Investigating students' use of mathematical tools and representations in undergraduate physics problem-solving

by

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B.S., University Of Kelaniya, Sri Lanka, 2011

M.S., Kansas State University, 2015

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Physics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas

2019

Abstract

One of the major focuses in the PER community has been on understanding student learning in order to improve how we teach undergraduate physics courses. Problem-solving receives significant emphasis across the undergraduate physics curriculum, and students' ability to solve physics problems has been researched extensively. Studying student problem-solving processes in upper-division physics courses is a growing area of research. The research presented in this dissertation is done in an effort to understand student problem-solving processes at the upper-division level. In particular, we investigate students' use of mathematical tools across the undergraduate physics curriculum, in particular, at intermediate mechanics students' written homework solutions in order to better understand how students use these tools as they solve traditional mechanics problems. We use a modified version of the ACER (Activation-Construction-Execution-Reflection) framework to analyze students' solutions and to identify patterns of mathematical skill use on traditional problems. We apply techniques borrowed from network analysis and the Resources Framework to build a "fingerprint" of students' mathematical tool use. In this study, we present preliminary findings on patterns that we have identified in students' problem solving.

Then we shift our focus to investigate how students use multiple representations as they solve problems. First, we study how students translate among different representations. Data for this study is drawn from an upper-division Electromagnetism I course, where students engaged in individual oral exams. We do a moment-by-moment analysis of students' problem solving to see how they translate among durable representations (diagrams, written mathematical equations) with the help of evanescent representations (gestures, words), and how students build up durable representations where they can "stand fast" later. In this study, we present the case of Larry as an exemplary case for problem solving. Larry starts from a durable representation (a diagram). He then builds from there, using evanescent

representations (gestures and words) and standing fast on the diagram. He later translates to a different kind of durable representation (mathematics), where he reasons and answers the original problem.

In our investigation to understand how students use multiple representations while solving problems, we further investigate how students construct spontaneous representations and also make connections among these representations. We use a social semiotic perspective to sketch a theoretical framework that accounts for how semiotic resources might be combined to solve problems. We identify the semiotic and conceptual resources that Larry uses, and we use a resource-graph representation to show Larry's coordination of resources in his problem-solving activity. Larry's case exemplifies coordination between multiple semiotic resources with different disciplinary affordances to build up compound representations. Our analysis of this case illustrates a novel way to think about how students use semiotic resources as they solve physics problems.

After that, we revise our theoretical framework to better describe students connecting multimodal representations. The revised framework focuses on the semiotic modes and it describes how students coordinate among different semiotic resources with different modes. In this study, we use multiple cases of students solving oral-exam problems to generalize the approach.

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Acknowledgments

I am very fortunate to have the support of many people during my time at K-State. I am grateful to my PhD advisor, Dr. Eleanor Sayre, for her tremendous support and guidance during the course of this research study. She provided me with lots of encouragement and sound advice whenever I needed them.

I would like to thank my masters advisor Dr. N. Sanjay Rebello for introducing me to the Physics Education Research field and to the Mathematization project at KSUPER.

I am grateful to Dr. Dean Zollman for his mentorship in the Mathematization project and his contribution to this work is invaluable. I am also grateful to Dr. James Laverty for the support he gave throughout my graduate career.

I would also like to thank the other members of my committee: Dr. Brett DePaola, Dr. Andrew Bennett and Dr. Steve Warren.

I am deeply grateful to the members of the KSUPER group, our copyeditors, and the anonymous reviewers whose comments improved my work. I would also like to thank Jeremy Smith and Judy Ball for their help.

Finally, this work is supported by National Science Foundation grant 1430967.

Dedication

To my parents, my brother *(Anil)*, my lovely wife *(Nirasha)* and my son *(Nelith)*.

Chapter 1

Introduction

Most prior studies in Physics Education Research (PER) have focused on understanding student learning in order to improve how we teach undergraduate physics courses. Most of that prior work has been done on introductory level physics courses, but studying student learning and problem-solving processes in upper-division physics courses is a growing area of research in the field of physics education. In order to improve student learning in upper-division physics courses, it is necessary to study and understand more about students' learning practices at this level. This thesis has two main sections: (1) Study how students use mathematical tools during physics problem-solving. (2) Characterize students' use of representations (including construction and making connections) while solving physics problems.

Problem-solving receives a significant emphasis across the undergraduate physics curriculum, and students' ability to solve physics problems has been researched extensively. Among various approaches, one way that researchers have studied students' learning practices has been to identify student difficulties in physics. We first describe the development and theoretical grounding of a new analytical framework designed to characterize how students use mathematical tools during physics problem solving. Then we apply this framework along with social network analysis to investigate student written homework solutions.

After that, we describe students' use of representations while solving problems; specifically, we describe the development of a novel way for describing students' construction of representations. This study seeks to understand how students construct representations to solve problems and what factors may influence whether they succeed or fail to connect among representations as they solve problems at the upper-division level.

This dissertation is organized as follows. Chapter 2 starts by describing previous studies that have been done on undergraduate students to understand their problem solving behaviors and use of mathematics in physics courses. Then we introduce an analytical tool (the ACER framework) for investigating upper-division students' use of mathematical tools in their physics homework solutions. After that, we redirect our focus toward students' use of representations and social networks in physics.

Chapter 3 explores how the modified ACER framework might be used to analyze the written evidence of typical problem solving in a traditional course. In this study, we use the modified ACER framework to identify three kinds of differences in student homework solutions: differences in solution paths within one problem; differences among multiple problems of the same type, and differences in students' initial problem-solving steps as a function of the type of problem.

Chapter 4 shifts focus to how students coordinate among multiple representations while solving problems. In this analysis, we present the case of Larry as an exemplary case. We suggest problem solving as a process that involves translations between durable representations (diagrams, written mathematical equations) with the help of evanescent representations (gestures, words). In addition, we maintain close attention to student reasoning.

Having established students' use of representations while solving physics problems, Chapter 5 describes a novel way to describe students' construction of representations while solving problems. We present an initial framework, which we suggest shows how students coordinate among different semiotic resources to construct representations that they use to solve problems.

Chapter 6 describes how can we better describe students' problem-solving processes when they connect multimodal representations while solving problems. We use multiple cases of students solving oral-exam problems to generalize the approach we proposed in our previous study. We explain the underlying mechanism behind students' construction of spontaneous representations as they coordinate among different semiotic resources with different modes. Finally, in chapter 7, we summarize the findings from the studies described in this thesis and suggest a potential avenue for further research. Chapter 8 summarizes the research questions and the answers.

Chapter 2

Literature Review Overview

2.1 Problem Solving in the Physics Curriculum

Before we discuss problem solving, it is necessary to have a definition of the word "problem". In common language, a problem can be a situation that causes difficulties, something that must be solved, or a condition that must be corrected. Researchers in Physics Education Research (PER) define a problem as "a task which requires one to devise a sequence of actions leading from some initial situation to some specified goal" [(1)]. Then the process of solving problems is a behavioral process, and problem solving is a strategy to select the most effective response from all the available responses [(2)].

One major focus of the PER community has been understanding student learning in order to improve the ways we teach undergraduate physics courses. Solving problems receives a significant emphasis across the undergraduate physics curriculum, and students' ability to solve physics problems has been studied intently in research. In early work at the undergraduate introductory physics level, researchers compared the problem-solving behaviors between expert and novice problem solvers [(3;4;5;6)]. This work led to identifying differences between expert and novice problem solving approaches. In particular, novices tend to pay attention only to the surface features of a problem while experts focus on deep structural features. In addition to that, experts seem to employ efficient problem solving approaches as a reason of their knowledge organization. These studies imply that instructional strategies should be developed that encourage students to develop the techniques used by expert problem solvers by influencing students to consider physics concepts and efficient procedures while solving problems.

2.2 Students' use of Mathematics in Problem Solving

Even though most prior work in physics problem-solving was done on introductory level physics courses, studying student-learning and problem-solving processes in upper-division physics courses is a growing area of research in the field of Physics Education. Students in upper-division physics courses are required to manipulate mathematical tools that have a higher degree of complexity (e.g. choosing an appropriate coordinate system, computing derivatives, evaluating integrals, using differential equations and approximations, etc.) when compared with students at the introductory level.

Also, students in advanced courses are often required to connect their mathematical ideas and expressions to physics concepts. It is well noticed that physicists use mathematics in a slightly different way from mathematicians[7;8;9;10;11]. Even though these required mathematical tools and the techniques associated with these tools are covered in mathematics classes, where students learn how to successfully solve mathematical problems, many students still struggle to apply mathematical tools appropriately in their physics courses to solve problems[12;13;14;15;16].

2.2.1 Prescriptive Problem-Solving Strategies

As an approach to finding ways to help students overcome these issues and to improve students' learning in upper-division physics courses, researchers investigated how students integrate their mathematical ideas with their available conceptual knowledge while solving problems. Investigating and comparing between the problem-solving behaviors of experts and novices is a well-established traditions in PER. As a starting point to compare how well algebra-based introductory physics students would use mathematics in their physics courses, Redish[9] developed a cyclic model to describe how experts use mathematics in physics (and in other sciences as well) to describe physical systems. This cyclic model consists of four steps that guide the process (Figure 2.1[9]). In this diagram, the boxes represent goal points, and the arrows represent the process. First, the physical system is described and modeled. Second, a translation into a mathematical representation follows, and are mathematical manipulations are performed. Third, obtained mathematical results are interpreted within the physical model. Finally, the physical results are evaluated against the problem statement and the physical system.

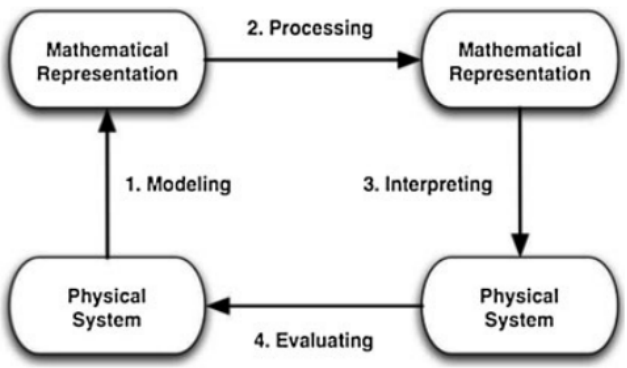


Figure 2.1: *The model proposed by Redish for the use of mathematics in physics*

Redish's modeling cycle can be useful for describing students' mathematical reasoning in physics; this cycle differentiates between students' abilities to mathematize the problem, perform mathematical manipulations, and interpret findings in terms of the physical system. These activities require a deeper understanding of physics concepts and also a familiarity with mathematical techniques. A majority of the students try to memorize and apply physical principles without fully understanding the underlying concepts however. Ultimately, this makes students unable to relate their understanding of concepts to their existing knowledge, and they do not know how to choose methods most suited to solve a given problem. These reasons affect students' problem-solving approaches; researchers have found that novice students' problem solving approaches do not align with Redish's model[17;18].

Novice students' problem-solving approaches seem inefficient to experts. In addition, educators believe that, if they do not teach students a problem-solving method, those students will not be able to fully understand what a given question means or what approaches are required to solve problems [19;20;21]. Researchers have sought ways to lead students toward more efficient problem-solving approaches. For example, as an attempt to teach their introductory physics students a problem-solving strategy, Wright and Williams [22] came up with a strategy (WISE). The WISE strategy has a consistent sequence of four problem-solving steps, which are more aligned with the expert approach. Each letter of **WISE** stands for a part of the problem-solving approach to be used in each problem.

1. **W**hat's Happening - Identify the physical principle; make a sketch or a diagram.
2. **I**solate the unknown - Select an equation and solve symbolically; if one equation is not enough look for others to solve simultaneously.
3. **S**ubstitute - Solve equations with both numbers and units.
4. **E**valuate - Check for correctness of signs, units, and magnitude; does the answer make sense within the context of the physical system?

Most of the time, WISE seemed to help students solve traditional textbook problems. This approach gave both students and instructors a common framework to discuss physics concepts as well as how these concepts are applied in solving physics problems.

Similarly, Heller [19] developed another problem-solving strategy to help introductory students to associate their conceptual understandings with the procedural aspects of problem solving that go beyond just solving the end-of-chapter problems in textbooks. This particular strategy included five steps:

1. Visualize the problem - Visualize the events described in the problem statement.
2. Physics description - Convert the qualitative understanding of the problem into the language of physics.
3. Plan the solution - Create an outline of equations to be solved algebraically to produce an answer.

4. Execute the plan - Substitute the known quantities from the problem statement into the final algebraic expression constructed in the third step.

5. Check and evaluate - Check if the answer is properly stated. Check units, and whether the answer is reasonable.

We classify all the above-mentioned problem-solving strategies as prescriptive strategies, that are well aligned with expert-like approaches. These strategies seem to match well with the kinds of problems students are asked to solve at the introductory level. Also, these strategies were found to help students improve their attitudes toward solving physics problems as well as to improve their problem-solving skills. In contrast to these prescriptive strategies, there also exist descriptive problem-solving strategies. Working from the resources framework, Tuminaro[23] developed a framework for the introductory level, and Bing[24] took this idea to develop the existing framework to apply it at the upper-division level.

2.2.2 Resources Framework

Studies on expert vs. novice problem solving have revealed that experts approach problems differently. The knowledge structure of the experts organized information in a different and better way from the novices. In order to understand and describe this difference between novices and experts, researchers have looked for a fine-grained model that would allow them to describe observed student behaviors and the underlying mechanisms that are responsible for those behaviors. In previous research on modeling student ideas, DiSessa described small chunks of student ideas as phenomenological primitives (p-prims)[25] that are used in students' intuitive reasoning. DiSessa argued that these p-prims are primitive reasonings that are abstracted from everyday phenomena (phenomenological) and are irreducible and undetectable (primitive) to the students using them. On the other hand, the resource-based model provides a fine-grained structure that can be used to describe student behaviors, despite diSessa's p-prims. Resources are reusable thoughts that can be broad and even have an internal structure. The resource-based model of cognition, introduced by Hammer,[26;27;28] is a framework that describes individual cognition as an in-the-moment, contextually dependent utilization and combination of fine grained bits of information, named "resources." Hammer's work is based on the previous work of diSessa; Hammer's defined resources could range in grain size from small, basic elements like diSessa's phenomenological primitives, or p-prims (e.g.,"closer means stronger"), to more complex conceptual structures such as coherent theories about physical phenomena. In the resources framework, resources (student idea pieces) are conceptual in nature, independent, and productive (neither correct or incorrect) and are used for solving a problem. Resources are highly context dependent (determined by the students' view of the context), so activating a particular resource may activate a network of resources. Usually many resources are applied to solve a single problem. Resources can have different types, and the existence of conceptual, epistemological, procedural, metacognitive, mathematical, and other resources (such as semiotic resources) has been proposed and empirically validated [26;29;30;31;32;33].

Educators are developing the resource-based model of cognition into a tool and they have used the developed tool to better understand ways they can improve instruction and encourage student learning of physics [34;35;26;36;37]. In order to answer the question, "How can we describe students' behavior and why it is different from the way experts behave?", researchers using the resources framework have investigated how students construct their knowledge by combining the idea pieces and reasoning while solving problems [38;39;40;41]. Further broad research work has investigated the organizational structure of students' resources including framing [31;42] and epistemological games [30;29].

2.2.3 Descriptive Problem-Solving Strategies

We categorize the frameworks developed by Tuminaro [23] and Bing [24] as descriptive problem-solving strategies because both these frameworks focus on explaining why students solve physics problems the way that they do. To understand the underlying mechanism, researchers looked at students' in-the-moment reasoning. First, Tuminaro [23] developed his theoretical framework to describe students' use of mathematics in introductory level physics courses. Tuminaro's framework consists of three aspects,

1. Mathematical resources - Abstract student ideas/ elements involved in mathematical thinking including students' understanding of mathematical symbols and strategies students use to extract information from mathematical expressions.

2. Epistemic games - Patterns of students' activities observed during problem solving. Epistemic games combine and structure resources to solve a problem. There exist several epistemic games. Each game is characterized by a different sequence of moves and different types of resources used by the students.

3. Epistemic frames - The epistemic game that students play is governed by the frame in which they are operating. Epistemic frames are associated with deciding what tools and skills are needed in a particular context or situation.

In order to analyze students' use of mathematics at the upper-division level, Bing[(24)] built on Tuminaro's theoretical construct. Bing argued that student framing can be identified by examining their solution for mathematical claims other than the moves that they make. Bing[(24)] used Tuminaro's theoretical construct only with mathematical resources and epistemic frames. Within Bing's study, he identified four epistemic frames that upper-division students engaged in while solving physics problems: invoking authority, physical mapping, calculation, and math consistency.

2.2.4 ACER Framework

Students' actual problem-solving processes are found to deviate from the expert-like approaches. Among those available strategies, prescriptive problem-solving strategies focused on explicitly teaching students to solve problems in a way that is well-aligned with expert-like approaches. Then descriptive problem-solving strategies focused on explaining why students solve physics problems in a particular way.

When we consider available data streams, homework is a key element in every undergraduate-level physics course for practicing problem solving, and student homework solutions provide a rich source of data for studying how students use mathematics in their physics courses. But descriptive strategies, however, use student in-time reasoning to understand and explain student problem-solving behaviors and this prevents the use of students' written work for further exploration. Both these approaches (prescriptive and descriptive) are found to have limitations and the researchers looked for ways to overcome those limitations. The ACER theoretical framework (Figure 2.2[14;43]) bridges the gap between the descriptive and the prescriptive approaches.

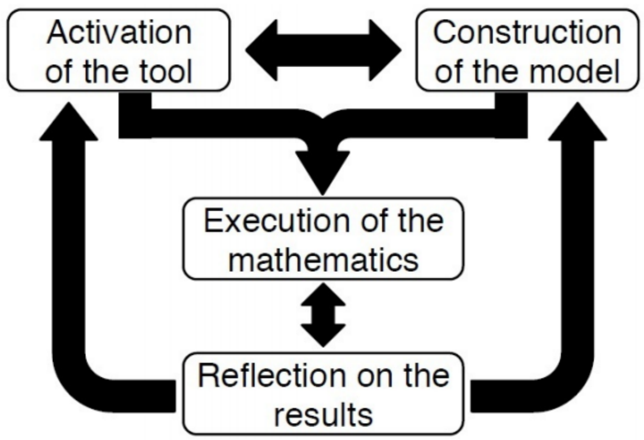


Figure 2.2: *Visual representation of the ACER framework developed by researchers at CU Boulder*

Researchers developed the ACER framework to identify a coherent set of student difficulties with mathematics in upper-division physics courses that the researchers could not learn from other sources such as informal observations and discussions with students and instructors. The ACER framework consists of four components: activation, construction, execution and reflection. A convenient visualization of this framework is given in Figure 2.2[43]. The ACER framework provides a logical flow and a way of organizing the problem solving process so that students might solve problems step by step[44].

The first component of the ACER framework is **activation of the tool**. In order to start solving a given problem, students have to decide on and choose appropriate mathematical tools too. Students start by reading the given problem statement and identifying the explicit and/or implicit cues integrated in the problem statement. These cues may include symbols that are used to represent physical quantities or language that directs the solution process (such as “Find the initial and final velocities”, “Find the angle of projection” etc.). Reading the problem statement may activate resources or mathematical ideas [26;27;28] that are related to the problems’ context and to students’ prior knowledge. Then students must pick the proper mathematical tool from among the available mathematical ideas.

Often in physics problem solving, students are required and encouraged to build models using different representations. These representations include diagrams and mathematical models that can be used to represent the actual physical situation presented in the problem. In the second component of the ACER framework, **construction of the model**, students construct a mathematical model that uses the previously chosen mathematical tools to describe the given physical system. Students develop their model to perform mathematical steps during the next component.

The third component of the ACER framework, **execution of the mathematics**, involves students performing mathematical steps to get to the final solution. Within this step, students execute procedural mathematical steps that were originally planned during the previous component of construction.

Self-reflection could be an excellent opportunity for students to think about their meta-thinking on how they solved problems. The fourth component of the ACER framework, **reflection on the results**, involves students comparing their solution with known or expected results. This component includes everything from basic checking such as ‘check the units of the final answer’ to very important activities that students could perform to develop and extend their understanding of the underlying physics concepts. As an example, after developing a final equation that expresses electric potential as a function of radius, a student could input values to see how the electric potential varies at the boundaries.

2.2.5 Previous Applications of the ACER Framework to Identify Student Difficulties

Researchers have applied the ACER framework to understand and characterize student difficulties related to specific mathematical tools such as Taylor series, Multivariable integration, and Dirac delta functions in upper- division classical mechanics and electrostatics courses[45;14;16;43]. Data for studies involving Multivariable integration and Dirac delta functions, were collected from the two-semester Electricity and Magnetism (EM 1) course at the University of Colorado Boulder (CU) that covers the first six chapters of Griffiths[46]. The course employed research-based teaching practices such as clickers[47] as well as problem-solving worksheets[48] both in and out of class.

Data for these studies were collected from three main sources: student solutions to traditional midterm exams, responses to specific questions on the multiple-choice Colorado Upper-division Electrostatics (CUE) Diagnostic[49], and think-aloud student interviews. Importantly, the think-aloud interviews were designed to target the student difficulties observed on student solutions to midterm exams. Data for the Taylor series study was collected from a second year classical mechanics (CM 1) course that uses the textbooks Boas[50] along with Taylor[51] and covers up to but does not include the calculus of variations. Data for the study were collected from midterm exam solutions and think-aloud interviews. The student population for both courses was composed of physics, engineering physics, and astrophysics majors.

These studies consisted of analyzing students' written solutions and interviews. Both the written solutions for exams and the written work embedded with the audio from the interviews were coded separately using the components of ACER. Each data type had two stages of analysis. First, students' solutions were coded by identifying key elements under the four main components (Activation, Construction, Execution, and Reflection) in the framework. The second stage involved further detailed coding of those identified key elements to further develop sub-codes under each component. The coding process was continued until the emergent codes covered important and unanticipated aspects of student solutions.

First, the study focused on student difficulties associated with multivariable integration in the context of the integral form of Coulomb's law [14]. Specifically, students were asked to use the integral form of Coulomb's law to find the electric field or electric potential from a continuous charge distribution. The exam problem asked students to calculate the electric potential from a disk with a given charge density along an axis of symmetry. The interview data was collected from a different student group (from the exam students) who were asked to calculate the potential from two parallel disks of charge by direct integration.

The researchers were able to identify common student difficulties with Coulomb's law integrals under each component of ACER. Under activation, nearly 75% of the students were able to successfully identify the appropriate tool to solve the problem, namely the Coulomb's law integral. However, researchers also found evidence of misapplications of Gauss's law to solve the problem, in both exam solutions and interviews, that is consistent with previous research at the junior-level [15]. Researchers found that students had many difficulties when attempting to generate appropriate mathematical expressions to represent the physical scenarios.

Common difficulties under the Construction component of ACER were related to articulating the differential charge element and to vector quantities in the mathematical model (mathematical equation). Some students in the interview section appropriately used diagrams to label vectors that they then successfully used to articulate vector quantities in their mathematical equations. Also, despite other difficulties, most of the students in this study were found to be comfortable selecting an appropriate coordinate system to solve the given problem, a skill valued at the upper-division level.

Under the Execution component, students were found not to make mathematical errors either with integrals or algebraic manipulations that are specific to solving Coulomb's law problems. There were a few significant mathematical errors, however, which are expected to be obvious to students at this level, such as pulling integration variables outside of integrals. In addition to, there were some cases of common minor mathematical errors such as plugging in limits incorrectly. Under the Reflection component, researchers found that only a small number of students spontaneously reflected on their final results. During the interviews, researchers encouraged students to reflect on their results. These students then reflected on their answers by checking units or by checking the behavior of the electric potential at the boundaries. When students engaged in reflection, they found inconsistencies that were due to mistakes that occurred during the construction and/or execution components.

The researchers then moved their attention in their next study to another mathematical tool commonly used in undergraduate physics curricula - the Dirac delta function[(16)]. In order to identify the kinds of challenges that students face when manipulating δ-functions, researchers in this study, presented problems that use δ-functions to express volume charge (or mass) densities of charge distributions such as the volume-charge density of a line charge running parallel to the z-axis. Once again, exam solutions, student responses to the multiple-response CUE[(49)], and interview data were collected using different groups of students. Under the Activation overall, the researchers found that many students had difficulty recognizing when the δ-function is an appropriate mathematical tool even immediately after completing an upper-division electrostatics course. During the interview, however, when students were asked for a mathematical expression for the potential of a finite square well in the limit that the well became very narrow and very deep, students used the words 'very narrow and very deep' as cues to choose the δ-function as the appropriate mathematical tool.

Under the Construction component, researchers found students had more difficulty when asked to translate a verbal description of a charge distribution into a mathematical formula for volume charge density than when doing the reverse process. In addition, most of the students misidentified the 1-D or 2-D volume charge densities as point charges when setting up the mathematical equation. About one fourth of the students made significant mathematical errors related to the δ-function while executing this integral in their exam solutions. In response to the CUE question, approximately one third of students made mistakes by selecting an incorrect value for the integral of a point mass density. During the interviews, two out of five students were found to have difficulties integrating the δ-function when asked to calculate the total charge on a uniformly charged spherical shell. Results show that most of the students did reflect at least somewhat on their final answer because around two-thirds of the students provided the correct units. One reason for this reflection may be that the prompts used in the exam explicitly asked students to comment on the units.

Like the Dirac delta function, Partial Differential Equations (PDEs) are also a common mathematical tool used in undergraduate physics curricula. Often students employ separation of variables (SoV) to solve PDEs. The next study used the ACER framework to identify and categorize common student difficulties associated with SoV in the context of Laplace's Equation[(52)]. Researchers used problems that require SoV in both Cartesian and spherical coordinates in exam questions. As an example, in Cartesian coordinates, a problem described a rectangular pipe with given values for the voltage on each side; students were asked to find an expression for the voltage inside the pipe. In spherical coordinates, a problem was presented with an azimuthally symmetric expression for the voltage on the surface of a spherical shell and asked for an expression for the voltage valid both inside and outside the shell.

Almost all the students were able to correctly operationalize under the activation component in both the Cartesian and the spherical SoV problems and they were able to recognize SoV as the correct approach. Researchers reasoned, however, that this high achievement may have occurred because of the distinctive nature of Cartesian SoV questions and the different types of prompts contained in the problems. In the Cartesian SoV problems, students did not demonstrate much difficulty within the construction component; they were successfully able to identify boundary conditions and to build a general expression for the voltage to satisfy those boundary conditions. In the spherical SoV problems, however, some students had difficulty with setting up the equations to solve for the unknown constants as they made mistakes when matching the boundary condition at the surface.

In contrast, students in the interviews showed fluency in moving quickly back and forth between identifying boundary conditions and setting up equations. Under Execution, for both Cartesian and spherical SoV problems, more than half of the students were able to successfully separate Laplace's equation into ODEs. Under the Reflection component, there were very few students in exams who tried to reflect on their final answer to check for units; this trend was same for the interviews.

The next study was conducted to identify student difficulties with Taylor series applications in physics problems[43]. Two of the exam problems used in this study were from the topics of motion and energy. One asked students to perform a Taylor series expansion on an expression for the 1-D position of a particle moving against linear drag; the other one asked students to find an approximate expression for the gravitational potential energy for a bead sliding inside a frictionless cylinder. During the interviews, students were asked to solve a number of Taylor series problems that included formal math and physics questions.

Researchers were able to identify few challenges when students were asked to solve Taylor approximation problems. With explicit cues, most of the students in both exam problems and interviews successfully identified the Taylor expansions under the Activation component. They had difficulty when the prompts were less explicit, however. Under the Construction and Execution components, students often demonstrated their identification of the variable and showed relatively good skills with expansions around zero; most of the difficulties came when they were asked to perform Taylor expansions around nonzero expansion points. Under the reflection component, many students struggled to make meaningful statements about the physics of their solutions.

The components of the ACER framework are designed to address and capture the critical aspects of the process that students use to develop solutions to physics problems; the above studies summarize the successful use of it. In addition, as discussed above, the studies show the flexibility of the ACER framework as it can be operationalized to identify student difficulties around a wide range of mathematical tools.

Researchers developed the ACER framework to identify student difficulties on mathematical applications in upper-division physics courses. They used a combination of conceptually rich research-based problems in think-aloud interviews and other data from classroom observations. In most undergraduate physics courses, however, there exists a ubiquitous data source for education research, which is student homework. In fact, student-written homework solutions are common at the upper-division level. Often instructors incorporate back-of-the-chapter style problems in their homework assignments to give their students practice on problem-solving techniques and to give an engaging opportunity for students to understand underlying physics concepts by solving problems.

2.3 Social Network Analysis

2.3.1 Social Networks

We can describe a network as a group or a system of interconnected objects. As an example, a computer network is composed of different devices and computer systems connected by physical and/or wireless connections. We can also observe networks within the teaching-learning environment that we could observe the teacher and the student are connected as a group. As an example let's consider a network-based classroom(a computer lab). In this setting each computer is connected through a local area network to the server. Then the teacher and each student sitting in front of a computer comprise a network that allows the teacher to engage with the whole class at the same time. In comparison, a social network is a group of connected people in a particular club, small town or, as in our case, a connected group of students inside and outside of classes. There exist online versions of social networks such as Facebook, Twitter, LinkedIn, Google+, YouTube, Pinterest, and Instagram etc., that allow people to build social relations, real-life connections with other people who share similar personal interests and backgrounds.

Networking is important to daily human activities. If we go back to the previous example about computer networks, the networking of devices and computer systems helps the network users to share data files and devices like printers or scanners. Then within the networked classroom, the network can facilitate communication for both students and the teacher. The teacher can communicate with the classroom by sharing the screen, monitoring student activities, and by participating in written conversations to give instructions by composing messages in real-time for the students in need. The students can share ideas and resources as well as help each other.

In a broader sense, global social networks allow network members to share ideas, opinions, and photos/ videos as well as to inform others about real-world activities and events with people in their network. In addition, businesses use social networks as a major tool for reaching consumers and promoting their brands.

There are many interesting things we can learn about networks. Social network analysis can provide information about the structure of a network (behavioral patterns), the position of members within a network (identify change agents, identify key players, direct resources), and visual representations of data to gain insights (to identify hidden patterns). Analysis of social networks can lead to new realizations about culture, politics, history, and lots of other interesting topics by overlaying different networks that connect and transfer people, friendships, information, money, and power. In that sense, businesses can target a specific audience based on business related information like personal interests and demographical backgrounds within the networks and their statistics.

2.3.2 What is social network analysis?

Social network analysis (SNA) is the study of social relations among a set of members within a social group using quantitative and qualitative methods; it has been developed under the theoretical constructs of network theory and graph theory [53;54;55;56;57;58]. Social network analysis gives us the tools to quantify the connections between individuals within a network. As an example, we can find out how one person is connected or disconnected with people, groups, or trends in a population.

Research in a number of academic and non-academic fields has shown that social networks operate at many levels; SNA, as an analytical tool, has been used in research studies in education [59;60], economics [61;62;63], computer science [64;65;66], health care and medicine [67;68], social and behavioral sciences [69;70;71], work and organizations [72;73], and crime and terrorism [74;75;76] studies.

SNA can be used to generate graphical representations of networks (network graphs) that we can use to find patterns in the interactions that connect individuals together as a society. Networks are often visualized using network graphs (Figure 2.3) in which networks are represented as a collection of points, called nodes, and lines connecting these points, called edges (ties). When the direction of a connection matters, we have a *directed graph*. In such cases, an edge from node A to B ($A \to B$) means something different than an edge from B to A ($B \to A$). If the direction of a connection does not contain any information about the relation between nodes (i.e, $A - B$ has the same meaning as $B - A$), a graph is called *undirected*.

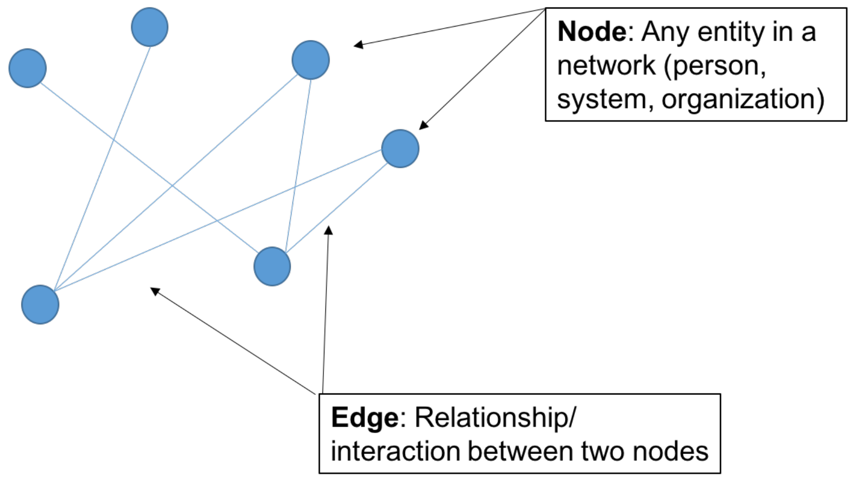


Figure 2.3: *Undirected network graph: nodes and edges.*

For simple networks, we can realize some network information just by looking at the network graph. But when more actors or nodes are involved in a network there can be many kinds of ties between the nodes. As a result, the related network graphs often get very complex and we cannot get the network information just by inspecting those graphs. In that sense, SNA measures are a vital tool for understanding the properties of networks.

2.3.3 Network Measurements

Each actor has a specific role to play within a network and the functions of each player coincide with the position or location of that player within the network. In terms of network graphs, nodes coincide to the structural positions of the actor relative to each other. In order to understand a network's participants and properties, SNA measurements assess the location and grouping of the nodes in the network. These measures give us information about the roles, functions, and groupings in a network such as who are the key players, who is in the core of the network; who are the connectors, mavens, leaders, bridges, and isolates; where are the clusters and who is in them?

A network graph can be characterized using various measures. Centrality measures the importance of a node's position in a network and plays an important role in the field of network analysis. Many different centrality measures that have been proposed over the years, including Degree, Betweenness, Closeness, and Eigenvector [77;78]. **Degree centrality** is the simplest measure of node connectivity; it describes how well connected an actor is in a network. Degree centrality indicates the importance of a node by the number of nodes connected to it. For the simplest form of networks, within un-directed graphs, the degree centrality is merely a count of the number of ties for every node. For directed networks, a given node can have both in-degree and out-degree centrality measures. The in-degree centrality is the count of inward ties (inbound ties) to the node, and the out-degree centrality is the count of outward ties (outbound ties). Degree centrality indicates the power and access to resources and information within the network. This measurement is used to identify those actors with the most direct contacts and with the least dependence on other actors while holding the most information in the network.

Betweenness centrality shows the nodes that act as 'bridges' between nodes in a network by calculating the number of times a node lies on the shortest path between other nodes. When calculating the betweenness centrality, the mathematical function first identifies all the shortest paths and then counts how many times each node falls on one. This measurement considers the relative importance of nodes by measuring the degree to which information (or relationships) have to flow through a particular node in the network. Betweenness centrality is useful for analyzing communication dynamics within networks to find actors who influence flow around the network. In that sense, actors who have higher betweenness centrality operate as key players by acting as the 'quickest' bridge for the flow of resources or information between other actors.

We can use the **Closeness centrality** measurement to identify actors who are very closely connected to other actors. Actors with high closeness have the best ability or position with the network to influence the whole network with minimum effort. Closeness centrality measures how many steps (ties) are required for a particular node (actor) to access every other node (actor) in the network. When calculating closeness centrality, the mathematical function first calculates the shortest paths between all nodes, and then assigns each node a value based on its sum of shortest paths. Nodes with higher closeness centrality are in a great position to monitor the information flow in the network, and they can easily and quickly transfer resources and coordinate tasks within the network.

Eigenvector centrality is similar to degree centrality as it measures the importance of a given node by its number of ties. Eigenvector centrality, however, specifically measures the degree to which a given node is connected to other well-connected nodes in the network. Eigenvector centrality can be more useful centrality measure than degree centrality because it does not only consider the nodes that are directly connected to a given node, but also takes into account the extended connections of a node. This improvement reveals how well connected an actor is to other well-connected actors in the network and therefore, how much influence the actor has over the whole network.

Centrality measurements are popular and well-accepted network measurements when it comes to identifying the most important nodes within a graph which corresponds to the key players (actors) within a network. There are other useful network measurements, however, aside from centrality measurements, including network density and network diameter. **Network density** is a measure of the connectedness, relatedness, and effectiveness in a network. Network density is defined as the proportion of ties that are connected out of all ties that could possibly be connected. In that sense, higher network density means that we have a more connected network. Network density ranges from 0 to a maximum of 1.0. A network with a density close to 1.0 is said to be dense; a network with a density closer to 0 otherwise is scattered. **Network diameter** gives us an idea about the linear size of a network. Network diameter is defined as the longest graph distance between any two nodes in the network. The network measurements network diameter and closeness centrality have opposite meanings and significances. Network diameter is interested in the maximum distance between any pair of nodes in the network; the closeness centrality measures how many steps (ties) are required to reach other nodes (actors) in the network. In that sense, when two nodes have a larger value for network diameter they have a smaller value for the closeness centrality.

Recently, the use of SNA in physics education research has sought to advance understanding of how students' collaborative approaches influence student learning trends within physics classrooms. SNA has been used to investigate student peer networks [79;80;81], as well as patterns of student interactions in computer supported collaborative learning environments [82;83;84;85]. With our interest in investigating student use of mathematical tools in undergraduate physics courses, in the study described in chapter 3, we look at the complex system of interconnected mathematical steps and tools in student problem solving, through the lens of SNA. In our analysis, each mathematical tool used by students represents a node in a network, and the order of the mathematical tool represents the tie in our network graphs. We use SNA as an analytical tool to identify patterns in students' use of mathematical tools and the relations/ interactions between those mathematical tools as they solve undergraduate physics homework problems.

Among other available data streams, I had access to videos of students solving problems in group and individual oral exams in both upper-division Electromagnetic Field 1 and Mechanics courses. When I analyzed those videos, I found that students coordinate among multiple representations as they solve problems. As a result, I decided to make an additional analysis. In seeking such an additional analysis, I reviewed the existing work on students' use of representation in PER.

2.4 Representations

Among the extensive research done in PER on student problem-solving processes in PER, researchers have studied student understanding of representations, which has been an integral part of the PER that aims to enhance student-learning outcomes [86;87]. In PER, the term *Representations* refers to the many ways in which students can express ideas, concepts, processes, and relationships. Sketches, diagrams, pictures, graphs, tables, and mathematical equations are some of the commonly used representations in undergraduate physics courses [88;89;90;91;92;93;94;95;96]. Diagrammatic representations can play a particularly important role in the initial stages of conceptual analysis and planning of the problem solution.

Diagrammatic representations have been shown to be superior to exclusively employing verbal representations when solving problems[97;98;99]. Students' use of gestures in physics has also received attention[95]. In that sense, gestures are a productive medium for understanding, constructing, representing, visualizing and communicating scientific concepts[100;101], particularly in chemistry problem solving. For example, students' use of gestures to describe molecular geometry in their own style can develop into a technical language[96]. Physics uses mathematical modeling to describe phenomena and mathematical symbols to explain relations between variables that represent the properties of a system. Further, some researchers have investigated how students work with mathematical representations and how students coordinate between different mathematical representations such as graphs, equations, vector manipulations and algebra[102;103;104;105;106;107]. In upper-division electricity and magnetism courses, students find that vector-field manipulation is easier when they use an algebraic representational approach[89], though students need to use graphical representations to interpret the difference between components and coordinates.

2.4.1 Classification of Representations

At their most basic, representations are either external or internal objects that stand for something else. Internal representations refer to mental models that problem solvers need to construct within the problem-solving process[94]. External representations exist in the physical world and include, but are not limited to, sketches, diagrams, graphs, mathematical equations, and words[108]. A descriptive representation consists of symbols that describe an object. Spoken and written texts as well as mathematical equations are all examples of descriptive representations.

Alternately, we can categorize representations as to whether they are durable or evanescent. A durable representation "leaves a trace of its production in the medium in which it was produced"[109]. Durable representations stay in place and can be easily returned to. They include diagrams, graphs, written text, and written mathematical equations. In contrast, evanescent representations are temporary and disappear when not in use, and include verbal and facial expressions, as well as physical gestures.

2.4.2 Different Stages of Student Reasoning

In the context of problem solving, we observe students switching back and forth between explanations that they build using p-prims, conceptual and/or mathematical resources (26;27;25), and the arguments built using the developed representations until they find it is insufficient for a complete answer. As students progress and develop their solution and the reasoning behind it, there is a point that students no longer hesitate and the instructor or other peers can accept their solution or the explanation without any further discussion. Fredlund uses this concept of agreement between the peers involved in a discussion/ reasoning situation to define 'stand fast' and relate this idea of standing fast to the process of students' reasoning while solving problems. While solving problems, students stand fast on "something that a person or group of people can call upon and use without hesitation or further questioning" (109). This is when, in a conversation, all the participants agree on a presented idea or an explanation (110).

Study presented in chapter 4 explores how students coordinate among multiple representations while solving problems. In this analysis we present the case of Larry as an exemplary case. We suggest problem solving as a process that involves translations between durable representations (diagrams, written mathematical equations) with the help of evanescent representations (gestures, words). In addition, we keep a closer attention to student reasoning to identify the agreeing or 'standing fast' moments.

2.4.3 Multiple Representations in Problem Solving

Results from previous research (4;5;111;112) show that experts' knowledge structure consists of many representations that are directly connected to physics concepts as experts tend to understand abstract physics concepts in some form of representation. In the context of physics education, instructional strategies focus on improving students' physics knowledge structure and aligning it with the way physics is represented in the minds of experts. In that sense, representations play a major role in effective problem solving at the undergraduate level, and they help students focus on the conceptual aspects of physics problems. This implies that being able to represent physical phenomena using several representations and the ability to coordinate among representations one features of expertise in physics problem solving. Based on this idea, in order to improve students' knowledge structure, we must have them learn and practice how to use multiple representations in their physics courses. The purpose of this study was to understand student use of multiple representations in undergraduate upper-division level courses. In order to ground this study within the existing literature, this literature review will next describe what has been learned through research on learning with multiple representations, including theoretical explanations for the observed challenges and benefits.

2.4.4 Student Challenges in Using Multiple Representations

Physics often involves the modeling of real world physical phenomena using different representations, but learning physics with the multiple representations is found to be challenging for students[113;114;115;116;117;118]. One important challenge that students have with multiple representations is revealed by the studies that have investigated the difference between expert and novice use of representations. These studies reveal that, students tend to focus on representations differently compared to experts[119;120;121;122].

In particular, students' prior knowledge seemed to influence which features they paid attention to. Students with higher prior knowledge paid attention to the deeper, conceptual meanings embedded in representations, while students with lower prior knowledge focused on surface features of those representations. This allowed researchers to observe a variation between the students as they moved between representations. The students with higher prior knowledge transitioned more frequently between the molecular representations, but the other group made transitions between the macroscopic representations[123]. Studies found that there is a strong tendency among students to view graphs as pictures rather than as symbolic representations. One of the studies investigated the connection between the symbolic representations of functions and their graphs. They found that middle school students found it challenging to interpret the negative slope of a graph as the value on the y-axis is decreasing, instead they viewed it as a hill[115]. In this example, students viewed the graph as a picture (hill) instead of a useful symbolic representations that shows how the object is moving with time. Investigations of the effect of a problem statement on student performances found that novices tend to focus on surface features of the problem statement, whereas experts tend to focus on deep relational features[124].

The ability to make translations among representations is a major advantages in physics problem solving[125]. The process of making connections between representations requires students to know how to negotiate each separate representation, and student challenges can occur due to many reasons. Students often struggle to identify meaningful differences between representations or to identify shared meaning between representations. Instead, they view each representation as separate and distinct in meaning[113;121;126]. For example, Figure 2.4 shows the graphical (energy-level diagram) and pictorial (spectral-line diagram) representations of the spectroscopy problem of the Balmer series[127]. As the initial step to build connections between representations, students should first be able to negotiate each separate representation. For the spectroscopy problem in Figure 2.4, a student needs to understand what is represented in the energy-level diagram. Energy level diagrams are a means of analyzing the energies electrons can accept and release as they transition from one orbital to another. A student also must know what is being represented by the lines (wavelengths) of a spectral line diagram. The correspondence of the wavelengths in the spectral line diagram to the energy differences in the energy level diagram is related to the transition of electrons between orbitals in the atom. In order for students to identify the shared meaning between the representations, they must connect the features from each representation. In the energy level diagram, the value of E, or energy, represents the difference between the energies of two energy levels when an electron goes through de-excitation. Then in the spectral line diagram, the value of wavelength corresponds to the wavelength of the emitted photon.

Different types of representations may be useful for different purposes[128]. If the context of a problem has to be represented, the best representations to use are text or pictures.

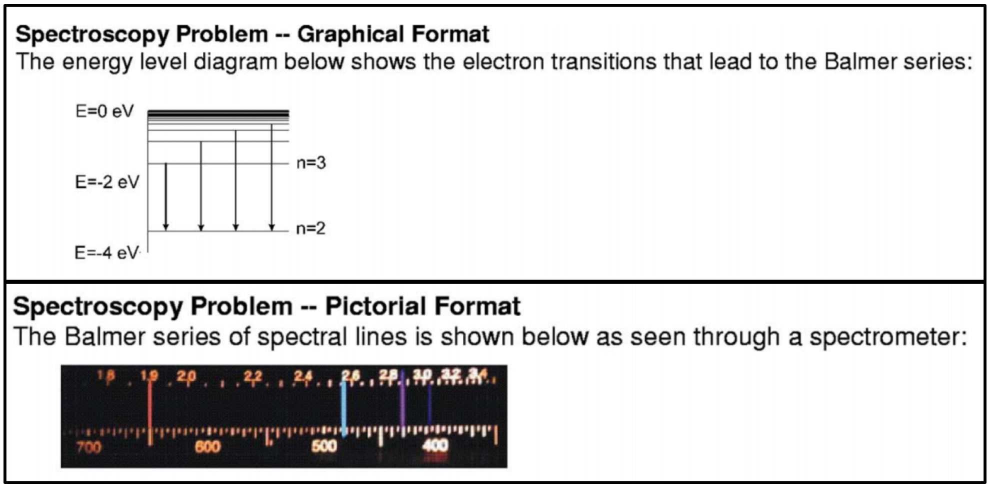


Figure 2.4: *Graphical and pictorial representations of the spectroscopy problem*

Other representations, such as graphs or tables, are less useful for representing this type of information. If qualitative information has to be shown, diagrams are the best representations. For showing quantitative information diagrams are less useful; graphs or formulas are better representations for this type of content. Lastly, students also struggle when using multiple representations because they do not understand the affordances and constraints of the representations being used [120;129;130]. For example, the energy level diagram has the affordance of being very specific about certain features of transitions (e.g., energies of energy levels) while it has the limitation of not specifying the features (wavelength, frequency) of the emitted spectrum. The spectral line diagram, on the other hand, affords one the ability to represent the wavelengths corresponding to the possible transitions while limiting the information about the energies related to each transition. I will talk about the affordances of the representations in detail later in this chapter within the sections on social semiotics. The challenges I discussed above limits students' ability to learn physics content with the use of multiple representations and to understand the basic concepts and principles of physics which in turn, limits students' ability to apply the understanding of concepts and principles through a broad range of applications that involve problem-solving.

2.4.5 Benefits of using Multiple Representations

Though there are clearly challenges associated with learning with multiple representations, studies have also found there to be benefits connected to learning with multiple representations[113;131;132;126]. The studies investigating the connection between students' use of representations and physics problem solving revealed that students who are consistent across different representations perform better on problem-solving tasks[133;92;88;134;135;136;137;138;139]. For example, if a student is having trouble understanding the concepts behind a given problem just by using a single representation, the other representations can be used to help the student get a firmer grasp on the concept with which he or she is struggling[113]. Teaching students to represent problems in different ways has a significant influence in moving novice-like formula-centered problem-solving approaches towards effective inquiry-based approaches.

When students are taught problem-solving strategies that emphasize use of different representations of knowledge, they construct higher quality and more complete representations and exhibit better performance than students who are taught traditional problem-solving approaches[108;140;97]. Outside of physics, the use of multiple representations is shown to increase students' conceptual understanding in courses including calculus[141] as well as other science courses where their use has also been investigated[142], such as chemistry[117]. In addition to improving students' performance in problem-solving activities, using multiple representations plays a critical role in the effectiveness of the interactive engagement between students and instructors in learning environments[88;109]. Translating between different representations may enhance students' sense-making abilities[89].

There are three primary explanations for the benefits of learning from multiple representations. First consider the argument about students' cognitive load when working with multiple representations. The amount of information that must be processed at any given time while engaged in problem solving in order to move forward with a solution is known as 'cognitive load'. Sweller[143] developed cognitive load theory in an effort to explain how people learn and extend their knowledge. Problem solving is a cognitive process which takes place in Short-Term Memory. Short-Term memory (STM) is one of two broad components of memory, the other being Long Term Memory (LTM). The term working memory is often used interchangeably with STM and working memory acts as a kind of notepad for temporary recall of the information which is being processed at any point in time. Working memory has a finite capacity to store information through 'slots'. During the problem-solving process, the working memory receives inputs from sensory buffers (eyes, ears, hands) and information from the LTM, which needs to be distinguished from the vast amount of other information that is stored in LTM. Since the amount of information that can be processed at any given time in working memory is finite, one must carefully process the particular information that makes the end goal easier to reach.

Studies by Rosengrant, Van Heuvelen, Etkina[144;88] found that when university physics students created multiple representations, their test scores increased. The first study[144] found that more students who correctly drew free-body diagrams (a diagram which represents an object as a point and the forces acting on the object as arrows) answered test questions correctly than those who did not draw correct free-body diagrams. As a result, these researchers indicated that it is critical for physics students to be able to use, create, and understand words, diagrams, sketches, equations, and graphs. In the second study, Rosengrant and colleagues[88] found that using multiple representations appeared important to students' ability to solve physics problems correctly. The most successful students in this study drew a picture and then drew a free-body diagram. They were able to use this free-body diagram to set up their mathematical equations and evaluate their answers.

The hypothesis of Rosengrant, Van Heuvelen and Etkina[88] is that students are probably aware intuitively that they do not have the mental capacity to remember all the information in the problem statement, and thus use the representations to visualize an abstract problem situation. in effect, they used the picture to help lessen the burden on their working memory. In another study[145], students were given a problem which required an addition of two vectors in two dimensions. Students who explicitly drew the components of the vectors performed better than students who did not. The researchers generated a hypothesis: the students who did not draw the components had to keep more information in their working memory while engaged in problem solving. This may have increased the students' cognitive load and ultimately led them to be unsuccessful.

When engaged in problem solving, physics experts often group several pieces of information together into a single chunk, which would take up one slot in working memory; but, for a novice student, those pieces of information could seem disparate and require different slots in order to be processed. Then due to their reduced information processing capabilities, students can experience cognitive overload when solving problems. For example, while engaged in problem solving and processing information in working memory, an expert could group together information about a vector such as its magnitude, direction, and components into one single memory slot because of the relationships that connect them. In contrast, a novice could perceive these pieces of information as distinct and require one slot of working memory for processing each. Thus, the amount of information that a novice can process at any given time while engaged in problem solving is reduced compared to an expert.

Second, Ainsworth's Functional Taxonomy framework [146] can be used to help students understand how multiple external representations function with each other and help them translate among multiple external representations. Several studies have indicated that using multiple external representations has the potential to generate deeper conceptual understanding than a single representation [146;113;120;147;148]. For example, Ainsworth [146;113] proposed a Functional Taxonomy of multiple external representations, which posits that different external representations have different functions in facilitating student learning. Ainsworth has categorized the multiple purposes of representations into three classes: representations that serve complementary roles, representations that constrain interpretation, and representations that help construct deeper understanding. As an example of representations being used in a complementary role, students can often find several pathways to a solution by using multiple representations, like using a graph rather than an equation or table. Using different representations can be especially productive if there are multiple tasks or if students prefer one representation over another since each representation (i.e., graph, table, equations) highlights different salient features of the situation that may not be obvious to the student from the other representations.

Third, some researchers argue that the increase in student learning from multiple representations is because learning is the product of creating representations [149;150;151;152;153]. Thus, as students are engaged in creating representations (sign-making) their learning is increased because they are constructing and clarifying knowledge [154;151;155].

2.5 Broadening the Scope of PER

Within a teaching/ learning environment, both the students and the teacher bring in and use a variety of resources to learn and understand physics concepts and scenarios that include, words (orally or in written form), extra-linguistic modes of expression (gestures, gazes ...), different types of inscriptions (sketches, graphs, formulas ...) and different instruments (from pen and pencil to the most sophisticated computers and electronics) that go beyond just the external representations used by students as counted in the representational approach. In order to overcome this limitation in understanding student meaning-making and to broaden the boundaries in physics as a discipline, Airey & Linder [156] use a social semiotic perspective and describe student meaning-making as the use of semiotic resources to realize and communicate physics knowledge.

2.5.1 Social Semiotics

As an approach to understanding how people communicate a variety of means in particular social settings, the central concern of social semiotic is how meanings are generated [32;157;158;159;153;152]. As the term suggests, social semiotics focuses on how people use social interaction to construct meaning.

