

A non-exceptional cyclic point configuration and Floer theory of the
non-exact torus

by

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Abstract

The main results of this thesis are twofold: one concerns the existence of a non-exceptional three-dimensional cyclic point configuration, while the other concerns the Floer-theoretic structure of the non-exact torus.

Part 1

The Fukaya category has emerged as one of the central objects in modern symplectic geometry and homological mirror symmetry. In particular, the wrapped Fukaya category of a cotangent bundle encodes deep geometric and algebraic information about the underlying manifold and its symplectic structure. The foundational work of^{1, 2} and others has established a rich relationship between Floer-theoretic invariants of cotangent bundles and categories of coherent sheaves on the mirror side, under the assumption that the symplectic form is exact.

The purpose of this work is to investigate how this picture changes when the cotangent bundle is equipped with a non-exact symplectic form. Specifically, we study the Lagrangian Floer theory of a cotangent fiber inside the cotangent bundle of the two-torus endowed with a perturbed symplectic form of the type $\omega_0 + \pi^*\theta$, where ω_0 is the standard exact symplectic form and θ is a closed two-form on the base. This perturbation changes the cohomology class of the symplectic form while preserving the underlying smooth structure of the cotangent bundle.

Our first goal is to analyze the resulting moduli spaces of pseudo-holomorphic curves and to establish structural properties of the associated Floer complex. In particular, we show that the Floer differential vanishes for the cotangent fiber, so that the Floer cohomology coincides with the underlying complex. We then proceed to compute the A_∞ -structure on this complex, with particular emphasis on the product μ_2 .

Proposition 1. *Let $(T^*\mathbb{T}^2, \omega)$ be the cotangent bundle of the torus equipped with a symplectic form $\omega = \omega_0 + \pi^*\theta$ where ω_0 is the tautological form on the cotangent bundle, θ is the area form on torus with area h and π is the projection onto the torus. If q is a point on the base, then $\text{End}(T_q^*\mathbb{T}^2)$ is well defined in a wrapped Fukaya setting and*

$$\text{End}(T_q^*\mathbb{T}^2) \simeq \mathbb{C} \langle X^{\pm 1}, Y^{\pm 1} \rangle$$

where the angular brackets represent X and Y do not commute but rather satisfy the relation: $XY = q^h YX$.

The main result of this work is an explicit computation of the Floer product for the cotangent fiber in the presence of a perturbed symplectic form. While in the exact case this algebra is known to be the commutative ring of Laurent polynomials in two variables, we show that the perturbation produces a *non-commutative deformation* whose commutation relations are determined directly by the cohomology class of the base symplectic form θ . The result included here is folklore and experts have speculated on it. However, a specific proof using the methods described here has not been written.

The paper is organized as follows. In Section 2.1 we construct the Floer complex for the cotangent fiber and analyze the moduli spaces governing its differential and product. In Section 2.2 we introduce the perturbed symplectic structure in $T^*\mathbb{T}^2$, compute the Hamiltonian flow explicitly, and derive a concrete formula for the Floer product by computing the symplectic area of holomorphic triangles. This leads to the explicit non-commutative algebra structure on $CF(T_q^*\mathbb{T}^2, T_q^*\mathbb{T}^2)$.

Part 2

This work studies secondary polytopes and regular subdivisions associated with cyclic point configurations. A central question is whether cyclic configurations satisfy exceptional generality, a property asserting that the dimensions of spaces of piecewise affine functions on subdivisions behave in the expected generic manner. While cyclic configurations in dimension two exhibit this property, the situation changes in higher dimensions.

We construct an explicit three-dimensional cyclic point configuration that is not ex-

exceptionally general. More precisely, we exhibit a regular subdivision of the configuration for which the space of piecewise affine functions has dimension larger than the dimension predicted by genericity. The construction is analyzed through the combinatorics of the subdivision and the affine compatibility conditions imposed on its maximal cells. This provides a concrete obstruction to extending the two-dimensional exceptional generality phenomenon to three-dimensional cyclic configurations.

The example shows that the combinatorial structure of a cyclic configuration alone does not force exceptional generality and highlights additional geometric constraints that arise in the study of secondary polytopes in dimension three.

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The purpose of this work is to investigate how this picture changes when the cotangent bundle is equipped with a non-exact symplectic form. Specifically, we study the Lagrangian Floer theory of a cotangent fiber inside the cotangent bundle of the two-torus endowed with a perturbed symplectic form of the type $\omega_0 + \pi^*\theta$, where ω_0 is the standard exact symplectic form and θ is a closed two-form on the base. This perturbation changes the cohomology class of the symplectic form while preserving the underlying smooth structure of the cotangent bundle.

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Proposition 2. *Let $(T^*\mathbb{T}^2, \omega)$ be the cotangent bundle of the torus equipped with a symplectic form $\omega = \omega_0 + \pi^*\theta$ where ω_0 is the tautological form on the cotangent bundle, θ is the area form on torus with area h and π is the projection onto the torus. If q is a point on the base, then $\text{End}(T_q^*\mathbb{T}^2)$ is well defined in a wrapped Fukaya setting and*

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We construct an explicit three-dimensional cyclic point configuration that is not ex-

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The example shows that the combinatorial structure of a cyclic configuration alone does not force exceptional generality and highlights additional geometric constraints that arise in the study of secondary polytopes in dimension three.

Contents

List of Figures	xiii
1 Background on Symplectic Geometry	1
1.0.1 Symplectic Manifolds	1
1.0.2 Lagrangian Submanifolds	15
1.0.3 Symplectic Complements	16
1.0.4 Isotropic, Coisotropic and Symplectic Subspaces	18
1.0.5 Lagrangian Subspaces	21
1.0.6 Lagrangian Submanifolds	29
1.1 Hamiltonian dynamics	34
1.1.1 Hamiltonian vector fields	34
1.1.2 Hamiltonian Flows	39
1.1.3 Hamiltonian Isotopies and Hamiltonian Diffeomorphisms	46
1.1.4 Hamiltonian Chords and Lagrangian Boundary Conditions	52
1.1.5 Hamiltonian Dynamics on Cotangent Bundles	58
1.2 Almost Complex Structures	62
1.2.1 Almost Complex Structures	62
1.2.2 Compatible and Tame Almost Complex Structures	65
1.2.3 The Associated Riemannian Metric	68
1.2.4 Existence of Compatible Almost Complex Structures	69
1.2.5 Almost Complex Structures on Cotangent Bundles	72
1.2.6 Pseudo-Holomorphic Curves	73
1.3 Floer Cohomology	75

1.3.1	Pseudo-Holomorphic Strips with Lagrangian Boundary Conditions . .	75
2	Computations on cotangent bundle of torus	80
2.1	Moduli Spaces on $T^*\mathbb{T}^2$	80
2.1.1	Notations	80
2.1.2	Compatible structures	82
2.1.3	Energy functional	84
2.2	Calculations on $T^*\mathbb{T}^2$ with a deformed symplectic form	86
2.2.1	Hamiltonian flow lines	87
2.2.2	Grading on Floer complex	91
2.2.3	Computing $\text{End}(T_0^*\mathbb{T}^2, T_0^*\mathbb{T}^2)$	94
2.3	Generalizations to higher genus	97
2.3.1	Hamiltonian flow in upper half plane	98
3	Background on Cyclic Polytopes and Secondary Polytopes	100
3.1	Cyclic Polytopes	100
3.1.1	Point Configurations and Cyclic Polytopes	100
3.1.2	Combinatorics of Cyclic Polytopes	103
3.1.3	Polyhedral and Coherent Subdivisions	106
3.1.4	The Secondary Fan	110
3.1.5	The Secondary Polytope	113
3.1.6	Exceptional Generality in Dimension Two	116
4	A non-exceptional 3-dimensional cyclic point configuration	120
4.1	Introduction	120
4.2	Vertex figures	122
4.2.1	Subdivisions of vertex figures	125
4.3	A non-exceptional cyclic point configuration of the vertex figure	125
4.4	Conclusion	129

Bibliography	132
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List of Figures

2.1	Product in the Fukaya category via counting pseudoholomorphic triangles. .	94
2.2	Lifted triangle in $T^*\mathbb{R}^2$ to compute the product	95

Chapter 1

Background on Symplectic Geometry

1.0.1 Symplectic Manifolds

Symplectic geometry is the natural mathematical framework for classical Hamiltonian mechanics and serves as the geometric foundation for Floer theory and the Fukaya category. Unlike Riemannian geometry, whose primary object is a positive-definite metric, symplectic geometry studies smooth manifolds equipped with a closed, non-degenerate differential two-form. The non-degeneracy of this form gives rise to a correspondence between smooth functions and vector fields, which in turn produces Hamiltonian dynamics.

We begin with the fundamental definition.

Definition 3. Let M be a smooth manifold. A *symplectic form* on M is a differential 2-form

$$\omega \in \Omega^2(M)$$

satisfying the following two conditions:

1. ω is closed,

$$d\omega = 0;$$

2. ω is non-degenerate, that is, for every point $p \in M$,

$$\omega_p(v, w) = 0 \quad \text{for all } w \in T_p M \implies v = 0. \quad (1.1)$$

The pair (M, ω) is called a *symplectic manifold*.

The first condition is a global topological condition, while the second is purely linear-algebraic. Together they endow the tangent spaces of M with the structure of symplectic vector spaces. The non-degeneracy condition implies that the map

$$\begin{aligned} T_p M &\longrightarrow T_p^* M \\ v &\longmapsto \omega_p(v, \cdot), \end{aligned}$$

is an isomorphism for every $p \in M$. Consequently, every covector determines a unique tangent vector. This fact plays a fundamental role in Hamiltonian mechanics, where the differential of a smooth function determines its Hamiltonian vector field.

The following theorem is one of the first structural properties of symplectic manifolds.

Theorem 4. *Every symplectic manifold has even dimension.*

Proof. Let (M, ω) be a symplectic manifold and fix a point $p \in M$. Choose a basis for $T_p M$, and let A denote the matrix of ω_p with respect to this basis. Since ω_p is skew-symmetric,

$$A^T = -A.$$

Suppose $\dim T_p M = n$ is odd. Then

$$\det(A) = \det(A^T) = \det(-A) = (-1)^n \det(A) = -\det(A),$$

which implies that

$$\det(A) = 0.$$

Thus A is singular, contradicting the non-degeneracy of ω_p . Therefore $\dim T_p M$ must be even. Since p was arbitrary, every connected component of M has even dimension. $\square$

Consequently, if (M, ω) is symplectic, we write

$$\dim M = 2n,$$

where n is called the *symplectic dimension* of M .

