

BUCKLING ANALYSIS OF FOLDED PLATES

by 6791

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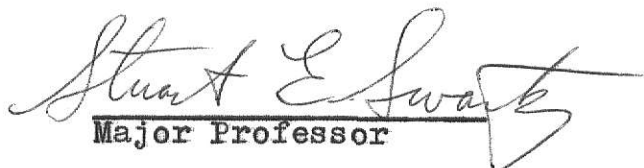
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CHAPTER I

INTRODUCTION

During recent years folded plate construction has found increasing application for roofs of industrial buildings and hangers. Such construction is particularly well-suited to fairly long spans, possessing some of the attributes of thin shell construction with the added advantage of somewhat simpler fabrication or forming. The materials required are usually much less than needed for flat slab, beam and slab or other conventional systems and are little more than required for continuously curved shells with the advantage of utilizing relatively simple form work.

Several procedures for the analysis of folded plate structures have been developed for the determination of stresses throughout these structures. In addition, several experimental studies^{4,7,10} * have been reported which indicate satisfactory agreement between measured and predicted values of stresses induced in such structures by the presence of transverse applied loads. Because information concerning the possibility of local or general instability of folded plate structures has been lacking, it has not ordinarily been possible in the past to predict the buckling load. To the

* Superscripts refer to references listed in the References.

writer's knowledge, however, before Swartz's⁶ analysis there has been little or no work published which deals with the buckling behaviour of folded plate structures. This report is concerned with the possibility of buckling of an individual plate element of a folded plate structure. Such a plate element is considered to be elastically supported along its longitudinal edges and simply supported along its transverse edges. Following Bleich,² the buckling may be considered to be caused essentially by the in-plane forces. Swartz^{6,7,8} introduced a buckling analysis of the folded plate and considered two possible types of buckling behaviour. The first type of buckling is that caused by transverse in-plane forces and it is treated separately from the second type of buckling caused by the longitudinal and shear in-plane forces acting in combination. In reality, as indicated by Swartz and Mikhail⁷ all of these in-plane forces should be taken into account at the same time.

The method used herein to determine the critical load is based upon an energy approach. An analysis considering transverse, longitudinal and shearing in-plane forces acting at the same time which utilizes an energy approach⁹ is presented in Chapter II. A necessary deflection field satisfying the boundary conditions used in this analysis is that given by Lundquist and Stowell,⁸ The scope of this report is confined to the buckling analysis of folded plate structures composed of rectangular thin plates rigidly connected along their

common ridges. The structure is supported on two end diaphragms perpendicular to the longitudinal axis and is acted upon by a uniformly distributed load.

By assuming pure compression, pure shear and simple supports on all sides, some numerical results for buckling loads are obtained in Chapter III. Assuming the plates are elastically supported along their longitudinal edges and simply supported along their transverse edges, a computer program was used to apply this analysis to two types of models of folded plate structures. The buckling results for these two types of models are listed in Chapter III and are compared with the results obtained from (1) the analysis of Swartz and Guralnick,^{6,8} (2) an analysis without considering the effect of shearing forces, and (3) experimental model tests.¹⁰

CHAPTER II

BUCKLING ANALYSIS OF FOLDED PLATES

In the calculation of critical values of forces applied in the middle plane of a plate at which the flat form of equilibrium becomes unstable and the plate begins to buckle, the same methods as in the case of compressed bars can be used.

By assuming that from the beginning the plate has some initial curvature or some lateral loading, the critical values of the forces acting in the middle plane of a plate can be obtained. Then those values of the forces in the middle plane at which deflections tend to grow indefinitely are usually the critical values.

Another way of investigating such a stability problem is to assume that the plate buckles slightly under the action of forces applied in its middle plane and then to calculate the magnitudes that the forces must have in order to keep the plate in such a slightly buckled shape. Referring to Fig. 1, the differential equation of the deflection surface in this case is obtained by Timoshenko:⁹

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{1}{D} \left(N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2 N_{xy} \frac{\partial^2 w}{\partial x \partial y} \right) \dots \dots \dots 2 \cdot 1$$

in which

$w = w(x,y)$, the deflection function

N_x = longitudinal in-plane forces

N_y = transverse in-plane forces

N_{xy} = shearing in-plane forces

$$D = \frac{E h^3}{12 (1 - \nu^2)}$$

ν = poisson's ratio

The simplest case is obtained when the forces N_x , N_y , and N_{xy} are constant throughout the plate. Assuming that there are given ratios between these forces so that $N_y = aN_x$ and $N_{xy} = bN_x$, and solving Eqn. 2.1 for the given boundary conditions, one finds that the assumed buckling of the plate is possible only for certain definite values of N_x . The smallest of these values is chosen to be the desired critical value.

But folded plates have variable forces N_x , N_y , and N_{xy} throughout each plate in the structure. The problem becomes more involved, since Eqn. 2.1 has in this case variable coefficients, but the general conclusion remains the same. In such case it may be assumed that the expressions for the forces N_x , N_y , and N_{xy} have a common factor k , so that a gradual increase of loading is obtained by an increase of this factor. From the investigation of Eqn. 2.1, together with the given boundary conditions, it will be concluded then

that curved forms of equilibrium are possible only for certain values of the factor k and that the smallest of these values will define the critical loading.

The energy method also can be used in investigating buckling of plates. This method is especially useful in those cases where a rigorous solution of Eqn. 2.1 is unknown or where the plate is reinforced by stiffeners and it is required to find only an approximate value of the critical load. In applying this method it is assumed that the plate, which is stressed by in-plane forces, undergoes some small lateral bending consistent with the given boundary conditions. Such limited bending can be produced without stretching of the middle plane, and one needs consider only the energy of bending and the corresponding work done by the in-plane forces of the plate. If the work done by these forces is smaller than the strain energy of bending for every possible shape of lateral buckling, the flat form of equilibrium of the plate is stable. If the same work becomes larger than the energy of bending for any shape of lateral deflection, the plate is unstable and buckling occurs.

Denoting the work done by external forces by T_c and the strain energy of bending by V_c , the critical values of forces may be found from the equation⁹

$$T_c = V_c \dots \dots \dots 2 \cdot 2$$

Based upon numerical results for end supported folded

plates, ^{1,3,4,5,8} it is assumed as shown in Fig. 4, Fig. 5, and Fig. 6 that

1. The variation of N_x is linear in the plate transverse direction and is parabolic in the longitudinal direction.

2. The variation of N_y is constant in the longitudinal direction and is parabolic in the plate transverse direction.

3. The variation of N_{xy} is linear in the longitudinal direction and is parabolic in the plate transverse direction.

Based upon the above assumptions, the in-plane forces on a plate element due to a unit load acting over the entire structure can be represented by:*

$$N_x^{q=1} = \frac{1}{2} \left(A_{01} + A_{11} \frac{x}{a} + A_{21} \frac{x^2}{a^2} \right) + \frac{y}{b} \left(A_{02} + A_{12} \frac{x}{a} + A_{22} \frac{x^2}{a^2} \right) \dots 2.3$$

in which A_{01} , A_{11} , A_{21} , A_{02} , A_{12} , and A_{22} are defined in Appendix I;

$$N_y^{q=1} = B_0 + B_1 \frac{y}{b} + B_2 \frac{y^2}{b^2} \dots 2.4$$

in which B_0 , B_1 , and B_2 are defined in Appendix I;

* The derivations of $N_x^{q=1}$, $N_y^{q=1}$, and $N_{xy}^{q=1}$ are outlined in Appendix I.

$$N_{xy}^{q=1} = K_0 + K_1 \frac{x}{a} + \left(K_2 + K_3 \frac{x}{a} \right) \frac{y}{b} \\ + \left(K_4 + K_5 \frac{x}{a} \right) \frac{y^2}{b^2} \dots \dots \dots 2 \cdot 5$$

in which K_0 , K_1 , K_2 , K_3 , K_4 , and K_5 are defined in Appendix I.

A sketch of a typical plate element with internal forces and associated deflections is shown in Fig. 1. Each plate is assumed to be supported on end diaphragms which are perfectly flexible normal to the plane of the diaphragms and perfectly rigid parallel to the plane of the diaphragms. The thickness of the plate is small compared to the other dimensions and small deflections are assumed throughout. The material is assumed to be homogeneous, isotropic and elastic.

A plate is divided longitudinally and transversely into panels. Each panel is assumed to be simply supported along its transverse edges and to be elastically supported with regard to rotation along its longitudinal edges. The deflections, w_i , along all edges are assumed to be zero.

