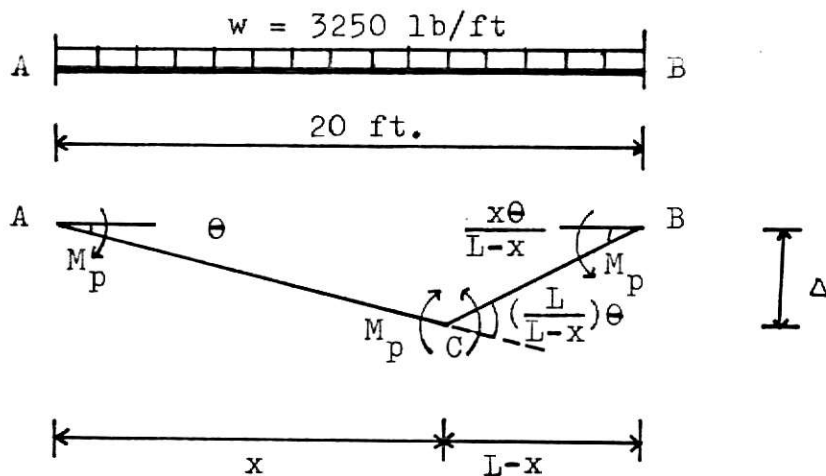


Fig.5-4 Collapse Mechanism of a Fixed-End Beam

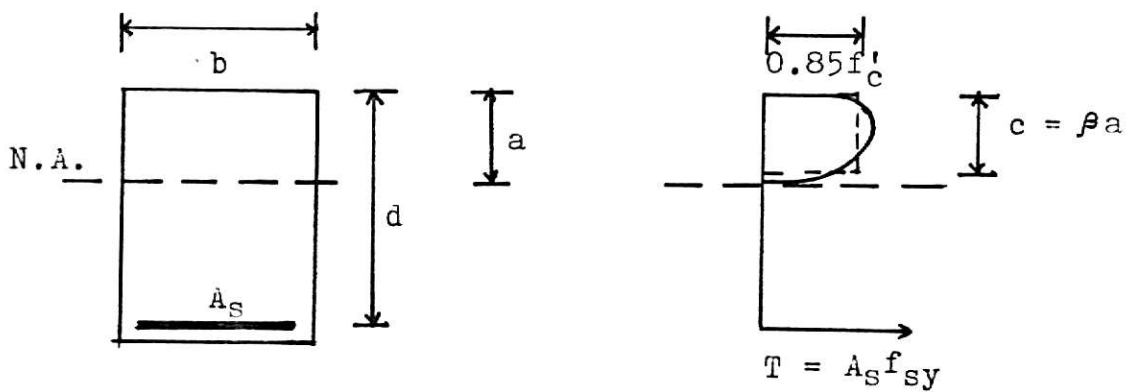


Fig.5-5 Stress Distribution at Ultimate Load

Substituting the value of x into Eq.2

$$w = M_p \frac{4}{(\frac{1}{2}L)(\frac{1}{2}L)} = \frac{16M_p}{L^2}$$

$$M_p = (1/16)wL^2 = (1/16)(3250)(20)^2 = 81250 \text{ lb-ft}$$

Theoretically we can choose any ratio of $M_A : M_B : M_C$, however some proportions will lead to an unsatisfactory design at working loads. For example, we choose $M_A : M_B : M_C = 0 : 0 : 1$, then from the same procedures we will find

$$M_C = (1/8)wL^2 = 162500 \text{ lb-ft}$$

$$M_A = M_B = 0.$$

Since M_A and M_B are equal to zero, hinges with severe cracking will form at ends A and B at a load much less than the working load. It is necessary and important to make sure of the serviceability of a designed structure.

Design of Cross-Section:

Referring to Fig.5-5, the moment is:

$$\text{External Moment } (M_E) = \text{Internal Moment } (M_I)$$

$$M_I = \phi_a A_s f_{sy} (d - \frac{1}{2}c)$$

$$\phi_a = 0.9, \quad c = \beta a$$

$$\beta = 0.85 - \left(\frac{f'_c - 4000}{1000} \right) (0.05). \quad (\text{when } f'_c \geq 4000 \text{ psi})$$

$$\beta = 0.85 \quad (\text{when } f'_c \leq 4000 \text{ psi})$$

or

$$c = \frac{A_s f_{sy}}{0.85 f'_c b}$$

$$\begin{aligned} M_I &= 0.9 A_s f_{sy} \left(d - \frac{A_s f_{sy}}{2(0.85 f'_c b)} \right) \\ &= 0.9 A_s f_{sy} d \left(1 - \frac{p f_{sy}}{2(0.85 f'_c)} \right) \dots\dots\dots (3) \end{aligned}$$

Assume $p_{allow} = 0.75 p_b$, $f'_c = 3000 \text{ psi} < 4000 \text{ psi}$

so $\beta = 0.85$

$$p_b = \frac{0.85 \beta f'_c}{f_{sy}} \frac{87000}{87000 + f_{sy}} = 0.0373$$

$$p_{allow} = (0.75)(0.0373) = 0.028$$

p_b = reinforcement ratio to produce balanced conditions.

ϕ_a = capacity reduction factor.

Substituting $p = 0.028$ into Eq.3

$$\begin{aligned} M_E = M_I = M_P &= 81250 \text{ lb-ft} = 977000 \text{ lb-in} \\ &= (0.9)(0.028)(bd)(40000)(d) \left(1 - \frac{(0.028)(40000)}{2(0.85)(3000)} \right) \end{aligned}$$

Let $b = \frac{1}{2}d$

$$977000 = 1008(0.5)(d^3)(1 - 0.22) = 393.12(d^3)$$

$$d^3 = 2485 , \quad d = 13.52"$$

use 8" x 18" section

$$d = 18 - 2 - 0.5 = 15.25 \text{ (in)}$$

From Eq. 3

$$977000 = (0.9)(A_s)(40000)\left(15.25 - \frac{(A_s)(40000)}{2(0.85)(3000)8}\right)$$

$$A_s = 2.05 \text{ in}^2$$

use 3 #8 $A_s = 2.37 \text{ in}^2$

$$p = \frac{2.37}{8(15.25)} = 0.0194 < 0.028$$

$$\begin{aligned} M_p &= (0.9)(2.37)(40000)\left(15.25 - \frac{(2.37)(40000)}{2(0.85)(3000)8}\right) \\ &= 1100000 \text{ lb-in} \end{aligned}$$

Check the Compatibility of Rotation:

Using Eq.3-12 and Eq.3-13, the calculations are as follows:

$$E_s = 29000000 \text{ psi}$$

$$E_c = 57000 \sqrt{f'_c} = (57000)(3000)^{\frac{1}{2}} = 3130000 \text{ psi}$$

$$n = E_s/E_c = 9.3$$

$$\frac{1}{2}b(kd)^2 = nA_s(d-kd)$$

$$\frac{1}{2}(8)x^2 = 9.3(2.37)(15.25 - x)$$

$$x = kd = a = 6.83"$$

Substituting $a = 6.83"$ into Eq.3-13

$$\begin{aligned} EI &= 3130000 \left[\left(\frac{1}{3} \right) (8) (6.83)^3 + 9.3(2.37)(15.25 - 6.83)^2 \right] \\ &= (7.5)(10)^9 \text{ lb-in}^2 \end{aligned}$$