From a linguistic point of view, social semiotics is a synthesis of contemporary approaches to the social production of meaning, and is based on formal, or mainstream semiotics, a theoretical approach to the study of signs and sign systems. Signs are the smallest unit in meaning making and anything that can be used to communicate is called a sign. A sign is some physical thing (such as written letters or a spoken sound) that stands for, or refers to, something else. As examples: a painting or photograph is an iconic sign; colors and gestures can also be signs. Words are linguistic signs. The physical form of the sign is termed the signifier, and the concept that it refers to, the signified (160). The main focus of social semiotics is about how meaning is made through signs. One can articulate signifying practices as the nature of the relation between signified and signifier as the processes of meaning-making. Signification, or meaning making, is the process of creating and refining "the relationship of a sign or sign system to its referential reality" (161).

In reality, the relation of signified to signifier is not purely linguistic. Rather, this relation also manifests as a social relation; social semiotics is concerned with the social "act of meaning making" (162). The particular social setting or social group that we are interested in consists of those involved in the discipline of physics. Clearly, physics as a discipline comprises many systems of signs with which people (both students and instructors) make sense of the world.

2.5.2 Semiotic Resources

Semiotic resources is a term used in social semiotics to refer to a means for meaning making. From a social semiotic perspective, all meaning is realized, constructed, and developed using social conventions through the production of semiotic resources. Van Leeuwen (32) defines previously recognized signs as semiotic resources such resources are "the actions and artifacts we use to communicate, whether they are produced physiologically - with our vocal apparatus; with the muscles we use to create facial expressions and gestures, etc. - or by means of technologies; with pen, ink and paper; with computer hardware and software; with fabrics, scissors and sewing machines, etc." According to this definition, the words, gestures, sketches, graphs, formulas and different instruments that are used in physics classrooms can be labeled as semiotic resources. These resources are then the material result of the students' meaning-making process (153). Both students and teachers bring in and use a variety of semiotic resources to learn and teach physical concepts and scenarios. These resources include, words, extra-linguistic modes of expression, different types of inscriptions, and different instruments. This is what makes the social semiotics approach an appropriate perspective for my aim to explore students' use of semiotic resources for meaning-making in undergraduate physics courses.

From a cognitive perspective, David Hammer [26] uses the term 'resources' to describe an individual's cognition as an in-the-moment. According to Hammer's resource-based model of cognition, resources are fine-grained bits of information that are highly context dependent (determined by the student's view of context). These conceptual resources are isolated, independent, productive student ideas that can be used for solving a problem.

Conceptual resources look very similar to semiotic resources, but they are not the same. Semiotic resources also could be seen as resources that students bring in to solve a problem, but the semiotic resources are not independent (like conceptual resources). Each is tightly affiliated to a certain representation. Conceptual resources, on the other hand can be named using descriptive names for the thing resources represent [163]. "Ampere's law" as an example, in the semiotic resource model, we have a resource that is Ampere's law written algebraically in integral form (semiotic resource: Integral form of Ampere's law). Here "written algebraically" is necessarily a part of the semiotic resource accounting and it shows us the affiliated representation (inscription- fromula/ mathematics). But in the conceptual resource model, the resource Ampere's law could just be an idea or it could be (not required) that specific algebraically written integral form. Similarly, the "semiotic resource: arrow as vector" [164] says that students are drawing arrows to represent vectors and, in this case, includes the affiliated representation (inscription- diagram). This suggests that even though both conceptual resources and semiotic resources look similar, in the case of semiotic resources, we necessarily have to go through the affiliated representation to solve the problems.

2.5.3 Modality and Multimodality

As described in the previous section, semiotic resources are people's ideas that are made up of small re-useable idea pieces that are tightly linked to a given representation. These representations are created within modes; a representation is made utilizing a mode or a combination of modes. According to Kress, all communication is understood to employ a variety of ways of representing messages [152] and these different ways are usually referred as modes. In most cases, a representation is multimodal in nature.

In social semiotics, a modality is a particular way in which information is to be encoded for presentation to humans. Lemke's [165] categorization of modes of representation is more specific: (a.) natural language (words, whether written or spoken), (b.) mathematical modes (including all the symbols of math), (c.) visual modes (such as images, graphs, tables), and (d.) actional modes (such as gesture). Others [166] have used just three groups: (a.) linguistic (including words), (b.) visual (images and pictures), and (c.) actional (such as gestures). Van Leeuwen [32] points out that "multimodality means the combination of different semiotic modes; for example, language and music in a communicative artifact or event".

A more specific definition is that "multimodal discourse involves the interaction of multiple semiotic resources such as language, gesture, dress, architecture, proximity lighting, movement, gaze, camera angle, and so on" [167]. Multimodality and social semiotics, together, make it possible to ask questions around meaning and meaning-making, the agency of meaning-makers, the constitution of identity in sign- and meaning-making, the (social) constraints faced in making meaning, social semiosis and knowledge, how knowledge is produced, shaped and constituted distinctly in different modes, and by whom. Multimodality includes questions around the capabilities/ potentials of the affordances of the resources that are available in any one society for the making of meaning; and how, therefore, knowledge appears differently in different modes.

2.5.4 Idea of Affordance

Each mode has affordances (strengths) and limitations (weaknesses), which dictate when a mode of representation is most apt to communicate messages [152;153]. Affordances are the characteristics of a mode that give it a specific advantage over another mode of representation; limitations are those characteristics that make using that mode of representation less beneficial than others. For example, when sending a message of warning to a person nearby, one would be inclined to use verbal speech rather than handwritten words because of the affordance of being able to quickly transmit the message across a distance.

According to the literature, the concept of "affordance" represents the relationship between human perceivers and their environment. In seeking to move beyond a focus only on an individual's mental processes to explain perception, Gibson[(168)] theorized that individuals interact with the physical environment in terms of 'affordances' that support their goals or intentions. Individuals recognize a required potential action that the environment both prompts as well as supports. This account of affordances has been generative across various domains, especially in computer program design, and problem solving. Seeking to clarify this construct further, Norman[(169)] claimed that all affordances are perceived affordances, in that the enabling feature in the environment needs to be noticed to be enabling. He considered that affordances are best understood as physical enablers and constraints, such as the design of computer programs where it is impossible to move a cursor outside the visible computer screen. Both Gibson and Norman were more concerned with explaining purposeful perception rather than accounting for exploratory or learned behavior with cultural tools, whether material or symbolic.

However, we argue that this idea of affordances as enabling constraints can be applied productively to understanding how and why generating representations supports learning in physics as a discipline. This entails an extension of the idea of affordances as perceptual interactions with the environment to include learned behaviors and strategies in physics classrooms and in problem solving. We suggest that use of ideas as semiotic resources offers specific affordances as students construct representations to make a claim about physics concepts, principles, or processes.

2.5.5 Affordances of Semiotic Resources

The social semiotic approach focuses on all types of meaning-making practices that are accomplished through different semiotic resources in different semiotic modes that include visual, verbal (or oral), written, and gestural modes. Semiotic resources have a meaning potential, based on their past uses, and a set of affordances based on their possible uses; these will be actualized in concrete social contexts where their use is subject to some form of semiotic regime [32]. Each semiotic resource offers a unique capability to create meaning. Researchers interpret this as *meaning potential* [32] and *affordances* of semiotic resources. Meaning potential refers to different ways that meaning is constructed and communicated using a semiotic resources. It is based on the way the person has used this resource in the past. Affordances constrain what ideas are possible to express using each semiotic resource [168;32], as perceived by the person [169] using the resource. Kress [153] expanded the idea of semiotic resources to include *modal affordances*, which constrain different modes of semiotic resources (e.g. verbal resources, gestural resources, etc), noting that there may be two or more communication modes within each artifact (e.g. a diagram may be married to a verbal description).

Recently, Fredlund [130] expanded the idea of affordances further. The *disciplinary affordance* of a given semiotic resource is "the inherent potential of that (semiotic resource) to provide access to disciplinary knowledge." This expansion allows researchers to investigate how semiotic resources' affordances connect to disciplinary ideas. Taking up disciplinary affordances allows us to focus on knowledge production and communication within the discipline (here, physics) more than focusing on the view or the experience of an individual student.

Studies indicated that multiple external representations have the potential to allow students to generate deeper conceptual understanding [146;113;119] of the physics concepts and principles. Previously, Fredlund [130] used two versions of basic RC-circuit (resistor-capacitor circuit) diagrams to show the importance of unpacking the disciplinary affordance of semiotic resources for effective learning in student laboratories. Researchers used the data collected in an introductory university course on electromagnetism in which students worked with electric circuits in the laboratory. In order to assemble the circuit appropriately in the laboratory, students needed to take into consideration a number of aspects that are not directly discernible in the circuit diagram. As an example, students could connect the coaxial cable to the function generator or the oscilloscope, but it's important to understand the functions of the two split wires at the two ends of the coaxial cable. By convention, red is connected to the inner conductor of the coaxial cable, which carries the signal, while black is connected to the common ground potential carried by the outer shield. Thus, the signal is measured with reference to this common ground. The first version of the diagram, the standard RC circuit, does not have any information in the diagram to indicate how students could connect the red and black probes correctly. In contrast the second version of the diagram (modified RC circuit diagram) contains the colored dots to indicate where to connect the components in the circuit.

As the author indicates in that article, the additional information included in the second diagram used to unpack the important aspects of the RC circuit representation helped students to understand the given representation better and to engage in the activity skillfully. Inclusion of the extra information in the RC circuit diagram or unpacking of representation helps students to come to 'see' the parts of intended meaning that are not directly discernible in the (initial) representation. According to Fredlund, instructors need to "unpack" the representations in order to help students access the intended meanings associated with the representations that are not directly noticeable.

In another attempt to build on the idea of disciplinary affordances, Fredlund[(109)] used a refraction of light experiment to explain student problem-solving behaviors. One aspect of this research was to identify the semiotic resources students use to explain the refraction of light. As students discuss about the refraction of light, the ray diagram is the most frequent semiotic resource that they find in their textbooks. With wavefront diagrams appearing less frequently, surprisingly, however, Fredlund found that the group of students used a wavefront diagram. In fact, students used both semiotic resources - a wavefront diagram and a ray diagram - to explain the scenario of refraction of light. Student reasoning showed that the disciplinary affordances of these two semiotic resources helped them access different aspects of physics knowledge when explaining the refraction of light. The ray diagram could help students to reason about the refraction angles at the boundary and also about the direction of propagation, but it could not help students reason about speed changes in the two media.

In contrast, the wavefront diagram promoted reasoning about speed changes but obscured reasoning about angles and directionality.

In both of these studies Fredlund [130;135] provided students with two different semiotic resources in the same semiotic mode, showing that different semiotic resources have different disciplinary affordances. But researchers did not consider how students might spontaneously construct representations to solve a given problem. In our approach to describe student problem-solving process at the upper-division level, we focus on spontaneous construction of representations. The study we put forward in chapter 5 presents the initial framework, which we suggest shows how students coordinate among different semiotic resources to construct representations that they use to solve problems.

In our previous study [170], we described how students' construction of spontaneous representations is achieved by coordinating among different semiotic resources with different disciplinary affordances. Even though we paid attention to the disciplinary affordances of the semiotic resources that students connect to construct representations, we did not pay attention to the modes of those semiotic resources. In particular, we did not consider how representations in different modes get connected to each other, or what factors may influence how students succeed or fail to connect among the representations that they spontaneously construct (that are in multiple modes).

Within these previous studies [130;135] researchers did not consider how students first spontaneously construct representations, or how students connect among representations that they use to solve a given problem. In chapter 6, we use multiple cases of students solving oral exam problems to generalize the approach we proposed in our previous study to explain the underlying mechanism behind students' constructing spontaneous representations by coordinating among different semiotic resources with different modes. I suggest a new way of explaining how students connect among representations (representational spaces) and reasons why students might get stuck or unstuck while solving problems due to the features of the representations they generate spontaneously.

Chapter 3

Varied reasoning schema in students' written solutions

3.1 Introduction

[1] Homework is a key element in every undergraduate-level physics course for practicing problem solving. Students often have difficulties combining physics ideas with mathematical calculations and selecting among known mathematical tools. Therefore, it is important to explore how students employ mathematics when approaching physics homework problems.

In this study, we focus on the mathematical tools required in upper-division physics courses, such as evaluating integrals, using differential equations and approximations, or choosing an appropriate coordinate system. Although these techniques are covered in mathematics classes, where students can successfully solve problems, many students still struggle to apply these mathematical tools to problems in physics. This is true especially at the upper-division when problems are more complicated and mathematics more tightly entwined with problem solving [(13;15)]. In our study, we use the ACER (Activation-Construction-Execution-Reflection) framework [(14;43)] combined with network analytic methods [(78)] to investigate how upper-division students' written work shows their mathematical tool use on typical home-

[1]This analysis was submitted to arXiv repository of electronic preprints [(171)]

work problems.

The ACER framework was developed using think-aloud interview data on conceptually-rich research-based problems. These problems are different in character from the more traditional problems in our study. Our data are wholly written accounts with sparse reasoning evidence. However, because students' written homework solutions are common at the upper-division level, they may represent a ubiquitous and easy-to-acquire data source for education research. Within these constraints, this analysis explores how, if at all, the modified ACER framework can be used to analyze written evidence of typical problem solving in a traditional course. We investigate three kinds of differences: differences in solution paths within one problem; differences among multiple problems of the same type; and differences in students' initial problem solving steps as a function of kind of problem.

3.2 Context

Our data are drawn from an undergraduate course in Classical Mechanics taught at a large land-grant university in the central United States. As at many similar institutions, our Classical Mechanics course is a textbook-centric 4 credit-hour course with a solid foundation in the basics of theoretical physics taught in a predominately lecture format. It is usually taken as the fourth physics course for physics majors and minors in the spring of their second year. Typically, the students are concurrently enrolled in a differential equations course. Our course uses *Classical Mechanics* by J. R. Taylor[(51)]; the lectures cover the first 10 chapters, including topics such as Newton's laws of motion, momentum, angular momentum, energy, oscillations, Lagrange's equations, two-body central-force problems and rotational motion of rigid bodies.

All homework problems were selected from the textbook and were chosen to encourage students to practice solving problems. In return, students could earn extra points towards their final grade. Students were encouraged to work on problems in groups but, for pedagogical reasons, they were to write solutions independently. Students in this class were not taught an explicit procedure to solve problems. Rather, they were free to solve them any way that they could, which gave us the opportunity to gain insight into their reasoning when left to their own devices.

In spring 2015, 12 students were enrolled in the class. They had 13 weekly homework assignments (each including 10-15 problems). A complete solution to each homework problem was expected to be about half a page long and include diagrams, mathematics, and verbal statements. We scanned students' submitted homework for analysis before passing them to the grader. All students in the course consented to participate in our research study.

3.3 Analytic Framework

Several attempts to model physics problem solving break it into discrete steps to be followed in a linear order[19;9]. However, unless students are explicitly taught to follow these algorithms, their approaches to problem solving rarely fit these linear models. In contrast, studies using knowledge-in-pieces frameworks to understand students' non-linear problem solving "in the wild" (e.g., Ref.[38]) rely on the richness of interview or video data to make inferences about student reasoning.

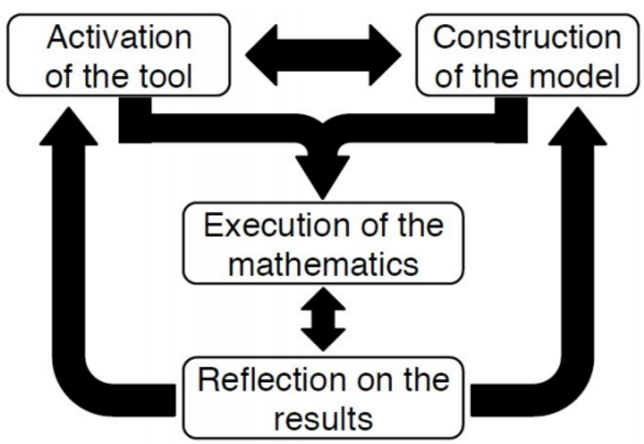


Figure 3.1: *A visual representation of the ACER framework*

The ACER framework bridges the gap between prescriptive problem solving and knowledge-in-pieces[14;43]. It was developed to analyze the use of mathematics in upper-division physics courses. The ACER framework (Figure 3.1) is organized around four components: *Activation* (determining the proper mathematical tool); *Construction* (making a mathematical model); *Execution* (performing mathematical steps); and *Reflection* (checking the final solution).

3.3.1 Sub-Codes Under ACER Components

Within the codebook, each component of the ACER framework may include a series of sub-codes that helps to describe student work. Under the **activation**, previous research on student difficulties included codes such as **CA1** - The problem asks for the potential or electric field, and **CA2** - The problem gives a charge distribution[(43)]. Ideas behind all these sub-codes explicitly exist in the problem statement.

As an example, previous research generated sub-codes under **construction** such as **CC1** - Use the geometry of the charge distribution to select a coordinate system. **CC2** - Express the differential charge element (dq) in the selected coordinates[(43)].

Then under **execution**, previous research included sub-codes under this component such as **CE2** - Execute integrals in the selected coordinate system, **CE3** - Manipulate the resulting algebraic expressions into a form that can be readily interpreted[(43)]. These sub-codes or steps are generated from the procedural mathematical steps that are required to solve mechanics problems.

Finally, previous research generated sub-codes under **reflection** such as **CR1** - Verify that the units are correct, **CR2** - Check the limiting behavior to ensure it is consistent with the total charge and geometry of the charge distribution[(43)].

3.3.2 Operationalizing the modified ACER Framework

Prior work using ACER framework has focused mostly on topics in electricity and magnetism to uncover student difficulties related to specific mathematical tools like Taylor series, Coulomb's law, Dirac delta functions in upper-division classical mechanics and electrostatics courses[(45;14;16;43)]. In this study, our focus is on the traditional Classical Mechanics course. As we coded students' problem solutions we came across the need of extending the standard sub-codes set to include codes appropriate for themechanicscourse. That way we developed a modified version of the ACER framework to analyze written evidence. As an example, the codebook contains sub-codes like **A2** - Student uses a general form of an equation, **A3** - Student uses a less-general form of an equation etc. under the activation component. It is obvious that the idea behind all these sub-codes is typically existing in the problem statement. Then, **C1** - Pick a coordinate system, **C2** - Visualize the problem, **C5** - Set the limits of the integration etc. are the examples of sub-codes for construction that we came up in our study. Under the execution component, Sub-codes like **E1** - Do the integration, **E2** - Take the derivatives, **E3** - Make the substitutions etc. were generated and these sub-codes or the steps are generated from the procedural mathematical steps that are required to solve mechanics problems. Finally, under reflection we generated sub-codes like **R1**- Check the units, **R2** - Check the limits of the final answer, **R3** - Does this answer make sense? etc. In addition to the modification we did to match the sub-codes to fit with the application in mechanics course, we extended the extent set of sub-codes to cover student errors and mistakes (e.g., C5X - mistakes did whensetting the limits of the integration,E2X - mistakes did when taking derivatives, etc.). A detailed version of the codebook is provided in Appendix A.

We coded students' problem solutions, iteratively seeking new content codes and combining proposed codes until our code book covered $> 95\%$ of student work. As the codebook approached stability, eight people participated in inter-rater reliability testing to assure the codebook was a valid representation of student problem solving. When the codebook was stable, multiple graders looked at different problems to confirm the reliability of coding. Once two graders coded individual problems and achieved $> 90\%$ agreement on student work, the codebook was established.

In this study, we investigate the research question: what, if anything, could be learned about students' problem solving from their homework solutions? Then, we report on three problems selected from homework assignments across all students. We compare both across students and across problems to find patterns of mathematical tools used within each problem.

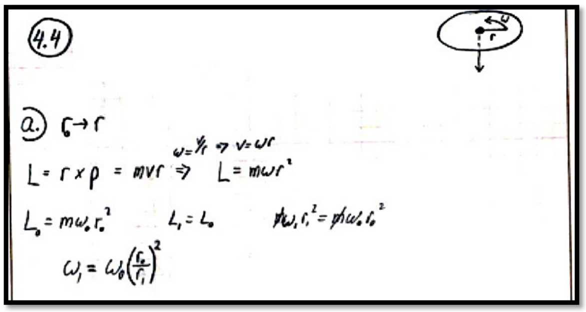
3.3.3 Application of Social Network Analysis

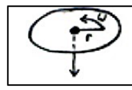
To compare patterns in students' solutions, we used techniques borrowed from network analysis[78]. A network graph is a collection of points called nodes, and lines connecting these points, called edges. When the direction of a connection matters, we have a *directed graph*. In such cases, an edge from node A to B ($A \to B$) means something different than an edge from B to A ($B \to A$). If the direction of a connection does not contain any information about the relationship between nodes (i.e, $A - B$ has the same meaning as $B - A$), a graph is called *undirected*. A graph can be characterized using various measures, such as node and edge count, diameter or density. Node and edge count correspond to the number of nodes and edges, respectively. Diameter is the longest graph distance between any two nodes in the network and it represents the linear size of a network. Density, on the other hand, is the ratio of the number of existing edges to the number of all possible edges and can be thought of as a measure of network effectiveness. To determine the most important nodes in a network, one can perform centrality analysis. Depending on the nature of a network, as well as on the category of the "importance of a node" one is interested in, there are multiple centrality measures that can be used.[78] Degree centrality indicates the importance of a node by the number of nodes connected to it, where the larger the degree, the more important the node is.

Within this framework, nodes represent ACER sub-codes and the directed edges represent the order of steps students took to solve a given problem. When looking at the network created from all solutions to a given problem, we weight the edges of the graph by how many times students connected a given two nodes to emphasize which connections between nodes are important. Edges with higher weights represent common connections within a network.

3.4 Variation in solutions within one problem

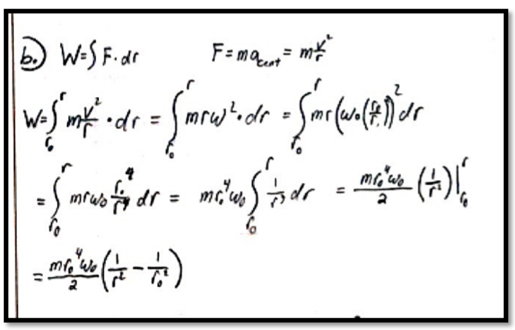
Problem 1: A particle of mass m is moving on a frictionless horizontal table and is attached to a massless string, whose other end passes through a hole in the table, where I'm holding it. Initially, the particle is moving in a circle of radius r_0 with angular velocity ω_0, but I now pull the string down through the hole until a length r remains between the hole and the particle. (a.) What is the particle's angular velocity now? (b.) Assuming that I pull the string so slowly that we can approximate the particle's path by a circle of slowly shrinking radius, calculate the work I did pulling the string. (c.) Compare your answer to part (b.) with the particle's gain in kinetic energy.





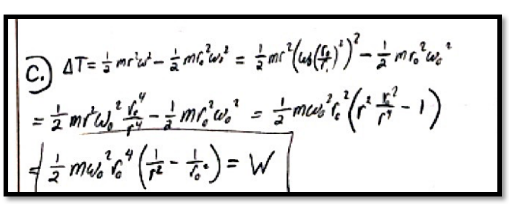
	C2 - Visualize the problem C1 - Pick a coordinate system

$L = r \times p$	A2 - Student uses a general form of an equation
$= mvr$	E3 - Make substitutions
$\omega = \frac{v}{r}$	A2 - Student uses a general form of an equation
$L = m\omega r^2$	E3 - Make substitutions
$L_0 = m\omega_0 {r_0}^2$	C3 - Use an equation (specific to the particular problem)
$L_1 = L_0$	C3 - Use an equation (specific to the particular problem)
$m\omega_1 {r_1}^2 = m\omega_0 {r_0}^2$	E3 - Make substitutions
$\omega_1 = \omega_0 \left(\frac{r_0}{r_1}\right)^2$	E5 - Doing algebra



$W = \int F . dr$	A2 - Student uses a general form of an equation
$F = ma_{cent}$	A2 - Student uses a general form of an equation
$= m\frac{v^2}{r}$	C3 - Use an equation (specific to the particular problem)

$W = \int_{r_0}^{r} m\frac{v^2}{r}\, dr$	C5 – Set the limits of the integration R4 - Does this answer fit in the next part E3 - Make substitutions
$= \int_{r_0}^{r} mr\omega^2\, dr$	E3 - Make substitutions
$= \int_{r_0}^{r} mr\left(\omega_0\left(\frac{r_0}{r_1}\right)^2\right)^2 dr$	R4 - Does this answer fit in the next part E3 - Make substitutions
$= mr_0^4\omega_0 \int_{r_0}^{r} \frac{1}{r^3} dr$	C4 – Understand the meaning of symbols in equation E5 - Doing algebra
$= \frac{mr_0^4\omega_0}{2}\left(\frac{1}{r^2}\right)_{r_0}^{r}$	E1 - Do the integration
$= \frac{mr_0^4\omega_0}{2}\left(\frac{1}{r^2} - \frac{1}{{r_0}^2}\right)$	E10 – Evaluate the integration



$\Delta T = \frac{1}{2}mr^2\omega^2 - \frac{1}{2}m{r_0}^2{\omega_0}^2$	C3 - Use an equation (specific to the particular problem)

$= \frac{1}{2}mr^2\left(\omega_0\left(\frac{r_0}{r_1}\right)^2\right)^2 - \frac{1}{2}m{r_0}^2{\omega_0}^2$	R4 - Does this answer fit in the next part E3 - Make substitutions
$= \frac{1}{2}mr^2{\omega_0}^2\frac{r_0^4}{r^4} - \frac{1}{2}m{r_0}^2{\omega_0}^2$	E5 - Doing algebra
$= \frac{mr_0^4\omega_0}{2}\left(\frac{1}{r^2} - \frac{1}{{r_0}^2}\right) = W$	E5 - Doing algebra

Figure 3.2: *A solution to Problem 1 with a response coded according to the modified ACER framework.*

3.4.1 What was expected

A solution for the sample problem is presented in Figure 3.2. This particular problem was not presented with a diagram. We expected that in order to visualize the physical system, the students would start with drawing one. In our ACER framework, that would be represented by codes *pick a coordinate system* (C1) and *visualize the problem* (C2). This problem has three parts and for all of them, we expected students to begin with a general form of an equation, which would be coded as *use a general form of an equation* (A2). In part (b.), as they have to do integration, we expected codes *set the limits of the integration* (C5), *do the integration* (E1) and *evaluate the integration* (E10) to occur. Both (b.) and (c.) require substitutions from the answer obtained in part (a.); therefore, we also expected codes *does this answer fit in the next part* (R4) and *make substitutions* (E3) to be used.

As students perform mathematical operations in their solutions, we expect to see *Execution* codes E3 and E5 (*make substitutions* and *do algebra*). When we consider the problem as a whole, we expect *Activation* code A2 (*use a general form of an equation*), *Construction* codes C1, C2 and C5 (*pick a coordinate system*, *visualize the problem* and *set the limits of the integration*), *Execution* codes E1, E3, E5 and E10 (*do the integration*, *make substitutions*, *do algebra* and *evaluate the integration*, respectively) and *Reflection* code R4 (*does this answer fit in the next part*) to be dominating codes.

3.4.2 What was found

When we used the ACER framework alone, we saw *Activation* code A2, *Construction* codes C1, C3 (*use an equation specific to the particular problem*) and C5, *Execution* codes E1, E3, E5 and E10 and *Reflection* code R4 as the most commonly used, dominant codes. Surprisingly, despite the lack of a figure in the problem statement, not all of the students started with visualizing the problem. They used both the general form of the equation and the problem related non-general form.

When we applied network analysis (NA) along with centrality measures to the ACER codes, we found that some of the nodes in the network, identified as important by the

ACER framework (by the number of appearance for particular node), were not significant when looked at using the NA approach. On the other hand, network analysis identified nodes as more central which a frequency analysis using ACER alone would have overlooked.

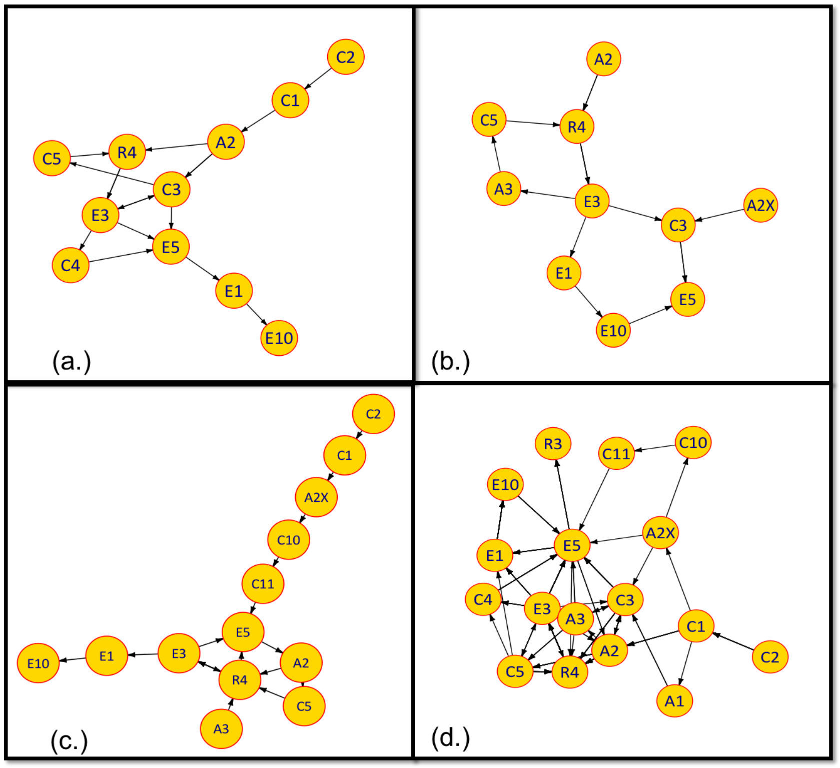


Figure 3.3: *(a.) Network graph representation of the solution presented in Figure 3.2.,(b.), (c.) network graphs representing two other students' solutions to Problem 1 (students B, C),(d.) all students' responses to this problem merged into a single network*

Figure 3.2 shows a single student's solution coded according to the ACER framework. In Figure 3.3, the network graph for that student (Fig. 3.3a), two other students (Fig. 3.3b, 3.3c), all students' responses combined into a single graph (Fig. 3.3d).

Even though this problem was a well-structured problem, different students had different approaches, as seen in the differences in their starting codes. As students in Figure 3.3a.) and 3.3c.) took a diagram approach while the student in Figure 3.3b.) used equations at the start. The graph for all students (Figure 3.3d) is denser than for any one student, and the specific patterns of each student are lost in the concatenation.

Table 3.1 presents a summary of the basic network descriptives for all networks presented in Figure 3.3. When we look at sizes as measured by the diameter, we see the average diameter for the students was 5.82 ($SD = 1.59$). The average density of students' solutions was 0.16 ($SD = 0.05$) with values for individual student's ranging from 0.09 to 0.23.

When we look at centrality indices for this problem, the node E3 (*making substitutions*) has the highest degree, indicating that it most frequently connects other steps in the problem solution. This is evidence that the mathematical operation of substitution is a crucial element in the solving process for this particular problem. Moreover, almost all the homework problems had multiple parts within the problem and often subsequent parts required a substitution found in a previous section. When we look at the network created by combining solutions to this problem from all students, we found that the top three nodes for all students are the same as the top nodes for both sample students (though their order is different).

These network graphs show that students use two different approaches to solve this problem: they may start with visualizing the problem (such as with a diagram) before using equations in either a general or specific form, or they may omit the visualization and move directly to using equations. While this particular problem does not explicitly ask for a graphical representation of the physical situation, drawing appropriate diagrams is a useful problem solving strategy [128;172], with the possibility that specific instruction on this topic may increase students' use of diagrams.

Table 3.1: *Problem 1: Network characteristics for solutions from three students, for a network created by combining all students' solutions to this problem.*

	Student A	Student B	Student C	All Students
Diameter	6	5	10	7
Density	0.155	0.167	0.115	0.632
Nodes w/max. degree	C3 and E3	R4 and E3	R4	E3
	E5 and R4	C3	E3	R4
	A2	E5	E5	E5

3.5 The effects of problem statement type on students' solution graph

After comparing students' network graphs (different problems by same student and same problem by different student), we found that there is a dependence on the original problem statement reflected on the network graphs. The first few problem solving steps that students use – the first few nodes in the network – depend on the kind of prompt that the problem uses. In particular, we identified three different types of prompts that suggest to students how they should start the problem:

1. The problem statement is straightforward and asks students to perform specific mathematical operations, including trivial math procedures. An example statement looks like "take the derivative/solve the above position equation for acceleration".

2. The prompt directs students towards the physical system or a diagram as the problem statement asks for an explanation of the physical system. An example prompt may include "start with a diagram/free body diagram".

3. The problem statement directs students to think about what equations or conceptual resources[(173)] they should bring together to get an equivalent expression along with the physical system. To do so requires physics knowledge along with the correct mathematical steps. An example statement looks like "show that/prove that... with the help of physical system".

In order to determine whether students take the same approach when working on problems that are structured similarly, we identified three problems that belong to the same category (the second kind mentioned above) and we studied networks created by combining solutions to these three problems for each individual student as well as across all students.

3.5.1 What was expected

Since we compare problems structured similarly, we expect the crucial components of students' solutions to be similar across all students. Allowing for some slight variations, the core of the solutions should remain unchanged. The problems that we chose for our analysis include the one discussed in the previous section and thus we expected *Activation* code: A2 (*use a general form of an equation*), *Construction* codes: C1 (*pick a coordinate system*), C2 (*visualize the problem*), C3 (*use an equation specific to the particular problem*) and C5 (*set the limits of the integration*), *Execution* codes: E1 (*do the integration*), E3 (*make substitutions*), E5 (*do algebra*), E10 (*evaluate the integration*) and *Reflection* code: R4 (*does this answer fit in the next part*) to be the dominating codes.

3.5.2 What was found

Table 3.2 shows characteristics of networks created by combining three structurally similar problems for three individual students, as well as of a network representing solutions from all students in the class combined. Nodes A2 (*use a general form of an equation*), C3 (*use an equation specific to the particular problem*), C5 (*set the limits of the integration*), E3 (*make substitutions*) and E5 (*do algebra*) are most central for these networks.

Table 3.2: *Network characteristics for solutions from three students and for a network created by combining all students' solutions to three problems that were classified as structurally similar.*

	Student C	Student D	Student E	All Students
Diameter	13	11	10	11
Density	0.133	0.277	0.211	0.309
Nodes w/max. degree	E5	E3	E3	E5
	A2	E5	C5	E3
	C3	C5 and A2	E5	C5 and C3

As we expected based on the structure similarity, students used the general (A2) or non-general (C3) form of an equation to start the problem. Nodes E3 (*make substitutions*) and E5 (*do algebra*) again are key players in the network; however, since the problems include integrals, the node C5 (*set the limits of the integration*) becomes more prominent. By comparing networks across different problems, we can see whether students are consistent in their approaches to solving problems with a similar structure. In all cases, node E5 (*do algebra*) was the most central node. Also, the four most prominent nodes (A2, C3, E3, E5) are the same for all students and for the merged network, although their order differs slightly. This alone could suggest that students take similar approaches to solve all three problems and that these approaches are quite similar among students.

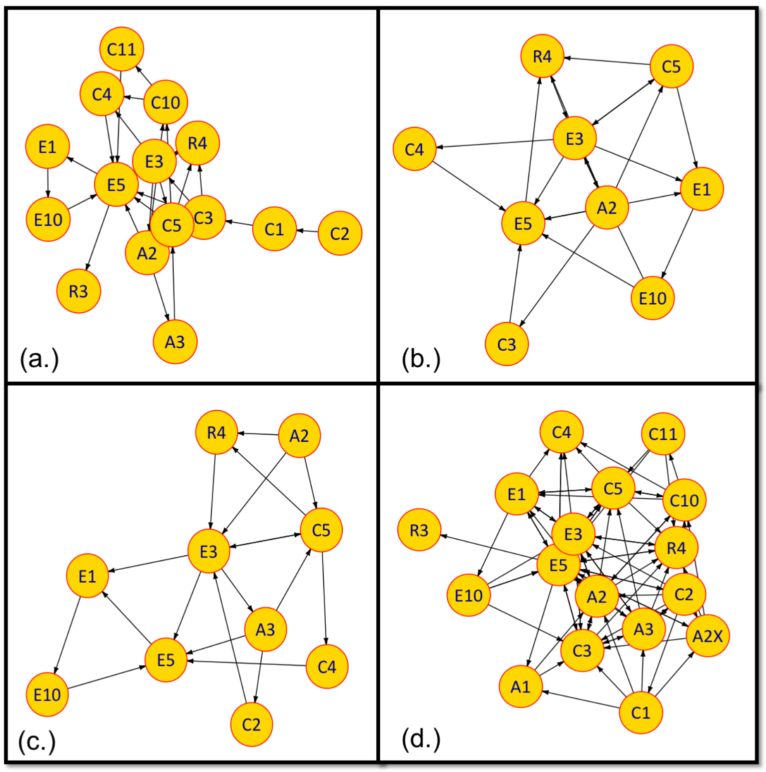


Figure 3.4: *Network graphs created by combining solutions to three problems from three individual students (a. – c.) and from all students' responses merged into a single network (d.).*

However, when we look at the network representing students' solutions in Figure 3.4 we can see that the network presented in Fig. 3.4a is significantly different from the other two networks (Fig. 3.4b and 3.4c). Also, the network characteristics in Table 3.2 reveal that the network for student C has a greater diameter and lower density (a less connected graph with more nodes), indicating that this student got sidetracked in his solution. This could not be seen by the ACER framework alone, as the significant codes for this student were the same as for students who took a more direct approach.

3.6 Problem statements and network complexity

Further analysis suggests that problems consisting of more subsections lead to relatively more complex networks. Here we compare three problems that fall into different categories. Problem 2 belongs to the first type where students are required to perform trivial math procedures. Problem 3, on the other hand, falls into the second type and requires students to start with a diagram.

Problem 2: The shortest path between two points on a curved surface, such as the surface of a sphere, is called a geodesic. To find a geodesic, one has to first set up an integral that gives the length of a path on the surface in question. This will always be similar to the integral

$$L = \int_1^2 \mathrm{d}s = \int_{x1}^{x_2} \sqrt{1 + y'^2(x)}\mathrm{d}s \tag{3.1}$$

but may be more complicated (depending on the nature of the surface) and may involve different coordinates than x and y. To illustrate this, use spherical polar coordinates (r, θ, ϕ) to show that the length of a path joining two points on a sphere of radius R is

$$L = R \int_{x_1}^{x_2} \sqrt{1 + y'^2(x)}\mathrm{d}s \tag{3.2}$$

Problem 3: Consider the pendulum of Figure 3.5, suspended inside a railroad car that is being forced to accelerate with a constant acceleration a. (a.) Write down the Lagrangian for the system and the equation of motion for the angle ϕ. Use a trick similar to the one used in $x(t) = A\cos(\omega\mathrm{t}-\delta)$ to write the combination of $\sin(\phi)$ and $\cos(\phi)$ as a multiple of $\sin(\phi+\beta)$ (b.) Find the equilibrium angle ϕ at which the pendulum can remain fixed (relative to the car) as the car accelerates. Use the equation of motion to show that this equilibrium is stable. What is the frequency of small oscillation about this equilibrium position?

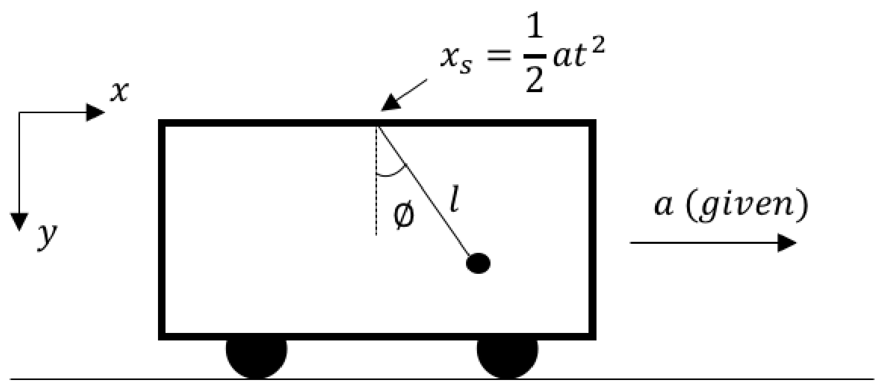


Figure 3.5: *Diagram for Problem 3.*

3.6.1 The comparison

In order to find the relation between the problem statement and the shape of the network here, we compare networks for three problems by all students. First, we compare the size of the networks and then we compare the most central nodes of each network.

The size of a network is a measure of the number of nodes within a network. The network graph discussed in Section 3.4 (Problem 1) has 16 nodes and 40 edges with eight strongly connected nodes. The network graph for Problem 2 has 14 nodes and 32 edges along with five strongly connected nodes and Problem 3 has a network graph with 11 nodes and 31 edges with two strongly connected nodes. The network graph of Problem 1 has a relatively large network size compared to the other two problems. This may be because Problem 1 has three subsections and the other two problems have fewer subsections.

Network analysis shows that the most central nodes for the network graph for Problem 1 (Figure 3.3c – all student responses) are E3 (*make substitutions*), R4 (*does this answer fit in the next part*), E5 (*do algebra*) and A2 (*use a general form of an equation*) with a network diameter of 5. According to the problem statement, students first have to understand the physical system and then use relevant equations to solve the problem. The network graph clearly shows that the students start with *Construction* codes, where they build a visual representation and then use the *Activation* codes, where they use appropriate equations. Because the problem has three subsections, students might recall an answer from a previous part and substitute it in the next part. Thus, R4 (*does this answer fit in the next part*) and

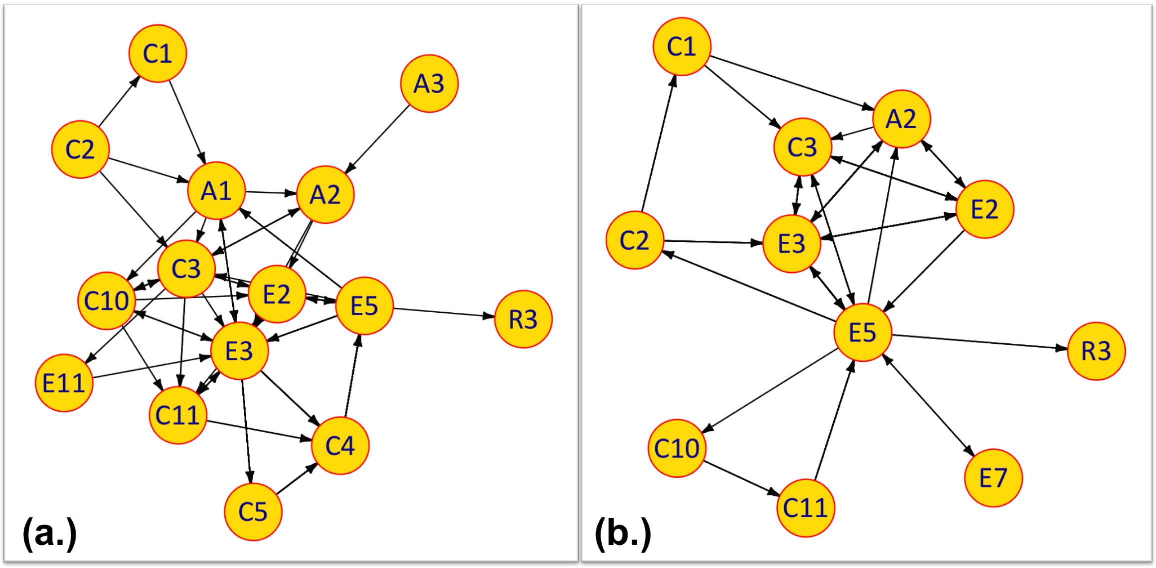


Figure 3.6: *Network graph for all responses to Problem 2. (b.): Network graph for all responses to Problem 3.*

E3 (*make substitutions*) are the most central nodes.

E5 (*do algebra*), E3 (*make substitutions*), C5 (*set the limits of the integration*) and E2 (*take the derivatives*) are the most central nodes in the network graph for Problem 2 (Figure 3.6a – all students' responses). The network has a diameter of 5. The phrasing of Problem 2 also requires a visual representation. The students start with *Construction* codes and with a diagram. Because the problem does not have subsections, R4 (*does this answer fit in the next part*) is not a central node.

In contrast, E3 (*make substitutions*), E5 (*do algebra*), E2 (*take the derivatives*) and A2 (*use a general form of an equation*) are the central nodes in Problem 3 (Figure 3.6b – all students' responses), with a network diameter of 4. Some different nodes appear in this network graph compared to the previous two problems. E2 (*take the derivatives*) becomes a central node because students were asked to find the equation of motion and students take derivatives. Within their solutions, students use boundary conditions to show that this equilibrium is stable; this behavior shows up with two *Construction* codes: C10 (*state boundary conditions*) and C11 (*apply boundary conditions*).

Comparison between problem statements and the network graphs shows that there is a dependence of problem statement on the shape of the network. The problems which consist of more subsections also lead to relatively complex networks.

3.7 Limitations and Future work

Our research question in this study was to see what, if anything, could be learned about students' problem solving given their homework solutions, as they represent a ubiquitous data source. The information about students' performance revealed by the network graphs could be used to enhance the quality and consistency of the homework or quiz problems.

However, coding homework solutions is a laborious process that is impractical for ordinary classroom use; instead, these recommendations suggest that instructors compare students' solutions for elements in common (or unique to each student) as a pedagogical tool to gauge problem difficulty and divergent thinking. The NA approach is still more practical than interviewing and videotaping every single student, as it was done in the original studies [14;43].

Another limitation of this data source is that we can't know if students go back and add more information later because the only thing we have is the written solution. We coded students' solutions by assuming they wrote this strictly linearly; this is reasonable, as we have observational data of different students writing problems in mostly the order that they solve them in. In future work, we could more carefully test this assumption.

Our data are drawn from one highly typical course at a single university. This approach could be used to compare active-learning classrooms to traditional ones, or real-world problem sets to ones drawn from a textbook. Though our data cover 23 questions overall, they are limited to classical mechanics. To expand our study more broadly across the upper division, we are currently collecting data on an Electricity and Magnetism (EM) course, and augmenting students' written work with video recordings of them solving homework problems.

We could use a fine-grained ACER framework to focus on different parts of student solutions. For instance, we could use the fine-grained activation and construction to answer a research question like, what is the effect of different problem statements to students' approach to solving problems. To do this we may have to carefully design the problem statements. In our study, we did not see large differences in students' approaches, perhaps because the textbook problems were not designed with that goal.

3.8 Implications for instruction

3.8.1 Help/guide the students

The network graphs by all students and by each individual student can be directly used to estimate the students understanding of the subject material. More importantly the individual network graphs could be used to help or guide the students who might have stuck at a certain step with in the problem solving process.

3.8.2 Choosing problems

Also, network graphs from both individual and all students together can be used to understand the difficulty level of the problems. If the students are taking more steps to reach the final answer than expected, there might be a issue with the way the problem statement was written.

Thus, the information about students performance revealed by the network graphs could be used to enhance the quality of the homework or quiz problems.

3.9 Conclusion

Network analysis has common applications in analyzing social interactions in groups of people, complex ecosystems and biological systems, and information transfer systems. In this study, we use network analytic tools to represent the relationships between the knowledge elements and steps of problem solving. We use the ACER framework to identify and code these elements and steps. Together, network analysis and the ACER framework can model how students connect ideas to solve problems and allow for quantitative comparisons among students and problems. Network analysis identifies the most important mathematical operations in the solving process, as interpreted through the ACER codes.

Network analysis indicates that in most of the analyzed cases, the most important nodes are E3 (*make substitutions*), E5 (*do algebra*), A2 (*use a general form of an equation*), R4 (*does this answer fit in the next part*). Substitution and algebra, in general, are crucial tools for solving these problems.

Most of the homework problems had few parts within the problem and in these cases, students may have to go back and check the previous solution and then substitute it in the next part. While many problems have similar central nodes, the exact nodes and their order are different across students and across problems. The shape of the network graphs varies with different problem statements. As a set, the homework problems are well designed homework problem, different student have different approach to start the problem and different step patterns in the construction section but in most of the cases the math execution steps are the same.

Further research is needed to see exactly what properties of problem statements prompt students to use specific mathematical tools. Nonetheless, network analysis of students' problem solutions reveals patterns in their tool use on typical problems.

Chapter 4

Standing fast: Translation among durable representations using evanescent representations in upper-division problem solving

4.1 Introduction

[1] In the course of problem solving, students use a series of representations to make sense of the problem, build a solution, and communicate it. In this study, we are concerned with how students build up and use representations in problem solving, focusing on a single case at the upper-division level.

At their most basic, representations are either external or internal objects that stand for something else. A descriptive representation consists of symbols describing an object. Spoken or written texts and mathematical equations are examples of descriptive representations. On the other hand, a depictive representation consists of iconic signs. Pictures or physical models are examples of depictive representations. Most of the prior research in student un-

[1] This analysis was published in the Proceedings of the 2017 Physics Education Research Conference [(174)]

derstanding and use of multiple representations (e.g.[88;96;89]) takes up this categorization scheme, either implicitly or explicitly. Using multiple representations plays a critical role in the effectiveness of the interactive engagement between students and instructors in learning environments[88;109]. Gestures can be a productive medium for representation, visualization, and interaction in chemistry problem solving, particularly in how students use gestures to describe molecular geometry in their own style to develop a technical language[96]. In upper-division electricity and magnetism classes, students find vector field manipulation is easier with algebraic representational approach[89], though students need to use graphical representations to interpret the difference between components and coordinates. In addition, translating between different representations may enhance students' sense-making abilities[89].

4.1.1 Classification of representations and stages of student reasoning

Alternately, we can categorize representations as to whether they are durable or evanescent. A persistent (durable) representation "leaves a trace of its production in the medium in which it was produced"[109]. Durable representations stay in place and can be easily returned to, such as diagrams, graphs, written text, and written mathematical equations. In contrast, evanescent representations are temporary and disappear when not in use, such as verbal and facial expressions, as well as physical gestures. Herein, the terms durable and evanescent are used with argument, does the used representation is persist within a given problem solving process.

We take up the language of "standing fast" to describe how students use durable and evanescent representations in their problem solving. The term "stand fast" refers to remaining firmly in the same position or keeping the same opinion. In the context of problem solving, people stand fast on "something that a person or group of people can call upon and use without hesitation or further questioning"[109]. This is when in a conversation all the participants agree on a presented idea or an explanation[110]. We relate this idea of standing fast to the process of students' reasoning while solving problems. As students progress and develop their solution and the reasoning behind it, there is a point that students no longer hesitate and the instructor or other peers can accept his/her solution or the explanation without any further discussion. We see students switch back and forth between explanations to stand fast to a representation until they find it is insufficient for a complete answer.

Fredlund[109] claims that durable representations can help students to stand fast, as students keep returning to them for sense-making while they build evanescent representations on top of these durable representations.

4.1.2 Research questions

In this study, we explore two research questions: how can problem solving be described as a process of transitioning between durable representations to make new meanings? How do students build meanings onto durable representations using the evanescent representations? To explore these questions, we present the case study of a student solving a problem during an oral exam in an upper-division Electromagnetism I course.

4.2 Context and Method

This research was carried out at Kansas State University and the data for this study were collected from an upper-division Electromagnetism I course, which had about twenty students enrolled. As at many similar institutions, our course is a textbook-centric 4 credit-hour course with a solid foundation in the basics of theoretical physics. This course covers the first seven chapters of Introduction to Electrodynamics (3rd Edition) by David J. Griffiths[46]. The class meets four hours per week and students work on tutorials and small group problem solving. In this course, students are highly encouraged to work in groups and think aloud while solving problems; class time is divided about equally among small group problem solving, interactive lecture, and problem-solving worksheets[42]. This class has a strong focus on problem-solving and sense-making, and we observe that the instructor engages and gives hints in a way that aligns with the sense-making goals.

As a part of the course assessments, students are required to complete two 20 to 30-minute individual oral exams with the instructor. These oral exams are used to assess students' conceptual understanding, problem-solving and scientific communication skills[(175)]. In this study, we analyze the case of "Larry" (a pseudonym), who works on an oral exam problem that takes place in the later part of the course. We select Larry as an exemplary case because he is unusually verbal in oral exams. Larry is a strong student whose marks are near the top of his class, and his group discussions are robust and far-ranging. Larry's approach and reasoning in the oral exam are typical to a student at this level. We select Larry as an exemplary case because he is unusually verbal in oral exams compared to his peers; we get a lot of information on his problem solving activity as a result. This then gives us lots of insight into how students at this level might solve the problem more generally.

In order to answer our research questions, we do a moment-by-moment analysis of students' videos to see how Larry builds up and translate between representations and how he stand fast.

4.3 Larry's representations

Larry is working on a problem for his oral exam in his Electromagnetism I course. During this episode, Larry starts by developing a diagram (durable representation) and spends time building meaning through gestures and words (evanescent representations). After recording the developed meaning on the diagram, Larry moves to mathematics (durable representation) in the later parts of his solution to calculate the magnitude. The episode starts with instructor posing the problem, *Suppose you had an infinite sheet which carries current k equal to some constant ($k = \alpha\hat{x}$). What's that look like? What kind of a physical scenario is that?*

To solve this problem, one can use the right-hand rule to find the direction of the magnetic field created by the sheet and to find the magnitude we could use the Ampere's law (equation 4.1).