The non-degeneracy of the symplectic form also determines a canonical orientation on the manifold.

Theorem 5. *Let (M^{2n}, ω) be a symplectic manifold. Then*

$$\omega^n = \underbrace{\omega \wedge \cdots \wedge \omega}_{n \text{ times}}$$

is a nowhere-vanishing $2n$ -form. Thus, every symplectic manifold is orientable.

Proof. Fix a point $p \in M$. Since ω_p is a non-degenerate skew-symmetric bilinear form, there exists a basis

$$e_1, \dots, e_n, f_1, \dots, f_n$$

of $T_p M$ satisfying

$$\omega_p(e_i, f_j) = \delta_{ij},$$

while

$$\omega_p(e_i, e_j) = \omega_p(f_i, f_j) = 0.$$

Let

$$x_1, \dots, x_n, y_1, \dots, y_n$$

denote the dual basis. Then

$$\omega_p = \sum_{i=1}^n x_i \wedge y_i.$$

Expanding the n -fold wedge product yields

$$\omega_p^n = n! x_1 \wedge y_1 \wedge \cdots \wedge x_n \wedge y_n,$$

which is nonzero because the covectors form a basis of T_p^*M . Hence ω^n never vanishes on M . Therefore ω^n defines a smooth volume form, which determines an orientation on M . $\square$

The previous theorem shows that every symplectic manifold carries a canonical volume form

$$\frac{\omega^n}{n!},$$

called the *Liouville volume form*. This volume form plays an important role in Hamiltonian dynamics, where it is preserved by Hamiltonian flows. We conclude this subsection with several fundamental examples.

Example 6. The Euclidean space $\mathbb{R}^{2n}$ equipped with coordinates

$$(x_1, \dots, x_n, y_1, \dots, y_n)$$

and the differential 2-form

$$\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$$

is a symplectic manifold.

Example 7. Every open subset of $(\mathbb{R}^{2n}, \omega_0)$ is naturally a symplectic manifold by restriction of the standard symplectic form.

Proposition 8. *Let Q be a smooth manifold of dimension n , and let*

$$\pi : T^*Q \longrightarrow Q$$

be its cotangent bundle. Then T^*Q admits a canonical exact symplectic structure. More precisely, there exists a globally defined one-form

$$\lambda \in \Omega^1(T^*Q), \tag{1.2}$$

called the tautological one-form, whose exterior derivative

$$\omega_0 = d\lambda \tag{1.3}$$

is a symplectic form on T^*Q .

Proof. A point of the cotangent bundle is an ordered pair

$$(q, p),$$

where $q \in Q$ and $p \in T_q^*Q$. The cotangent bundle comes equipped with the natural projection

$$\pi : T^*Q \longrightarrow Q,$$

which sends (q, p) to q . For each point $(q, p) \in T^*Q$, define a linear functional

$$\lambda_{(q,p)} : T_{(q,p)}T^*Q \longrightarrow \mathbb{R}$$

$$\lambda_{(q,p)}(v) = -p(d\pi(v)).$$

Since the definition depends only on the intrinsic projection map, this defines a globally

smooth one-form

$$\lambda \in \Omega^1(T^*Q).$$

Let

$$(x_1, \dots, x_n)$$

be local coordinates on Q . These induce coordinates

$$(x_1, \dots, x_n, y_1, \dots, y_n)$$

on T^*Q , where every covector is written uniquely as

$$p = \sum_{i=1}^n y_i dx_i.$$

A direct computation shows that

$$\lambda = - \sum_{i=1}^n y_i dx_i.$$

Define

$$\omega_0 = d\lambda.$$

Using the identity

$$d(dx_i) = 0,$$

we obtain

$$\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i.$$

Since

$$\omega_0 = d\lambda,$$

it follows immediately that

$$d\omega_0 = d^2\lambda = 0,$$

so ω_0 is closed. It remains to prove non-degeneracy.

Suppose

$$v = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} + \sum_{i=1}^n b_i \frac{\partial}{\partial y_i}$$

is a tangent vector satisfying

$$\omega_0(v, w) = 0$$

for every tangent vector

$$w \in T_{(q,p)}T^*Q.$$

Evaluating against the coordinate vector fields gives

$$\omega_0\left(v, \frac{\partial}{\partial x_j}\right) = -b_j$$

and

$$\omega_0\left(v, \frac{\partial}{\partial y_j}\right) = a_j.$$

Since these vanish for every j , we conclude that

$$a_j = b_j = 0$$

for all j . Hence

$$v = 0.$$

Therefore ω_0 is non-degenerate. Consequently,

$$(T^*Q, \omega_0)$$

is a symplectic manifold. Finally, because

$$\omega_0 = d\lambda,$$

the symplectic form is exact. Therefore every cotangent bundle carries a canonical exact symplectic structure. $\square$

The examples above illustrate the ubiquity of symplectic structures in geometry. In contrast with Riemannian geometry, however, symplectic geometry has no local invariants. This remarkable phenomenon is expressed by Darboux's Theorem, which states that every symplectic manifold is locally symplectomorphic to the standard symplectic vector space. The theorem shows that all local symplectic geometry is identical and that the richness of the subject arises entirely from global phenomena.

Darboux's Theorem

The local structure of a symplectic manifold is described by Darboux's Theorem. It states that every symplectic form can be written locally in the same standard form. Thus, unlike a Riemannian metric, a symplectic form has no local invariants analogous to curvature. We first recall the standard symplectic form on $\mathbb{R}^{2n}$. In coordinates

$$(x_1, \dots, x_n, y_1, \dots, y_n),$$

it is given by

$$\omega_{\text{std}} = \sum_{i=1}^n dx_i \wedge dy_i.$$

Definition 9. Let (M, ω_M) and (N, ω_N) be symplectic manifolds. A diffeomorphism

$$\varphi : M \longrightarrow N$$

is called a *symplectomorphism* if

$$\varphi^* \omega_N = \omega_M.$$

If φ is a diffeomorphism from an open subset of M onto an open subset of N and satisfies the same identity, then φ is called a *local symplectomorphism*.

We shall prove Darboux's Theorem using Moser's path method. The basic idea is to interpolate between two symplectic forms and construct an isotopy that identifies them.

Lemma 10 (Local Moser Lemma). *Let $U \subseteq \mathbb{R}^{2n}$ be an open neighborhood of the origin, and let ω_0 and ω_1 be symplectic forms on U such that*

$$(\omega_0)_0 = (\omega_1)_0.$$

Then, after possibly replacing U by a smaller neighborhood of the origin, there exists a diffeomorphism

$$\varphi : U \longrightarrow \varphi(U)$$

such that

$$\varphi(0) = 0$$

and

$$\varphi^* \omega_1 = \omega_0.$$

Proof. Consider the family of closed two-forms

$$\omega_t = (1-t)\omega_0 + t\omega_1, \quad t \in [0, 1].$$

At the origin we have

$$(\omega_t)_0 = (\omega_0)_0 = (\omega_1)_0$$

for every $t \in [0, 1]$. In particular, $(\omega_t)_0$ is non-degenerate. Non-degeneracy is an open

condition. Since the parameter interval $[0, 1]$ is compact, we may shrink U so that ω_t is non-degenerate on U for every $t \in [0, 1]$. Hence each ω_t is a symplectic form. Set

$$\eta = \omega_1 - \omega_0.$$

Since both ω_0 and ω_1 are closed,

$$d\eta = 0.$$

By the Poincaré Lemma, after shrinking U if necessary, there exists a one-form α on U such that

$$\eta = d\alpha.$$

Moreover, because

$$\eta_0 = 0,$$

the primitive α may be chosen so that

$$\alpha_0 = 0.$$

We seek a time-dependent vector field X_t whose local flow

$$\varphi_t$$

satisfies

$$\varphi_t^* \omega_t = \omega_0.$$

Differentiating this expression with respect to t , we obtain

$$\frac{d}{dt} (\varphi_t^* \omega_t) = \varphi_t^* \left(\frac{\partial \omega_t}{\partial t} + \mathcal{L}_{X_t} \omega_t \right).$$

Since

$$\frac{\partial \omega_t}{\partial t} = \omega_1 - \omega_0 = d\alpha,$$

and since ω_t is closed, Cartan's formula gives

$$\mathcal{L}_{X_t}\omega_t = d(\iota_{X_t}\omega_t) + \iota_{X_t}d\omega_t = d(\iota_{X_t}\omega_t).$$

Therefore,

$$\frac{d}{dt}(\varphi_t^*\omega_t) = \varphi_t^*d(\alpha + \iota_{X_t}\omega_t).$$

We define X_t by the equation

$$\iota_{X_t}\omega_t = -\alpha.$$

Because ω_t is non-degenerate, this equation determines a unique vector field X_t for every t . With this choice,

$$\frac{d}{dt}(\varphi_t^*\omega_t) = 0.$$

Thus,

$$\varphi_t^*\omega_t = \varphi_0^*\omega_0 = \omega_0.$$

Since $\alpha_0 = 0$, the defining equation for X_t implies

$$X_t(0) = 0.$$

Consequently, the flow fixes the origin:

$$\varphi_t(0) = 0.$$

After shrinking U once more, the flow exists for all $t \in [0, 1]$. Setting

$$\varphi = \varphi_1,$$

we obtain

$$\varphi^* \omega_1 = \omega_0.$$

□

We now state the principal local normal-form theorem of symplectic geometry.