From Eq.5-4

$$\begin{aligned} -M_{AC}^F &= -M_{CB}^F = M_{CA}^F = M_{BC}^F = (1/12)wL^2(\frac{1}{4}) \\ &= (1/12)(3250)(10)^2 = 325000 \text{ lb-in} \end{aligned}$$

$$M_{AB} = M_{BA} = -M_{BC} = -M_{CB} = M_p = -1100000 \text{ lb/in}$$

$$\begin{aligned} \theta_{AC} &= \frac{\Delta}{120} + \frac{20}{EI} \left[2(-1100000 + 325000) - (-1100000 - 325000) \right] \\ &= \frac{\Delta}{120} + \frac{20}{EI} (-125000) \end{aligned}$$

$$\begin{aligned} \theta_{CA} &= \frac{\Delta}{120} + \frac{20}{EI} \left[2(-1100000 - 325000) - (-1100000 + 325000) \right] \\ &= \frac{\Delta}{120} + \frac{20}{EI} (-2075000) \end{aligned}$$

$$\begin{aligned} \theta_{CB} &= \frac{-\Delta}{120} + \frac{20}{EI} \left[2(1100000 + 325000) - (1100000 - 325000) \right] \\ &= \frac{-\Delta}{120} + \frac{20}{EI} (2075000) \end{aligned}$$

$$\begin{aligned} \theta_{BC} &= \frac{-\Delta}{120} + \frac{20}{EI} \left[2(1100000 - 325000) - (1100000 + 325000) \right] \\ &= \frac{-\Delta}{120} + \frac{20}{EI} (125000) \end{aligned}$$

$$\theta_{RA} = -\theta_{AC} = \frac{-\Delta}{120} + \frac{20}{EI} (125000) \quad (-)$$

$$\theta_{RC} = \theta_{CA} - \theta_{CB} = \frac{\Delta}{60} - \frac{20}{EI} (4150000) \quad (+)$$

$$\theta_{RB} = \theta_{BC} = \frac{-\Delta}{120} + \frac{20}{EI} (125000) \quad (-)$$

$$\text{Let } \theta_{RA} = 0 \quad (\text{same as } \theta_{RB} = 0)$$

$$\frac{\Delta}{120} = \frac{20}{EI} (125000)$$

$$\theta_{RA} = 0$$

$$\theta_{RC} = \frac{20}{EI} (250000 - 4150000) = \frac{20}{EI} (-390000) \quad (-)$$

$$\theta_{RB} = 0$$

NO GOOD

$$\text{Let } \theta_{RC} = 0$$

$$\frac{\Delta}{60} = \frac{20}{EI} (4150000)$$

$$\theta_{RA} = \frac{20}{EI} (-2075000 + 125000) = \frac{20}{EI} (-1950000) \quad (-)$$

$$\theta_{RC} = 0$$

$$\theta_{RB} = \frac{20}{EI} (-1950000) \quad (-)$$

ALL RIGHT

$$\theta_{RA} = \theta_{RB} = \frac{(20)(1950000)}{7500000000} = 0.00520 \text{ rad.}$$

Then using Eq.3-8 and Eq.3-9

$$\theta_T = \frac{(\epsilon_{cu} - \epsilon_{cy})K_1K_3f'_c}{p d f_{sy}} L_p$$

$$\epsilon_{cu} - \epsilon_{cy} = 0.0035 - 0.0015 = 0.002$$

$$p = 0.0194, \quad d = 15.25"$$

$$f_{sy} = 40000 \text{ psi}, \quad f'_c = 3000 \text{ psi}$$

$$K_1K_3 = \frac{3900 + (0.35)(3000)}{3200 + 3000} = 0.8$$

$$L_P = R_1 R_2 R_3 (z/d)^{\frac{1}{4}} d$$

From Fig.5-6

$$1100000 - \frac{(3250)(20)}{2} z + \frac{3250}{12} (\frac{1}{2} z^2) = 0$$

$$z = 40.4"$$

$$R_1 = 0.75 , \quad R_2 = 0.8 , \quad R_3 = 1.25$$

$$L_P = (0.75)(0.8)(1.25)(\frac{40.4}{15.25})^{\frac{1}{4}}(15.25) = 14.75 \text{ (in)}$$

then

$$\theta_T = \frac{(0.0035 - 0.0015)(0.8)(3000)}{(0.0194)(15.25)(40000)} (14.75)$$

$$= 0.00598 \text{ rad.} > 0.00520 \text{ rad.}$$

Check the Serviceability of This Beam:

At working load the elastic bending moments are:

$$M_A = M_B = (1/12)(D.L. + L.L.)(L^2) = \frac{(2000)(20)^2}{12} \quad (12)$$

$$= 800000 \text{ lb-in}$$

$$M_C = (1/24)(D.L. + L.L.)(L^2) = \frac{(2000)(20)^2}{24} \quad (12)$$

$$= 400000 \text{ lb-in}$$

Hence this design has a safety factor:

$$\frac{M_P}{M_A} = \frac{1100000 \text{ lb-in}}{800000 \text{ lb-in}} = 1.375$$

The maximum deflection at working load is:

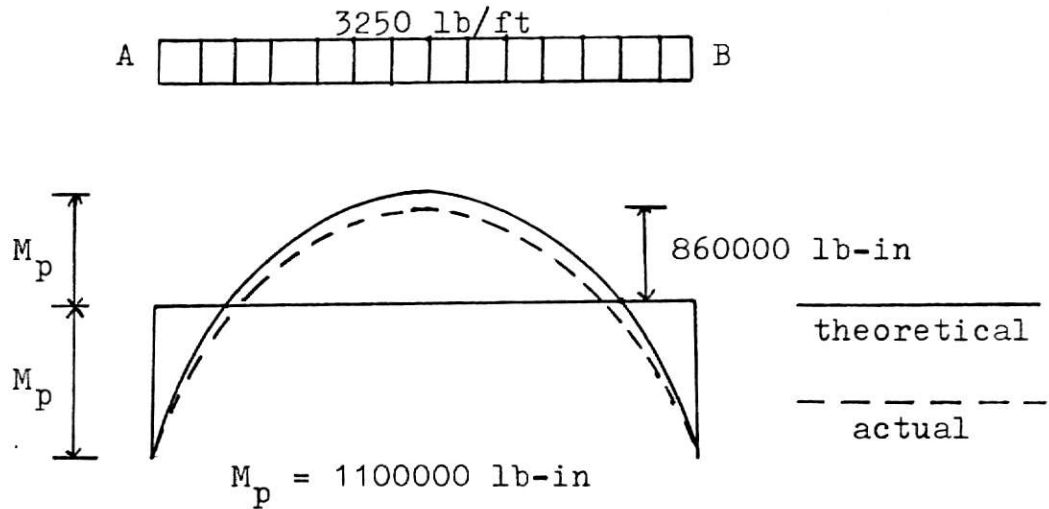


Fig.5-6 Bending Moment at Collapse

$$\Delta_{\max} = \frac{wL^4}{384EI}$$

$$E = 57000 (f'_c)^{\frac{1}{2}} = 313000 \text{ psi}$$

$$I = I_e = \left(\frac{M_{cr}}{M_a}\right)^3 I_g + \left(1 - \left(\frac{M_{cr}}{M_a}\right)^3\right)(I_{cr}) \quad (\text{see Reference 7})$$