The possibility of local buckling of the individual plate was demonstrated by tests⁷ carried out on symmetrical folded plates under the action of uniformly distributed loads. This local buckling was observed to start in an intermediate zone symmetrically located about the midspan of the structure. It is then assumed that the buckling load of the plate is that of an intermediate plane panel of initially indetermi-

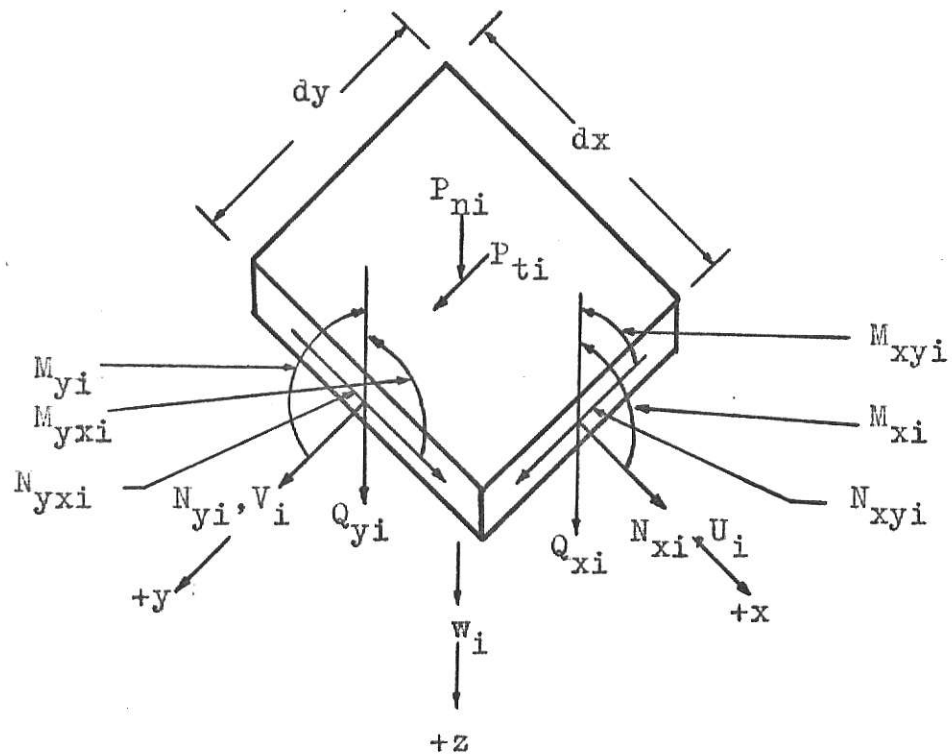


Fig. 1 Element of a folded plate structure

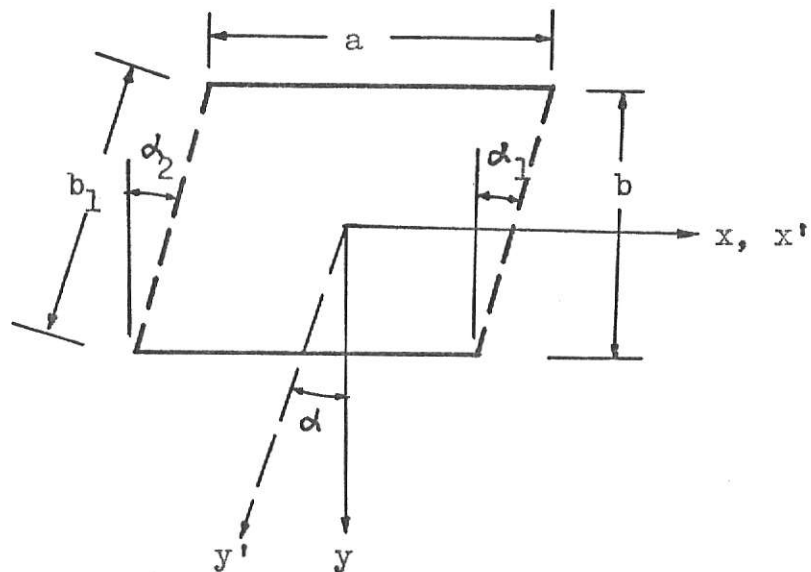


Fig. 2 The oblique coordinate system of plate panel

nate length a and width b .

If such a panel is subjected to all the in-plane forces at the same time, the panel may undergo a deformation along its transverse edges as shown in Fig. 2.

Setting $\alpha_1 = \alpha_2 = \alpha$, the relationships between the orthogonal and the oblique coordinate systems are

$$x = x' - y' \sin \alpha$$

$$y = y' \cos \alpha$$

$$dx = dx'$$

$$dy = \cos \alpha \, dy'$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial x'}$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial x'^2}$$

$$\frac{\partial w}{\partial y} = \frac{1}{\cos \alpha} \left(\frac{\partial w}{\partial y'} + \sin \alpha \frac{\partial w}{\partial x'} \right)$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{1}{\cos^2 \alpha} \left(\frac{\partial^2 w}{\partial y'^2} + 2 \frac{\partial^2 w}{\partial x' \partial y'} \sin \alpha + \frac{\partial^2 w}{\partial x'^2} \sin^2 \alpha \right)$$

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{1}{\cos \alpha} \left(\frac{\partial^2 w}{\partial x' \partial y'} + \frac{\partial^2 w}{\partial x'^2} \sin \alpha \right) \dots 2.6$$

For the oblique coordinate system, the equations of $N_x^{q=1}$, $N_y^{q=1}$, and $N_{xy}^{q=1}$ can be rewritten as:

$$\begin{aligned}
N_x^{q=1} = & \frac{1}{2} \left(A_{01} + A_{11} \frac{x'}{a} - A_{11} \frac{y'}{a} \sin \alpha + A_{21} \frac{x'^2}{a^2} \right. \\
& - 2 A_{21} \frac{x'y'}{a^2} \sin \alpha + A_{21} \frac{y'^2}{a^2} \sin^2 \alpha \left. \right) + \frac{y'}{b_1} \left(A_{02} \right. \\
& + A_{12} \frac{x'}{a} - A_{12} \frac{y'}{a} \sin \alpha + A_{22} \frac{x'^2}{a^2} - 2 A_{22} \frac{x'y'}{a^2} \sin \alpha \\
& \left. + A_{22} \frac{y'^2}{a^2} \sin^2 \alpha \right) \dots \dots \dots 2.7
\end{aligned}$$

$$N_y^{q=1} = B_0 + B_1 \frac{y'}{b_1} + B_2 \frac{y'^2}{b_1^2} \dots \dots \dots 2.8$$

$$\begin{aligned}
N_{xy}^{q=1} = & K_0 + K_1 \frac{x'}{a} - K_1 \frac{y'}{a} \sin \alpha + \left(K_2 + K_3 \frac{x'}{a} \right. \\
& - K_3 \frac{y'}{a} \sin \alpha \left. \right) \frac{y'}{b_1} + \left(K_4 + K_5 \frac{x'}{a} \right. \\
& \left. - K_5 \frac{y'}{a} \sin \alpha \right) \frac{y'^2}{b_1^2} \dots \dots \dots 2.9
\end{aligned}$$

Referring to Fig. 2, the boundary conditions for a panel which is assumed to be simply supported along its transverse edges and to be elastically supported along its longitudinal edges are as follows:

$$\text{At the transverse edges, } x' = \pm \frac{a}{2},$$

$$w = 0$$

$$M_{x'} = -D \left(\frac{\partial^2 w}{\partial x'^2} + \frac{\partial^2 w}{\partial y'^2} \right) = 0$$

and along the longitudinal edges, at $y' = -b_1/2$,

$$w = 0.$$

If the stiffness against rotation is $S_0 = \frac{M_{y'}}{4\theta}$, then for an elastic support against rotation

$$M_{y'} = 4 S_0 \theta.$$

Taking the positive transverse moment causing a positive rotation at $y' = -b_1/2$ and noting that $\theta = \frac{\partial w}{\partial y'}$, the boundary condition in this case is

$$D \left(\frac{\partial^2 w}{\partial y'^2} + \nu \frac{\partial^2 w}{\partial x'^2} \right) - 4 S_0 \frac{\partial w}{\partial y'} = 0$$

and at $y' = +b_1/2$,

$$w = 0,$$

$$\text{and } D \left(\frac{\partial^2 w}{\partial y'^2} + \nu \frac{\partial^2 w}{\partial x'^2} \right) + 4 S_0 \frac{\partial w}{\partial y'} = 0.$$

A deflection function w satisfying the above boundary conditions as presented by Lundquist and Stowell is given by Swartz.⁸

$$w = B \frac{\pi e}{2} \left(\frac{y'^2}{b_1^2} - \frac{1}{4} \right) + \left(1 + \frac{e}{2} \right) \cos \frac{\pi y'}{b_1} \cos \frac{\pi x'}{a}$$

$$\dots \dots \dots 2 \dots 10$$

in which

$$e = \frac{4 S_0 b_1}{D}$$

B = undetermined constant.

The energy approach is used here to determine the buckling load. When a deflection field which satisfies only the boundary conditions at the edges of the plate is used to calculate the energies T_c and V_c , a possible solution to the buckling problem will be obtained if Eqn. 2.2 is satisfied. The solution obtained will be the correct one if the deflection field also satisfies the equilibrium equation, Eqn. 2.1, for the true buckled shape. Otherwise, the buckling load obtained by using the energy approach will be merely an upper bound.

Let T_c be the work done by the in-plane forces, V_{1c} be the strain energy of bending in the plate, and V_{2c} be the strain energy in the elastic supporting medium. From Eqn. 2.2, the requirement for buckling becomes

$$T_c = V_{1c} + V_{2c} \dots \dots \dots 2.11$$

where for the orthogononal coordinate system and a rectangular plate panel

$$T_c = - \frac{q}{2} \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} \left[N_x^{q=1} \left(\frac{\partial w}{\partial x} \right)^2 + N_y^{q=1} \left(\frac{\partial w}{\partial y} \right)^2 \right. \\ \left. + 2 N_{xy}^{q=1} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right] dx dy \dots \dots \dots 2.12$$

$$V_{1c} = \frac{D}{2} \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} \left\{ \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 + 2 \left(1 - \nu \right) \left[\left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right] \right\} dx dy \dots 2.13$$

$$V_{2c} = \frac{1}{2} \int_{-a/2}^{a/2} \left[(M_y \theta)_{y=b/2} + (M_y \theta)_{y=-b/2} \right] dx \dots 2.14$$

where M_y and θ are the transverse moment and rotation in the panel.