Theorem 11 (Darboux's Theorem). *Let (M^{2n}, ω) be a symplectic manifold and let $p \in M$. Then there exists a coordinate neighborhood U of p with coordinates*

$$(x_1, \dots, x_n, y_1, \dots, y_n)$$

such that

$$\omega|_U = \sum_{i=1}^n dx_i \wedge dy_i.$$

Equivalently, every point of a symplectic manifold has a neighborhood that is symplectomorphic to an open subset of

$$(\mathbb{R}^{2n}, \omega_{\text{std}}).$$

Proof. Fix a point $p \in M$. By the Symplectic Basis Theorem, there exists a basis

$$e_1, \dots, e_n, f_1, \dots, f_n$$

of $T_p M$ satisfying

$$\omega_p(e_i, e_j) = 0, \quad \omega_p(f_i, f_j) = 0,$$

and

$$\omega_p(e_i, f_j) = \delta_{ij}.$$

Choose a coordinate chart

$$\psi : U \longrightarrow V \subseteq \mathbb{R}^{2n}$$

such that

$$\psi(p) = 0$$

and such that the differential

$$d\psi_p : T_p M \longrightarrow \mathbb{R}^{2n}$$

takes the chosen symplectic basis to the standard basis of $\mathbb{R}^{2n}$. Let

$$\tilde{\omega} = (\psi^{-1})^* \omega$$

be the symplectic form induced on V . By the choice of $d\psi_p$, we have

$$\tilde{\omega}_0 = (\omega_{\text{std}})_0.$$

The Local Moser Lemma therefore provides, after shrinking V , a diffeomorphism

$$\varphi : V \longrightarrow \varphi(V)$$

such that

$$\varphi(0) = 0$$

$$\varphi^* \tilde{\omega} = \omega_{\text{std}}.$$

The map

$$\Phi = \psi^{-1} \circ \varphi$$

is then a local diffeomorphism from a neighborhood of the origin in $\mathbb{R}^{2n}$ to a neighborhood of p in M . Moreover,

$$\Phi^* \omega = \varphi^* (\psi^{-1})^* \omega = \varphi^* \tilde{\omega} = \omega_{\text{std}}.$$

Thus the components of Φ^{-1} define local coordinates

$$(x_1, \dots, x_n, y_1, \dots, y_n)$$

near p in which

$$\omega = \sum_{i=1}^n dx_i \wedge dy_i.$$

□

Definition 12. A coordinate system

$$(x_1, \dots, x_n, y_1, \dots, y_n)$$

in which the symplectic form is written as

$$\omega = \sum_{i=1}^n dx_i \wedge dy_i$$

is called a system of *Darboux coordinates*.

Darboux's Theorem shows that all symplectic manifolds of the same dimension are locally indistinguishable. This is in sharp contrast with Riemannian geometry. A Riemannian metric can generally be made Euclidean at a single point, but its derivatives contain curvature information that cannot be removed by a change of coordinates. A symplectic form, on the other hand, can be made exactly standard throughout an entire neighborhood. Consequently, the essential information in symplectic geometry is global rather than local. Examples of such global information include the de Rham cohomology class

$$[\omega] \in H_{\text{dR}}^2(M),$$

the existence and topology of Lagrangian submanifolds, and invariants arising from pseudo-holomorphic curves and Floer theory.

Remark 13. Darboux's Theorem does not imply that two symplectic manifolds of the same dimension are globally symplectomorphic. For example, the cohomology class of the symplectic form is preserved by a symplectomorphism. Indeed, if

$$\varphi : (M, \omega_M) \longrightarrow (N, \omega_N)$$

is a symplectomorphism, then

$$\varphi^*[\omega_N] = [\omega_M].$$

Thus global cohomological data may distinguish symplectic manifolds even though their local models are identical.

Remark 14. The deformation considered later in this thesis is

$$\omega = \omega_0 + \pi^*\theta$$

on a cotangent bundle, where θ is a closed two-form on the base. By Darboux's Theorem, this perturbed form is locally equivalent to the standard symplectic form. Nevertheless, when

$$[\pi^*\theta] \neq 0,$$

the perturbed form need not be globally exact. The distinction between the exact and non-exact forms is therefore global and is not detected by Darboux coordinates.

1.0.2 Lagrangian Submanifolds

Lagrangian submanifolds occupy a central position in symplectic geometry and Hamiltonian mechanics. They are the primary objects of study in Floer cohomology and the Fukaya category, and they play a fundamental role in homological mirror symmetry. Throughout this thesis, we shall be primarily concerned with cotangent fibers, which form one of the most

important classes of Lagrangian submanifolds. We begin with the corresponding notions in symplectic linear algebra.

1.0.3 Symplectic Complements

Let (V, ω) be a symplectic vector space of dimension $2n$.

Definition 15. Let $W \subseteq V$ be a vector subspace. The *symplectic complement* of W is the subspace

$$W^\omega = \{v \in V : \omega(v, w) = 0 \text{ for every } w \in W\}.$$

The symplectic complement is analogous to the orthogonal complement in Euclidean geometry. Unlike the Euclidean case, however, it is possible for a subspace to intersect its symplectic complement nontrivially.

Example 16. Consider the standard symplectic vector space $(\mathbb{R}^{2n}, \omega_0)$, where

$$\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i.$$

Let

$$W = \text{span} \left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k} \right\}.$$

Then

$$W^\omega = \text{span} \left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_{k+1}}, \dots, \frac{\partial}{\partial y_n} \right\}.$$

In particular,

$$W \subseteq W^\omega.$$

The next proposition is one of the fundamental results of symplectic linear algebra. Let (V, ω) be a symplectic vector space and let $W \subseteq V$ be a subspace. Then

$$\dim W + \dim W^\omega = \dim V.$$

In particular, if $\dim V = 2n$, then

$$\dim W + \dim W^\omega = 2n.$$

Proof. Since ω is non-degenerate, it induces a linear isomorphism

$$\Phi : V \longrightarrow V^*$$

given by

$$\Phi(v) = \omega(v, \cdot).$$

Let

$$W^\circ = \{\alpha \in V^* : \alpha(w) = 0 \text{ for every } w \in W\}$$

denote the annihilator of W . By construction,

$$W^\omega = \Phi^{-1}(W^\circ).$$

Since Φ is an isomorphism,

$$\dim W^\omega = \dim W^\circ.$$

From elementary linear algebra,

$$\dim W^\circ = \dim V - \dim W.$$

Therefore,

$$\dim W^\omega = \dim V - \dim W,$$

which proves the result. □

An immediate consequence is that taking the symplectic complement twice returns the original subspace. For every subspace $W \subseteq V$,

$$(W^\omega)^\omega = W.$$

Proof. Since $W \subseteq (W^\omega)^\omega$, it suffices to compare dimensions. Using the previous proposition,

$$\begin{aligned} \dim(W^\omega)^\omega &= \dim V - \dim W^\omega \\ &= \dim V - (\dim V - \dim W) \\ &= \dim W. \end{aligned}$$

Hence the two subspaces coincide. □

1.0.4 Isotropic, Coisotropic and Symplectic Subspaces

The symplectic complement gives rise to several distinguished classes of subspaces.

Definition 17. Let (V, ω) be a symplectic vector space and let $W \subseteq V$ be a subspace.

1. W is called *isotropic* if

$$W \subseteq W^\omega.$$

2. W is called *coisotropic* if

$$W^\omega \subseteq W.$$

3. W is called *symplectic* if

$$W \cap W^\omega = \{0\}.$$

These three classes of subspaces play important roles throughout symplectic geometry. Isotropic subspaces are those on which the symplectic form vanishes identically. Coisotropic subspaces arise naturally in symplectic reduction, while symplectic subspaces inherit a non-degenerate symplectic form from the ambient vector space. The following proposition characterizes isotropic subspaces.

Proposition 18. *A subspace $W \subseteq V$ is isotropic if and only if*

$$\omega(u, v) = 0$$

for every $u, v \in W$.

Proof. Suppose first that W is isotropic. Then

$$W \subseteq W^\omega.$$

Hence every vector of W is symplectically orthogonal to every other vector of W , so

$$\omega(u, v) = 0$$

for all $u, v \in W$. Conversely, suppose

$$\omega(u, v) = 0$$

for every pair of vectors $u, v \in W$. Then every vector of W belongs to W^ω , implying

$$W \subseteq W^\omega.$$

Thus W is isotropic. □

The dimension of an isotropic subspace is bounded above by one-half of the dimension of the ambient symplectic vector space.

Proposition 19. *If W is an isotropic subspace of a symplectic vector space of dimension $2n$, then*

$$\dim W \leq n.$$

Similarly, if W is coisotropic, then

$$\dim W \geq n.$$

Proof. Suppose first that W is isotropic. Since

$$W \subseteq W^\omega,$$

we have

$$\dim W \leq \dim W^\omega.$$

Using Proposition 1.2.1,

$$\dim W^\omega = 2n - \dim W.$$

Therefore,

$$2 \dim W \leq 2n,$$

which implies

$$\dim W \leq n.$$

The proof for coisotropic subspaces is analogous. □

The previous proposition shows that isotropic subspaces can have dimension at most n . Those isotropic subspaces having maximal possible dimension are of particular importance and are called *Lagrangian subspaces*. They will be studied in the next subsection.

1.0.5 Lagrangian Subspaces

Lagrangian subspaces are the maximal isotropic subspaces of a symplectic vector space. They occupy a distinguished position in symplectic geometry because they provide the linear-algebraic model for Lagrangian submanifolds, which later serve as the objects of the Fukaya category. Let (V, ω) be a symplectic vector space of dimension $2n$.

Definition 20. A subspace $L \subseteq V$ is called *Lagrangian* if

$$L = L^\omega.$$

Thus, a Lagrangian subspace is simultaneously isotropic and coisotropic. The following proposition gives several equivalent characterizations.

Proposition 21. *Let $L \subseteq V$ be a subspace. The following conditions are equivalent:*

1. *L is Lagrangian;*
2. *L is isotropic and $\dim L = n$;*
3. *$\omega|_L = 0$ and $\dim L = n$;*
4. *L is a maximal isotropic subspace of V .*

Proof. Suppose first that L is Lagrangian. By definition,

$$L = L^\omega.$$

In particular, $L \subseteq L^\omega$, so L is isotropic. Moreover, the dimension formula for symplectic complements gives

$$\dim L + \dim L^\omega = 2n.$$

Since $L = L^\omega$, we obtain

$$2 \dim L = 2n,$$

and therefore $\dim L = n$. This proves that the first condition implies the second. The second and third conditions are equivalent because L is isotropic if and only if the restriction of the

symplectic form to L vanishes:

$$L \subseteq L^\omega \iff \omega|_L = 0.$$

Suppose now that L is isotropic and $\dim L = n$. Since $L \subseteq L^\omega$, the dimension formula gives

$$\dim L^\omega = 2n - \dim L = n.$$

Thus L and L^ω have the same dimension. Since one is contained in the other, they are equal:

$$L = L^\omega.$$

Hence L is Lagrangian.