$$M_{cr} = \frac{f_r I_g}{Y_t} \quad (\text{see Reference 7})$$

$$f_r = 7.5 (f'_c)^{\frac{1}{2}} \quad (\text{see Reference 7})$$

$$M_a = M_A = 800000 \text{ lb-in}$$

$$f_r = (7.5)(3000)^{\frac{1}{2}} = 410 \text{ psi}$$

$$I_g = (1/12)(8)(18)^3 = 3900 \text{ in}^4$$

$$I_{cr} = (1/3)(8)(6.83)^3 + (9.3)(2.37)(15.25 - 6.82)^2$$

$$= 2495 \text{ in}^4$$

$$Y_t = 9"$$

$$M_{cr} = \frac{(410)(3900)}{9} = 177000 \text{ lb-in}$$

$$\frac{M_{cr}}{M_a} = \frac{177000}{800000} = 0.222$$

So

$$I = I_e = (0.222)^3(3900) + (1 - (0.222)^3)(2495)$$

$$= 2508 < 3900$$

$$\Delta_{max} = \frac{1}{(384)(313000)(2508)} (2000)(20)^4(12)^3 = 0.183"$$

$$0.183" = \frac{1}{1315} \text{ span} < \frac{1}{180} \text{ span} \quad \text{check}$$

f_r = modulus of rupture of concrete, psi.

I_{cr} = moment of inertia of cracked section transformed to concrete.

$I = I_e$ = effective moment of inertia for computation of deflection.

I_g = moment of inertia of gross concrete section about the centroidal axis, neglecting the reinforcement.

M_a = maximum moment in member at stage for which deflection is being computed.

M_{cr} = cracking moment.

Y_t = distance from centroidal axis of gross section,

neglecting the reinforcement, to extreme fiber in tension.

Comparison of Limit Design⁺ and Ultimate Strength Design

Ultimate Strength Design:

$$M_A = M_B = (1/12)wL^2 = (1/12)(3250)(20)^2(12) = 1300000$$

$$1300000 = (0.9)(0.028)(bd)(40000)(d)\left(1 - \frac{(0.028)(40000)}{2(0.85)(3000)}\right)$$

Assume $b = \frac{1}{2}d$

$$1300000 = 1008(0.5)d^3(1 - 0.22) = 393.12 d^3$$

$$d^3 = 3320, \quad d = 14.95"$$

use 9" x 20" section $d = 20 - 2 - 0.5 = 17.25$ (in)

$$1300000 = (0.9)A_s(40000)\left(17.25 - \frac{(A_s)(40000)}{2(0.85)(3000)9}\right)$$

$$A_s = 2.32 \text{ in}^2$$

use 3 #8 $A_s = 2.37 \text{ in}^2$

	b	d	A_s *
the Cambridge Method	8"	15.25"	2.37 in ²
Ultimate Strength Design	9"	17.25"	2.37 in ²

+ Using Cambridge Method.

* A_s : Actual Area of Steel.

(3). Baker's Method¹

Determining the Number of Degrees of Statical Indeterminacy of a Structure N.

The degree of indeterminacy of a structure N can be calculated by

$$N = NU - NA \quad \dots\dots\dots(5-7)$$

NU = number of unknown reactions.

NA = number of equations available.

The Basic Elastic Equation and the General Equation of Baker's Limit Design.

In a structure with N degree of indeterminacy, then N hinges have to be assumed at some suitable points to make this structure statically determinate. This means N unknown equal and opposite bending moments $X_1, X_2, X_3, \dots, X_n$ are assumed to act at either side of assumed hinges 1, 2,n respectively. These bending moments are equal in magnitude to the elastic moments acting at these sections of assumed hinges due to applied loads. Consider a section i in the structure, the total moment acting at this section is

$$M_i = M_{i0} + m_{i1}X_1 + m_{i2}X_2 + \dots\dots\dots + m_{in}X_n \quad \dots\dots(5-8)$$

M_{i0} = the moment at section i due to the external loads on the statically determinate structure.

$m_{i1}, m_{i2}, m_{i3}, \dots, m_{in}$ = the moment at section i due to unit load acting at assumed hinges 1,2,.....n.

$m_{i1}X_1, m_{i2}X_2, \dots, m_{in}X_n$ = the moment acting at section i
 due to the unknown moments $X_1, X_2, \dots, X_n$ acting
 at the assumed hinges 1, 2, 3,n respectively.

The strain energy stored in the structure is

$$U = \sum \int_0^L \frac{M_i^2}{2EI} ds \quad \dots\dots\dots(5-9)$$

The integral is carried over each member and all the members of the structure are included in the summation. The structure has to maintain the compatibility of displacements and hence applying the principal of minimum strain energy, the partial derivative of U with respect to the assumed moment X_i at hinge i has to be zero.

$$\begin{aligned} \frac{\partial U}{\partial X_i} &= 0 \quad i = 1, 2, 3, \dots\dots\dots n \\ &= \sum \int \frac{\partial}{\partial X_i} \left(\frac{M_i^2}{2EI} \right) ds = \sum \int \frac{M_i}{EI} \frac{\partial M_i}{\partial X_i} ds = 0 \quad \dots\dots(5-10) \end{aligned}$$

Then differentiating Eq.5-8

$$\begin{aligned} \frac{\partial M_i}{\partial X_i} &= m_{ii} \\ \frac{\partial U}{\partial X_i} &= \sum \int \frac{M_i}{EI} m_{ii} ds \end{aligned}$$

Since $M_i = M_{i0} + m_{i1}X_1 + m_{i2}X_2 + \dots\dots\dots + m_{in}X_n$

$$\begin{aligned} \frac{\partial U}{\partial X_i} &= \sum \int_0^L (M_{i0} + m_{i1}X_1 + m_{i2}X_2 + \dots\dots\dots + m_{in}X_n) \frac{m_{ii}}{EI} ds \\ &= 0 \end{aligned}$$

$$\Sigma \int_0^L \frac{M_{io} m_{ii}}{EI} ds +$$

$$\Sigma \int_0^L (m_{i1} X_1 + m_{i2} X_2 + \dots + m_{in} X_n) \frac{m_{ii}}{EI} ds = 0 \dots (5-11)$$

Let

$$s_{io} = \Sigma \int_0^L \frac{M_{io} m_{ii}}{EI} ds = \text{the relative rotation at hinge } i, \text{ due to external load.}$$

$$s_{ij} = \Sigma \int_0^L \frac{m_{ii} m_{ij}}{EI} ds = \text{the relative rotation at hinge } i, \text{ due to unit moment at hinge } j.$$

Eq.5-11 can be rewritten as

$$s_{io} + \sum_{j=1}^n s_{ij} X_j = 0$$

$$i = 1, 2, 3, \dots, n$$

$$j = 1, 2, 3, \dots, n$$

There is no unique set of s_{ij} value for any given structure because the structure could be made statically determinate in a number of different ways by assuming different hinge positions. For an N degree statically indeterminate structure, an equation is obtained for each hinge.