Substituting Eqns. 2.7, 2.8, 2.9, and 2.6 for the oblique case into the expressions of the internal strain energy of bending and the external energy of all in-plane forces obtains: (They are derived in Appendix II.)

$$\begin{aligned} T_c = & - \frac{q}{2} \left\{ F_1 \frac{B^2 \pi^2}{4} \frac{b_1}{a} \left[\cos \alpha \left(A_{01} + \frac{A_{21}}{12} + \frac{A_{21}}{2 \pi^2} \right) \right. \right. \\ & \left. \left. + \frac{2 \sin^2 \alpha}{\cos \alpha} B_0 + 4 \sin \alpha K_0 \right] \right. \\ & + F_2 B^2 \pi^2 \sin \alpha \frac{b_1}{a} \left[\cos \alpha \frac{b_1}{a} \left(\frac{A_{21}}{4} \frac{b_1}{a} \sin \alpha - \frac{A_{12}}{2} \right) \right. \\ & \left. + \frac{\sin \alpha}{2 \cos \alpha} B_2 + K_4 - K_3 \sin \alpha \frac{b_1}{a} \right] + F_3 \frac{B^2 \pi^2}{2} \frac{a}{b_1} \frac{B_0}{\cos \alpha} \\ & \left. + F_4 \frac{B^2 \pi^2}{2} \frac{a}{b_1} \frac{B_2}{\cos \alpha} \right\} \dots 2.15 \end{aligned}$$

$$V_{1c} = B^2 \frac{\pi^4 D}{4 b_1 a \cos^3 \alpha} \left[\left(\frac{b_1}{a} \right)^2 F_1 + \left(\frac{a}{b_1} \right)^2 F_5 \right. \\ \left. + 2 (1 + 2 \sin^2 \alpha) F_3 \right] \dots \dots \dots 2 \cdot 16$$

$$V_{2c} = B^2 \frac{\pi^2 D a e}{2 b_1^3 \cos^3 \alpha} \dots \dots \dots 2 \cdot 17$$

in which the coefficients F_1 , F_2 , F_3 , F_4 , and F_5 are defined in Appendix II.

For the buckling criteria, Eqn. 2.11 gives the critical load as:

$$q_{cr} = \frac{-\pi^2 D}{2 b_1 a \cos^3 \alpha} \left\{ \left(\frac{b_1}{a} \right)^2 F_1 + \left(\frac{a}{b_1} \right)^2 F_5 \right. \\ \left. + 2 (1 + 2 \sin^2 \alpha) F_3 + \frac{2 e}{\pi^2} \left(\frac{a}{b_1} \right)^2 \right\} \\ \div \left\{ \frac{F_1}{4} \frac{b_1}{a} \left[\cos \alpha \left(A_{01} + \frac{A_{21}}{12} + \frac{A_{21}}{2 \pi^2} \right) + \frac{2 \sin^2 \alpha}{\cos \alpha} B_0 \right. \right. \\ \left. \left. + 4 \sin \alpha K_0 \right] + F_2 \sin \alpha \frac{b_1}{a} \left[\cos \alpha \frac{b_1}{a} \left(\frac{A_{21}}{4} \frac{b_1}{a} \sin \alpha \right. \right. \right. \\ \left. \left. - \frac{A_{12}}{2} \right) + \frac{\sin \alpha}{2 \cos \alpha} B_2 + K_4 - K_3 \sin \alpha \frac{b_1}{a} \right] \right. \\ \left. + F_3 \frac{a}{b_1} \frac{B_0}{2 \cos \alpha} + F_4 \frac{a}{b_1} \frac{B_2}{2 \cos \alpha} \right\} \dots \dots \dots 2 \cdot 18$$

As the plate starts to buckle, the plate undergoes

transverse deformation caused by shear forces which may affect the critical load.

Let q_{cr} be $\frac{U(\alpha)}{V(\alpha)}$, then

$$\frac{\partial q_{cr}}{\partial \alpha} = \frac{V \frac{\partial U}{\partial \alpha} - U \frac{\partial V}{\partial \alpha}}{V^2}$$

and minimizing $\frac{\partial q_{cr}}{\partial \alpha}$ gives α_{cr} when the plate buckles.

$$V \frac{\partial U}{\partial \alpha} - U \frac{\partial V}{\partial \alpha} = 0 \dots\dots\dots 2.19$$

CHAPTER III

NUMERICAL RESULTS OF BUCKLING ANALYSIS

When the plate is elastically supported, the computer is very useful for calculating the buckling load because of the complexity of the equation. However if the plate is simply supported at both ends, ie. $e = 0$, Eqn. 2.18 can be simplified. By assuming pure compression, pure shear and simple supports on all sides, some numerical results for buckling loads are given in this chapter.

Case 1. Uniaxial compression acting in the x direction and all sides simply supported,

$$e = 0 ; \alpha = 0$$

$$N_{x1} = N_{x2} = N_{x3} = - q h = - h$$

$$N_y = N_{xy} = 0$$

The coefficients needed for the buckling equation are

$$A_0 = - h$$

$$A_1 = A_2 = 0$$

$$A_{01} = A_0^t + A_0^b = - 2 h$$

$$A_{02} = A_{11} = A_{12} = A_{21} = A_{22} = 0$$

$$B_0 = B_1 = B_2 = K_0 = K_1 = K_2 = K_3 = K_4 = K_5 = 0$$

$$F_1 = F_3 = F_5 = \frac{1}{2}$$

$$\therefore \frac{(N_x)_{cr}}{h} = \frac{\pi^2 D}{b^2 h} \left(\frac{b^2}{a^2} + \frac{a^2}{b^2} + 2 \right) \dots\dots\dots 3.1$$

This is the equation for $(N_x)_{cr}$ obtained by Timoshenko.⁹

Case 2. Uniaxial compression in the y direction and all sides simply supported.

$$e = 0 ; \alpha = 0$$

$$N_y^t = N_y^m = N_y^b = -q h = -h$$

$$N_x = N_{xy} = 0$$

and the coefficients are

$$B_0 = -h$$

$$B_1 = B_2 = 0$$

$$A_{01} = A_{02} = A_{11} = A_{12} = A_{21} = A_{22} = 0$$

$$K_0 = K_1 = K_2 = K_3 = K_4 = K_5 = 0$$

$$F_1 = F_3 = F_5 = 1/2$$

$$\therefore \frac{(N_y)_{cr}}{h} = \frac{\pi^2 D}{a^2 h} \left(\frac{b^2}{a^2} + \frac{a^2}{b^2} + 2 \right) \dots\dots\dots 3.2$$

Case 3. Pure shear is assumed.

$$N_{xy} = -h$$

$$N_x = N_y = 0$$

If e and α are not zero, the coefficients are

$$K_0 = -h$$

$$K_1 = K_2 = K_3 = K_4 = K_5 = 0$$

$$A_{01} = A_{02} = A_{11} = A_{12} = A_{21} = A_{22} = 0$$

$$B_0 = B_1 = B_2 = 0$$

F_1 , F_3 , and F_5 can not be simplified in this case. Knowing that $b_1 = b/\cos \alpha$, the critical shear force is obtained from Eqn. 2.18.

$$\begin{aligned} \frac{(N_{xy})_{cr}}{h} = & \frac{\pi^2 D}{b^2 h \sin 2\alpha} \left(\frac{b}{a} \right)^2 \frac{1}{\cos^2 \alpha} \\ & + \left(\frac{a}{b} \right)^2 \frac{\cos^2 \alpha}{F_1} \left(F_5 + \frac{2e}{\pi^2} \right) + 2 \left(1 \right. \\ & \left. + 2 \sin^2 \alpha \right) \frac{F_3}{F_1} \dots \dots \dots 3.3 \end{aligned}$$

which agrees with the equation given by E. Z. Stowell.⁸

In the case of pure shear, an equation to calculate α when the plate buckles is obtained by E. Z. Stowell⁸ by minimizing Eqn. 3.3 with respect to α . Such an equation is

$$\cos \alpha = \sqrt{C_3 + \sqrt{C_3^2 + C_4}} \dots \dots \dots 3.4$$

in which

$$C_3 = \frac{\frac{3}{2} C_2 - 2 \left(\frac{b}{a} \right)^2}{4 C_2 + C_1 \left(\frac{a}{b} \right)^2}$$

$$C_4 = \frac{3 \left(\frac{b}{a} \right)^2}{4 C_2 + C_1 \left(\frac{a}{b} \right)^2}$$

and

$$C_1 = \frac{F_5 + \frac{2e}{\pi^2}}{F_1}$$

$$C_2 = \frac{2 F_3}{F_1}$$

If we further assume that both ends of the plate are simply supported and $a = b$ for the square plate, then the coefficients to calculate α_{cr} are

$$e = 0$$

$$F_1 = F_3 = F_5 = 1/2$$

$$C_1 = 1$$

$$C_2 = 2$$

$$C_3 = 1/9$$

$$C_4 = 1/3$$

Eqn. 3.4 gives

$$\cos \alpha = 0.838$$

$$\alpha_{cr} = 33^\circ$$

Putting α_{cr} into Eqn. 3.3 determines the critical shear forces as

$$\begin{aligned} \frac{(N_{xy})_{cr}}{h} &= \frac{\pi^2 D}{b^2 h 0.914} \left[\frac{1}{(0.838)^2} + (0.838)^2 + 2 \right. \\ &\quad \left. + 4 (0.545)^2 \right] \\ &= 5.815 \frac{\pi^2 D}{b^2 h} \end{aligned}$$