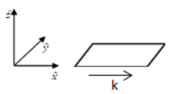
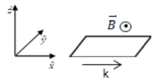
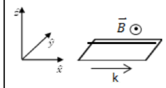
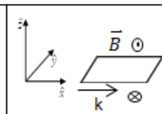
Type of Representation	Time →					
Diagram	 record	 record		 record		 record
Kinesthetic		Right hand rule sense-make	Pretend play - paper for sheet Gesture – direction of current		Right hand rule communicate	
Mathematics						
Words	"if the current is in x hat"	"above the sheet it will point out of the board from one wire "	"if this is the sheet the current is that way and the magnetic field ...." communicate		"so the current above the wire, my hand curls, points back in me"	"Below it goes in"
Column Number	**1**	**2**	**3**	**4**	**5**	**6**

Figure 4.1: *Larry works to figure out the magnetic field direction (in this figure: words in green color show Larry's purpose of each representational type and each column represents a segment of Larry's problem solving activity, that column number increases with time)*

$$\oint B \cdot \mathrm{d}l = \mu_0 I_{enc} \tag{4.1}$$

However, Larry starts by drawing the sheet of current along with a Cartesian coordinate system to represent the given current with direction, noting that "so if the current is in x hat ($\hat{x}$)..." Larry picks the mathematical information (direction of the current: $\hat{x}$) and uses it to develop a diagram (durable representation) to start with (**Figure 4.1: column 1**). After recording the sheet of current, Larry employs the right-hand rule (**Figure 4.1: column 2**) with the current on the diagram to come to a conclusion and records the direction of the magnetic field direction on the board (**Figure 4.1: column 2**).

Larry: Above the sheet it would be pointing out of the board. So I think it would be true for the rest of the sheet as well.

Larry is not provided with a diagram, so he has to come up his own to progress. He uses evanescent representation gestures to improve his diagram and to build new meaning and then records the developed meaning on his diagram. At this point, we see Larry uses the developed diagram to reason with no hesitation or further questions. He uses the improved durable representation as his first place to stand fast.

Up to this point, Larry has worked on the problem using the diagram on board, but for further clarification, the instructor brings in a different representation: a sheet of paper. The physical paper is durable; the hand gestures showing the direction of the current and verbal accompaniment are evanescent. Throughout the solution we see Larry keep referring to the same sheet of paper as a sheet of current, so we interpret the representation as a combination of the physical paper and the meanings associated with it as a sheet of current. This whole representation persists throughout the problem. According to our definition mentioned in section one, within the context of this problem we assign the sheet of paper to be a durable representation.

Instructor: Ok, So if this is the sheet, if the current is this way (uses the hand to show the direction)

Larry: Okay, so if it is the sheet, the current is going this way and looking at a point above it, then from one wire, magnetic field will point in that way.

Larry starts using the new durable representation, the sheet of paper, to figure out the direction of the magnetic field created by a single wire (**Figure 4.1: column 3**). Where Larry builds new meaning on the top of durable representation, using evanescent representations, gestures, and words.

After determining the magnetic field created by a single wire, Larry moves to consider a different reference point for a current carrying wire. To do so he goes back to his own on-board problem (his original durable representation) and draws (**Figure 4.1: column 4**) a line right in the middle of the sheet (to represent a current carrying wire).

Larry then stretches his arm on the board to show a point in the y direction, then applies the right-hand rule again (**Figure 4.1: column 5**). Here, Larry goes back to his original durable representation and improves his diagram. Then he builds the evanescent representation, gestures on top of it to determine the magnetic field created by a different current wire.

The instructor introduces the idea of a mirror current,

Instructor: What's about the one, that mirror, so that wire way far back, the mirror wire way far forwards?

Larry replies quickly that "[his] finger is pointing up", concluding that the direction of magnetic field is "out" above and "in" below and records that on the board (**Figure 4.1: column 6**).

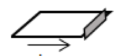
Type of Representation	Time				
Diagram			 k record		
Kinesthetic		Pretend play - paper for sheet Gesture - hand for loop			
Mathematics	$\oint \boldsymbol{B}d\boldsymbol{l} = \mu_0 I_{enc}$ record			$\oint \boldsymbol{B}d\boldsymbol{l} = \mu_0 \int k\, da$ record	$2BL = \mu_0 k \int da$ $2BL = \mu_0 k\, L$ $B = \frac{\mu_0 k}{2}$ record
Words		"I would pick a square loop sits like this" communicate			
Column Number	**7**	**8**	**9**	**10**	**11**

Figure 4.2: *Larry works to figure out the magnitude of magnetic field (in this figure: words in green color show Larry's purpose of each representational type and each column represents a segment of Larry's problem solving activity, that column number increases with time)*

Larry does lots of sense-making using the evanescent representations to develop his original durable representation. Though the sheet of paper is durable, Larry does not stand fast on it; rather he uses it to improve upon and make more sense of his diagram. So far we do not see Larry translate between different types of durable representations but Larry uses the newly introduced sheet of paper and builds the evanescent representations (gestures and words) on top of it to improve the original stand fast durable representation (diagram on board). Then he uses the developed diagram to stand fast for the second time and answer the original question about the direction of the magnetic field by the sheet of current.

After figuring out the direction of the magnetic field, the instructor asks Larry to calculate the magnitude. Because Larry could not remember the equation for the magnetic field from a single wire, the instructor suggests using Ampere's law. Larry right away starts by recording the formula (**Figure 4.2: column 7**). So far during the first section of his solution, Larry built and used his diagram for reasoning, but just the diagram is insufficient to calculate the magnitude. Larry translates his prior work to mathematics, which is a new durable representation.

The instructor gives him a hint to pick an Amperian loop, and Larry uses his hand to describe the loop (**Figure 4.2: column 8**), but he switches back to the sheet of paper to do so,

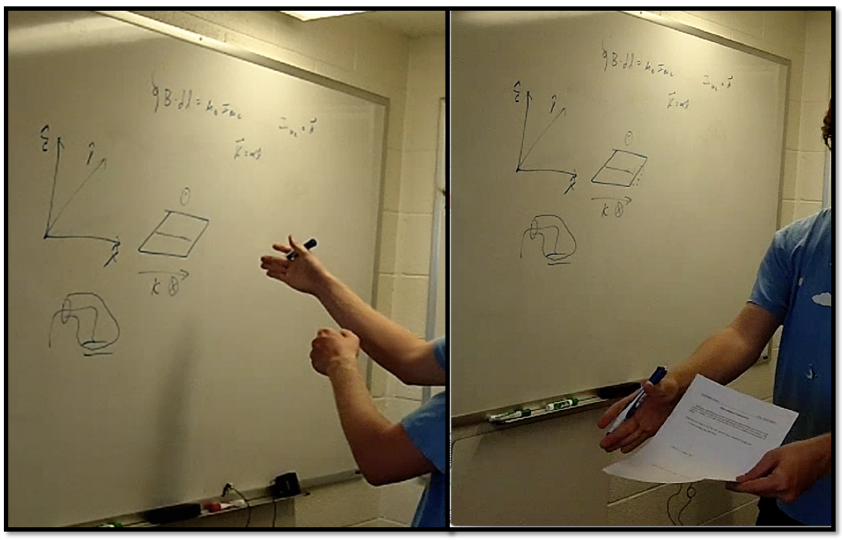


Figure 4.3: *Larry gestures on the board and uses the sheet of paper to show how he picks the Amperian loop*

Larry: Uh...yes. And I would pick a loop. Uh...I guess I would pick a square loop sits like this (again uses the sheet of paper and use fingers to show a cut at the middle of the sheet).

Larry gestures to show a loop which is parallel to the edge of the paper. Here we see Larry is using evanescent representations along with the sheet of paper to progress and build up mathematics.

The instructor suggests that he draw the Amperian loop on the board, asking "So how big is your loop? Draw your loop." Even though Larry gestures to make the size and orientation of his chosen loop successfully (Figure 4.3), he is not sure how to draw it on his diagram.

To figure out how to draw it, Larry uses gestures and even comes back to the sheet of paper to help him with drawing. After a while, Larry records the loop on the diagram (**Figure 4.2: column 9**) and labels the sides of the loop with l and w. Here we see Larry is referring back to the sheet of paper and improves his diagram (durable representation) using gestures and words (evanescent representations) to finish the mathematical calculation for the magnitude of the magnetic field.

Once Larry draws the loop, the instructor asks a question about the current piercing the loop. Larry answers those questions by developing the mathematics on the board (**Figure 4.2: column 10**) and reasons that for the right hand side of the integral (equation 4.1).

$$\oint B \cdot \mathrm{d}l = \mu_0 \int k \cdot \mathrm{d}a \tag{4.2}$$

Communication with the instructor leads Larry to picks a direction for his loop and records it as counter-clockwise. After this step, Larry takes a little time to compute the right side. With a little help from the instructor, Larry reaches the final answer (**Figure 4.2: column 11**). Here Larry stands fast on the mathematics to answer about the magnitude of the magnetic field.

$$2B\,L = \mu_0\,k\,L \tag{4.3}$$

$$B = \mu_0\,\alpha/2 \qquad (with\;\; \alpha = k) \tag{4.4}$$

During the second half of this problem, Larry translates from diagram to mathematics. He uses the durable representation, sheet of paper and evanescent representations, gestures, and words to amend mathematics. After developing the mathematics representation, he uses it to stand fast in the later part of his solution.

4.4 Limitations

The case study approach enables researchers to perform in-depth investigations of the behavior of single participants or group of individuals. As a research method case studies are often used in exploratory research projects. Because of the exploratory and detailed nature of this approach, case studies help us to generate new ideas about human thinking and behavior. The study we conducted in this chapter is a pilot study. We choose the case study method as it makes it easy to analyze the behavior of a single person than dealing with a large number of research participants.

One of the main limitations in the case study approach is the influence that may cause due to the researchers own subjective feeling. The example we picked better suited the theory and our claim, but within this episode. We select Larry as an exemplary case because he is unusually verbal in oral exams compared to his peers. But the way Larry engaged in this problem allows us to get lots of insight into how students at this level might solve the problem more generally.

In this study, we used interviews data which is an effective method for obtaining information about the student thinking. Within the interview, there are some limitations to the instructor's approach while engaging with Larry. She provides hints to Larry in a certain way that aligns with other practices used in the classroom. But we think this interaction does not importantly change the way Larry translate between durable representations to solve this problem. In addition to that, We were not paying attention to power dynamics between the student and the instructor. We only looked at how the student communicates with the instructor but we were not paying attention to the two-way nature of that communication.

Further research could involve interviews or classroom observations to further explore how students construct spontaneous representations and also make connections among these representations.

4.5 Discussion and Conclusion

The case of Larry shows a good example of how students translate between durable representations, with the help of evanescent representations. In summary, we see Larry starts using one kind of durable representation (diagram), then translates to a different kind of durable representation (mathematics) using the evanescent representations (gestures and words) to reason better and answer the original problem. Larry stands fast on the first durable representation (diagram), which he builds in the beginning. Later in his solution, Larry stands fast on mathematics.

This particular problem required the student to start from the diagram and move to mathematics, so is a great case to investigate how a student may transition between two durable standing fast representations. There exist other, simpler problems which may only require one representation to stand fast upon; conversely, there are more complicated, research-like problems which may require a long series of representations on which to stand fast.

We have evidence for Larry building new meaning on durable representations using evanescent representations. Larry brings in evanescent representations to amend his first durable representation: diagram because it was not sufficient for further reasoning. Larry spends most of his time building his durable representation: the diagram where he stands fast before translating to other durable representation: mathematics when calculating the magnitude of the magnetic field. Larry uses the sheet of paper along with gestures and words (evanescent representations) to build up mathematics and also to bridge the two durable representations.

Translating between different classes of representations might improve student understanding and how the combinations of representations could be effective for visualization and interaction in other disciplines(96). As we think about classifying representations, our choice of categorization scheme impacts what translating between representations entails. Building meaning on durable representations using evanescent ones can be fruitful for problem solving; choosing it as a categorization scheme focuses our efforts on the flow of representations in problem solving and how students stand fast.

Chapter 5

Case Study: Coordinating Among Multiple Semiotic Resources to Solve Complex Physics Problems

5.1 Introduction

[1] In the science education literature, 'representations' refer to many ways in which students can communicate ideas, concepts, processes, and relationships. Sketches, diagrams, pictures, graphs, tables and mathematical equations are some of the commonly used representations in undergraduate physics courses (88;89;90;91;92;93;94;95;96). In addition to use of these artifacts, students engage with peers and instructors using physical objects ranging from pen and pencil to experimental apparatus and bodily actions, such as taking measurements, peer interactions and class discussions. These objects and actions play an important role in student meaning-making practices (176;177;178;179) and we interpret them as part of the representational landscape of physics problem solving.

Work by Airey & Linder (180) brings new insight to student meaning-making and broadens the boundaries in physics as a discipline. They use a social semiotic perspective and

[1] This analysis was published in European Journal of Physics (170)

describe student meaning-making as the use of semiotic resources to realize and communicate physics knowledge. By definition[32], semiotic resources are "the actions, materials, and artifacts used for communication purposes". The social semiotic approach focuses on all types of meaning-making practices that are accomplished through different semiotic modes that include visual, verbal (or aural), written and gestural modes. In this sense, the symbolic tools such as language, text, algebra, diagrams, sketches, graphs, body movements, signs, and gestures are semiotic resources. Even though words or the verbal language act as the key method of conveying ideas while constructing and sharing knowledge[181;182], use of gestures[183;184;96] is also useful in the process of meaning-making. Extending beyond words or gestures, semiotic resources such as diagrams, sketches, graphs, written text, and mathematical formulas also serve a critical role in visualizing physical concepts and mathematical relationships[185;186;187;188].

While researchers highlight the importance of building mental models, schema and organizing strategies in the development of student problem solving capabilities[26;27;31;30;189], much remains to be studied about how students can engage in successful problem-solving practices through the use of all available artifacts, objects, and actions[190]. In this study, we are interested in student meaning-making while solving physics problems using multiple semiotic resources.

5.2 Background

Even though semiotic resources play important roles while planning, executing and communicating solutions, these individual semiotic resources are incomplete just by themselves. Therefore, using a single semiotic resource to understand or to present a given situation does not illustrate a robust understanding[154]. Each semiotic resource offers a unique capability to create meaning and researchers interpret this as the *meaning potential*[32] and the *affordances* of semiotic resources. The meaning potential refers to different ways that the meaning is constructed and communicated using semiotic resources and is based on the way the person has used this resource in the past. The affordances constrain what ideas are possible to express using each semiotic resource[168;32], as perceived by the person[169] using the resource. Kress[153] expanded the idea of semiotic resources to include modal affordances, which constrain different modes of semiotic resources (e.g. verbal resources, gestural resources, etc), noting that there may be two or more communication modes within the same artifact (e.g. a diagram may be married to a verbal description). Recently, Fredlund[130] expanded the idea of affordances further. The *disciplinary affordance* of a given semiotic resource is "the inherent potential of that [semiotic resource] to provide access to disciplinary knowledge." This expansion allows researchers to investigate how semiotic resources' affordances connect to disciplinary ideas.

Together with modal affordances, disciplinary affordances allow us to connect ideas in physics with the kinds of representations that best express them. As an example, spoken language is better for certain tasks and diagrams are better for other tasks; different semiotic resources access and fabricate different aspects of physics knowledge. Taking up disciplinary affordances allows us to focus on knowledge production and communication within the discipline (here, physics) more than focusing on the view or the experience of an individual student.

Among the research done on the disciplinary affordances of different semiotic resources, Fredlund[130] uses two versions of basic RC-circuit diagrams to show the importance of unpacking the disciplinary affordance of semiotic resources for effective learning in student laboratories. First, the students were given a circuit diagram that can be connected in eight possible ways, but only one way of connecting correctly. It was difficult for students to connect the circuit and to get an appropriate output. To help students, researchers introduced a modified circuit diagram that shows the positions for connecting the signal input and the ground cable with color-coded dots. The modified circuit diagram was a semiotic resource with different disciplinary affordances than the original circuit diagram. These new disciplinary affordances foregrounded disciplinary relevant aspects, helping students make better connections among different semiotic resources and allowing the students to make meaning of the circuit.

In another attempt to visualize the effect of disciplinary affordances of semiotic resources, Fredlund[135] investigated a group of third-year physics undergraduates who are selecting among semiotic resources as they describe the refraction of light. During the task, students produced two semiotic resources: a ray diagram and a wavefront diagram. The diagrams have different potential to provide access to different aspects of disciplinary knowledge. The ray diagram could help students to reason about the refraction angles at the boundary and also about the direction of propagation, but it could not help students reason about speed changes in the two media. In contrast, the wavefront diagram promoted reasoning about speed changes but obscured reasoning about angles and directionality.

In both of these studies[130;135], Fredlund, et al. provided students with two different semiotic resources in the same semiotic mode, showing that different semiotic resources have different disciplinary affordances. In addition to the use of individual semiotic resources or selecting among two semiotic resources in student meaning making practices, research in science classrooms[93;156;130;191;96] highlights the importance of using several semiotic resources to mediate classroom interactions[192;193]. Additionally, getting students to use multiple semiotic resources helps shift their focus towards understanding the scientific processes and concepts[132;118].

5.3 The Study

As one of the most frequently utilized procedures in physics problem-solving, multiple semiotic resources are very useful in translating the initial, mostly verbal description of a problem into a representation more suitable for further analysis and mathematical manipulations. On the other hand, the ability to construct representations plays an important role in helping students to make physics knowledge and communicate[155]. The process of constructing a representation of a problem makes it easier for the problem solver to make appropriate decisions about the solution process.

We argue that to better represent an idea or a concept, students should be able to strategically combine multiple semiotic resources. Expanding on this idea, we adopt a social semiotic perspective to sketch a theoretical framework (Figure 5.1) that accounts for how

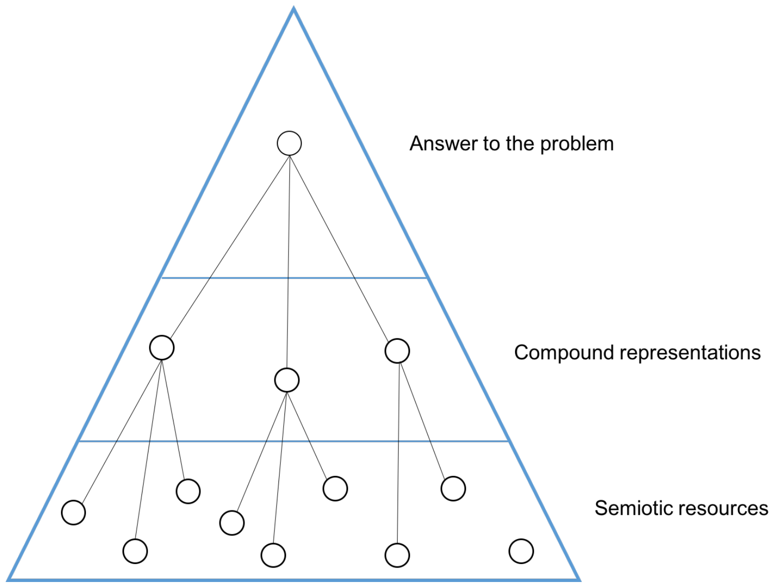


Figure 5.1: *Theoretical framework: Each semiotic resource has a semiotic mode and connects conceptual information that contributes to meaning-making. These semiotic resources are coordinated to build compound representations. Then the compound representations are used to solve the problems (in this figure: circles represent elements in each level. As an example, in lowest level, each circle represents a semiotic resource. Then the lines represent the coordination between elements in one level to generate another element which is in the next level. As an example, semiotic resources in the lowest level are coordinated to build a compound representation, which is in the next level). In this framework, the triangle is the whole solution of the problem, including reasoning. Solution to the problem is thus "made up of" all the lower parts of this framework.*

semiotic resources should be combined to solve problems. In our work, we take up the idea of different semiotic resources having different disciplinary affordances. Under this framing, the process of making meaning is about coordinating different semiotic resources with different disciplinary affordances across multiple semiotic modes. By coordination, we mean how the disciplinary affordances of each semiotic resource reinforce each other or hinder the use of other semiotic resources towards building representations. We introduce the idea of compound representations, composed of two or more parts or semiotic modes and linking at least two semiotic resources in the same representation. Then these compound representations can be used to solve problems.

In this study, we are interested in how students coordinate among multiple semiotic resources. In particular, we consider how students *construct* compound representations using semiotic resources, not how they select between researcher-provided representations. In this sense, our work extends Fredlund et al's prior work to be closer to authentic classroom practice and student problem solving.

We explore the research question: How do students coordinate among different semiotic resources to build up compound representations while solving complex physics problems?

5.4 Context and Method

5.4.1 Context

Data for this study is from an upper- division Electromagnetism I course. In this analysis, we present the case of Larry[1], a white male student, who engages in problem solving for his course requirements. Larry is the same student that we present in the study in chapter 4.

5.4.2 Physics problem

We selected a problem which is canonical at the upper-division level: *Suppose you had an infinite sheet which carries current k equal to some constant (records on the board, $k = \alpha\hat{x}$). What's the magnetic field look like?*. First, Larry determines the direction of the magnetic field created by the current sheet, then the instructor asks Larry to find the magnitude of the magnetic field,how big is it?. To solve this problem, one can use the right hand rule to find the direction of the magnetic field created by the sheet, and use Ampere's law ($\oint B \cdot \mathrm{d}l = \mu_0 I_{enc}$) to find the magnitude.

5.4.3 Method

This study has three stages of analysis. Our analysis is mainly driven by the discourse analysis. Discourse analysis is defined as the analysis of language that goes beyond the sentences. Discourse analysis requires to consider the larger discourse context in order to understand

[1]All student names are pseudonyms

the whole picture. Dividing the whole activity into segments of events or reframing is the way of re-interpreting the meaning of sentences. Then the frame analysis is a type of discourse analysis that takes into account, what activity does speaker perform when he says something.

The first stage involves transcribing Larry's oral exam and dividing the problem-solving activity into segments of events (frames). From the transcript and the video of the oral exam, each frame is described without explicitly mentioning the resources (semiotic or conceptual) to generate a narrative description of the event. Within each event, we identify key elements using semiotic modes: different types of inscriptions (diagrams, mathematical formulas); extra-linguistic modes of expression (gestures); words (oral); and objects (used by Larry and the instructor).

For the second stage of analysis, we classify semiotic and conceptual resources within each semiotic mode. First, we identify semiotic resources within inscription, verbal, and gestural modes, naming each resource descriptively. Conceptual resources are identified using published guidelines(163), as augmented by work in semiotic resources(130) and procedural resources(194).

Resources are named using descriptive names for the things resources represent(38). For this study, we used this idea to name resources and link them with their semiotic modes. For example, in the process of solving the problem, when the student uses a right-hand rule gesture (connecting the directions of the inputs of the cross product with the fingers on their right-hand, and making a gripping gesture to perform the cross product), we coded it as the semiotic resource (gestural) *right-hand grip rule*. Similarly, we identify mathematical resources by naming them - e.g. *integral form for Ampere's law* - for the formulas or mathematical actions they describe. Notationally, we denote resource names in *italics*.

Finally, we make a list of conceptual and semiotic resources. The third stage of analysis involves comparing the resources' frequency and connections across episodes and connecting the semiotic resources to their disciplinary affordances (Tables 5.1, 5.2 and 5.3).

We present Larry's oral exam using three episodes. We chose each episode with our research focus to show how the coordination between resources can be used to build different stages of the compound representation. We use the resource graph representation to show Larry's coordination of resources; and in our resource graphs, each circle represents either a semiotic or a conceptual resource. Within Larry's problem solving activity, we also look at the extent of his confidence using the voice level, choice of words and speed of hand gestures.

We adopt a case study methodology because it allows us to gain an understanding of the subject (Larry), events and processes that are involved in the data through a detailed analysis. Three researchers worked together in the process to establish the reliability of coding and to list the disciplinary affordances of each semiotic resource that Larry uses.

In our previous study, presented in the 4th chapter, we focused on how Larry coordinate among different types of representations to solve the Amperian loop problem. Then, in this current study, we investigate how Larry bring different elements (semiotic resources) together to build up representations.

5.5 Analysis

In this section, we present Larry's oral exam using three episodes and we use the resource graph representation to show Larry's coordination of resources to build different stages of the compound representations. During the first half of his oral exam, Larry works to determine the direction of the magnetic field created by the current sheet and then continues to find the magnitude.

5.5.1 Episode 1

The episode starts with the instructor presenting the problem.

Instructor: Suppose you had an infinite sheet which carries current k equal to some constant (records on the board, $k = \alpha\hat{x}$). What's the magnetic field look like?

The problem statement activates the conceptual resources *current sheet* and *magnetic field direction*. This leads Larry to represent the current sheet on the board (Figure 5.2.a) using the semiotic resource *parallelogram as current sheet*. After visually representing the current sheet, Larry looks at the mathematical equation ($k = \alpha\hat{x}$) on the board. The directional information ($\hat{x}$) embodied in the equation prompts Larry to think of a way to represent this detail. He uses the visual semiotic resource *3D vector coordinates* to add the directional information on to his diagram and draws three arrows with their tails together, labeling them $\hat{x}$, $\hat{y}$, and $\hat{z}$.

Larry: So the coordinate . . . x hat, y hat, z hat, xyz I mean.

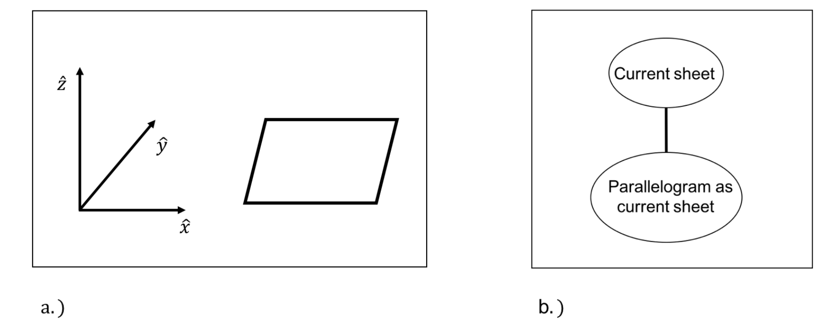


Figure 5.2: *a.) Initial compound representation b.) Resource graph for the initial compound representation*

Figure 5.2.a shows the initial compound representation which has 3D vector coordinates and a parallelogram representing the current sheet. Figure 5.2.b shows the resource graph, the resources (*current sheet* and *parallelogram as current sheet*) that are connected to produce the initial compound representation. In our representation of resource graphs, each circle represents either a semiotic or a conceptual resource. After generating the initial compound representation, Larry moves to describe how he imagines the current sheet is built up from a collection of wires. Larry has the idea of focusing on a single wire to find the magnetic field that can represent the effect of the whole current sheet (*part of a whole*).

Larry: Um...m okay, so think of this, a sheet is kind of having a bunch of infinite wires (up and down movement of hand) one next to each other and the current from a single wire curls around this (applies right-hand grip rule).

Larry starts using the gestural semiotic resource *hand for wires* in free space to show adjacent current wires. Then he focuses on a single wire and applies the gestural semiotic resource *right-hand grip rule* to figure out and then to demonstrate the *magnetic field direction* from a single *current wire*. While applying the right-hand grip rule, Larry does not indicate a specific current direction (whether or not he uses the given current direction). Soon after, Larry decides to use the diagram on the board (Figure 5.2.a) to continue with this argument and concentrates on a point above the current sheet by using the semiotic resource *pinpointing gesture*.

Larry: So, I think that like if you look at a point above it (pinpoints to a location), then uh .. from a single wire .... will pointing ...

After pinpointing to a location above the sheet, Larry considers an imaginary wire to apply the *right-hand grip rule*. He applies the *right-hand grip rule* for the second time without specifying the current direction and on this occasion, we observe Larry gets stuck. Instead of making a clear conclusion, he ends up repeatedly changing the orientation of his right-hand gesture.

In order to apply the right-hand grip rule, Larry has to have a certain current direction; but when he applies the right-hand grip rule in free space, he should not have to specifically mention the current direction. Because at that point the semiotic resource *hand for wires* allowed Larry to show the existence of a current wire in space, the right-hand grip rule helps him to get the *magnetic field direction* using an arbitrary current direction. When Larry moves to build his argument using the diagram on the board, the diagram itself contains 3D vector coordinates that affords to define the direction in space. So, unlike using free space, Larry has to specify the current direction before applying the right-hand rule. The missing detail of specific current direction leads Larry to decide between the orientations of his gesture. Here the semiotic resource *3D vector coordinates* hinders the use of gestural semiotic resource *right-hand grip rule* without a specific current direction.

Finally, Larry decides to add the current direction information to his diagram and uses the semiotic resources *arrow as vector* to represent the current direction.

Larry: So, if the current is in x hat (records an arrow on diagram)

3D vector coordinates on the compound representation (Figure 5.2.a) allows Larry to show the direction in space (Table 5.1). That also permits representing the given current direction along the x-axis (Figure 5.3.a) using the semiotic resource *arrow as vector* along with the mathematical symbol (k).

Larry: Then above the sheet it would be pointing out of the board (records on diagram), from one wire, so above one wire. So, I think it would be true for the rest of the sheet as well.

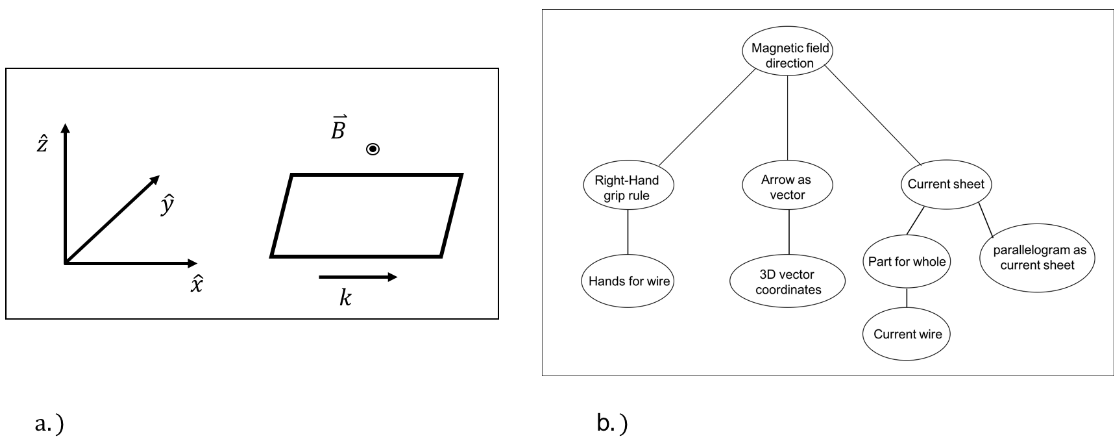


Figure 5.3: *a.) Compound representation at the end of episode 1 b.) Resource graph for this compound representation*

After recording the current direction, the rest is straightforward for Larry. He repeats the same procedure and focuses on a single wire above sheet to apply *right-hand grip rule* once more. Then he uses the semiotic resource *arrow as vector* to record the resulting *magnetic field direction* on the diagram (Figure 5.3.a). Finally, Larry refers back to his initial assumption (*part of a whole*) and concludes “I think it would be true for the rest of the sheet as well.”

Figure 5.3.a shows the compound representation at the end of first episode, which has 3D vector coordinates, a parallelogram to represent the current sheet, arrows to represent

current direction and the net magnetic field direction above the current sheet. Figure 5.3.b shows the resources that are connected to produce the compound representation in Figure 5.3.a. We can see Larry brings in and combines more resources as he progresses with the problem at hand. At the beginning of this episode, Larry does not have a diagram to start with; but after few steps, Larry builds up a compound representation that contains 3D vector coordinates and a parallelogram for the current sheet. Then he adds an arrow to represent the current direction with the help of 3D vector coordinates. Larry builds up his compound representation to include more details in it. By the end of this episode, Larry adds the magnetic field direction onto his diagram by thinking of a single wire and applies the right-hand grip rule to figure out the direction.

5.5.2 Episode 2

After finding the *magnetic field direction* above the current sheet, Larry could have continued with the same argument using the diagram on board to figure out the direction below the sheet. However, the instructor introduces a sheet of paper to represent the current sheet. Then Larry switches to the sheet of paper and successfully reconstructs the magnetic field direction above sheet from a single current wire.

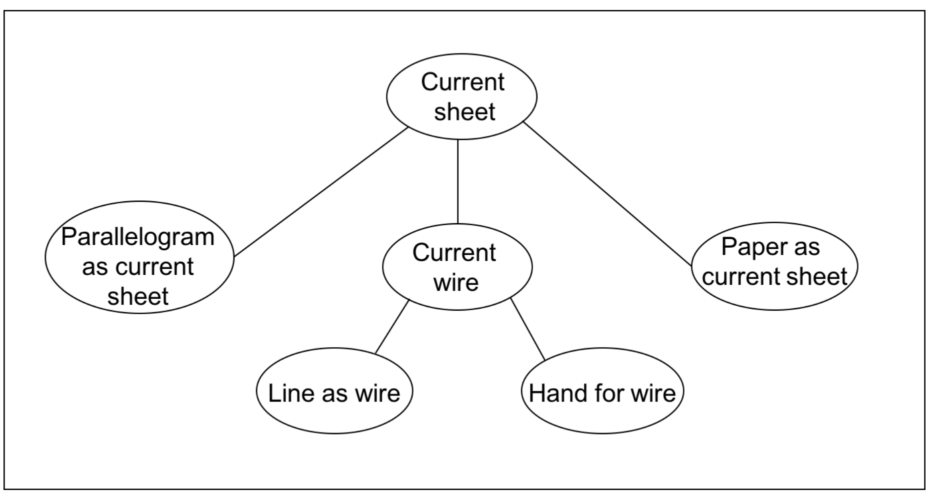


Figure 5.4: *The compound representation of the current sheet is built and represented through the gesture, sheet of paper and a drawing of a parallelogram on the board with a line to represent the current wire*

The current sheet (Figure 5.4) is represented using a parallelogram with a line to represent the current wire, a sheet of paper and some hand gestures. In this compound representation, the parallelogram and the sheet of paper (Table 5.1) allow Larry to generate visual representations of the current sheet. The inclusion of hand gestures helps to simplify the structure of the current sheet and the inclusion of the line to represent current wire further develops the idea into a visual representation.

Table 5.1: *Disciplinary affordances of the semiotic resources coordinated to build the compound representation of the current sheet*

Semiotic Resource	Disciplinary Affordance
Parallelogram as current sheet	This allows Larry to generate a visual representation of the current sheet in the space of the board. Larry keeps referring back to it as he progresses on this task and it allows Larry to add features (current direction, current wire, Amperian loop) and findings (magnetic field direction) on to his compound representation. These recording steps also help Larry to keep track of his problem solving activities.
Paper as current sheet	The paper allows Larry to generate a visual representation of the current sheet in the free space. Larry refers to this while considering multiple current wires in free space and also while gesturing for the loop in episode 3.
Hands for wire	Larry repeats the up and down movement of his hand to show the current wires in free space. This gesture allows Larry to simplify the sheet to a bunch of wires and focus on a single current wire to apply the right-hand rule to get the magnetic field from a single current carrying wire.
Line as wire	This helps Larry to visually represent the current wire in the space of the sheet in the diagram (on the board) and allows him to locate the wire on the sheet. It also helps Larry to keep track of his reference current wires (with embodied current direction) while applying the right-hand grip rule, which leads Larry's effort to a successful conclusion.

Larry: So, if this is the sheet (refers to the sheet of paper), the current is going this way (across the surface of sheet- right to left), and looking at a point above it, then from one wire, magnetic field will be point in that way (away from him).

Larry first uses the semiotic resource *paper as current sheet* to represent the current sheet in free space. Then he uses the gestural semiotic resource *finger pointing in direction* to specify the current direction and directs his index finger along the current direction. Finally, he applies the *right-hand grip rule* by considering an imaginary current wire that is located above the sheet. While using the sheet of paper to figure out the *magnetic field direction* above the sheet, we can observe Larry's confidence as he reasons smoothly and his voice stays around the same level. This may be because this step is all about repeating the same procedure, even though now he is using a different semiotic resource.

Next, Larry decides to take his argument to the next level and starts to reason for the net magnetic field direction above the current sheet. Larry moves to consider multiple current wires and he starts by locating two imaginary current wires using the semiotic resource *pinpointing gesture*: one closer to the surface of the sheet and the other little above the surface of the sheet. Then he applies the *right-hand grip rule* by considering one wire at a time and soon we observe Larry gets stuck.

Larry: So like from this wire (above the surface of sheet), the current is gonna ... the magnetic field points that way (away from him), but from a wire over here (closer to the surface of sheet), it would be pointing more ..uh .. this way (pointing towards him), uh .. so it seems like they are gonna superposition, kind of complicated.

After applying the *right-hand grip rule* for the second wire, Larry realizes that the resulting magnetic field directions are in opposite directions, which contradicts his previous conclusion ("I think it would be true for the rest of the sheet as well"). Previously he concluded that above the current sheet, the *magnetic field direction* from all the individual current wires should be in the same direction. Larry's explanation about the net effect reaches a dead end ("uh ... so it seems like it's gonna superposition, kind of, complicated.") At this point, Larry's voice level indicates a confusion. We can observe Larry's strong voice starts to fade after considering the second wire, his pauses between words increase and the speed of his hand movements decreases. In addition, his words "it would be **pointing more** ... uh ... this way ..... **kind of complicated**" show his uncertainty about the result.

As a result of this confusion, Larry returns to the diagram on the board to consider multiple current wires. First, he uses the semiotic resource *line as wire* (Figure 5.6.a) to add a current wire on to his diagram.

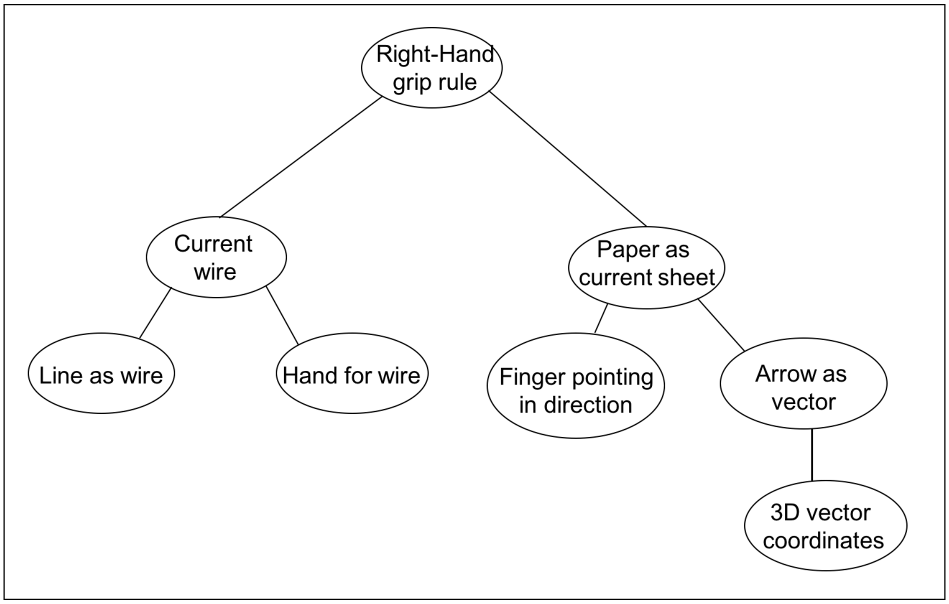


Figure 5.5: *The compound representation of the right-hand grip rule is developed using gesture and drawing of a line to represent the current wire with an arrow to represent given current direction and is applied using the right-hand gesture*

The compound representation of right-hand grip rule (Figure 5.5) allows him to figure out the resulting direction of the magnetic field lines from a current carrying wire. In some occasions, gesture (Table 5.2) allows Larry to specify the current direction before applying the right-hand grip rule. Later 3D vector coordinates (Table 5.2) allows Larry to represent the given current direction (using an arrow), and the line to represent current wire allows Larry to apply right-hand grip rule to find the magnetic field direction.

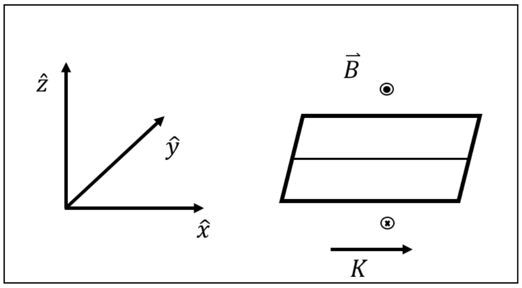
Larry: If we are looking at a wire right here (draws a line at middle of the sheet), and then, so the current above the wire (applies right-hand grip rule) my hand curls, points back in me. Uh ... if we look at a wire, like this is an infinite sheet so way back here in the y direction, uh then my fingers at that point are more pointing in down than they are towards me.

Larry considers the current wire he just recorded on diagram to apply the *right-hand grip*

Table 5.2: *Disciplinary affordances of the semiotic resources coordinated to build the compound representation of the right-hand grip rule*

Semiotic Resource	Disciplinary Affordance
3D vector coordinates	A coordinate system enables Larry to define the location of a point, distance between points, and direction in space (on a planar surface). This allows Larry to represent the given current direction (using an arrow) that he uses to find the magnetic field direction.
Arrow as vector	Arrow as vector permits Larry to visually represent the vector direction. As 3D vector coordinates permit showing the direction in space, using an arrow to represent a vector allows Larry to represent the given current direction along the x-axis. In this situation, visually representing the current direction helps Larry to apply the right-hand rule towards determining the magnetic field direction. Later, the same semiotic resource allows Larry to visualize the resulting magnetic field directions.
Finger pointing in direction	Larry uses the index finger to the side while gesturing for direction. This allows him to show the direction of the current on the surface of paper and also in the space of the sheet in the diagram (on the board) that helps to apply right-hand grip rule.

rule and successfully reproduces the same result. Then Larry considers a second current wire located far back in y direction and applies the *right-hand grip rule.* Switching to the diagram on the board and recording a line to represent the current wire allows Larry to consider two current wires when drawing a conclusion about the net *magnetic field direction* above sheet. The semiotic resource *line as wire* helps Larry to explicitly indicate the location of the current wire on the diagram. In addition, this inclusion allows Larry to incorporate and specify the current direction (which is already recorded on diagram) while applying the right-hand grip rule. When Larry switches to the diagram on the board, the semiotic resources *line as wire* and *arrow as vector* allow Larry to keep track of his reference current wires along with the current direction. But while using the sheet of paper to consider multiple wires, Larry works in free space using the gestural semiotic resources *pinpointing gesture* and *finger pointing in direction* to locate imaginary current wires and to show the current direction. While working in free space, in addition to keep tracking of the wire he is considering, Larry has



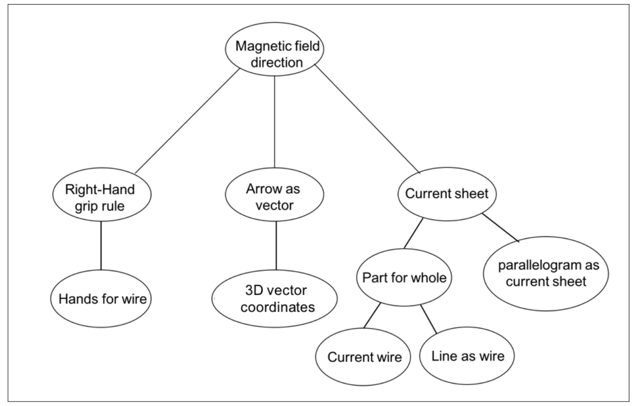


Figure 5.6: *a.) Compound representation at the end of episode 2 b.) Resource graph for this compound representation*

to remember more than one direction at a time (current and resulting magnetic fields), this leads Larry's effort to a conflict.

The instructor's follow-up question leads Larry to consider another wire located far forward in y direction (*mirror current wire*) to make a conclusion about the net effect above the sheet.

Instructor: What about one, that mirror, so that wire way far back, the mirror wire way far forwards?

After reapplying the *right-hand grip rule*, Larry reasons for the net direction above the current sheet He says "Uh, my finger is pointing up". This result aligns with his earlier conclusion. Then the instructor suggests that Larry consider below the sheet as well, asking "So, it came out above and below it goes?" Larry continues with the same argument and reapplies the *right-hand grip rule*. Then he concludes that "so below they go in" and uses the semiotic resource *arrow as vector* to record the magnetic field direction below the sheet on diagram (Figure 5.6.a).

At the beginning of episode 2, the compound representation (Figure 5.3.a) has 3D vector coordinates, a parallelogram to represent the current sheet, arrows representing the magnetic field direction above the sheet and the current direction. Now by the end of episode 2, Figure

5.6.a has all the required information about *magnetic field direction* above and below the current sheet. Larry has added a line to represent a current wire and an arrow to represent net magnetic field direction below the current sheet. Figure 5.6.b shows the resources that are connected to produce this compound representation and we see Larry continues with the same combination of resources as he is still looking for the magnetic field direction. As the only modification, he adds the *line as wire* to represent the current wire and further builds up his compound representation to include magnetic field direction below the current sheet.

5.5.3 Episode 3

Larry completes the first task by finding the magnetic field direction and now he has to figure out the magnitude of the magnetic field. To start the process, Larry records the semiotic resource of mathematical formula for the *integral form of the Ampere's law* ($\oint B \cdot \mathrm{d}l = \mu_0 I_{enc}$). This mathematical formula visualizes the relationships between the integrated magnetic field around a closed loop and the electric current passing through the loop. In order to continue with the mathematical manipulations, Larry has to pick an Amperian loop, pick the dimensions of his loop and then pick a direction for his loop. Instead, Larry does this as a two-step process. First he gestures for the loop orientation and tries to advance with mathematical manipulations. Later as he gets stuck, he picks the dimensions and a direction for his loop.

After recording the mathematical formula for the Ampere's law, the instructor suggests Larry pick a loop,

Instructor: Cool, now you need to pick an Amperian loop.

Larry: I'll pick a loop. Current is in this way (pointing on the surface of paper). I think the loop is like this (hand shows the loop perpendicular to the edge of the paper).

Within this step Larry combines several semiotic resources. First, he returns to the semiotic resource *paper as current sheet* and uses it along with the gestural semiotic resources *finger pointing in direction* and *hand for loop* to show how he picks the loop. The semiotic resource *paper as current sheet* allows him to represent the existence of the current sheet in free space, *finger pointing in direction* helps to indicate the current direction. Then the semiotic resource *hand for loop* helps to visualize the orientation of the loop with respect to the sheet of current. But it does not help Larry figure out the vector orientation between the magnetic field and the unit length on the Amperian loop that is required to manipulate the integral in the Ampere's law equation. Later we observe this limitation of the semiotic resource *hand for loop*, leads Larry to get stuck and then Larry uses another semiotic resource to represent the orientation of loop.

Larry's intention is to continue with the mathematical manipulation, so after gesturing for the loop, he moves to figure out the *current enclosed* (current flowing through the Amperian loop).

Larry: My current enclosed is gonna be ... said got (records equation) ($k = \alpha\hat{x}$), so my current is just, my current enclosed is just gonna be uh ... k dl right (records $I_{enc} = k...$). But it can't just be k times dl cause that's a vector and it not be a vector. I mean two end up is not a vector. But, could it be k dot dl, you like the sound of that?

$$k = \alpha\hat{x} \tag{5.1}$$

$$I_{enc} = k... \tag{5.2}$$

Although he is ultimately unsuccessful, Larry expresses certainty that he could use a mathematical formula to get the current flowing through the loop. So far, Larry has not picked dimensions for his Amperian loop, instead he just uses the hand to show the orientation of the loop. This limitation prevents his moving forward with the mathematical manipulation. As Larry gets stuck, we observe Larry's voice keeps fading and he pauses between words more than usual. Finally, he asks for the instructor's feedback to move forward: "could it be .... you like the sound of that?". Instead of answering Larry's question, the instructor suggests he add the loop to his diagram, asking him "So how big is your loop?

Draw your loop." After the instructor's suggestion, Larry uses the semiotic resource *square as loop* to add a loop on to his diagram (Figure 5.8.a).

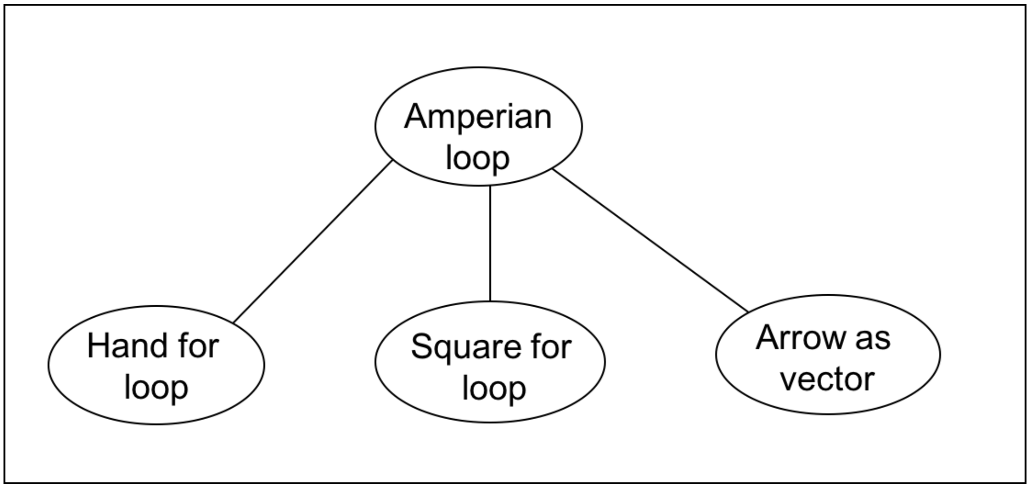


Figure 5.7: *The compound representation of the Amperian loop is built and represented through the gesture and a drawing of a square on the board with an arrow to represent the direction on the loop*

In the compound representation of the Amperian loop (Figure 5.7) gesturing and drawing (square) on the board (Table 5.4) allows Larry to visually represent the existence of the loop with respect to the orientation of the current sheet. Then the inclusion of the direction on the loop helps to figure out the relative orientation with the magnetic field direction.

Before moving forward, Larry decides to explicate the orientation of the loop because he doubts his drawing ability. To gesture the orientation of the loop, he uses the semiotic resources *paper as current sheet*, *finger pointing in direction* and *hand for loop*. Next, the instructor reminds Larry to pick the dimensions for his loop.

Instructor: Okay, good. And how wide is your loop? And what is the other dimension?

Larry labels the loop dimensions as "l" and "w" (Figure 5.5.a) and this step allows Larry to figure out the current flowing through the loop that he could not do earlier. He records the semiotic resource of mathematical formula for *current enclosed*, $I_{enc} = kl$ and substitutes the given current information ($k = \alpha\hat{x}$) for current density $I_{enc} = kl = l\alpha\hat{x}$. Even though Larry correctly reasons that "it (I_{enc}) not be a vector. I mean two end up is not a vector",

Table 5.3: *Disciplinary affordances of the semiotic resources coordinated to build the compound representation of the Amperian loop*

Semiotic Resource	Disciplinary Affordance
Hand for loop	Larry uses the hand to represent the location and orientation of the loop with respect to the orientation of the paper sheet. Even after recording the loop on the diagram, Larry reuses the hand for loop to best communicate his idea to the instructor.
Square for loop	This helps to generate a visual representation of the loop in the space of the board (diagram on the board) and allows Larry to show the orientation of the loop with respect to the orientation of the current sheet (parallelogram for sheet). This recording step helps Larry to continue with the mathematical manipulations as he labels the loop dimensions and to figure out the current enclosed in the Amperian loop.
Arrow as vector	Arrow as vector permits Larry to visually represent the given current direction and the resulting magnetic field directions. While Larry is trying to simplify the left-hand side of the integral form of the Ampere's law equation during the latter part of episode 3, either gesturing for the loop or just recording the loop does not help Larry to figure out the relative orientation between the magnetic field direction and the unit length on the Amperian loop. Then the use of an arrow to represent the direction on the loop helps Larry to build a complete argument and advance with the mathematical manipulation.

his mathematical formula for current through the loop still has the vector information ($\hat{x}$). Later, Larry is going to realize that he does not need to have $\hat{x}$ anymore because he does the dot product.

$$I_{enc} = kl \tag{5.3}$$

$$I_{enc} = kl = l\alpha\hat{x} \tag{5.4}$$

After figuring out the current through the loop (*current enclosed*), Larry moves towards simplifying the left-hand side of the Ampere's law formula. First, he records the semiotic resource of mathematical formula $\oint B \cdot \mathrm{d}l = \mu_0 \int k \cdot \mathrm{d}a$.

Larry: So (records on the board) integral B dot dl equals μ_0 integral k dot da. And then ... for this (left hand side) I can do four separated integrals right? Since its square (show by hand) and these two ("w" legs on loop) do not end up mattering.

Larry calls out loud the names of the mathematical symbols while recording the formula and then uses the gestural semiotic resource *hand for loop*, while talking about the left-hand side of the Ampere's law equation. Larry gestures that the loop has four sides and then uses the semiotic resource *pinpointing gesture* to indicate the two "w" legs (Figure 5.8.a). Without providing complete reasoning, Larry talks about how the integral manipulation regarding two "w" legs cancels out. Then the instructor helps Larry to build up his argument.