It remains to establish the equivalence with maximal isotropicity. If L is Lagrangian and $L \subseteq W$ for some isotropic subspace W , then

$$W \subseteq W^\omega \subseteq L^\omega = L.$$

Thus $W = L$, showing that L is maximal isotropic. Conversely, suppose that L is maximal isotropic. Since every isotropic subspace has dimension at most n , it is enough to show that $\dim L = n$. If $\dim L < n$, then

$$\dim L^\omega = 2n - \dim L > n.$$

In particular, L is a proper subspace of L^ω . Choose $v \in L^\omega \setminus L$. Since ω is skew-symmetric, $\omega(v, v) = 0$, and because $v \in L^\omega$, we have

$$\omega(v, \ell) = 0$$

for every $\ell \in L$. Therefore

$$L + \text{span}\{v\}$$

is an isotropic subspace strictly containing L , contradicting maximality. Hence $\dim L = n$, and L is Lagrangian. $\square$

The preceding proposition shows that a Lagrangian subspace has precisely half the dimension of the ambient symplectic vector space. However, the half-dimensionality condition alone is not sufficient: the restriction of the symplectic form must also vanish.

Example 22. Consider the standard symplectic vector space $(\mathbb{R}^{2n}, \omega_0)$, where

$$\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i.$$

The coordinate subspaces

$$\begin{aligned} L_x &= \text{span} \left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\}, \\ L_y &= \text{span} \left\{ \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n} \right\} \end{aligned}$$

are Lagrangian. Indeed, the symplectic form vanishes on each subspace, and both have dimension n .

The two Lagrangian subspaces in the preceding example are complementary:

$$\mathbb{R}^{2n} = L_x \oplus L_y.$$

More generally, a pair of complementary Lagrangian subspaces provides a linear model for position and momentum coordinates.

Definition 23. Two Lagrangian subspaces $L_0, L_1 \subseteq V$ are called *transverse* if

$$L_0 \cap L_1 = \{0\}.$$

Since both subspaces have dimension n , this condition is equivalent to

$$V = L_0 \oplus L_1.$$

Transversality will later be important in Floer theory. The generators of the Floer complex associated to two Lagrangian submanifolds are typically their transverse intersection points, or equivalently Hamiltonian chords after one of the Lagrangians has been perturbed. The next result shows that every Lagrangian subspace admits a complementary Lagrangian subspace.

Proposition 24. *Let L be a Lagrangian subspace of a finite-dimensional symplectic vector space (V, ω) . Then there exists a Lagrangian subspace L' such that*

$$V = L \oplus L'.$$

Proof. Choose a basis $e_1, \dots, e_n$ of L . Since ω is non-degenerate, the linear functionals

$$\omega(e_1, \cdot), \dots, \omega(e_n, \cdot)$$

are linearly independent. Therefore, one may choose vectors $f_1, \dots, f_n \in V$ such that

$$\omega(e_i, f_j) = \delta_{ij}.$$

The vectors f_i need not initially span an isotropic subspace. Define modified vectors

$$f'_i = f_i - \frac{1}{2} \sum_{j=1}^n \omega(f_i, f_j) e_j.$$

Since L is isotropic, the modification does not alter the pairings with the vectors e_i :

$$\omega(e_k, f'_i) = \delta_{ki}.$$

A direct computation using skew-symmetry shows that

$$\omega(f'_i, f'_j) = 0$$

for all i, j . Hence

$$L' = \text{span}\{f'_1, \dots, f'_n\}$$

is isotropic. Since $\dim L' = n$, it is Lagrangian. Finally, the matrix of pairings

$$(\omega(e_i, f'_j))_{i,j}$$

is the identity matrix, so $L \cap L' = \{0\}$. Since both subspaces have dimension n , it follows that

$$V = L \oplus L'.$$

□

A choice of complementary Lagrangian subspaces identifies a symplectic vector space with a cotangent-type model.

Proposition 25. *Let L and L' be complementary Lagrangian subspaces of (V, ω) . Then the*

bilinear pairing

$$\begin{aligned} L' \times L &\longrightarrow \mathbb{R}, \\ (v, \ell) &\longmapsto \omega(\ell, v) \end{aligned}$$

is non-degenerate. Consequently, there is a natural vector-space isomorphism

$$L' \cong L^*.$$

Under the resulting identification

$$V \cong L \oplus L^*,$$

the symplectic form is given by

$$\omega((x, \alpha), (y, \beta)) = \beta(x) - \alpha(y).$$

Proof. Suppose $v \in L'$ satisfies

$$\omega(\ell, v) = 0$$

for every $\ell \in L$. Then $v \in L^\omega$. Since L is Lagrangian, $L^\omega = L$, and hence

$$v \in L \cap L' = \{0\}.$$

Thus the pairing is non-degenerate in the first variable. Since L and L' have the same finite dimension, it is non-degenerate in both variables.

Define

$$\begin{aligned}\Psi : L' &\longrightarrow L^*, \\ v &\longmapsto \omega(\cdot, v)|_L.\end{aligned}$$

The non-degeneracy of the pairing implies that Ψ is an isomorphism. Because both L and L' are isotropic, the only nonzero terms in the symplectic pairing on $L \oplus L'$ are the mixed terms. Under the identification $L' \cong L^*$, one obtains

$$\omega((x, \alpha), (y, \beta)) = \beta(x) - \alpha(y).$$

□

This cotangent-type description explains why cotangent bundles form the canonical geometric model for Lagrangian submanifolds. At each point of a Lagrangian submanifold, the ambient tangent space is locally modeled on the direct sum of the tangent space of the Lagrangian and its dual. This linear observation is globalized by the Weinstein Lagrangian Neighborhood Theorem, which will be stated later in this section.

Definition 26. Let (V, ω) be a symplectic vector space of dimension $2n$. The set of all Lagrangian subspaces of V is called the *Lagrangian Grassmannian* and is denoted by

$$\text{LGr}(V, \omega).$$

The topology of the Lagrangian Grassmannian is important in the construction of the Maslov index and in the grading of Lagrangian Floer complexes. In particular, a path of Lagrangian subspaces determines a Maslov index after suitable endpoint conditions are imposed. This will be discussed later when gradings in Floer theory are introduced.

1.0.6 Lagrangian Submanifolds

We now pass from symplectic linear algebra to symplectic manifolds. The definition of a Lagrangian submanifold is obtained by requiring that each of its tangent spaces be a Lagrangian subspace of the corresponding ambient tangent space.

Definition 27. Let (M^{2n}, ω) be a symplectic manifold. A smooth embedded submanifold $L \subseteq M$ is called a *Lagrangian submanifold* if $T_p L$ is a Lagrangian subspace of $(T_p M, \omega_p)$ for every $p \in L$. Equivalently,

$$\omega_p|_{T_p L} = 0$$

for every $p \in L$, and

$$\dim L = n.$$

If $i : L \hookrightarrow M$ denotes the inclusion map, then the vanishing of ω on the tangent spaces of L is equivalent to

$$i^* \omega = 0.$$

Thus, a submanifold $L \subseteq M$ is Lagrangian if and only if

$$i^* \omega = 0$$

and

$$\dim L = \frac{1}{2} \dim M.$$

Proposition 28. Let (M^{2n}, ω) be a symplectic manifold and let $L \subseteq M$ be a smooth sub-

manifold. Then L is Lagrangian if and only if

$$i^*\omega = 0$$

and

$$\dim L = n,$$

where $i : L \hookrightarrow M$ is the inclusion.

Proof. Suppose first that L is Lagrangian. Then, for every $p \in L$, the tangent space $T_p L$ is a Lagrangian subspace of $T_p M$. Hence

$$\omega_p(v, w) = 0$$

for all $v, w \in T_p L$, and

$$\dim T_p L = n.$$

The first condition is precisely

$$i^*\omega = 0,$$

while the second implies $\dim L = n$.

Conversely, suppose that $i^*\omega = 0$ and $\dim L = n$. Then

$$\omega_p|_{T_p L} = 0$$

for every $p \in L$, so $T_p L$ is isotropic. Since $T_p L$ has dimension n , it is a maximal-dimensional isotropic subspace of the $2n$ -dimensional symplectic vector space $T_p M$. Therefore $T_p L$ is

Lagrangian for every $p \in L$, and hence L is a Lagrangian submanifold. $\square$

The half-dimensionality condition is essential. A submanifold on which the symplectic form vanishes but whose dimension is smaller than n is merely isotropic. Thus every Lagrangian submanifold is isotropic, but not every isotropic submanifold is Lagrangian.

Definition 29. Let (M, ω) be a symplectic manifold.

1. A submanifold $N \subseteq M$ is called *isotropic* if

$$\omega|_{TN} = 0.$$

2. A submanifold $N \subseteq M$ is called *coisotropic* if

$$(T_p N)^\omega \subseteq T_p N$$

for every $p \in N$.

3. A submanifold $N \subseteq M$ is called *symplectic* if the restriction

$$\omega|_{TN}$$

is non-degenerate.

Lagrangian submanifolds are simultaneously isotropic and coisotropic. Indeed, if L is Lagrangian, then

$$(T_p L)^\omega = T_p L$$

for every $p \in L$. The condition of being Lagrangian is preserved by symplectomorphisms.

Proposition 30. *Let*

$$\varphi : (M, \omega_M) \longrightarrow (N, \omega_N)$$

be a symplectomorphism, and let $L \subseteq M$ be a Lagrangian submanifold. Then $\varphi(L) \subseteq N$ is Lagrangian.

Proof. Since φ is a diffeomorphism,

$$\dim \varphi(L) = \dim L = \frac{1}{2} \dim M = \frac{1}{2} \dim N.$$

Let $p \in L$ and let $v, w \in T_p L$. Since $\varphi^* \omega_N = \omega_M$, we have

$$\omega_N(d\varphi_p(v), d\varphi_p(w)) = \omega_M(v, w).$$

Because L is Lagrangian, the right-hand side vanishes. Hence

$$\omega_N|_{T_{\varphi(p)}\varphi(L)} = 0.$$

Therefore $\varphi(L)$ is Lagrangian. □

This proposition allows one to construct new Lagrangian submanifolds from known ones by applying symplectomorphisms. In particular, Hamiltonian diffeomorphisms preserve Lagrangian submanifolds. Two Lagrangian submanifolds $L_0, L_1 \subseteq M$ are said to intersect *transversely* at a point $p \in L_0 \cap L_1$ if

$$T_p L_0 + T_p L_1 = T_p M.$$

Since both tangent spaces have dimension n , transversality is equivalent to

$$T_p L_0 \cap T_p L_1 = \{0\}.$$

Definition 31. Two Lagrangian submanifolds $L_0, L_1 \subseteq M$ are called *transverse*, written

$$L_0 \pitchfork L_1,$$

if they intersect transversely at every point of $L_0 \cap L_1$.