Case 4. The longitudinal and shearing in-plane forces act in combination. Assume $N_x = N_{xy} = -h$, $a = b$, and both ends of the plate are simply supported. The coefficients needed for the buckling equation are

$$A_{01} = -2h$$

$$A_{02} = A_{11} = A_{12} = A_{21} = A_{22} = 0$$

$$K_0 = -h$$

$$K_1 = K_2 = K_3 = K_4 = K_5 = 0$$

$$B_0 = B_1 = B_2 = 0$$

$$F_1 = F_3 = F_5 = 1/2$$

With these coefficients, Eqn. 2.18 becomes

$$q_{cr} = \frac{\frac{\pi^2 D}{2 b^2} \left(\frac{1}{2 \cos^4 \alpha} + \frac{1}{2} + \frac{1}{\cos^2 \alpha} + 2 \frac{\sin^2 \alpha}{\cos^2 \alpha} \right)}{\frac{h}{8} \left(2 + 4 \frac{\sin \alpha}{\cos \alpha} \right)} \quad \dots 3.5$$

The partial derivatives of the numerator and the denominator of the right side of Eqn. 3.5 are

$$\frac{\partial U}{\partial \alpha} = \frac{\pi^2 D \sin \alpha}{b^2 \cos^3 \alpha} \left(\frac{1}{\cos^2 \alpha} + 3 \right)$$

$$\frac{\partial V}{\partial \alpha} = \frac{h}{2} \frac{1}{\cos^2 \alpha}.$$

From Eqn. 2.19, an equation to solve for α_{cr} is

$$\frac{\sin \alpha}{\cos^3 \alpha} + 3 \frac{\sin \alpha}{\cos \alpha} + 2 \frac{\sin^2 \alpha}{\cos^4 \alpha} + 4 \frac{\sin^2 \alpha}{\cos^2 \alpha} - \frac{1}{2 \cos^4 \alpha}$$

$$- \frac{1}{\cos^2 \alpha} - 0.5 = 0$$

Solving the above equation numerically gives

$$\alpha_{cr} = 19.62^\circ$$

Substituting α_{cr} into Eqn. 3.5 yields

$$q_{cr} = 2.938 \frac{\pi^2 D}{b^2 h}$$

For a panel acted upon by constant shearing and longitudinal stresses τ_{xya} and σ_{xa} the requirement for buckling is determined from an interaction formula.^{7,8}

$$\left(\frac{\tau_{xya}}{\tau_{xycr}} \right)^2 + \frac{\sigma_{xa}}{\sigma_{xcr}} \geq 1 \quad 3.6$$

Applying the numerical results previously obtained, Eqn. 3.6 becomes

$$\left(\frac{2.938}{5.815} \right)^2 + \frac{2.938}{4} = 0.99$$

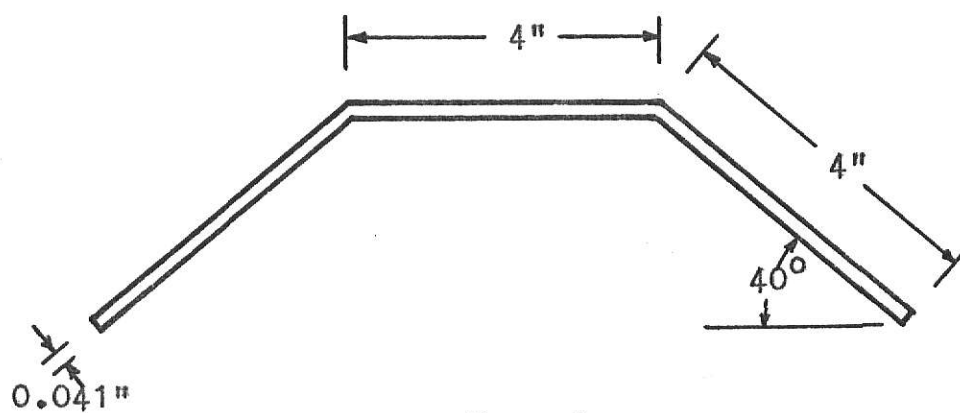
which shows that the buckling load obtained by using Eqn. 2.18 is quite satisfactory.

A computer program was written for the IBM 360-50 computer to apply this analysis to particular models of folded plate structures. The two types of models used here are shown in Fig. 3.

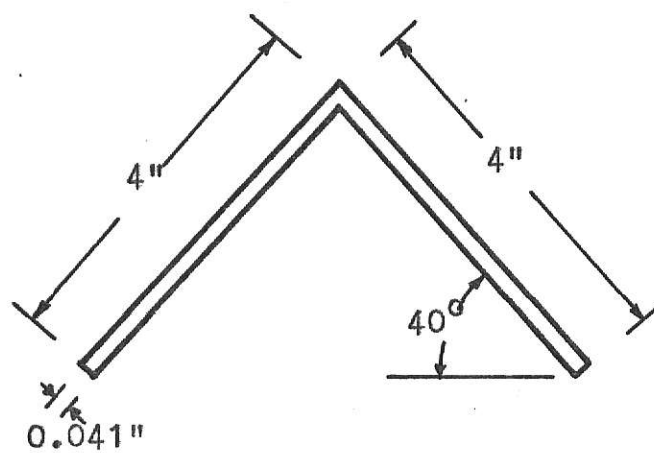
The following data are to be read in :

1. The number of plates.
2. The number of stiffeners.
3. The number of longitudinal data points.
4. Span length of the structure between the end diaphragms.
5. Young's modulus of material.
6. Poisson's ratio of material.
7. Width of each plate.
8. Thickness of each plate.
9. The maximum number of panels.
10. The number of deflection values in the plate transverse direction.
11. $N_x^{q=1}$, $N_y^{q=1}$, and $N_{xy}^{q=1}$ at all data points.
12. The transverse moments and deflections at all data points due to a unit load acting over the entire structure.

The buckling results for the models are listed in Table 1 and are compared with the results obtained from (1) the analysis of Swartz and Guralnick,⁸ (2) an analysis without considering the effect of shearing forces, and (3) experimental model tests.¹⁰



a. Type 1



b. Type 2

* No stiffener in the plates.

Plate length = 54"

Poisson's ratio = 0.333

Fig. 3 The Cross Section of The Models

Table 1. Comparison of Different Methods
of Analysis with Experimental Results.

	$\frac{q_{cr}}{E} \times 10^{-6}$			
	Analysis of Swartz ^{6,8}	Analysis without effect of N_{xy}	Analysis with effect of N_{xy}	Experimental results ¹⁰
Type 1	0.110	0.0524	0.0520	0.0441
Type 2	0.454	0.210	0.210	0.0894

E = Young's Modulus

The critical loads obtained in Table 1 first occurred at the center of the longitudinal span, where the shear forces are very small or zero. The shear forces in the Type 1 model and the Type 2 model seem to have had no effect on the buckling loads. In fact, the shear forces did affect the buckling loads for panels off the center of the span, but this effect was not great enough to control the critical buckling load. Although the results obtained in this analysis are much lower than the results given by the method of Swartz and Guralnick, the critical value of the buckling load for the Type 2 model is more than twice as great as the value obtained by the experimental test. Further study of the buckling problem of folded plates is greatly encouraged.

CHAPTER IV

CONCLUSIONS

An analysis of folded plate structures to predict loads at which local elements of plates will buckle has been presented. A plate element is divided longitudinally and transversely into panels. Each panel is assumed to be simply supported along its transverse edges and to be elastically supported with regard to rotation along its longitudinal edges. The deflections along all edges are assumed to be zero. If such a panel is subjected to all the in-plane forces at the same time, the panel may undergo a deformation along its transverse edges. Assuming the panel may buckle as it deforms transversely, an oblique coordinate system is employed throughout this analysis which makes use of the energy approach.

The following conclusions may be drawn from this study:

1. If the shear forces are small compared with the other forces, they will not affect the buckling load. This may be seen from the examples wherein the plate buckled first at the center of the span. If the shear forces are large compared with the other forces, they will affect the buckling load and the plate may initiate buckling off the center of the span. For the latter case, the analysis without considering the effects of shearing forces will result in mislea-

dingly high values for the buckling loads.

2. Since this analysis is concerned with all in-plane forces acting on the plates at the same time, any case of the possibility of local buckling of folded plate structures can be approximately predicted from this analysis. This can be seen from those numerical results obtained in Chapter III by assuming the cases of pure compression, pure shear and simple supports on all sides of the plates, and also can be seen from the buckling results obtained by applying this analysis to the models of folded plate structures and assuming all in-plane forces acting on the plates at the same time.