Instructor: Because?

Larry: Because they're perpendicular to the ... Am I ... Is it these ones that I am not gonna concern about? I am pretty sure it is.

Instructor: Yes. Because the direction of dl and the direction of B are ...

Larry: .. are perpendicular.

The above conversation between the instructor and Larry shows the argument behind the integral manipulation. Because the orientations of the "w" legs are perpendicular to the magnetic field direction, when Larry takes the dot product they cancel out. Even though Larry's argument is correct, so far he has not picked a direction for his loop. Just the existence of loop without direction on it could not allow Larry to see the relative vector orientations and this prevents his making a complete argument. Even though Larry seems to be building an argument just by completing the missing bit of the instructor's statement, we can observe that Larry's argument is not yet complete as he moves to consider the other two legs ("l" legs).

Instructor: Okay. What's about the other two legs?

Larry: Uh ... So for the other two legs, uh ... all right, so this one is kind of above it in the z, which means that the field is coming out at me (finger pointing in direction gesture). ... So, I need to pick a direction for my loop, don't I?

Instead of continuing with the same argument, Larry uses the semiotic resource *finger pointing in direction* to reason about the orientation of the loop in three-dimension (3D). Larry stops without making complete reasoning. Larry then immediately, without prompting from the instructor, realizes that he needs to pick a direction for his loop (*"so I need to pick a direction for my loop, don't I?"*) and uses the semiotic resource *arrow as vector* to record the loop direction (Figure 5.5.a).

Larry: Uh ... Let's say it all goes this way (counter clockwise). Okay. So, above it B is coming out and the way I draw my loop is coming out at me so they are parallel and then below it B is going in and my loop is going that way so they're parallel again.

Instructor: Cool. OK. So, you get for that integral?

Larry: Uh... so $2Bl$ (records on board)

$$2B\,L = \mu_0 \int k \cdot \mathrm{d}a \tag{5.5}$$

Either picking the loop using the semiotic resource *hand for loop*, or recording the loop using semiotic resource *square for loop*, does not help Larry to figure out the relative orientation between the magnetic field and the unit length on the Amperian loop. This prevents Larry from advancing with the mathematical manipulations. But the inclusion of *arrow as vector* to represent the direction on the loop helps Larry to continue with a valid argument. Larry reasons how the *unit length* (on "l" legs) on the loop is parallel to the magnetic field direction. Also, we can see Larry's confidence level as he reasons with a strong voice but with no pauses between words.

After few communication steps with the instructor, Larry recalls the mathematical formula for the current through the loop and finds the magnitude of the magnetic field to finish the task.

Instructor: How much current pierces this loop? You worked this out not two lines ago.

Larry: Right. $l\alpha\hat{x}$

Instructor: Yup. Though you do not need the x hat anymore because you have done the dot product.

Larry: All right. (Completes the right side by $\mu_0 l\alpha$ and records on the board: $B = \mu_0\alpha/2$).

$$2B\,L = \mu_0\,k\,L \tag{5.6}$$

$$B = \mu_0\,\alpha/2 \qquad (with\ \ \alpha = k) \tag{5.7}$$

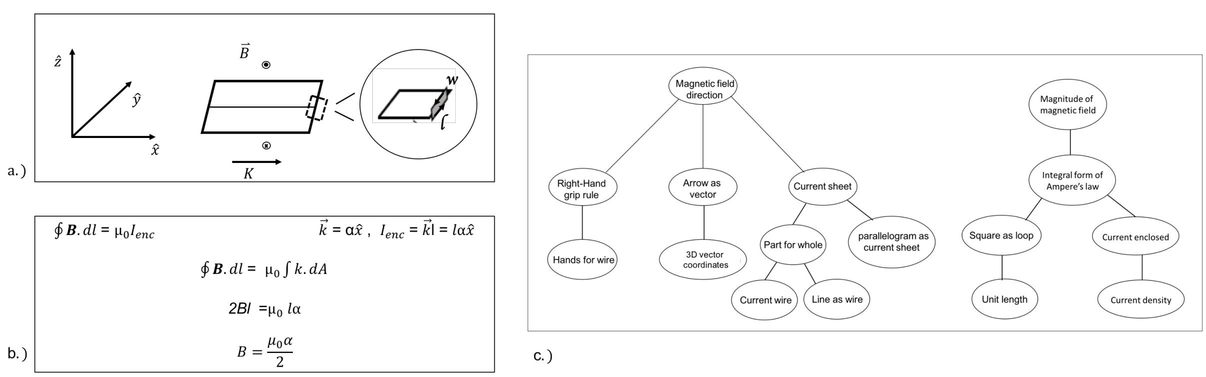


Figure 5.8: *a.) Compound representation at the end of episode 3 b.) Larry's use of mathematical formulas c.) Resource graph for the final compound representation and mathematical formulas*

Finding the *magnitude of the magnetic field* is a new task for Larry. At the beginning of episode 3 Larry has a compound representation that contains 3D vector coordinates, arrows to represent current and the magnetic field directions, a line to represent a current wire and a parallelogram to represent the current sheet. Larry starts by recording the mathematical formula for the *integral form of the Ampere's law* and combines it with the other mathematical relations to get the current through loop (*current enclosed*). In order to manipulate the integral on left hand side of the *integral form of the Ampere's law* formula, Larry starts by recording an Amperian loop on his digram and later he adds the details of Amperian loop dimensions and direction to his compound representation (Figure 5.8.a). This recording step helps Larry to manipulate the integral and find the magnitude of magnetic field (Figure 5.8.b). In the resource graph (Figure 5.8.c), we see Larry coordinate between conceptual resource *unit length* and semiotic resource *square as loop* along with additional resources related to mathematical formulas: *integral form of the Ampere's law*, *current enclosed*, *current density* and *magnitude of the magnetic field*.

5.5.4 Connections among the episodes

We see the disciplinary affordances of some semiotic resources that Larry uses to reinforce the use of forthcoming semiotic resources. Larry starts his oral exam by combining semiotic resources *3D vector coordinates* and *parallelogram as sheet* to build up his compound representation. The *3D vector coordinates* allows Larry to represent the current direction using *arrow as vector*. This inclusion allows Larry to focus on a single current carrying wire to apply the *right-hand grip rule* to get the magnetic field direction. This finding leads Larry to further develop his compound representation by using *arrow as vector* to include the *magnetic field direction* above the sheet. Moving on to consider multiple current wires using *paper as current sheet* results in Larry switching to his diagram on the board to use *line as wire* to represent the current wire. The inclusion of *line as wire* and *arrow as vector* (the current direction information which is already available in the diagram) allows Larry to consider multiple current wires to make a conclusion about magnetic field direction above and below the sheet. At the end of episode 2, Larry uses *arrow as vector* to further build up his compound representation to include the magnetic field direction. After determining the magnetic field direction, Larry moves to find the magnitude of the magnetic field in episode 3. Larry starts with the mathematical formula *integral form of the Ampere's law* and combines it with the given information of current density to get the current through the loop (*current enclosed*) to simplify the right-hand side of the integral. After manipulating the left-hand side of the Ampere's law integral, Larry further adds the details of the Amperian loop to his compound representation. He uses *square as loop* and *arrow as vector* to represent and to pick a direction for the loop. This recording step helps Larry to manipulate the left-hand side of the integral and finally to obtain the magnitude of the magnetic field.

Our study also shows the disciplinary affordances of some semiotic resources hindering the use of other semiotic resources. In episode 1, Larry first applies the *right-hand grip rule* in free space, without specifying the current direction to figure out the magnetic field direction. Then he continues to apply the *right-hand grip rule* considering the diagram on the board again without specifying the current direction. The directional information embodied on the diagram on the board by *3D vector coordinates* obstructs the use of *right-hand grip rule* and Larry ends up changing his gestural orientation. Later the inclusion of the current direction information helps Larry to figure out the magnetic field direction.

During the first two episodes, Larry works to get the magnetic field direction and we observe him using the semiotic resource *right-hand grip rule*. Once he gets the magnetic field direction and builds onto his compound representation, he no longer uses the *right-hand grip rule*. As Larry moves to use the mathematical formulas in episode 3 to get the magnitude, the disciplinary affordance of mathematical formulas *integral form of the Ampere's law*, *current enclosed*, *current density* and *magnitude of the magnetic field* prevents the use of semiotic resource *right-hand grip rule*. It was important but no longer useful for Larry to get the magnitude of the magnetic field in episode 3.

Likewise, Larry uses the semiotic resource *hand for loop* to gesture how he picks the Amperian loop in third episode. Using *hand for loop* helped Larry to show the orientation of the Amperian loop relative to the orientation of the current sheet. As Larry switches to work on mathematical formulas to get the magnitude of magnetic field and he ends up associating mathematical formulas: *integral form of the Ampere's law*, *current enclosed* and *current density*. The inclusion of mathematical formulas constrains the use of semiotic resource *hand for loop*. We observe Larry switching to the diagram on the board to represent the loop using a *square as loop*. Just the hand gesture is not helpful in determining the current piercing the loop or in determining the left-hand side of the Ampere's law integral.

5.6 Discussion

The process of student meaning-making is not a purely cognitive one; and in physics classes, both students and teachers use a number of semiotic resources in addition to speech and writing. The approach[180] to consider the involvement of all artifacts, objects, and actions to describe student meaning-making broadens the boundaries in physics as a discipline. The idea of disciplinary affordance allows us to connect ideas in physics with the kinds of representations which best express them.

Fredlund[130;135] used the affordances approach to show that different semiotic resources have different disciplinary affordances. They interviewed students on particular problems which had two specialized visual representations, each of which has a strong set of disciplinary affordances. Previous work on disciplinary affordances did not investigate how students could combine multiple semiotic resources within classroom problem-solving. In our study, we have actual classroom data, where the representations were developed, were determined to be insufficient and were replaced or augmented by new ones brought in by the student.

The problem Larry solves is a canonical problem at the upper-division level. Larry solving this problem gives us lots of insight into how students at this level might solve this problem. But, we are not looking for prevalence, and we are not trying to make a normative argument that all the students should solve this problem in the same way. This particular problem required Larry to start from a diagram and move to the mathematics at the later part of his solution; this process required Larry to coordinate between multiple semiotic resources. We observe Larry combine a series of semiotic resources with other available conceptual resources, some of which he keeps coming back to and some of which he discards after using.

There are three compound representations feeding in to give the features (direction and magnitude) of the magnetic field to solve the given problem (Figure 5.9). First, Larry builds up the compound representation of the current sheet (episodes 1 and 2) that he uses to buildup the direction and the magnitude of the magnetic field. The iconic parts of the current sheet are represented using a parallelogram with a line to represent the current wire, a sheet of paper and some hand gestures. In this compound representation, the parallelogram and the

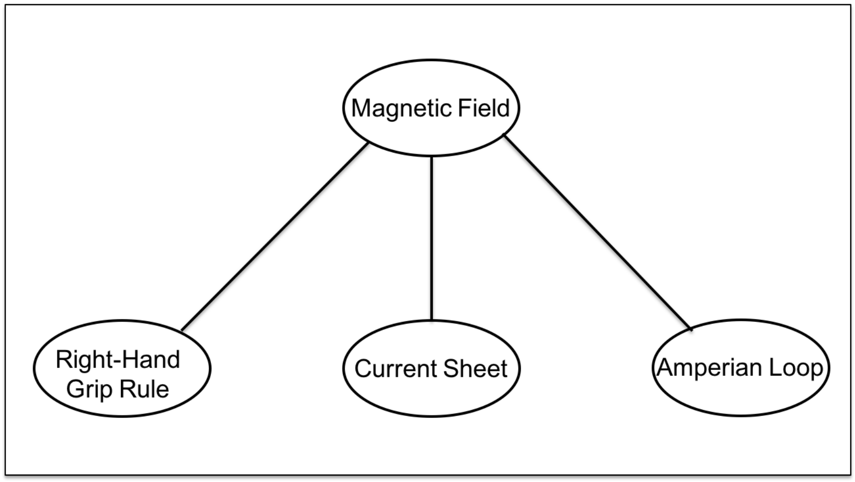


Figure 5.9: *Three compound representations from Larry's previous episodes combined to give the features (direction and magnitude) of the magnetic field*

sheet of paper (Table 5.4) allow Larry to visually represent the current sheet. The inclusion of hand gesture and line to represent wire helps to simplify the structure of the current sheet. Then he does some different things with the right-hand grip rule to buildup magnetic field direction. First, the compound representation of the right-hand grip rule (episodes 1 and 2) is developed using gesture, drawing of a line to represent the current wire with an arrow to represent given current direction. Then the curled right-hand gesture visualizes the application of the right-hand grip rule. Later, Larry builds up the compound representation of the Amperian loop (episode 3) using gesture and a drawing of a square on the board with an arrow to represent the direction on the loop. In this compound representation, gesturing and drawing (square) on the board (Table 5.4) allow Larry to visually represent the existence of the loop with respect to the orientation of the current sheet. Then the inclusion of the direction on the loop helps him figure out the relative orientation with the magnetic field direction.

The compound representation of right-hand grip rule allows Larry to figure out the resulting direction of the magnetic field lines from a current carrying wire. In order to apply the right-hand grip rule, Larry has to have a specified current wire with a certain current direction. Larry's approach to first, simplify the current sheet into individual wires and then to focus on a single wire to apply the right-hand grip rule helps to build a valid argument

Table 5.4: *Disciplinary affordances of three representations incorporated to find direction and magnitude of the magnetic field*

Compound Representation	Disciplinary Affordance
Current sheet	In this compound representation, the sheet of paper and the drawing of the parallelogram on the board generate visual representations of the current sheet. Larry keeps referring back to the diagram on the board as he progresses on this task and it allows Larry to add different features: current direction (episode 1), current wire (episode 2), Amperian loop (episode 3) and findings: magnetic field direction (episodes 1 and 2) on to his compound representation.
Right-hand grip rule	Right-hand rule helps to define the magnetic field. It reveals the connection between current direction and the magnetic field lines in the magnetic field created by a current. Right-hand grip allows Larry to manipulate this phenomenon by hand and to visualize the resulting magnetic field direction. In episode 1, missing detail of current direction prevents Larry from making a conclusion while reasoning using the diagram on the board. But the inclusion of the current direction and the line to specify the current wire in episode 2 helps Larry to figure out the magnetic field direction.
Amperian loop	Gesturing and drawing (square) on the board allow Larry to visually represent the existence of the loop with respect to the orientation of the current sheet. The inclusion of loop direction helps Larry to figure out the relative orientation of the unit length on the loop with the magnetic field direction. This leads Larry to find the magnitude of the magnetic field.

towards finding the net magnetic field direction above and below the sheet. The use of the parallelogram (episode 1) to visually represent the loop in the space of the diagram (on the board) helps Larry to include the arrow to represent the direction on the Amperian loop. This allows Larry to figure out the relative orientation between the magnetic field direction and the unit length on the Amperian loop. Finally, Larry builds a complete argument and advances with the mathematical manipulation to simplify the left-hand side of the integral form of the Ampere's law equation. Then he completes the oral exam after recording the magnitude of the magnetic field.

Research has shown that the use of multiple representations can greatly enhance students understanding of mathematical and physical concepts and in physics problem-solving. Students can connect these multiple representations to build meaning and construct representations that help them to make appropriate decisions about the solution process. The case of Larry exemplifies the coordination between multiple semiotic resources with different disciplinary affordances to build up compound representations to solve complex physics problems. Our analysis of this case illustrates a novel way of thinking about what it means to solve physics problems and, we hope, contributes to the application of social semiotics to the teaching and learning of university physics[154;180]. Having a single student and an instructor is the limitation of our study. The instructor involved in this study conducts the oral exam and provides hints to Larry in a certain way, but we think this interaction does not materially change our argument about how semiotic resources can come together to build representations and arguments to solve problems.

5.7 Conclusion

As part of a larger project to investigate problem-solving processes among upper-division physics students, we investigate how students coordinate among multiple representations while solving problems. We analyzed the case of Larry solving the Amperian loop problem for his oral exam using two approaches. In the study presented in chapter 4, we used the classification of representations into durable (diagrams, written mathematical equations) and evanescent (gestures, words) in our exploratory research project to suggest how students might translate among representational types to solve physics problems.

Later, the study presented in this chapter is different from the previous study because we use a separate theory, a social semiotic approach to perform a fine-grained approach of developing representations. We investigated how Larry brings in semiotic resources (elements of the representation) to build up representations. In this study instead of talking about the specific representational types (diagrams, mathematical equations, and gestures) that Larry used we focused how Larry builds up three compound representations ("Current sheet", "Right-hand grip rule", and "Amperian loop") to solve the problem. Also, we observe Larry going back and forth between some representations depending on which is more useful at each time. Further, sometimes Larry coordinates among representations and sometimes he uses several in series as he looks for one representation which he can use for a given situation.

The aim of this study is to show how the coordination of multiple semiotic resources give rise to physics meaning-making, especially, how the disciplinary affordances of each semiotic resource reinforce each other or make up for the constraint of other resources in the process of meaning-making. Our semiotic approach describes an important phenomenon about coordinating among different semiotic resources. This analysis highlights how the disciplinary affordances of different semiotic resources which make up for the constraint of other resources or reinforce each other to enhance meaning-making possibilities, and most importantly, it highlights important features of semiotic resources that should be taken into account by teachers during the meaning-making process.

5.8 Implications

The process of constructing an effective representation of a problem makes it easier for the problem solver to make appropriate decisions about the solution process. In addition to that, this process symbolizes the student's work on that particular problem. If the student constructs an effective representation then the student is more likely to progress towards solving the problem[195], but if the student constructs an inappropriate representation then the process is unlikely to make any progress until the student re-represents the problem accurately. This is evident in Larry's case, he could not continue to consider multiple current wires while using the sheet of paper representation (episode 2) but his decision to switch to the diagram on board helps to consider multiple wires and this leads Larry to draw a conclusion about the net magnetic field above and below the sheet.

Further, during episode 3, either gesturing for the loop or recording the loop does not help

Larry to figure out the relative orientation between the magnetic field direction and the unit length on the Amperian loop until he adds the direction on the loop. In some cases, students get stuck and cannot identify the nature of the sticking point. We observe on some occasions, Larry gets stuck and his voice level, choice of words and the speed of gestures indicates his confusion. But after adding new features or switching to a different semiotic resource to re-represent the idea or concept Larry's reasoning goes back to normal and he moves forward to solve the problem. As the case of Larry demonstrates, one of the important problem-solving skills is that of effectively representing and re-representing problems. This skill includes both students and teachers being aware of the nature of the disciplinary affordances of semiotic resources that students bring together to construct representations[(135)].

The research findings presented above suggest that it is important to highlight the complex use of multiple semiotic resourses in student problem solving. An implication is that instructors need to identify the disciplinary affordances of the different semiotic resources in different modalities so that instructors could demonstrate and help students to become better problem solvers. Further research could involve interviews or classroom observations with a series of typical problems to further explore how the representations are developed, how those representations are determined to be insufficient and replaced, or how those representations are augmented by new ones brought in by the students.

Chapter 6

Semiotic Resources Approach: Students Connecting Among Representational Spaces to Solve Physics Problems

6.1 Introduction

Learning with multiple representations has often found to be difficult for students and can even lead to lower learning outcomes than learning with single representations [(113)]. Previous research on students' use of multiple representations has revealed two problems. First, it has been shown that students often do not make spontaneous use of multiple representations. Second, the translation process (integration between representations) is often shown to be difficult for students [(117;120)]. Students might not spontaneously use (alternate) multiple representations on their own because novices have great difficulty connecting representations to the physical phenomena they stand for. The integration process can be seen as a process of mapping corresponding elements between different representations. In order to translate between representations, students must clearly identify the corresponding elements between different representations.

We can categorize existing work in PER that examines student use of multiple representations into two types. The first type of study examines the effects on problem-solving activities and on the success of problem representations provided by the instructor [128;196;197;127]. The second type of study examines the effects of student generated representations on problem-solving [198;199;195]. In this study, we focus on students' spontaneous construction of representations and the connections that they make among those representations while engaged in physics problem-solving.

6.1.1 Previous approaches: Cognitive Load

Within the previous work on student use of spontaneous representations, researchers used the Cognitive Load theory [143] to explain the advantage of using spontaneous representations while solving problems. Rosengrant [144] found most successful students in this study drew a sketch of the situation and then drew a free-body diagram. They were able to use this free-body diagram to set up their mathematical equations and evaluate their answers. Rosengrant's hypothesis is that students are probably aware intuitively that they do not have the mental capacity to remember all the information in the problem statement, and thus use representations to visualize problem situation. In other words, students used the diagram to help lessen the burden of their working memory. In another study [200], students were given a problem which required an addition of two vectors in two-dimensions. Students who explicitly drew components of vectors performed better than students who did not. The researchers generated a hypothesis - students who did not draw vector components had to keep more information in their working memory while engaged in problem-solving; this might have increased their cognitive load and ultimately lead them to become unsuccessful.

The hypotheses used in these studies were able to explain why students' spontaneous use of representations helped them to become successful but could not explain how students connect among representations (such as using the free-body diagram to set up mathematical equations). Especially, they were not able to explain what factors or features of representations influence how students succeed or fail in connecting among representations.

In addition, according to Cognitive Load Theory (CLT)[143], when students try to keep track of too many things they get stuck. The hypothesis is that students use representations to keep records of their findings, allowing them to lower their cognitive load, which ultimately allows them to perform better. Cognitive load theory is a leading learning theory, and it might be one possible framework that can explain why students use multiple representations. It is still criticized for a number of reasons, however, and researchers find it difficult to operationalize[201;202;203]. The main problem associated with operationalizing cognitive load theory is measuring the cognitive load. Because it is difficult to measure the cognitive load, it is difficult to generate evidence in support of the theory. Difficulties in measuring cognitive load increase further if we assume that cognitive load can vary during the learning process[204] and the researchers have to measure cognitive load multiple times and calculate an average value.

6.1.2 Previous approaches: Social Semiotic

From a social semiotic perspective, all meaning is realized, constructed, and developed using social conventions through the production of semiotic resources[153] . According to Van Leeuwen's definition of semiotic resources, the words, gestures, sketches, graphs, formulas, and different instruments that are used in physics classrooms can be labeled as semiotic resources[32]. Researchers documented that these semiotic resources have different strengths and limitations. Fredlund[130] relates this idea to define the disciplinary affordances of semiotic resources - "the disciplinary affordances of a given semiotic resource as the inherent potential of that semiotic resource to provide access to disciplinary knowledge."

In previous studies that used the idea of disciplinary affordances to explain student problem-solving behaviors, first, Fredlund with the refraction of light experiment[109], argued that two semiotic resources (wavefront diagram and ray diagram) allow students to access different areas of physics knowledge. Then Fredlund with the RC circuit diagram[130], showed when the representation is unpacked (when the representation is simplified to illustrate more details), students were able to access knowledge that is highlighted by the given representation and successfully complete the activity. Within these previous studies, researchers did not consider spontaneous construction of representations by students and they did not study how students connect among representations that they use to solve given problem.

In our attempt to better understand how students use representations while solving problems, in our first study [174] we looked at how students would translate among representation types to solve upper-division level physics problems. At the same time, we observed students decide whether existing representations are adequate; if not then they bring in ideas to bring more meaning to those representations. As a result of these observations, we change our focus to investigate how students go about building those representations. It turned out that students bring in ideas/ resources to build meaning on to existing representations. These resources turned out to be explicitly connected to a representation, however different from conceptual resources. In our previous study [170], we picked a new theoretical approach: Social semiotics to describe the underlying mechanism behind students' spontaneous construction of representations. Students coordinate among different semiotic resources with disciplinary affordances to construct compound representations. Even though we paid attention to the disciplinary affordances of the semiotic resources that students connect to construct representations in our previous study [170], we did not pay attention to the modes of those semiotic resources.

In real-world classroom settings or real problem-solving settings we observe students either draw a diagram, use gestures or use mathematical equations. Within this study we still focus on how students construct spontaneous representations but we specifically pay attention on how representations in different modes get connected to each other and what factors may influence how students succeed or fail when connecting among the representations that they spontaneously construct (that are in multiple modes).

6.2 Theory

6.2.1 Conceptual resources vs. Semiotic resources

From a cognitive perspective, David Hammer[173] uses the term 'resources' to describe individual's cognition as in-the-moment. According to Hammer's resource-based model of cognition, resources are fine grained bits of information that are highly context dependent (determined by the students' view of context). Actually, these conceptual resources are isolated, independent, productive student ideas that are used for solving a problem. Conceptual resources look very similar to semiotic resources, but they are not the same thing. Semiotic resources also could be seen as resources that students bring in to solve a problem but semiotic resources are not independent (like conceptual resources); each is tightly affiliated to a specific representation. Among the properties of the conceptual resources, they can be named using descriptive names for the thing the resources represent[205]. Let's take "Ampere's law" as an example; in the semiotic resource model, we need a resource that is actually Ampere's law written algebraically in integral form (semiotic resource: Integral form of Ampere's law). Here written algebraically is necessarily a part of the semiotic resource accounting and it shows us the affiliated representation (inscription- formula/ mathematics). But in the resource model, the resource Ampere's law could just be an idea or it could be (but may not be) that specific algebraically written integral form. Similarly, the "semiotic resource: arrow as vector" says that students are drawing arrows to represent vectors and in this case the affiliated representation (inscription- diagram)[164]. This suggests that even though both conceptual resources and semiotic resources look similar, in the case of semiotic resources we necessarily have to go through the affiliated representation to solve problems.

These representations are in different modes or modalities. In social semiotics, a modality[32] is a particular way in which information is to be encoded for presentation to humans. According to Lemke, different modes can be listed as: (a.) visual modes (such as sketches, images, graphs, tables), (b.) actional modes (body movements, gestures), (c.) mathematical modes (mathematical symbols, equations) and (d.) natural language (words, whether written or spoken).

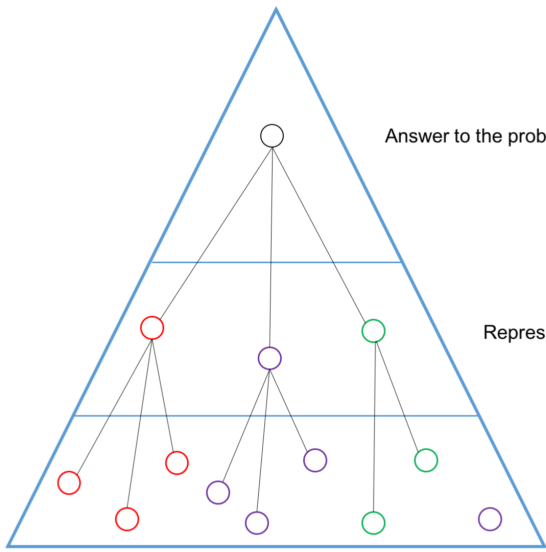


Figure 6.1: *Theoretical framework: Each semiotic resource (small circles) has a semiotic mode (visual in red, algebraic in green, and actional in purple). These semiotic resources are coordinated to build representational spaces. Then students connect representational spaces to answer problems. In this framework, the triangle is the whole solution of the problem, including reasoning. Solution to the problem is thus "made up of" all the lower parts of this framework.*

Expanding on our previous idea[170], we modify our theoretical framework (Figure 6.1) that accounts for how semiotic resources should be combined to solve problems. In this study we introduce the idea of representational spaces (Diagrammatic, Gestural and Algebraic...), composed of two or more semiotic resources in the same semiotic mode. Then these representational spaces can be used to solve problems.

6.2.2 Research questions

What is a better mechanism to explain how students build and connect together among different representations to solve physics problems? What are the factors that influence how students succeed or fail in connecting among representations?

6.3 Method and Data sources

The research presented in this chapter is part of an ongoing research project examining student meaning-making in undergraduate physics problem-solving. We used the case study methodology from our pilot study [170] presented in a previous chapter (chapter 5) to explain how students construct representations to solve physics problems. The analysis presented in this chapter uses multiple videos of individual students solving physics problems (oral exams) as a part of their upper-division physics courses. We use this study to generalize and to suggest a better way of explaining the underlying mechanism behind students' coordinating among different semiotic resources in different modes to construct spontaneous representations.

6.3.1 Data sources

The primary data set for this analysis was collected from upper-division Electromagnetism I (2013 fall semester) and Classical Mechanics courses taught by a white female instructor at two different institutions. The instructor involved in this study is the same instructor involved in our previous studies presented in chapters 4 and 5. This particular instructor used a similar approach in both courses and she used oral exams as a method to assess students' conceptual understanding, problem-solving, and scientific communication skills in her courses [175]. During the oral exams, students were given problems of similar difficulty level as the examples discussed in class. Students enrolled in these courses were required to complete 20- to 30-minute individual oral exams with the instructor. Students enrolled in Electromagnetism I course completed two oral exams, and Classical Mechanics students completed just one.

First, the data of students solving electric and magnetic field problems was collected from the same Electromagnetism I course presented in chapters 4 and 5. This is an upper-division course covering the first seven chapters of Griffiths' textbook [46].

Second, the case of Zeke[1], the student solving a mechanics oral exam problem, was

[1] All student names are pseudonyms

collected from an upper-division Classical Mechanics course was that taught at a different smaller institution. The class meets three times per week for an hour and had a small enrollment of about 4-8 students.

In addition to using oral exams in her teaching approach, this particular instructor used similar practices in both these classrooms. These classes have a strong focus on problem-solving and sense-making, and we observe that the instructor engages and gives hints in a way that aligns with the sense-making goals that students exercised in the classroom.

6.3.2 Analysis method

Here in this study we use a similar approach to that presented in chapter 5. For this analysis, we transcribed students' problem-solving videos. The videos were then divided into segments of events where each event is selected based on the semiotic and conceptual resources used by the students. For this initial round, we also identify key elements within each event using semiotic modes in general: different types of inscriptions (diagrams, mathematical formulas); extra-linguistic modes of expression (gestures); words (oral); and objects (used by students and the instructor).

In a second analysis round, we classify emerging semiotic and conceptual resources within each semiotic mode. This classification involves naming each resource descriptively [38]. We use a similar approach to our case study (chapter 5) to identify and name semiotic, conceptual resources [163;194;130]. Within the analysis section, we denote resource (semiotic and conceptual) names in italics.

Finally, the third analysis round involves comparing the resources' frequency and connections across different students and problems, and connecting the semiotic resources to their disciplinary affordances. In this analysis, three researchers worked together in the process to establish the reliability.

6.4 Analysis

In this section, we choose and present three problems, two of them solved by multiple students. We use this analysis to show how students coordinate between semiotic resources to build representational spaces (diagrammatic, gestural and algebraic) and how the identified disciplinary affordances of the semiotic resources affect the student problem-solving process.

6.4.1 Problem 1: Loosely wound solenoid

The first problem we discuss here is from Electromagnetism I course that asks students to determine the magnetic field inside and outside of a loosely wound solenoid. The term "loose winding" suggests that though the wires are next to each other, there is a current component going around while another current component is going down the solenoid. Two students Alan and Danny solve this problem individually during the oral exams conducted around the later part of the semester. All three sessions start with the instructor posting the problem.

Instructor: Suppose you have a solenoid, which is infinitely long and it is loosely wound so that there is some component of the current goes down the solenoid and some component of the current goes around the solenoid. We want to consider there to be a component of the current that travels down and a component of the current that travels around. What would be the magnetic field inside and outside?

To solve this problem, one can use the right-hand rule to find the direction of the magnetic field created by each current component, and use Ampere's law ($\oint B \cdot \mathrm{d}l = \mu_0 I_{\mathrm{enc}}$) to find the magnitude.

Student 1: Alan

Alan is the first student to take this problem and he is not provided with diagram so while the instructor was explaining, Alan starts to draw on the board. First, he uses the semiotic resource *cylinder for solenoid* to represent the solenoid on the space of the board and had a small conversation with the instructor to understand how to represent the loosely wound nature of the solenoid.

Alan: Oh, I see... so then... where we have previously ignored the fact that current is got some upward direction... okay.

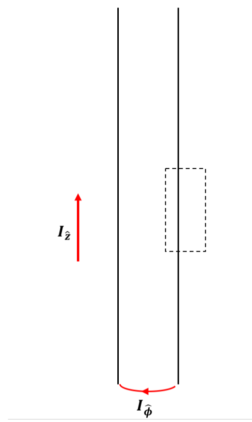


Figure 6.2: *Alan further develops his diagrammatic space*

With this clarification Alan takes few quick steps without even sharing his thoughts with the instructor. First, he decides to focus on the current components and he further develops the diagrammatic space using the semiotic resource *arrow as vector* to represent the current components going up and around. Alan labels the current component going up to be $I_{\hat{z}}$ and the current going around to be $I_{\hat{\phi}}$. Then he uses the semiotic resource *square for loop* to pick an *Amperian loop* that is parallel to the axis of his solenoid (Figure 6.2).

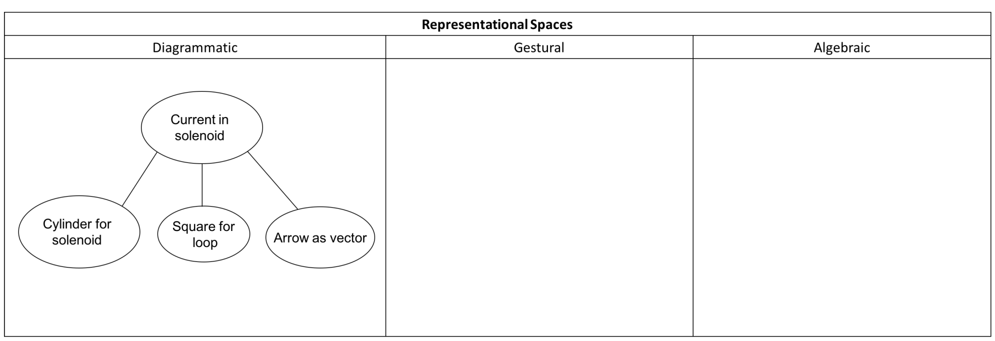


Figure 6.3: *Initial stage of Alan's representational spaces*

In this analysis, we divided the students' problem-solving activities into different stages of developing representational spaces. In our descriptions, we use the resource graph representation to show student's coordination of resources to build different stages of the representational spaces. At each stage, we chose semiotic resources students to bring in to build representational spaces with our research focus to show how the coordination between resources can be used to build representational spaces.

Figure 6.3 shows the initial stage of Alan's representational spaces. Here Alan lays the initial foundation in the diagrammatic space before moving to the algebraic space to do some manipulations to find the magnitude of the magnetic field. As a result, we do not see any of the semiotic resources being used in either gestural or algebraic spaces yet. The semiotic resource *cylinder for solenoid* on the diagrammatic space allows Alan to show the presence of the solenoid in space of the board (table 6.1). That also permits his representing the current components going up and clockwise using the semiotic resource *arrow as vector* along with the mathematical symbols $I_{\hat{z}}$ and $I_{\hat{\phi}}$ that he returns to while manipulating mathematics. With his work to come figuring out the the magnetic field inside and outside the solenoid, the step Alan has taken to include the *Amperian loop* using the semiotic resource *square for loop* allows him to figure out the amount of current piercing through the loop.

Table 6.1: *Disciplinary affordances of the semiotic resource coordinated to build the initial stage of Alan's representational spaces*

Semiotic Resource	Disciplinary Affordance
Cylinder for solenoid	Vertical cylinder allows Alan to generate a visual representation of the solenoid in the diagrammatic space of the board (diagram on the board). Alan refers to this while specifying the current components before moving to figure out the magnetic field.
Arrow as vector	This allows Alan to visually represent the vector direction. Spatial extent used to represent both distances in coordinate space and magnitudes of electric field vectors. Here, this visual representation of a straight line with an arrow head allows Alan to show the current components going up inside the solenoid and then a curved line with an arrow head allows him to represent the current components going around the solenoid.
Square for loop	This helps to generate a visual representation of the loop in the diagrammatic space of the board. Further this allows Alan to show the orientation of the loop with respect to the orientation of the current on his diagram that will allow him to figure out the mathematical manipulations while working with *Ampere's law* in the algebraic space.

After all of this work he starts to talk about the magnetic field that he relates to *Ampere's law* and the *Amperian loop* that he just picked.

Alan: Okay, the only way this differs is that the B dot dl contribution is... (while talking he moves to check his notes) we can't say it's a zero for the... box of that...

The instructor jumps in to get Alan to talk more about what he just mentioned about the magnetic field manipulation.

Instructor: Why not?

Alan keeps talking while he is off the frame.

Alan: Well, there is a wrote component... since we do B dot dl ... dot product is going to be the cosine of the two.

He comes back and starts referring to his diagrammatic space while talking more about manipulating the mathematics that is related to *Ampere's law*.

Alan: If we are looking at (pause, keep talking to self). Let's consider it separately... we might do B dot dl twice... (Alan records Ampere's law) ... I wanna say... we do B_z and the dot product being the... um... the cosine component... B cosine dl ... would be... cosine zero is one... So, then this is going to determine our vertical component... where $\mu_0 I_{enclosed}$ (records mathematics) and then... current is constant and then basically... B L (Alan picks the direction on his loop and labels the dimensions) ... L here... $\mu_0 I$ ... there is a coil density... in... per length... times that L (continues to record mathematics) so then B_z component is going to be $\mu_0 I$ over... that doesn't look right. What I've done? (talking to self and completes the mathematics)

Alan has the idea of considering current components separately "we might do B dot dl twice...". Alan starts by recording the semiotic resource in the mathematical form *integral form of the Ampere's law* and at the beginning he starts saying he is going to figure out the z component of the magnetic field but his mathematical symbol for current I was representing a more general current until he specifies the current components using his previous mathematical symbols $I_{\hat{\phi}}$. While manipulating the left hand side of the *integral form of the Ampere's law* Alan further develops his diagrammatic space using the semiotic resource *arrow as vector* to indicate the direction on his *Amperian loop* and the mathematical symbol 'L' to indicate the side length of his loop. After resolving the left hand side of the mathematical equation, he moves to consider the right hand side where he brings in the new idea of "coil density n".

$$\oint B_z \cdot \mathrm{d}l = \mu_0 I_{enc} \tag{6.1}$$

$$B_z.L = \mu_0 In.L \tag{6.2}$$

Then moving to specify the z component of the magnetic field, Alan gets confused and starts to talk slowly to himself "that doesn't look right. What I've done?". After few seconds, he continues to complete the mathematics; but he keeps looking at his final mathematical equation. This gives us an idea that Alan does not feel confident about his answer. One reason for this may be because he has not yet specified the current component in his mathematical equation, instead he still has I.

$$B_z = \frac{\mu_0 I}{n}\hat{\phi} \tag{6.3}$$

In addition, Alan makes a mistake while recording his final equation that prompts the instructor to jump in. Soon Alan realizes the mistake and corrects it.

Instructor: Why is n in the denominator?

Alan: Yeah... um... because I stopped what I'm doing... I think about why it looks different... no... no. It's right the z component is (corrects his mathematics and further modifies it) that's from the *theta* ... $\phi - hat$ component... I guess.

$$B_z = \mu_0 I n \hat{\phi} \tag{6.4}$$

$$B_z = \mu_0 I_\phi n \hat{\phi} \tag{6.5}$$

Even though Alan now specifies the current component ("that's from the $\hat{\phi}$ component...) that he is considering his mathematical equation does not show the correct direction of the magnetic field. The magnetic field created by the I_ϕ should be along the $\hat{z}$ direction. The instructor interrupts one more time concerning this issue.

Instructor: Okay, why would you call this B_z ...?

Alan replies quickly with the use of a gesture.

Alan: Because the... B is in the direction (gestures) cross with the direction of I. So...

He uses the semiotic resource *finger pointing in direction* to indicate the resulting magnetic field direction. Still Alan does not realize the $\hat{\phi}$ direction in his answer. So the instructor goes further to ask,

Instructor: Okay... so, why is it in the *phi* ($\hat{\phi}$) direction?

Alan quickly changes his mathematical equation so that it aligns with what he showed using the gesture.

Alan: What is in the *phi* direction? ... that I or ... (goes back to mathematical equation and modifies it). Yeah. I know... I'm sorry... I'm (modifies the mathematical equation) ... there we go... that's what I meant to do.

$$B_z = \mu_0 I_\phi n \hat{z} \tag{6.6}$$

So far Alan specified the current component he is considering and found the resulting magnetic field direction. But he has not mentioned if this resulting magnetic field is inside or outside the solenoid. Instructor asks him to talk about that "So that's good for inside and outside?".

Alan: That's only inside. Because... it's zero outside...

Here Alan talks further about how he disagreed during the class to have the magnetic field outside the solenoid to be zero. Then the instructor spends a little time explaining how he could build a really long solenoid and measure the magnetic field outside close to zero.

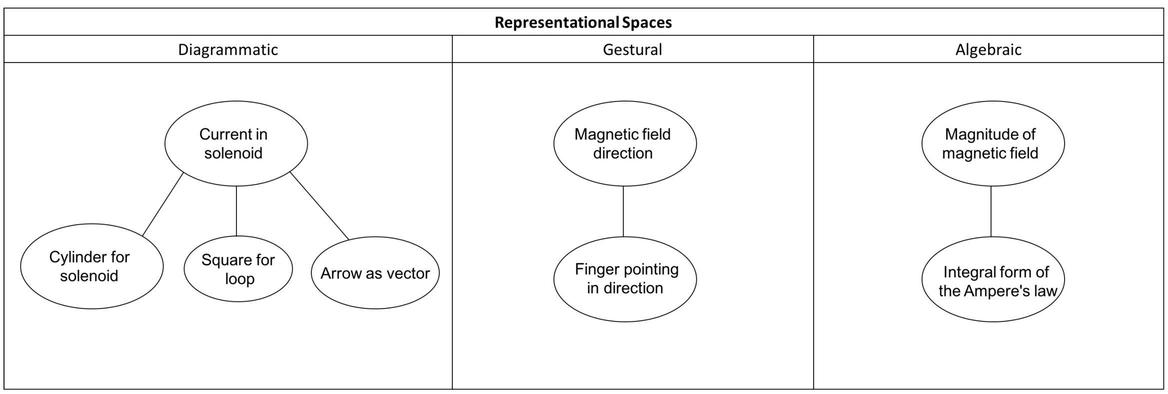


Figure 6.4: *Intermediate stage of Alan's representational spaces*

Figure 6.4 shows the intermediate stage of Alan's representational spaces. We observe him using the semiotic resource *arrow as vector* to further develop his diagrammatic space and getting started on the algebraic space. Alan just uses the semiotic resource *finger pointing in direction* that allows him to represent the resulting magnetic field direction. In general, students in this course use the semiotic resource *right-hand grip rule* to figure out the magnetic field direction of a current carrying wire. But in the case of Alan, we do not observe him using the gestural space; instead we see him approach to work in algebraic space with the predetermined magnetic field direction.

Within the algebraic space, Alan uses the semiotic resource in the mathematical form *integral form of the Ampere's law* and *magnetic field*. The semiotic resource *integral form of the Ampere's law* allows him to go about figuring out the magnetic field created by the certain current component. Then the semiotic resource *magnetic field* allows him to represent the specific magnetic field component with the directionality information (equation 6.3).

Alan approaches to develop the diagrammatic space and then moves to work on algebraic space separately. But while continuing with the mathematics, he returns to the diagrammatic space and we observe Alan translating between representational spaces. Within the diagrammatic space, first he uses the semiotic resource *arrow as vector* which allows him to indicate the direction on his *Amperian loop* that later helps him figure out the vector orientation between the magnetic field (B) and unit length (dl) while simplifying the left hand side of the *integral form of the Ampere's law*.

Table 6.2: *Disciplinary affordances of the semiotic resource coordinated to build Alan's intermediate representational spaces*

Semiotic Resource	Disciplinary Affordance
Integral form of the Ampere's law	This physical law shows the relationship between the integrated magnetic field around a closed loop and the electric current passing through the loop. It states that the magnetic field around an electric current is proportional to the current. In relation to the *Amperian loop* each segment of current loop produces a magnetic field like that of a long straight wire, and the total field of any shape current is the vector sum of the fields due to each segment. In this mathematical representation, the left hand side allows us to show the relative vector directions (do the dot product) between magnetic field and the unit length on an Amperian loop. Where, right hand shows the two scalar quantities, permeability and current, passing through the loop.
Arrow as vector	Here he uses the arrow as vector to represent the direction on the loop that helps Alan to build a complete argument and advance with the mathematical manipulation while working with the left hand side of the *integral form of the Ampere's law* equation.
Finger pointing in direction	Alan uses the index finger to the side while gesturing for direction. This allows him to show the direction of the resulting magnetic field while answering instructor's question.

After figuring out the magnitude and the direction of the magnetic field created by the current component that goes around the solenoid then Alan moves to considers the effect by the current going up (axial current component).

Alan: Okay there is the z component of the B field... so then doing the B_r component... (starts recording the mathematical equation but stops half way through $\oint B_r \cdot \mathrm{d}l = \ldots$)

Instructor's interruption about the loop "Are you doing the same loop or a different loop?" makes him stop and think a little bit before he started to move forward again.

Alan: I guess... do the same loop... um... let me think about that. Though... um... no... because it's gonna be... It's like a cylinder carrying a surface charge, surface current I mean (Alan draws a cylinder and picks a loop) and there is my new dl ... um... my new Amperian loop.

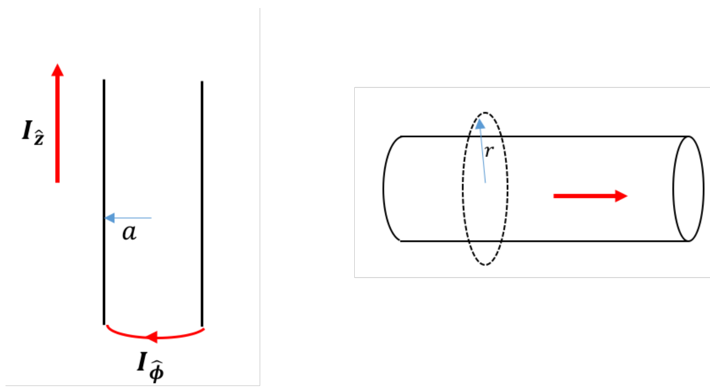


Figure 6.5: *Alan develops a separate diagram to represent his new Amperian loop*

Alan starts developing a separate diagram (Figure 6.5) using the same semiotic resource *cylinder for solenoid*. But this time the solenoid is horizontal and he reuses the semiotic resource *arrow as vector* to represent the current going up inside the solenoid. He keeps talking and thinking after stepping back while working on his new diagram. After a little pause, he uses the semiotic resource *circle for loop* to represent how he picks the *Amperian loop* for this new case.

Now Alan is considering the current component (axial current) going up which is along the $\hat{z}$ direction. Then the magnetic field created by this current component should be in $\hat{\phi}$ direction. According to Alan's notation so far, this should be B_ϕ. But at beginning of this section, Alan talked about how he is going to find the r component of the magnetic field

("B_r component") which is notationally incorrect. The Instructor pushes Alan to rethink this and soon he corrects the mistake.

Instructor: Is this going to give you the B in the r direction?

Alan: Nope. I would like to revise that... so, it's going to be the *phi* component or the *phi* direction of B ... again $\mu_0 I_{enclosed}$ ... (completes the *Ampere's law* equation) this will do it ... easier.

$$\oint B_\phi \cdot \mathrm{d}l = \mu_0 I_{enc} \tag{6.7}$$

After revising his mathematical equation, Alan turns towards the instructor and starts to talk about the possible current densities - linear, surface and volume currents - in general and how to represent those using mathematical symbols. This gives us an idea (actually he is) that he is going to relate this into his calculations even though the problem does not talk about the current densities. The following section shows Alan is thinking about how much current is going through his *Amperian loop*. To represent that in his mathematics, first he uses *sigma* ("*sigma* times *da*") and then another symbol k (may be a surface current density) that shows us he thinks about a surface current here.

Alan: ... then $I_{enclosed}$ is going to be... um... μ_0 ... total $I_{enclosed}$ is going to be (continues recording mathematics) *sigma* times *da* ... um... *k da* right? which is just...

$$\oint B_\phi \cdot \mathrm{d}l = \mu_0 I_{enc} = \mu_0 \int k \cdot \mathrm{d}a \tag{6.8}$$

Instructor: So, you want to know how much current is piercing your loop?

Alan: Right, so it's going to be *sigma* times V and how we get there? ... so there going to be *sigma* times...

Earlier Alan related *sigma* to the infinitesimal area *da* that indicated he is thinking about a surface current but right here he started to relate *sigma* with volume ("*sigma* times V") that shows us he is confused. His behavior and the pauses between his words show us that this is difficult for him. Alan talks to himself ("how much current is piercing the loop?... how we get there?") and the instructor jumps in to help him resolve this confusion.

Instructor: Where is *sigma* in this problem?

Alan: Isn't *sigma* is the... so the total current is just I, the z component of I ... ... can I leave it like that?

Instructor: Yup, we haven't specified how big is I_z compared to I_ϕ yet.

Instructor's questioning helped Alan to rethink and remind him about the current component that goes up inside the solenoid. Alan returns to his former mathematical symbol for current component $I_{\hat{z}}$ and moves forward with the mathematical calculation.

Alan: Okay... and this is going to be (further modifies his mathematics) B_ϕ ... two *pi* ... (goes back to his diagram and labels r) ... r ... μ_0 ... um... again just I ...

While working on the mathematics, Alan goes back to his diagram and uses the mathematical symbol r to label the radius of his new *Amperian loop* (Figure 6.5). This time Alan does not talk much about how he will simplify the left and the right hand sides of the *integral form of the Ampere's law*. instead, he quickly manipulates the mathematics and gets the magnetic field component along the $\hat{\phi}$ direction.

$$\oint B_\phi \cdot \mathrm{d}l = \mu_0 I_{enc} \tag{6.9}$$

$$B_\phi 2\pi r = \mu_0 I_z \tag{6.10}$$

$$B_\phi = \frac{\mu_0 I_z}{2\pi r}\hat{\phi} \tag{6.11}$$

Then the instructor's question ("Is that good inside and outside?") leads Alan to specify the regions using the mathematical symbols r for radius of his new *Amperian loop* and a for radius of the solenoid (Figure 6.5). Finally, Alan returns to the algebraic space where he relates the specified regions in space to conclude whether his findings of magnetic field components are inside or outside the solenoid.

$$B_z = \mu_0 I_\phi n \hat{z} \qquad \rightarrow r < a \tag{6.12}$$

$$B_\phi = \frac{\mu_0 I_z}{2\pi r} \hat{\phi} \qquad \rightarrow r > a \tag{6.13}$$

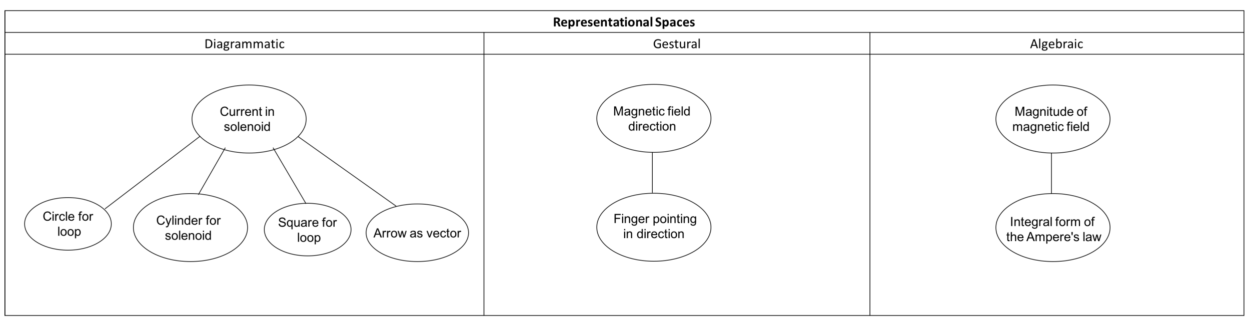


Figure 6.6: *Final stage of Alan's representational spaces*

Figure 6.6 shows the final stage of Alan's representational spaces. In addition to the reuse of the semiotic resources that are already being used in all three representational spaces, the only modification is to add the semiotic resource *circle for loop* into his diagrammatic space. There are two things someone needs to consider when picking an *Amperian loop*. First, magnetic field should be parallel to the infinitesimal length on the loop because then the dot product turns into multiplication. Second, magnetic field should be constant so we can pull it out of the integral while manipulating the mathematics. Here Alan is considering the axial component of the current and it produces a constant magnetic field which is going around it. Considering the requirements when picking the *Amperian loop* in this situation, the semiotic resource *circle for loop* helps Alan to pick the best shape for his *Amperian loop* that encloses the axial current inside the solenoid.

Overall we do not observe Alan spending much time translating between representational spaces. Throughout this activity Alan spends much time developing a certain representational space and then moves to the next. The time he spends in the first representational space helps him to build background knowledge that he is going to use in the next representational space. Students who solve problems that use *Amperian law* often use the *right-hand grip rule* while figuring out the magnetic field direction from a current wire. Interestingly, Alan in this case has already determined the resulting magnetic field direction as he moves to consider a certain current component and relates it to the *integral form of the Ampere's law.*

Table 6.3: *Disciplinary affordances of the semiotic resource coordinated to build the final stage of Alan's representational spaces*

Semiotic Resource	Disciplinary Affordance
Arrow as vector	Similar to the previous case of Alan using an arrow to show the current direction that is going around the solenoid, this time it allows him to represent the axial current which is inside (along $\hat{z}$) the solenoid.
Circle for loop	This helps Alan to generate a visual representation of the loop and also to figure out the mathematical manipulations. There are two things someone needs to consider when picking an *Amperian loop*. First, magnetic field should be parallel to the infinitesimal length on the loop. Second, magnetic field should be constant. As Alan is going to consider the current component going up the best shape for his *Amperian loop* is a circle.