So

$$\epsilon_{10} + \epsilon_{11}X_1 + \epsilon_{12}X_2 \dots\dots\dots + \epsilon_{1n}X_n = 0$$

$$\epsilon_{20} + \epsilon_{21}X_1 + \epsilon_{22}X_2 \dots\dots\dots + \epsilon_{2n}X_n = 0$$

$$\dots\dots\dots$$

$$\dots\dots\dots$$

$$\epsilon_{n0} + \epsilon_{n1}X_1 + \epsilon_{n2}X_2 \dots\dots\dots + \epsilon_{nn}X_n = 0$$

In the above equation ϵ_{i0} , ϵ_{ij} are known and $X_1, X_2, \dots\dots X_n$ are unknown, that is, there are N unknowns in N equations.

The analysis of a statically indeterminate structure within the elastic limit is discussed in the previous paragraphs. When the loads are increased further, the section subjected to maximum stress in the elastic stage will become plastic and will behave like hinges with known constant bending moments acting on each side of the hinge. The plastic deformation of this hinge will introduce an angle of discontinuity between adjacent members which is referred to as plastic rotation of the hinge.

The plastic moments assumed at hinge 1, 2, 3,n are denoted by $\bar{X}_1, \bar{X}_2, \bar{X}_3, \dots\dots \bar{X}_n$ and they remain constant through the further deformation of the structure. Since we have a situation similar to the elastic analysis described above, the difference is that all the unknown moments $X_1, X_2, \dots\dots X_n$ are replaced by the full plastic moment of the sections $\bar{X}_1, \bar{X}_2, \bar{X}_3, \dots\dots \bar{X}_n$. But the compatibility of

displacement is not satisfied at plastic hinges because of the plastic rotation. Therefore, $\frac{\partial U}{\partial X_i}$ is made equal to the inelastic rotation at hinge i instead of zero.

$$\frac{\partial U}{\partial X_i} = -\theta_i \quad \dots\dots\dots(5-13)$$

So

$$\sum \int_0^L \frac{M_{i0} m_{ii}}{EI} ds = -\theta_i$$

$$\begin{aligned} \sum \int_0^L \frac{M_{i0} m_{ii}}{EI} ds + \sum \int_0^L (m_{i1} X_1 + m_{i2} X_2 \dots\dots + m_{in} X_n) \frac{m_{ii}}{EI} ds \\ = -\theta_i \end{aligned}$$

By replacing X_i by $\bar{X}_i$

$$\begin{aligned} \sum \int_0^L \frac{M_{i0} m_{ii}}{EI} ds + \sum \int_0^L (m_{i1} \bar{X}_1 + m_{i2} \bar{X}_2 \dots\dots + m_{in} \bar{X}_n) \frac{m_{ii}}{EI} ds \\ = -\theta_i \quad \dots\dots\dots(5-14) \end{aligned}$$

Similar to Eq.5-12, the general equation for Baker's method can be written as:

$$\left. \begin{aligned} s_{10} + s_{11} \bar{X}_1 + s_{12} \bar{X}_2 + \dots\dots\dots + s_{1n} \bar{X}_n &= -\theta_1 \\ s_{20} + s_{21} \bar{X}_1 + s_{22} \bar{X}_2 + \dots\dots\dots + s_{2n} \bar{X}_n &= -\theta_2 \\ \dots\dots\dots \\ \dots\dots\dots \\ s_{n0} + s_{n1} \bar{X}_1 + s_{n2} \bar{X}_2 + \dots\dots\dots + s_{nn} \bar{X}_n &= -\theta_n \end{aligned} \right\} \dots\dots(2-1)$$

A trial and error method for solving this equation suggested by Baker is:

- (1). Choose arbitrary values for plastic moments

$$\bar{X}_1, \bar{X}_2, \dots, \bar{X}_n.$$

- (2). The plastic rotations $\theta_1, \theta_2, \dots, \theta_n$ necessary for the full moment redistribution are calculated from Eq.2-1.

If the calculated θ_i is positive and less than the available rotation θ_{Ti} , then the chosen values of $\bar{X}_i$ can be used in design. Otherwise, the values of $\bar{X}_i$ must be revised until the value of θ_i are positive and less than the available rotations θ_{Ti} .

Choice of Hinge Position

It is not necessary to determine the position of hinges by the method used in the plastic design of steel structures. The plastic hinges could be assumed at sections of maximum moments in the elastic stage. The support sections of beam and junction sections of columns usually have maximum moments in the elastic stage. Such an arbitrary choice of hinge location is possible because reinforced concrete structures can always be made to fail in a desired manner by designing the member between the selected hinges to resist the bending moment without yielding of steel or fracture of concrete until the formation of the Nth hinge.

The compatibility of Rotation at a Plastic Hinge

By using Eq.3-8 and Eq.3-9, we can find the value of available rotation θ_{Ti} . And from Eq.2-1, we can get the value of required rotation θ_{Ri} . Then check whether $\theta_{Ti} \geq \theta_{Ri}$ or not. When $\theta_{Ri} > \theta_{Ti}$, it is necessary to reassume $\bar{X}_i$ and recalculate θ_{Ri} as mentioned in the previous paragraph or to consider increasing the available rotation capacity at that hinge.

Increase of Rotation Capacity

From the discussion in Chapter 4, we know it is possible to get sufficient rotation capacity by using suitable secondary reinforcement. Baker has recommended the following empirical expressions¹:

$$\text{For circular binder: } p'' = 5000(\epsilon'_{cu} - \epsilon_{cu})^3, \dots(5-15)$$

For rectangular binder:

$$p'' = 14600(\epsilon'_{cu} - \epsilon_{cu})^3, \dots(5-16)$$

ϵ'_{cu} = the ultimate strain of bound concrete.

ϵ_{cu} = the ultimate strain of unbound concrete.

(usually equal to 0.0035 in/in)

$$p'' = \frac{\text{volume of binder}}{\text{volume of bound concrete}} \cdot$$

ϵ_{sy} = .0013 in/in = the strain of steel at yield.

ϵ_{cy} = the strain in concrete at extreme fiber at start of steel yielding.

$$\epsilon_{cy} = \frac{(K_u)\epsilon_{sy}}{1 - K_u} \dots\dots\dots(3-8a)$$

Then

$$\theta_{Ti} = \frac{(\epsilon'_{cu} - \epsilon_{cy})K_1K_3f'_c}{pdf_{sy}} L_p \dots\dots\dots(3-10)$$

Calculating of Flexural Rigidity EI

The value of EI used in calculation of the rotation of plastic hinges in Baker's method can be derived from the theory of simple bending. In the theory of simple bending,

we have $\frac{M}{I} = \frac{E_c}{\rho}$.