If L_0 and L_1 are transverse, then their intersection is discrete. Moreover, if both Lagrangians are compact, then $L_0 \cap L_1$ is finite. These intersection points later serve as generators of the Lagrangian Floer complex.

Proposition 32. *Let L_0 and L_1 be transverse Lagrangian submanifolds of a symplectic manifold (M^{2n}, ω) . Then $L_0 \cap L_1$ is a zero-dimensional submanifold. If L_0 and L_1 are compact, then $L_0 \cap L_1$ is finite.*

Proof. By the transverse intersection theorem, the intersection $L_0 \cap L_1$ is a submanifold whose dimension is

$$\begin{aligned} \dim(L_0 \cap L_1) &= \dim L_0 + \dim L_1 - \dim M \\ &= n + n - 2n \\ &= 0. \end{aligned}$$

Thus, $L_0 \cap L_1$ is discrete. If both Lagrangians are compact, then their intersection is compact as well. A compact discrete space is finite, so $L_0 \cap L_1$ is finite. $\square$

The study of intersections of Lagrangian submanifolds is one of the central themes of symplectic topology. Floer theory refines ordinary intersection theory by associating a chain

complex to a pair of Lagrangians and defining its differential through counts of pseudo-holomorphic strips.

1.1 Hamiltonian dynamics

1.1.1 Hamiltonian vector fields

Let (M, ω) be a symplectic manifold. Since ω_p is non-degenerate, the following map:

$$\omega_p^\# : T_p M \longrightarrow T_p^* M, \quad (1.4)$$

$$v \longmapsto \omega_p(v, \cdot) \quad (1.5)$$

is an isomorphism for each $p \in M$. Thus, we have the following bundle isomorphism

$$\omega^\# : TM \longrightarrow T^*M, \quad (1.6)$$

$$X \longmapsto \iota_X \omega, \quad (1.7)$$

where $\iota_X \omega$ denotes the contraction of ω with the vector field X .

Definition 33. Let (M, ω) be a symplectic manifold and let $H \in \mathcal{C}^\infty(M, \mathbb{R})$. The *Hamiltonian vector field* associated to H is the unique vector field X_H satisfying

$$\iota_{X_H} \omega = dH. \quad (1.8)$$

The function H is called a *Hamiltonian*.

Remark 34. There are two common sign conventions for the Hamiltonian vector field. Some authors define X_H by

$$\iota_{X_H} \omega = -dH.$$

Throughout this thesis, we use the convention

$$\iota_{X_H}\omega = dH.$$

All formulas for Hamiltonian vector fields and Hamiltonian flows are written according to this convention.

The existence and uniqueness of a Hamiltonian vector field follow directly from the non-degeneracy of the symplectic form.

Proposition 35. *Let (M, ω) be a symplectic manifold. For every smooth function $H : M \rightarrow \mathbb{R}$, there exists a unique smooth vector field X_H satisfying*

$$\iota_{X_H}\omega = dH. \tag{1.9}$$

Proof. The map Eq. (1.6) gives us a unique way to construct X_H . Take $X_H = (\omega^\#)^{-1}(dH)$. □

The assignment $H \mapsto X_H$ is $\mathbb{R}$ -linear.

Proposition 36. *Let $F, H \in C^\infty(M, \mathbb{R})$ and let $a, b \in \mathbb{R}$. Then*

$$X_{aF+bH} = aX_F + bX_H.$$

In particular, every constant function has zero Hamiltonian vector field.

Proof. Using Eq. (1.9), we have

$$\begin{aligned} \iota_{X_{aF+bH}}\omega &= d(aF + bH) \\ &= a dF + b dH \\ &= a \iota_{X_F}\omega + b \iota_{X_H}\omega \\ &= \iota_{aX_F+bX_H}\omega. \end{aligned}$$

Eq. (1.1) implies

$$X_{aF+bH} = aX_F + bX_H.$$

If H is constant, then $dH = 0$, so

$$\iota_{X_H}\omega = 0.$$

Eq. (1.1) gives us $X_H = 0$. □

We next compute the Hamiltonian vector field in standard symplectic coordinates. Let (M, ω) be locally identified with an open subset of $\mathbb{R}^{2n}$ with Darboux coordinates

$$(x_1, \dots, x_n, y_1, \dots, y_n)$$

and symplectic form

$$\omega = \sum_{i=1}^n dx_i \wedge dy_i.$$

Let

$$X_H = \sum_{i=1}^n \left(a_i \frac{\partial}{\partial x_i} + b_i \frac{\partial}{\partial y_i} \right). \quad (1.10)$$

Using the contraction formula, we obtain

$$\iota_{X_H}\omega = \sum_{i=1}^n \iota_{X_H}(dx_i \wedge dy_i) \quad (1.11)$$

$$= \sum_{i=1}^n (a_i dy_i - b_i dx_i). \quad (1.12)$$

$$dH = \sum_{i=1}^n \left(\frac{\partial H}{\partial x_i} dx_i + \frac{\partial H}{\partial y_i} dy_i \right). \quad (1.13)$$

Comparing Eq. (1.11) and Eq. (1.13),

$$\begin{aligned} a_i &= \frac{\partial H}{\partial y_i}, \\ b_i &= -\frac{\partial H}{\partial x_i}. \end{aligned}$$

We proved the following coordinate formula.

Proposition 37. *In Darboux coordinates for which*

$$\omega = \sum_{i=1}^n dx_i \wedge dy_i,$$

the Hamiltonian vector field associated to H is

$$X_H = \sum_{i=1}^n \left(\frac{\partial H}{\partial y_i} \frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i} \frac{\partial}{\partial y_i} \right).$$

Example 38. Consider the standard symplectic plane $(\mathbb{R}^2, dx \wedge dy)$ and the Hamiltonian

$$H(x, y) = \frac{1}{2}(x^2 + y^2).$$

The Hamiltonian vector field is

$$\begin{aligned} X_H &= \frac{\partial H}{\partial y} \frac{\partial}{\partial x} - \frac{\partial H}{\partial x} \frac{\partial}{\partial y} \\ &= y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}. \end{aligned}$$

Thus, the integral curves of X_H move along circles with the origin as the center. With the sign convention used in this thesis, the resulting rotation is clockwise.

Example 39. Let (T^*Q, ω_0) be a cotangent bundle, and let

$$(x_1, \dots, x_n, y_1, \dots, y_n)$$

denote canonical local coordinates, where the x_i are base coordinates and the y_i are fiber coordinates. Suppose the Hamiltonian depends only on the fiber coordinates:

$$H = H(y_1, \dots, y_n).$$

Then,

$$X_H = \sum_{i=1}^n \frac{\partial H}{\partial y_i} \frac{\partial}{\partial x_i}. \quad (1.14)$$

The fiber coordinates remain constant along the Hamiltonian trajectories.

Example 40. On $T^*\mathbb{R}^n$ with its standard symplectic form, consider the following Hamiltonian

$$H(x, y) = \frac{1}{2} \sum_{i=1}^n y_i^2.$$

Then,

$$X_H = \sum_{i=1}^n y_i \frac{\partial}{\partial x_i}.$$

The corresponding Hamiltonian trajectories satisfy

$$\dot{x}_i = y_i,$$

$$\dot{y}_i = 0.$$

Thus the fiber coordinates are constant and the base projection of every trajectory is a

straight line. This is an instance of the relation between quadratic Hamiltonians on cotangent bundles and geodesic flow.

The Hamiltonian vector field depends on both the Hamiltonian function and the symplectic form. Let Q be a smooth manifold, let θ be a closed two-form on Q , and equip T^*Q with

$$\omega_\theta = \omega_0 + \pi^*\theta.$$

The same function $H : T^*Q \rightarrow \mathbb{R}$ generally determines different Hamiltonian vector fields with respect to ω_0 and ω_θ . We compute the Hamiltonian vector field associated with ω_θ later in the thesis (Chapter 2).

Definition 41. A *time-dependent Hamiltonian* is a smooth function

$$\begin{aligned} H : [0, 1] \times M &\longrightarrow \mathbb{R}, \\ (t, p) &\longmapsto H_t(p). \end{aligned}$$

For each fixed t , the associated time-dependent Hamiltonian vector field X_{H_t} is defined by

$$\iota_{X_{H_t}}\omega = dH_t.$$

Thus, a time-dependent Hamiltonian determines a smooth family of vector fields $\{X_{H_t}\}_{t \in [0, 1]}$. Their integral curves and associated diffeomorphisms will be introduced in the following subsection.

1.1.2 Hamiltonian Flows

A Hamiltonian vector field determines a one-parameter family of diffeomorphisms obtained by the integral flows of the vector field. These diffeomorphisms are called Hamiltonian flows.

They preserve the symplectic form and act by symplectomorphisms.

Let (M, ω) be a symplectic manifold and let $H : M \rightarrow \mathbb{R}$ be a smooth Hamiltonian. Recall that the associated Hamiltonian vector field X_H is defined by Eq. (1.9).

Definition 42. An integral curve of the Hamiltonian vector field X_H through a point $p \in M$ is a smooth curve

$$\gamma_p : I \longrightarrow M \tag{1.15}$$

satisfying

$$\frac{d}{dt}\gamma_p(t) = X_H(\gamma_p(t)) \tag{1.16}$$

$$\gamma_p(0) = p. \tag{1.17}$$

The Eq. (1.16) defines a system of ordinary differential equations in a local chart at p . The local existence and uniqueness theorem for ordinary differential equations guarantees that for every $p \in M$ there exists a unique maximal integral curve through p . This defines a smooth family of maps φ_H^t as follows.

Definition 43. The *Hamiltonian flow* generated by H is the family of maps

$$\varphi_H^t : M \longrightarrow M$$

defined by

$$\varphi_H^t(p) = \gamma_p(t), \tag{1.18}$$

where γ_p is the integral curve of X_H through p .

Equivalently, φ_H^t satisfies

$$\frac{d}{dt}\varphi_H^t(p) = X_H(\varphi_H^t(p)) \quad (1.19)$$

$$\varphi_H^0(p) = p \quad (1.20)$$

If the vector field X_H is complete, then φ_H^t is defined for all $t \in \mathbb{R}$. In this case, the flow satisfies the group property

$$\begin{aligned} \varphi_H^{t+s} &= \varphi_H^t \circ \varphi_H^s \\ (\varphi_H^t)^{-1} &= \varphi_H^{-t}. \end{aligned}$$

An important property of Hamiltonian flows is that they preserve the symplectic form.