3. For the cases of pure compression, pure shear and simple supports on all sides of the plates, the buckling results obtained from this analysis agree with the results obtained from other analyses.^{8,9} The buckling results obtained for the models of folded plate structures are lower than those obtained from the analysis of Swartz and Guralnick⁶ and higher than those obtained from experimental tests.¹⁰ However, it is very hard to calculate the critical values of loads without using a computer if the plates are assumed to be elastically supported along the longitudinal edges.

4. There still are many factors that affect the buckling strength which should be considered in addition to those used in this study. Further research for the buckling of folded plates is needed.

ACKNOWLEDGEMENTS

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NOTATION

$A_{01}, A_{11}, A_{21},$	
A_{02}, A_{12}, A_{22}	Coefficients of $N_x^{q=1}$
B	Undetermined constant of w
B_0, B_1, B_2	Coefficients of $N_y^{q=1}$
D	$\frac{E h^3}{12 (1 - \nu^2)}$, plate constant
E	Young's modulus
F_1, F_2, F_3, F_4, F_5	Buckling coefficients
$K_0, K_1, K_2, K_3, K_4, K_5$	Coefficients of $N_{xy}^{q=1}$
M_x	The plate longitudinal moment
M_y	The plate transverse moment
N_x	Longitudinal in-plane forces/unit length
N_y	Transverse in-plane forces/unit length
N_{xy}	Shearing in-plane forces/unit length
$N_x^{q=1}, N_y^{q=1}, N_{xy}^{q=1}$	N_x, N_y, N_{xy} due to unit uniform vertical live load
S_0	Plate stiffness against rotation
T_c	The work done by external forces
V_{1c}	The strain energy of bending in the plate

V_{2c}	The strain energy in the elastic supporting medium
a	Panel length
b	Panel width
b_1	Width of the plate panel after transverse deformation
e	$\frac{4 S_0 b_1}{D}$
q	Uniform vertical live load
h	Thickness of the plate
w	Deflection function
x, y	Directions of orthogonal coordinate system
x', y'	Directions of oblique coordinate system
α	The angle between orthogonal and oblique coordinate of the plate panel
θ	Rotation of the plate panel
ν	Poisson's ratio

APPENDIX I

THE DERIVATIONS OF $N_x^{q=1}$, $N_y^{q=1}$, AND $N_{xy}^{q=1}$

For the derivations of $N_x^{q=1}$, $N_y^{q=1}$, and $N_{xy}^{q=1}$ in this appendix, a unit uniformly distributed load ($q=1$) acting over the entire structure of folded plate is assumed throughout.

a. The derivation of $N_x^{q=1}$.

As shown in Fig. 4a, N_{xn}^t and N_{xn}^b are the longitudinal forces at the top and the bottom of the n th edge of the panels respectively. If the variation of N_x is linear in the plate transverse direction, then the N_x at any point can be represented as:

$$N_x = \frac{N_x^t + N_x^b}{2} + \frac{y}{b} (N_x^b - N_x^t) \quad \dots \quad A . 1$$

in which N_x^t and N_x^b are the longitudinal forces at the top edge and bottom edge in the transverse direction at that point.

Referring to Fig. 4b, the variation of N_x is parabolic in the longitudinal direction. It then can be assumed that

$$N_x = a_0 + a_1x + a_2x^2 \quad \dots \quad A . 2$$

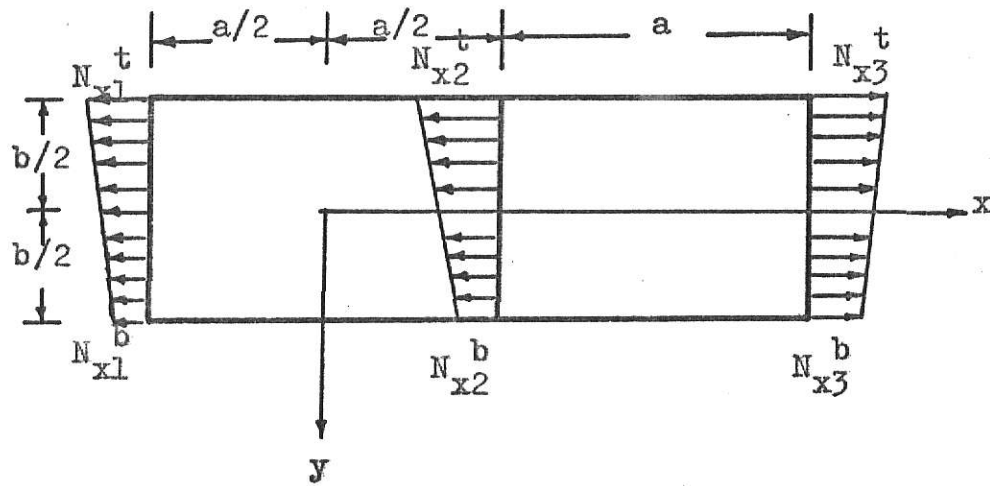


Fig. 4a The Variation of N_x in Y Direction

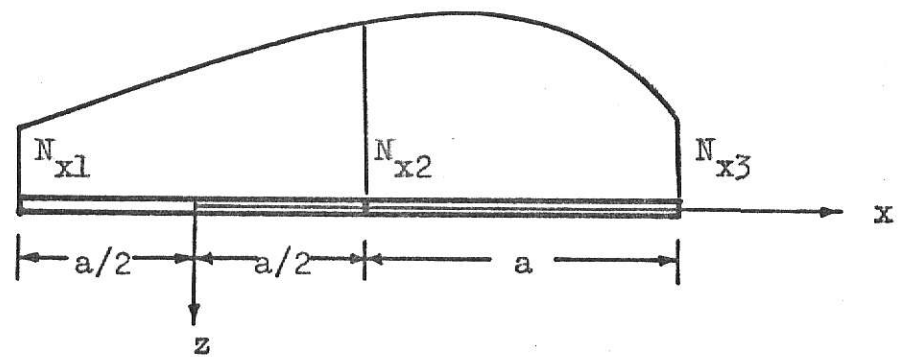


Fig. 4b The Variation of N_x in X Direction

For the boundary conditions at $x = -a/2$, $N_x = N_{x1}$, that is

$$N_{x1} = a_0 - \frac{a}{2} a_1 + \frac{a^2}{4} a_2 \dots \dots \dots A . 3$$

At $x = a/2$, $N_x = N_{x2}$, that is

$$N_{x2} = a_0 + \frac{a}{2} a_1 + \frac{a^2}{4} a_2 \dots \dots \dots A . 4$$

and at $x = 3a/2$, $N_x = N_{x3}$,

$$N_{x3} = a_0 + \frac{3a}{2} a_1 + \frac{9a^2}{4} a_2 \dots \dots \dots A . 5$$

Solving Eqns. A.3, A.4, and A.5 gives

$$a_0 = \frac{3 N_{x1} + 6 N_{x2} - N_{x3}}{8}$$

$$a_1 = \frac{N_{x2} - N_{x1}}{a}$$

$$a_2 = \frac{N_{x1} - 2 N_{x2} + N_{x3}}{2 a^2}$$

Setting

$$A_0 = \frac{3 N_{x1} + 6 N_{x2} - N_{x3}}{8}$$

$$A_1 = N_{x2} - N_{x1}$$

$$A_2 = \frac{N_{x1} - 2 N_{x2} + N_{x3}}{2}$$

Eqn. A.2 becomes

$$N_x = A_0 + A_1 \frac{x}{a} + A_2 \frac{x^2}{a^2}$$

In particular,

$$N_x^t = A_0^t + A_1^t \frac{x}{a} + A_2^t \frac{x^2}{a^2} \dots \dots \dots A . 6$$

$$N_x^b = A_0^b + A_1^b \frac{x}{a} + A_2^b \frac{x^2}{a^2} \dots \dots \dots A . 7$$

where

$$A_0^t = \frac{3 N_{x1}^t + 6 N_{x2}^t - N_{x3}^t}{8}$$

$$A_1^t = N_{x2}^t - N_{x1}^t$$

$$A_2^t = \frac{N_{x1}^t - 2 N_{x2}^t + N_{x3}^t}{2}$$

$$A_0^b = \frac{3 N_{x1}^b + 6 N_{x2}^b - N_{x3}^b}{8}$$

$$A_1^b = N_{x2}^b - N_{x1}^b$$

$$A_2^b = \frac{N_{x1}^b - 2 N_{x2}^b + N_{x3}^b}{2}$$

Substituting Eqn. A.6 and Eqn. A.7 into Eqn. A.1 gives

$$N_x = \frac{1}{2} \left[A_0^t + A_0^b + (A_1^t + A_1^b) \frac{x}{a} + (A_2^t \right.$$

$$+ A_2^b \Big) \frac{x^2}{a^2} \Big] + \frac{y}{b} \left[A_0^b - A_0^t + (A_1^b - A_1^t) \frac{x}{a} \right. \\ \left. + (A_2^b - A_2^t) \frac{x^2}{a^2} \right]$$

Setting

$$A_{01} = A_0^t + A_0^b$$

$$A_{02} = A_0^b - A_0^t$$

$$A_{11} = A_1^t + A_1^b$$

$$A_{12} = A_1^b - A_1^t$$

$$A_{21} = A_2^t + A_2^b$$

$$A_{22} = A_2^b - A_2^t$$

and noting that the expression of N_x is for the case of $q=1$, we then have

$$N_x^{q=1} = \frac{1}{2} \left(A_{01} + A_{11} \frac{x}{a} + A_{21} \frac{x^2}{a^2} \right) \\ + \frac{y}{b} \left(A_{02} + A_{12} \frac{x}{a} + A_{22} \frac{x^2}{a^2} \right) \quad \dots \quad A \cdot 8$$

b. The derivation of $N_y^{q=1}$.