At the beginning of this session, Alan does not have a diagram to start with; but after a few steps, Alan brings in semiotic resources to build up his diagrammatic space that contains a cylinder to represent the solenoid, arrows to represent the current components and square to represent the *Amperian loop*. The moves to the algebraic space to find the magnitude of the magnetic field by each current component.

We observe the background knowledge that is already being built on one representational space helps him to work on the second representational space. The diagrammatic space allows Alan to move forward with the diagrammatic space with ease. During both cases when considering current components going around and going up, the specified current component (direction) and the *Amperian loop* on the diagrammatic space helps Alan to progress with the mathematical manipulations.

The case of Alan exemplifies the coordination between multiple semiotic resources with different disciplinary affordances to build up representational spaces and then using those representational spaces. The information that is built on those representational spaces allows Alan to translate among representational spaces to solve the problems.

Student 2: Danny

Danny is the second student to take this problem and the instructor draws a diagram (Figure 6.7) while presenting the problem.

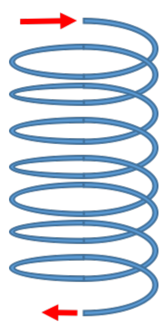


Figure 6.7: *Instructor provided diagram for Danny: helix for solenoid and arrows for current*

Danny starts by asking a question from the instructor to make sure he understands what the instructor wants him to do.

Danny: Infinitely long... but... you want me to consider current going down (gesture)?

Instructor: Yes, there is current going around and current going down.

The initial diagram that is provided by the instructor already contains the semiotic resources *helix for solenoid* and *arrow as vector* to indicate the current entering from the top and the current leaving from the bottom. Right here we observe Danny using the gestural semiotic resource *finger pointing in direction* while talking about the axial current component that is going down. Then he moves to add the current feature to the diagram.

Danny: If we just call the current going down (draws an downward arrow) as... let's call it I.

Danny moves to consider the current component going down first and he uses the semiotic resource *arrow as vector* to represent it in the diagrammatic space and he labels it using the mathematical symbol I. After this step, he steps back and keep looking at the diagram for some time before moving to talk about how he will go about finding the magnetic field of this current component.

Danny: So, then my first thought could be to put an Amperian loop (gesture- hands to show a loop) around so... like this (adds a loop to his diagram) ... so, then you would have a current going down through it.

Unlike Alan, Danny talks about the *Amperian loop* even before talking about *Ampere's law*. Here he uses the gestural semiotic resource *hand for loop* to show how he would pick it and soon after he uses the semiotic resource *circle for loop* in his diagrammatic space.

Both these students do not spend much time (compared to Larry presented in chapters 4 and 5) when picking the *Amperian loop*. This may be because the case of straight current carrying wire is the simplest case related to *Ampere's law*. In addition to that, they might have seen this scenario in their lectures or even in their text book.

Next Danny looks into an equation sheet and starts to record the mathematical equation for *Ampere's law*. While doing it, he seems to think whether he need include the Maxwell's correction while recording the *integral form of Ampere's law*. To clarify this, he communicates with the instructor.

Danny: So... what was the exact question again?

Instructor: What's the magnetic field inside and outside?

Danny: Okay and the current... there is no varying currents here.

Instructor: Time varying currents? ... no.

He uses the algebraic semiotic resource *Maxwell's correction to Ampere's law* as the starting point to calculate the magnetic field but his words to follow and the pauses between his words show the uncertainty in his mind.

$$\oint B_\phi \cdot \mathrm{d}l = \mu_0 I_{enc} + \mu_0 \epsilon_0 \int \frac{\partial E}{\partial t} \cdot \mathrm{d}s \tag{6.14}$$

Danny: Umm... since there is no... (long pause) seems like it... trying to think specially... it may be something tricky... I don't know why.

Just then the instructor jumps in and asks him to talk about the simplest possible case as he was thinking.

Instructor: Okay... if there's the case as simple as you think, what you think it would be?

Danny: It would be this (points to the equation) is just zero and you just get... I ... this I (points to the diagram) over two *pi* r (records the mathematical equation for magnetic field). That's for the outside and zero inside.

$$B = \frac{\mu_0 I}{2\pi r} \tag{6.15}$$

Danny jumps into work in his algebraic space and first shows that he is going to cancel the Maxwell's correction term in his Ampere's law equation. Then he uses the algebraic semiotic resource *magnetic field* (equation 6.15) to show the resulting magnetic field outside the solenoid. Within this step, he does not show or talk about how he did the mathematical manipulations. Even though Danny used the equation 6.15 to represent how big the magnetic field outside the solenoid is, he failed to specify the direction.

Instructor: What's the direction of the B, on the outside then?

Danny: Then... if we call this... z (add more parts to his diagram) and then this angle would be *pi* (modifies his mathematical equation) and this (points to mathematical equation) would be $phi - hat$

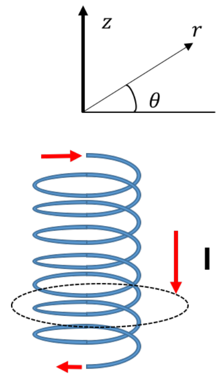


Figure 6.8: *Danny decides to further develop his diagrammatic space*

Instructor's question about the direction leads Danny to further develop his diagrammatic space (Figure 6.8). He uses the semiotic resource *3D vector coordinates* to specify the directions in his diagrammatic space and finally he encodes the directional information into his mathematical equation for *magnetic field*. Danny even goes for further explanations where he uses the gestural semiotic resource *finger pointing in direction* to show the current component going down.

Danny: If you define like... oh... okay... I've already said this (current direction - points downward) was down. So... this B field equation be minus *phi* − *hat* (modifies the equation).

$$B = \frac{\mu_0 I}{2\pi r}\,(-\hat{\phi}) \tag{6.16}$$

After modifying the mathematical equation to show the magnetic field is going around but pointing downwards $(-\hat{\phi})$, Danny quickly applies the *right-hand grip rule*. At this point, Danny has already figured out the resulting magnetic field direction and he did not use the *right-hand grip rule* for the usual purpose of finding magnetic field direction or double checking his direction. Also he gestured towards to instructor and the idea behind that was to better communicate his magnetic field direction to the instructor.

Figure 6.9 shows the initial stage of Danny's representational spaces. Within the diagrammatic space, Danny has the semiotic resources *arrow as vector*, *3D vector coordinates*,

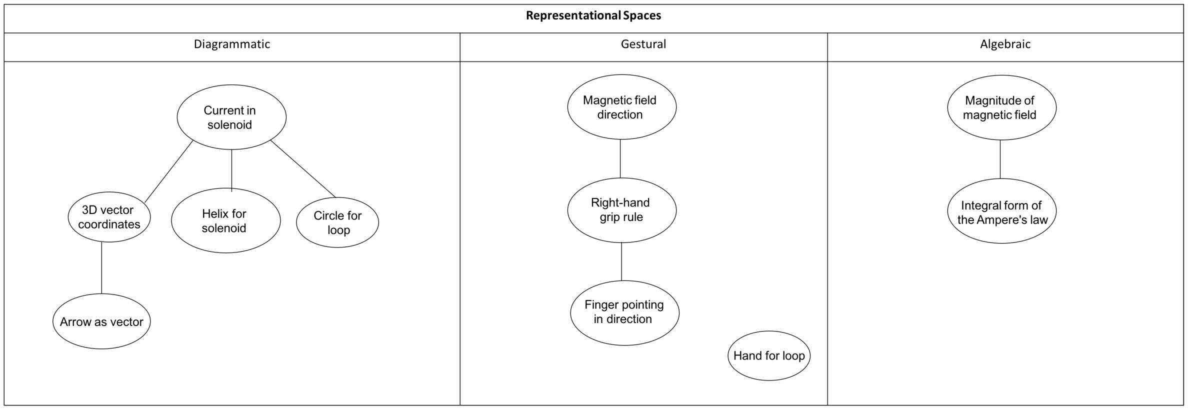


Figure 6.9: *Initial stage of Danny's representational spaces*

helix for solenoid and *circle for loop*. Then the semiotics resource *finger pointing in direction*, *right-hand grip rule* and *hand for loop* are used in the gestural space while *Maxwell's correction to Ampere's law* and *magnetic field* occupy the algebraic space. So far we observe Danny not doing much about coordinating between the representational spaces other than using the axial labels in diagrammatic space associated with the gestures.

Even though the initial semiotic resource *helix for solenoid* is used by the instructor, Danny then uses it as he moves forward within this problem. While it allows showing the presence of solenoid in the diagrammatic space of the board, the semiotic resource *3D vector coordinates* allows specifying the directional information in the diagrammatic space. We observe Danny coordinating between the semiotic resources *3D vector coordinates* and *arrow as vector*. This allows Danny to talk about the current with a definite or specified direction ($-\hat{z}$).

Then within the gestural space, even though he uses the *right-hand grip rule* it is not for the purpose of figuring out the magnetic field direction but to show the resulting magnetic field direction by the downward axial current component. Further he uses the semiotic resource *finger pointing in direction* and *hand for loop* that allows him to best communicate his idea to the instructor.

Maxwell's correction to Ampere's law in algebraic space allows Danny to show how he will find the magnitude of the magnetic field. In this case Danny has a additional term in his mathematical equation compared to Alan's *Ampere's law* equation. This is because initially

Danny is not sure whether the current component going down is steady. But the instructor confirmed it and that may have helped him to ignore this additional term while doing the mathematical manipulations. Then he does a similar thing like Alan in that he uses the semiotic resource *magnetic field* to specify the resulting magnetic field.

Table 6.4: *Disciplinary affordances of the semiotic resource coordinated to build Danny's initial stage of representational spaces*

Semiotic Resource	Disciplinary Affordance
3D vector coordinates	Similar to the previous case of Larry (chapters 4, 5) this visual representation allows Danny to define the direction in his diagrammatic space.
Hand for loop	Like in the case of Larry here Danny uses his hand to represent how he would go about picking an *Amperian loop*.
Circle for loop	Similar to the previous case of Alan, this helps Danny to generate a visual representation of the loop in the diagrammatic space of the board. Having said that he is going to consider the current component going down, the best shape for his *Amperian loop* is a circle.

Even though Danny made a detailed explanation about the magnetic field direction the instructor asks Danny to take her through the mathematical manipulation steps that he skipped earlier.

Instructor: So, you went from... the closed integral of B dot dl to B equals two *pi* r ... really quickly. How did that happen?

Danny: Um... (points to his mathematical equation) a line integral closed ... so... a loop right... and then um... (long pause) because this is still... we are considering a cylinder right (gesture - both hands holding like a cylinder) (long pause) ... okay...

Going back to explain how he did those mathematical manipulations makes Danny think for a long time and then talk a little bit then pause and use the gestural semiotic resource *hand for solenoid* but still he is not able to make a complete statement. After another long pause, Danny continues and this time he brings in a new idea "wire of charge".

Danny: I see what you were saying... because I assumed that was like a line, a wire of charge... because what I did... was assume that B is always in the direction of my loop points.

Finally, Danny is getting there and while talking he keeps pointing towards the diagram on the board. Then the instructor asks him to talk more about it and this leads Danny to develop a separate diagram in his diagrammatic space for further explanation.

Instructor: Okay, and is it?

Danny: Well... it's going to be perpendicular to the current (points to the diagram) ... okay... So... I was thinking like... a cylinder (draws on the board) and I was thinking like a instead of a sheet of charge coming down (gesture - both hands holding like a cylinder), just think like one line coming down... and so that would be like a... part of the current... and then that would be exactly like line charge... So, I'm kind of imagining like... yeah. that should work... I think it's good.

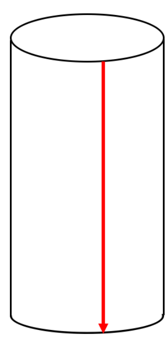


Figure 6.10: *Danny represents a wire of charge*

This new diagram (Figure 6.10) helps Danny to better explain how he would have simplified the mathematics. He starts talking about how he picked the *Amperian loop* being perpendicular to the current component that he considered ("it's going to be perpendicular to the current"). Here we also observe Danny taking a *part of a whole* approach similar to Larry 9in chapter 4 and 5) that he uses to simplify the current component to consist of surface *current wires* that is a part of the whole or net current going down ("that would be like a... part of the current... "). This gives the idea that the magnetic field created by the current component is going to be parallel to his *Amperian loop*.

With this satisfying explanation, the instructor asks Danny to move on to consider the other current component. We observe Danny using this same *part of a whole* argument of *current wires* when he moves to consider the current component that is going around.

Instructor: Okay... so that gives us the B due to the component of the current that's going down the outside. There should be some B due to the component of the current that's going around. What's that?

Danny moves on to develop another diagram (Figure 6.11) with a side on view of the solenoid that he uses to find the other magnetic field component.

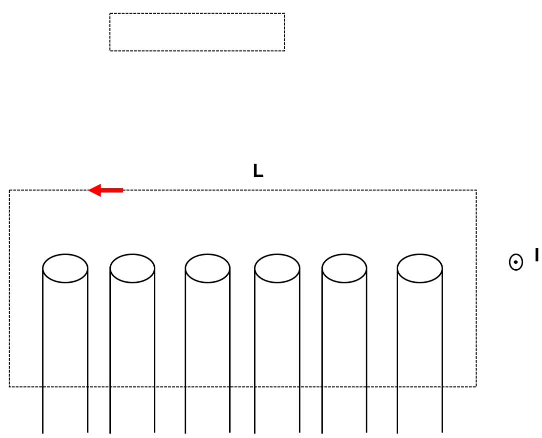


Figure 6.11: *Danny visualizes the side on view*

Danny: And this is still infinite... so, we would look at a loop... so... I turn this onto side (draws on the board) and say that there was... um... current coming in out of the board, so... then this is inside... I draw my Amperian loop like this then I do B dot dl ... and I end up with (continues to work in mathematics) ... yeah... this (labels on the diagram) is L (records mathematics).

$$B = \frac{\mu_0 N I}{L} \tag{6.17}$$

First, he uses the semiotic resource *line as wire* with a magnification to it that looks like cylinders (Figure 6.11). Then he uses the semiotic resource *arrow as vector* to represent the current that is coming out of the board in his side-on view. Then he moves to pick the *Amperian loop* and he uses a different semiotic resource *square for loop* which is the appropriate shape in this situation. Further he uses the mathematical symbol L to label the side of his loop.

Again Danny skips the mathematical manipulation steps but the instructor asks him to show how he got that.

Instructor: Where did that come from?

Danny: So... B ... dot *dl* ... so, this is ... I'm going to go... around this way (picks the direction on his loop) ... so... We make a loop out here (draws a loop far away) and there is no flux... Yeah... so there is no flux through a field out here (points to the loop that is far away) ... through the loop out here and there can be any flux through... um... a loop at infinity. We know that the field goes to zero at infinity. So... if the field is not charging it has to be zero at infinity and then it's zero all the way up to for an infinite solenoid. So... that means we don't consider this side of the loop (points to the loop) ... because B is zero. So B dot *dl* would be zero. These sides (points to the sides of the loop) won't contribute so, you just left with L ... that's why there's one L here (points to his mathematical equation) ... so it would be B times L equal μ_0 I but there is N of them. So you get whatever up here.

Unlike Alan in the previous case but somewhat similar to Larry, Danny gives a detailed explanation of how he simplifies the left-hand side of the *Amperian law* equation with the use of coordinating between diagrammatic and algebraic spaces. He further adds the missing steps in his mathematical manipulation but still he misses the directional information. Then instructor asks him for clarification purposes.

Instructor: So you have outside B in the $phi - hat$ direction, inside you have B in the z direction?

Danny: Yes.

Even though Danny does not specify the regions like what Alan did, he concludes with a short answer because he already mentioned most of the details while talking about the mathematical manipulation steps. One major difference between Alan's solution and Danny?s is that Danny does not specify the current components using mathematical symbols. Instead he calls both current components I.

In summary, the mathematical manipulation steps associated with the *Ampere's law* equation seem to be obvious and easy for Danny and this aligns with what we saw previously with Alan. First, Danny records the final mathematical equations for the *magnetic field* then later he further develops representational spaces and coordinates between them to explicitly explain it to the instructor.

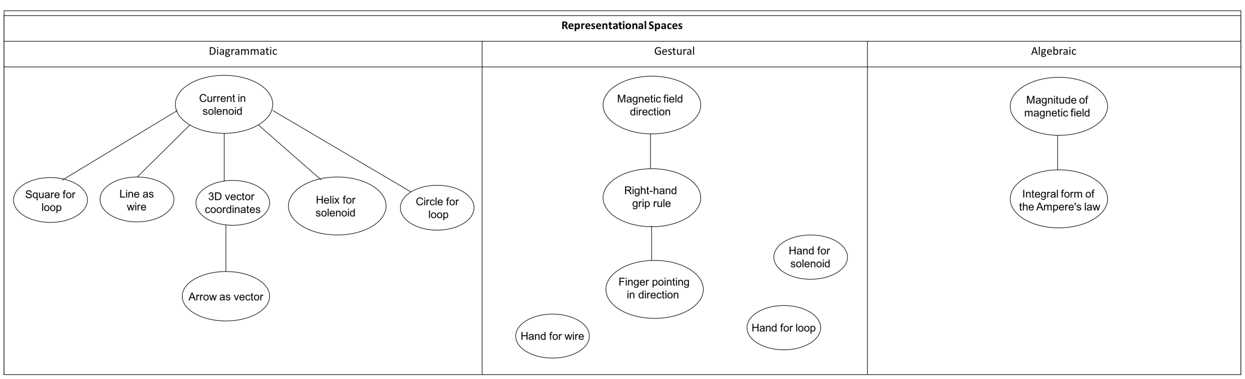


Figure 6.12: *Final stage of Danny's representational spaces*

Figure 6.12 shows the final stage of Danny's representational spaces. In addition to reusing the semiotic resources that are already in the representational space, Danny brings in a few new semiotic resources. The semiotic resources *line as wire* and *square for loop* are being used in the diagrammatic space while semiotic resources *hand for solenoid* and *hand for wire* come into the gestural space. Within the algebraic space, Danny uses the semiotic resource *Integral form of Ampere's law*

Table 6.5: *Disciplinary affordances of the semiotic resource coordinated to build Danny's final stage of representational spaces*

Semiotic Resource	Disciplinary Affordance
Line as wire	This helps Danny to visually represent the current that helps him to get an idea about the directional relation between the resulting magnetic field and the direction on the loop allowing him to simplify the mathematics.
Square for loop	With the newly generated side-on view, this visual representation allows Danny again to pick the most appropriate shape for the *Amperian loop* for the current going around the solenoid.

The semiotic resource *line as wire* allows Danny to first represent the current in a different orientation that eventually helps him to figure out and explain how he solves the left hand side of the *Integral form of the Ampere's law* equation. Then both the semiotic resources *hand for solenoid* and *hand for wire* in the gestural space allows him to better communicate his ideas to the instructor. Finally Danny decides to discard *Maxwell's correction to Ampere's law* and he uses the semiotic resource *Integral form of the Ampere's law* in the algebraic space.

Overall we observe Danny doing a better job compared to Alan when coordinating among representational spaces. Danny often coordinates between the diagrammatic and algebraic spaces while explaining the mathematical manipulations related to the left-hand side of the *Integral form of the Ampere's law* equation. Also he used the gestural space with the other two representational spaces during explanation related to the current component going around the solenoid.

Similar to the case of Alan, the case of Danny also shows us how students might coordinate among semiotic resources to build up representational spaces that they use to solve problems. Similar to Alan we observe Danny working on the diagrammatic space to build the background knowledge before moving to algebraic space. Even though Danny quickly completes the mathematical manipulation he was not able to successfully reason how he arrived at the answer. Here we see the disciplinary affordance of the current representational spaces (Figure 6.9) prevent Danny connecting among representational spaces.

When Danny tries to use the existing diagrammatic space (Figure 6.8) to reason for his mathematical manipulations the information that is already available does not seem to allow him to make a complete argument. Then added feature of current wire (Figure 6.10) brings in the missing information and the updated representational spaces allows Danny to make connection among diagrammatic and algebraic spaces to successfully reason for his mathematical manipulations.

6.4.2 Comparison between students: Loosely wound solenoid

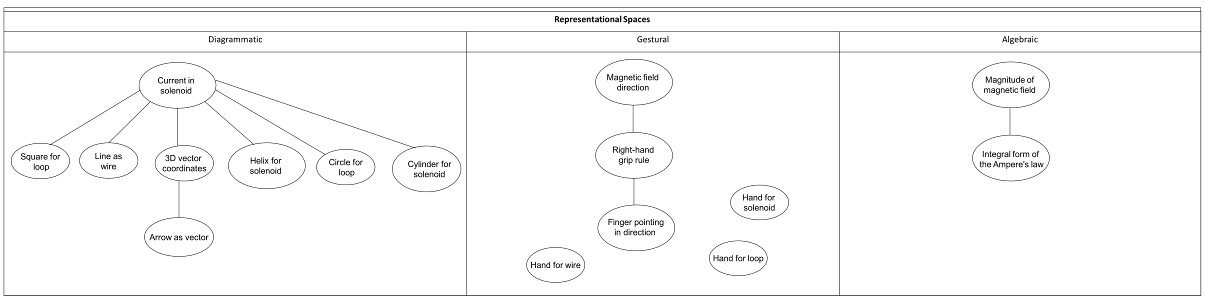


Figure 6.13: *Combined representational spaces for Alan and Danny*

Figure 6.13 shows the combined representational spaces for Alan and Danny. Alan and Danny use different semiotic resources to represent the presence of the solenoid in the diagrammatic space. One major reason for this is because Danny decides to use the instructor-provided initial diagram while Alan develops his own diagram. in addition to that, Danny uses the semiotic resources *3D vector coordinates* and *line as wire* in his diagrammatic space that Alan does not use in his representational space. These additional semiotic resources allow Danny to talk about directions more specifically while making more detailed explanations about how he simplifies the *Integral form of the Ampere's law* equation that is less detailed in the case of Alan. Other than these differences, both the students use similar semiotic resources within the diagrammatic space mostly for same purposes.

Then in the gestural space, Danny uses quite a few semiotic resources compared to Alan using only the *finger pointing in direction* just to show the resulting magnetic field direction. Having more tools within the gestural space gives Danny the option to use frequent gestures while explaining things and as well in coordinating between other representational spaces.

Within the algebraic space, the only difference we observe is that Danny adds more detail when he uses the *Integral form of the Ampere's law* in that he uses Maxwell's correction. Other than that, again both these students use the same other two semiotic resources for calculation purposes in the algebraic space.

In comparison with Larry solving a problem that also uses the same mathematical equation, the mathematical manipulation steps associated with the *Integral form of the Ampere's law* seem to be obvious and easy for both Danny and Alan compared to Larry. Students would often use this equation while figuring out the magnetic field direction from a current wire. Also we observe Larry, who is in the same class, frequently employing the *right-hand grip rule* while solving the problem. But we do not observe Alan even use the *right-hand grip rule*. Instead Alan in this case has already determined the resulting magnetic field direction as he moves to consider a certain current component and goes on to relate it to the *integral form of the Ampere's law*. Similarly, Danny also does not make much use of *right-hand grip rule* while solving this problem.

Reason for these differences may be that Larry's problem is a long and a little complicated problem that requires him to do much work to determine how the sheet is made before finding the magnetic field, compared to the problem these two students solving. Also they might have previously seen this type of question as it involves the most basic application of *Ampere's law* to find the magnetic field of a current carrying wire.

6.4.3 Problem 2: Two charges on a line

Also, the second problem we discuss here is from the Electromagnetism I course that asks students to determine the electric field generated by two point charges on x-axis. Three students Larry (same student), Oliver and Charlie solve this problem individually during the oral exams conducted in the early part of the semester. The instructor presents the problem with a diagram (Figure 6.14) at the beginning of each session.

Instructor: Suppose (draws on the board) We have two point charges, one at $-a$ & the other at $+a$ and this one (at $-a$) has charge $-q$ and this one (at $+a$) has charge $+q$. What's the electric field looks like along the x-axis?

Figure 6.14: *Instructor-provided initial diagram for two charges*

In order to solve this problem one could use to use Coulomb's law ($E = [\frac{kq}{r^2}](\hat{r})$) to find the magnitude and the superposition principle to find the net electric field in each region (left of $-q$, between charges and right of $+q$).

Student 1: Larry

Larry solving this problem is the same student who solved the Amperian loop problem presented in previous chapters. In this case, Larry starts by predicting how he is going to solve this problem.

Larry: Alright, so I suppose let's start by finding the electric field for each of them... then summing...

Then he moves to specify the point of interest (*field point*) in the far right region (right of $+q$ that he labels region 3 later).

Larry: Um... so let's put down a point I guess (marks point p on the axis right of $+q$) and then I've to get the distance from the charge to the point... and for each of them. I think it's different in each of the three regions. I've to write a different... expression for... the distance.

After mentioning his plan, he moves to figure out the electric field for each charge.

Larry: Okay... well... alright... so the electric field from this guy (draws a circle around $+q$ charge) is gonna be... I'll call him $E{-}plus$ (starts recording mathematical equation and calls out the names of the symbols while recording) it's gonna be k q over... um... I'll just call $r - plus$ and that's gonna be...

$$E_+ = \frac{kq}{(r_+)^2} \tag{6.18}$$

Larry starts on the diagrammatic space and picks a point of interest in far-right region. Then he moves to work in the algebraic space to write a generic equation for the electric field from the $+q$ charge. Moving to describe the distance associated with his mathematical equation leads him to return to the diagrammatic space to specify the directional information.

Larry: Since it goes outward... uh... like here it's going to be this way (records on the board - a rightward arrow to right from $+q$ charge) and from here (points to the region left of $+q$) it's going to be that way (records on the board - a leftward arrow to left from $+q$ charge) and here (points to the region right of $-q$) it looks like that (records on the board - a leftward arrow towards $-q$ charge).

Within the diagrammatic space, he uses the semiotic resource *arrow as vector* to represent the direction of electric field created by each individual charge and further he labels the regions (Figure 6.15).

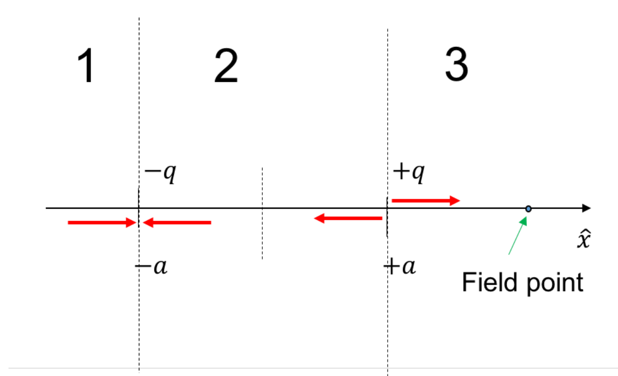


Figure 6.15: *Larry uses arrows to represent the electric field direction from each charge*

Then he continues with region 3.

Larry: Um... so... for region 3 ... uh... my $E - plus$ would be k q over... my r will be (recording mathematics) uh... x minus a ... yeah and that would be in the positive $x - hat$ direction. Then $E - minus$ will be k q over uh... x plus a ... yeah... in the minus $x - hat$ direction.

Larry starts recording mathematical equations and calls out the names of the symbols while recording. Even though Larry does not specify the coordinate of his *field point*, he seems to assume that is to be "x" as he uses it in his mathematical equations.

$$E_+ = [\frac{kq}{(x-a)^2}]\hat{x} \tag{6.19}$$

$$E_- = [\frac{kq}{(x+a)^2}](-\hat{x}) \tag{6.20}$$

Electric field direction in the diagram aligns with what Larry has in his mathematical equations. Soon Larry thinks about the sign of the $-q$ charge and realizes there is a mistake in his second equation (for $-q$) and goes to correct it.

Larry: Oh... that's a minus q as well (modifies the mathematical equation) so they cancels... never mind they points the same direction... they will always add.

$$E_- = [\frac{-kq}{(x+a)^2}](-\hat{x}) \tag{6.21}$$

Now Larry has two minus signs in his equation for $-q$ and when they are combined the mathematical equation says the electric field created by the $-q$ charge in region 3 will be along (). This does not match with what he already has on his diagram. With the instructor?s question about this inconsistency, Larry moves to reconsider it.

Instructor: So... before you had... in that region minus q charge had a... electric field to the left and now you wanted it to be to the right?

Larry: Uh... so... $E-minus$ will be minus $k\ q$ over (modifies the mathematical equation) because that minus is encoded there and so... they will end up subtracting from each other... and so... the total is the sum of these two (records a mathematical equation) and that's gonna be true for all areas.

$$E_- = [\frac{kq}{(x+a)^2}](-\hat{x}) \tag{6.22}$$

$$E_{total} = E_+ + E_- \tag{6.23}$$

Larry shows his confidence about his diagram ("I feel good about the arrows") and decides to stick to his initial equation. Larry was thinking about the negative sign of the of the $-q$ charge and he wanted to insert that into his equation but finally he realizes that the negative sign is already being included in his initial equation (-). In addition to that, Larry talks about the total electric field at the *field point* and uses the semiotic resource *net electric field* to represent it in the algebraic space.

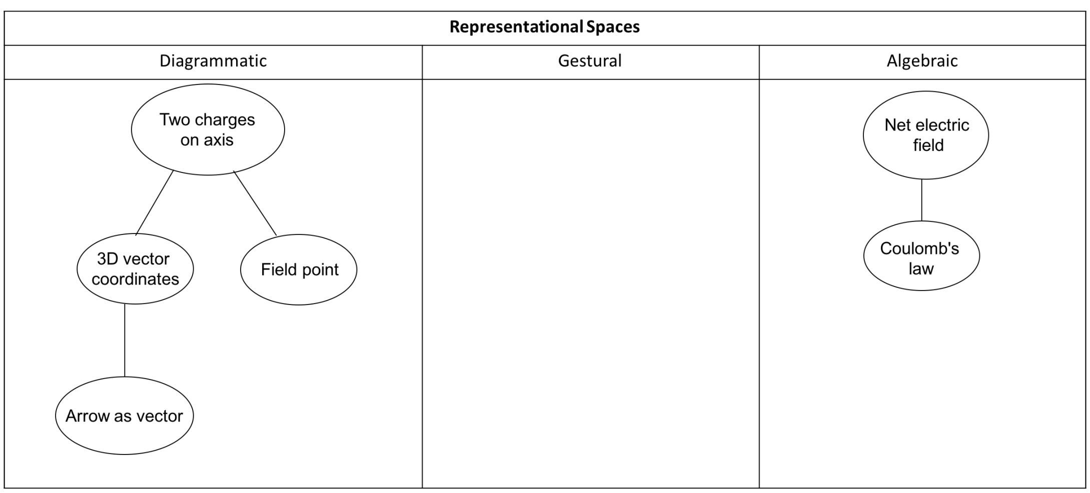


Figure 6.16: *Current stage of Larry's representational spaces*

Figure 6.16 shows the current stage of Larry's representational spaces. The semiotic resource *coordinate axis* is already included in the diagrammatic space as the initial diagram is provided by the instructor. As usual the semiotic resource *arrow as vector* allows Larry to include the directional information into his diagrammatic space and this time it is used to specify electric field direction. This semiotic resource allows Larry to visually represent the physics concept into his diagrammatic space that says the positive charge is distributing radially away from the charge while negative charge creates a field that points radially towards it. Also here we see the semiotic resources *arrow as vector* and *coordinate axis* reinforcing each other within the diagrammatic space.

So far, we do not observe Larry using any of the gestural semiotic resources. Next Larry starts by using the mathematical semiotic resource *Coulomb's law* in algebraic space to represent the magnitude of the electric field from each charge. Then he uses the mathematical semiotic resource *net electric field* to represent how he will find the net electric field in region 3.

In addition to that, we see Larry coordinating between representational spaces as he goes to imbed the electric field direction into his mathematical equations. He looks at and uses the directional information that is already being developed in the diagrammatic space.

Table 6.6: *Disciplinary affordances of the semiotic resource coordinated to build the current stage of Larry's representational spaces*

Semiotic Resource	Disciplinary Affordance
Coordinate axis	Here the one dimensional coordinate system or the coordinate axis allows Larry to determine the distances (separation between field point and charge) that then needs to be plugged into the *Coulomb's law* equation. Then this also helps him to interpret the resulting electric fields with a definite direction $((\hat{x})$ or $(-\hat{x}))$.
Arrow as vector	This allows Larry to visually represent the electric field direction. In this case this representation allows Larry to visually represent positive charges radially away and negative charges radially towards.
Field point	This allows Larry to specify the region that he is considering. Then having the *field point* labeled on the *coordinate axis* allows Larry to visualize the resulting electric field directions for a given region.
Coulomb's law	This represents the electrostatic force between charges. That is directly proportional to the product of the charges and inversely to the square of the distance between them.

Then Larry moves to consider region 2, which is the middle region, and he predicts that the process is going to be the same as the previous case.

Larry: Then for region 2... it's basically the same except... both of these are gonna be... uh... the distance from both of these gonna be... a minus x. I'm pretty sure... no I don't think that's right... they are both in the minus $x - hat$ this time.

On the bright side, the semiotic resource *arrow as vector* helps him figure out the electric field direction and Larry concludes correctly that the electric field created by both the charges in middle region will be in the (-)) direction. While considering the previous region 3, the first thing Larry did was to pick a *field point*. That allows him to determine the relative distances between the charge and *field point*. But in region 2 he has not picked a *field point* yet and we observe Larry getting confused while talking about the distances. So, he uses generic distances r_+ and r_- while representing the magnetic field in his algebraic space.

$$E_+ = [\frac{kq}{{r_+}^2}](-\hat{x}) \tag{6.24}$$

$$E_- = [\frac{kq}{{r_-}^2}](-\hat{x}) \tag{6.25}$$

After that, he moves to pick a *field point* in region 2 (Figure 6.17 a.) and then he specifies the distances in his mathematical equations.

Larry: This time the distance... if I pick a point here (points to the field point marked in the middle region) the distance is a minus x for this guy (E_+) and the distance here would be... still a plus x or x plus a.

Picking a *field point* made Larry's life much easier and we see him moving forward within the algebraic space.

$$E_+ = [\frac{kq}{(a - x)^2}](-\hat{x}) \tag{6.26}$$

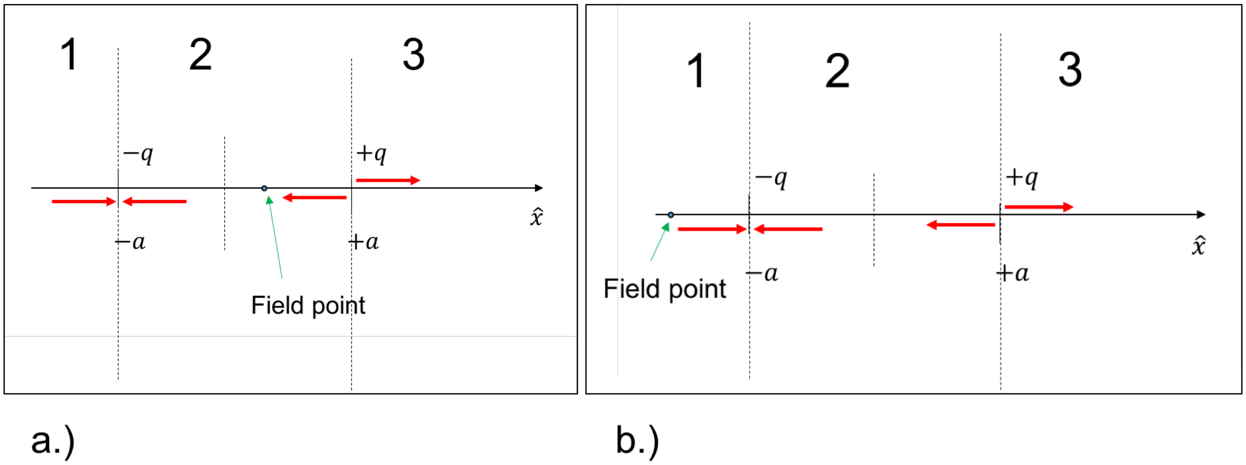


Figure 6.17: *Larry picks field points: a.) in region 2 b.) in region 1*

$$E_{-} = [\frac{kq}{(x+a)^2}](-\hat{x}) \tag{6.27}$$

With positive feedback from the instructor ("you are doing fine"), Larry moves to consider the far-left region 1.

Larry: Um... still got (calls out the names of the symbols while recording the math equation) k q over... this time... it's still going to be a minus x down here. Yeah... since x is gonna be negative... and that's in the minus $x - hat$ direction... and this (points to E_{-} equation) is going to be k q over... uh... a plus x ... no ... a minus x ... positive $x - hat$ direction.

$$E_{+} = [\frac{kq}{(a-x)^2}](-\hat{x}) \tag{6.28}$$

$$E_{-} = [\frac{kq}{(a-x)^2}](\hat{x}) \tag{6.29}$$

Again, we see Larry defining the resulting electric field directions with ease. This is because his diagrammatic space helps him see the directions clearly. Even though Larry picked a *field point* in region 1, we still see him hesitating while figuring out the distances.

Still he seems to be concerned about the distances in his mathematical equations ("No that's not right").

Instructor: Which part of it's not right?

Larry: Um… this (points to E_- equation) should be plus (modifies the equation) yeah… that effectively shortens the distance… yeah.

$$E_- = [\frac{kq}{(x+a)^2}](\hat{x}) \tag{6.30}$$

Going on to explain how and why he changed the sign in his mathematical equation, Larry starts to use the diagram on the board and uses the gestural semiotic resource *finger pointing in direction* but eventually he finds the best way to do that is by labeling the distances on the diagram.

Larry: So… like intuitively I know this point (points to field point in region 1) closer in region 1. So… the a and the x should interfere with each other and since x is negative then it should be positive… um… I mean if you just look at it … your distance is this (gesture- fingers at $-q$ and field point) your total distance is this (gesture- fingers at mid point and field point) and this is your a (draws lines to specify the distances on the diagram) and this is your x … this is much better way to explain it.

First, he moves to label distances in region 1.

Larry: I change my mind… and this (draws lines to specify the distances on the diagram) is your x … and this is your a and your total distance… is wait… I'll call this minus x I guess.

Even with this new approach, Larry seems to be confused about whether or not to consider the negative sign when he considers the *field point* on the negative side of the *coordinate axis.* Even though ultimately he decides "I'll call this minus x," he still seems to be confused.

Because of this confusion, Larry skips region 1 for now and returns to label the distances in region 3 and 2 respectively. Then he rechecks his new approach with region 3. He considers the electric field of the $+q$ charge and gets the same expression as what he already has. With satisfaction and with confidence, he returns to consider region 1.

Larry goes on to figure out the electric field in region 1 from the $+q$ charge.

Larry: And then here we got x and then a. So that should be x plus a (modifies the electric field equation for region 1)

$$E_+ = [\frac{kq}{(a+x)^2}](-\hat{x}) \tag{6.31}$$

Then he moves to figure out the electric field of the $-q$ charge.

Larry: I've got x and a ... so it should be x minus a ... uh... or if I wrote it a minus x ...

$$E_- = [\frac{kq}{(a-x)^2}](\hat{x}) \tag{6.32}$$

Here he again calls the coordinate of the *field point* in region 1 x while developing both equations E_+ and E_-. But two steps earlier, Larry himself decided that he is going to call it $-x$. Soon he realizes the mistake.

Larry: No it doesn't still come out right. I'm getting hung up on how the ... in here (points to region 1) my x is always going to be negative ... so... yeah it should be... I'll write it negative x plus a ... so a minus x.

$$E_- = [\frac{kq}{(-x+a)^2}](\hat{x}) \tag{6.33}$$

Even though he mistakenly called it just x while developing both equations E_+ and E_-, he makes only the correction for E_- equation.

After talking to the instructor further about it, Larry again changes his mathematical equation. But he does this only for E_-.

$$E_- = [\frac{kq}{(x+a)^2}](\hat{x}) \tag{6.34}$$

Larry kept the E_- term change and now he has both E_+ and E_- as the same and opposite (equations 6.31 and 6.34) that cancel each other. This leads Larry to make another sign change.

Larry: No they shouldn't ... they should be ... (modifies the mathematical equations) it should be x minus a.

$$E_+ = [\frac{kq}{(x-a)^2}](-\hat{x}) \tag{6.35}$$

This time he changes the sign of E_+ but it is not necessary. After these changes, Larry seems to be happy with his final expressions. In the next few minutes of this session, the instructor asks Larry to compare his expressions for all three regions.

This particular problem does not require much use of the gestural space compared to Larry's previous problem or the loosely wound solenoid problem. But this problem requires Larry to coordinate between representational spaces - diagrammatic and algebraic. It requires students to refer to the diagrammatic space for two reasons - to determine the distances and to determine the electric field directions while working on the algebraic space.

Figuring out the electric field on the positive side of the coordinate axis (regions 2, 3) seems to be relatively easy for Larry compared to doing it on the negative side. The reason behind this is the disciplinary affordance of the semiotic resource *Coulomb's law*. One reason is that the students are required to consider the field direction but not the charge signs. Then when considering the negative side of the coordinate axis and figuring out the distances, students need to consider the negative coordinate of the *field point*.

Student 2: Oliver

Oliver is the second student to solve this problem and the instructor draws a similar diagram (Figure 6.14) while presenting the problem. He starts by asking a question to make sure what the instructor is looking for.

Oliver: Okay… So you just … you just want an expression for it or …

Instructor: Sure.

Oliver: So … the total field is going to equal the contribution from each of them, because of the superposition (records math on board) … I want to just consider along the x-axis?

$$\vec{E} = E_{+}q + E_{-}q \tag{6.36}$$

Similar to Larry, Oliver also starts to solve this problem by talking about how the electric field from each charge is going to be added or subtracted to figure out the *net electric field*.

Oliver: Cool … so for … (records on the board) $E_{+}q$ … (steps away from the board) it's going to equal … (he calls out the names of the symbols while recording the math equation) one over four pi epsilon note … times … q … um… curl r squared … r hat.

$$E + q = \frac{1}{4\pi\epsilon_0}\frac{q}{\Gamma^2}\hat{\Gamma} \tag{6.37}$$

Oliver moves to consider the electric field created by the $+q$ charge. Since he has not yet specified any region, he uses a generic mathematical symbol r for his distance in the*Coulomb's law* equation. Even though the instructor presents the problem along with a diagram, both these students approach solving the problem by using mathematics. Maybe they are trying to get a mathematical expression for electric field. But eventually Larry moved to the diagrammatic space relatively early and kept referring back to it while working on the algebraic space.

Now Oliver moves to specify the distances.

Oliver: Um… so… um… I'm trying to think. I'm sorry… this r is going to equal… um… (long pause).

Instructor: You look like you are stuck. Where are you stuck?

Oliver: Um… I'm … figuring out this (points to Γ on the board) it's (records on the board, $\Gamma = (a - x)$) and then x is anywhere here (indicate the region between a and $-a$ on the axis) right? … no I don't know if that works.… that's what we have in notes.

$$\Gamma = (a - x) \tag{6.38}$$

Oliver starts to move forward by figuring out the distance in his mathematical equation but still he shows some uncertainty about it. This is because so far he has not specified a region or picked a *field point*. He refers back to his notes and he talks to the instructor to do a clarification about 'script dr (Γ)' in his equation ("I'm forgetting is it $r - prime$ minus r or r minus $r - prime$?"). Finally Oliver understands that it should not make any difference because he is going to get Γ^2.

Oliver: Okay, that's what I was trying to figure out … so, x minus a … and so … then we substitute that in (modifies the math equation on the board).

$$E + q = \frac{1}{4\pi\epsilon_0}\frac{q}{(x-a)^2}\hat{\Gamma} \tag{6.39}$$

So far Oliver has not mentioned the direction of this electric field. Then he moves to talk about the electric field of the $-q$ charge.

Oliver: Then E for negative q equals... (records on the board) I'm going to change this ($\frac{1}{4\pi\epsilon_0}$) to k because it's easier... negative q... and then we are going to have ... x plus a squared and then r-hat...

Oliver worked to develop two mathematical equations that represent the electric fields of $+q$ and $-q$. In his equation, he has generic for the direction and then, according to Oliver, x that could be any point along the axis. Also he substitutes for his mathematical symbols ($k = \frac{1}{4\pi\epsilon_0}$).

$$E + q = \frac{kq}{(x-a)^2}\hat{\Gamma} \tag{6.40}$$

$$E - q = \frac{k(-q)}{(x+a)^2}\hat{\Gamma} \tag{6.41}$$

Here Oliver starts to talk about the electric field direction and he also gives a hint about a *field point* ("it depends on the point we pick...").

Oliver: Um...so... r-hat just um r-hat... to any region in pointing on the x-axis (points to x-axis on the diagram) on that direction and I just think it depends on the point we pick or we want the whole axis. right? So we just change that to x-hat, does that work?

Then the instructor asks a follow-up question that leads Oliver to expand on his idea and soon he switches to the diagrammatic space.

Instructor: Well... that says the charge from the plus q charge or the electric field from the plus q is always in the x-hat direction.

Oliver: Um... but it could be negative (points the index finger along the negative x direction on the diagram) ... and so that would work.

Here Oliver identifies that his earlier statement about changing r-hat to x-hat would not work and he further associates the gestural semiotic resource *finger pointing in direction* while reasoning.

Oliver: So... the field is going to point out here... and here (draws arrows)...

Then he moves to the diagrammatic space and uses the semiotic resource *arrow as vector* to represent the electric field direction from each charge (Figure 6.18). Oliver's next step is very similar to what Larry did in early parts of his solution. He divides the diagram into regions (which is the same as what Larry did).

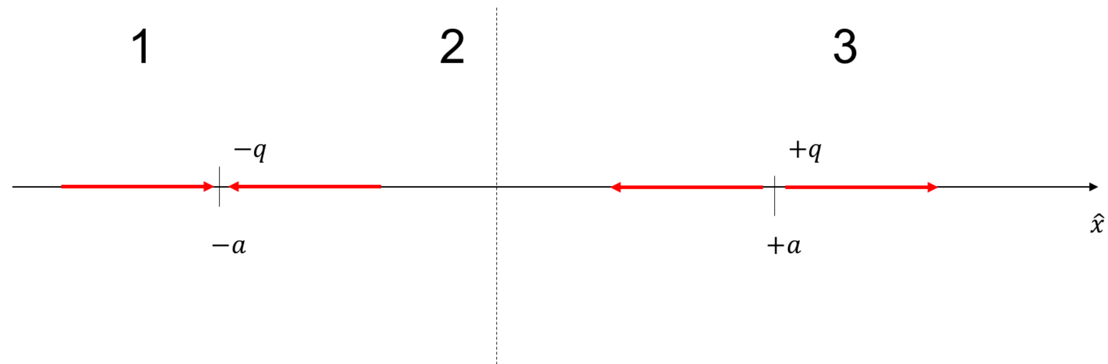


Figure 6.18: *Oliver uses arrows to represent the electric field direction from each charge*

Now he moves to talk about the electric field in each region.

Oliver: So... Um... one and three are zero... because they... they are going to (gestures-index fingers towards each other) be destructive... or is... I guess... out here (points to region 3) ... does this one (points to $-q$ charge) ... this is gonna have... um... a greater effect than the contribution from this (points to $+q$ charge) or just...

Oliver reasons the *net electric field* in regions 1 and 3 to be zero because the electric fields of each charge are in opposite directions. Even though he considers the direction of the electric fields, he does not seem to think about the magnitudes that his mathematical equations *Coulomb's law* represent. But during the later part of his statement, he seems to have understood the effect of the distance.

Different from Larry, Oliver at this very moment uses the gestural semiotic resource *finger pointing in direction* to show the electric field direction.

He moves to talk specifically about region 1.

Oliver: so... in area 1... it's gonna be (points to the mathematical equation $\vec{E} = E_+q + E_-q$). This one (E_-q) minus this one ($\vec{E} = E_+q$)... right?

Oliver is eventually getting to the correct equations, and he now incorporates the directional information embedded in the diagrammatic space into his mathematics. But he does not mention the *net electric field* direction yet.

Instructor: So... okay... and what will be the overall direction of E?

Oliver: E is going to be to the right (index finger moves rightward).

Instructor: Okay, is that plus x-hat or minus x-hat?

Oliver: Plus x-hat.

Oliver concludes the *net electric field* direction in region 1 (far left region) to be in the x-hat direction. While doing so, Oliver again uses the gestural semiotic resource *finger pointing in direction.* Next he moves to the algebraic space to represent the *net electric field.*

Oliver: So... E is going to equal (records on the board) ... k times negative q ... x plus a squared ...minus q ... that doesn't seem right though.

$$E_1 = k[\frac{-q}{(x+a)^2} - \frac{q}{(x-a)^2}]\hat{x} \tag{6.42}$$

A few steps ago, Oliver reasoned that the electric fields from the two charges are going to be opposite to each other. But his mathematical equation shows that they are in the same direction, the - direction. Importantly, Oliver's diagrammatic space already has opposite information. Right here Oliver seems to be coordinating between his diagrammatic and his algebraic spaces and soon he realizes the mistake. Also, he seems to believe more in his diagrammatic spaces, exactly same as what Larry previously did in this problem.

Oliver: Well... if these are both negative (points to the two terms in his mathematical equation) then looks like this... I'm not sure about that. It looks like this... would add together… but I have different denominators… so I can't say for sure. Um… because the field from this guy (points to $+q$ charge) in here (points to region 1) is going to reduce the field from this guy (points to $-q$ charge) out here (points to region 1).

While talking, Oliver refers back to his diagram on the board. The Instructor tries to help him sort out this issue.

Instructor: Okay and right now they have the same sign as each other or they have a different sign from each other?

Oliver: Um… right now it looks like they have same as each other. But… yeah… the only sign difference is going to be the cross product of the binomial part.

Instructor: Okay… is that going to give you an overall sign changed to your term?

Oliver: I don't think so. I guess you could pull the sign off… but…does it?

Instructor: No it doesn't.

Oliver: That's not right.

Instructor: So, you should have opposite signs from each other.

Oliver: Right.

Instructor: But your math doesn't say that.

At the end of this conversation between the instructor and Oliver, he agrees that the terms have to have different signs from each other, and he moves to change the signs. One reason for this agreement may be use of his previous conclusion about the electric fields being in opposite directions.

Oliver: Right… so I'm not wrong probably am I?… oh… because this (points to the second term in his mathematical equation) should be a plus (makes changes in his mathematical equation) because this still… it shouldn't be minus. That's my bad.

$$E_1 = k[\frac{-q}{(x+a)^2} + \frac{q}{(x-a)^2}]\hat{x} \tag{6.43}$$

The sign change does not make Oliver's mathematical equation correct but the two terms are still opposite in direction. Then the instructor decides to ask a follow-up question.

Instructor: So... the contribution from the electric field from the plus q charge points in the plus x-hat direction?

Oliver: No... oh man... this should be minus x-hat. you see then this (points to the first term in his mathematical equation) should be plus.

Instructor: They should both be plus?

Even though Oliver reasons correctly, his mathematical equation shows the exact opposite.

Oliver: No, one should be minus... this... this... okay... so... this (points to the second term in his mathematical equation) is this one (points to $+q$ charge) right... this points the negative x-hat direction out here (points to region 1). And then this one (points to the first term in his mathematical equation) points opposite right here (points to region 1) because it's pointing in...

Instructor: Good. So, what does your math say about that?

Oliver: It says that's positive.

Instructor: How would you fix that?

Oliver: Doing the opposite (changes the sign in his mathematical equation) that (first term) should be a plus and this (second term) should be a minus.

$$E_1 = k[\frac{q}{(x+a)^2} - \frac{q}{(x-a)^2}]\hat{x} \tag{6.44}$$

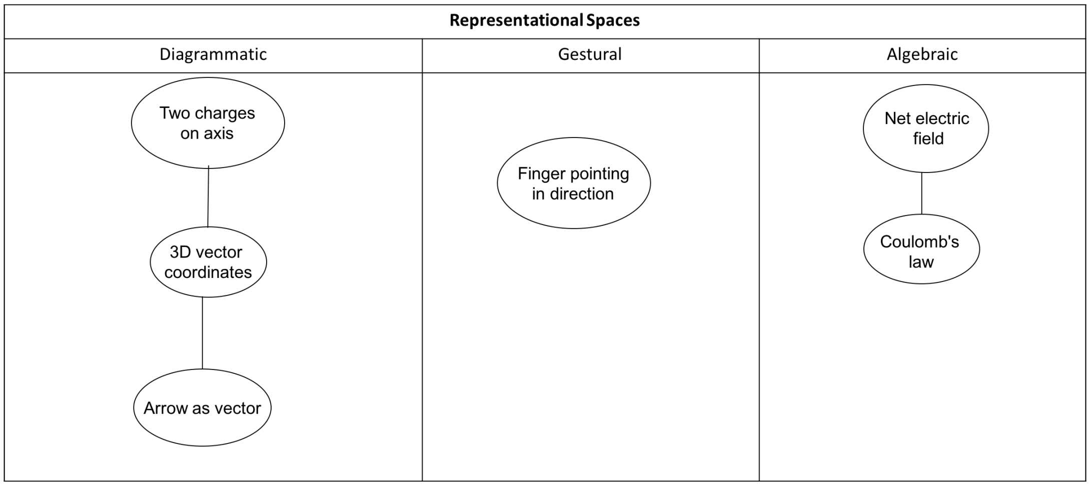


Figure 6.19: *Current stage of Oliver's representational spaces*

Figure 6.19 shows the current stage of Oliver's representational spaces. Similar to the case of Larry, the semiotic resource *coordinate axis* is already included in the diagrammatic space and the other semiotic resource *arrow as vector* allows him to specify electric field directions. The difference between Larry and Oliver is that Oliver does not pick a *field point*. Instead he keeps referring to the whole region. This might prevent him from considering a point on the axis while considering the electric field direction and the distance.

