STRUCTURAL ANALYSIS OF A REINFORCED CONCRETE CANTILEVER STAIRCASE

by

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SYNOPSIS

Methods of analyzing a concrete cantilever staircase comprising two straight flights and a landing and supported only on the upper and the lower floors are presented herein.

The method developed by A. Siev includes the determination of the primary moments considering the whole structure as a simple statically indeterminate system and the secondary moments resulting from the consistent deformation at the intersection between the flights and the landing.

A. R. Cusens and J. G. Kuang developed a method of analysis by assuming that the space structure, which is composed of plates or slabs, can be replaced by straight beams and that the method of least work can be used to solve this highly indeterminate structure.

W. Fuchsteiner's assumptions are similar to those of Cusens and Kuang with the exception that the landing slab is replaced by a curved bar element.

The results from previously reported tests are in good agreement with the above-mentioned methods which have already been employed in the practical design of staircases of this type.

INTRODUCTION

The use of cantilever straight multiflight staircases has become popular with architects in the past few years. The stress analysis of this type of structure is of considerable interest to structural engineers.

Theoretical analyses have been published by W. Fuchsteiner (1), G. Szabo (2), A. C. Liebenberg (3), A. Siev (4), P. L. Gould (5), A. R. Cusens and J. G. Kuang (6), and F. Sauter (7).

Liebenberg first introduced the concept of space interaction of plates. His method of analysis is based on the staircase treated as a statically indeterminate structure with the assumption that the torsional restraining moment in the landing may be neglected.

Siev has extended Liebenberg's theory to include the determination of the torsional restraining moment resulting from the compatibility of deformations between the flights and the landing. His method shows that this moment is usually small and may be considered as a secondary effect.

Cusens and Kuang's method is based on the application of the principle of least work with the assumptions that the flight plates can be reduced to bar elements which coincide with their longitudinal axes. The landing bar

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Numbers in parentheses refer to references listed in the Bibliography.

element will be a straight line to be located in a position near the line of intersection.

Fuchsteiner is the first person who suggested the space bar method which is similar to that of Cusens and Kuang with the only difference being that the landing is replaced by a curved bar instead of a straight one.

Siev, Cusens and Kuang, and Fuchsteiner's approaches have shown good correlation with test results and hence only these three methods will be described in detail in the following sections. A numerical example is also given to illustrate the procedure of calculation for each approach.

NOTATION

a,b,c,g,h,l = Dimensions of the staircase

A = Cross-sectional area of the flight

E = Modulus of elasticity

G = Modulus of elasticity in shear

H = Reaction in the X-direction

I = Moment of inertia

I_b = Moment of inertia of beam C-H about the X-axis

J = Polar moment of inertia (For a rectangular section, the value depends on the magnitude of the width-depth ratio of the section)

M_o = Redundant moment at point 0 about the X-axis

M_r = Bending moment about the horizontal axis of the section

M_s = Bending moment about the vertical axis of the section

M_t = Torsional moment of the section

M_x = Total torsional restraining moment in the landing or moment about the X-axis

M_y = Moment about the Y-axis

M_z = Moment about the Z-axis

$M_{\bar{x}}$ = Torsional moment in the flight

$M_{\bar{z}}$ = Bending moment in the flight about the axis perpendicular to the flight surface

r = Radius

R	=	Total reactive force or reactive force in the Y-direction
R'	=	Reactive force resisted by primary stresses
R''	=	Reactive force resisted by secondary stresses
S, S'	=	Stress
t	=	Overall depth of slab
U	=	Total strain energy
V	=	Reactive force in the Z-direction
w	=	Uniformly distributed load in psf.
W	=	Displacement normal to the flight surface
W_1, W_2	=	Dead load of flight and landing
W_3, W_4, W_5	=	Live load of lower and upper flight and landing
X, Y, Z	=	Direction of the axes
$\bar{X}$	=	Direction of the axis parallel to the longitudinal axis of the flight
$\bar{Z}$	=	Direction of the axis perpendicular to the flight surface
$\bar{X}_u$, $\bar{X}_1$	=	Thrust in upper and lower flight
α	=	Angle of slope of the flight
θ	=	Angle
δ, Δ	=	Vertical deflection
ε	=	Strain
τ	=	Torsional shear stress
δ'	=	Elongation or contraction of end fiber

SIEV'S ANALYTICAL METHOD

Free straight multiflight stairs without a landing support are attractive (Fig. 1) and have become increasingly prevalent as a design feature with architects during the past few years. The elimination of columns under the landing frequently has both structural and aesthetic advantages. In the following the analysis of such stairs by Siev's method is first introduced.

Siev approaches the problem of the free straight multiflight staircase in a procedure similar to that of folded plate analysis. The stress analysis for this structure will be accomplished in stages and only the case of symmetrical loading is considered.

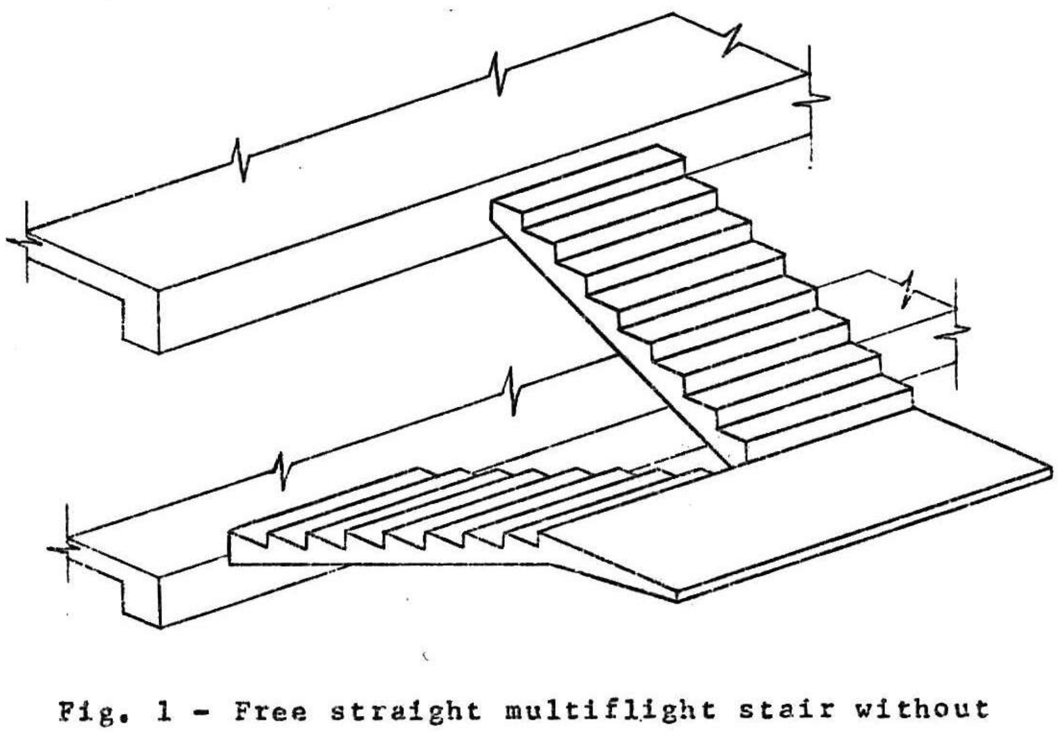


Fig. 1 - Free straight multiflight stair without landing support.

Initially, as in the case of folded plates, an imaginary support is assumed along the line C-D and G-H of Fig. 2 (i.e., the line of intersection between the flights and the landing.) The stair is then analyzed as two separate slabs being fixed at one end and hinged at the intersection. It is obvious that the support moments of the slabs under various possible loading conditions can be easily obtained by using any classical method of solving statically indeterminate structure. The reactions at C-D and G-H are then determined.

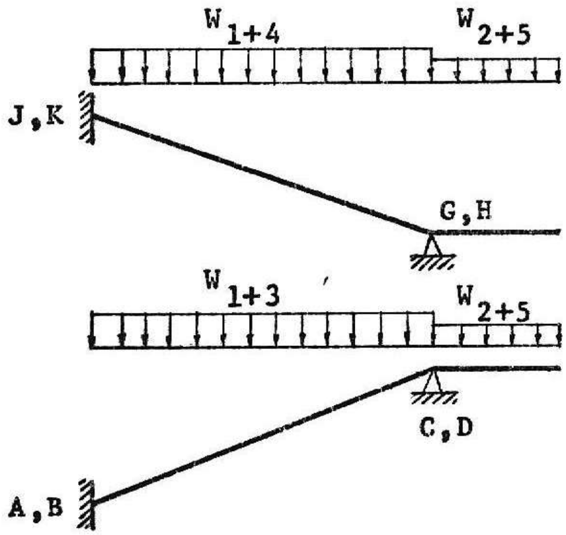


Fig. 2 - Loading on slabs with imaginary supports

Since the calculation of the support moments is an elementary problem, the details of finding them will be left out of this discussion. It is necessary to consider, however, the behavior of the structure as a whole when subjected to the reactive forces. (The reactions at the imaginary supports due to external loads will be regarded as loads acting along the line of intersection of the slab structure.)

Siev conceived the total reactive force R to be composed of two kinds of forces: 1) The forces resisted by primary stresses as denoted by R' and 2) The forces resisted by secondary stresses as denoted by R". This relation can be written as

$$R = R' + R'' \tag{1}$$

However, it will be shown later that R" is much less than R' and therefore

$$R \cong R' \tag{2}$$

Then, in the second step, assume a load R' acting along CD and GH as shown in Fig. 3. (8) The resultant of this load 2bR' is

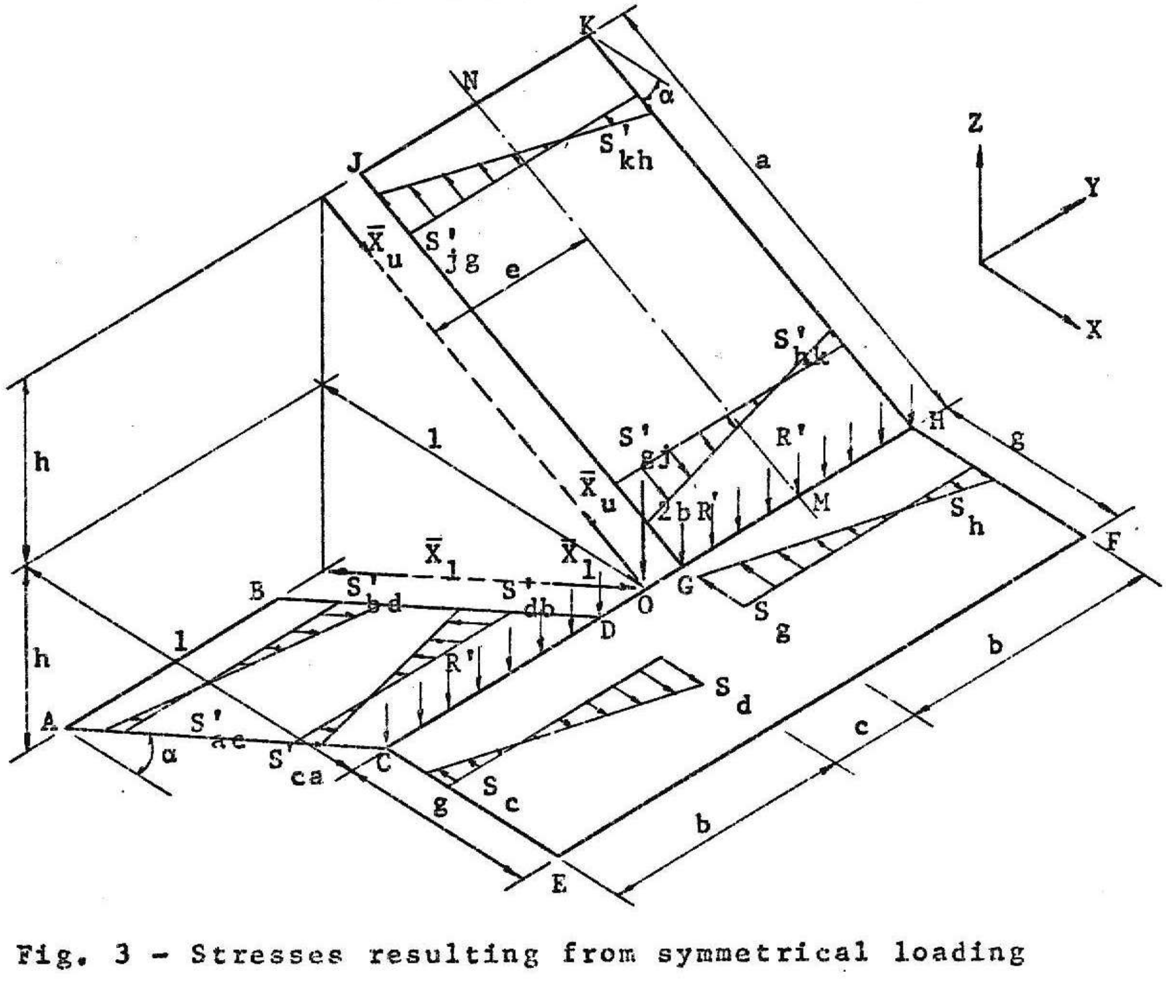


Fig. 3 - Stresses resulting from symmetrical loading

resolved into forces which act on a section through the center of the landing parallel to the longitudinal axis of each flight or through point 0. Thus each component will have the value

$$\overline{X} = \overline{X}_1 = \overline{X}_u = \frac{b\ R'}{\sin\alpha} \tag{3}$$