Therefore

$$I = \frac{M\rho}{E_c} .$$

From Fig.5-7 and Fig.5-8

$$\frac{ds}{\rho} = \frac{(\epsilon_c)ds}{kd} , \quad \frac{1}{\rho} = \frac{\epsilon_c}{kd} ,$$

$$I = \frac{Mkd}{\epsilon_c E_c} ,$$

$$E_c I = \frac{Mkd}{\epsilon_c} \dots\dots\dots(5-17)$$

$$M = \alpha f_c b a (d - u a) .$$

So

$$EI = \frac{\alpha f_c b a^2 (d - u a)}{\epsilon_c} \dots\dots\dots(5-18)$$

$a = kd$ = the depth of neutral axis.

$$k = \frac{1}{1 + (\epsilon_s / \epsilon_c)} ,$$

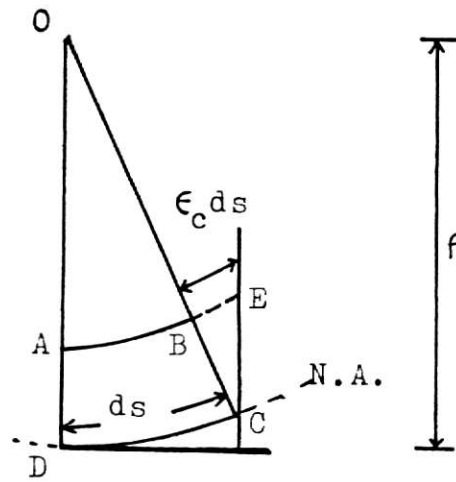


Fig.5-7 Compression Zone of a Member Subjected to Bending

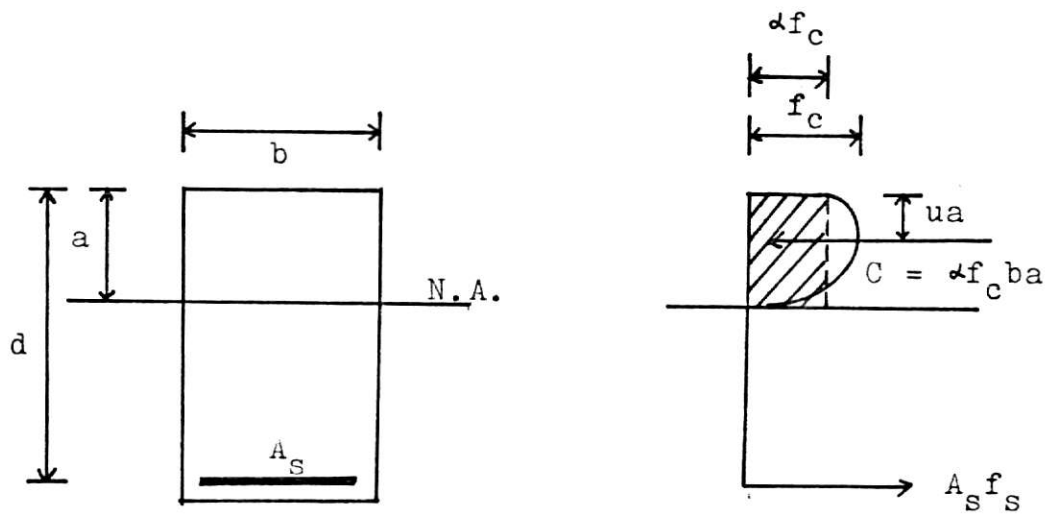


Fig.5-8 Stress-Block of a Reinforced Concrete Section at Any Stage

ρ = the radius of curvature of the neutral axis.

$$f_c = 0.85f'_c \left[2 \frac{\epsilon_c}{\epsilon_o} - \left(\frac{\epsilon_c}{\epsilon_o} \right)^2 \right]$$

$\epsilon_o = 0.002$ in/in = strain in concrete at f'_c .

ϵ_c = the compressive strain in concrete at the extreme fiber.

αf_c = the average compression stress in concrete distributed over the compression zone.

ua = the depth to the center of compression force.

Usually $\alpha = 0.67$, $u = 0.4$ (see Reference 1)

At every stage the total tension force T will equal to the total compression force C in concrete. Hence at the stage of the yield of steel we will have

$$A_s f_{sy} = \alpha f_c ba = \alpha f_c bkd$$

$$k = \frac{1}{1 + (\epsilon_s / \epsilon_c)}$$

From these equations and with the trial and error method, we can find the correct ϵ_c , k , and kd . Then substituting these values into Eq.5-18, we can obtain a value of EI .

(4). Numerical Example

A four equal-span continuous beam with simply supported ends loaded with a uniform load 2100 lb/ft (including the self weight of the beam) as shown in Fig.5-9. The span is 13 ft. and a load factor of 2 is used in this design.

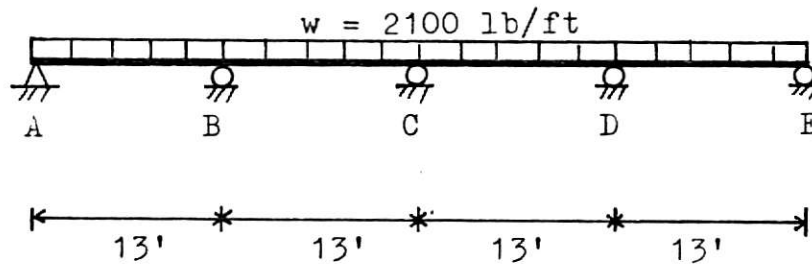


Fig.5-9 Beam with Uniform Load

$$f'_c = 3000 \text{ psi} , \quad \epsilon_{cu} = 0.0035 \text{ in/in}$$

$$f_{sy} = 35000 \text{ psi} , \quad \epsilon_{sy} = 0.0012 \text{ in/in}$$

$$n = 9.2$$

It will be economical to use a uniform section throughout and approximately equal design moments at the mid-span and support points. So the resisting moment at each section will be approximately equal to half of the bending moment of a simple beam at the mid-span.

$$\frac{1}{2} \left(\frac{1}{8} \right) wL^2 = \left(\frac{1}{16} \right) (2) (2100) (13)^2 (12) = 532000 \text{ lb-in}$$

Design of Cross-Section

$$M = \phi_a f_{sy} p b d^2 \left(1 - \frac{p f_{sy}}{2(0.85) f'_c} \right)$$

$$\phi_a = 0.9$$

$$p = 0.75 p_b = 0.75 \frac{(0.85)(0.85)(3000)}{35000} - \frac{87000}{(87000 + 35000)}$$

$$= 0.0334$$

Assume $b = \frac{1}{2}d$

$$532000 = (0.9) \left(\frac{1}{2}d \right) (0.0334) d^2 \left(1 - \frac{(0.0334)(35000)}{2(0.85)(3000)} \right) (35000)$$

$$532000 = 526 (1 - 0.229) d^3 = 405.5 d^3$$

$$d^3 = 1315, \quad d = 10.95"$$

use depth $d = 14"$, $b = 6"$.

$$532000 = 0.9 A_s (35000) \left(14 - \frac{35000 A_s}{2(0.85)(3000)6} \right)$$

$$A_s = 1.28 \text{ in}^2$$

use 2 #8 $A_s = 1.58 \text{ in}^2$

$$p = \frac{1.58}{(6)(14)} = 0.0188 < 0.0334$$

$$M = (0.9)(1.58)(35000) \left(14 - \frac{(1.58)(35000)}{2(6)(0.85)(3000)} \right)$$

$$= 608000 \text{ lb-in}$$

Calculating EI

Assume, as an initial trial, that the steel yields at a strain of 0.0012 in/in (ϵ_{sy}). The corresponding strain in concrete (ϵ_c) is 0.001 in/in.