Oliver has the same semiotic resources as Larry in his algebraic space and both of them make the same use of it. This may be because this problem requires a limited number of resources in algebraic space.

Unlike Larry, Oliver used the gestural space quite a few times. Importantly, Oliver uses the gestural space along with other two representational spaces. He makes frequent translations between representational spaces.

With the instructor's suggestion, Oliver moves to consider region 2 and he starts by talking about the relative electric field directions and he further uses the semiotic resource *finger pointing in direction* to support his explanation.

Oliver: Region 2, they are going to be constructive... So, E-two... by constructive, I mean they both pointing in same direction (gestures-index fingers in same direction). So... it would be...

Table 6.7: *Disciplinary affordances of the semiotic resource coordinated to build the current stage of Oliver's representational spaces*

Semiotic Resource	Disciplinary Affordance
Finger pointing in direction	It allows Oliver to show the electric field direction. Importantly it helps him to represent the destructive nature of the superposition in the gestural space. Also, this gestural activity helps him best communicate his idea to the instructor. It also helps him specify different regions on the coordinate axis.

Again in this region, Oliver moves on to work on mathematics without picking a *field point*. We again observe him getting stuck half way through his explanation.

Oliver: Um... k um... so... this (points to $-q$ charge) is my x charge but it has negative x-hat direction (gestures-index fingers along negative x). So this is going to be (starts recording mathematical equation for the electric field in region 2) q over x plus a squared. This (points to $+q$ charge) um... wait a second (steps away from the board) ... that one is going to be negative then... that does not make sense.

He decides to consider one charge at a time.

Oliver: Yeah, um... let's just do both separately. (Oliver calls out the names of the symbols while recording mathematics o the board) negative q over x plus a squared and that's going to be negative x-hat. Since it's backward... plus kqx minus a squared... negative x-hat.

$$E_2 = k\frac{-q}{(x+a)^2}-\hat{x} + k\frac{q}{(x-a)^2}-\hat{x} \tag{6.45}$$

Again, Olive's mathematical equation contradicts what he mentioned when he moved to consider this region. But this time he identifies his issue.

Oliver: I do not worry about the sign (charge sign) but do worry about the field points at. that's what I'm keeping up on.

Quickly he modifies his mathematical equation.

$$E_2 = k\frac{q}{(x+a)^2} - \hat{x} + k\frac{q}{(x-a)^2} - \hat{x} \tag{6.46}$$

$$E_2 = kq[\frac{1}{(x+a)^2} + \frac{1}{(x-a)^2}] - \hat{x} \tag{6.47}$$

Instructor: Okay... what's region 3 look?

Having identified his mistake in region 2, Oliver moves really quickly in this region without any mistakes.

Oliver: Region 3... is going to be much like region 1 except exactly the opposite. so... (records on the board) kq over x plus a squared and this is going to be negative x-hat... and plus q... x minus a squared and going to be positive x-hat, right?... so... for this charge ($-q$) the field is pointing inwards (gesture-index finger to the left) because this charge ($+q$) is pointing out. So that's going to be k minus q x plus a squared plus q over x minus a squared and x-hat for both of them.

$$E_3 = k[\frac{q}{(x+a)^2} - \hat{x} + \frac{q}{(x-a)^2}\hat{x}] \tag{6.48}$$

$$E_3 = k[\frac{-q}{(x+a)^2} + \frac{q}{(x-a)^2}]\hat{x} \tag{6.49}$$

Even with Oliver, we observe the same issue as Larry came across when figuring out the electric field on the negative side of the *coordinate axis*. The reason behind this is the disciplinary affordance of the semiotic resource *Coulomb's law* (detailed description is provided in section 6.4.4).

Overall, because of the nature of this problem, both of the students Larry and Oliver seem to be using a limited number of semiotic resources in all three representational spaces compared to what Larry uses when he solves the *Amperian loop* problem. Also, we observe that Oliver's use of gestural semiotic resources stands out when we compare his and Larry's approaches in this problem.

Student 3: Charlie

Charlie solves this problem as our third participant and he also gets the diagram (Figure 6.14) with the problem explanation. Charlie's approach to solving this problem is different from both Larry's and Oliver's. Instead of getting started on the algebraic space, Charlie uses the instructor-provided diagram and uses the semiotic resource *arrow as vector* along the *coordinate axis* to represent the electric field direction of each charge on the diagrammatic space.

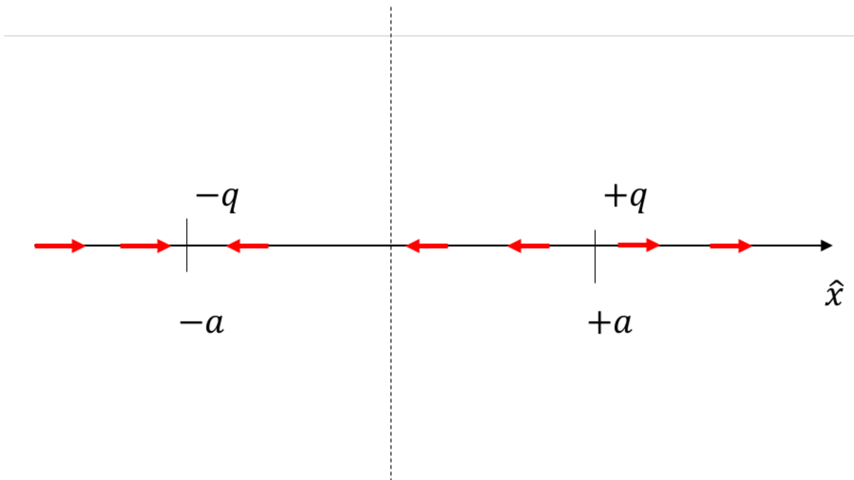


Figure 6.20: *Charlie uses arrows to represent the electric field direction from each charge*

Charlie: Positive radiates out and negative radiates in and in the middle, it's... coming over to the negative

After the instructor asks "shall we do some math?" Charlie moves to work on the algebraic space and records *Coulomb's law.*

Charlie: Um... we wanna do the total charge... r squared r-hat.

$$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r} \tag{6.50}$$

While recording the equation, Charlie specified the Q as total charge and right after recording the equation he defines his total charge.

Charlie: And then Q is going to be $2q$ negative $2q$ … with… along x direction. So… x-hat … with r (points to r in the equation) being a distance of a.

$$Q = -2q \tag{6.51}$$

$$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{x} \tag{6.52}$$

Even though Charlie seems to define the electric field direction; so far, he has not mentioned any region.

Charlie: So, the… just the electric field… I don't know… I guess… just generally… I guess this is the electric field.

$$E = \frac{-1}{4\pi\epsilon_0}\frac{2q}{r^2}\hat{r} \tag{6.53}$$

Charlie states that this equation represent the electric field in space "just anywhere". And then relating it to this problem he says,

Charlie: Just all along the axis… So, it's going to be in the x direction

$$E = \frac{-1}{4\pi\epsilon_0}\frac{2q}{r^2}\hat{x} \tag{6.54}$$

The instructor's follow-up question leads Charlie to move from a general electric field to specifying the electric field for each region (Figure 6.20).

Instructor: Always in the plus or the minus x?

Charlie: Um… well… for over here (points to the left of $-q$) it's in the positive x direction, then between here and here (region between two charges) it's in the negative x direction and here (points to the right of $+q$) to the positive x direction again.

Instructor: Okay. It sounds like you should write electric field for each region.

Charlie starts to record the electric field for each region, starting from far left, middle and far right regions. Here we observe that Charlie uses his general equation for electric field and then inserts the directional information that is available in the diagram to generate specific electric field equations for each region.

$$E_{left} = \frac{1}{4\pi\epsilon_0}\frac{2q}{r^2}\hat{x} \tag{6.55}$$

$$E_{middle} = \frac{-1}{4\pi\epsilon_0}\frac{2q}{r^2}\hat{x} \tag{6.56}$$

$$E_{right} = \frac{1}{4\pi\epsilon_0}\frac{2q}{r^2}\hat{x} \tag{6.57}$$

While recording the electric field for the far-right region, Charlie talks about the superposition and he starts to think about the contribution from charges separately.

Charlie: However... am I going to be getting a different electric field from this charge (points to $-q$ since it's further away? So, this $2q$ (points to mathematical equation recorded for far-left region) going to be something different. So... am I suppose to do superposition too?

Then he modifies the electric field equation for far-left region.

Charlie: So, rather than this (points to mathematical equation recorded for far left region) we could have something like (calls out the names of the symbols while recording the math equation) E equal k positive q over $2a$ squared.

$$E = k[\frac{q}{(2a)^2}] \tag{6.58}$$

Charlie seems to think of the contribution from one charge as he uses q instead of $2q$ but still he has not thought about the distance to that charge and the contribution from the other charge. Then the instructor's question leads him to pick a *field point* which he has not considered yet.

Instructor: Okay, so, where are you looking at in this region?

Charlie: Will see. Could we look right here (points to the same position as $-q$ is on the axis) or right beyond it?

Instructor: We could, but we should.

Charlie: Then this one (points to mathematical equation recorded for far left region) could be... it could be some distance x. What I'm referring is x (labels the distance from $+q$ to the field point in left of $-q$). So just x squared (modifies the mathematical equation).

$$E = k[\frac{q}{(x)^2}] \tag{6.59}$$

Charlie picks a *field point* in the far left region and then he goes on to define the distance from the positive charge to the *field point* to be x and he modifies the mathematical equation. Picking the *field point* has allowed him to think of how the distances are related in the mathematical representation of the electric field. Soon he rethinks (Figure 6.21) how he measures the distance x; and for the first time, he talks about the *net electric field* in this region as a contribution from each charge.

Charlie: And then, it also... I'm going to re-define my x. I'm going to say my x is from this point (points to mid point on axis). Makes this (modifies the mathematical equation) x plus a squared. Sounds like a good idea? And this is going to be... um... x minus a squared.

$$E = k[\frac{q}{(x+a)^2} + \frac{-q}{(x-a)^2}]\hat{x} \tag{6.60}$$

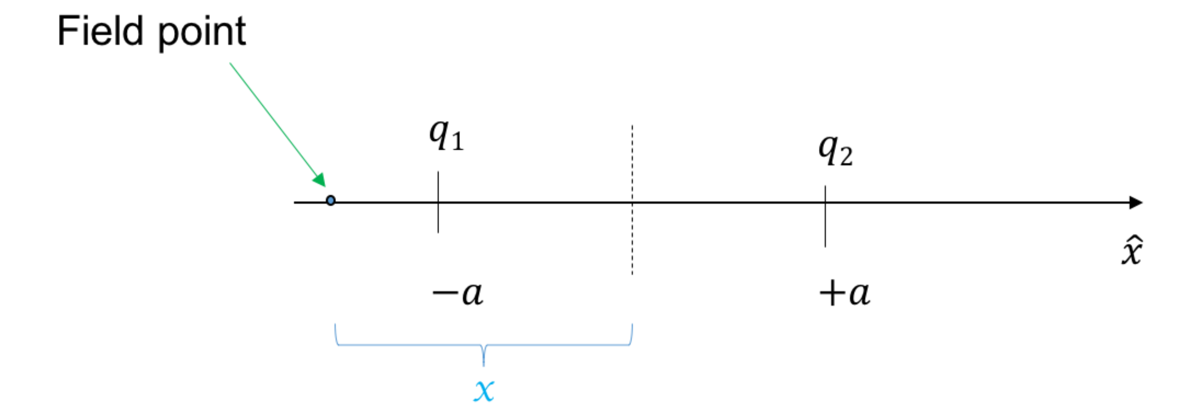


Figure 6.21: *Charlie picks a field point in far-left region*

Here we observe Charlie coordinating between his diagrammatic and algebraic spaces. On his diagrammatic space shows the electric field (*net electric field*) to be on the direction. Then he thinks his mathematical equation on the algebraic space does not agree with that.

Charlie: Now we got a whole different... Now that says it's going to be negative.

Instructor: What's negative?

Charlie: The field over here (points to the left of $-q$) because this (points to the term $\frac{-q}{(x-a)^2}$ of the mathematical equation) is going to be bigger than this (points to the term $\frac{q}{(x+a)^2}$). Because the bottom is going to e smaller than that (points to the term $(x+a)^2$).

Instructor: Okay. How would you adjust this?

Charlie: Change my negative sign (laughs)?

So far we observe Charlie developed diagrammatic and algebraic spaces to represent the electric field in far-left region. But when he coordinate between these to representational spaces he identifies a mismatch. Charlie also has a problem working in the negative side of the *coordinate axis* (that is common for previous two students Larry and Oliver) but there is no argument behind his suggestion to fix the issue.

Then with the instructor's suggestion to identify contribution from each charge in his mathematical equation, charlie relabels the charges. He labels $-q$ charge as q_1 and $+q$ charge as q_2 and then modifies the mathematical equation.

$$E = k[\frac{q_2}{(x+a)^2} - \frac{-q_1}{(x-a)^2}]\hat{x} \tag{6.61}$$

then he combines the two negative signs.

$$E = k[\frac{q_2}{(x+a)^2} + \frac{q_1}{(x-a)^2}]\hat{x} \tag{6.62}$$

This mathematical equation shows that the contribution from both charges to be in the same direction which does not agree with Charlie's previous conclusion about the electric field directions. So he realizes this issue and modifies the equation.

Charlie: we just do that (modifies the mathematical equation) because that has a negative direction. Now this one is bigger as earlier said and would be positive.

$$E = k[\frac{-q_2}{(x+a)^2} + \frac{q_1}{(x-a)^2}]\hat{x} \tag{6.63}$$

Instructor: Okay.

Charlie justifies his answer using diagram along with the gestural space.

Charlie: Because that one (points to $+q$) should be pushing it (points to $-q$) away (gesture-hand moves leftward) over here (points to left of $-q$).

Charlie successfully argues for the electric field directions using the semiotic resource *finger pointing in direction* but representing that in the algebraic space is tricky and difficult (for the negative side). This is agin due to the nature of the disciplinary affordances of the semiotic resources *Coulomb's law* and the *coordinate axis* (see section 6.4.4).

Then he moves to consider the middle region and starts with the mathematical equation. As Charlie has not yet picked a *field point* in middle region he gets stuck. After a long pause and a conversation with the instructor Charlie picks a *field point* and labels the distances.

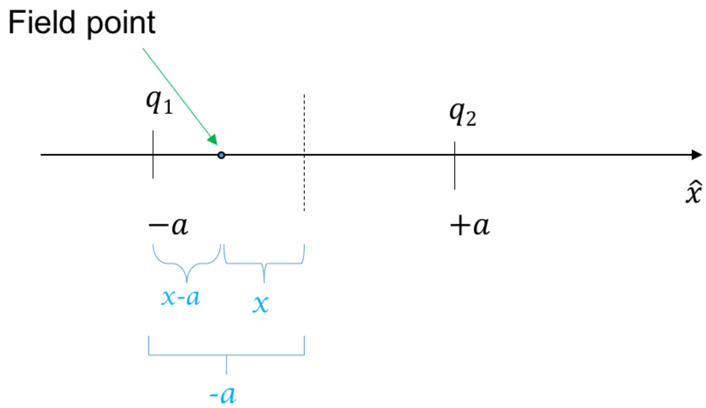


Figure 6.22: *Charlie picks a field point in middle region*

First Charlie labels the distance from midpoint to the *field point* as x but soon after he changes his mind and express the uncertainty of whether to label it $-x$. Charlie picks his *field point* in middle region to be on the negative side of the *coordinate axis*. He again comes across the same difficulty as in the far-left region. This is exactly the opposite of Larry picking his *field point* in middle region to be on the positive side of the *coordinate axis* and he does not came across any difficulty.

Right here, Charlie's words also express the level of difficulty working on the negative side of the *coordinate axis*. He says "I got negative and positive signs switched around" and later "But then my negative is over in the x positive direction. I need to switch them".

He starts over on the far-left region with labeling x to be $-x$.

Charlie: Okay... over here (marks the field point in the region to the left of $-q$) this is going to be negative x (Charlie re-defines the distance on the diagram). So the negative x plus a... that has to be minus a because it's going in that direction too. Should make this... negative x minus a (modifies the equation). That's fair.

$$E = k[\frac{-q_2}{(-x-a)^2} + \frac{q_1}{(-x-a)^2}]\hat{x} \tag{6.64}$$

Here we observe using the coordinate of the *field point* available in diagrammatic space into the algebraic space. But he only considers and modifies the contribution term from q_2

charge but ignores the contribution term from q_1 charge. This makes his correct approach ineffective. Ultimately, Charlie develops a mathematical equation that has both denominator terms looks similar which is incorrect.

Then the instructor asks about this issue.

Instructor: That means they both the same as each other?

Charlie: Um... plus a (modifies the equation) negative x minus negative a. Should be minus x and plus this a to get that distance.

$$E = k[\frac{-q_2}{(-x-a)^2} + \frac{q_1}{(-x+a)^2}]\hat{x} \tag{6.65}$$

Ultimately his approach becomes effective as he treats both charges to get the net contribution. Then he moves to consider the middle region. He goes back to call it x instead of $-x$.

Charlie: Still over here (points to the region between q_1 and q_2) oh man... um... may be I'll just make my positive x over here (points to the diagram). I kind of lost my chain of thoughts... where I was?

First Charlie gets confused but the instructor reminds him how he tackled the issue in previous far-left region. Still he seems to be confused working in this region and gets the electric field to be,

$$E = \frac{1}{4\pi\epsilon_0}[\frac{-q_1}{(x+a)^2} + \frac{q_2}{(-x+a)^2}] \tag{6.66}$$

Instructor: Are both these in minus x-hat or positive x-hat or one in each?

While reasoning for the *net electric field* direction, Charlie successfully uses the semiotic resource *finger pointing in direction* on the space of the diagram on the board.

Charlie: Um... (records x-hat and modifies the equation) oh yeah. because they should... they both pushing the negative x-hat direction. So the field should be going in the negative x-hat direction (gesture-right hand moves leftward). Yes.

$$E = \frac{1}{4\pi\epsilon_0}[\frac{-q_1}{(x+a)^2} + \frac{-q_2}{(-x+a)^2}]\hat{x} \tag{6.67}$$

Instructor: Okay. What's the third region look like?

Working on the far-right region looks starlight forward for Carlie and without much effort he gets the electric field.

Charlie: This is going to be the same as that (points to the electric field equation in far left region) with (records equation) ... yeah.

$$E = k[\frac{-q_1}{(x+a)^2} + \frac{q_2}{(x-a)^2}]\hat{x} \tag{6.68}$$

In summary, Charlie's use of semiotic resources looks very similar to that's of Oliver and his representational spaces looks similar to Figure 6.19. Importantly, Charlie gets the use of the gestural space often than Larry while coordinating between diagrammatic and algebraic spaces.

While working on the negative side of the *coordinate axis*, only Charlie seems to understand how to insert coordinate information decided from the *coordinate axis* into the distances associated with *Coulomb's law*. He is successful when considering the far-left region but again he gets lost when working in the middle region. But working in the far-right region is very easy for Carlie that aligns with what we observe for other two students as well.

6.4.4 Comparison between students: Two charges on a line

Within this problem we observe students use diagrammatic and algebraic spaces with totally different approaches and this is true for all the three students. While working on the diagrammatic space students seems to think of a *Universal electric field*. They think of as positive charge is distributing radially away from the charge while negative charge creates field that points radially towards it. Then students use the semiotic resource *arrow as vector* to represent electric field in the space. In contrast, while working on the algebraic space students seems to think of a *Regional electric field*. Where we observe students using the mathematical semitic resource to *Coulomb's law* represent the electric field by each large and then combine those equations to *net electric field* in each region.

Charlie's approach to solving this problem is different compared to the other two students. Carlie stars on the diagrammatic space and then he moves to some mathematical calculations. But, both Larry and Oliver stars on the algebraic space with more generic mathematical equations for the electric field but eventually, they have to switch to the diagrammatic space.

Larry uses a generic distance in his mathematical equation and going to specify the distances makes him move to diagrammatic space to specify the electric field directions and to pick *field point*s in each region. Oliver goes further than Larry within the algebraic space to space to specify the distances from each charge without picking *field point*s in each region. Even though Charlie does not start with math, when he moves to algebraic space he starts with the idea of net charge Q $(2q)$. Finally, he moves to think of the contribution from each charge and moves to pick *field point*s.

Out of the two students pick *field point*s, Larry picks two of them to be on the positive side of the *coordinate axis* (that are in middle and far-right regions) compared to Charlie picking two of them to be on the negative side of the *coordinate axis* (that are in middle and far-left regions). Both Larry and Charle figure out the electric field in the positive side of the *coordinate axis* easily compared to the *field point*s on the other side.

Overall, all three students use semiotic resources (even though the number is small compared to in other problems presented in this dissertation) to buildup representational spaces and connection among those spaces to solve this problem. While working on the algebraic space, we observe students often translating back and forth to the diagrammatic space to check the consistency. For Larry and Charle the background knowledge (electric field direction from each charge in regions, field point in each region) that is already developed on the diagrammatic space helps them to make these transitions with ease. But in Oliver's case, we observe the missing information of *field point*s in the diagrammatic space preventing him from making productive connections between the representational spaces.

Figuring out the electric field in the positive side of the *coordinate axis* seems to be relatively easy for all three students compared to doing it in the negative side. The reason behind this is the disciplinary affordance of the semiotic resources *Coulomb's law* and *coordinate axis*.

Disciplinary affordance of *coordinate axis*

Choosing a coordinate system for 1-D space that is, for a straight line - involves choosing an origin (mid-point), a unit of length, and an orientation for the line. An orientation choice which of the two half-lines determined by mid-point is the positive, and negative, we then call the line is oriented or points from negative half towards the positive half. According to this convention, positive numbers always lie on the right-side of mid-point on side. Then we can write the coordinate of the $+q$ charge in this problem as $x_{+q} = a(\hat{x})$. Negative numbers always lie on the left-side of mid-point on - side. Similarly we can write the coordinate of the $-q$ charge as $x_{-q} = a(-\hat{x})$.

Disciplinary affordance of *Coulomb's law*

Coulomb's law says the force on a test charge (Q) due to a point charge (q) at a rest rest distance Γ away is given by,

$$F = \frac{1}{4\pi\epsilon_0}[\frac{Qq}{\Gamma^2}]\hat{\Gamma} \tag{6.69}$$

Then the electric field of the source charge q is

$$|F| = Q|E| \tag{6.70}$$

Then the magnitude of the electric field is given by,

$$|E| = \frac{1}{4\pi\epsilon_0}[\frac{q}{\Gamma^2}] \tag{6.71}$$

To get the direction we have to consider the charge sign. Same charges repel each other while opposite charges attract each other.

Here $\vec{\Gamma}$ is the separation vector from $\vec{r'}$ (source point or the location of q) to $\vec{r}$ (field point

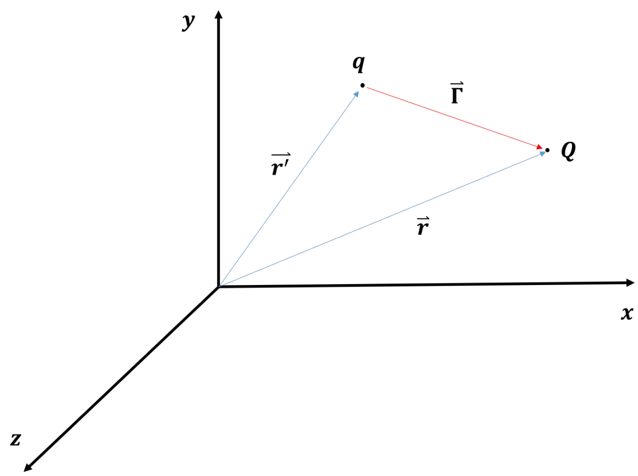


Figure 6.23: *Separation vector*

or the location of test charge Q). Then,

$$\vec{r} = \vec{r'} + \vec{\Gamma}(6.72)$$

$$\vec{\Gamma} = \vec{r} - \vec{r'} \tag{6.73}$$

Relating this in to our 1-D problem and using the separation in the electric field equation,

$$\vec{x_{seperation}} = \vec{x_{field\,point}} - \vec{x_{source\,point}}(6.74)$$

Field points in regions and the electric field

Some one going to find the electric field along the axis have to take in to consideration of the above mentioned disciplinary affordances.

Case 1: *field point* in far-right region

Contribution from $+q$ charge,

$$\vec{x_{seperation+}} = x(\hat{x}) - a(\hat{x}) = (x - a)\hat{x} \tag{6.75}$$

Then we need the $|\vec{x_{seperation+}}|$ to substitute in *Coulomb's law* equation and then we get the electric field,

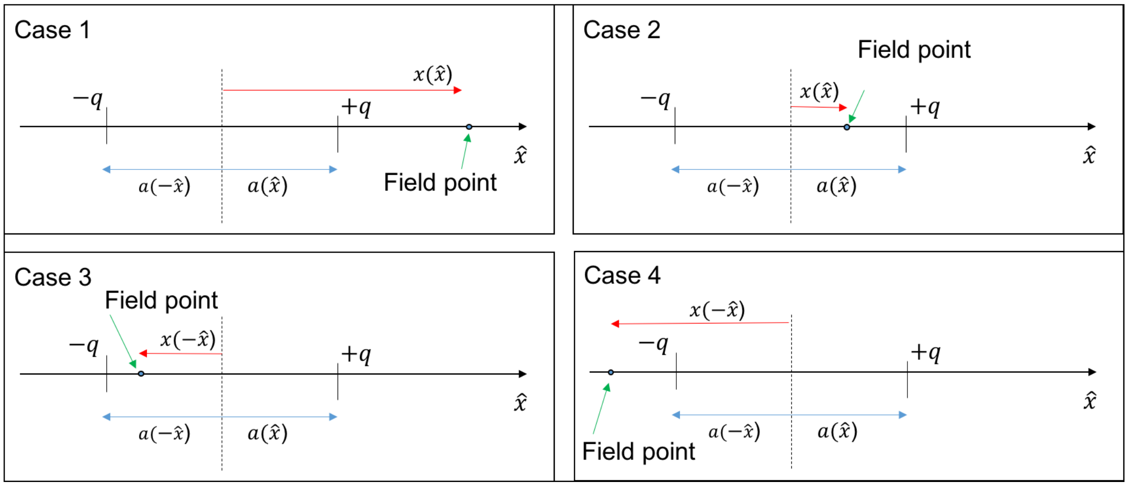


Figure 6.24: *Field points in regions*

$$\vec{E_+} = \frac{1}{4\pi\epsilon_0}\frac{q}{(x-a)^2}\hat{x} \tag{6.76}$$

Contribution from $-q$ charge,

$$\vec{x_{seperation-}} = x(\hat{x}) - a(-\hat{x}) = (x+a)\hat{x} = -(x+a)-\hat{x} \tag{6.77}$$

Similarly with $|\vec{x_{seperation-}}|$ we get,

$$\vec{E_-} = \frac{1}{4\pi\epsilon_0}\frac{q}{(x+a)^2}-\hat{x} \tag{6.78}$$

Then the *net electric field*,

$$\vec{E_{net}} = \vec{E_+} + \vec{E_-} \tag{6.79}$$

$$\vec{E_{net}} = \frac{1}{4\pi\epsilon_0}[\frac{q}{(x-a)^2} - \frac{q}{(x+a)^2}]\hat{x} = \frac{1}{4\pi\epsilon_0}[\frac{q}{(x+a)^2} - \frac{q}{(x-a)^2}]-\hat{x} \tag{6.80}$$

<u>Case 2: *field point* in middle region - positive side</u>

Contribution from $+q$ charge,

$$\vec{x_{seperation+}} = x(\hat{x}) - a(\hat{x}) = (x-a)\hat{x} = (a-x) - \hat{x} \tag{6.81}$$

$$\vec{E_+} = \frac{1}{4\pi\epsilon_0}\frac{q}{(a-x)^2} - \hat{x} = \frac{1}{4\pi\epsilon_0}\frac{q}{(x-a)^2} - \hat{x} \tag{6.82}$$

Contribution from $-q$ charge,

$$\vec{x_{seperation-}} = x(\hat{x}) - a(-\hat{x}) = (x+a)\hat{x} = -(x+a) - \hat{x} \tag{6.83}$$

$$\vec{E_-} = \frac{1}{4\pi\epsilon_0}\frac{q}{(x+a)^2} - \hat{x} \tag{6.84}$$

Net electric field,

$$\vec{E_{net}} = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{(a-x)^2} + \frac{q}{(x+a)^2}\right] - \hat{x} = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{(x-a)^2} + \frac{q}{(x+a)^2}\right] - \hat{x} \tag{6.85}$$

Case 3: *field point* in middle region - negative side

Contribution from $+q$ charge,

$$\vec{x_{seperation+}} = x(-\hat{x}) - a(\hat{x}) = (x+a) - \hat{x} = -(x+a)\hat{x} \tag{6.86}$$

$$\vec{E_+} = \frac{1}{4\pi\epsilon_0}\frac{q}{(x+a)^2} - \hat{x} \tag{6.87}$$

Contribution from $-q$ charge,

$$\vec{x_{seperation-}} = x(-\hat{x}) - a(-\hat{x}) \tag{6.88}$$

$$\vec{E_-} = \frac{1}{4\pi\epsilon_0}\frac{q}{(x-a)^2} - \hat{x} \tag{6.89}$$

Net electric field,

$$\vec{E_{net}} = \frac{1}{4\pi\epsilon_0}[\frac{q}{(x+a)^2} + \frac{q}{(x-a)^2}]-\hat{x} \tag{6.90}$$

<u>Case 4: *field point* in far-left region</u>

Contribution from $+q$ charge,

$$\vec{x_{seperation+}} = x(-\hat{x}) - a(\hat{x}) = (x+a)-\hat{x} = -(x+a)\hat{x} \tag{6.91}$$

$$\vec{E_+} = \frac{1}{4\pi\epsilon_0}\frac{q}{(x+a)^2}-\hat{x} \tag{6.92}$$

Contribution from $-q$ charge,

$$\vec{x_{seperation-}} = x(-\hat{x}) - a(-\hat{x}) \tag{6.93}$$

$$\vec{E_-} = \frac{1}{4\pi\epsilon_0}\frac{q}{(x-a)^2}\hat{x} \tag{6.94}$$

Net electric field,

$$\vec{E_{net}} = \frac{1}{4\pi\epsilon_0}[\frac{q}{(x+a)^2} - \frac{q}{(x-a)^2}]-\hat{x} = \frac{1}{4\pi\epsilon_0}[\frac{-q}{(x+a)^2} + \frac{q}{(x-a)^2}]\hat{x} \tag{6.95}$$

6.4.5 Mechanics problem - Student: Zeke

This example comes from an oral exam in the second month of Classical Mechanics course. The episode starts with the instructor posting the problem with a diagram on the board (Figure 6.25).

Instructor: If you have a platform which rotates, and on the platform is a mass on a spring...the mass is constrained in the frame of the platform so that it can only go back and forth... does the mass undergo simple harmonic motion?

Zeke is working on a much longer problem and he has to consider all the forces on the mass. Within his approach we observe Zeke taking time to build each representational space and to frequently coordinating among representational spaces to check for the consistency.

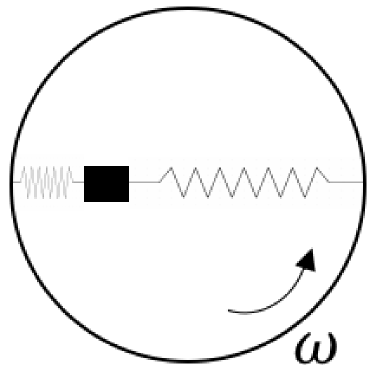


Figure 6.25: *Instructor provided diagram for Zeke: mass spring system on a rotating platform*

Zeke starts using the given diagram and he confirms with the instructor that both springs have the same spring constant k. Then he moves to consider the forces on the mass exerted by the springs. First, he talks about the force on the left spring. Even though Zeke looks like he is about to record something, he takes a step backward and moves towards the diagram on the board.

Zeke: OK, the real forces would be... OK, let's say, we'll call that x... let's say equilibrium (records a dotted line on the diagram).

While talking about the force on the left spring, Zeke goes to the diagram on the board and uses the semiotic resource *midline for equilibrium* to represent the equilibrium position of the mass. After recording a vertical dotted line to the right of the mass (Figure 6.26a) Zeke talks about the left spring.

Zeke: OK. So, in that case, um. The force on this (force on left spring). Um. I'm going to write something down but I don't think it's right. I'll have to think about it, negative kx (records on the board $-kx$ along with a leftward arrow on the diagram).

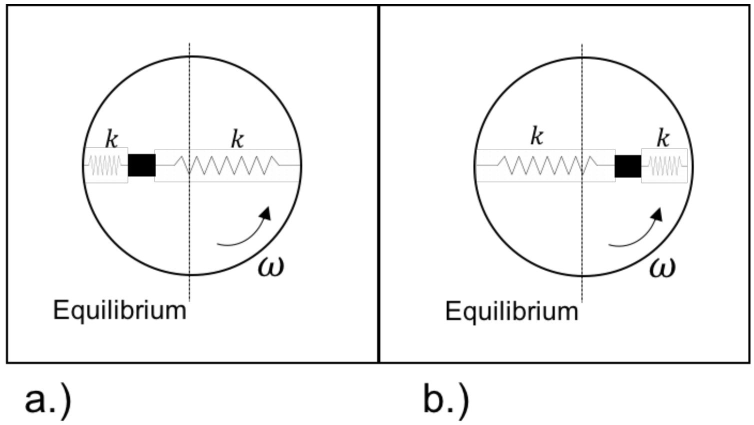


Figure 6.26: *Equilibrium position: a.) Equilibrium to the left of the mass b.) Equilibrium to the right of the mass*

$$F = -\,kx \tag{6.96}$$

After using the semiotic resource *arrow as vector* to record the force on the left spring, Zeke shows the uncertainty of his own work. *"I'm going to write something down but I don't think it's right"*. He does show this kind of behavior often during this problem-solving activity. First, he predicts or records something (may be a mathematical equation or adds a feature onto the diagram) on the board and then he steps back from the board to make sense of his work. Here, at this very moment, Zeke steps back and starts talking through the mathematical equation (*restoring force on the spring*) $-kx$ that he just recorded.

Zeke: So, if x is negative, then we expect k to push (gesture - *finger pointing in direction*) ... um, OK. So in this situation. Can I? ... I'm going to put the mass over here, just for sign. Just to make the sign make more sense (moves the mass to the right of the equilibrium).

Zeke refers to the diagram on the board (Figure 6.26a) and uses the gestural semiotic resource *finger pointing in direction* to specify the direction of restoring force on the left spring. Zeke picks the leftward *displacements* to be negative (*"so if x is negative"*) and rightward to be positive. Figure 6.26a, shows a compressed left spring with a negative *displacement* of the mass. According to the diagram (Figure 6.26a), the compressed left spring should be pushing, but rightward, with a force $F = kx$. Even though Zeke makes one thing right *"then we expect (left spring) to push"*, his work on the board ($-kx$ along with an leftward arrow) does not agree with his work on the board. Soon Zeke realizes this mismatch and decides to change the position of the mass to the right of equilibrium position (Figure 6.26b)

After changing the position of mass (Figure 6.26b), Zeke takes a step back to check the consistency between the diagram on the board and the mathematical equation.

Zeke: We have positive k ... So negative kx makes sense because we have positive *displacement*, and it's gonna be pulling backwards in the negative direction (gesture - *finger pointing in direction*).

Figure 6.26b shows a stretched left spring with a positive *displacement* of the mass. So, the left spring should be pulling, but leftward, with a force $F = -kx$. This agrees with Zeke's mathematical equation (*restoring force on the spring*). We can observe this agreement supports Zeke's confidence in that he reasons with a steady voice without any pauses. Then he continues to use the gestural semiotic resource *finger pointing in direction* to indicate the force direction on the mass.

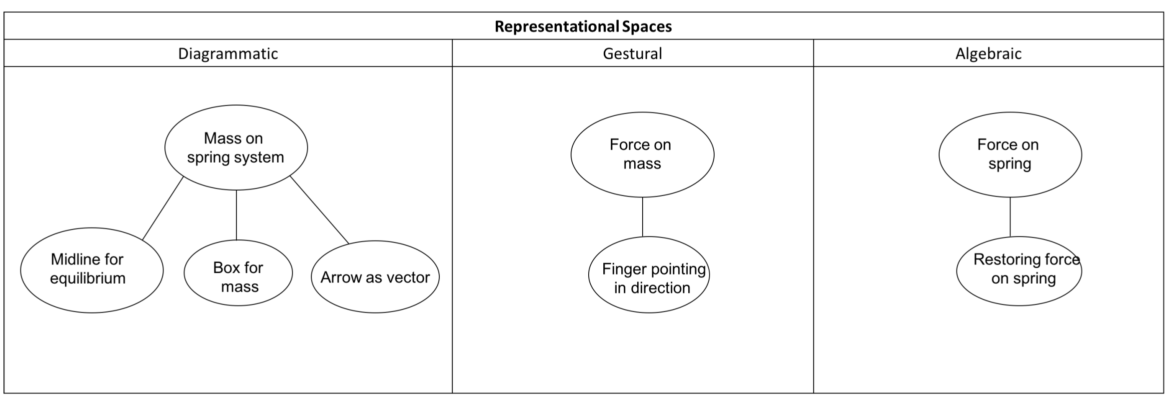


Figure 6.27: *Initial stage of Zeke's representational spaces*

Figure 6.27 shows the initial stage of Zeke's representational spaces. Within the diagrammatic space, Zeke coordinates among the semiotic resources *box for mass*, *midline for* *equilibrium* along with *arrow as vector*. At this initial stage the gestural space consists of a single semiotic resource *finger pointing in direction*. Zeke has taken the initial step to find the net force on the mass and he so far considered only the left spring. As a result of this, his algebraic space also has one semiotic resource *restoring force on the spring ($F = -kx$)*. So far, we observe Zeke is coordinating among representational spaces that he checks for consistency between his diagrammatic and algebraic spaces almost without the use of gestural space. Even though Zeke uses the gestural space, he uses the *finger pointing in direction* just to indicate the direction of the displacement and force on spring. Within the diagrammatic space, while the semiotic resource *box for mass* allows Zeke to represent the presence of mass on the space of the board, both (*Midline for equilibrium* and *box for mass*) together allow Zeke to represent the *displacement* of the mass while the *arrow as vector* represents the force direction. Algebraic space (*restoring force on the spring ($F = -kx$)*) allows him to get an idea about the directional relationship between the *displacement* of the mass and the restoring force on the spring.

Here we see the disciplinary affordance (Table 6.8) of the semiotic resource *restoring force on the spring ($F = -kx$)* in the algebraic space, first hinders and then reinforces the use of the semiotic resource *midline for equilibrium* in the diagrammatic space. According to Hooke's law, if the mass is displaced sideways, such that the spring becomes compressed (or extended), then the mass experiences a horizontal force given by $F(x) = -kx$, where k is the spring constant and the negative sign in the expression indicates that $F(x)$ is a restoring force (if the *displacement* is positive then the force is negative and vice versa). The semiotic resource *midline for equilibrium* helps to visualize the *displacement* of the mass from its equilibrium position. Zeke has already predicted the *restoring force on the spring* (left) to be leftward $F = -kx$; and according to his choice of coordinates, the leftward *displacements* are negative. The Figure 6.26a shows a negative *displacement* of the mass. Hooke's law for this situation, indicates the compressed left spring should be pushing, but rightward, with a force $F = kx$. Then the Figure 6.26b shows a positive *displacement* of the mass and for tis situation, Hooke's law indicates the stretched left spring should be pulling the mass leftward with a force $F = -kx$. Only the *restoring force on the spring* according to Figure 6.26b

Table 6.8: *Disciplinary affordances of the semiotic resources coordinated to build Zeke's initial representational spaces*

Semiotic Resource	Disciplinary Affordance
Box for mass	This allows Zeke to generate a visual representation of the mass in the space of the board. He uses it to record the direction of restoring forces on the mass by each spring.
Midline for equilibrium	This helps Zeke to visualize the displacement of the mass from its equilibrium position. The view of the compressed or stretched springs allows him to decide the direction of the restoring force that is very important in this situation. As Zeke steps back from the board to check the consistency between the diagram and the mathematics, initially this (midline) helps Zeke to check the validity of his mathematical equation for the force by the left spring and then later to check the sign on the mathematical equation for the force by the right spring.
Arrow as vector	In this case, this visual representation of a line with an arrow head allows Zeke to show the restoring force direction from each spring to figure out the net force by both springs.
Restoring force on the spring ($F = -kx$)	According to Hooke's law, if the mass is displaced sideways, such that the spring becomes compressed (or extended), then the mass experiences a horizontal force given by $F(x) = -kx$, where k is the spring constant and the negative sign in the expression indicates that $F(x)$ is a restoring force (if the displacement is positive then the force is negative and vice versa). The idea that the direction of the restoring force is opposite to that of the displacement of the mass, helps Zeke to amend his diagram according to the mathematical equation that he recorded.

aligns with Zeke's prediction. The mismatch in Figure 6.2a results in the diagram being inconsistent and the semiotic resource *restoring force on the spring ($F = -kx$)* stops the use of the semiotic resource *midline for equilibrium* in Figure 6.26a. Then later after the modification (Figure 6.26b) we can see the semiotic resource *restoring force on the spring ($F = -kx$)* reinforces the use of the semiotic resource *midline for equilibrium.*

With the satisfied agreement in his revised work, Zeke moves to consider the second (right) spring.

Zeke: This one [the spring on the right], it's going to be pushing, so it's going to be pushing at... kx.

$$F = \ kx \tag{6.97}$$

After he identifies the direction of the *restoring force on the spring* (right), Zeke steps back to do more sense-making.

Zeke: As x is bigger (fist moves rightward), this (force on right spring), the force gets bigger by the factor of kx, so it's gonna push this way at kx. That's correct.

Right here for the first time Zeke uses the semiotic resource *fist for mass* to represent the presence of mass in free space. Even though Zeke concludes his work to be correct, Figure 6.26b shows the mass has a positive *displacement*, and by Hooke's law, the compressed right spring should be pushing the mass leftward with a force $F = -kx$.

Now Zeke has two mathematical equations to represent the *restoring force on the spring* from both springs. Then he moves to connect these two equations. While doing so, Zeke decides to change the direction of the force on right spring (uses the semiotic resource *arrow as vector*) and then uses a minus sign in his mathematics to *"make it come out of the mass"*. Then he steps back again to check his work. Zeke uses the gestural semiotic resources *fist for mass* and *finger pointing in direction*, but soon he realizes that his work is inconsistent with the diagram.

Zeke: No, I have done something silly.

Zeke goes back to the board and changes the force on the right spring to be leftward *"we want to push back"*. For one last time, he returns to considers both springs.

Zeke: If m goes this way (fist moves rightwards) then we have a positive *displacement* ($+x$)...this ($-kx$) is negative, that (the force from the left spring) points that way (leftwards), that makes sense. (then talks about the spring on right) We've got a positive x (fist moves rightward), this (equation for right spring) is negative (finger points leftward) and (force) points that way, makes sense.

$$F = -\ 2kx \tag{6.98}$$

After finding an agreement on his work, Zeke concludes *"I am happy now"* and he gets the *force by two springs* is leftward with a magnitude $F = -2kx$.

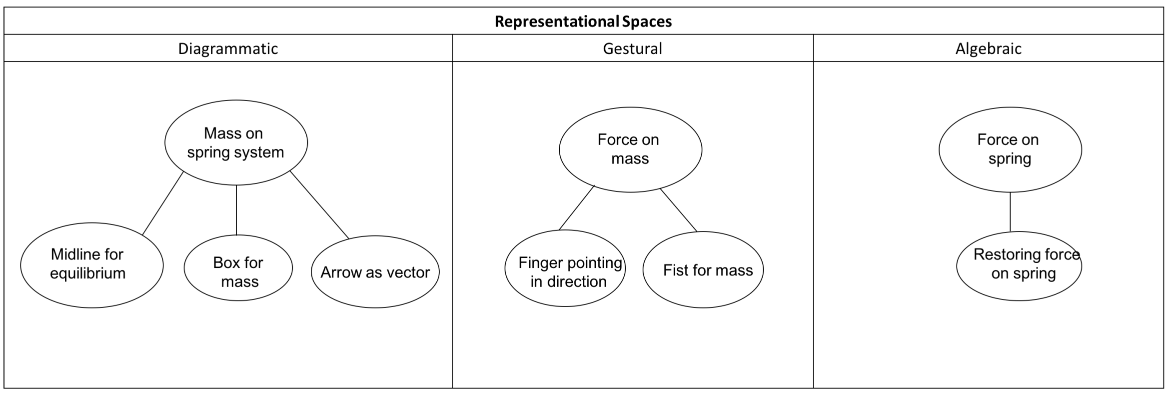


Figure 6.28: *Intermediate stage of Zeke's representational spaces*

Figure 6.28 shows the intermediate stage of representational spaces. Even tough Zeke has not added any new semiotic resource to his diagrammatic space, we observe him making modifications to it. Zeke brings in the new semiotic resource *fist for mass* into his gestural space. while working to figure out the net force on the mass, Zeke continuously translates between the representational spaces: diagrammatic and algebraic to checks for the consistency. This allows him to further develop his algebraic space and Zeke ends up adding the semiotic resource *force by two springs* as a result of considering the effect from both springs. Zeke seems to be confident on his mathematics. Then with much effort, he develops the diagrammatic space to match with what he has in the algebraic space.

Here for the first time we observe Zeke using the gestural space (*fist for mass* and *finger pointing in direction*) to sort out the contradiction between his diagrammatic and algebraic spaces. The gestural semiotic resource *fist for mass* allows him to represent the mass in free space. By doing so, Zeke can represent the displacement of the mass in free space while getting a sense of the restoring forces on the springs (to be either pushing or pulling). Zeke starts by considering just one spring and the semiotic resource *fist for mass* allows him to figure out the force direction on that spring. He then goes back and makes modifications in his diagrammatic space. After resolving this disagreement Zeke continues to consider both springs together. Again he returns to the gestural spaces and keep using semiotic resources *fist for mass* and *finger pointing in direction* until he concludes (*"I am happy now"*) and record the *force by two springs* to be directing in the same direction with a magnitude of $F = -2kx$.

Table 6.9: *Disciplinary affordances of the semiotic resource coordinated to build Zeke's intermediate representational spaces*

Semiotic Resource	Disciplinary Affordance
Fist for mass	This allows Zeke to generate a visual representation of the mass in the free space. Zeke holds the fist while gesturing for mass displacement and keeps referring back as he progresses in this task. The process of sense-making using the *fist for mass* helps Zeke to figure out the direction of the restoring forces that he could mostly do using the diagram on board.
Finger pointing in direction	Zeke uses the index finger to the side while gesturing for direction. This allows him to show the direction of the displacement and the direction of the restoring force in the free space.

Similar to the previous cases presented in this analysis, we see Zeke connecting among semiotic resources to build the representational spaces. Initially, he could not connect among these representational spaces but then the use of additional representational space allows him to make that connection. So far we observe Zeke working on diagrammatic and algebraic space. Then we observe the limitations of these representational spaces prevent him from connecting these representational spaces to check for the consistency. In order to overcome this limitation, Zeke moves to the other available representational space, the gestural space. From then we observe the semiotic resources *fist for mass* and *finger pointing in direction* in gestural space bringing the missing information that allows Zeke to connect coming to the diagrammatic and algebraic spaces.

After figuring out the *force by two springs*, Zeke moves to consider the rotation of the platform to find the effective forces on the mass. When Newton's laws are transformed to a rotating frame of reference, the Coriolis force and Centrifugal force appear. One can represent the effective force on the rotating mass by the following mathematical equation.

$$F = \underbrace{m[2]\vec{r}t}_{Inertial\,force} - \underbrace{2m\vec{\omega}\times\vec{r}t}_{Coriolis\,force} - \underbrace{m\vec{\omega}\times(\vec{\omega}\times\vec{r})}_{Centrifugal\,force} - \underbrace{m\vec{\omega}t\times\vec{r}}_{Angular\,acceleration\,force}$$

But in this case, it is clear that we do not have an angular acceleration force because the platform rotates with a constant ω. But, Zeke starts by figuring out the *direction of the angular velocity* and to do so he applies the *right-hand grip rule.*

Zeke: Okay. Now ω (angular velocity) is going to be (applies the right-hand grip rule) pointing out of the board (records on the board).

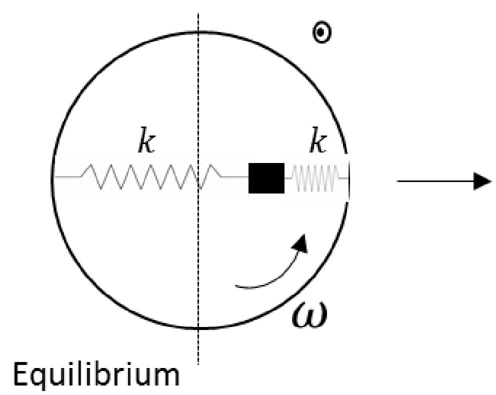


Figure 6.29: *Zeke further develops the diagram to include the direction of angular velocity (arrow head) and the direction of centrifugal force (rightward)*

The angular velocity is a vector and its direction is perpendicular to the plane of rotation. Using the *right-hand grip rule*, the *direction of the angular velocity* is defined as the direction in which the thumb points when Zeke curls his fingers in the direction of rotation. Once he figures out the direction, then Zeke uses the semiotic resource *arrow as vector* to add the *direction of the angular velocity* onto the diagrammatic space (Figure 6.29). Then Zeke moves to talk about the inertial force and quickly concludes it to be zero.

Zeke: So, the inertial force. Frame is not accelerating, there is no inertial force.

Zeke continues to consider the next force, which is the *centrifugal force.*

Zeke: Centrifugal force ... um .. we have ω cross r (applies the right-hand grip rule). So, we are going to have that $(\vec{\omega} \times \vec{r})$ pointing up. And then ω cross ω cross r $(\vec{\omega} \times (\vec{\omega} \times \vec{r}))$ is (applies the right-hand grip rule) pointing in negative, is going to point out and sense because centrifugal force always points out of the circle.

Here Zeke first uses the *right-hand grip rule* to get the direction of *direction of the angular velocity* and re-applies it to get the cross product $(\vec{\omega} \times \vec{r})$. then he uses the semiotic resource *finger pointing in direction* to indicate the resulting direction to the instructor. Next he moves to consider the cross product of $(\vec{\omega} \times \vec{r})$ with $\vec{\omega}$ and re-applies the *right-hand grip rule.* Zeke's finding agrees with what he learned in the class and we can see his confidence as he comments right away about his finding *" sense because centrifugal force always points out of the circle".* The instructor's follow up question about the direction of the *centrifugal force* leads Zeke to comment *"yes, it's pointing this way (rightward)"* and use the semiotic resource *arrow as vector* to record it on the diagram on the board (Figure 6.29).

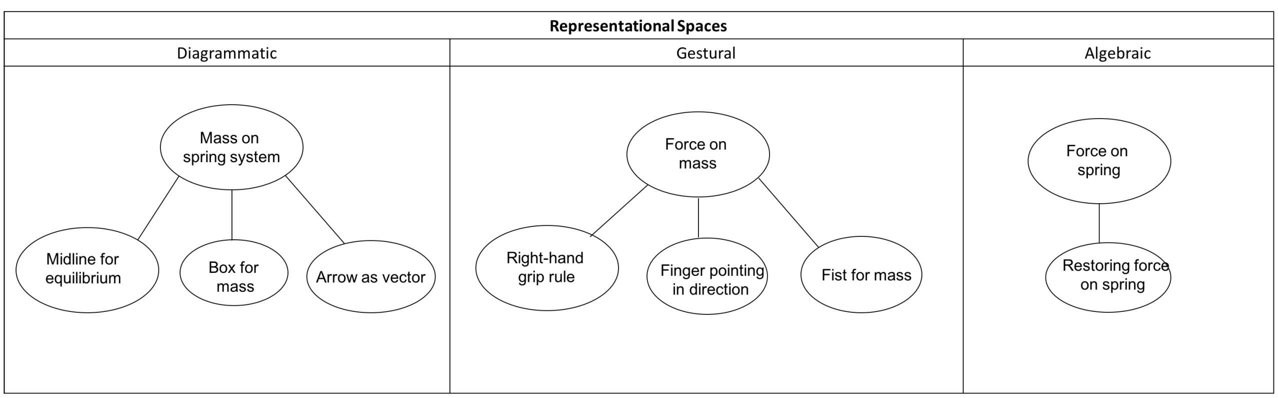


Figure 6.30: *Zeke includes centrifugal force into his representational spaces*

Figure 6.30 shows the current stage of Zeke's representational spaces, that he included the information of the *direction of the angular velocity* and *centrifugal force.* Since the intermediate stage, Zeke has not added any new semiotic resource to both diagrammatic and algebraic spaces. Zeke mostly works on the gestural space and then he reuses the semiotic resource *arrow as vector* to add the directional information into his diagrammatic space.