It is obvious that the upper flight is subjected to an axial tensile force in addition to a bending moment about its own plane. The lower flight is subjected to an axial compressive force plus a bending moment about its own plane. The corresponding fiber stresses at the end of each flight will be

$$\begin{aligned} S'_{gj} = - S'_{db} &= \frac{\overline{X}}{A} + \frac{\overline{X}(b+c)b}{4\ I_{\overline{z}}} \\ &= \frac{\overline{X}}{b\ t}\left(1 + 3\ \frac{b+c}{b}\right) \\ &= \frac{R'}{t\ \sin\alpha}\left(1 + 3\ \frac{b+c}{b}\right) \end{aligned} \tag{4}$$

and

$$\begin{aligned} - S'_{ca} = + S'_{hk} &= \frac{\overline{X}}{A} - \frac{\overline{X}(b+c)b}{4\ I_{\overline{z}}} \\ &= \frac{R'}{t\ \sin\alpha}\left(1 - 3\ \frac{b+c}{b}\right) \end{aligned} \tag{5}$$

In which t represents the thickness of the flight plate, A is the cross-sectional area of the flight and $I_{\overline{z}}$ denotes the moment of inertia about the $\overline{Z}$ axis (the axis perpendicular to the plate surface). The resultant of the vertical components of these stresses gives the reaction on the landing CEFH which can be considered as a beam (beam CH) because the slab is usually so designed that the section tapered to the end will have the centroid close to the inner edge. The reactions at points D and C, respectively, are

$$R' \left(1 + 3 \frac{b + c}{b}\right)$$

and

$$R' \left(-1 + 3 \frac{b + c}{b}\right)$$

It is now necessary to consider the primary bending moments in Beam C-H. As shown in Fig. 4, in addition to the load R', the beam is subjected to the reactive forces from the flights. It is apparent that the resultant of these forces will pass through point O. Owing to the symmetrical forces on the beam, it is therefore possible to calculate the bending moment in beam C-H taking it as free at the ends and fixed at point O. That is

$$M'_o = \frac{1}{2}\, b\ 3R' \left(\frac{b+c}{b}\right) \left(\frac{b}{3} + \frac{c}{2}\right)$$

$$- \frac{1}{2}\, b\ 3R' \left(\frac{b+c}{b}\right) \left(\frac{2b}{3} + \frac{c}{2}\right)$$

$$= - \frac{1}{2} R'\, b\, (b + c) \qquad (6)$$

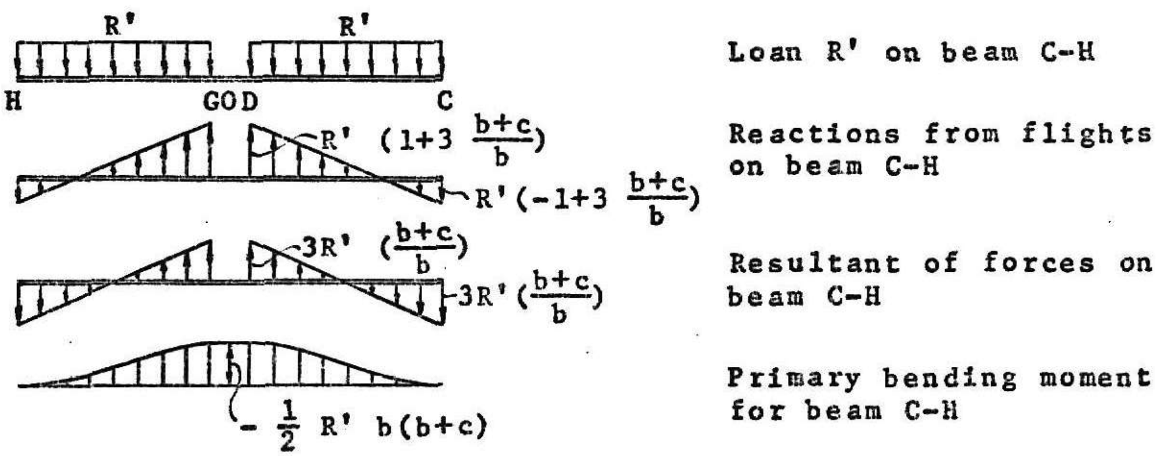


Fig. 4 - Loading and primary bending moment on beam C-H

The horizontal component of $\bar{X}$ is $\bar{X}$ cosα. From Eq. 3

$$X = \bar{X}\cos\alpha = \frac{b\ R'}{\sin\alpha}\cos\alpha = b\ R'\cot\alpha \tag{7}$$

Substituting X into Eqs. similar to Eqs. 4 and 5, and multiplying by t, the horizontal loads on the landing at points D and C (Fig. 3), respectively, are

$$-\ t\ S_d = R'\cot\alpha\ (1 + 3\ \frac{b+c}{b})$$

and $$t\ S_c = R'\cot\alpha\ (-1 + 3\ \frac{b+c}{b})$$

where a positive sign represents the tensile force and a negative sign the compressive force.

It is seen that, from symmetry in loading, M_z and M_y (moments about Z and Y axis) are both equal to zero.

At this stage, all primary moments have been known. Subsequently, the secondary moments will be calculated and shown to be small. Therefore, the calculations to the present stage are sufficient for most practical design use, especially if an approximate result is desired.

The displacements caused by the primary stresses are produced by deformation of the flights and the landing. For simplification, the effect of shear deformation will be neglected in the following calculations.

Flight ends undergo displacements in the Y and $\bar{X}$ directions (directions of the longitudinal axis of the landing and each flight). From Fig. 5, if the flights are equal in length,

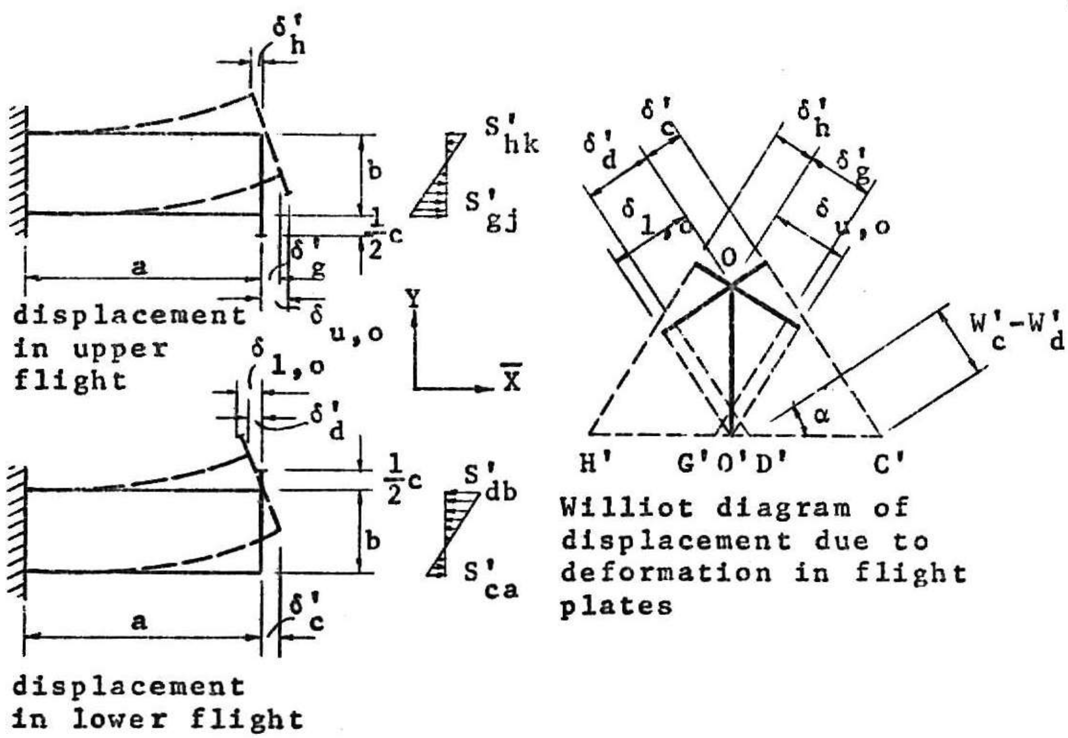


Fig. 5 - Stresses and displacements resulting from deformation in flight plates

both flight ends have equal displacements in the Y direction, there will be no change in stresses in the plate system and therefore, the magnitude of this deformation is of no further interest to the discussion. It is worth while to note, however, the strains of the end fibers of each flight in the $\overline{X}$ direction. From Hooke's law

$$\varepsilon'_g = - \varepsilon'_d = S'_{gj} \;/\; E$$

and

$$\varepsilon'_c = - \varepsilon'_h = S'_{ca} \;/\; E$$

in which E is Young's Modulus of Elasticity. Hence the total elongations and contractions of the end fibers are

$$\delta'_g = - \delta'_d = (S'_{gj} \;/\; E)\; a = \frac{R'\; a}{t\; E\; \sin\alpha}\,(1 + 3\,\frac{b+c}{b}) \tag{8}$$

and

$$\delta'_c = -\delta'_h = \left(\frac{S'_{ca}}{E}\right) a = \frac{R' a}{t E \sin\alpha} \left(-1 + 3 \frac{b+c}{b}\right) \qquad (9)$$

If the deformed lines C-D and G-H are extended to the central point, the additional extensions $\delta_{u,o}$ and $\delta_{1,o}$ at point O can be determined by simple geometric relations. That is

$$\frac{|\delta'_h| + |\delta'_g|}{|\delta'_h| + |\delta_{u,o}|} = \frac{b}{b + \frac{1}{2} c} \qquad (10)$$

Substituting the corresponding values of δ'_g and δ'_h in Eq. 10,

$$\delta_{u,o} = -\delta_{1,o} = \frac{R' a}{t E \sin\alpha} \left[1 + 3\left(\frac{b+c}{b}\right)^2\right] \qquad (11)$$

A Williot diagram (Fig. 5) is now drawn for the projection of all point displacements in the flights on the X-Z plane, and the vertical deflection OO' of point O is found to be

$$\delta_{oo'} = \delta_{u,o} / \sin\alpha \qquad (12)$$

Introducing the known value of $\delta_{u,o}$ in Eq. 12,

$$\delta_{oo'} = \frac{R' a}{t E \sin^2\alpha} \left[1 + 3\left(\frac{b+c}{b}\right)^2\right] \qquad (13)$$