As shown in Fig. 5, N_{yn}^t , N_{yn}^m , and N_{yn}^b are the plate transverse forces at $y = -b/2$, 0 , and $b/2$ respectively. The variation of N_y is constant in the longitudinal direction; that is

$$N_{y1}^t = N_{y2}^t = N_y^t$$

$$N_{y1}^b = N_{y2}^b = N_y^b$$

$$N_{y1}^m = N_{y2}^m = N_y^m$$

In the plate transverse direction, a parabolic variation of N_y is assumed.

$$N_y = b_0 + b_1 y + b_2 y^2 \quad \text{-----} \quad A.9$$

Applying the boundary conditions:

$$\text{at } y=0, N_y = N_y^m$$

$$\therefore b_0 = N_y^m$$

$$\text{at } y = b/2, N_y = N_y^b$$

$$b_0 + \frac{b}{2} b_1 + \frac{b^2}{4} b_2 = N_y^b$$

$$\text{at } y = -b/2, N_y = N_y^t$$

$$b_0 - \frac{b}{2} b_1 + \frac{b^2}{4} b_2 = N_y^t$$

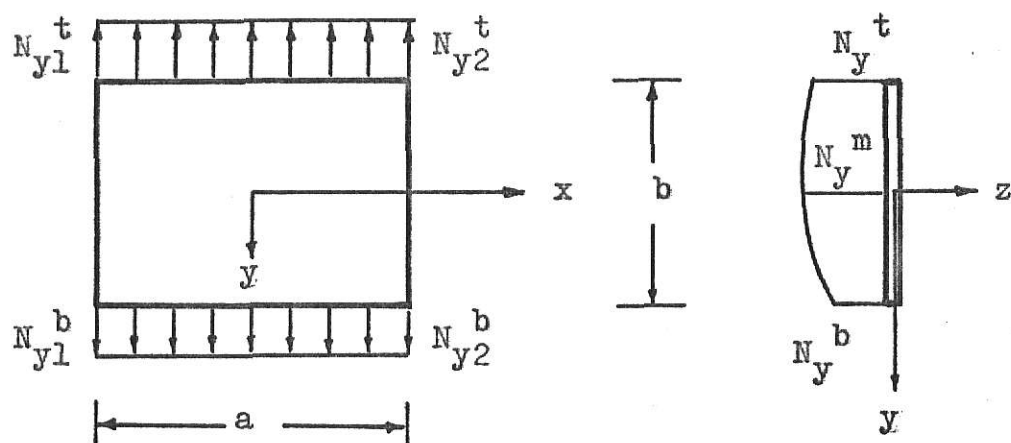


Fig. 5 The Variation of N_y

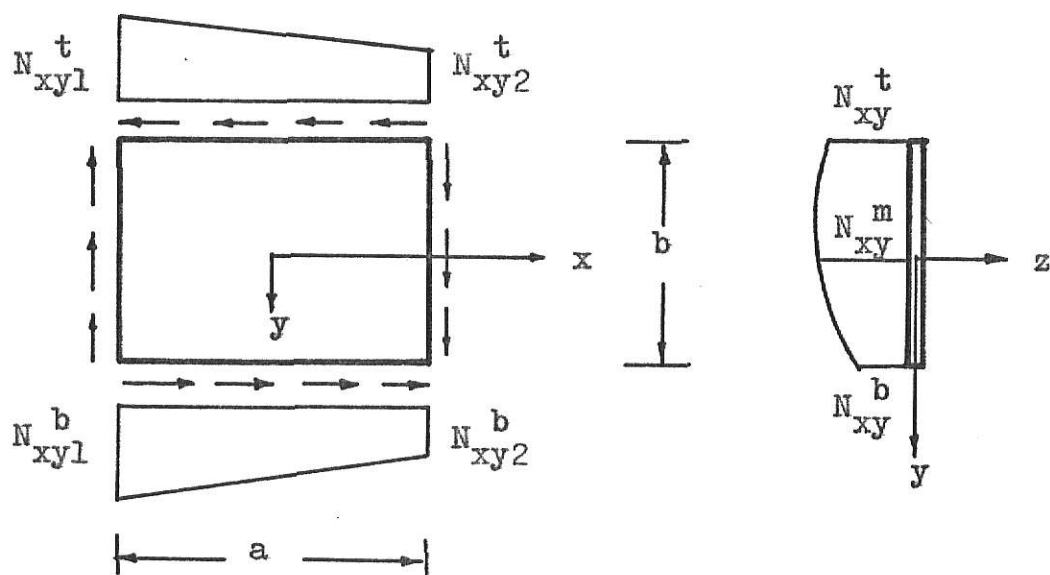


Fig. 6 The Variation of N_{xy}

gives

$$b_0 = N_y^m$$

$$b_1 = \frac{N_y^b - N_y^t}{b}$$

$$b_2 = \frac{2 (N_y^b + N_y^t)}{b^2} - \frac{4 N_y^m}{b^2}$$

Letting

$$B_0 = N_y^m$$

$$B_1 = N_y^b - N_y^t$$

$$B_2 = 2 (N_y^b + N_y^t) - 4 N_y^m$$

and substituting into Eqn. A.9 gives

$$N_y^{q=1} = B_0 + B_1 \frac{y}{b} + B_2 \frac{y^2}{b^2} \dots \dots \dots A. 10$$

c. The derivation of $N_{xy}^{q=1}$

As shown in Fig. 6, N_{xyn}^t , N_{xyn}^m , and N_{xyn}^b are the shear forces at $y = -b/2$, 0 , and $b/2$ of n th edge of the panels. If the variation of N_{xy} is linear in the longitudinal direction of the plate, any N_{xy}^t , N_{xy}^m , and N_{xy}^b between the edge 1 and the edge 2 are related to N_{xy1}^t , N_{xy1}^m , N_{xy1}^b ,

N_{xy2}^t , N_{xy2}^m , and N_{xy2}^b as

$$N_{xy}^t = k_1^t + k_2^t \frac{x}{a}$$

$$N_{xy}^b = k_1^b + k_2^b \frac{x}{a}$$

$$N_{xy}^m = k_1^m + k_2^m \frac{x}{a} \quad \text{----- A . 11}$$

in which

$$k_1^t = \frac{N_{xy2}^t + N_{xyl}^t}{2}$$

$$k_2^t = N_{xy2}^t - N_{xyl}^t$$

$$k_1^b = \frac{N_{xy2}^b + N_{xyl}^b}{2}$$

$$k_2^b = N_{xy2}^b - N_{xyl}^b$$

$$k_1^m = \frac{N_{xy2}^m + N_{xyl}^m}{2}$$

$$k_2^m = N_{xy2}^m - N_{xyl}^m$$

Again, the variation of N_{xy} is parabolic in the plate transverse direction.

$$N_{xy} = C_0 + C_1 y + C_2 y^2 \quad \text{----- A . 12}$$

Applying boundary conditions

$$\text{at } y = 0, N_{xy} = N_{xy}^m$$

$$\text{at } y = b/2, N_{xy} = N_{xy}^b$$

$$\text{at } y = -b/2, N_{xy} = N_{xy}^t$$

gives

$$C_0 = N_{xy}^m$$

$$C_1 = \frac{N_{xy}^b - N_{xy}^t}{b}$$

$$C_2 = \frac{2 (N_{xy}^b + N_{xy}^t)}{b^2} - \frac{4 N_{xy}^m}{b^2}$$

Substituting Eqn. A.11 and the coefficients C_0 , C_1 , and C_2

into Eqn. A.12 gives

$$\begin{aligned} N_{xy}^{q=1} = & K_0 + K_1 \frac{x}{a} + (K_2 + K_3 \frac{x}{a}) \frac{y}{b} \\ & + (K_4 + K_5 \frac{x}{a}) \frac{y^2}{b^2} \dots \dots \dots \text{A. 13} \end{aligned}$$

in which

$$K_0 = k_1^m$$

$$K_1 = k_2^m$$

$$K_2 = k_1^b - k_1^t$$

$$K_3 = k_2^b - k_2^t$$

$$K_4 = 2 (k_1^t + k_1^b - 2 k_1^m)$$

$$K_5 = 2 (k_2^t + k_2^b - 2 k_2^m)$$

APPENDIX II

DETAILS OF BUCKLING ANALYSIS

The expressions for the total internal strain energy of bending for a panel of the plate, $V_{1c} + V_{2c}$, and the total external energy of all in-plane forces, T_c , are developed in this appendix. The general expressions are Eqns. 2.12, 2.13, and 2.14.

A function w which satisfies the boundary conditions given in Chapter II is Eqn. 2.10:

$$w = B \left[\frac{\pi e}{2} \left(\frac{y'^2}{b_1^2} - \frac{1}{4} \right) + \left(1 + \frac{e}{2} \right) \cos \frac{\pi y'}{b_1} \right] \cos \frac{\pi x'}{a}$$

..... 2 . 10

in which

$$e = \frac{4 S_0 b_1}{D}$$

$$D = \frac{E h^3}{12 (1 + \nu^2)}$$

ν = Poisson's ratio

B = undetermined constant.