Proposition 44. *Let (M, ω) be a symplectic manifold and let $H : M \rightarrow \mathbb{R}$ be a smooth Hamiltonian. Then the Hamiltonian flow generated by H , φ_H^t is a symplectomorphism for all $t \in I$.*

$$(\varphi_H^t)^*\omega = \omega. \quad (1.21)$$

Proof. The standard formula for differentiation along a flow gives

$$\frac{d}{dt}(\varphi_H^t)^*\omega = (\varphi_H^t)^*(\mathcal{L}_{X_H}\omega), \quad (1.22)$$

where $\mathcal{L}_{X_H}$ denotes the Lie derivative along X_H .

By Cartan's formula,

$$\mathcal{L}_{X_H}\omega = d(\iota_{X_H}\omega) + \iota_{X_H}(d\omega). \quad (1.23)$$

Since ω is closed,

$$d\omega = 0,$$

and since X_H satisfies

$$\iota_{X_H}\omega = dH,$$

we obtain

$$\mathcal{L}_{X_H}\omega = d(dH) = 0.$$

Therefore,

$$\frac{d}{dt}(\varphi_H^t)^*\omega = 0.$$

It follows that $(\varphi_H^t)^*\omega$ is independent of t . At $t = 0$, the flow is the identity map, so

$$(\varphi_H^0)^*\omega = \omega.$$

Hence,

$$(\varphi_H^t)^*\omega = \omega$$

for every t for which the flow is defined. □

Thus, Hamiltonian flows form a class of symplectomorphisms. This implies that if $L \subseteq M$ is a Lagrangian submanifold, then

$$\varphi_H^t(L)$$

is a Lagrangian for every t .

For an autonomous Hamiltonian, the Hamiltonian function itself is conserved along its flow.

Proposition 45. *Let $H : M \rightarrow \mathbb{R}$ be an autonomous Hamiltonian. Then H is constant along every Hamiltonian trajectory,*

$$H \circ \varphi_H^t(p) = H(p). \quad (1.24)$$

for all $p \in M$.

Proof. Let $\gamma(t)$ be an integral curve of X_H . Then,

$$\begin{aligned} \frac{d}{dt}H(\gamma(t)) &= dH_{\gamma(t)}\left(\frac{d\gamma}{dt}\right) \\ &= dH(X_H) \\ &= \omega(X_H, X_H) = 0. \end{aligned}$$

Therefore, H is constant along the trajectory. □

This implies that Hamiltonian trajectories remain inside level sets of H . In classical mechanics, the Hamiltonian represents the total energy of a system, and Eq. (1.24) is the conservation of energy.

Example 46. Consider $(\mathbb{R}^2, dx \wedge dy)$ with Hamiltonian

$$H(x, y) = \frac{1}{2}(x^2 + y^2).$$

The corresponding Hamiltonian vector field X_H is

$$X_H = y\partial_x - x\partial_y.$$

The Hamiltonian equations are as follows:

$$\begin{aligned}\dot{x} &= y, \\ \dot{y} &= -x.\end{aligned}$$

The solutions are

$$\begin{aligned}x(t) &= x_0 \cos t + y_0 \sin t, \\ y(t) &= -x_0 \sin t + y_0 \cos t.\end{aligned}$$

Hence, the Hamiltonian flow is given by

$$\varphi_H^t(x_0, y_0) = (x_0 \cos t + y_0 \sin t, -x_0 \sin t + y_0 \cos t).$$

The trajectories lie on the level sets

$$x^2 + y^2 = \text{constant},$$

as in Eq. (1.24).

Example 47. Consider the cotangent bundle $T^*\mathbb{R}^n$ with coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$ and Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^n y_i^2.$$

As computed previously,

$$X_H = \sum_{i=1}^n y_i \frac{\partial}{\partial x_i}.$$

Thus the Hamiltonian equations are

$$\dot{x}_i = y_i,$$

$$\dot{y}_i = 0.$$

The solution through (x_0, y_0) is given by,

$$x_i(t) = x_i(0) + t y_i(0),$$

$$y_i(t) = y_i(0).$$

Therefore, the Hamiltonian flow is

$$\varphi_H^t(x_0, y_0) = (x_0 + t y_0, y_0).$$

The base projection of each Hamiltonian trajectory is a straight line.

We also encounter Hamiltonians that depend on time. Let,

$$H : [0, 1] \times M \longrightarrow \mathbb{R}$$

be a smooth time-dependent Hamiltonian, and write

$$H_t(p) = H(t, p).$$

For each t , the Hamiltonian vector field X_{H_t} is defined by

$$\iota_{X_{H_t}} \omega = dH_t.$$

Definition 48. The flow of a time-dependent Hamiltonian H_t is the family of maps φ_H^t

satisfying

$$\frac{d}{dt}\varphi_H^t(p) = X_{H_t}(\varphi_H^t(p)) \quad (1.25)$$

$$\varphi_H^0(p) = p. \quad (1.26)$$

Unlike the flow of an autonomous Hamiltonian, a time-dependent Hamiltonian flow generally does not satisfy

$$\varphi_H^{t+s} = \varphi_H^t \circ \varphi_H^s.$$

However, each time- t map still preserves the symplectic form.

Proposition 49. *Let H_t be a smooth time-dependent Hamiltonian. Then*

$$(\varphi_H^t)^*\omega = \omega$$

for every t for which the flow is defined.

Proof. The proof technique is exactly the same as that of Proposition 44. □

The time-one map φ_H^1 of a time-dependent Hamiltonian flow is called a Hamiltonian diffeomorphism. Hamiltonian isotopies and Hamiltonian diffeomorphisms will be discussed in the next subsection. They provide the natural class of deformations of Lagrangian submanifolds used in Floer theory. Hamiltonian chords between Lagrangians can be interpreted as intersection points after applying a Hamiltonian flow.

1.1.3 Hamiltonian Isotopies and Hamiltonian Diffeomorphisms

Hamiltonian flows give rise to a class of isotopies through symplectomorphisms. These isotopies play a central role in symplectic topology and Floer theory, where Lagrangian submanifolds are perturbed by Hamiltonian diffeomorphisms to make them intersect transversally.

Let (M, ω) be a symplectic manifold and let $H : [0, 1] \times M \rightarrow \mathbb{R}$ be a smooth time-dependent Hamiltonian. For each $t \in [0, 1]$, let X_{H_t} denote the Hamiltonian vector field defined by

$$\iota_{X_{H_t}} \omega = dH_t.$$

The associated flow will be denoted by φ_H^t .

Definition 50. A smooth family of diffeomorphisms

$$\varphi_t : M \longrightarrow M, \quad t \in [0, 1],$$

is called a *Hamiltonian isotopy* if $\varphi_0 = \text{id}_M$ and there exists a smooth time-dependent Hamiltonian H_t such that

$$\frac{d}{dt} \varphi_t = X_{H_t} \circ \varphi_t. \tag{1.27}$$

Equivalently, a Hamiltonian isotopy is the flow of a time-dependent Hamiltonian vector field. As shown in the previous subsection, every Hamiltonian flow preserves the symplectic form. Hence, each φ_t is a symplectomorphism:

$$\varphi_t^* \omega = \omega.$$

Definition 51. A diffeomorphism

$$\varphi : M \longrightarrow M$$

is called a *Hamiltonian diffeomorphism* if there exists a Hamiltonian isotopy φ_t such that

$$\varphi_0 = \text{id}_M$$

$$\varphi_1 = \varphi.$$

The set of Hamiltonian diffeomorphisms of (M, ω) is denoted by

$$\text{Ham}(M, \omega).$$

Hamiltonian diffeomorphisms form a subgroup of the group of symplectomorphisms $\text{Symp}(M, \omega)$. Indeed,

$$\text{Ham}(M, \omega) \subseteq \text{Symp}(M, \omega).$$

Locally, every symplectic vector field is Hamiltonian. However, a symplectic isotopy does not always arise from a global Hamiltonian function.

Definition 52. A vector field X on (M, ω) is called *symplectic* if

$$\mathcal{L}_X \omega = 0.$$

Using Cartan's formula (Eq. (1.23)) and the fact that $d\omega = 0$, this condition becomes

$$d(\iota_X \omega) = 0.$$

Thus, X is symplectic if and only if the one-form $\iota_X \omega$ is closed. By contrast, X is Hamiltonian if there exists a smooth function $H : M \rightarrow \mathbb{R}$ such that,

$$\iota_X \omega = dH.$$

Thus, every Hamiltonian vector field is symplectic, but the converse may not be true.

Proposition 53. *Every Hamiltonian vector field is symplectic.*

Proof. Let X_H be the Hamiltonian vector field associated to H . Using Cartan's formula (Eq. (1.23)),

$$\begin{aligned}\mathcal{L}_{X_H}\omega &= d(dH) \\ &= 0.\end{aligned}$$

Hence X_H is symplectic. □

The obstruction to a symplectic vector field being Hamiltonian is measured by the de Rham cohomology class

$$[\iota_X\omega] \in H_{\text{dR}}^1(M).$$

In particular, if

$$H_{\text{dR}}^1(M) = 0,$$

then every symplectic vector field is Hamiltonian.

Proposition 54. *Suppose*

$$H_{\text{dR}}^1(M) = 0.$$

Then every symplectic vector field on (M, ω) is Hamiltonian.

Proof. Let X be a symplectic vector field. Then,

$$\mathcal{L}_X\omega = 0.$$

Using Cartan's formula (Eq. (1.23)),

$$d(\iota_X \omega) = 0.$$

Thus $\iota_X \omega$ is closed. Since $H_{\text{dR}}^1(M) = 0$, there exists a smooth function $H : M \rightarrow \mathbb{R}$ such that

$$\iota_X \omega = dH.$$

Therefore $X = X_H$. □

Hamiltonian isotopies preserve Lagrangian submanifolds.

Proposition 55. *Let $L \subseteq (M, \omega)$ be a Lagrangian submanifold and let φ_t be a Hamiltonian isotopy. Then*

$$L_t = \varphi_t(L)$$

is Lagrangian for every t .

Proof. Since each φ_t is a symplectomorphism,

$$\varphi_t^* \omega = \omega.$$

L_t is a Lagrangian submanifold. □

This construction provides the most important class of deformations of Lagrangian submanifolds in Floer theory.