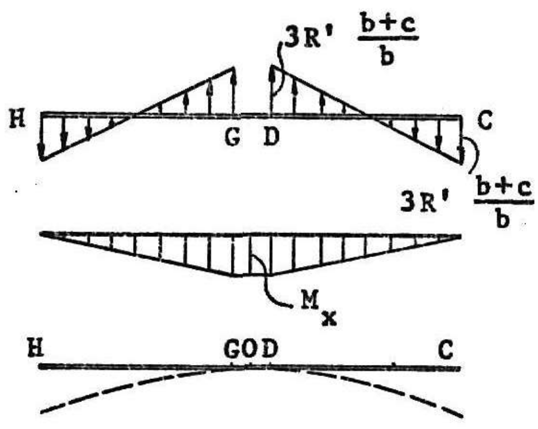
The final step in the analysis of the cantilever staircase is to calculate the torsional restraining moment by substituting all related displacements of the flights and the landing into the compatibility equation. As shown in Fig. 5, the difference between the displacements of point C and point D normal to the flight is

$$W'_c - W'_d = (|-\delta'_d| + |\delta'_c|) \tan\alpha \qquad (14)$$

Substituting Eqs. 8 and 9 into 14,

$$W'_c - W'_d = \frac{6\ R'\ a\ (b+c)}{E\ b\ t\ \cos\alpha} \tag{15}$$

The difference between the vertical displacements of points C and D in the landing will now be considered. As previously stated, the beam C-H is subjected to the load R' and the vertical reactions from the flights; thus the beam will deflect as though it were fixed at midspan or point O and free at either end as shown in Fig. 6. However, this clearly shows that the deflection of point C is greater than that of point D, the flight plate is therefore twisted, and a torsional moment $M_{\bar{x}}$ is introduced therein. As a result of this effect, it can be visualized that the beam C-H, in addition to the negative bending, is restrained at the center by a positive moment M_x which tends to decrease the deflection of the beam shown in Fig. 6.



Resultant of forces on beam

Positive restraining moment M_x on beam C-H

Deflection of beam C-H

Fig. 6 - Loading, torsional restraining moment and deflection of beam C-H

Thus the difference between the vertical deflections of points C and D may be obtained as

$$\Delta_c'' - \Delta_d'' = \frac{R' \, b^2 (b+c)}{4 \, E \, I_b} (c + 0.7b) - \frac{b \, M_x}{6EI_b} (3c + 2b) \qquad (16)$$

where I_b is the moment of inertia of beam CH. The relative displacement, W'', in the direction normal to the flight plane (Fig. 7) is

$$W_c'' - W_d'' = \frac{\Delta_c'' - \Delta_d''}{\cos\alpha} \qquad (17)$$

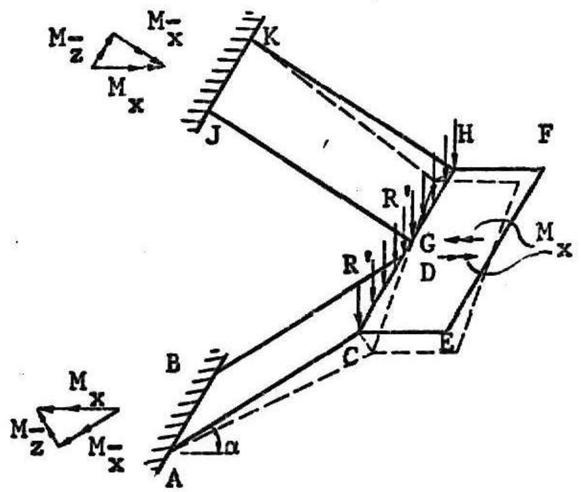


Fig. 7 - Displacement of the plates due to bending of beam CH

The third term of the relative displacement, W''', caused by the torsional moment $M_{\bar{x}}$ (component of M_x) in the flight, can be easily found by Castigliano's theorem

$$W_c''' - W_d''' = \frac{M_{\bar{x}}}{G\,J}\,b\,a = \frac{M_x \cos\alpha\, ba}{G\,J} = \frac{M_x\, b\, l}{G\,J} \tag{18}$$

in which G is the modulus of elasticity in shear, GJ is the torsional rigidity and l is the horizontal projection of the plate length.

The last relative displacement W^{IV} in the flight plate caused by $M_{\bar{z}}$, the component of restraining moment M_x, will be obtained in a way similar to the first term, but in the opposite sense. The stresses due to $M_{\bar{z}}$ are

$$S_{ca}^{IV} = -\,S_{db}^{IV} = S_{gj}^{IV} = -\,S_{hk}^{IV} = -\,\frac{6\,M_{\bar{z}}}{tb^2} \tag{19}$$

and the total elongations and contractions of the end fibers are

$$\delta_c^{IV} = -\,\delta_d^{IV} = \delta_g^{IV} = \delta_h^{IV} = -\,\frac{6\,M_{\bar{z}}}{E\,t\,b^2} \tag{20}$$

The relative deflection W^{IV} is

$$W_c^{IV} - W_d^{IV} = (|\delta_c^{IV}|+|-\delta_d^{IV}|)\tan\alpha = -\,\frac{12\,M_{\bar{z}}\,a}{E\,t\,b^2}\tan\alpha\,,$$

but, since $M_{\bar{z}} = M_x \sin\alpha$, and $h = a\sin\alpha$,

therefore $$W_c^{IV} - W_d^{IV} = -\,\frac{12\,M_x\,h}{E\,t\,b^2}\tan\alpha \tag{21}$$

At this stage, all displacements in the same direction are known. All that must be done in the final step is to apply the compatibility condition which can be accounted for as follows: along the line of intersection, the deflection of each flight and the landing, which are caused by both the primary and the secondary stresses, should coincide with each other. Thus the compatibility equation will be of the form

$$(W_c' - W_d') + (W_c''' - W_d''') + (W_c^{IV} - W_d^{IV}) = (W_c'' - W_d'') \qquad (22)$$

As W'', W''' and W^{IV} are represented in terms of the restraining moment M_x, solve the above equation and the moment M_x is then obtained.

The effect of the vertical deflection $\delta_{oo'}$ obtained from Eq. 13 is similar to that of the settlement of the supports and is governed by specific conditions. If the flights are completely fixed at both floors, this effect may be considered by introducing an additional load R" acting on the line of intersection, producing the same amount of deflection $\delta_{oo'}$ in the cantilevered plates. Hence

$$\delta_{oo'} = \frac{R''b\ a^3}{3\ E\ \frac{bt^3}{12}} = \frac{4\ R''\ a^3}{E\ t^3} \qquad (23)$$

By equating Eq. 23 with Eq. 13 and rearranging,

$$R'' = \frac{t^2}{4\ a^2\ \sin^2 \alpha}\ [1 + 3(\frac{b+c}{b})^2]R' \qquad (24)$$

and the additional negative bending moment at the floor support is

$$M = R''\ b\ 1 \qquad (25)$$

As t is much smaller than a, it can be concluded that the fraction t^2/a^2 in Eq. 24 will lead R'' to be only a very small portion of R'. Thus, Eq. 2 is a reasonably good approximation of Eq. 1.

Numerical Example 1

The concrete staircase shown in Fig. 8 will be analyzed for the full-load condition. Nominal dimensions are given as follows:

a = 9.503 ft. l = 8.5 ft. h = 4.25 ft. g = 3.5 ft.

b = 4.0 ft. c = 1.0 ft. t = 4.5 in. for flights

α = 26°34' t = 8.0 in. tapered to 4.0 in. for landing

$\sin\alpha$ = 0.4472 $\cos\alpha$ = 0.8944

$\tan\alpha$ = 0.5000

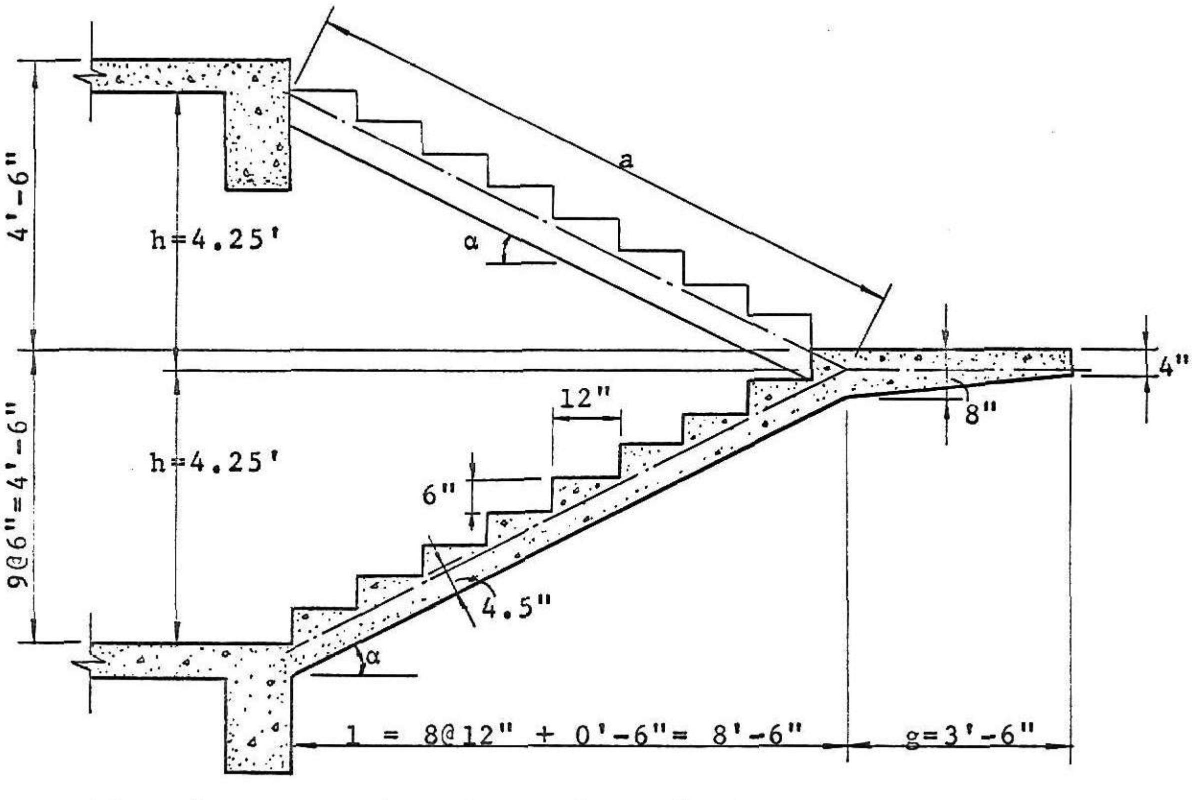


Fig. 8 - Elevational section of the staircase

1. Loading

Live load = 100 psf for both flights and landing

Total weight of one flight

$$= 9.503 \times \frac{4.5}{12} \times 4 \times 150 + 8 \times \frac{6 \times 12}{2 \times 144} \times 4 \times 150$$

$$= 2{,}140 + 1{,}200 = 3{,}340 \text{ lbs.}$$

Dead load of the flight $= \frac{3340}{8.5(4)} = 98.2$ psf

say 98 psf

Dead load of the landing $= \frac{6}{12} \times 150 = 75$ psf

w_1 = D.L. of the flight = 98 psf

w_2 = D.L. of the landing = 75 psf

$w_3 = w_4$ = L.L. on lower or upper flight = 100 psf

w_5 = L.L. on the landing = 100 psf

w_{1+3} = D.L. + L.L. on the lower flight = 198 psf

w_{1+4} = D.L. + L.L. on the upper flight = 198 psf

w_{2+5} = D.L. + L.L. on the landing = 175 psf

2. Moment of inertia of the flights and the landing

$$I_1 = \frac{1}{12} \times 42 \times 6^3 = 756 \text{ in.}^4$$

$$I_1' = \frac{1}{12} \times 42^3 \times 6 = 37{,}044 \text{ in.}^4$$

$$I_2 = \frac{1}{12} \times 48 \times 4.5^3 = 365 \text{ in.}^4$$ (neglect the effect of step portions)

$$I_2' = \frac{1}{12} \times 48^3 \times 4.5 = 41{,}472 \text{ in.}^4$$

where I_1, I_1' = moment of inertia of the landing section about horizontal and vertical axes, respectively

I_2, I_2' = moment of inertia of flight section about horizontal and vertical axes, respectively.