In undergraduate mathematics and physics classrooms, the direction of the cross product is often found using the *right-hand grip rule*. So far we see students using the semiotic resource *right-hand grip rule* for sense making and communication purposes. Larry (in chapters 4 and 5) uses the *right-hand grip rule* to figure out the magnetic field generated by a current carrying wire. Similarly, during the loosely wound solenoid problem, Danny uses *right-hand grip rule* to indicate the magnetic field generated by the axial current component.

Similarly, the gestural semiotic resource *right-hand grip rule*, first, allows Zeke to assign a vector (*direction of the angular velocity*) to the rotation of the platform. Right here he makes a unique use of *right-hand grip rule* that stands out from other students in this analysis. He keeps using it to figure out more complicated mathematical manipulation. Zeke further employ *right-hand grip rule* to figure out the cross product between ω and r ($\vec{\omega} \times (\vec{\omega} \times \vec{r})$) to figure out the direction of *centrifugal force*. A given student could use the algebraic space to figure out the direction of *centrifugal force* with some effort along with mathematical calculations. Instead, on this occasion the semiotic resource *right-hand grip rule* allows Zeke to skip all those manipulation steps and he gets to the final result result with much ease.

But, here we do not observe Zeke coordinating between representational spaces like what he did so far. Instead, he uses one semiotic resource in one representational space to figure out something and then he uses a separate semiotic resource to include his findings onto a different representational space.

Then Zeke moves to consider the *Coriolis force* and we can observe Zeke takes relatively more time to get going. In addition to that his voice level and choice of words *"This is going to be tricky ... how do we want to do this?"* indicate that this is a tough task for him. Finally, Zeke begins to move forward and he starts with another prediction.

Zeke: Well, imagine at this point just for a moment and then we will generalize it in a second. We will say at this point (refers to the diagram on board) that x dot ($\dot{x}$ - *linear velocity*) is pointing out and it is accelerating this way (rightward arrow).

First he refers to the diagram on board and talks about the *linear velocity* ($\dot{x}$ or $\vec{r}t$) of the mass when the mass is displaced by x (rightward) from the equilibrium. Here Zeke uses the

Table 6.10: *Disciplinary affordances of the semiotic resource coordinated to include centrifugal force into Zeke's representational spaces*

Semiotic Resource	Disciplinary Affordance
Right-hand grip rule	On this occasion, Zeke first uses the right-hand grip rule to figure out the rotation vector to represent the rotation of the platform, where Zeke's thumb of right-hand points when he curls his fingers in the direction of rotation. Then Zeke uses the right-hand grip rule to find the direction of the centrifugal force that he uses to manipulate the cross product of the three vectors $(\vec{\omega} \times (\vec{\omega} \times \vec{r})$. By definition the right-hand grip rule states the orientation of the vectors' cross product $(\vec{u} \times \vec{v})$, by placing u and v tail-to-tail, flattening the right hand, extending the hand in the direction of u, and then curling the fingers in the direction of the angle v makes with u. Then the thumb points in the direction of $\vec{u} \times \vec{v}$. Zeke uses this argument first to find the direction of $\vec{\omega} \times \vec{r}$ and then to find the direction of $\vec{\omega} \times (\vec{\omega} \times \vec{r})$. Later, Zeke uses the right-hand grip rule to find the direction of the Coriolis force that he re-uses to manipulate the cross product of the vectors $\omega \times \vec{r}t$. Zeke flattens the right hand, extending the direction of $\dot{r}$ (that is $\vec{r}t$), and then curls the fingers in the direction that the angle ω makes with $\dot{r}$. Then Zeke's thumb points in the direction of $\omega \times \vec{r}t$. In addition to that Zeke uses the direction of angular velocity that is originally figured out using the right-hand grip rule.
Arrow as vector	Again Zeke uses the Arrow as vectors visual representation to show vector direction. Here he uses the same semiotic resource to visualize the direction of angular velocity and the direction of centrifugal force. Later, He uses the same semiotic resource to visualize the direction of the Coriolis force.

semiotic resource *arrow as vector* to add the direction (rightward) of the *linear velocity* onto the diagram. Then he moves to figure out the direction of the *Coriolis force* by considering the cross product $(\vec{\omega} \times \vec{r}t)$.

Zeke: The case then ... ω cross $\dot{r}$ points up (records on the board). Let's see if sense. Well ... because ω goes this way (anti-clockwise) ... oh negative. It's negative ω cross $\dot{r}$. So, it's this way (downward arrow records on the board).

While applying the *right-hand grip rule*, even though Zeke mentions $\dot{r}$; he refers to the *linear velocity* ($\dot{x}$). Soon Zeke realizes that the direction should be downwards and uses the semiotic resource *arrow as vector* to add the direction of the *Coriolis force* onto the diagram. Zeke's regains his confidence. We can see that as he further comments on the *Coriolis force* with a steady tone with no pauses.

Zeke: This is Coriolis and that makes sense because as if you look at it from the inertial frame as m comes this way (hand moves leftward) and the platform moves behind (hand moves anti-clockwise). It's going to end up ... if it weren't constrained, it would end up further down the platform. Because the platform rotates under it.

Then Zeke quickly comments on the angular acceleration force likewise he did for the inertial force.

Zeke: Okay. Transverse force (angular acceleration force) does not exist because there is no omega dot ($\dot{\omega}$ - angular acceleration).

From here, we observe Zeke moves to consider the constrained motion of the mass (mass is constrained in the frame of the platform so that it can only go back and forth). He starts by ignoring all the forces other than the *net force by springs* and *centrifugal force*. So far he figured out the direction but now moves to work on the mathematics to figure out the magnitude of the *centrifugal force*.

Zeke: Now, we have centrifugal force which was ω cross ω cross r ($\vec{\omega} \times (\vec{\omega} \times \vec{r})$). So that would be (records on the board) $m\omega^2 x$ and r which in this case is x.

$$F = m\omega^2 x \tag{6.99}$$

Zeke refers back to the relationship ($\vec{\omega} \times (\vec{\omega} \times \vec{r})$) that he used to determine the direction of the *centrifugal force* and replaces r by the displacement x accordingly to the diagram on the board. After figuring out the magnitude of the *centrifugal force*, Zeke talks about the *net force on mass* along the direction of the motion. Zeke did a substantial amount of work to figure out the direction of the *Coriolis force* and when he considers the constraint in the motion, the *Coriolis force* is perpendicular to the direction of motion. Then Zeke continues to consider only the forces along the motion.

Zeke: So, we were to add up forces in this direction (rightward) which is the only direction we care about because of the constraint (and records $\Sigma F = (-2k + m\omega^2)x$ on the board).

$$F = (-2k + m\omega^2)x \tag{6.100}$$

Zeke adds up all the forces on another part of the board, where he has the *net force on mass* as $\Sigma F = (-2k + m\omega^2)x$. Zeke accomplishes this step really quickly with no hesitations or pauses thanks to all the recording steps he has made before of the direction and the formula for each force.

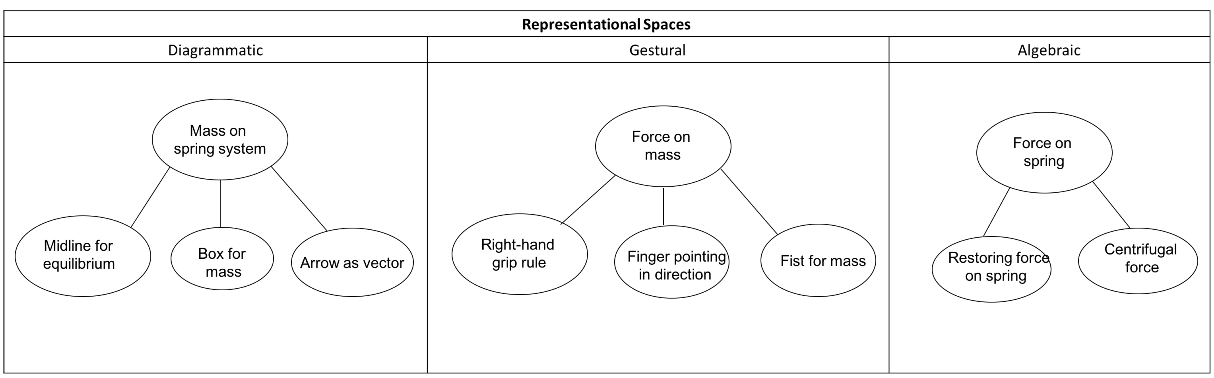


Figure 6.31: *Final stage of Zeke's representational spaces*

Figure 6.31 shows the final stage of Zeke's representational spaces before he moves to make some assumptions about the value of the constants. We see Zeke recording some mathematics but they are mostly simplifications steps. From now we observe Zeke using his finding the *net force on mass* ($\Sigma F = (-2k + m\omega^2)x$) to go about considering the possible cases to see if the mass undergoes simple harmonic motion.

Zeke: So, if we have ω is such that this term $m\omega^2$ not bigger than $2k$ (and records $m\omega^2 < 2k$ on the board).

Zeke starts by checking for the first condition ($m\omega^2 < 2k$) to see the possibility of the simple harmonic motion.

Zeke: If that is the case then this ($-2k + m\omega^2$) going to be constant and stays negative. So we can define an effective k (k^1) which would be $k^1 = 2k - m\omega^2$ (records on the board) and we get $F = -k^1 x$ (records on the board). So, in that case it would be simple harmonic motion because k^1 being a constant and they have the same form.

$$k^1 = 2k - m\omega^2 \tag{6.101}$$

For the first condition, he defines a new quantity "effective spring constant" effective k. He records all of them on the board and turns to the instructor as he confirms that it is the form of simple harmonic motion. As he gets the instructor's confirmation then Zeke continues his analysis to check for the second condition ($m\omega^2 > 2k$).

Zeke: Now, if $m\omega^2$ is greater than $2k$ (and records on the board) $m\omega^2 > 2k$. That would be interesting, because in that case then we can have like a kappa (records on the board $\kappa = m\omega^2 - 2k$) equal to the other way. Just to keep it positive. Then we have the sum of the forces equal to kappa x (records on the board $F = \kappa x$), which is funny.

$$\kappa = m\omega^2 - 2k \tag{6.102}$$

While testing the second condition he brings in another symbol κ, which replaces $m\omega^2 - 2k$. Then he starts to comment on the situation.

Zeke: It won't be simple harmonic motion because it doesn't have a restoring force. Because there is no negative sign. With x gets bigger positive force continues to be positive. So, it's going to continue come this way (rightward hand movement) until something stops it. Probably the wall or what ever the spring is tight to.

Then he continues to talk through and explains this solution to the instructor.

Zeke: What you will have an unstable equilibrium at the center, because at $x = 0$ you will have no force (points to $F = \kappa x$) . But if you give a little jolt in the negative direction, then you have a negative x and it will force it this way (leftward) that will keep it forcing until the normal force is counteracted. If you give a little jolt in the positive direction, then it will continue going until it hits a normal force.

Zeke talks about the unstable equilibrium and the motion of the mass in both possible initial *displacements* to either to left or right. Then he moves to summaries of what he has done so far.

Zeke: So I guess the summary is then, if ω is such that $m\omega^2 < 2k$, it would exhibit simple harmonic motion with a slightly different frequency of oscillation. if $m\omega^2 > 2k$ then no simple harmonic motion and it would just accelerate to the edge of the circle.

Here we observe Zeke talks about the *frequency of oscillation* while summarizing his analysis. The instructor asks about the third possible condition, which Zeke has not yet covered.

Instructor: What happens if $m\omega^2 > 2k$.

Zeke replies quickly, but as he moves forward, we see frequent pauses between his words. This may be because he has not thought about this condition until the instructor asks.

Zeke: In this case, this $-2k + m\omega^2$ becomes zero and this is ... a ... that's interesting .... no force, why would be no force. ...

After a long pause Zeke starts to reason again,

Zeke: Okay, you would have where ever your mass was it stays there and that could be because, if you imagine as the picture is drawn (points to the diagram) the spring wants to pull it back towards the equilibrium but the exact amount of spring wants to pull it back. The centrifugal force wants to pull it out.

Even though he comes to a point where he finds the new situation, which he had not considered before, he uses the body parts to make sense and explain it to the instructor. The instructor's follow up question leads Zeke to explain more.

Instructor: Does it matter where it is?

Zeke: No, because as you go further the restoring force is going to become larger because the springs displaces more. But at the same time, you are also closer to the outside, so you are experiencing a bigger centrifugal force.

This concludes Zeke's problem-solving activity. Zeke is working on a much longer problem which has an early component of looking at the signs of the forces on a box on two springs. In order to solve this problem, Zeke has to consider all the forces on the mass, the forces exerted by the springs and the effective forces on the rotating mass (Coriolis force, centrifugal force and angular acceleration force). Then he has to consider only the forces along the direction of motion (because the mass is constrained to move only back and forth) before moving to consider the possible cases to see if the mass undergoes simple harmonic motion.

Zeke spends a lot of time worrying about this problem and starts with a simpler situation - a single spring. He breaks it into the cases of expansion and compression. Later he considers the rotation to see if the mass undergoes simple harmonic motion under different conditions. We observe as a habit, Zeke does predictions before he moves on to consider the situations and he does much sense-making. Unlike the other cases presented in this study, we find Zeke talking, trying things most of the time rather than waiting for instructor to give frequent feedback or some times helpful cues.

During the episode, Zeke starts using the given diagram and coordinates among other semiotic and conceptual resources to further develop the diagram. Then he starts to develop a mathematical relationship. In order to answer the problem at hand, Zeke further uses the gestural space to check the consistency between the diagrammatic and mathematical spaces. First, Zeke works to figure out the net force on the mass by the springs. Later he moves to consider the rotation of the platform to find the effective forces on the mass.

We observe Zeke doing more coordinating between representational spaces during the first parts of his problem-solving activity. But after figuring out the force by springs on the mass he moves to consider effective forces on the mass, where he mostly works with one representational space before adding originated information into another representational space. Then he returns to the gestural space while talking about the possibilities of the mass going on simple harmonic motion or not.

6.5 Discussion

In order to generalize our approach to describe the underlying mechanism behind students constructing spontaneous representations to solve physics problems by coordinating semiotic resources, using our revised framework, first, we analyze multiple cases of individual students solving two physics problems in an Electromagnetism I course. Then to show the applicability/ durability of this approach in other physics courses, we further analyze the case of a student (Zeke) solving another problem in his Classical Mechanics course.

In our approach, we suggest that students' approach to solving physics problems is by first, coordinating among semiotic resources to build representational spaces (diagrammatic, gestural, and algebraic) and then coordinating among those representational spaces to solve problems. By coordination, we mean how the disciplinary affordance of a particular semiotic resource/ representational space hinders or reinforces the use of other semiotic resources/ representational spaces.

Talking about the disciplinary affordance of semiotic resources, especially the second problem (two charges on a line) appears to be easy because all they have to do is to figure out the electric field from each charge and then think about the superposition. But it turns out that students have a hard time figuring out how to use the coordinate information (in the diagram) within the mathematical calculations (*Coulombs law*). As shown in section 6.4.4, this is due to the the disciplinary affordances of those semiotic resources. In that sense, teaching our students about the disciplinary affordances of semiotic resources ahead of time could make them better prepared to tackle complex problems at this level.

In addition, according to previous research, the process of translating between representations was found to be difficult for students. In order to overcome this challenge, students need to identify the corresponding elements between different representations. The approach described in our study shows that this could be accomplished by understanding the disciplinary affordances of the representational spaces.

Within the first problem, the loosely wound solenoid, both Alan and Danny seem to use the diagrammatic space to first develop a background understanding that will help them continue with mathematical manipulations while working in the algebraic space. Perhaps because of the nature of this problem, we do not observe much coordination between representational spaces compared to the other two problems.

Next we consider the two representational spaces that students develop as they solve the two charges on a line problem. The algebraic space shows a *Regional electric field* as a reason for the contribution from each charge. Then the diagrammatic space shows a *Universal electric field* that is generated by each charge. When students get stuck while working on the algebraic space, they return to the diagrammatic space. Even though they have a *Regional electric field* idea while working on the mathematics, the coordination between the diagrammatic space seems to help Larry, Oliver, and Charlie.

On the mechanics problem, Zeke first uses the diagrammatic space to represent the force on the mass, both when he is considering a single spring and when considering both springs. Then he moves to the algebraic space to represent this force mathematically. Even though both these representational spaces help Zeke to gets a sense of the physical scenario, they both also seem to have limitations. Zeke moves to the gestural space, which allows him to get a feel for the push and pull from each spring on his fist that he uses to replace the mass. Moving on to a different representation helps Zeke to get a better understanding of the physical scenario. He is then able to coordinate between the representational spaces to move forward.

Cases presented this study show students getting to a point where they can not move forward (gets stuck) with a particular representational space. They then moves to a different representational space and compare between representational spaces. Such a change in direction seems to help students move forward. Within the first problem, we observe Danny's attempt to connect between algebraic and diagrammatic spaces to reason for his quick mathematical manipulations is not successful until he decides to use another semiotic resource (*line as wire*) to further develop his diagrammatic space.

Then within the two charges on a line problem we observe the missing background information in Oliver's diagrammatic space due to not picking the *field point* preventing him connect between the algebraic and diagrammatic spaces. This situation compared to Larry and Charlie, they pick *field point*s in each region and the developed diagrammatic spaces allows them to move forward without getting stuck.

On the Mechanics problem, again we observe Zeke could not make the connection between the algebraic and diagrammatic spaces in his first attempt. Then the gestural space allows him to feel for the forces that help him to make his algebraic space align with the diagrammatic space.

Overall, the cases presented in this analysis exemplifies the coordination between multiple semiotic resources with different disciplinary affordances to build up representational spaces and then using those representational spaces to solve physics problems. Also, we can conclude that the disciplinary affordances of those representational spaces influencing how students connecting among those spaces.

Different representations are used in science learning communities for different purposes. Within undergraduate classrooms, representations are mostly used for communication and problem-solving purposes. Studies have shown the importance of improving students' representational practices not only in physics but in other disciplines as well. Some of these studies focus on how we can facilitate students' development of representational fluency [206;207;208;209]. Other studies focused on how we can measure students' representational fluency [210;211;212;213].

One of the important goals of undergraduate physics instruction is to help students develop expertise in problem-solving [214;215;216] and one part of that is by improving the representational fluency [127;217;207;218]. The researchers who have tried to investigate and improve students' representational skills in physics as well as in other disciplines have come across student difficulties related to conceptualization, interpretation of diagrams and spatial reasoning[219;86;87;220;221]. In his study Fredlund[130] also suggested that one approach to overcoming students' difficulty with this issue is by having instructors unpack or simplify representations for the students.

Instructors can use the ideas presented in this study to make students more aware of the disciplinary affordance of semiotic resources or the components of the representations that they use to build up representational spaces (diagrams, gestures, and mathematics). This can then be used is to help students to become better problem solvers. In addition, instructors becoming aware of the semiotic resources students bring in and the disciplinary affordances of those semiotic resources can help instructors understand why students have difficulties with solving problems that are associated with certain representations. Using this understanding, instructors can help students get un-stuck, which will better motivate students.

Altogether, this approach of understanding how students solve problems through coordination between semiotic resources and representations, can be used to change the way students think about physics and physics problem solving, which then show students that their thinking about physics can grow and change and need to be fixed[222;223].

6.5.1 Connection between studies

In our previous work we focused on how students translate among representations (diagrams, mathematical equations, gestures, and spoken words) in order to solve physics problems[174]. We categorized representations as follows: durable representations, which include diagrams and written mathematical equations, andevanescent representations, which include gestures and spoken words. In our study, Larry starts from a durable representation (diagram) and builds up from there using evanescent representations (gestures and words). He later translates to a different kind of durable representation (mathematics), where he reasons and solves the original problem.

Then we focused on how students build up these representations[170]. In this study, we used a social semiotic approach that focuses on all types of meaning-making practices that are accomplished through different semiotic modes. These modes, which are examples of semiotic resources, include visual, verbal (or aural), written, and gestural modes such as language, text, algebra, diagrams, sketches, graphs, body movements, signs, and gestures. Here we use the developed theoretical framework to investigate how semiotic resources might be combined to build up compound representations. These compound representations can then be used to solve problems.

In this approach, we introduce the idea of compound representations, which are composed of two or more parts or semiotic modes and which link at least two semiotic resources in the same representation. As an example, Larry builds the compound representation "current sheet" using semiotic resources in different modes. He uses the "parallelogram as current sheet" on the whiteboard (visual-diagram) and "paper as current sheet" on the gestural space. Later, he uses "hand for wire" on the gestural space and "line as wire" on the whiteboard (visual-diagram) to represent the current sheet.

In this study, we investigated the underlying mechanisms behind students' constructing spontaneous representations as they coordinated among different semiotic resources with disciplinary affordances. Even though we paid attention to the disciplinary affordances of the semiotic resources in this study, we did not pay attention to semiotic modes.

In the real world, we observe that students draw diagrams, record mathematical equations, use body parts (gestures), and use words as they solve problems. This leads us to continue on our social semiotic approach to investigate how students build up representations. In this study, we modified our theoretical framework that accounts for how semiotic resources should be combined to solve problems.We also introduced the idea of representational spaces (such as diagrammatic, gestural, and algebraic), composed of two or more semiotic resources in the same semiotic mode. Then these representational spaces can be used to solve problems.

Even though these studies were published separately and used different theories and approaches to analyze different cases, they are strongly related. We could say that any given concept can be divided among representational spaces. Students bring small, reusable elemental ideas (semiotic resources) that are tightly connected to a particular representation to build these representational spaces. Figure 6.32 shows how ideas, semiotic resources, compound representations,representational spaces and durable/evanescent representations come together to help us understand how students solve physics problems.

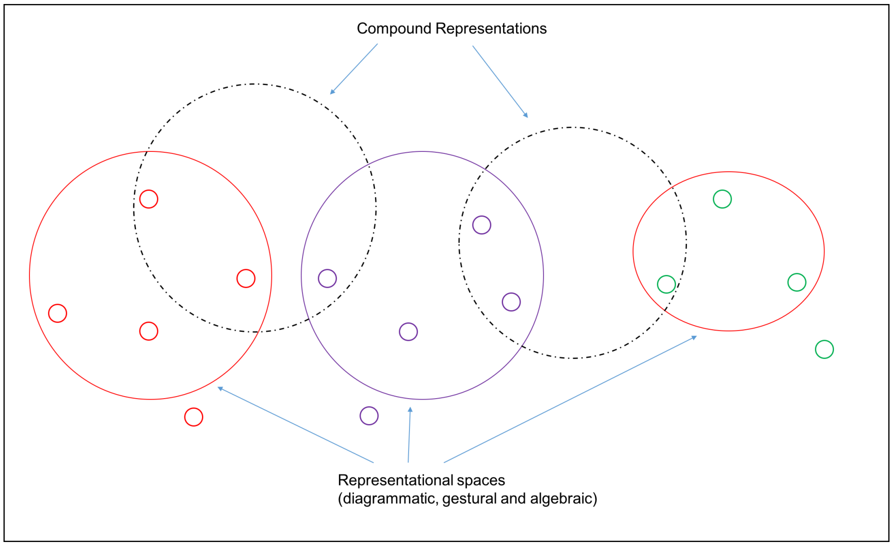


Figure 6.32: *Connection between studies: Each semiotic resource (small circles) has a semiotic mode (visual: diagram in red and algebraic in green, actional: gesture in purple) that contributes to meaning-making. These semiotic resources are coordinated to build either representational spaces (diagrammatic, gestural, algebraic) or compound representations*

As an example, let's think of a situation where a force acts upon an object (a mass). We will have a set of semiotic resources that are in different modes to construct representational spaces that deal with Newton's second law. The diagrammatic space would be a free body diagram that consists of semiotic resources like:"block as object", "arrow as vector" to represent forces on the object. Then gestural space would consist of semiotic resources like: "hand for object", "finger pointing in direction" to represent the force direction or acceleration direction. Then the algebraic space would consist of the semiotic resource "mathematical equation F=ma" that shows the object will accelerate in the direction of the force, with acceleration proportional to the magnitude of the force. Then we observe how students translate among those developed representational spaces to solve physics problems[(174)].

But when we talk about the compound representations, still students bring those elemental idea pieces (semiotic resources) together but they may not be in the same mode as we see in the representational spaces. As an example, a compound representation named "moving object” could be built by coordinating visual semiotic resources“block as object”, “arrow as vector” and gestural semiotic resources “hand for object”, “finger pointing in direction”.

Above mentioned Newton’s second law example is directly related to physics. But these approaches to develop representations and translating among those representations to build meaning to solve problems can be used in other disciplines (such as chemistry, engineering, biology) to improve student understanding and visualization of concepts[224;132;225]. Recent research identifies the process of students constructing representations as an effective method to support students’ visual sense-making[226;227] that results in enhancing students’ learning of domain knowledge in STEM courses[228;229].

Within this context, the approach to solving problems presented can be helpful for effective student-student and student-instructor interactions in undergraduate classrooms[230;231;232;233;234]. This approach can be very helpful for instructors to create more inclusive environments to facilitate student learning with different knowledge backgrounds and levels. Importantly, such a detailed approach of student-generated representations can facilitate collaborative problem solving that allows students to consider the affordances of each component of the representation[235;236;237]. Also, this allows students to share and modify representations with feedback from peers who sometimes have limited ability to describe their emerging understandings and ideas just by using one particular representational type[238;239]. This approach to make the classroom environment to be engaged and collaborative would help us as instructors to foster a *Growth mindset* in students.

Chapter 7

Discussion and Future Work

The goal of this dissertation is to increase understanding of students' use of mathematical tools and representations across the undergraduate physics curriculum. First, I used a modified version of the ACER framework, to make better sense of the ways that students use mathematical tools in their homework solutions in upper-division mechanics courses. Then I investigated how students use representations while solving problems. We combined these results to build better ways of explaining how students connect among different semiotic resources as they build up representations while solving problems. In this chapter, I briefly summarize the studies that were presented in the preceding chapters and discuss potential future research directions for each study.

As I was trying to understand undergraduate students' problem solving approaches, my intention was not to develop a new theory but to use an existing theoretical framework to guide my analysis about what and how students might use mathematical tools to solve physics problems. In ACER, we find a theoretical framework[45] that provides a structure to organize analysis of complex, mathematical problem solving. Previously the ACER framework was used to identify student problem-solving difficulties during think-aloud interviews andclassroom activities. Instead of looking for what students do wrong (finding student difficulties), we looked for the steps and processes that students actually use as they solve problems. In addition, we identified student homework solutions as an alternate stream ofdata, beyond think-aloud interviews and classroom observations.

In order to identify the most important mathematical operations in the problem-solving process, I modified the existing ACER framework and used it to code students' homework solutions. We could then use these coded solutions to identify the most frequent steps that students took in a solution. The process of comparing all the solutions in a given homework even for one student would be prohibitively time consuming if we attempted to extract this information manually. Instead, we employed a well-known method that is mostly applied in biological information systems to obtain the information contained in social networks, known as social network analysis (SNA). We used the visual representations of networks to represent the relations among students' ideas/knowledge components. These knowledge components were organized within the ACER framework, using the connections between those components, which were decided by the order of emergence of the knowledge components in the coded solutions. In the network graph representation, the knowledge components are referred to as nodes and the links between them as edges.

Finally, we were able to identify important mathematical steps/operations in student solutions that we had interpreted through the ACER codes. We found many problems to have similar central nodes. This similarity may be because all the problems analyzed in this study were similar well-designed problems that were picked from end-of-chapter textbook problems. But, further analysis showed the exact nodes and their order are different across students and across problems. This difference may be due to different students having different approaches to starting and working through each problem.

The analysis presented in the 3rd chapter represents the data collected from a traditionally taught mechanics course at a single university. We used a commonly available data source at the undergraduate level (homework) and applied a very powerful analytical tool (SNA) to visualize students' problem-solving approaches that are not immediately apparent in their written solutions. Instructors can use this method to visualize student problem solving in a way that allows for comparisons between students across a problem, a homework, or even across an entire course or multiple courses. In addition, this method helped us visualize student problem-solving in a way that allowed us to identify common mistakes that students make while solving problems. It takes much labor to code student solutions in this way, however. Future work with this approach might include comparisons within just one course - like analyzing real-world problem sets along with ones drawn from textbooks. Or we could expand this approach to compare between courses across the upper-division level. We could also analyze student solutions using this approach to compare different teaching styles, such as active-learning classrooms versus traditional ones.

At the beginning of my research work, we looked at upper-division mechanics students' written homework solutions(171) using ACER(45) to model student problem-solving. Even though student homework solutions provide a record of the steps used to solve a given problem, these solutions still fail at some points. Using only a student's written solution, we cannot identify which step was completed first. Also, written solutions often tell us very little about why a student chose a particular process or substitution. We almost always fail to find any reasoning in student solutions other than straightforward mathematical steps.

In order to obtain more information about student problem-solving processes, we asked upper-division Electromagnetism I students to generate problem solving videos ("pencasts")(240) as they solved homework problems. In addition, I made analyzing students' problem solving videos (oral exams) a part of their upper-division Electromagnetism I course. I was able to observe interesting moves and processes of students engaging in problems that we could not observe in written homework solutions.

But, as a project, we are curious about the student process of problem solving. As I analyze students' written homework solutions(171), I found that it is just a record of what students are doing than a process of solving problems. Then pencasts(240) which are student generated videos felt like a presentation of a problem that students already solved than actual problem solving. Then I moved to look in to upper division physics students and we look in to their videos of problem solving as a part of the regular course work (oral exams). In these videos, students were found to be using different representations (diagrams, kinesthetic (gestures), mathematics, words) for multiple purposes (recording, sense-making, and communicating) as they solved their oral exam problems. This study led us to investigate how students use representations to solve physics problemsas well as how students bring up and use those representations.

The case study presented in Chapter 4 shows an example of phases in problem solving. The case presented in this study, Larry, solves a physics problem by coordinating and translating between different durable and evanescent representations. Larry starts by using a diagram (durable representation), but he moves to a different type of durable representation (mathematics) to better progress and solve the problem. Larry also uses a sheet of paper along with gestures (evanescent representations) to make sense of his diagram and mathematical equations. As an extension of this project, we were curious to learn how students construct representations, and we wanted to look into more cases/students across different problems in different physics courses at the upper-division level.

In previous work on students' use of representations to solve physics problems, researchers associated the Cognitive Load Theory [143] to explain the advantage of using spontaneous representations. The researchers [145] suggested that the use of representations reduces cognitive load. The students who did not employ representations had to keep more information in their working memory while also reasoning about the problem. This may have increased their cognitive load and ultimately led them to be unsuccessful in deriving a correct solution to their problem. One major limitation of this approach is that It does not allow us to get an accurate view of the actual cognitive load that leads to further difficulties with measuring the cognitive load. On the other hand, the researchers did not focus on how students use these representations. By use we mean how do students construct representations and make connections among those representations.

Fredlund [109;130] took a different approach and used the idea of disciplinary affordances of semiotic resources to explain student problem-solving behaviors using representations.First, Fredlund [130] with the refraction of light experiment, argued that two semiotic resources, wavefront diagram and ray diagram, allow students to access different areas of physics knowledge. Then Fredlund [109] with the RC circuit diagram, showed that when students unpack a representation, they are able to access knowledge that is highlighted by that representation. Within these previous studies, researchers did not consider spontaneous construction of representations by students, and they did not look at how students connect among representations that they use to solve a problem.

In the next study, I used a social semiotic perspective to construct a theoretical framework to tell us how students might construct compound representations as they solved physics problems. In the study presented in Chapter 5, we use the same case of Larry. Instead of focusing on how he translates between representation categories, however, we focused on how Larry coordinates among different semiotic resources with different disciplinary affordances to build compound representations and to solve a problem. In general, most physics problems are represented with verbal description (maybe along with a diagram). In order to start the solving process, students must convert the word problem into a form that is more suitable to further analysis and mathematical manipulation. In our study, we have observations of the classroom, where the representations were developed, determined to be insufficient, and replaced with new representations brought in by the student.

In the theoretical framework we proposed in the previous study, we did not pay attention to the modes of the semiotic resources. In particular, we did not pay attention to how representations in different modes get connected to each other or to what factors might influence how students succeed or fail to connect among representations that they spontaneously construct (that are in multiple modes). In the study presented in chapter 6, we revise our proposed theoretical framework to describe the underlying mechanism behind how students construct spontaneous representations by coordinating among different semiotic resources with disciplinary affordances. Then we used multiple cases of students solving oral exam problems to generalize the approach we proposed in our previous study. We did this in order to explain the underlying mechanism behind how students construct spontaneous representations by coordinating among different semiotic resources in different modes.

Studies presented in Chapters 5 and 6, use a social semiotics approach and we analyze the same case of Larry. The study presented in chapter 5 is our pilot study that we investigate the case of Larry as an initial approach to social semiotics. Later, we identified the limitations of the theoretical framework used in this study and we analyze and present the case of Larry again with a revised theoretical framework in chapter 6. Overall, the social semiotics approach used in the studies presented in chapters 5 and 6 illustrates an analysis that gives us a novel way to understand the functionality of each elemental idea (semiotic resource) as they are used to develop representations to aid in solving problems. Studies[200;231] show that physics students need explicit help with understanding how to construct representations and how to coordinate among representations as they solve physics problems.

Researchers who use the social semiotic approach have shown the importance ofunpacking representations as a way to help students gain a better understanding of the use of these representations as well as the underlying physics concepts[130]. In that sense, understanding the fine-grain nature of each element of each representation that is illustrated in our analysis allows the instructor to better show students how to build and use effective representations when solving physics problems.

Students' use of representations to solve physics problems were both investigated at the introductory level and at the upper-division level. Studies conducted at the introductory level focused on how students use representations to solve physics problems[88] as well as on the effect of instructor-provided representations on students' success at solving physics problems[241;127;242;243]. The work done at the upper-division level that used the social semiotic approach focused on students' approaches to solving physics problems with instructor-provided representations[109;130].

Often times researchers treat students' representations as static ones[241;127;242;243;88]. In fact these representations are built up over time. Thinking about how students construct and connect among representations in these two approaches can help us as educators pay attention to what are the things that actually students are doing while solving problems. For example often times in problem solving we pay a lot of attention to students' diagrams and algebra. But paying attention to their gestures is just as important for understanding how students sense making. Moving forward, I am interested in investigating how do students mediate between diagrams and algebra using gestures while solving physics problems.

Work presented in this thesis at the upper-division level focused on the ways individual students develop representations and how those students use their developed representations to solve physics problems. We have shown the reliability of our approach by using it across multiple cases (students) and across different courses. We can predict its applicability across different physics courses at different levels. Moving forward, I am excited to look more deeply at how introductory-level students build up and use representations to solve physics problems in both individual and group settings. There is a clear difference in conceptual understanding between introductory-level and upper division students. In addition, students at the introductory level get only limited exposure to problem-solving and solve less number of quantitative problems compared to students at the upper-division level. Students at the upper-division level also learn new and advanced problem-solving skills and approaches compared to students at the introductory level.

The interviewer involved in these oral exams are the same person who is the instructor for this Electromagnetism I course. This instructor conducts her classes, oral exams and provides hints to students in a certain way, but we think this interaction does not materially change our argument about how semiotic resources can come together to build representations and arguments to solve problems. But, we understand that other instructors might emphasize different semiotic resources, therefore students might have experienced a different mix of semiotic resources that they might use while solving problems. This might be completely different for institutions where there is a math methods course. Along this lines, it would also be interesting to investigate how students develop and use representations at the introductory level at different institutions. More specifically, it would be interesting to compare those institutions that use more traditional instruction methods with those institutions that use research-based transformed approaches for instruction.

In summary, overall this thesis presented two novel approaches. First, we presented the use of rich source of data that is students' homework solutions along with social network analysis for studying how students use mathematics in their physics courses. Later, we proposed and further improved an approach to describe how do students develop representations and what are the features of how students connect among these representations. It is important that these representations are developed by students but not ones that are given to them by researchers.

Chapter 8

Research Questions and Answers

In chapter 3 we present our attempt to analyze the written evidence of students' problem-solving in a traditional physics course. In this study, we investigate the research question: what, if anything, could be learned about students' problem solving from their homework solutions? Using the modified ACER framework and the social network analysis to compare students and problems, we were able to identify three kinds of differences in student homework solutions. Differences in solution paths within one problem - even though this problem was a well-structured problem, different students had different approaches. Differences in students? initial problem-solving steps as a function of the type of problem - we found that there is a dependence on the problem statement reflected on the network graphs that is because of the cues provided on the problem statement. Additionally, we found differences among multiple problems of the same type - the problems which consist of more subsections also lead to relatively complex networks.

The study presented in chapter 4 focuses on students' use of multiple representations. In this study, we explore research questions: how can problem-solving be described as a process of transitioning between durable representations to make new meanings? How do students build meanings onto durable representations using the evanescent representations? We found that problem-solving can be described as a cyclic process of building new meaning onto durable representations using evanescent representations.

In our initial approach to describing students' construction of representations, the study presented in chapter 5 we explore the research question: How do students coordinate among different semiotic resources to build up compound representations while solving complex physics problems? We found that in the process of solving physics problems students make-meaning by coordinating multiple semiotic resources. This analysis highlights how the disciplinary affordances of different semiotic resources which make up for the constraint of other resources or reinforce each other to enhance meaning-making possibilities.

Later in our final study presented in chapter 6, we further study how students construct representations. In this study, we explore research questions: What is a better mechanism to explain how students build and connect together among different representations to solve physics problems? What are the factors that influence how students succeed or fail in connecting among representations? We found the need for revising our theoretical framework presented in chapter 5 to better describe students connecting multimodal representations. In our new framework, we described how students coordinate among different semiotic resources (which are in different modes) with different disciplinary affordances to construct representational spaces (diagrammatic, gestural and algebraic) instead of compound representations to solve complex physics problems.

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Appendix A

Appendix A - Codebook of the modified ACER framework

1. *Activation* of the tool

    - A1 - Identify the target (quantity/ value)
    - A2 - Student uses a general form of an equation
    - A3 - Student uses a less-general form of an equation

    ANX (N=1, 2, 3) used strategy is unhelpful or incorrect

    - A2X Student uses an unhelpful /or other form of an equation

2. *Construction* of the model

    - C1 - Pick a coordinate system
    - C2 Visualize the problem
    - C3 use an equation specific to the particular problem
    - C4 understand the meaning of symbols in equation
    - C5 set the limits of the integration
    - C6 determine at which point the derivatives should evaluate

- C8 label forces on the free body diagram
- C9 make assumptions
- C10 stating boundary conditions
- C11 applying boundary conditions

CNX (N=1, 2,..7) used strategy is unhelpful or incorrect

- C1X - mistakes made whenpicking a coordinate system
- C5X - mistakes made whensetting the limits of the integration

3. *Execution* of the mathematics

- E1 do the integration
- E2 take the derivatives
- E3 make the substitutions
- E4 student draw graph(s)
- E5 doing algebra
- E7 Approximations
- E8 evaluate the derivatives
- C9 make assumptions
- E9 [previously C7] take the cross product
- E10 evaluate the integration

ENX (N=1, 2,..4) used strategy is unhelpful or incorrect

- E1X - mistakes made whendoing the integration
- E2X - mistakes made when taking derivatives
- E3X - mistakes made when making the substitutions

4. *Reflection* on the result

- R1 check the units
- R2 check the limits of the final answer
- R3 does this answer make sense?
- R4 does this answer fit in the next part?
- R5 Comparing cases

RNX (N=1, 2 ...4) used strategy is unhelpful or incorrect

- R1X - mistakes made whenchecking the units
- R2X - mistakes made when checking the limits of the final answer

Appendix B

Appendix B - TRANSCRIPTS

B.1 Larry - Chapter 4 and Chapter 5

Instructor: Suppose you had an infinite sheet which carries current k equal to some constant (records on the board, $k = \alpha\hat{x}$). What's the magnetic field look like?

Larry: (draws a parallelogram to represent the sheet) So the coordinate $\ldots x$ hat, y hat, z hat, xyz I mean. Uh... and you want to know like what is a real life infinite sheet...?

Instructor: Yea. What's it look like?

Larry: (long pause) Uh... I know capacitors like infinite planes of charges but I am pretty sure there is no current across, like actually across the surfaces of capacitors. Umm... and solenoids are more of just a bunch of rings. And sheets be assumed there is no gap...

Instructor: Okay.

Larry: I don't think you can do a math for them, consider them to be a sheet.

Instructor: Well take one sheet and bend to a circle, not one flat sheet.

Larry: Right...

Instructor: Okay. Shall we keep going?

Larry: Yes.

Instructor: Okay. What the magnetic field looks like, If you have this sheet of current?

Larry: Um...m okay, so think of this, a sheet is kind of having a bunch of infinite wires (up and down movement of hand) one next to each other and the current from a single wire curls around this (applies right-hand grip rule), and so, I think that like if you look at a point above it (pinpoints to a location), then uh .. from a single wire .... will point in ... So, if the current is in x hat (records an arrow on diagram). Then above the sheet it would be pointing out of the board (records on diagram), from one wire, so above one wire. So, I think it would be true for the rest of the sheet as well.

Instructor: So, if this is the sheet (refers to the sheet of paper), the current is going this way (indicates the current direction on the sheet)...

Larry: So, if this is the sheet (refers to the sheet of paper), the current is going this way (across the surface of sheet- right to left), and looking at a point above it, then from one wire, magnetic field will be point in that way (away from him).

Instructor: Okay.

Larry: So. I would think, ah well, but from... so like from this wire (above the surface of sheet), the current is gonna ... the magnetic field points that way (away from him), but from a wire over here (closer to the surface of sheet), it would be pointing more ..uh .. this way (pointing towards him), uh .. so it seems like they are gonna superposition, kind of complicated. Because like, so far, we were looking at a wire right here (draws a line at middle of the sheet) and then the current above the wire (applies right-hand grip rule) my hand curls ... points back in me. Uh ... if we look at a wire, like this is (points to the diagram) an infinite sheet so way back here in the y direction, uh then my fingers at that point will more point in down than they are towards me.

Instructor: What about one, that mirror, so that wire way far back, the mirror wire way far forwards?

Larry: Uh ... my finger is pointing up.

Instructor: Okay. Does that up-ness cancel the down-ness.

Larry: Yeah.

Instructor: We went out with a net of?

Larry: uh .. out. So I was right the first time?

Instructor: Yes. So it came out above and below it goes...?

Larry: So ... below it goes in (records on diagram).

Instructor: Okay. How big is it?

Larry: uh... So, I don't remember the magnetic field from a single wire. Could you tell me or I will look it up really quick.

Instructor: I could but I think you should use Ampere's law here instead.

Larry: uh... alright, so it looks like the integral B dot dl equals μ_0 I enclosed (records the Ampere's law equation $\oint B \cdot \mathrm{d}l = \mu_0 I_{enc}$).

Instructor: Cool, now you need to pick an Amperian loop.

Larry: I'll pick a loop. Current is in this way (pointing on the surface of paper). I think the loop is like this (hand shows the loop perpendicular to the edge of the paper).

Instructor: Okay. Which way is the current in now?

Larry: Uh ... rather cause current is in this way (points on the paper from left to right), I think the loop is like this (gestures the loop perpendicular to the edge of the paper).

Instructor: Okay. Good.

Larry: I can't ever feel like drawing 3D. Okay. And so? my current enclosed is gonna be ... said got (records equation) ($k = \alpha\hat{x}$), so my current is just, my current enclosed is just gonna be uh ... k dl right (records $I_{enc} = k$...). But it can't just be k times dl cause that's a vector and it not be a vector.

Instructor: Right.

Larry: I mean two end up is not a vector. But, could it be k dot dl, you like the sound of that?

Instructor: Well, you have to be careful about what direction dl in that case?

Larry: Because, I was good to opposites cancel out if...

Instructor: So, usually if we have an integral that is dl around a ... boundary, on the other side we have an integral of an area

Larry: Yeah, Right. I think about like we have circuit, we draw a loop which's like a soap bubble (draws on the board) ... uh, but circuit is just linear current like wire current, we have surface current

Instructor: So how big is your loop? draw your loop.

Larry: Alright. I am not really sure (draws loop), no it's not right (erases it). I committed to an idea of (draw the loops, then take the paper and gestures for the loop) if this is the current (left to right), it's gonna be like this. I think that...

Instructor: And you are wondering how to draw it?

Larry: Yeah. Did I communicated it properly?

Instructor: Sure, okay. So one of those dashed lines is sort of above in z, the other one is below in z.

Larry: And the other two are in the... and so this one (points to the drawing) comes up and down. It is in yz plane.

Instructor: Okay, good. And how wide is your loop?

Larry: When you say "wide", do you mean this dimension or this dimension (points to the loop on board)?

Instructor: It doesn't matter cause I am about to ask you about the other dimension so denote and tell me about the other one.

Larry: l (records next to one leg of the loop).

Instructor: Cool. And what is the other dimension?

Larry: Um... w.

Instructor: Okay, so now you are curious about how much current goes through you loop.

Larry: I am gonna say that these (points to w legs on the diagram) are not really important because they are perpendicular with the direction.

Instructor: the w lines?

Larry: Yes.

Instructor: Okay. But the l line is?

Larry: Yes.

Instructor: How much current goes through your loop?

Larry: So, we got the surface current k and then it is going across the whole entire thing and uh... Let's see. I think it would be (long pause), it would be uh... kl?

Instructor: Yes.

Larry: equals $l\alpha\hat{x}$

Instructor: Good. Okay. So we did k dot it with dA?

Larry: Um...

Instructor: And k and dA are in the same direction?

Larry: Yes.

Instructor: So you have no further vector information?

Larry: So (records on the board) integral B dot dl equals μ_0 integral k dot da. And then ... for this (left hand side) I can do four separated integrals right? Since its square (show by hand) and these two ("w" legs on loop) do not end up mattering.

Instructor: Because?

Larry: Because they're perpendicular to the ... Am I ... Is it these ones that I am not gonna concern about? I am pretty sure it is.

Instructor: Yes. Because the direction of dl and the direction of B are ...

Larry: .. are perpendicular.

Instructor: Okay. What's about the other two legs?

Larry: Uh ... So for the other two legs, uh ... all right, so this one is kind of above it in the z, which means that the field is coming out at me (finger pointing in direction gesture). ... So, I need to pick a direction for my loop, don't I?

Instructor: Yes.

Larry: Uh ... Let's say it all goes this way (counter clockwise). Okay. So, above it B is coming out and the way I draw my loop is coming out at me so they are parallel and then below it B is going in and my loop is going that way so they're parallel again.

Instructor: Cool. OK. So, you get for that integral?

Larry: Uh... so $2Bl$ (records on the board).

Instructor: Excellent and the right hand side?

Larry: μ_0 and the integral of k dot dA, wait just be kwl.

Instructor: No.

Larry: Okay. k is constant, so I can pull it out?

Instructor: No.

Larry: It's not?

Instructor: It's true that k is constant, but k is not everywhere in your dA.

Larry: Oh... uh...

Instructor: How much current pierces this loop? You worked this out not two lines ago.

Larry: Right. $l\alpha\hat{x}$

Instructor: Yup. Though you do not need the x hat anymore because you have done the dot product.

Larry: All right. (Completes the right side by $\mu_0 l\alpha$ and records on the board: $B = \mu_0\alpha/2$).

Instructor: Good.

Appendix C

Appendix C - TRANSCRIPTS

C.1 Chapter 6

C.1.1 Problem 1: Loosely wounded solenoid

<u>**Student 1: Alan**</u>

Instructor: Let's talk about the solenoid. I premised you that you will get to do a problem about loosely wounded solenoid. Well, let's do it. So, **you have a solenoid, which is infinitely long and it is loosely wounded so that there is some component of the current goes down the solenoid and some component of the current goes around the solenoid. We want to consider there to be a component of the current that travels down and a component of the current that travels around.**

Alan: (while the instructor was explaining, Allan draws on the board) Oh, I see… so then… where we have previously ignored the fact that current is got some upward direction… okay.

Instructor: Okay, find the magnetic field inside and outside.

Alan: (Alan labels the current component going up to be $I_{\hat{z}}$ and the current going around to be $I_{\hat{\phi}}$) Okay, the only way this differs is that the B dot dl contribution is... we can't say it's a zero for the... box of that...

Instructor: Why not?

Alan: Well, there is a wrote component... since we do B dot dl ... dot product is going to be the cosine of the two. If we are looking at (pause, keep talking to self) let's consider it separately... we might do B dot dl twice... (Alan records Ampere's law) ... I wanna say... we do B_z and the dot product being the... um... the cosine component... B cosine dl ... would be... cosine zero is one... So, then this is going to determine our vertical component... where $\mu_0 I_{enclosed}$ (records mathematics) and then... current is constant and then basically... B L (Alan picks the direction on his loop and labels the dimensions) ... L here... $\mu_0 I$ ... there is a coil density... in... per length... times that L (continues to record mathematics) so then B_z component is going to be $\mu_0 I$ over... that doesn't look right. What I've done (talking to self and completes the mathematics)

$$\oint B_z \cdot \mathrm{d}l = \mu_0 I_{enc} \tag{C.1}$$

$$B_z.L = \mu_0 In.L \tag{C.2}$$

$$B_z = \frac{\mu_0 I}{n}\hat{\phi} \tag{C.3}$$

Instructor: Why is n in the denominator?

Alan: Yeah... um... because I stopped about what I'm doing... I think about why it looks different... no... no. It's right the z component is (corrects his mathematics and further modifies it) that's from the *theta* ... $\phi - hat$ component... I guess.

$$B_z = \mu_0 I n \hat{\phi} \tag{C.4}$$

$$B_z = \mu_0 I_\phi n \hat{\phi} \tag{C.5}$$

Instructor: Okay, why would call this B_z ...?

Alan: Because the... B is in the direction (gestures) cross with the direction of I. So...

Instructor: Okay... so, why is it in the *phi* direction?

Alan: What is in the *phi* direction? ... that I or ... (goes back to mathematical equation and modifies it)

$$B_\phi = \mu_0 I n_z \hat{z} \tag{C.6}$$

Instructor: You had $phi - hat$ in your math.

Alan: Yeah. I know... I'm sorry... I'm (modifies the mathematical equation) ... there we go... that's what I meant to do.

$$B_z = \mu_0 I_\phi n \hat{z} \tag{C.7}$$

Instructor: Okay.

Alan: So then the... a...

Instructor: So that's good for inside and outside?

Alan: That's only inside. Because... it's zero outside... arguably... again... I don't... I still don't understand why exactly that is... it seems... you remember that big conversation... it's zero outside because it's zero at infinity... and difference is zero... therefore.

Instructor: It turns out that you can do this empirically... you can make a really long one... and measure what the magnetic field is really close to it and gets zero.

Alan: Okay. That works for me. Okay there is the z component of the B field... so then doing the B_r component... (starts recording the mathematical equation)

$$\oint B_r \cdot \mathrm{d}l = \tag{C.8}$$

Instructor: Are you doing the same loop or a different loop?

Alan: I guess... do the same loop... um... let me think about that. Though... um... no... because it's gonna be... It's like a cylinder carrying a surface charge, surface current I mean (Alan draws a cylinder and picks a loop) and there is my new *dl* ... um... my new Amperian loop.

Instructor: Is this going to give you the B in the r direction?

Alan: Nope. Is that you originally asked me? ... I'm sorry.

Instructor: No, but you wrote down B in the r direction.

Alan: I would like to revise that... so, it's going to be the *phi* component or the *phi* direction of B ... again $\mu_0 I_{enclosed}$ ... (continues recording mathematics) this will do it ... easier.

$$\oint B_\phi \cdot \mathrm{d}l = \mu_0 I_{enc} \tag{C.9}$$

Alan: ... then $I_{enclosed}$ is going to be... um... μ_0 ... total $I_{enclosed}$ is going to be (continues recording mathematics) *sigma da* ... um... *k da* right? which is just...

$$\oint B_\phi \cdot \mathrm{d}l = \mu_0 I_{enc} = \mu_0 \int k \cdot \mathrm{d}a \tag{C.10}$$

Instructor: So, you want to know how much current is piercing your loop?

Alan: Right, so it's going to be *sigma* times V and how we get there? . . . so there going to be *sigma* times. . .

Instructor: Where is *sigma* in this problem?

Alan: Isn't *sigma* is the. . . so the total current is just I, the z component of I . . . . . . can I leave it like that?

Instructor: Yup, we haven't specified how big is I_z compared to I_ϕ yet.

Alan: Okay. . . and this is going to be (further modifies his mathematics) B_ϕ . . . two *pi* . . . (goes back to his diagram and labels r) . . . r . . . μ_0 . . . um. . . again just I . . .

$$\oint B_\phi \cdot \mathrm{d}l = \mu_0 I_{enc} \tag{C.11}$$

$$B_\phi 2\pi r = \mu_0 I_z \tag{C.12}$$

$$B_\phi = \frac{\mu_0 I_z}{2\pi r}\hat{\phi} \tag{C.13}$$

Instructor: Is that good inside and outside?