From Eqn. 2.10, the following expressions are obtained:

$$\frac{\partial w}{\partial x'} = - \frac{B \pi}{a} \left[\frac{\pi e}{2} \left(\frac{y'^2}{b_1^2} - \frac{1}{4} \right) + \left(1 + \frac{e}{2} \right) \cos \frac{\pi y'}{b_1} \right] \sin \frac{\pi x'}{a}$$

$$+ \frac{e}{2} \cos \frac{\pi y'}{b_1} \Big] \sin \frac{\pi x'}{a}$$

$$\frac{\partial^2 w}{\partial x'^2} = - \frac{B \pi^2}{a^2} \left[\frac{\pi e}{2} \left(\frac{y'^2}{b_1^2} - \frac{1}{4} \right) + \left(1 + \frac{e}{2} \right) \cos \frac{\pi y'}{b_1} \right] \cos \frac{\pi x'}{a}$$

$$+ \frac{e}{2} \cos \frac{\pi y'}{b_1} \Big] \cos \frac{\pi x'}{a}$$

$$\frac{\partial^2 w}{\partial x' \partial y'} = - \frac{B \pi}{a} \left[\frac{\pi e}{b_1^2} y' - \frac{\pi}{b_1} \left(1 + \frac{e}{2} \right) \sin \frac{\pi y'}{b_1} \right] \sin \frac{\pi x'}{a}$$

$$\frac{\partial w}{\partial y'} = B \left[\frac{\pi e}{b_1^2} y' - \frac{\pi}{b_1} \left(1 + \frac{e}{2} \right) \sin \frac{\pi y'}{b_1} \right] \cos \frac{\pi x'}{a}$$

$$\frac{\partial^2 w}{\partial y'^2} = B \left[\frac{\pi e}{b_1^2} - \frac{\pi^2}{b_1^2} \left(1 + \frac{e}{2} \right) \cos \frac{\pi y'}{b_1} \right] \cos \frac{\pi x'}{a}$$

..... A . 14

Determining T_c , Eqn. 2.12 is rewritten as:

$$- \frac{2}{q} T_c = \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} N_x^{q=1} \left(\frac{\partial w}{\partial x} \right)^2 dx dy$$

$$+ \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} N_y^{q=1} \left(\frac{\partial w}{\partial y} \right)^2 dx dy$$

$$+ \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} 2 N_{xy}^{q=1} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} dx dy \dots A. 15$$

From Eqns. 2.7 and 2.6

$$\begin{aligned} & \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} N_x^{q=1} \left(\frac{\partial w}{\partial x} \right)^2 dx dy \\ &= \int_{-b_1/2}^{b_1/2} \int_{-a/2}^{a/2} N_x^{q=1} \left(\frac{\partial w}{\partial x'} \right)^2 \cos \alpha dx' dy' \\ &= \int_{-b_1/2}^{b_1/2} \int_{-a/2}^{a/2} \frac{\cos \alpha}{2} \left[\left(A_{01} + A_{11} \frac{x'}{a} - A_{11} \frac{y'}{a} \sin \alpha \right. \right. \\ &\quad \left. \left. + A_{21} \frac{x'^2}{a^2} - 2 A_{21} \frac{x' y'}{a^2} \sin \alpha + A_{21} \frac{y'^2}{a^2} \sin \alpha \right) \right. \\ &\quad \left. \times \left(\frac{\partial w}{\partial x'} \right)^2 dx' dy' + \frac{\cos \alpha}{b_1} \int_{-b_1/2}^{b_1/2} \int_{-a/2}^{a/2} y' \left[A_{02} \right. \right. \\ &\quad \left. \left. + A_{12} \frac{x'}{a} - A_{12} \frac{y'}{a} \sin \alpha + A_{22} \frac{x'^2}{a^2} - 2 A_{22} \frac{x' y'}{a^2} \sin \alpha \right. \right. \\ &\quad \left. \left. + A_{22} \frac{y'^2}{a^2} \sin^2 \alpha \right] \left(\frac{\partial w}{\partial x'} \right)^2 dx' dy' \dots A. 16 \right. \end{aligned}$$

Substituting Eqn. A.14 into Eqn. A.16 and integrating yields

$$\int_{-b/2}^{b/2} \int_{-a/2}^{a/2} N_x^{q=1} \left(\frac{\partial w}{\partial x} \right)^2 dx dy$$

$$\begin{aligned}
&= \frac{B^2 \pi^2}{4} \cos \alpha \frac{b_1}{a} \left[\frac{\pi^2 e^2}{120} - \frac{4 e \left(1 - \frac{e}{2} \right)}{\pi^2} \right. \\
&\quad \left. + \frac{\left(1 + \frac{e}{2} \right)^2}{2} \right] \left(A_{01} + \frac{A_{21}}{12} + \frac{A_{21}}{2 \pi^2} \right) \\
&\quad + B^2 \pi^2 \cos \alpha \sin \alpha \left(\frac{b_1}{a} \right)^2 \left[\frac{\pi^2 e^2}{3360} + e \left(1 + \frac{e}{2} \right) \left(\frac{48}{\pi^4} \right. \right. \\
&\quad \left. \left. - \frac{5}{\pi^2} \right) + \left(1 + \frac{e}{2} \right)^2 \left(\frac{1}{24} - \frac{1}{4 \pi^2} \right) \right] \left(\frac{A_{21}}{4} \frac{b_1}{a} \sin \alpha \right. \\
&\quad \left. - \frac{A_{12}}{2} \right) \dots \dots \dots A . 17
\end{aligned}$$

From Eqns. 2.8 and 2.6

$$\begin{aligned}
&\int_{-b/2}^{b/2} \int_{-a/2}^{a/2} N_y^{q=1} \left(\frac{\partial w}{\partial y} \right)^2 dx dy \\
&= \int_{-b_1/2}^{b_1/2} \int_{-a/2}^{a/2} N_y^{q=1} \frac{1}{\cos^2 \alpha} \left(\frac{\partial w}{\partial y'} + \sin \alpha \frac{\partial w}{\partial x'} \right)^2 \cos \alpha dx' dy' \\
&= \frac{1}{\cos \alpha} \int_{-b_1/2}^{b_1/2} \int_{-a/2}^{a/2} \left(B_0 + B_1 \frac{y'}{b_1} + B_2 \frac{y'^2}{b_1^2} \right) \left(\frac{\partial w}{\partial y'} \right)^2 dx' dy' \\
&\quad + \frac{2 \sin \alpha}{\cos \alpha} \int_{-b_1/2}^{b_1/2} \int_{-a/2}^{a/2} \left(B_0 + B_1 \frac{y'}{b_1} \right. \\
&\quad \left. + B_2 \frac{y'^2}{b_1^2} \right) \left(\frac{\partial w}{\partial x'} \right) \left(\frac{\partial w}{\partial y'} \right) dx' dy'
\end{aligned}$$

$$\begin{aligned}
& + \frac{\sin^2 \alpha}{\cos \alpha} \int_{-b_1/2}^{b_1/2} \int_{-a/2}^{a/2} \left(B_0 + B_1 \frac{y'}{b_1} \right. \\
& \left. + B_2 \frac{y'^2}{b_1^2} \right) \left(\frac{\partial w}{\partial x'} \right)^2 dx' dy' \dots \dots \dots A . 18
\end{aligned}$$

Substituting Eqn. A.14 into Eqn. A.18 and integrating gives

$$\begin{aligned}
& \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} N_y^{q=1} \left(\frac{\partial w}{\partial y} \right)^2 dx dy \\
& = \frac{1}{\cos \alpha} \left\{ \frac{B^2}{2} B_0 \frac{a}{b_1} \left[\frac{\pi^2 e^2}{120} - 4 e \left(1 + \frac{e}{2} \right) \right. \right. \\
& \quad \left. \left. + \frac{2 \left(1 + \frac{e}{2} \right)^2}{2} \right] + \frac{B^2}{2} B_2 \frac{a}{b_1} \left[\frac{\pi^2 e^2}{80} + \left(1 + \frac{e}{2} \right) \left(\frac{24}{2} \right. \right. \right. \\
& \quad \left. \left. - 3 \right) + \pi^2 \left(1 + \frac{e}{2} \right)^2 \left(\frac{1}{24} + \frac{1}{4 \pi^2} \right) \right] \\
& \quad + \frac{\sin^2 \alpha}{\cos \alpha} \frac{B^2 \pi^2}{2} B_0 \frac{b_1}{a} \left[\frac{\pi^2 e^2}{120} - \frac{4 e \left(1 + \frac{e}{2} \right)}{2} \right. \\
& \quad \left. + \frac{\left(1 + \frac{e}{2} \right)^2}{2} \right] + \frac{B^2 \pi^2}{2} B_2 \frac{b_1}{a} \left[\frac{\pi^2 e^2}{3360} + e \left(1 \right. \right. \\
& \quad \left. \left. + \frac{e}{2} \right) \left(\frac{48}{\pi^4} - \frac{5}{\pi^2} \right) + \left(1 + \frac{e}{2} \right)^2 \left(\frac{1}{24} - \frac{1}{4 \pi^2} \right) \right] \left. \right\} \\
& \dots \dots \dots A . 19
\end{aligned}$$