For finding the torsional rigidity of a rectangular section, use Saint-Venant formula (10) when b/t>2.5

$$G\,J_1 = \frac{1}{3}\,b\,t^3\,(1 - 0.63\,t/b)G = \frac{1}{3} \times 42 \times 6^3(1-0.63\,\frac{6}{42})G$$

$$= 2{,}752 \text{ G lb-in.}^2$$

$$G\,J_2 = \frac{1}{3} \times 48 \times 4.5^3\,(1 - 0.63\,\frac{4.5}{48})G = 1{,}372 \text{ G lb-in.}^2$$

where GJ_1, GJ_2 = torsional rigidity of the landing and the flight sections, respectively.

3. Moments in slab structure

a) Maximum cantilever moment in the landing

$$M_1 = -\frac{1}{2}\,w_{2+5}\,(b + \frac{1}{2}c)\,g^2$$

$$= -\frac{1}{2}\,(175)(4 + \frac{1}{2})(3.5)^2 = -\,4{,}825 \text{ ft-lb}$$

b) Minimum cantilever moment in the landing

$$M_2 = -\frac{1}{2}\,w_2\,(b + \frac{1}{2}c)\,g^2$$

$$= -\frac{1}{2}\,(75)(4.5)(3.5)^2 = -\,2{,}070 \text{ ft-lb}$$

c) Maximum negative moment at the floor supports assuming that the flights are completely fixed at floor beams

$$M_3 = -\frac{1}{8}\,w_{1+3}\,b\,l^2 + \frac{1}{2}\,M_2$$

$$= -\frac{1}{8} \times 198 \times 4 \times 8.5^2 + \frac{1}{2}\,(2{,}070)$$

$$= -7{,}150 + 1{,}035 = -6{,}115 \text{ ft-lb.}$$

d) Negative moment at the floor supports for full load in the landing

$$M_4 = -\frac{1}{8} w_{1+3}\, b\, l^2 + \frac{1}{2} M_1$$

$$= -7{,}150 + \frac{1}{2}(4{,}825) = -4{,}738 \text{ ft-lb.}$$

e) Maximum positive moment in each flight

By using the notations shown in Fig. 9

$$V_l = \frac{1}{2} W\, l + \frac{M_l - M_r}{l}$$

$$x = \frac{V_l}{W}$$

$$M_5 = \frac{1}{2}\, \frac{V_l^2}{W} - M_l$$

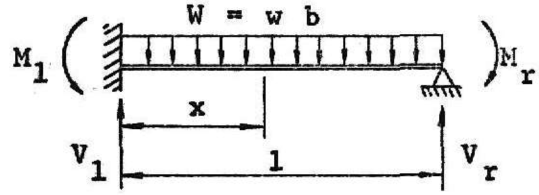


Fig. 9 - Notations for computing positive moments in the flights

For both flights and landing loaded

$$M_l = M_4 = 4{,}738 \text{ ft-lb}$$

$$M_r = M_1 = 4{,}825 \text{ ft-lb}$$

$$W = w_{1+3}\, b = 792 \text{ plf}$$

$$V_l = \frac{792 \times 8.5}{2} + \frac{4{,}738 - 4{,}825}{8.5} = 3{,}356 \text{ lbs}$$

$$V_r - 792 \times 8.5 - 3{,}356 = 3{,}376 \text{ lbs}$$

$$M_5 = \frac{(3{,}356)^2}{2 \times 792} - 4{,}738 = 2{,}373 \text{ ft-lb}$$

the moment occurs at $x = \frac{3{,}356}{792} = 4.24$ ft. from left support.

For dead load on landing only

$$M_1 = M_3 = 6{,}115 \text{ ft-lb}$$

$$M_r = M_2 = 2{,}070 \text{ ft-lb}$$

$$V_1 = \frac{792 \times 8.5}{2} + \frac{6{,}115 - 2{,}070}{8.5} = 3{,}842 \text{ lbs}$$

$$V_r = 6{,}732 - 3{,}842 = 2{,}890 \text{ lbs}$$

$$M_5' = \frac{(3{,}842)^2}{2 \times 792} - 6{,}115 = 3{,}204 \text{ ft-lb}$$

the moment occurs at $x = \frac{3{,}842}{792} = 4.85$ ft. from the left support.

f) Maximum reaction along the line of intersection

$$b\ R = V_r + w_{2+5}\ g\ (b + \tfrac{1}{2}c)$$

$$= 3{,}376 + 175 \times 3.5 \times 4.5 = 6{,}132 \text{ lbs}$$

Therefore $R = 6{,}132/(4 \times 12) = 128$ lbs per in.,

where R is the total reactive force in the fictitious support and is equal to the sum of R' and R". In the case of symmetrical loading, from Eq. 24

$$R'' = \frac{t^2}{4\ a^2\ \sin^2\alpha}\ [1 + 3(\frac{b+c}{b})^2]R'$$

$$= \frac{4.5^2}{4(114)^2(0.4472)^2}\ [1 + 3(\frac{5}{4})^2]R'$$

$$= 0.00195(1 + 4.69)\ R' = 0.0111\ R'$$

the ratio between the secondary and the total reactive force is

$$\frac{R''}{R} = \frac{R''}{R' + R''} = \frac{0.0111\ R'}{1.0111\ R'} = 0.0110$$

or $R'' = 1.1\ \%$ of R

g) Minimum reaction along the line of intersection

$$b\ R = V_r + W_2\ g\ (b + \frac{1}{2}\ c)$$

$$= 2,890 + 75\ (3.5)\ (4.5) = 4,071\ \text{lbs}$$

$$R = 4,071/48 = 85\ \text{lb/in.}$$

h) Additional negative moment at the floor support

- Due to full load:

$$M_6 = -\ 0.011\ R\ b\ l$$

$$= -\ 0.011\ (6,132)\ (8.5) = -\ 573\ \text{ft-lb}$$

- Due to live load on flights only:

$$M_6' = -\ 0.011\ (4,071)\ (8.5) = -\ 381\ \text{ft-lb}$$

i) Total negative moment at floor support

$$M_7 = M_4 + M_6 = -\ 4,738 - 573 = -\ 5,311\ \text{ft-lb}$$

$$M_7' = M_3 + M_6' = -\ 6,115 - 381 = -\ 6,496\ \text{ft-lb}$$

4. Computation of the torsional restraining moment M_x

From item 3(f), it is clear that R" is only a very small portion of R; therefore neglecting the effect of R" is permissible. The following calculations are thus based on the full load R equal to 128 lb per in. acting on the plate system. Replacing R' by R in those equations containing R', the displacement terms can be obtained as follows:

a) Due to deformation of the flight plates caused by primary stresses

From Eq. 15

$$E(W'_c - W'_d) = \frac{6\ R'a\ (b + c)}{b\ t\ \cos\alpha}$$

$$= \frac{6 \times 128 \times 114 \times (5 \times 12)}{(4 \times 12)(4.5)(0.8944)} = 27{,}192 \text{ lb/in.}$$

b) Due to twist of the flights caused by the torsional moment $M_{\bar{x}}$. From Eq. 18

$$(W'''_c - W'''_d) = \frac{b\ l}{G\ J_2} M_x$$

For concrete Poisson's ratio $\mu = 0.15$

$$E^{(9)} = 2\ G(1 + \mu) = 2\ (1 + 0.15)\ G = 2.3\ G$$

$$\text{or } G = (1/2.3)\ E = 0.435\ E$$

$$E(W'''_c - W'''_d) = \frac{(4 \times 12)(8.5 \times 12)}{(0.435)(1{,}372)} M_x = 8.21\ M_x \text{ lb/in.}$$

c) Due to bending of the flights caused by the moment $M_{\bar{z}}$. From Eq. 21

$$E(W^{IV}_c - W^{IV}_d) = -\frac{12\ h\ \tan\alpha}{b^2\ t} M_x$$

$$= -\frac{12(4.25 \times 12)(0.5000)}{(4 \times 12)^2\ (4.5)} M_x$$

$$= -0.0295\ M_x$$

d) Due to bending of the landing caused by load R' and M_x. From Eq. 17

$$E(W''_c - W''_d) = \frac{1}{\cos\alpha}\left[\frac{R'b^2(b + c)}{4\ I_1}(c + 0.7b) - \frac{b(3c+2b)}{6\ I_1} M_x\right]$$

$$= \frac{1}{0.8944}\left[\frac{128(48)^2(5 \times 12)}{4 \times 756}(12 + 0.7 \times 48)\right.$$

$$\left. - \frac{48(3 \times 12 + 2 \times 48)}{6 \times 756} M_x\right]$$

$$= 298{,}328 - 1.562\ M_x$$

Substituting the corresponding terms into the compatibility equation 22, thus

$$27,192 + 8.21\ M_x - 0.0295\ M_x = 298,328 - 1.562\ M_x$$

$$9.7425\ M_x = 271,136$$

and $$M_x = 27,830 \text{ in-lb} = 2,320 \text{ ft-lb}$$

The torsional moment $M_{\bar{x}} = M_x \cos\alpha$

$$= 27,830 \times 0.8944 = 24,891 \text{ in-lb}$$

By using Saint-Venant's formula (10) for less narrow cross-sections, the maximum torsional shear stress on the flight is

$$\tau = \frac{M_T}{k_2\ b\ t^2}$$

in which M_T is the torsional moment and k_2 is a numerical factor which is related to the b/t ratio.

For $\frac{b}{t} = \frac{48}{4.5} = 1.068$, $k_2 = 0.212$

therefore $$\tau = \frac{24,891}{(0.212)(48)(4.5)^2}$$

$$= 121 \text{ psi.}$$

The computation above is based on full load condition.

3. Computation of torsional restraining moment in case of live load on flights only

a) $E(W'_c - W'_d) = \frac{85}{128}(27,192) = 18,035$ lb/in.

b) $E(W'''_c - W'''_d) = 8.21\ M_x$

c) $E(W_c^{IV} - W_d^{IV}) = -0.0295\ M_x$

d) $E(W_c'' - W_d'') = \frac{1}{0.8944}\ [\frac{85}{128}(266,825) - 1.397\ M_x]$

$$= 197,689 - 1.562\ M_x$$

Substituting the corresponding terms into the compatibility equation, thus

$$18,035 + 8.21\ M_x - 0.0295\ M_x = 197,689 - 1.562\ M_x$$

$$9.7425\ M_x = 179,654$$

and

$$M_x = 18,440 \text{ in-lb}$$

6. Bending moments in the landing

The total bending moment about the horizontal axis in the landing section at point 0 is equal to the primary moment (Eq. 6) minus the torsional restraining moment M_x

-For full load condition:

$$M_o = -\frac{1}{2}\ R'b\ (b + c) + M_x$$

$$= -\frac{1}{2}\ (128)(48)(5 \times 12) + 27,830$$

$$= -\ 156,490 \text{ in-lb} = -\ 13,041 \text{ ft-lb.}$$

-For live load on flights only:

$$M_o = -\frac{1}{2}(4,071)(60) + 18,440$$

$$= -\ 103,700 \text{ in-lb} = -\ 8,642 \text{ ft-lb.}$$

7. Bending moments in the flights about the vertical axis

The total bending moment about the vertical axis in the flight section at point M in Fig. 3 is equal to the primary moment minus the moment $M_{\bar{z}}$