Then

$$\begin{aligned} f_c &= 0.85f'_c \left[2\left(\frac{\epsilon_c}{\epsilon_o}\right) - \left(\frac{\epsilon_c}{\epsilon_o}\right)^2 \right] \\ &= 0.85(3000) \left[2\left(\frac{0.001}{0.002}\right) - \left(\frac{0.001}{0.002}\right)^2 \right] = 1910 \text{ psi} \\ k &= \frac{1}{1 + \frac{0.0012}{0.001}} = 0.455 \end{aligned}$$

$$T = A_s f_{sy} = (1.58)(35000) = 55500 \text{ lb}$$

$$\begin{aligned} C &= \alpha f_c b k d, \quad \alpha = 0.67 \\ &= (0.67)(1910)(6)(0.455)(14) = 49000 \text{ lb} \neq 55500 \text{ lb} \end{aligned}$$

Try $\epsilon_c = 0.00112 \text{ in/in}$

$$f_c = (0.85)(3000) \left[2\left(\frac{0.00112}{0.002}\right) - \left(\frac{0.00112}{0.002}\right)^2 \right] = 2050 \text{ psi}$$

$$k = \frac{1}{1 + \frac{0.0012}{0.00112}} = 0.484$$

$$C = (0.67)(2050)(0.484)(6)(14) = 55750 \text{ lb} \approx 55500 \text{ lb}$$

$$EI = \frac{\alpha f_c b (k d)^2 (d - u k d)}{\epsilon_c}, \quad u = 0.4, \quad k d = 0.484(14)$$

$$\begin{aligned} EI &= \frac{(0.67)(2050)(6)((0.484)(14))^2 (14 - (0.4)(0.484)(14))}{0.00112} \\ &= 3800000000 \text{ lb-in}^2 \end{aligned}$$

Calculation of the Plastic Moment $\bar{X}$

$$\begin{aligned}\bar{X}_1 = \bar{X}_2 = \bar{X}_3 &= 0.9A_s f_{sy} d \left(1 - \frac{P f_{sy}}{2(0.85)(f'_c)}\right) \\ &= 608000 \text{ lb-in}\end{aligned}$$

Calculation of s_{io} and s_{ij}

Since the continuous beam is statically indeterminate to the third degree, three hinges are inserted at the interior supports B, C, and D to transform the beam into a statically determinate one. By using Table 5-1 and referring to Fig.5-10

$$s_{io} = \sum \int_0^L \frac{M_{io} m_{ii}}{EI} ds$$

and

$$s_{ij} = \sum \int_0^L \frac{m_{ii} m_{ij}}{EI} ds$$

$$i = 1, 2, 3, \dots, n$$

$$j = 1, 2, 3, \dots, n$$

can be given by the following procedures and shown in Table 5-2

AB span

$$s_{11}^{AB} = \int_0^L \frac{m_{11} m_{11}}{EI} ds = \frac{Lac}{3EI} = \frac{(13)(1)(1)}{3EI} = \frac{13}{3EI}$$

BC span

$$s_{11}^{BC} = \int_0^L \frac{m_{11} m_{11}}{EI} ds = \frac{Lac}{3EI} = \frac{(13)(1)(1)}{3EI} = \frac{13}{3EI}$$

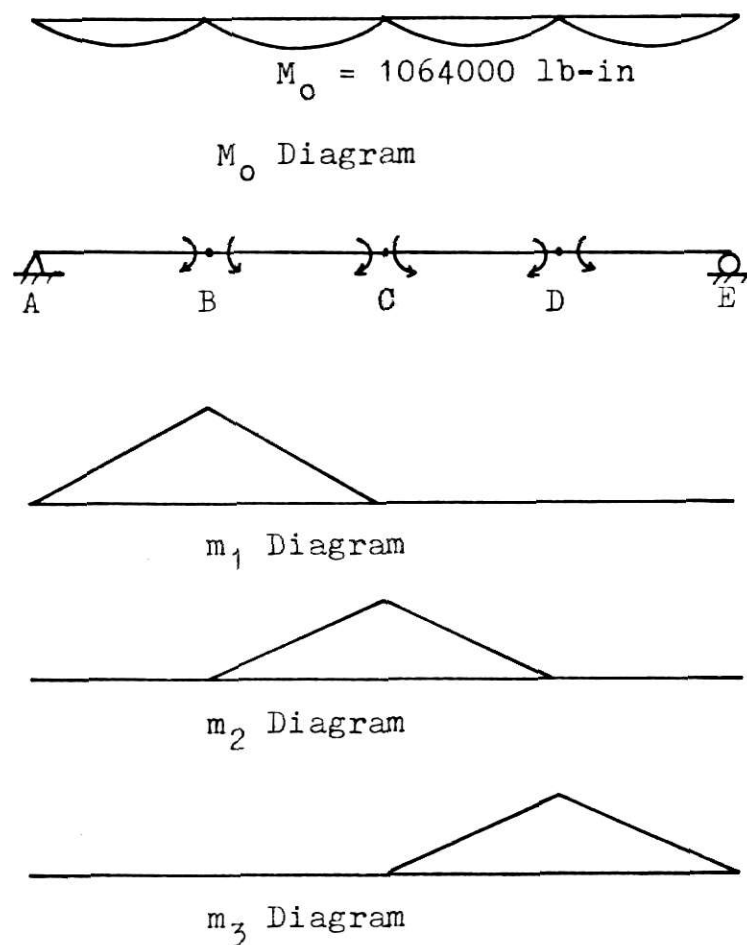


Fig.5-10 Moment Distribution for Checking Rotation Compatibility

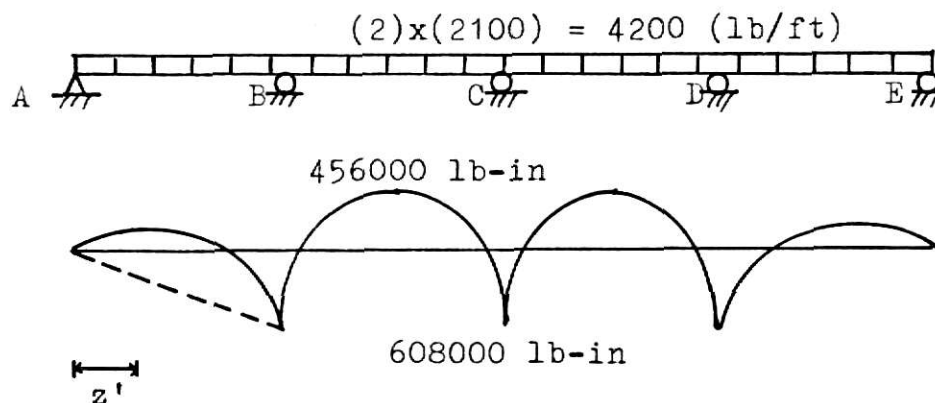
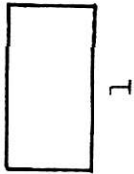
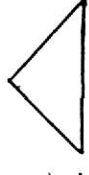
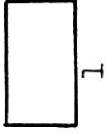