Definition 56. Two Lagrangian submanifolds $L_0, L_1 \subseteq M$ are called *Hamiltonian isotopic*

if there exists a Hamiltonian isotopy φ_t such that

$$\varphi_1(L_0) = L_1.$$

Hamiltonian isotopy is stronger than both smooth isotopy and isotopy through Lagrangian submanifolds. Floer-theoretic invariants are invariant under Hamiltonian isotopy, which is one of the main reasons that this is a fundamental example of deformations in this theory.

An application is the perturbation of a Lagrangian submanifold to obtain transverse intersections. Suppose $L_0, L_1 \subseteq M$ are Lagrangian submanifolds that do not intersect transversely; for instance, when $L_0 = L_1$. One may choose a suitable Hamiltonian H_t and replace L_0 by $\varphi_H^1(L_0)$.

For a generic choice of perturbation, one expects

$$\varphi_H^1(L_0) \pitchfork L_1.$$

The intersection points are then used as generators of a Lagrangian Floer complex.

Example 57. Consider the zero section $Q \subseteq T^*Q$ and let $f : Q \rightarrow \mathbb{R}$ be a smooth function. Define a Hamiltonian on T^*Q by

$$H(q, p) = f(q).$$

In local coordinates,

$$X_H = - \sum_{i=1}^n \partial_{x_i}(f) \partial_{y_i}.$$

Thus, the Hamiltonian flow translates the fiber coordinates according to df . Up to signs,

the time-one image of the zero section is the graph of df :

$$\varphi_H^1(Q) = \Gamma_{\pm df}.$$

Hence graphs of exact one-forms are Hamiltonian isotopic to the zero section.

Example 58. Let $L = T_q^*Q$ be a cotangent fiber. Since Hamiltonian diffeomorphisms preserve Lagrangian submanifolds, the image

$$\varphi_H^1(L)$$

is again Lagrangian. The intersection points

$$\varphi_H^1(L) \cap L$$

are in bijection with Hamiltonian trajectories whose endpoints lie on L . This correspondence will be described more precisely in the next subsection and forms the basis for the Hamiltonian-chord model of Lagrangian Floer theory used later in this thesis.

1.1.4 Hamiltonian Chords and Lagrangian Boundary Conditions

Let (M, ω) be a symplectic manifold, let $L_0, L_1 \subseteq M$ be Lagrangian submanifolds, and let

$$H : [0, 1] \times M \longrightarrow \mathbb{R}$$

be a smooth time-dependent Hamiltonian. We denote by X_{H_t} the Hamiltonian vector field determined by

$$\iota_{X_{H_t}} \omega = dH_t.$$

Definition 59. A *Hamiltonian chord* from L_0 to L_1 associated to H is a smooth path $x : [0, 1] \longrightarrow M$ satisfying,

$$\dot{x}(t) = X_{H_t}(x(t)),$$

$$x(0) \in L_0,$$

$$x(1) \in L_1.$$

We denote the set of such Hamiltonian chords by

$$\mathcal{X}_H(L_0, L_1).$$

A Hamiltonian chord is therefore an integral trajectory of the Hamiltonian vector field subject to Lagrangian boundary conditions. Since the solution of the Hamiltonian equation is uniquely determined by its initial point, the set of Hamiltonian chords may be identified with the intersection of the chosen pair L_0, L_1 .

Proposition 60. *Let φ_H^t denote the Hamiltonian flow generated by H . There is a natural bijection*

$$\mathcal{X}_H(L_0, L_1) \cong \varphi_H^1(L_0) \cap L_1.$$

Proof. Let $x \in \mathcal{X}_H(L_0, L_1)$. Since x satisfies the Hamiltonian equation and $x(0) \in L_0$, uniqueness of solutions to the Hamiltonian equation implies

$$x(t) = \varphi_H^t(x(0)).$$

In particular,

$$x(1) = \varphi_H^1(x(0)).$$

Since $x(0) \in L_0$ and $x(1) \in L_1$, we have

$$x(1) \in \varphi_H^1(L_0) \cap L_1.$$

Conversely, let

$$p \in \varphi_H^1(L_0) \cap L_1.$$

Since $p \in \varphi_H^1(L_0)$, there exists $q \in L_0$ such that

$$p = \varphi_H^1(q).$$

$$x(t) := \varphi_H^t(q).$$

Then,

$$\dot{x}(t) = X_{H_t}(x(t)),$$

$$x(0) = q \in L_0$$

$$x(1) = p \in L_1.$$

Hence x is a Hamiltonian chord from L_0 to L_1 . These two constructions are inverse to each other. Thus, we have a bijection. □

Depending on convention, one may instead identify Hamiltonian chords with

$$L_0 \cap (\varphi_H^1)^{-1}(L_1).$$

Indeed, if $x(0) = q$, then

$$\varphi_H^1(q) \in L_1 \iff q \in L_0 \cap (\varphi_H^1)^{-1}(L_1).$$

The two descriptions are equivalent.

The case in which

$$L_0 = L_1 = L$$

is particularly important. The importance lies in the fact that in the case of exact cotangent bundles over a smooth manifold, the set $\mathcal{X}(\mathcal{L}, \mathcal{L})$ when L is the cotangent fiber over a base point homologically generates the Fukaya category. A Hamiltonian chord with endpoints on L is a path satisfying

$$\begin{aligned} \dot{x}(t) &= X_{H_t}(x(t)), \\ x(0), x(1) &\in L. \end{aligned}$$

The corresponding set of chords is naturally identified with

$$\varphi_H^1(L) \cap L.$$

This is precisely the situation that arises when studying the endomorphism algebra of a single Lagrangian object in the Fukaya category.

Definition 61. A Hamiltonian chord $x \in \mathcal{X}_H(L_0, L_1)$ is called *non-degenerate* if the corre-

sponding intersection point of $\varphi_H^1(L_0)$ and L_1 is transverse.

$$d\varphi_H^1(T_{x(0)}L_0) \cap T_{x(1)}L_1 = \{0\}.$$

Since both subspaces are Lagrangian and therefore have dimension $\frac{1}{2} \dim M$, non-degeneracy is equivalent to

$$d\varphi_H^1(T_{x(0)}L_0) + T_{x(1)}L_1 = T_{x(1)}M.$$

When all Hamiltonian chords are non-degenerate, the set of chords is discrete. Under suitable compactness assumptions, it is finite. In wrapped Floer theory, where a generating set of Lagrangians are often non-compact (for instance, the cotangent fiber over a base point), the set of chords may instead be infinite. There are only finitely many chords, however, in the pre-image of a compact set under the action functional.

Proposition 62. *A Hamiltonian chord x from L_0 to L_1 is non-degenerate if and only if the intersection point*

$$x(1) \in \varphi_H^1(L_0) \cap L_1$$

is transverse.

Proof. The tangent space of the image $\varphi_H^1(L_0)$ at the point $x(1)$ is

$$T_{x(1)}\varphi_H^1(L_0) = d\varphi_H^1(T_{x(0)}L_0).$$

Thus transversality of the intersection is equivalent to

$$d\varphi_H^1(T_{x(0)}L_0) + T_{x(1)}L_1 = T_{x(1)}M.$$

Since both summands are Lagrangian and therefore have half the dimension of M , this is equivalent to

$$d\varphi_H^1(T_{x(0)}L_0) \cap T_{x(1)}L_1 = \{0\}.$$

This is the non-degeneracy condition. □

Hamiltonian chords form the generators of the Floer complex associated to a pair of Lagrangian submanifolds. That is,

$$CF(L_0, L_1; H) = \bigoplus_{x \in \mathcal{X}_H(L_0, L_1)} \mathbb{k} \cdot x,$$

where $\mathbb{k}$ is a chosen coefficient ring or field. In Lagrangian-Floer theory, $\mathbb{k}$ is often chosen as the Novikov ring. The differential and the higher A_∞ operations are then defined by counting solutions of perturbed Cauchy–Riemann equations with asymptotic conditions determined by these Hamiltonian chords.

Remark 63. In the setting of this thesis, the principal Lagrangian is a cotangent fiber

$$L = T_q^*\mathbb{T}^2.$$

For an appropriate Hamiltonian H , the generators of the wrapped Floer complex are Hamiltonian chords satisfying,

$$x(0), x(1) \in L.$$

Equivalently, they correspond to intersection points

$$\varphi_H^1(L) \cap L.$$

The projection of such a chord to the base,

$$\pi \circ x : [0, 1] \longrightarrow \mathbb{T}^2,$$

is a loop based at q . Hence every Hamiltonian chord determines a homotopy class in $\pi_1(\mathbb{T}^2, q)$. Since the fundamental group,

$$\pi_1(\mathbb{T}^2, q) \cong \mathbb{Z}^2,$$

the Hamiltonian chords occurring in the cotangent-fiber Floer complex can be indexed by $\mathbb{Z}^2$.

Thus, in the notation used later in this thesis, one writes

$$\gamma_{ij}, \quad (i, j) \in \mathbb{Z}^2,$$

for the Hamiltonian chord whose projection represents the corresponding element of the fundamental group. This identification between Hamiltonian chords and homotopy classes of based loops is a special feature of cotangent bundles and will play an essential role in the computation of the endomorphism algebra of the cotangent fiber.

1.1.5 Hamiltonian Dynamics on Cotangent Bundles

Cotangent bundles provide the natural phase spaces for Hamiltonian mechanics. Their canonical symplectic structure allows the Hamiltonian equations to be written explicitly in local coordinates, and this coordinate description will be used repeatedly later in the thesis.

Let Q be a smooth manifold of dimension n , and let

$$\pi : T^*Q \longrightarrow Q$$

denote its cotangent bundle. Choose local coordinates $(x_1, \dots, x_n)$ on Q and corresponding fiber coordinates $(y_1, \dots, y_n)$ on T^*Q . The canonical symplectic form was discussed in Proposition 8 Eq. (1.3). In local coordinates, it can be expressed as follows. The Liouville one form is

$$\lambda = - \sum_{i=1}^n y_i dx_i,$$

and the canonical symplectic form is

$$\omega_0 = d\lambda = \sum_{i=1}^n dx_i \wedge dy_i.$$

Let $H : T^*Q \rightarrow \mathbb{R}$ be a smooth Hamiltonian. We have the computation for the corresponding Hamiltonian vector X_H in Eq. (1.14).