- For full load condition:

$$X = X_1 = X_u = \frac{b\ R'}{\sin\alpha}$$

$$= \frac{6,132}{0.4472} = 13,712 \text{ lbs.}$$

$$M_m = X\ (\tfrac{1}{2}b + \tfrac{1}{2}c) - M_z$$

$$= 13,712\ (2 + \tfrac{1}{2}) - 2,320(0.4472)$$

$$= 33,243 \text{ ft-lb.}$$

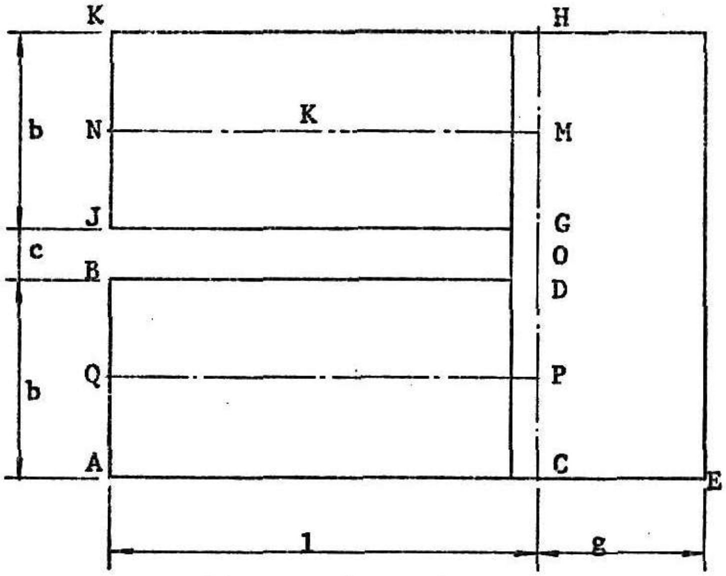
- For live load on flights only:

$$X = X_1 = X_u = \frac{4,071}{0.4472} = 9,100 \text{ lbs}$$

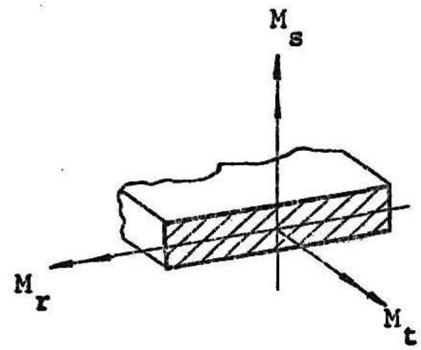
$$M_m = 9,100\ (2.5) - 1,537(0.4472)$$

$$= 22,063 \text{ ft-lb.}$$

A summary of the moments computed for the more important sections of the stair is shown in Table 1. The position of the points at which the moments are computed and the sign convention used for all moments are shown in Fig. 10.



Plane of stair



Positive vectors for moments with right-hand rule

Fig. 10 - Plane of stair and sign convention for moments.

Moments	M_r (ft-lb)				M_s (ft-lb)	M_t (ft-lb)
Points	N	M	O	K	M-N	M-N
With both flights and landing loaded	-5,311	-4,825	-13,041	+2,373	+33,243	-2,074
With live load on both flights only	-6,496	-2,070	- 8,642	+3,204	+22,063	-1,375

Table 1 - Primary and secondary moments in staircase

ANALYSIS OF THE STAIRCASE BY THE METHOD OF LEAST WORK BASED ON CUSENS AND KUANG'S ASSUMPTIONS

It is widely known that the principle of least work is a powerful tool in solving statically indeterminate structural problems. This is true especially when the structure is a three-dimensional frame of which the members are subjected to torsional stresses in addition to the conventional bending and the axial stresses.

The basic structure of the staircase, as previously described, with two flights fixed at the floors and an unsupported intermediate landing, is indeterminate to the sixth degree (three reactions and three moments at one end). According to Cusens and Kuang's assumptions (6), the staircase can be analyzed by reducing the plates to beam elements. Thus the stair will be in the form of a space frame consisting of the beams located in a position coincident with their longitudinal axes. The following analysis will be based on the application of these assumptions and the method of least work.

Fig. 11 shows the staircase with the beam elements represented by heavy lines. A statically determinate base structure is then selected by removing all the redundant forces and moments at end Q, which in turn will be regarded as external loads acting on the base structure when the moment expressions for each member are computed. Notation and sign conventions for all moments are shown in Fig. 12. The bending and torsional

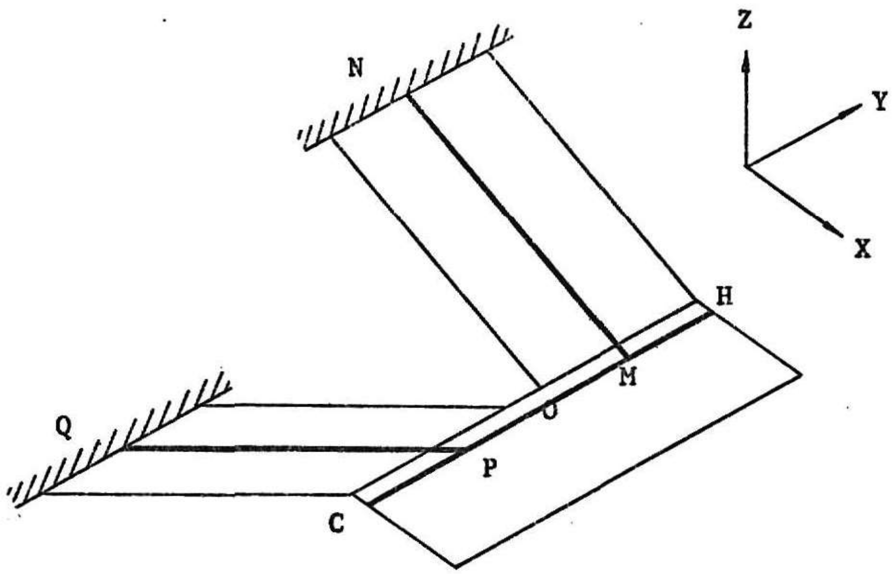


Fig. 11 - Skeletal rigid frame representing the cantilever staircase

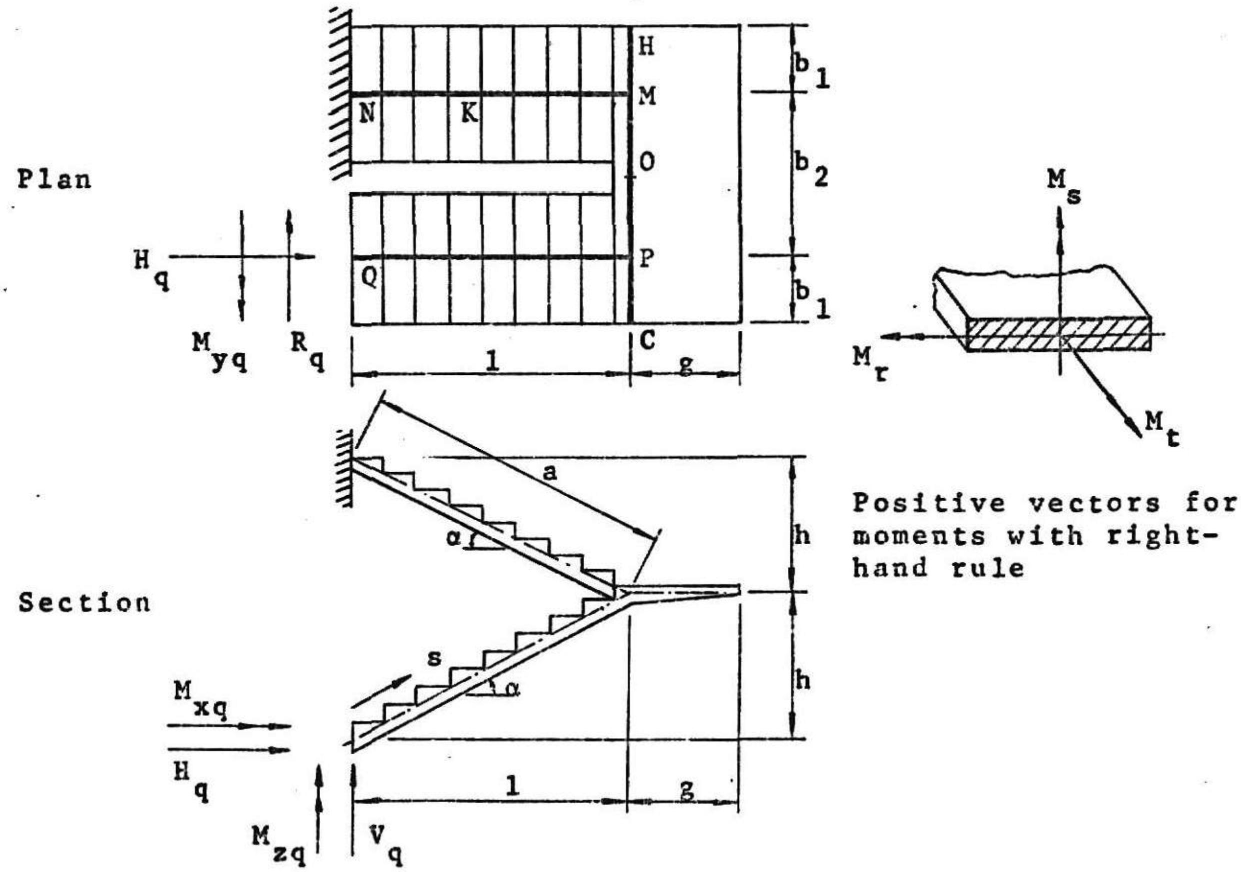


Fig. 12 - Plan, Section and Sign conventions for moments of the staircase

moments along the members of the space frame are listed in Table 2 and Table 3, respectively.

Member	Q-P	C-P	P-M
Origin	Q	C	C
Limit	$s=0$ to $s=a$	$y=0$ to $y=b_1$	$y=b_1$ to $y=b_1+b_2$
M_r	$V_q(\cos\alpha\ s)-H_q(\sin\alpha\ s)-\frac{1}{2}W_{1+3}(\cos\alpha\ s)^2 - M_{yq}$	$-\frac{1}{2}W_{2+5}\ y^2$	$V_q(y-b_1) - R_q(h) - M_{xq} - W_{1+3}\ l(y-b_1) - \frac{1}{2}\ W_{2+5}\ y^2$
M_s	$- R_q(s) - M_{xq}\ \sin\alpha + M_{zq}\ \cos\alpha$	0	$H_q(y-b_1) - R_q(l) + M_{zq}$
M_t	$- M_{xq}\ \cos\alpha - M_{zq}\ \sin\alpha$	$-\frac{1}{2}(g)W_{2+5}y$	$- V_q(l) + H_q(h) + M_{yq} + \frac{1}{2}\ W_{1+3}(l)^2 - \frac{1}{2}\ W_{2+5}(g)\ y$

Table 2 - Bending and torsional moments of members Q-P, C-P and P-M

Member	M-H	M-N
Origin	C	M
Limit	$y=b_1+b_2$ to $y=b_2+2b_1$	$s=0$ to $s=a$
M_r	$-\frac{1}{2} W_{2+5}(2b_1+b_2-y)^2$	$-V_q(l) + H_q(\sin\alpha\ s) + V_q(\cos\alpha\ s) + H_q(h) + M_{yq} - W_{1+3}(l)(\cos\alpha\ s) - W_{1+3}(l)(\frac{1}{2}l) - \frac{1}{2} W_{1+4}(\cos\alpha\ s)^2 - W_{2+5}(b_2+2b_1)(\cos\alpha\ s) - W_{2+5}(\frac{1}{2}g)(b_2+2b_1)$
M_s	0	$-V_q(b_2 \sin\alpha) + H_q(b_2 \cos\alpha) + R_q(s) - R_q(l \cos\alpha) + R_q(h \sin\alpha) + M_{xq} \sin\alpha + M_{zq} \cos\alpha + W_{1+3}(l)(b_2 \sin\alpha) + W_{2+5}(b_2 + 2b_1)(\frac{1}{2}b_2)\sin\alpha$
M_t	$-\frac{1}{2} W_{2+5}(2b_1+b_2-y)$	$-V_q(b_2 \cos\alpha) - H_q(b_2 \sin\alpha) + R_q(l \sin\alpha) + R_q(h \cos\alpha) + M_{xq} \cos\alpha - M_{zq} \sin\alpha + W_{1+3}(l)(b_2 \cos\alpha) + W_{2+5}(b_2+2b_1)(\frac{1}{2}b_2)(\cos\alpha)$