Alan: Only outside. . . so if we draw our Amperian loop inside . . . then I_{enc} is zero (then Alan specifies the magnetic fields for each region)

$$B_z = \mu_0 I_\phi n\hat{z} \qquad \rightarrow r < a \tag{C.14}$$

$$B_\phi = \frac{\mu_0 I_z}{2\pi r}\hat{\phi} \qquad \rightarrow r > a \tag{C.15}$$

Student 2: Danny

Instructor: Okay, we have been talking about the solenoids. As if there were a sheet of current that got folded around a circle but it turns out that there is actually some current that goes in at the top and comes out at the bottom. So, there must be a component of the current goes in the down direction. So, **if you had an infinite long solenoid. what would be the magnetic field inside and outside?**

Danny: Infinitely long… but… you want me to consider current going down (gesture - hand moves down)?

Instructor: Yes, there is current going around and current going down.

Danny: If we just call the current going down (draws an downward arrow) as… let's call it I.

Danny: So, then my first thought could be to put an Amperian loop (gesture- hands to show a loop) around so… like this (adds a loop to his diagram) … so, then you would have a current going down through (gesture - downward hand movement)

Danny: (next Danny starts recording math without talking) So… what was the exact question again?

Instructor: What's the magnetic field inside and outside?

Danny: Okay (keeps recording math) and the current… there is no varying currents here.

Instructor: Time varying currents? … no.

Danny: … um… since there is no… (long pause) seems like it… trying to think specially… it may be something tricky… I don't know why.

$$\oint B_\phi \cdot \mathrm{d}l = \mu_0 I_{enc} + \mu_0 \epsilon_0 \int \frac{\partial E}{\partial t} \cdot \mathrm{d}s \tag{C.16}$$

Instructor: Okay… if there's the case as simple as you think, what you think it would be?

Danny: It would be this (points to the Maxwell's correction term in his mathematical equation) is just zero and you just get... I ... this I (points to the diagram) over two *pi* r (records the mathematical equation for magnetic field). That's for the outside and zero inside.

$$B = \frac{\mu_0 I}{2\pi r} \tag{C.17}$$

Instructor: What's the direction of the B, on the outside then?

Danny: Then... if we call this... z (refers to his coordinate system) and then this angle would be *pi* (Danny decides to add more parts to his diagram and modifies his mathematical equation) and this (points to mathematical equation) would be $phi - hat$

$$B = \frac{\mu_0 I}{2\pi r}\hat{\phi} \tag{C.18}$$

Danny: If you define like... oh... okay... I've already said this (current direction - points downward) was down. So... this B field equation be minus $phi - hat$ (modifies the equation).

$$B = \frac{\mu_0 I}{2\pi r}\left(-\hat{\phi}\right) \tag{C.19}$$

Instructor: So, you went from... the closed integral of B dot *dl* to B equals two *pi* r ... really quickly. How did that happen?

Danny: So, for outside... and this would be r (labels direction of r of his diagram) defined from z (gesture - up and down movement of hand) down the middle. Um... (points to his mathematical equation) a line integral closed ... so... a loop right... and then um... (long pause) because this is still... we are considering a cylinder right (gesture - both hands holding like a cylinder) (long pause) ... okay... I see what you were saying... because I assumed that was like a line, a wire of charge... because what I did… was assume that B is always in the direction of my loop points.

Instructor: Okay, and is it?

Danny: Well… it's going to be perpendicular to the current (points to the diagram) … okay… So… I was thinking like… a cylinder (draws on the board) and I was thinking like a instead of a sheet of charge coming down (gesture - both hands holding like a cylinder), just think like one line coming down… and so that would be like a… part of the current… and then that would be exactly like line charge… So, I'm kind of imagining like… yeah. that should work… I think it's good.

Instructor: Okay… so that gives us the B due to the component of the current that's going down the outside. There should be some B due to the component of the current that's going around. What's that?

Danny: And this is still infinite… so, we would look at a loop… so… I turn this onto side (draws on the board) and say that there was… um… current coming in out of the board (gesture - finger pointing out of the board) so… then this is inside… I draw my Amperian loop like this then I do B dot dl … and I end up with (continues to work in mathematics) … yeah… this (labels on the diagram) is L (then he record mathematics without any explanation).

$$B = \frac{\mu_0 N I}{L} \tag{C.20}$$

Instructor: Where did that come from?

Danny: So… B … dot dl … so, this is … I'm going to go… around this way (picks the direction on his loop) … so… do I need to argue like the full argument where… we make a loop out here (draws a loop far away) and there is no flux… like or can I just… well I'll just do it.

Instructor: Sounds like you did most of it already.

Danny: Yeah... so there is no flux through a field out here (points to the loop that is far away) ... through the loop out here and there can be any flux through... um... a loop at infinity. We know that the field goes to zero at infinity. So... if the field is not charging it has to be zero at infinity and then it's zero all the way up to for an infinite solenoid. So... that means we don't consider this side of the loop (points to the loop) ... because B is zero. So B dot dl would be zero. These sides (points to the sides of the loop) won't contribute so, you just left with L ... that's why there's one L here (points to his mathematical equation) ... so it would be B times L equal μ_0 I but there is N of them. So you get what ever up here.

$$\oint B \cdot \mathrm{d}l = \mu_0 I \tag{C.21}$$

$$B.L = \mu_0 I N \tag{C.22}$$

$$B.L = \frac{\mu_0 I N}{L} \tag{C.23}$$

Instructor: Okay... is that I same as the I in the other one?

Danny: Well... yeah... because... well... let's see... yeah.

Instructor: So you have outside B in the $phi - hat$ direction, inside you have B in the z direction?

Danny: Yes.

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C.1.2 Problem 2: Two charges on the line

<u>**Student 1: Larry**</u>

Instructor : So... suppose (draws on the board) We have two point charges, one at $-a$ & the other at $+a$ and this one (at $-a$) has charge $-q$ and this one (at $+a$) has charge $+q$. What's the electric field looks like along the x-axis?

Larry : Alright, so I suppose let's start by finding the electric field for each of them... then summing... um... so let's put down a point I guess (marks point p on the axis right of $+q$) and then I've to get the distance from the charge to the point... and for each of them. I think it's different in each of the three regions. I've to write a different... expression for... the distance. Are you allowed to tell me if what I'm saying is right or not.

Instructor : Yes.

Larry : And you haven't fell the need to mention anything so far?

Instructor : I want to see how far you can get.

Larry : Okay... well... alright... so the electric field from this guy (draws a circle around $+q$ charge) is gonna be... I'll call him $E{-}plus$ (starts recording mathematical equation and calls out the names of the symbols while recording) it's gonna be $k\ q$ over... um... I'll just call $r - plus$ and that's gonna be... since it goes outward... uh... like here it's going to be this way (records on the board - a rightward arrow to right from $+q$ charge) and from here (points to the region left of $+q$) it's going to be that way (records on the board - a leftward arrow to left from $+q$ charge) and here (points to the region right of $-q$) it looks like that (records on the board - a leftward arrow towards $-q$ charge).

Instructor : Do you want to divide this in to different regions?

Larry : That's what my intuition tells me.

Instructor : Okay, divide it in to regions.

Larry : Alright, we'll call this region 1, region 2, region 3 (labels on the diagram).

Instructor : Okay.

Larry : Um... so... for region 3 ... uh... my $E - plus$ would be k q over... my r will be (starts recording mathematical equation and calls out the names of the symbols while recording) uh... x minus a ... yeah and that would be in the positive $x - hat$ direction. Then $E - minus$ will be k q over uh... x plus a ... yeah... in the minus $x - hat$ direction.

$$E_{+} = [\frac{kq}{(x-a)^2}]\hat{x} \tag{C.24}$$

$$E_{-} = [\frac{kq}{(x+a)^2}](-\hat{x}) \tag{C.25}$$

Larry : Oh... that's a minus q as well (modifies the mathematical equation) so they cancels... never mind they points the same direction... they will always add.

$$E_{-} = [\frac{-kq}{(x+a)^2}](-\hat{x}) \tag{C.26}$$

Instructor : So... before you had... in that region minus q charge had a... electric field to the left and now you wanted it to be to the right?

Larry : No that's not right. Isn't it? um...

Instructor : Definitely inconsistent.

Larry : Yeah. I fell good about the arrows. Uh... so... $E - minus$ will be minus k q over (modifies the mathematical equation) because that minus is encoded there and so... they will end up subtracting from each other... and so... the total is the sum of these two (records a mathematical equation) and that's gonna be true for all areas.

$$E_- = [\frac{kq}{(x+a)^2}](-\hat{x}) \tag{C.27}$$

$$E_{total} = E_+ + E_- \tag{C.28}$$

Larry : ... and... then for region 2... it's basically the same except... both of these are gonna be... uh... the distance from both of these gonna be... a minus x. I'm pretty sure... no I don't think that's right... they are both in the minus $x - hat$ this time.

$$E_+ = [\frac{kq}{{r_+}^2}](-\hat{x}) \tag{C.29}$$

$$E_- = [\frac{kq}{{r_-}^2}](-\hat{x}) \tag{C.30}$$

Larry : ... and... this time the distance... if I pick a point here (points to the field point marked in the middle region) the distance is a minus x for this guy (E_+) and the distance here would be... still a plus x or x plus a.

$$E_+ = [\frac{kq}{(a-x)^2}](-\hat{x}) \tag{C.31}$$

$$E_- = [\frac{kq}{(x+a)^2}](-\hat{x}) \tag{C.32}$$

Larry : Then region 1 ... so I'm not doing anything terrible wrong so far?

Instructor : You are doing fine. You are more nerves than you need to be.

Larry : Um... still got (calls out the names of the symbols while recording the math equation) k q over... this time... it's still going to be a minus x down here. Yeah... since x is gonna be negative... and that's in the minus $x - hat$ direction... and this (points to E_- equation) is going to be k q over… uh… a plus x … no … a minus x … positive $x - hat$ direction.

$$E_+ = [\frac{kq}{(a-x)^2}](-\hat{x}) \tag{C.33}$$

$$E_- = [\frac{kq}{(a-x)^2}](\hat{x}) \tag{C.34}$$

Larry : No that's not right.

Instructor : Which part of it's not right?

Larry : Um… this (points to E_- equation) should be plus (modifies the equation) yeah… that effectively shortens the distance… yeah.

$$E_- = [\frac{kq}{(x+a)^2}](\hat{x}) \tag{C.35}$$

Instructor : Okay… how do you know it's a plus x?

Larry : So… like intuitively I know this point (points to field point in region 1) closer in region 1. So… the a and the x should interfere with each other and since x is negative then it should be positive… um… I mean if you just look at it … your distance is this (gesture- fingers at $-q$ and field point) your total distance is this (gesture- fingers at mid point and field point) and this is your a (draws lines to specify the distances on the diagram) and this is your x … this is much better way to explain it.

Larry : I change my mind… and this (draws lines to specify the distances on the diagram) is your x … and this is your a and your total distance… is wait… I'll call this minus x I guess.

Larry : Or may be… I shouldn't… (Larry moves to consider region 3 and draws lines to specify the distances on the diagram)

Larry : ... this should be x minus a for the region 3. Then for this one (Larry moves to consider region 2 and draws lines to specify the distances on the diagram) that's x and that's a and this should be a minus x ... which I have... and for this one (points to region 1) ... do you... am I being clear enough?

Instructor : I'll ask... if I needed to.

Larry : And then here we got x and then a. So that should be x plus a (modifies the electric field equation for region 1)

$$E_+ = [\frac{kq}{(a+x)^2}](-\hat{x}) \tag{C.36}$$

$$E_- = [\frac{kq}{(x+a)^2}](\hat{x}) \tag{C.37}$$

Larry : I really should have been more careful... when I did this the first time... and (Larry starts to reconsider his mathematical equations) for the negative ones... um... I've got a point here (points to region 3) so the big x and a and that should be x plus a, which it is. And then here (points to region 2) I've got little x ... still should be x plus a, which it is. And then here (points to region 1) ... I've got x and a ... so it should be x minus a ... uh... or if I wrote it a minus x ...

$$E_- = [\frac{kq}{(a-x)^2}](\hat{x}) \tag{C.38}$$

Larry : ... no it doesn't still come out right. I'm getting hung up on how the ... in here (points to region 1) my x is always going to be negative ... so... yeah it should be... I'll write it negative x plus a ... so a minus x.

$$E_- = [\frac{kq}{(-x+a)^2}](\hat{x}) \tag{C.39}$$

Instructor : So the negative x plus a ... says that... if x has a value ... has a negative value... suppose x was negative seven then you would over here in the left side... then negative x plus a would give you *seven* plus a ... is that the distance you got to express?

Larry : No, it's not... (then Larry modifies the equation) yup.

$$E_{-} = [\frac{kq}{(x+a)^2}](\hat{x}) \tag{C.40}$$

Instructor : Okay, now you have $E-plus$ and $E-minus$ as the same as each other for region 1.

Larry : Yes... um... and no where it looks alike.

Instructor : Should $E-plus$ and $E-minus$ be the same in region 1?

Larry : No they shouldn't ... they should be ... (modifies the mathematical equations) it should be x minus a.

$$E_{+} = [\frac{kq}{(x-a)^2}](-\hat{x}) \tag{C.41}$$

Instructor : So now, if we look across all three regions just $E-plus$ ... is it always the same?

Larry : Um ... no.

Instructor : What's different about it?

Larry : x minus a (points to $E-plus$ equation in region 3), a minus x (points to $E-plus$ equation in region 2). I like that one though ... then if my x is positive... then a minus x ... the number gets smaller the magnitude gets smaller and my distance is ... closer in this side (points to right side from mid point in region 2) of the x axis. Then if it's negative then a minus x gets bigger and that would probably be here (points to left side from mid point in region 2).

Instructor : Okay. Is a minus x quantity square the same as x minus a quantity square?

Larry : Um... yeah it is.

Instructor : Okay.

Larry then modifies the equations for region 2)

$$E_{+} = [\frac{kq}{(x-a)^2}](-\hat{x}) \tag{C.42}$$

$$E_{-} = [\frac{kq}{(x+a)^2}](-\hat{x}) \tag{C.43}$$

Instructor : Okay. If we look across regions 1,2 and 3 ... only $E - plus$ is always the same?

Larry : Um... except for the direction in region 3.

Instructor : Okay.

Larry : The magnitude is always the same.

Instructor : What about $E - minus$?

Larry : Um... except for the direction in region 1, yes.

Instructor : Okay.

Final versions of Larry's mathematical equations.

Region 3,

$$E_{+} = [\frac{kq}{(x-a)^2}](\hat{x}) \tag{C.44}$$

$$E_- = [\frac{kq}{(x+a)^2}](-\hat{x}) \tag{C.45}$$

Region 2,

$$E_+ = [\frac{kq}{(x-a)^2}](-\hat{x}) \tag{C.46}$$

$$E_- = [\frac{kq}{(x+a)^2}](-\hat{x}) \tag{C.47}$$

Region 1,

$$E_+ = [\frac{kq}{(x-a)^2}](-\hat{x}) \tag{C.48}$$

$$E_- = [\frac{kq}{(x+a)^2}](\hat{x}) \tag{C.49}$$

<u>Student 2: Oliver</u>

Instructor: Okay… So suppose … (draws on the board) We have two point charges, one at $-a$ & the other at $+a$ and this one (at $-a$) has charge $-q$ and this one (at $+a$) has charge $+q$. What's the electric field looks like along the x-axis?

Oliver: Okay… So you just … you just want an expression for it or …

Instructor : Sure.

Oliver : Okay.

Instructor : You can draw on the board. It's okay.

Oliver : So … the total field is going to equal the contribution from each of them, because of the superposition (records math on board) … I want to just consider along the x-axis?

$$\vec{E} = E_+q + E_-q \tag{C.50}$$

Instructor : Yup.

Oliver : Cool ... so for ... (records on the board) $E_{+}q$ ... (steps away from the board) it's going to equal ... (he calls out the names of the symbols while recording the math equation) one over four pi epsilon note ... times ... q ... um... curl r squared ... r hat.

$$E + q = \frac{1}{4\pi\epsilon_0}\frac{q}{\Gamma^2}\hat{\Gamma} \tag{C.51}$$

Oliver : Um... so... um... I'm trying to think. I'm sorry... this r is going to equal... um... (records $\Gamma = (\dots)$ on the board and a long pause. He could not move forward)

Instructor : You look like you are stuck. Where are you stuck?

Oliver : Um... I'm ... figuring out this (points to $\Gamma = (\dots)$ on the board) it's (records on the board, $\Gamma = (a - x)$) and then x is anywhere here (indicate the region between a and $-a$ on the axis) right? ... no I don't know if that works.... that's what we have in notes.

$$\Gamma = (a - x) \tag{C.52}$$

Instructor : Sure.

Oliver : I'm forgetting is it r-prime minus r or r minus r-prime...?

Instructor : You are going to square script dr right?

Oliver : Yeah.

Instructor : So, does it matter?

Oliver : Um... no... well... no... does it?

Instructor : No. So, it would be x minus a.

Oliver : Okay, that's what I was trying to figure out (goes back and changes the equation, $\Gamma = (x - a)$)... so, x minus a ... and so ... then we substitute that in (modifies the math equation on the board).

$$E + q = \frac{1}{4\pi\epsilon_0}\frac{q}{(x-a)^2}\hat{\Gamma} \tag{C.53}$$

Oliver : Then E for negative q equals... (records on the board) I'm going to change this (points to, $\frac{1}{4\pi\epsilon_0}$) to k because it's easier... negative q... and then we are going to have ... x plus a squared and then r-hat... um...so... r-hat just um r-hat... to any region in pointing on the x-axis (points to x-axis on the diagram) on that direction and I just think it depends on the point we pick or we want the whole axis. right?

$$E + q = \frac{kq}{(x-a)^2}\hat{\Gamma} \tag{C.54}$$

$$E - q = \frac{k-q}{(x+a)^2}\hat{\Gamma} \tag{C.55}$$

Instructor : Hmm...

Oliver : So we just change that to x-hat, does that work?

Instructor : Well... that says the charge from the plus q charge or the electric field from the plus q is always in the x-hat direction.

Oliver : Um... but it could be negative (points the index finger along the negative x direction on the diagram) ... and so that would work.

Instructor : Alright.

Oliver : So... the field is going to point out here... and here (points to $-q$ and draws arrows on the diagram to represent electric field direction from each charge)... oh...

hold... excuse me... that way (changes the direction of the arrow from $-q$ charge to be rightward on the far left-hand region).

Instructor : It's okay... so, you could divide this up in to regions.

Oliver : Yeah. We could do this region (labels the regions, region 3 - left of $-q$, region 2 - between $-q$ and $+q$, region 1 - right of $+q$) ... that will work.

Oliver : So... Um... one and three are zero... because they... they are going to (gestures-index fingers towards each other) be destructive... or is... I guess... out here (points to region 3) ... does this one (points to $-q$ charge) ... this is gonna have... um... a greater effect than the contribution from this (points to $+q$ charge) or just...

Instructor : Right.

Oliver : Because... but I guess... um... so... in area 1... it's gonna be (points to the mathematical equation $\vec{E} = E_{+}q + E_{-}q$). This one ($E_{-}q$) minus this one ($\vec{E} = E_{+}q$)... right.

Instructor : So... okay... and what will be the overall direction of E?

Oliver : E is going to be to the right (index finger moves rightward).

Instructor : Okay, is that plus x-hat or minus x-hat?

Oliver : Plus x-hat.

Instructor : Okay.

Oliver : Right... so... E is going to equal (records on the board while calling out the names of the symbols) ... k times negative q ... x plus a squared ... minus q ... that doesn't seem right though.

$$E_1 = k[\frac{-q}{(x+a)^2} - \frac{q}{(x-a)^2}]\hat{x} \tag{C.56}$$

Instructor : What's wrong about it?

Oliver : Well... if these are both negative (points to the two terms in his mathematical equation) then looks like this... I'm not sure about that. It looks like this... would add together... but I have different denominators... so I can't say for sure. Um... because the field from this guy (points to $+q$ charge) in here (points to region 1) is going to reduce the field from this guy (points to $-q$ charge) out here (points to region 1).

Instructor : Okay and right now they have the same sign as each other or they have a different sign from each other?

Oliver : Um... right now it looks like they have some as each other. But... yeah... the only sign difference is going to be the cross product of the binomial part.

Instructor : Okay... is that going to give you an overall sign changed to your term?

Oliver : I don't think so. I guess you could pull the sign off... but...does it?

Instructor : No it doesn't.

Oliver : That's not right.

Instructor : So, you should have opposite signs from each other.

Oliver : Right.

Instructor : But your math doesn't say that.

Oliver : Right... so I'm not wrong probably am I... oh... because this (points to the second term in his mathematical equation) should be a plus (makes changes in his mathematical equation) because this still... it shouldn't be minus. That's my bad.

$$E_1 = k[\frac{-q}{(x+a)^2} + \frac{q}{(x-a)^2}]\hat{x} \tag{C.57}$$

Instructor : So... the contribution from the electric field from the plus q charge points in the plus x-hat direction?

Oliver :

Instructor : They should both be plus?

Oliver : No, one should be minus... this... this... okay... so... this (points to the second term in his mathematical equation) is this one (points to $+q$ charge) right... this points the negative x-hat direction out here (points to region 1).

Instructor : okay.

Oliver : And then this one (points to the first term in his mathematical equation) points opposite right here (points to region 1) because it's pointing in...

Instructor : Good. So, what does your math say about that?

Oliver : It says that's positive.

Instructor : How would you fix that?

Oliver : Doing the opposite (changes the sign in his mathematical equation) that (first term) should be a plus and this (second term) should be a minus.

Oliver : Do I need to go more further than that?

Instructor : Nope. Tell me about region 2.

Oliver : Region 2, they are going to be constructive... So, E-two... by constructive, I mean they both pointing in same direction (gestures-index fingers in same direction). So... it would be...

Instructor : Okay.

Oliver : Um... k um... so... this (points to $-q$ charge) is my x charge but it has negative x-hat direction (gestures-index fingers along negative x). So this is going to be (starts recording mathematical equation for the electric field in region 2) q over x plus a squared. This (points to $+q$ charge) um... wait a second (steps away from the board) ... that one is going to be negative then... that does not make sense.

Instructor : You sounds like you are re-considering something.

Oliver : Yeah, um... let's just do both separately. (Oliver calls out the names of the symbols while recording mathematics o the board) negative q over x plus a squared and that's going to be negative x-hat. Since it's backward... plus kqx minus a squared... negative x-hat.

$$E_2 = k\frac{-q}{(x+a)^2} - \hat{x} + k\frac{q}{(x-a)^2} - \hat{x} \tag{C.58}$$

Instructor : So, you have minus q, minus x-hat and plus q, minus x-hat

Oliver : Yeah.

Instructor : So if I stick all those minus together... I get the first term is positive and the second term is negative.

Oliver : So, it would be zero. Well... no... may be not...

Instructor : Are they suppose to be in the same direction or in opposite directions?

Oliver : Same direction.

Instructor : And that direction should be?

Oliver : To the left (points his index finger to left along negative x-hat direction)

Instructor : Okay.

Oliver : So... negative x-hat... so... yeah. I'm confused.

Instructor : You resolved this in region one by saying that the term from the charge on the left ($-q$) should be positive because it's contribution to the electric field is in the positive direction and the term from the charge on the right ($+q$) should be negative because it's contribution is in the negative direction.

Oliver : Yup.

Instructor : So... now you're in region 2 and the contribution from charge on the left should be?

Oliver : I do not worry about the sign but do worry about the field points at. that's what I'm keeping up on.

Instructor : Yup.

Oliver : And so... just be... um... (makes changes to his mathematical equations)

$$E_2 = k\frac{q}{(x+a)^2} - \hat{x} + k\frac{q}{(x-a)^2} - \hat{x} \tag{C.59}$$

$$E_2 = kq[\frac{1}{(x+a)^2} + \frac{1}{(x-a)^2}] - \hat{x} \tag{C.60}$$

Instructor : Okay... what's region 3 look?

Oliver : Region 3... is going to be much like region 1 except exactly the opposite. so... (records on the board) kq over x plus a squared and this is going to be negative x-hat... and plus q... x minus a squared and going to be positive x-hat, right?... so... for this charge ($-q$) the field is pointing inwards (gesture-index finger to the left) because this charge ($+q$) is pointing out. So that's going to be k minus q x plus a squared plus q over x minus a squared and x-hat for both of them.

$$E_3 = k[\frac{q}{(x+a)^2} - \hat{x} + \frac{q}{(x-a)^2}\hat{x}] \tag{C.61}$$

$$E_3 = k[\frac{-q}{(x+a)^2} + \frac{q}{(x-a)^2}]\hat{x} \tag{C.62}$$

Instructor : Okay.

Oliver : That's what it should be.

<u>**Student 3: Charlie**</u>

Charlie : So, along this line be coming in this direction (records the electric field direction on the axis) . . . this side (right from the mid-line) and also be . . . like so.

Instructor : Okay, So, why does it change direction of that charge ($-q$) but not the other charge ($+q$)?

Charlie : Because with the negative charge it comes in. . . wait a second (goes back to the diagram and make changes to the electric field direction). Yeah, it suppose to go out this way (rightward hand movement) that's my bad.

Instructor : Okay.

Charlie : Positive radiates out and negative radiates in.

Instructor : So, why are they. . . why is it pointing to the left in the middle.

Charlie : Because. . . it's. . . coming over to the negative (points to the $-q$ charge)

Instructor : Okay.

Charlie : Is that not how it suppose to be?

Instructor : No. It looks good.

Charlie : Okay.

Instructor : Alright, shall we do some math?

Charlie : Yes, we should.

Instructor : What's this look like in math?

Charlie : (charlie starts recording a mathematical equation) Um... we wanna do the total charge... r squared r-hat.

$$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r} \tag{C.63}$$

Instructor : Okay.

Charlie : And then Q is going to be $2q$ negative $2q$ ... with... along x direction. So... x-hat ... with r (points to r in the equation) being a distance of a.

$$Q = 2q \rightarrow Q = -2q \tag{C.64}$$

$$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{x} \tag{C.65}$$

Instructor : Okay.

Charlie : I feel like... this (points to mathematical equation) is more for... like a point... right?

Instructor : Okay.

Charlie : So, the... just the electric field... I don't know... I guess... just generally... I guess this is the electric field (modifies the mathematical equation).

$$E = \frac{-1}{4\pi\epsilon_0}\frac{2q}{r^2}\hat{r} \tag{C.66}$$

Instructor : Where is that electric field?

Charlie : Just anywhere.

Instructor : Anywhere... it's always in minus r-hat?

Charlie : I mean… I don't know what… what do you want for electric field? Just an electric field?

Instructor : All along the axis.

Charlie : Just all along the axis… So, it's going to be in the x direction (modifies the mathematical equation).

$$E = \frac{-1}{4\pi\epsilon_0}\frac{2q}{r^2}\hat{x} \tag{C.67}$$

Instructor : Always in the plus or the minus x?

Charlie : Um…well… for over here (points to the left of $-q$) it's in the positive x direction, then between here and here (region between two charges) it's in the negative x direction and here (points to the right of $+q$) to the positive x direction again.

Instructor : Okay. It sounds like you should write electric field for each region.

Charlie : (charlie starts to record electric field for each region, starting from far left, middle and far right regions) So this going to be E is equal to…

(he calls out the names of the symbols while recording the math equation) one over four pi epsilon note $2q$ r squared x-hat.

$$E = \frac{1}{4\pi\epsilon_0}\frac{2q}{r^2}\hat{x} \tag{C.68}$$

$$E = \frac{-1}{4\pi\epsilon_0}\frac{2q}{r^2}\hat{x} \tag{C.69}$$

$$E = \frac{1}{4\pi\epsilon_0}\frac{2q}{r^2}\hat{x} \tag{C.70}$$

Charlie : However... am I going to be getting a different electric field from this charge (points to $-q$ since it's further away? So, this $2q$ (points to mathematical equation recorded for far left region) going to be something different. So... am I suppose to do superposition too? So, rather than this (points to mathematical equation recorded for far left region) we could have something like (calls out the names of the symbols while recording the math equation) E equal k positive q over $2a$ squared.

$$E = k[\frac{q}{(2a)^2}] \tag{C.71}$$

Instructor : Where?

Charlie : Over here (points to the left of $-q$) where you got... because a plus another a.

Instructor : Okay, so, where are you looking at in this region?

Charlie : Um... this one (points to the left of $-q$)?

Instructor : Yeah.

Charlie : Will see. Could we look right here (points to the same position as $-q$ is on the axis) or right beyond it?

Instructor : We could, but we should.

Charlie : Then this one (points to mathematical equation recorded for far left region) could be... it could be some distance x. What I'm referring is x (labels the distance from $+q$ to the field point in left of $-q$). So just x squared (modifies the mathematical equation).

$$E = k[\frac{q}{(x)^2}] \tag{C.72}$$

Instructor : Okay.

Charlie : And then, it also... I'm going to re-define my x. I'm going to say my x is from this point (points to mid point on axis). Makes this (modifies the mathematical equation) x plus a squared. Sounds like a good idea? And this is going to be... um... x minus a squared.

$$E = k[\frac{q}{(x+a)^2} + \frac{-q}{(x-a)^2}]\hat{x} \tag{C.73}$$

Charlie : Now we got a whole different... Now that says it's going to be negative.

Instructor : What's negative?

Charlie : The field over here (points to the left of $-q$) because this (points to the term $\frac{-q}{(x-a)^2}$ of the mathematical equation) is going to be bigger than this (points to the term $\frac{q}{(x+a)^2}$). Because the bottom is going to e smaller than that (points to the term $(x+a)^2$).

Instructor : Okay. How would you adjust this?

Charlie : Change my negative sign (laughs)?

Instructor : Okay.

Charlie : Or is this (points to the mathematical equation) would be... or this should be added right? I think... I think I'm missing something... that makes my negative sign... obviously something has to happen to it.

Instructor : Something has to happen yes. Okay

Charlie : These (points to the terms q and $-q$ in mathematical equation) shouldn't be subtracted from one other. Should they?

Instructor : Why not?

Charlie : Just because... I feel like it's... like the electric field from both of them added together (gesture-hands with fingers spread moves towards each other). In one case... just in this case it's negative (points to the arrows on diagram).

Instructor : So, in this region over here on the left... the contribution from the plus charge points in what direction?

Charlie : Points in the direction (gesture-right hand moves leftward).

Instructor : And the contribution from the minus charge points?

Charlie : That direction (gesture-right hand moves rightward).

Instructor : So these two contributions should add or subtract?

Charlie : Yes. they should subtract (modifies the mathematical equation) and that makes sense.

$$E = k[\frac{q}{(x+a)^2} - \frac{-q}{(x-a)^2}]\hat{x} \tag{C.74}$$

Instructor : Now you have two negative signs, which makes a plus says they are adding.

Charlie : Yeah. Because it will become a positive... yes. Alright, I agree with that.

Instructor : So... label me which of these terms belongs with the negative charge, which belongs to positive charge.

Charlie : (Charlie labels $-q$ charge as q_1 and $+q$ charge as q_2) So q_1 and q_2 (then modifies the mathematical equation).

$$E = k[\frac{q_2}{(x+a)^2} - \frac{-q_1}{(x-a)^2}]\hat{x} \tag{C.75}$$

Instructor : Okay and so, when I combine these two negative signs, it says there is a positive sign.

Charlie : Yeah (then modifies the mathematical equation).

$$E = k[\frac{q_2}{(x+a)^2} + \frac{q_1}{(x-a)^2}]\hat{x} \tag{C.76}$$

Instructor : And now I'm saying the direction... the contribution of the electric field from each of these is both in the plus x direction.

Charlie : Or we just do that (modifies the mathematical equation) because that has a negative direction. Now this one is bigger as earlier said and would be positive.

$$E = k[\frac{-q_2}{(x+a)^2} + \frac{q_1}{(x-a)^2}]\hat{x} \tag{C.77}$$

Instructor : Okay.

Charlie : Because that one (points to $+q$) should be pushing it (points to $-q$) away (gesture-hand moves leftward) over here (points to left of $-q$).

Instructor : Okay. Let's go to the middle region.

Charlie : Um... so... something... (Charlie starts recording mathematical equation without talking, but stops recording halfway through) just have a distance x that's just anywhere from the origin (points to mid-point on axis) ... I know I need that there... I know.

$$E = k[\frac{-q}{(x+a)^2} \tag{C.78}$$

Instructor : Why it is need to be there?

Charlie : Because, if it was over here (marks the field point in the region between $+q$ and $-q$) then you can't just say it's x away from that one (points to $+q$). So I need... so... flip side if it's over here (marks the field point between $+q$ and mid-point) I can't say it's x away from that (points to $-q$) one. That's going to be the x plus negative a.

So the distance right here is x (labels the distance from mid-point to the field point), but this is negative a and this should be x minus a (records on the diagram).

Charlie : Then when we extend it to... this way (right side of the mid-point) or I guess then x would become negative or if it was negative over here (left side of the mid-point) that should work then.

Instructor : Okay. So... x is positive going to the left?

Charlie : Yes. I got mine negative and positive signs switched around.

Instructor : Okay. Can you draw for me which direction is x positive?

Charlie : Well draw. But then my negative is over in the x positive direction. I need to switch them. That means... can I just put a negative (modifies the equation)?

$$E = k[\frac{-q_2}{(-x+a)^2} + \frac{q_1}{(x-a)^2}]\hat{x} \tag{C.79}$$

Charlie : Okay... over here (marks the field point in the region to the left of $-q$) this is going to be negative x (Charlie re-defines the distance on the diagram). So the negative x plus a... that has to be minus a because it's going in that direction too. Should make this... negative x minus a (modifies the equation). That's fair.

$$E = k[\frac{-q_2}{(-x-a)^2} + \frac{q_1}{(-x-a)^2}]\hat{x} \tag{C.80}$$

Instructor : That means they both the same as each other?

Charlie : Um... plus a (modifies the equation) negative x minus negative a.

$$E = k[\frac{-q_2}{(-x-a)^2} + \frac{q_1}{(-x+a)^2}]\hat{x} \tag{C.81}$$

Instructor : Negative x minus negative a ... so... um... q_1 over negative x plus a squared... when...

Charlie : Should be minus x and plus this a to get that distance.

Charlie : Can't we just keep it like this (points to the equation) and then... the negative sign with the x.

Instructor : Yes. You can do that.

Charlie : So this is fine?

Instructor : This looks good.

Charlie : Still over here (points to the region between q_1 and q_2) oh man... um... may be I'll just make my positive x over here (points to the diagram). I kind of lost my chain of thoughts... where I was?

Instructor : Why don't you erase things in the middle part and start a again with the middle part. You might want to erase all of your x-s ... they are confusing.

Charlie : Alright, (Charlie starts over to record the mathematical equation for electric field in the middle region) got a negative q up here and this is going to be some distance (re-defines the distance on the diagram) and this is the distance negative x ... See I feel like if I didn't put the negative in there. why am I now calling this negative x?

Instructor : I don't know why you are doing that.

Charlie : I'm not sure why am I either. I don't need to... do I?

Instructor : No.

Charlie : Where do I caught up last time?

Instructor : You changed the sign on the top one. You hit all the possible comments.

Charlie : Surely, one of them has to be correct. Okay, um... this (points to the diagram) is the distance x. (starts recording equation but stops halfway through) I guess that's minus a plus x. I don't know why I'm still caught up on this? This is just the same...

See what my thought is..., if this was x minus a, what suppose to be x minus negative a? (records equation) Okay that makes sense... I think this entire distance is x minus negative a (records on the diagram). Since that's negative a because then... if x is negative... as it approach to a (points to $-a$) it gets to zero. Yeah... negative q over a ... still it would be a positive distance from there.

$$E = \frac{1}{4\pi\epsilon_0}[\frac{-q_1}{(x+a)^2} \tag{C.82}$$

Instructor : Okay.

Charlie : Or on the positive x direction... so... and this... I think it's going to be negative x plus a (records equation) ... Yes.

$$E = \frac{1}{4\pi\epsilon_0}[\frac{-q_1}{(x+a)^2} + \frac{q_2}{(-x+a)^2}] \tag{C.83}$$

Instructor : Are both these in minus x-hat or positive x-hat or one in each?

Charlie : Um... (records x-hat and modifies the equation) oh yeah. because they should... they both pushing the negative x-hat direction. So the field should be going in the negative x-hat direction (gesture-right hand moves leftward). Yes.

$$E = \frac{1}{4\pi\epsilon_0}[\frac{-q_1}{(x+a)^2} + \frac{-q_2}{(-x+a)^2}]\hat{x} \tag{C.84}$$

Instructor : Okay. What's the third region look like?

Charlie : This is going to be the same as that (points to the electric field equation in far left region) with (records equation) ... yeah.

$$E = k[\frac{-q_1}{(x+a)^2} + \frac{q_2}{(x-a)^2}]\hat{x} \tag{C.85}$$

Instructor : So, it looks like in the denominator x plus a always goes with q_1 and x minus a always goes with q_2.

Charlie : That's what it seems like.

Instructor : Okay. So what are the things that change with these regions?

Charlie : The direction of the field.

Instructor : Okay. How does that show up in your math?

Charlie : Well this (points to middle region) direction would be a negative x-hat. This (points to far left region) would be a positive x-hat and this (points to far right region) would be a positive x-hat. Which this direction is positive x-hat and this direction is negative x-hat (records on the diagram).

Charlie : Because, this (points to term $\frac{q_1}{(x+a)^2}$ on electric field equation for the far left region) is going to be greater than this (points to term $\frac{-q_2}{(-x-a)^2}$ on electric field equation for the far left region) because of the negative and positive signs.

Instructor : Okay.

Charlie : Did I switch this? because that's not what it showing up. Um... I think... I want to go back to my original and (modifies the equation) because now this (points to term $\frac{q_1}{(x+a)^2}$) will be always be greater... that would be if it's over here (points to far left region).

$$E = k[\frac{q_2}{(-x-a)^2} + \frac{-q_1}{(x+a)^2}]\hat{x} \tag{C.86}$$

Instructor : You are confusing your self again.

Charlie : Okay (modifies the equation).

$$E = k[\frac{-q_2}{(-x-a)^2} + \frac{q_1}{(x+a)^2}]\hat{x} \tag{C.87}$$

Instructor : Okay. Now we have minus signs in all possible places. Are you going to settle on this one?

Charlie : I don't know... I'm debating. I'm thinking because if it was that... x was at negative $2a$ then that would still be negative three and makes nine which makes smaller. Yeah, I'm still with this one. Then this one (electric field equation for middle region), two negatives obviously giving us a negative. It's the same thing with this (electric field equation for far right region). So yes.

Instructor : x minus a is going to be smaller than x plus a.

Charlie : Yeah. Which means ...

Instructor : Okay. Good. We are done.

Charlie : Is that correct?

Instructor : Yeah.

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C.1.3 Mechanics problem

Student: ZEKE

Instructor: If you have a platform which rotates, and on the platform is a mass on a spring... the mass is constrained in the frame of the platform so that it can only go back and forth... does the mass undergo simple harmonic motion?

Zeke: Ok... they're both spring constants k? it's not like a half k, half k. It's k and k.

Zeke: Ok, so, we'll start by the real forces and do a free-body diagram. OK, the real forces would be... OK, let's say, we'll call that x... let's say equilibrium (records a dotted line on the diagram).

Zeke: Um… is it too much to say that, I don't think we necessarily need to say this, but can we say, just to be easier that when it's at equilibrium, they're at rest length?

Zeke: OK. So, in that case, um. The force on this (force on mass by left spring). Um. I'm going to write something down but I don't think it's right. I'll have to think about it, negative kx (records on the board $-kx$ along with a leftward arrow).

Zeke: So, if x is negative, then we expect k to push (gesture)… um, OK. So in this situation. Can I?… I'm going to put the mass over here, just for sign. Just to make the sign make more sense (moves the mass to the right of the equilibrium).

Zeke: We have positive k… So negative kx makes sense because we have positive displacement, and it's gonna be pulling backwards in the negative direction (gesture).

Zeke: This one (the spring on the right), it's going to be pushing, so it's going to be pushing at… kx.

Zeke: As x is bigger (fist moves rightward), this (force on right spring), the force gets bigger by the factor of kx, so it's gonna push this way at kx. That's correct.

Zeke: No, I have done something silly.

Zeke: If m goes this way (fist moves rightwards) then we have a positive *displacement* ($+x$)… this ($-kx$) is negative, that (the force from the left spring) points that way (leftwards), that makes sense. (then talks about the spring on right) We've got a positive x (fist moves rightward), this (equation for right spring) is negative (finger points leftward) and (force) points that way. Makes sense.

Zeke: I am happy now.

Zeke: Okay. Now ω (angular velocity) is going to be (applies the right-hand grip rule) pointing out of the board (records on the board).

Zeke: So, the inertial force. Frame is not accelerating, there is no inertial force.

Zeke: Centrifugal force ... um .. we have ω cross r (applies the right-hand grip rule). So, we are going to have that $(\vec{\omega} \times \vec{r})$ pointing up. And then ω cross ω cross r $(\vec{\omega} \times (\vec{\omega} \times \vec{r}))$ is (applies the right-hand grip rule) pointing in negative, is going to point out and this makes sense because centrifugal force always points out of the circle.

Zeke: This is going to be tricky... how do we want to do this?

Zeke: Well, imagine at this point just for a moment and then we will generalize it in a second. We will say at this point (refers to the diagram on board) that x dot ($\dot{x}$ - *linear velocity*) is pointing out and it is accelerating this way (rightward).

Zeke: The case then ... ω cross $\dot{r}$ points up (records on the board). Let's see if this makes sense. Well ... because ω goes this way (anti-clockwise) ... oh negative. It's negative ω cross $\dot{r}$. So, it's this way (downward and records on the board).

Zeke: This is Coriolis and that makes sense because as if you look at it from the inertial frame as m comes this way (hand moves leftward) and the platform moves behind (hand moves anti-clockwise). It's going to end up ... if it weren't constrained, it would end up further down the platform. Because the platform rotates under it.

Zeke: Okay. Transverse force (angular acceleration force) does not exist because there is no omega dot ($\dot{\omega}$ - angular acceleration).

Zeke: Now, we have centrifugal force which was ω cross ω cross r $(\vec{\omega} \times (\vec{\omega} \times \vec{r}))$. So that would be (records on the board) $m\omega^2 x$ and r which in this case is x.

Zeke: So, we were to add up forces in this direction (rightward) which is the only direction we care about because of the constraint (and records $\Sigma F = (-2k + m\omega^2)x$ on the board).

Zeke: So, if we have ω is such that this term $m\omega^2$ not bigger than $2k$ (and records $m\omega^2 < 2k$ on the board).

Zeke: If that is the case then this $(-2k + m\omega^2)$ going to be constant and stays negative. So we can define an effective k (k^1) which would be $k^1 = 2k - m\omega^2$ (records on the board) and we get $F = -k^1 x$(records on the board). So, in that case it would be simple harmonic motion because k^1 being a constant and they have the same form.

Zeke: Now, if $m\omega^2$ is greater than $2k$ (and records on the board) $m\omega^2 > 2k$. That would be interesting, because in that case then we can have like a kappa (records on the board $\kappa = m\omega^2 - 2k$) equal to the other way. Just to keep it positive. Then we have the sum of the forces equal to kappa x (records on the board $F = \kappa x$), which is funny.

Zeke: It won't be simple harmonic motion because it doesn't have a restoring force. Because there is no negative sign. With x gets bigger positive force continues to be positive. So, it's going to continue come this way (rightward hand movement) until something stops it. Probably the wall or what ever the spring is tight to.

Zeke: What you will have an unstable equilibrium at the center, because at $x = 0$ you will have no force (points to $F = \kappa x$) . But if you give a little jolt in the negative direction, then you have a negative x and it will force it this way (leftward) that will keep it forcing until the normal force is counteracted. If you give a little jolt in the positive direction, then it will continue going until it hits a normal force.

Zeke: So I guess the summary is then, if ω is such that $m\omega^2 < 2k$, it would exhibit simple harmonic motion with a slightly different frequency of oscillation. if $m\omega^2 > 2k$ then no simple harmonic motion and it would just accelerate to the edge of the circle.

Instructor: What happens if $m\omega^2 > 2k$.

Zeke: In this case, this $-2k + m\omega^2$ becomes zero and this is... a... that's interesting... no force, why would be no force.

Zeke: Okay, you would have where ever your mass was it stays there and that could be because, if you imagine as the picture is drawn (points to the diagram) the spring wants to pull it back towards the equilibrium but the exact amount of spring wants to pull it back. The centrifugal force wants to pull it out.

Instructor: Does it matter where it is?

Zeke: No,because as you go further the restoring force is going to become larger because the springs displaces more. But at the same time, you are also closer to the outside, so you are experiencing a bigger centrifugal force.

Appendix D

Appendix D - SEMIOTIC RESOURCES AND THEIR DISCIPLINARY AFFORDANCES

D.1 Chapter 6

D.1.1 Problem 1: Loosely wound solenoid

Student 1: Alan

Semiotic Resource	Disciplinary affordance	Location

Cylinder for solenoid	Vertical cylinder allows Alan to generate a visual representation of the solenoid in the diagrammatic space of the board (diagram on the board). Alan refers to this while specifying the current components before moving to figure out the magnetic field	Alan's initial stage, in Figure 6.3
Arrow as vector	This allows Alan to visually represent the vector direction. Spatial extent used to represent both distances in coordinate space and magnitudes of electric field vectors. Here, this visual representation of a straight line with an arrow head allows Alan to show the current components going up inside the solenoid and then a curved line with an arrow head allows him to represent the current components going around the solenoid	Alan's initial stage, in Figure 6.3

Square for loop	This helps to generate a visual representation of the loop in the diagrammatic space of the board. Further this allows Alan to show the orientation of the loop with respect to the orientation of the current on his diagram that will allow him to figure out the mathematical manipulations while working with *Ampere's law* in the algebraic space	Alan's initial stage, in Figure 6.3

Integral form of the Ampere's law	This physical law shows the relationship between the integrated magnetic field around a closed loop and the electric current passing through the loop. It states that the magnetic field around an electric current is proportional to the current. In relation to the *Amperian loop* each segment of current loop produces a magnetic field like that of a long straight wire, and the total field of any shape current is the vector sum of the fields due to each segment. In this mathematical representation, the left hand side allows us to show the relative vector directions (do the dot product) between magnetic field and the unit length on an Amperian loop. Where, right hand shows the two scalar quantities, permeability and current, passing through the loop	Alan's intermediate stage, in Figure 6.4

Arrow as vector	Here he uses the arrow as vector to represent the direction on the loop that helps Alan to build a complete argument and advance with the mathematical manipulation while working with the left hand side of the *integral form of the Ampere's law* equation	Alan's intermediate stage, in Figure 6.4
Finger pointing in direction	Alan uses the index finger to the side while gesturing for direction. This allows him to show the direction of the resulting magnetic field while answering instructor's question	Alan's intermediate stage, in Figure 6.4
Arrow as vector	Similar to the previous case of Alan using an arrow to show the current direction that is going around the solenoid, this time it allows him to represent the axial current which is inside (along $\hat{z}$) the solenoid	Alan's final stage, in Figure 6.6

Circle for loop	This helps Alan to generate a visual representation of the loop and also to figure out the mathematical manipulations. There are two things someone needs to consider when picking an *Amperian loop*. First, magnetic field should be parallel to the infinitesimal length on the loop. Second, magnetic field should be constant. As Alan is going to consider the current component going up the best shape for his *Amperian loop* is a circle	Alan's final stage, in Figure 6.6

Student 2: Danny

Semiotic Resource	Disciplinary affordance	Location
Coordinate system	Similar to the previous case of Larry (chapters 4, 5) this visual representation allows Danny to define the direction in his diagrammatic space	Danny's initial stage, in Figure 6.9

Hand for loop	Like in the case of Larry here Danny uses his hand to represent how he would go about picking an *Amperian loop*	Danny's initial stage, in Figure 6.9
Circle for loop	Similar to the previous case of Alan, this helps Danny to generate a visual representation of the loop in the diagrammatic space of the board. Having said that he is going to consider the current component going down, the best shape for his *Amperian loop* is a circle	Danny's initial stage, in Figure 6.9
Line as wire	This helps Danny to visually represent the current that helps him to get an idea about the directional relation between the resulting magnetic field and the direction on the loop allowing him to simplify the mathematics	Danny's final stage, in Figure 6.12

Square for loop	With the newly generated side-on view, this visual representation allows Danny again to pick the most appropriate shape for the *Amperian loop* for the current going around the solenoid	Danny's final stage, in Figure 6.12

D.1.2 Problem 2: Two charges on a line

Student 1: Larry

Semiotic Resource	Disciplinary affordance	Location
Coordinate axis	Here the one dimensional coordinate system or the coordinate axis allows Larry to determine the distances (separation between field point and charge) that then needs to be plugged into the *Coulomb's law* equation. Then this also helps him to interpret the resulting electric fields with a definite direction (($\hat{x}$) or ($-\hat{x}$))	Larry's current stage, in Figure 6.16

Arrow as vector	This allows Larry to visually represent the electric field direction. In this case this representation allows Larry to visually represent positive charges radially away and negative charges radially towards	Larry's current stage, in Figure 6.16
Field point	This allows Larry to specify the region that he is considering. Then having the *field point* labeled on the *coordinate axis* allows Larry to visualize the resulting electric field directions for a given region	Larry's current stage, in Figure 6.16
Coulomb's law	This represents the electrostatic force between charges. That is directly proportional to the product of the charges and inversely to the square of the distance between them	Larry's current stage, in Figure 6.16

Student 2: Oliver

Semiotic Resource	Disciplinary affordance	Location

Finger pointing in direction	It allows Oliver to show the electric field direction. Importantly it helps him to represent the destructive nature of the superposition in the gestural space. Also, this gestural activity helps him best communicate his idea to the instructor. It also helps him specify different regions on the coordinate axis	Oliver's current stage, in Figure 6.19

D.1.3 Mechanics problem - Student : ZEKE

Semiotic Resource	Disciplinary affordance	Location
Box for mass	This allows Zeke to generate a visual representation of the mass in the space of the board. He uses it to record the direction of restoring forces on the mass by each spring	Zeke's initial stage, in Figure 6.27

Midline for equilibrium	This helps Zeke to visualize the displacement of the mass from its equilibrium position. The view of the compressed or stretched springs allows him to decide the direction of the restoring force that is very important in this situation. As Zeke steps back from the board to check the consistency between the diagram and the mathematics, initially this (midline) helps Zeke to check the validity of his mathematical equation for the force by the left spring and then later to check the sign on the mathematical equation for the force by the right spring	Zeke's initial stage, in Figure 6.27

Arrow as vector	In this case, this visual representation of a line with an arrow head allows Zeke to show the restoring force direction from each spring to figure out the net force by both springs	Zeke's initial stage, in Figure 6.27

Restoring force on the spring ($F = -kx$)	According to Hooke's law, if the mass is displaced sideways, such that the spring becomes compressed (or extended), then the mass experiences a horizontal force given by $F(x) = -kx$, where k is the spring constant and the negative sign in the expression indicates that $F(x)$ is a restoring force (if the displacement is positive then the force is negative and vice versa). The idea that the direction of the restoring force is opposite to that of the displacement of the mass, helps Zeke to amend his diagram according to the mathematical equation that he recorded	Zeke's initial stage, in Figure 6.27

Fist for mass	This allows Zeke to generate a visual representation of the mass in the free space. Zeke holds the fist while gesturing for mass displacement and keeps referring back as he progresses in this task. The process of sense-making using the *fist for mass* helps Zeke to figure out the direction of the restoring forces that he could mostly do using the diagram on board	Zeke's intermediate stage, in Figure 6.28
Finger pointing in direction	Zeke uses the index finger to the side while gesturing for direction. This allows him to show the direction of the displacement and the direction of the restoring force in the free space	Zeke's intermediate stage, in Figure 6.28

Right-hand grip rule	On this occasion, Zeke first uses the right-hand grip rule to figure out the rotation vector to represent the rotation of the platform, where Zeke's thumb of right-hand points when he curls his fingers in the direction of rotation. Then Zeke uses the right-hand grip rule to find the direction of the centrifugal force that he uses to manipulate the cross product of the three vectors $(\vec{\omega}\times(\vec{\omega}\times\vec{r})$. By definition the right-hand grip rule states the orientation of the vectors' cross product $(\vec{u} \times \vec{v})$, by placing u and v tail-to-tail, flattening the right hand, extending the hand in the direction of u, and then curling the fingers in the direction of the angle v makes with u. Then the thumb points in the direction of $\vec{u} \times \vec{v}$. Zeke uses this argument first to find the direction of $\vec{\omega} \times \vec{r}$ and then to find the direction of $\vec{\omega} \times (\vec{\omega} \times \vec{r})$. Later, Zeke	Zeke includes *centrifugal force*, in Figure 6.30

Arrow as vector	Again Zeke uses the Arrow as vectors visual representation to show vector direction. Here he uses the same semiotic resource to visualize the direction of angular velocity and the direction of centrifugal force. Later, He uses the same semiotic resource to visualize the direction of the Coriolis force	Zeke includes *centrifugal force*, in Figure 6.30