Fig.5-11 Moment Distribution Used for Limit Design

TABLE 5-1. Product Integrals -- $\int m_i m_k ds$ value

$m_i \backslash m_k$				PARABOLIC 		
	lac	$\frac{1}{2} lac$	$\frac{1}{2} lac$	$\frac{2}{3} lac$	$\frac{1}{2} lac$	$\frac{1}{2} l(a+b)c$
	$\frac{1}{2} lac$	$\frac{1}{3} lac$	$\frac{1}{6} lac$	$\frac{1}{3} lac$	$\frac{1}{4} lac$	$\frac{1}{6} l(2a+b)c$
PARABOLIC 	$\frac{2}{3} lac$	$\frac{1}{3} lac$	$\frac{1}{3} lac$	$\frac{8}{15} lac$	$\frac{5}{12} lac$	$\frac{1}{3} l(a+b)c$
	$\frac{1}{2} lac$	$\frac{1}{4} lac$	$\frac{1}{4} lac$	$\frac{5}{12} lac$	$\frac{1}{3} lac$	$\frac{1}{4} l(a+b)c$
	$\frac{1}{2} la(c+d)$	$\frac{1}{6} la(2c+d)$	$\frac{1}{6} la(c+2d)$	$\frac{1}{3} la(c+d)$	$\frac{1}{4} la(c+d)$	$\frac{1}{6} l[a(2c+d) + b(2d+c)]$

CD span and DE span

$$m_{11} = 0$$

$$s_{11}^{CD} = s_{11}^{DE} = 0$$

$$s_{11} = \frac{13}{3EI} + \frac{13}{3EI} + 0 + 0 = \frac{26}{3EI}$$

s_{i0} and all other s_{ij} can be found by the same way, as shown in Table 5-2.

Calculation of Relative Rotation θ_1 , θ_2 , and θ_3 .

From Eq.2-1

$$s_{10} + s_{11}\bar{X}_1 + s_{12}\bar{X}_2 + s_{13}\bar{X}_3 = -\theta_1$$

$$s_{20} + s_{21}\bar{X}_1 + s_{22}\bar{X}_2 + s_{23}\bar{X}_3 = -\theta_2$$

$$s_{30} + s_{31}\bar{X}_1 + s_{32}\bar{X}_2 + s_{33}\bar{X}_3 = -\theta_3$$

$$\bar{X}_1 = \bar{X}_2 = \bar{X}_3 = M_p = 608000 \text{ lb-in}$$

s_{i0} and s_{ij} can be obtained from Table 5-2

$$-0.0292 + (0.00274)(10^{-5})M_p + (0.000685)(10^{-5})M_p = -\theta_1$$

$$\begin{aligned} -0.0292 + (0.000685)(10^{-5})M_p + (0.00274)(10^{-5})M_p \\ + (0.000685)(10^{-5})M_p = -\theta_2 \end{aligned}$$

$$-0.0292 + (0.00274)(10^{-5})M_p + (0.000685)(10^{-5})M_p = -\theta_3$$

TABLE 5-2. Calculation of Influence Coefficients

	$\int \frac{m_i m_k}{EI} ds$ value				$\Sigma \int \frac{m_i m_k}{EI} ds$
	Span AB	Span BC	Span CD	Span DE	Total Radians
δ_{11}	$\frac{13.1 \cdot 1}{3 EI}$	$\frac{13.1 \cdot 1}{3 EI}$	0	0	$\frac{26.12}{3 EI}^*$ $0.00274 \cdot 10^{-5}$
$\delta_{12} = \delta_{21}$	0	$\frac{13.1 \cdot 1}{6 EI}$	0	0	$\frac{13.12}{6 EI}$ $0.000685 \cdot 10^{-5}$
$\delta_{13} = \delta_{31}$	0	0	0	0	0
δ_{22}	0	$\frac{13.1 \cdot 1}{3 EI}$	$\frac{13.1 \cdot 1}{3 EI}$	0	$\frac{26.12}{3 EI}$ $0.00274 \cdot 10^{-5}$
$\delta_{23} = \delta_{32}$	0	0	$\frac{13.1 \cdot 1}{6 EI}$	0	$\frac{13.12}{6 EI}$ $0.000685 \cdot 10^{-5}$
δ_{33}	0	0	$\frac{13.1 \cdot 1}{3 EI}$	$\frac{13.1 \cdot 1}{3 EI}$	$\frac{26.12}{3 EI}$ $0.00274 \cdot 10^{-5}$
δ_{10}	$-\frac{13.1 \cdot 10.64 \cdot 10^5}{3 EI}$	$-\frac{13.1 \cdot 10.64 \cdot 10^5}{3 EI}$	0	0	$\frac{-26.12 \cdot 10.64 \cdot 10^5}{3 EI}$ -0.0292
δ_{20}	0	$-\frac{13.1 \cdot 10.64 \cdot 10^5}{3 EI}$	$-\frac{13.1 \cdot 10.64 \cdot 10^5}{3 EI}$	0	$\frac{-26.12 \cdot 10.64 \cdot 10^5}{3 EI}$ -0.292
δ_{30}	0	0	$-\frac{13.1 \cdot 10.64 \cdot 10^5}{3 EI}$	$-\frac{13.1 \cdot 10.64 \cdot 10^5}{3 EI}$	$\frac{-26.12 \cdot 10.64 \cdot 10^5}{3 EI}$ -0.0292

$$*EI = 3.8 \cdot 10^9 \text{ lb-in}^2$$

$$-0.0292 + (0.003425)(10^{-5})M_p = -\theta_1$$

$$-0.0292 + (0.00411)(10^{-5})M_p = -\theta_2$$

$$-0.0292 + (0.003425)(10^{-5})M_p = -\theta_3$$

$$-0.0292 + 0.0208 = -\theta_1$$

$$-0.0292 + 0.0249 = -\theta_2$$

$$-0.0292 + 0.0208 = -\theta_3$$

$$\theta_1 = 0.0084 \text{ rad.}$$

$$\theta_2 = 0.0043 \text{ rad.}$$

$$\theta_3 = 0.0084 \text{ rad.}$$

Calculation of Available Rotation

$$\theta_T = \frac{(\epsilon_{cu} - \epsilon_{cy})K_1K_3f'_c}{p d f_{sy}} L_p$$

$$\epsilon_{cu} - \epsilon_{cy} = 0.0035 - 0.002 = 0.0015 \text{ (in/in)}$$

$$p = 0.0188$$

$$f_{sy} = 35000 \text{ psi}$$

$$f'_c = 3000 \text{ psi}$$

$$K_1K_3 = 0.8$$

$$\begin{aligned} L_p &= R_1R_2R_3(z/d)^{\frac{1}{4}}d = (0.8)(1.25)(0.75)(z/d)^{\frac{1}{4}}d \\ &= 0.75(z/d)^{\frac{1}{4}}d \end{aligned}$$