Table 3 - Bending and torsional moments of member M-H and M-N (Continued)

where

W_{1+3} = dead load + live load on the lower flight (plf)

W_{1+4} = dead load + live load on the upper flight (plf)

W_{2+5} = dead load + live load on the landing (plf)

Since the floor supports are assumed to be perfectly rigid, it can be concluded that no deflections or rotations occur at the end support Q. Then from the theorem of least work it follows that

$$\frac{\partial U}{\partial V_q} = 0 \qquad \frac{\partial U}{\partial H_q} = 0 \qquad \frac{\partial U}{\partial R_q} = 0$$
$$\frac{\partial U}{\partial M_{xq}} = 0 \qquad \frac{\partial U}{\partial M_{yq}} = 0 \qquad \frac{\partial U}{\partial M_{zq}} = 0 \tag{26}$$

where U is the total strain energy due to flexure and torsion in all members (the strain energy due to shearing and direct forces may be neglected in this case(11)). The complete expression for the total strain energy of the frame will be

$$\begin{aligned} U = & \int_L \frac{M_r^2}{2EI_2}\,ds + \int_L \frac{M_r^2}{2EI_1}\,dy \\ & + \int_L \frac{M_s^2}{2EI_2'}\,ds + \int_L \frac{M_s^2}{2EI_1'}\,dy \\ & + \int_L \frac{M_t^2}{2GJ_2}\,ds + \int_L \frac{M_t^2}{2GJ_1}\,dy \end{aligned} \tag{27}$$

where E,G; I_1, I_2; I_1',I_2'; and J_1,J_2 have the same meaning as in the preceding section.

Differentiating the total strain energy U in Eq. 27 with respect to V_q, H_q, R_q, M_{xq}, M_{yq} and M_{zq}, respectively, and substituting the corresponding moment expressions shown in Table 2 and Table 3 into Eq. 26; integrating and simplifying, six linear equations will be obtained as follows:

$$\frac{\partial H}{\partial V_q} = \frac{1}{EI_2}\ [\frac{2}{3}\ V_q(a\ 1^2) - H_q(a\ h\ 1) - M_{yq}(a\ 1) + \frac{1}{24}(a\ 1^3)(-5W_{1+3}$$

$$+ W_{1+4}) + \frac{1}{2}\ W_{2+5}(a\ 1)(b_2+2b_1)(\frac{1}{3} + \frac{g}{2})]$$

$$+ \frac{1}{EI_1}\ [\frac{1}{3}\ V_q(b_2)^3 - \frac{1}{2}\ R_q(b_2^2\ h) - \frac{1}{2}\ M_{xq}(b_2)^2 - \frac{1}{3}\ W_{1+3}(b_2^3\ 1) -$$

$$\frac{1}{2}\ W_{2+5}(\frac{1}{2}\ b_1^2 b_2^2 + \frac{2}{3}\ b_1 b_2^3 + \frac{1}{4}\ b_2^4)]$$

$$+ \frac{1}{EI_2'}\ [V_q(\frac{b_2^2\ h^2}{a}) - H_q(\frac{b_2^2\ h\ 1}{a}) - \frac{1}{2}\ R_q(a\ b_2\ h) + R_q(\frac{b_2\ h\ 1^2}{a})$$

$$- R_q(\frac{b_2\ h^3}{a}) - M_{xq}(\frac{b_2\ h^2}{a}) - M_{zq}(\frac{b_2\ h\ 1}{a}) - W_{1+3}(\frac{b_2^2\ h^2\ 1}{a})$$

$$- \frac{1}{2}\ W_{2+5}(2b_1+b_2)(\frac{b_2^2\ h^2}{a})]$$

$$+ \frac{1}{GJ_1}\ [V_q(b_2\ 1^2) - H_q(b_2\ h\ 1) - M_{yq}(b_2\ 1) - \frac{1}{2}\ W_{1+3}(b_2\ 1^3) +$$

$$\frac{1}{4}\ W_{2+5}(g\ 1)(2b_1 b_2 + b_2^2)]$$

$$+ \frac{1}{GJ_2}\ [V_q(\frac{b_2^2\ 1^2}{a}) + H_q(\frac{b_2^2\ h\ 1}{a}) - 2\ R_q(\frac{b_2^2\ h\ 1^2}{a}) - M_{xq}(\frac{b_2\ 1^2}{a})$$

$$+ M_{zq}(\frac{b_2\ h\ 1}{a}) - W_{1+3}(\frac{b_2^2\ 1^3}{a}) - \frac{1}{2}\ W_{2+5}(2b_1+b_2)(\frac{b_2^2\ 1^2}{a})$$

$$= 0 \qquad (28)$$

$$\frac{\partial U}{\partial H_q} = \frac{1}{EI_2}\ [-\ V_q(a\ h\ 1) + \frac{8}{3}\ H_q(a\ h^2) + 2\ M_{yq}(a\ h) + \frac{1}{24}(a\ h\ 1^2)$$

$$(W_{1+3} - 7\ W_{1+4}) - \frac{1}{2}\ W_{2+5}(a\ h)(2b_1+b_2)(\frac{51}{3} + \frac{3g}{2})]$$

$$+ \frac{1}{EI_1'}\ [\frac{1}{3}\ H_q(b_2)^2 - \frac{1}{2}\ R_q(b_2^2\ 1) + \frac{1}{2}\ M_{zq}(b_2^2)]$$

$$+ \frac{1}{EI_2'}\ [-\ V_q(\frac{b_2^2\ h\ 1}{a}) + H_q(\frac{b_2^2\ 1^2}{a}) + \frac{1}{2}\ R_q(b_2\ a\ 1) - R_q(\frac{b_2\ 1^3}{a})$$

$$+ R_q(\frac{b_2\ h^2\ 1}{a}) + M_{xq}(\frac{b_2\ h\ 1}{a}) + M_{zq}(\frac{b_2\ 1^2}{a}) + W_{1+3}$$

$$\left(\frac{b_2^2\ h\ l^2}{a}\right) + \frac{1}{2} W_{2+5}(2b_1+b_2)\left(\frac{b_2^2\ h\ l}{a}\right)]$$

$$+ \frac{1}{GJ_1} [- V_q(b_2\ h\ l) + H_q(b_2\ h^2) + M_{yq}(b_2\ h) + \frac{1}{2} W_{1+3}(b_2\ h\ l^2)$$

$$- \frac{1}{4} W_{2+5}(g\ h)(2b_1b_2 + b_2^2)]$$

$$+ \frac{1}{GJ_2} [V_q\left(\frac{b_2^2\ h\ l}{a}\right) + H_q\left(\frac{b_2^2\ h^2}{a}\right) - 2\ R_q\left(\frac{b_2\ h^2\ l}{a}\right) - M_{xq}\left(\frac{b_2\ h\ l}{a}\right)$$

$$+ M_{zq}\left(\frac{b_2\ h^2}{a}\right) - W_{1+3}\left(\frac{b_2^2\ h\ l^2}{a}\right) - \frac{1}{2} W_{2+5}(2b_1+b_2)\left(\frac{b_2^2\ h\ l}{a}\right)]$$

$$= 0 \tag{29}$$

$$\frac{\partial\ U}{\partial\ R_q} = \frac{1}{EI_1} [- \frac{1}{2} V_q(b_2^2\ h) + R_q(b_2\ h^2) + M_{xq}(b_2\ h) + \frac{1}{2} W_{1+3}(b_2^2\ h\ l)$$

$$+ \frac{1}{6} W_{2+5}(h)((b_1+b_2)^3 - b_1^3)]$$

$$+ \frac{1}{EI_1'} [- \frac{1}{2} H_q(b_2^2\ l) + R_q(b_2\ l^2) - M_{zq}(b_2\ l)]$$

$$+ \frac{1}{EI_2'} [- \frac{1}{2} V_q(a\ b_2\ h) + V_q\left(\frac{b_2\ h\ l^2}{a}\right) - V_q\left(\frac{b_2\ h^3}{a}\right) + \frac{1}{2} H_q(b_2\ a\ l)$$

$$- H_q\left(\frac{b_2\ l^3}{a}\right) + H_q\left(\frac{b_2\ h^2\ l}{a}\right) + \frac{2}{3} R_q(a)^3 + R_q(a\ h^2) -$$

$$2\ R_q\left(\frac{h^2\ l^2}{a}\right) + R_q\left(\frac{h^4}{a}\right) + R_q\left(\frac{l^4}{a}\right) - R_q(a\ l^2) + M_{xq}(a\ h)$$

$$+ M_{xq}\left(\frac{h^3}{a}\right) - M_{xq}\left(\frac{h\ l^2}{a}\right) - M_{zq}\left(\frac{l^3}{a}\right) + M_{zq}\left(\frac{h^2\ l}{a}\right) + \frac{1}{2} W_{1+3}$$

$$(a\ b_2\ h\ l) - W_{1+3}\left(\frac{b_2\ h\ l^3}{a}\right) + W_{1+3}\left(\frac{b_2\ h^3\ l}{a}\right) + \frac{1}{4} W_{2+5}$$

$$(2b_1+b_2)(a\ b_2\ h) - \frac{1}{2} W_{2+5}(2b_1+b_2)\left(\frac{b_2\ h\ l^2}{a}\right) + \frac{1}{2} W_{2+5}$$

$$(2b_1+b_2)\left(\frac{b_2\ h^3}{a}\right)]$$

$$+ \frac{1}{GJ_2} [- 2\ V_q\left(\frac{b_2\ h\ l^2}{a}\right) - 2\ H_q\left(\frac{b_2\ h^2\ l}{a}\right) + 4\ R_q\left(\frac{h^2\ l^2}{a}\right) +$$

$$2\ M_{xq}(\frac{h\ 1^2}{a}) - 2\ M_{zq}(\frac{h^2\ 1}{a}) + 2\ W_{1+3}(\frac{b_2\ h\ 1^3}{a}) +$$

$$W_{2+5}(2b_1+b_2)(\frac{b_2\ h\ 1^2}{a})] = 0 \qquad (30)$$

$$\frac{\partial\ U}{\partial\ M_{xq}} = \frac{1}{EI_1}\ [-\frac{1}{2}\ V_q(b_2)^2 + R_q(b_2\ h) + M_{xq}(b_2) + \frac{1}{2}\ W_{1+3}(b_2^2\ 1) +$$

$$\frac{1}{6}\ W_{2+5}((b_1+b_2)^3 - b_1^3)]$$

$$+ \frac{1}{EI_2'}\ [-\ V_q(\frac{b_2\ h^2}{a}) + H_q(\frac{b_2\ h\ 1}{a}) + R_q(a\ h) - R_q(\frac{h\ 1^2}{a}) + R_q(\frac{h^3}{a})$$

$$+ 2\ M_{xq}(\frac{h^2}{a}) + W_{1+3}(\frac{b_2\ h^2\ 1}{a}) + \frac{1}{2}\ W_{2+5}(2b_1+b_2)(\frac{b_2\ h^2}{a})]$$

$$+ \frac{1}{GJ_2}\ [-\ V_q(\frac{b_2\ 1^2}{a}) - H_q(\frac{b_2\ h\ 1}{a}) + 2\ R_q(\frac{h\ 1^2}{a}) + 2\ M_{xq}(\frac{1^2}{a})$$

$$+ W_{1+3}(\frac{b_2\ 1^3}{a}) + \frac{1}{2}\ W_{2+5}(2b_1+b_2)(\frac{b_2\ 1^2}{a})] = 0 \qquad (31)$$

$$\frac{\partial\ U}{\partial\ M_{yq}} = \frac{1}{EI_2}\ [-\ V_q(a\ 1) + 2\ H_q(a\ h) + 2\ M_{yq}(a) + \frac{1}{6}\ (a\ 1^2)(W_{1+3} -$$