From Eqns. 2.6 and 2.9

$$\begin{aligned}
& \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} 2 N_{xy}^{q=1} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} dx dy \\
&= \int_{-b_1/2}^{b_1/2} \int_{-a/2}^{a/2} 2 N_{xy}^{q=1} \left[\frac{\partial w}{\partial x'} \frac{\partial w}{\partial y'} + \sin \alpha \left(\frac{\partial w}{\partial x'} \right)^2 \right] dx' dy'
\end{aligned}$$

..... A . 20

Substituting Eqn. A.14 into Eqn. A.20 and integrating gives

$$\begin{aligned}
& \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} 2 N_{xy}^{q=1} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} dx dy \\
&= B^2 \pi^2 \sin \alpha \frac{b_1}{a} \left[\frac{\pi^2 e^2}{120} - \frac{4 e (1 + \frac{e}{2})}{\pi^2} + \frac{(1 + \frac{e}{2})^2}{2} \right] K_0 \\
&+ B^2 \pi^2 \sin \alpha \frac{b_1}{a} \left[\frac{\pi^2 e^2}{3360} + e (1 + \frac{e}{2}) \left(\frac{48}{\pi^4} - \frac{5}{\pi^2} \right) \left(1 + \frac{e}{2} \right)^2 \left(\frac{1}{24} - \frac{1}{4 \pi^2} \right) \right] (K_4 - K_3 \sin \alpha \frac{b_1}{a}) \dots A . 21
\end{aligned}$$

Letting

$$\begin{aligned}
F_1 &= \frac{\pi^2 e^2}{120} - \frac{4 e (1 + \frac{e}{2})}{\pi^2} + \frac{(1 + \frac{e}{2})^2}{2} \\
F_2 &= \frac{\pi^2 e^2}{3360} + e (1 + \frac{e}{2}) \left(\frac{48}{\pi^4} - \frac{5}{\pi^2} \right) + \left(1 + \frac{e}{2} \right)^2 \left(\frac{1}{24} - \frac{1}{4 \pi^2} \right)
\end{aligned}$$

$$F_3 = \frac{e^2}{12} - \frac{4 e (1 + \frac{e}{2})}{\pi^2} + \frac{(1 + \frac{e}{2})^2}{2}$$

$$F_4 = \frac{e^2}{80} + e (1 + \frac{e}{2}) (\frac{24}{\pi^4} - \frac{3}{\pi^2}) + (1 + \frac{e}{2})^2 (\frac{1}{24} + \frac{1}{4 \pi^2})$$

and adding Eqns. A.17, A.19, and A.21 gives the total external energy T_c as

$$T_c = - \frac{q}{2} \left\{ F_1 \frac{B^2 \pi^2}{4} \frac{b_1}{a} \left[\cos \alpha (A_{01} + \frac{A_{21}}{12} + \frac{A_{21}}{2 \pi^2}) + \frac{2 \sin^2 \alpha}{\cos \alpha} B_0 + 4 \sin \alpha K_0 \right] + F_2 B^2 \pi^2 \sin \alpha \frac{b_1}{a} \left[\cos \alpha \frac{b_1}{a} (\frac{A_{21}}{4} \frac{b_1}{a} \sin \alpha - \frac{A_{12}}{2}) + \frac{\sin \alpha}{2 \cos \alpha} B_2 + K_4 - K_3 \sin \alpha \frac{b_1}{a} \right] + F_3 \frac{B^2 \pi^2}{2} \frac{a}{b_1} \frac{B_0}{\cos \alpha} + F_4 \frac{B^2 \pi^2}{2} \frac{a}{b_1} \frac{B_2}{\cos \alpha} \right\} \dots \dots \dots A.21$$

To determine the total initial strain energy of bending, V_{1c} can be rewritten as

$$V_{1c} = \frac{D}{2 \cos \alpha} \int_{-b_1/2}^{b_1/2} \int_{-a/2}^{a/2} \left\{ \left(\frac{\partial^2 w}{\partial x'^2} \right)^2 \right.$$

$$\begin{aligned}
& + \left(\frac{\partial^2 w}{\partial x'^2} \right)^2 \Big] \frac{1}{\cos^2 \alpha} + 2 (1 - \nu + 2 \tan^2 \alpha) \left(\frac{\partial^2 w}{\partial x' \partial y'} \right)^2 \\
& + 2 (\tan^2 \alpha + \nu) \frac{\partial^2 w}{\partial x'^2} \frac{\partial^2 w}{\partial y'^2} + \frac{4 \sin \alpha}{\cos^2 \alpha} \frac{\partial^2 w}{\partial x' \partial y'} \left(\frac{\partial^2 w}{\partial x'^2} \right. \\
& \left. + \frac{\partial^2 w}{\partial y'^2} \right) \Big\} dx' dy' \quad \dots \dots \dots \quad A . 23
\end{aligned}$$

Substituting Eqn. A.14 into Eqn. A.23 and integrating yields

$$\begin{aligned}
V_{1c} &= B^2 \frac{\pi^4 D}{4 b_1 a \cos^3 \alpha} \left\{ \left(\frac{b_1}{a} \right)^2 \left[e^2 \left(\frac{\pi^2}{120} + \frac{1}{8} - \frac{2}{\pi^2} \right) \right. \right. \\
& + e \left(\frac{1}{2} - \frac{4}{\pi^2} \right) + \frac{1}{2} \Big] + \left(\frac{a}{b_1} \right)^2 \left[e^2 \left(\frac{1}{8} - \frac{1}{\pi^2} \right) \right. \\
& + e \left(\frac{1}{2} - \frac{4}{\pi^2} \right) + \frac{1}{2} \Big] + 2 (1 + 2 \sin^2 \alpha) \left[e^2 \left(\frac{5}{24} \right. \right. \\
& \left. \left. - \frac{2}{\pi^2} \right) + e \left(\frac{1}{2} + \frac{4}{\pi^2} \right) + \frac{1}{2} \Big] \right\} \\
&= B^2 \frac{\pi^4 D}{4 b_1 a \cos^3 \alpha} \left[\left(\frac{b_1}{a} \right)^2 F_1 + \left(\frac{a}{b_1} \right)^2 F_5 \right. \\
& \left. + 2 (1 + 2 \sin^2 \alpha) F_3 \right] \quad \dots \dots \dots \quad A . 24
\end{aligned}$$

in which

$$F_5 = e^2 \left(\frac{1}{8} - \frac{1}{\pi^2} \right) + e \left(\frac{1}{2} - \frac{4}{\pi^2} \right) + \frac{1}{2}$$

Assume M_y and θ to vary in a similar manner with x and let $M_y = 4 S_0 \theta$ at the boundaries. Noting that $\theta = \frac{\partial w}{\partial y}$, Eqn. 2.14 becomes

$$\begin{aligned} V_{2c} &= \frac{1}{2} \int_{-a/2}^{a/2} \left\{ 4 S_0 \left[\left(\frac{\partial w}{\partial y} \right)_{y=b/2} \right]^2 \right. \\ &\quad \left. + 4 S_0 \left[\left(\frac{\partial w}{\partial y} \right)_{y=-b/2} \right]^2 \right\} dx \\ &= \frac{2 S_0}{\cos \alpha} \int_{-a/2}^{a/2} \left\{ \left[\left(\frac{\partial w}{\partial y'} + \sin \alpha \frac{\partial w}{\partial x'} \right)_{y'=b_1/2} \right]^2 \right. \\ &\quad \left. + \left[\left(\frac{\partial w}{\partial y'} + \sin \alpha \frac{\partial w}{\partial x'} \right)_{y'=-b_1/2} \right]^2 \right\} dx' \quad \dots A \quad 25 \end{aligned}$$

Substituting Eqn. A.14 into Eqn. A.25 and integrating gives

$$V_{2c} = B^2 \frac{\pi^2 D a e}{2 b_1^3 \cos^3 \alpha} \quad \dots \dots \dots A \quad 26$$

BUCKLING ANALYSIS OF FOLDED PLATES

by

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An energy approach for the buckling analysis of single cell folded plate structures composed of rectangular thin plates with rigid joints and subjected to uniformly distributed loads has been developed. The assumptions made for the variations of all in-plane forces of the folded plate are based upon numerous numerical results obtained from many different stress analyses. This buckling analysis is concerned with the problem of local buckling of individual plate elements of the structure and treats the case of all in-plane forces acting at the same time. The equations of the external energy of all in-plane forces and of the internal strain energy of bending are given by Timoshenko. A deflection function which satisfies all necessary boundary conditions of a panel taken from the plate element is given by Lundquist and Stowell.

Numerical results obtained from considering the cases of pure compression, pure shear on simply supported plate elements are obtained by this buckling analysis. For those cases the results obtained in this report agree with the results obtained by other buckling analyses. A computer program was written to apply this analysis to models of folded plate structures. The buckling results for two types of models obtained from this analysis were compared with the results obtained from (1) the analysis of Swartz and Guralnick, (2) an analysis without considering the effect of shearing forces, and (3) experimental model tests. The values of

critical loads obtained from this analysis are lower than those obtained from the analysis of Swartz and Guralnick, nearly the same as those obtained from the analysis without considering the effect of shearing forces, and higher than those obtained from experimental model tests.