From Fig.5-11

$$608000\left(\frac{z'}{(13)(12)}\right) = \frac{(4200)(13)}{2} z' - \frac{1}{2} \frac{4200}{12}(z')^2$$

$$(3900)(z') = (27300)(z') - (175)(z')^2$$

$$z' = 0, \quad z' = 134"$$

$$z = 156" - 134" = 22"$$

$$L_p = (0.75)\left(\frac{22}{14}\right)^{\frac{1}{4}}(14) = 11.7 \text{ (in)}$$

$$\theta_T = \frac{(0.0015)(0.8)(3000)}{(0.0188)(14)(35000)} (11.7) = 0.00456 \text{ rad.}$$

θ_T is the available rotation on one side of the hinge and therefore the total available rotation is

$$2(0.00456) = 0.00912 > 0.0084 \quad \text{Check}$$

Check the Serviceability of This Beam

At working load the maximum elastic moments are

$$\begin{aligned} M_B = M_D &= (-0.1071)wL^2 = (-0.1071)(2100)(13)^2(12) \\ &= 456000 \text{ lb-in} \end{aligned}$$

Hence this design has a safety factor

$$\frac{608000}{456000} = 1.33$$

At working load, the maximum stress in the steel is:

Referring to Fig.5-12

$$\frac{1}{2}ba^2 = (9.2)(1.58)(d - a)$$

$$3a^2 = 204 - 14.55a$$

$$a = 6.18 \text{ (in)} = kd$$

$$1.58f_s\left(14 - \frac{a}{3}\right) = 456000$$

$$f_s = \frac{456000}{(1.58)(11.94)} = 24800 \text{ psi} < 35000 \text{ psi}$$

ALL RIGHT

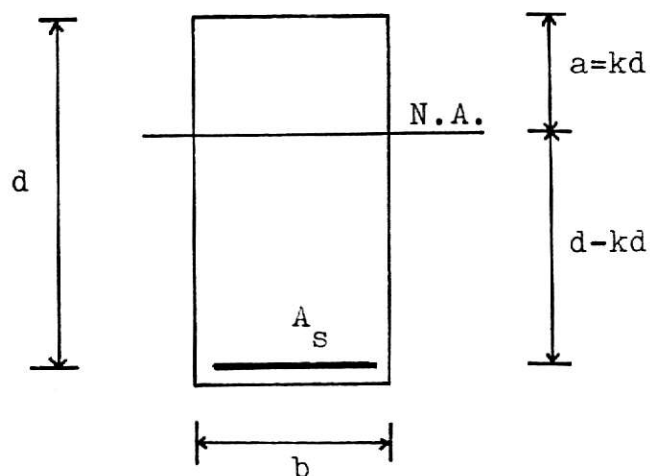


Fig.5-12 Section at Support

Comparison of Limit Design⁺ and Ultimate Strength Design

Ultimate Strength Design:

$$M_B = M_D = (-0.1071)wL^2 = (-0.1071)(2)(2100)(13)^2(12) \\ = 912000 \text{ lb-in}$$

$$912000 = (0.9)(0.0334)(bd)(35000)d\left(1 - \frac{(0.0334)(35000)}{(2)(0.85)(3000)}\right)$$

Assume $b = \frac{1}{2}d$

$$912000 = 1050(0.5)(d^3)(1 - 0.229) = 405d^3$$

$$d^3 = 2255, \quad d = 13.1"$$

use $d = 16", \quad b = 8"$

+ Using Baker's Method

$$912000 = 0.9A_s(35000)(16 - \frac{35000A_s}{2(0.85)(3000)8})$$

$$912000 = 31500A_s(16 - (0.855)A_s)$$

$$A_s = 2.04 \text{ in}^2$$

use 3 #8 $A_s = 2.37 \text{ in}^2$

	b	d	A_s *
Baker's Method	6"	14"	1.58 in ²
Ultimate Strength Design	8"	16"	2.37 in ²

* Actual area of steel

6. CONCLUSIONS

In limit design, the advantage of moment redistribution in calculating the maximum external moment acting at assumed hinges is utilized. By using limit design, we can make a more accurate estimate of the actual condition of a structure under loading and the maximum allowable load of a structure. And from a practical point of view, limit design is easy to carry out and will lead to a more economical design. After a structure has been analyzed and designed by using the limit design method, it is necessary and very important to check its serviceability, shear stress, bond slip and stability. Although there are many methods available for improving the ductility of concrete and increasing the rotation capacity of a hinge, an under-reinforced concrete beam may have enough rotation capacity without any special secondary reinforcement to permit other hinges to form, and finally to form a failure mechanism. This characteristic has been shown clearly in the two numerical examples. Both of the two designed beams have enough rotation capacity to meet the requirement of the rotation compatibility.

Two methods, the Cambridge method and Baker's method, have been discussed in this report. The Cambridge method is completely the same as the plastic design of steel structures.

It also has the same difficulty as plastic design has, ie. how to find the correct failure mechanism pattern. The value of the moment of inertia used in this method is calculated from the transformed section, and the value of E used in this method is the initial tangent modulus. In Baker's method, by assuming the locations of plastic hinges and the ultimate moments of assumed hinge-sections, the problem is reduced to solving N independent equations, each of which contains only one unknown θ_i . This assumption greatly simplifies Baker's method and also makes it become a more practical method of calculation. The value of EI used in Baker's method is calculated from the relationship of moment and curvature ($\frac{-M}{EI} = \frac{d^2y}{dx^2} = \phi$). One thing that should be mentioned here is that although many methods and theories have been developed for limit design, the most simple, practical and widely accepted one is the theory and method presented by Baker¹. The advantages of Baker's method are:

- 1). The rotations at hinges have been calculated and checked as being reasonable.
- 2). The hinge location can be fixed arbitrarily by the designer at the approximate sections having maximum moment in the elastic case.
- 3). The method can be used to design a structure to satisfy both the ultimate load and working load condition.

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ASPECTS OF LIMIT DESIGN OF
REINFORCED CONCRETE STRUCTURES

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ABSTRACT

The objective of this report is to introduce the basic idea of limit design of reinforced concrete structures and two design methods, the Cambridge method and Baker's method, corresponding to this idea. Limit design of reinforced concrete structures is similar to plastic design of steel structures and is an extension of ultimate strength design. There are four main parts in this report. They are:

- (1). The concept of limit design --- a review of plastic design of steel structures, the Cambridge method, Baker's method.
- (2). Moment curvature relationships --- a review of the compatibility of rotation, calculating the rotations required at a plastic hinge (reinforced concrete structures), calculating the rotation capacity available in a section.
- (3). Ways to increase the rotation capacity of a section --- a review of the effect of helical binders, the effect of rectangular spiral binders, the effect of compression steel and the effect of depth.
- (4). Design procedures and numerical examples of the

Cambridge method and Baker's method.