$$W_{1+4}) - \frac{1}{2}\ W_{2+5}(a)(2b_1+b_2)(g + 1)]$$

$$+ \frac{1}{GJ_1}\ [-\ V_q(b_2\ 1) + H_q(b_2\ h) + M_{yq}(b_2) + \frac{1}{2}\ W_{1+3}(b_2\ 1^2) -$$

$$\frac{1}{4}\ W_{2+5}(g)(2b_1b_2 + b_2^2)] = 0 \qquad (32)$$

$$\frac{\partial\ U}{\partial\ M_{zq}} = \frac{1}{EI_1'}\ [\frac{1}{2}\ H_q(b_2)^2 - R_q(b_2\ 1) + M_{zq}(b_2)]$$

$$+ \frac{1}{EI_2'}\ [-\ V_q(\frac{b_2\ h\ 1}{a}) + H_q(\frac{b_2\ 1^2}{a}) - R_q(\frac{1^3}{a}) + R_q(\frac{h^2\ 1}{a}) +$$

$$2\ M_{zq}(\frac{1^2}{a}) + W_{1+3}(\frac{b_2\ h\ 1^2}{a}) + \frac{1}{2}\ W_{2+5}(2b_1+b_2)(\frac{b_2\ h\ 1}{a})]$$

$$+ \frac{1}{GJ_2}\ [V_q(\frac{b_2\ h\ 1}{a}) + H_q(\frac{b_2\ h^2}{a}) - 2\ R_q(\frac{h^2\ 1}{a}) + 2\ M_{zq}(\frac{h^2}{a}) -$$

$$W_{1+3}(\frac{b_2\ h\ 1^2}{a}) - \frac{1}{2}\ W_{2+5}(2b_1+b_2)(\frac{b_2\ h\ 1}{a})] = 0 \qquad (33)$$

The details of the procedure will be illustrated in the following numerical example.

NUMERICAL EXAMPLE 2

The concrete cantilever staircase shown in Fig. 8 (Page 17) will be analyzed by the method described in this section. The additional dimensions for the landing are:

b_1 = 2 ft. b_2 = 5 ft.

1. Case 1: Unsymmetrical loading

a)-Loading

Consider the case where only the lower flight and the landing are subjected to an imposed load of 100 psf.

W_1 = 98 psf. x 4 = 392 plf.

W_2 = 75 psf. x 3.5 = 263 plf.

W_3 = 100 psf. x 4 = 400 plf.

W_4 = 0

W_5 = 100 psf. x 3.5 = 350 plf.

W_{1+3} = 392 + 400 = 792 plf.

W_{1+4} = 392 plf.

W_{2+5} = 263 + 350 = 613 plf.

b)-Moment of inertia of the flights and the landing

As previously shown in Pages 18 and 19

I_1 = 756 $in.^4$

I_1' = 37,044 $in.^4$

I_2 = 365 $in.^4$

$I_2' = 41{,}472 \text{ in.}^4$

$J_1 = 2{,}752 \text{ in.}^4$

$J_2 = 1{,}372 \text{ in.}^4$

c)-Computation of the redundants in the structure

Substituting the appropriate values into Eq. 28 through Eq. 33, six simultaneous equations will be obtained as follows:

$$306.5229V_q - 148.3711H_q - 97.0416R_q - 40.7742M_{yq} - 12.7748M_{xq} + 4.9838M_{zq} = 631769.64$$

$$-148.3711V_q + 224.6375H_q - 43.5909R_q + 37.9557M_{yq} - 4.9838M_{xq} + 2.7274M_{zq} = 1213694.4$$

$$-97.0416V_q - 43.5909H_q + 168.0524R_q + 0.\ M_{yq} + 21.7171M_{xq} - 8.9639M_{zq} = -920745.48$$

$$-40.7742V_q + 37.9557H_q + 0.\ R_q + 8.9308M_{yq} + 0.\ M_{xq} + 0.\ M_{zq} = 101153.85$$

$$-12.7748V_q - 4.9838H_q + 21.7171R_q + 0.\ M_{yq} + 5.1099M_{xq} + 0.\ M_{zq} = -121185.29$$

$$4.9838V_q + 2.7274H_q - 8.9639R_q + 0.\ M_{yq} + 0.\ M_{xq} + 1.0910M_{zq} = 47298.745$$

Solving,

$V_q = 8732.65$ lbs

$H_q = 10598.03$ lbs

$R_q = -648.15$ lbs

$M_{yq} = 6154.43$ ft-lb

$M_{xq} = 11206.92$ ft-lb

$M_{zq} = -28358.50$ ft-lb

2. Case 2: Symmetrical loading

a)-Loading

Consider the case where both flights and landing loaded.

$W_{1+3} = W_{1+4} = 792$ plf.

$W_{2+5} = 613$ plf.

b)-Computation of the redundants in the structure

The coefficients of the unknowns in the simultaneous equations are exactly the same as case 1. The constant terms for each equation, respectively, are:

589458.46 1361783.7 -920745.48 121065.01 -121185.29 47298.745

Solving the six linear equations

$V_q = 9490.50$ lbs

$H_q = 11826.91$ lbs

$R_q = 0.$

$M_{yq} = 6621.29$ ft-lb

$M_{xq} = 11543.82$ ft-lb

$M_{zq} = -29564.20$ ft-lb

3. Case 3: Symmetrical loading with the landing unloaded

a)-Loading

$W_{1+3} = W_{1+4} = 792$ plf.

$W_{2+5} = 263$ plf.

b)-Computation of the redundants in the structure

The constant terms for each equation, respectively, are

765009.12 777496.27 -768035.08 41108.27 -101095.66

39449.26

Solving the simultaneous equations

V_q = 7915.50 lbs

H_q = 7852.69 lbs

R_q = 0.

M_{yq} = 7367.96 ft-lb

M_{xq} = 7662.22 ft-lb

M_{zq} = -19629.70 ft-lb

The corresponding bending and torsional moments for the more important sections of the stair (Fig. 12) are listed in Table 4.

Points	Moments (ft-lb)	With both flights and landing loaded	With the landing unloaded	With upper flight and landing loaded	With lower flight and landing loaded
0	M_r	-10854	- 7366	- 9657	- 9657
	M_s	0	0	- 3646	+ 3646
	M_t	0	0	- 752	+ 752
M-N	M_r	- 4827	- 2070	- 5580	- 4075
	M_s	+31608	+20983	+24223	+32432
	M_t	- 2896	- 1925	- 2658	- 2530
N-M	M_r	- 6621	- 7368	- 6154	- 3083
	M_s	-31608	-20983	-30381	-26273
	M_t	- 2896	- 1925	- 2658	- 2530
K	M_r	+ 1429	+ 2665	+ 1285	- 39

Table 4 - Bending and torsional moments in staircase due to symmetrical and unsymmetrical loadings

From the results shown in Table 4, it is seen that the moments due to unsymmetrical loads are only slightly greater than those for full load at some particular sections. Consequently, it should be sufficiently conservative to consider only the symmetric loading condition for practical design purposes. Furthermore, since R_q, M_s and M_t at point 0 are all zero in the case of symmetry, the analysis can be simplified by cutting the frame into two equal halves at point 0, and each half of the structure will become a simpler indeterminate structure having two redundants only. This approach will be described in detail in the next section.

ALTERNATE METHOD OF ANALYSIS OF THE STAIRCASE BASED ON FUCHSTEINER'S ASSUMPTIONS

The method of analyzing the statically indeterminate staircase formed by a series of bar elements was first developed by W. Fuchsteiner (1). The main difference in his assumptions from those suggested by Cusens and Kuang (6) is that the landing slab can be represented by a curved bar element (Fig. 13) when the moments in the frame are computed.

It was previously found that the moments produced due to unsymmetrical loads at some sections of the staircase are only slightly greater than those for symmetrical loads. Therefore the following analysis will be based on Fuchsteiner's assumptions and only the cases of symmetrical loading conditions will be considered.

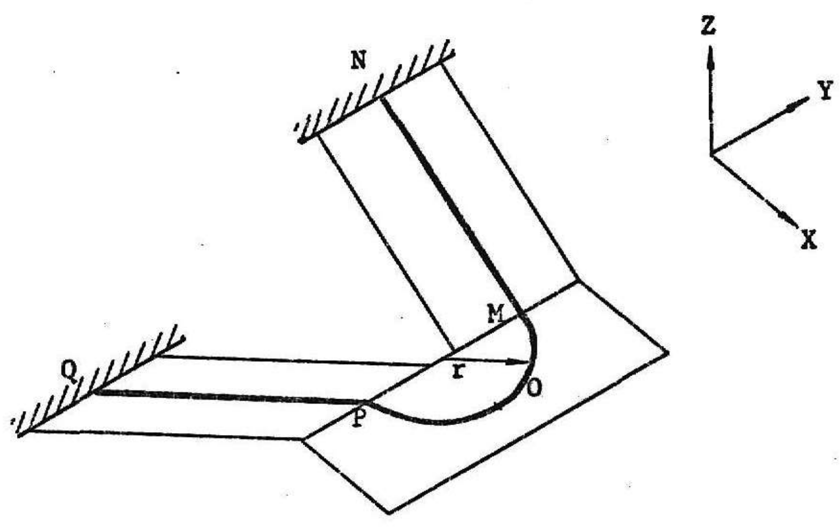


Fig. 13 - Fuchsteiner's assumed form for the cantilever staircase

From geometric and loading symmetry of the staircase, it is much simpler to solve this problem by cutting the whole frame at the mid-point of the landing into two equal halves, which will be treated as two separate cantilever beams, than it is to solve the original structure. Thus each half of the frame can be considered as a cantilever structure with only two unknown redundants, i.e., a bending moment M_o and a shearing force H, both acting along the cut section (Fig. 14).

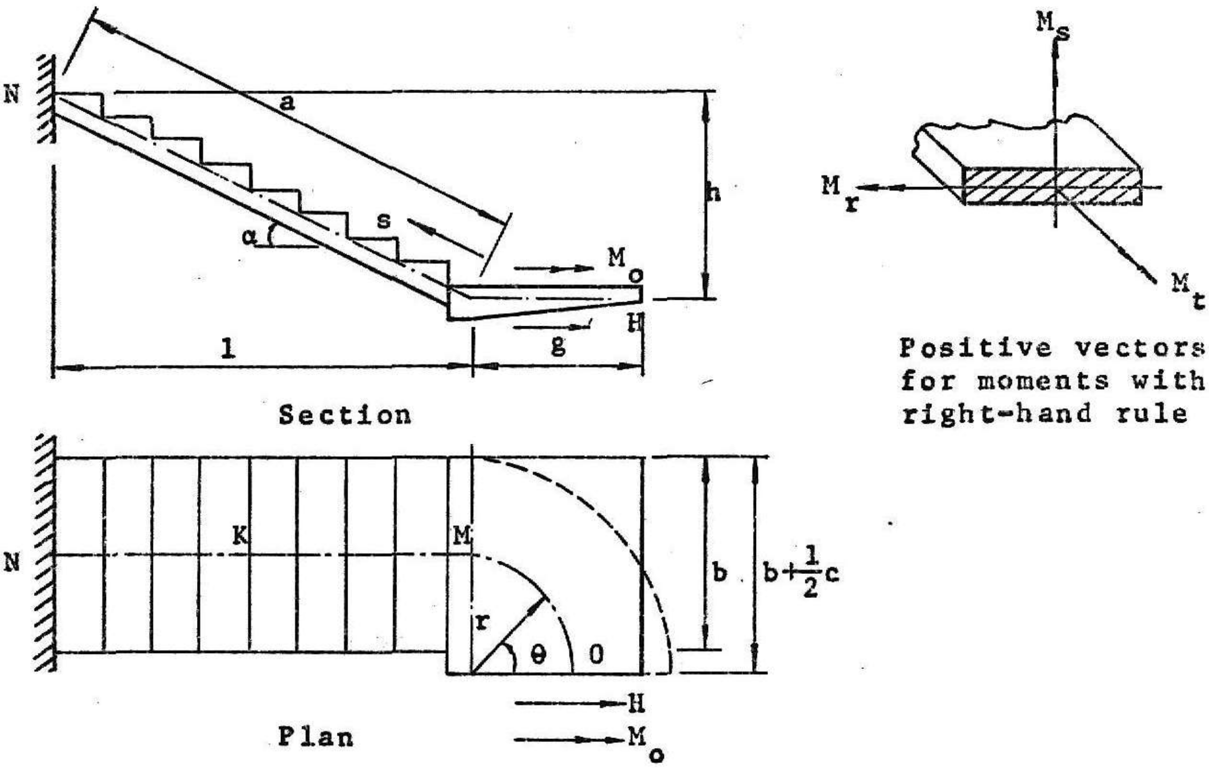


Fig. 14 - Plan, section and moment sign conventions of the half staircase

The bending and torsional moments along the members of the upper half of the frame are:

Member $\widehat{OM}$

Origin: Point O

Limit: $\theta = 0$ to $\theta = \frac{\pi}{2}$

$$M_r = - M_o \cos\theta - W_{2+5}(r\,\theta)(r \sin\frac{\theta}{2}) \tag{34}$$

$$M_s = H(r \sin\theta) \tag{35}$$

$$M_t = M_o \sin\theta + W_{2+5}(r\,\theta)(r - r\cos\frac{\theta}{2}) \tag{36}$$

Member $\overline{MN}$

Origin: Point M

Limit: $s = 0$ to $s = a$

$$M_r = H(s \sin\alpha) - \frac{\pi}{2}(r)\, W_{2+5}(\frac{\sqrt{2}}{2}\, r) - \frac{\pi}{2}\,(r)\, W_{2+5}(s \cos\alpha) - \frac{1}{2}\, W_{1+4}(s \cos\alpha)^2 \tag{37}$$

$$M_s = H(r \cos\alpha) + \frac{\pi}{2}\,(r)\, W_{2+5}(r - r\frac{\sqrt{2}}{2})\sin\alpha + M_o \sin\alpha \tag{38}$$

$$M_t = - H(r \sin\alpha) + \frac{\pi}{2}\,(r)\, W_{2+5}(r - r\frac{\sqrt{2}}{2})\cos\alpha + M_o \cos\alpha \tag{39}$$

where W_{1+4} = D.L. + L.L. per unit length of the upper flight

W_{2+5} = D.L. + L.L. per unit length along the curved beam in the landing

Since the stair is assumed to be completely fixed at the floor supports and the landing is symmetrically loaded, it is evident that there will be neither horizontal movement (X-direction) nor bar rotation at point O. Consequently, by applying the theorem of least work it follows that

$$\frac{\partial U}{\partial M_o} = 0 \qquad \text{and} \qquad \frac{\partial U}{\partial H} = 0 \tag{40}$$

Neglecting the effects of shearing and direct forces (11), the complete expression for the strain energy U of the half frame is

$$\begin{aligned} U = & \int_0^{\frac{\pi}{2}} \frac{M_r^2}{2EI_1} d\theta + \int_0^a \frac{M_r^2}{2EI_2} ds \\ & + \int_0^{\frac{\pi}{2}} \frac{M_s^2}{2EI_1'} d\theta + \int_0^a \frac{M_s^2}{2EI_2'} ds \\ & + \int_0^{\frac{\pi}{2}} \frac{M_t^2}{2GJ_1} d\theta + \int_0^a \frac{M_t^2}{2GJ_2} ds \end{aligned} \tag{41}$$

where E, G; I_1, I_2; I_1', I_2' and J_1, J_2 have the same meanings as before.

Differentiating the total strain energy U in Eq. 41 with respect to H and M_o, respectively, substituting the corresponding moment expressions in Eq. 34 through Eq. 39, and integrating and simplifying, two simultaneous equations will be obtained as follows:

$$\frac{\partial U}{\partial H} = \frac{1}{EI_2} [\frac{1}{3} H(a\ h^2) - \frac{\sqrt{2}\pi}{8} W_{2+5}(a\ h\ r^2) - \frac{\pi}{6} W_{2+5}(a\ h\ l\ r) -$$

$$\frac{1}{8} W_{1+4}(a\ h\ l^2)]$$

$$+ \frac{1}{EI_1'} [\frac{\pi}{4} H (r)^2]$$

$$+ \frac{1}{EI_2'} [H (\frac{1^2 r^2}{a}) + M_o (\frac{h\ 1\ r}{a}) + \frac{\pi}{4}(2 - \sqrt{2})\ W_{2+5} (\frac{h\ 1\ r^3}{a})]$$

$$+ \frac{1}{GJ_2} [H (\frac{h^2 r^2}{a}) - M_o (\frac{h\ 1\ r}{a}) - \frac{\pi}{4}(2 - \sqrt{2})\ W_{2+5} (\frac{h\ 1\ r^3}{a})] = 0 \qquad (42)$$

$$\frac{\partial U}{\partial M_o} = \frac{1}{EI_1} [\frac{\pi}{4} M_o + 0.22388008\ W_{2+5}\ (r)^2]$$

$$+ \frac{1}{EI_2'} [H (\frac{h\ 1\ r}{a}) + M_o (\frac{h^2}{a}) + \frac{\pi}{2}(1 - \frac{\sqrt{2}}{2})\ W_{2+5} (\frac{h^2 r^2}{a})]$$

$$+ \frac{1}{GJ_1} [\frac{\pi}{4} M_o + 0.16914005\ W_{2+5}\ (r)^2]$$

$$+ \frac{1}{GJ_2} [- H (\frac{h\ 1\ r}{a}) + M_o (\frac{1^2}{a}) + \frac{\pi}{2} (1 - \frac{\sqrt{2}}{2})\ W_{2+5} (\frac{1^2 r^2}{a})] = 0 \qquad (43)$$

The following example is worked out by the method discussed in this section.

NUMERICAL EXAMPLE 3

Analyze the same concrete staircase shown in Fig. 8 by the method of least work based on Fuchsteiner's assumptions. The additional dimension is: $r = 2.5$ ft.

Case 1: With both flights and landing loaded

a) Loading

W_{2+5} = 175 psf

Total loads on the landing = 175 x 9' x 3.5' = 5,512.5#

Total length of the curved beam = $\pi(2.5)$ = 7.8540 ft.

W_{2+5} = 5512.5/7.8540 = 702 plf.

W_{1+4} = 792 plf.

b) Moment of inertia of the flights and the landing

Neglecting the corner effect of the landing and using the same I and J values as shown before

$I_1 = 756 \text{ in}^4$

$I_1' = 37{,}044 \text{ in}^4$

$I_2 = 365 \text{ in}^4$

$I_2' = 41{,}472 \text{ in}^4$

$J_1 = 2{,}752 \text{ in}^4$

$J_2 = 1{,}372 \text{ in}^4$

c) Computation of the redundants in the structure

Substituting the appropriate values into Eq. 42 and Eq. 43, two simultaneous equations will be obtained as follows:

$$28.2523\,H - 2.4919\,M_o = 310728.09$$

$$-2.4919\,H + 2.2990\,M_o = -4402.25$$

Solving,

$$H = 11974.203 \text{ lbs}$$

$$M_o = 11063.954 \text{ ft-lb}$$

Case 2: With live load on both flights only

a) Loading

$$W_{1+4} = 792 \text{ plf.}$$

$$W_{2+5} = \frac{75}{175}(702) = 301 \text{ plf}$$

b) Computation of the redundants in the structure

Two simultaneous equations will be as follows:

$$28.2523\ H - 2.4919\ M_o = 205015.0100$$

$$-\ 2.4919\ H + 2.2990\ M_o = -\ 1887.5744$$

Solving,

$$H = 7943.5897 \text{ lbs}$$

$$M_o = 7788.9907 \text{ ft-lb}$$

The corresponding bending and torsional moments for the more important sections of the stair (Fig. 14) are listed in Table 5.

Moments	M_r (ft-lb)				M_s (ft-lb)	M_t (ft-lb)
Points	N	M	O	K	M-N	M-N
With both flights and landing loaded	-6,027	-4,873	-11,064	+1,703	+32,622	-1,687
With live load on both flights only	-6,985	-2,089	- 7,789	+2,819	+21,633	-1,140

Table 5 - Bending and torsional moments in staircase due to symmetrical loading

For comparison, a summary of the results computed by the three different methods discussed above are shown in the following table.

Moments		M_r (ft-lb)				M_s (ft-lb)	M_t (ft-lb)
Points		0	M	N	K	M-N	M-N
With both flights and landing loaded	Siev	-13,041	-4,825	-5,311	+2,373	+33,243	-2,074
	Cusens and Kuang	-10,854	-4,827	-6,621	+1,429	+31,608	-2,896
	Fuchs-teiner	-11,064	-4,873	-6,027	+1,707	+32,622	-1,687
With live load on both flights only	Siev	- 8,642	-2,070	-6,496	+3,204	+22,063	-1,375
	Cusens and Kuang	- 7,366	-2,070	-7,368	+2,665	+20,983	-1,925
	Fuchs-teiner	- 7,789	-2,089	-6,985	+2,819	+21,633	-1,140

Table 6 - Comparison of computed values of moments in the staircase by three different methods

CONCLUSIONS

From the summary of results shown in Table 6, it is apparent that there exist some discrepancies between the values obtained from the different methods. The reason for this may be attributed chiefly to the use of different assumptions made by each author. In reality, however, since the analysis of staircase structures of the type considered in this report involves the solution of a highly complex plate problem, any attempt to obtain a more precise result by using only elementary structural methods seems to be impracticable. Therefore, for the purpose of practical use, the application of a simple and approximate approach is usually necessary.

Siev's method of analysis is somewhat more difficult to comprehend and to apply in practice if secondary moments are considered. Moreover, since his method mainly deals with the symmetrical staircase by assuming that the whole structure can be treated as two separate plates with one end fixed at the floor and the other end hung over the fictitious support at the line of intersection, this apparently will become unjustifiable when the flights are unequal in length.

The application of the theorem of least work to the analysis of a staircase based on either Cusens and Kuang's or Fuchsteiner's assumptions is preferable. In the case of only symmetrical loads, both methods will be greatly simplified if the stair itself is also symmetrical in its geometry. The tedious work described in the second section is a necessary procedure when problems dealing with flexible supports and unequal flights are involved.

It is seen from Table 6 that the method proposed by Fuchsteiner gives results close to those of Siev with the exception that the former yields lower torsional moments in the flights than those of Cusens and Kuang.

A number of stairs of this type designed by the methods discussed in this report have been constructed. (7) (12) The behavior of cantilever staircases determined by these analytical methods also correlates satisfactorily with experimental results (13) (14).

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STRUCTURAL ANALYSIS OF A REINFORCED
CONCRETE CANTILEVER STAIRCASE

by

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AN ABSTRACT OF A MASTER'S REPORT

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ABSTRACT

The purpose of this report is to present the methods of analyzing a concrete cantilever staircase consisting of two straight flights and a landing and supported only on the upper and the lower floors.

Siev's analytical method is chiefly based on Liebenberg's early theory that the staircase can be treated as two separate plates with one end fixed on the floor and the other end hung over an imaginary support at the line of intersection. He also extended the method to include the determination of the torsional restraining moment in the landing resulting from the compatibility in deformation between the flights and the landing.

A. R. Cusens and J. G. Kuang developed the method of analysis by applying the principle of least work and assuming that the space structure, which is composed of plates or slabs, can be replaced by straight bar elements.

W. Fuchsteiner was the first person who suggested the space bar assumptions for the cantilever staircase. His assumptions are similar to those of Cusens and Kuang with the only difference being that the landing is replaced by a curved bar instead of a straight one.

The results from previously reported tests give good agreement with the above-mentioned methods of analysis. These methods have already been employed in the practical design of cantilever staircases.