Consequently, a Hamiltonian trajectory

$$\gamma(t) = (x_1(t), \dots, x_n(t), y_1(t), \dots, y_n(t))$$

satisfies Hamilton's equations

$$\begin{aligned} \dot{x}_i &= \frac{\partial H}{\partial y_i}, \\ \dot{y}_i &= -\frac{\partial H}{\partial x_i}. \end{aligned}$$

Proposition 64. *Let $H : T^*Q \rightarrow \mathbb{R}$ be a smooth Hamiltonian. In local coordinates on T^*Q , the Hamiltonian trajectories are precisely the solutions of*

$$\dot{x}_i = \frac{\partial H}{\partial y_i}, \tag{1.28}$$

$$\dot{y}_i = -\frac{\partial H}{\partial x_i}. \tag{1.29}$$

Proof. The result follows directly from the coordinate expression for X_H . Since an integral curve γ satisfies

$$\dot{\gamma}(t) = X_H(\gamma(t)), \quad (1.30)$$

the coefficients of $\partial/\partial x_i$ and $\partial/\partial y_i$ give the stated system of equations. $\square$

Example 65. We have computed Hamiltonian flow on $T^*\mathbb{R}^n$ with the standard quadratic Hamiltonian. The same computation descends to a flat torus. Let

$$T^n = \mathbb{R}^n / \Lambda$$

for a lattice $\Lambda \subseteq \mathbb{R}^n$. Then

$$T^*T^n \cong (\mathbb{R}^n / \Lambda) \times \mathbb{R}^n.$$

For the quadratic Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^n y_i^2, \quad (1.31)$$

the lifted flow on $T^*\mathbb{R}^n$ is

$$\widetilde{\varphi}_H^t(x, y) = (x + ty, y).$$

Passing to the quotient gives the Hamiltonian flow on T^*T^n .

This simple expression makes the relation between Hamiltonian chords and elements of the lattice transparent.

Proposition 66. *Let $L = T_q^*T^n$ be a cotangent fiber and let*

$$H = \frac{1}{2}|y|^2.$$

Then Hamiltonian chords from L to itself are indexed by elements of the lattice Λ .

Proof. Choose a lift $\tilde{q} \in \mathbb{R}^n$ of q . A Hamiltonian trajectory starting in the lifted cotangent fiber over $\tilde{q}$ has the form

$$x(t) = \tilde{q} + ty_0, \quad y(t) = y_0.$$

Its projection to T^n ends at q precisely when

$$x(1) - \tilde{q} = y_0 \in \Lambda.$$

Thus the initial fiber coordinate must be an element of the lattice, and each lattice element determines a unique chord. $\square$

Remark 67. In the case of the two-dimensional torus, one may choose generators v_0, v_1 of the lattice and write every lattice element as

$$iv_0 + jv_1, \quad (i, j) \in \mathbb{Z}^2.$$

The corresponding Hamiltonian chords may therefore be indexed by pairs of integers. This is the origin of the notation

$$\gamma_{ij}, \quad (i, j) \in \mathbb{Z}^2,$$

used later in this thesis for generators of the cotangent-fiber Floer complex.

Remark 68. More generally, let g be a Riemannian metric on Q . The metric induces an

isomorphism

$$T^*Q \cong TQ$$

and a fiberwise norm $|\cdot|_g$ on T^*Q . The associated kinetic-energy Hamiltonian is

$$H(q, p) = \frac{1}{2}|p|_g^2.$$

The Hamiltonian flow of this function is closely related to the geodesic flow of the metric g . This relation will be discussed in the following subsection.

1.2 Almost Complex Structures

1.2.1 Almost Complex Structures

Almost complex structures provide the required framework in which pseudo-holomorphic curves are defined. They are important in symplectic geometry because a symplectic form can be combined with an almost complex structure to produce a Riemannian metric. This interaction will be used later in the definition of Floer trajectories and in the analysis of their energy. In the setting of this thesis, an almost complex structure is constructed on the twisted cotangent bundle T^*T^2 and used to define the metric appearing in the Floer-theoretic computations.

Definition 69. Let M be a smooth manifold. An *almost complex structure* on M is a smooth bundle endomorphism

$$J : TM \longrightarrow TM$$

such that

$$J^2 = -\text{id}_{TM}.$$

Equivalently, for every $p \in M$, the linear map

$$J_p : T_p M \longrightarrow T_p M$$

satisfies

$$J_p^2 = -\text{id}_{T_p M}.$$

Thus, the almost complex structure J induces a complex vector bundle structure on TM . The multiplication by the complex number i is defined as follows,

$$iv = J_p v.$$

Indeed,

$$i^2 v = J_p^2 v = -v,$$

as required for a complex vector-space structure.

The existence of an almost complex structure immediately imposes a restriction on the dimension of the underlying manifold.

Proposition 70. *If a smooth manifold M admits an almost complex structure, then M has even real dimension.*

Proof. Let J be an almost complex structure on M . At every point $p \in M$, the tangent space $T_p M$ becomes a complex vector space under the action of J_p . If its complex dimension

is n , then its real dimension is $2n$. Therefore,

$$\dim_{\mathbb{R}} M = 2n.$$

□

Example 71. Consider $\mathbb{R}^{2n}$ with coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$. Define an endomorphism J_0 by

$$\begin{aligned} J_0 \left(\frac{\partial}{\partial x_i} \right) &= \frac{\partial}{\partial y_i}, \\ J_0 \left(\frac{\partial}{\partial y_i} \right) &= -\frac{\partial}{\partial x_i}. \end{aligned}$$

Then,

$$\begin{aligned} J_0^2 \left(\frac{\partial}{\partial x_i} \right) &= -\frac{\partial}{\partial x_i}, \\ J_0^2 \left(\frac{\partial}{\partial y_i} \right) &= -\frac{\partial}{\partial y_i}. \end{aligned}$$

Hence,

$$J_0^2 = -\text{id},$$

so J_0 is an almost complex structure. It is called the *standard almost complex structure* on $\mathbb{R}^{2n}$.

Every complex manifold has a natural almost complex structure. Indeed, if (M, J) is a complex manifold, multiplication by i on its complex tangent spaces defines an endomorphism satisfying $J^2 = \text{id}$. The converse, however, is not generally true: an almost complex structure need not arise from a complex coordinate atlas.

Definition 72. An almost complex structure J is called *integrable* if it is induced by the

complex structure of a complex manifold.

The obstruction to integrability is measured by the Nijenhuis tensor, and the Newlander–Nirenberg theorem states that an almost complex structure is integrable if and only if its Nijenhuis tensor vanishes. We will not require this result in what follows. In Floer theory, one works with almost complex structures without imposing integrability.

This flexibility is essential in symplectic geometry. Given a symplectic manifold (M, ω) , one seeks almost complex structures whose interaction with ω satisfies certain positivity and compatibility conditions. These conditions will be introduced in the next subsection.

1.2.2 Compatible and Tame Almost Complex Structures

Let (M, ω) be a symplectic manifold and let J be an almost complex structure on M . In order for J to interact well with the symplectic form, one imposes certain positivity conditions. The two most common notions are compatibility and tameness.

Definition 73. An almost complex structure J on (M, ω) is called *ω -compatible* if, for every $p \in M$ and every $u, v \in T_p M$,

$$\omega_p(Ju, Jv) = \omega_p(u, v), \quad (1.32)$$

$$\omega_p(v, Jv) > 0 \quad (1.33)$$

for every nonzero $v \in T_p M$.

The first condition says that J preserves the symplectic form, while the second provides a positivity condition. They imply that the bilinear form

$$g_J(u, v) = \omega(u, Jv) \quad (1.34)$$

is a Riemannian metric. This associated metric will be discussed in the next subsection. There is also a weaker notion which is sufficient for many applications in symplectic topology.

Definition 74. An almost complex structure J on (M, ω) is called ω -tame if

$$\omega(v, Jv) > 0$$

for every nonzero tangent vector v .

Thus, every compatible almost complex structure is tame, but the converse need not hold. The compatibility condition can also be expressed in terms of the symmetry of the bilinear form $\omega(\cdot, J\cdot)$.

Proposition 75. *Let J be an almost complex structure on (M, ω) . Suppose*

$$\omega(Ju, Jv) = \omega(u, v)$$

for all tangent vectors u, v . Then

$$\omega(u, Jv) = \omega(v, Ju).$$

Proof. Using $J^2 = -\text{id}$ and the invariance of ω under J , we have

$$\begin{aligned} \omega(Ju, v) &= \omega(Ju, J(-Jv)) \\ &= \omega(u, -Jv) \\ &= -\omega(u, Jv). \end{aligned}$$

By skew-symmetry of ω ,

$$\omega(Ju, v) = -\omega(v, Ju).$$

Comparing the two expressions gives

$$\omega(u, Jv) = \omega(v, Ju).$$

□

Hence, for an ω -compatible almost complex structure, the bilinear form g_J defined in Eq. (1.34) is symmetric.

Example 76. Consider the standard symplectic vector space $(\mathbb{R}^{2n}, \omega_0)$, where

$$\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i.$$

Let J_0 denote the standard almost complex structure

$$\begin{aligned} J_0 \left(\frac{\partial}{\partial x_i} \right) &= \frac{\partial}{\partial y_i}, \\ J_0 \left(\frac{\partial}{\partial y_i} \right) &= -\frac{\partial}{\partial x_i}. \end{aligned}$$

Then,

$$\omega_0(J_0u, J_0v) = \omega_0(u, v)$$

for all u, v , and

$$\omega_0(v, J_0v) > 0$$

for every nonzero v . Therefore J_0 is ω_0 -compatible.

The positivity condition is especially important in the study of pseudo-holomorphic curves. If J is ω -tame and u is a J -holomorphic map from a Riemann surface, then the symplectic area of u is nonnegative. This positivity is a key analytic feature that underlies

compactness and energy estimates in Floer theory. Throughout this thesis, we will use ω -compatible almost complex structures. For the twisted cotangent bundle considered later, we choose an explicit almost complex structure compatible with the deformed symplectic form and use the associated metric in the analysis of Floer trajectories.

1.2.3 The Associated Riemannian Metric

Let (M, ω) be a symplectic manifold and let J be an ω -compatible almost complex structure. The compatibility conditions Eq. (1.32) allow one to construct a Riemannian metric directly from ω and J .

Definition 77. The *Riemannian metric associated to ω and J* is the bilinear form

$$g_J(u, v) = \omega(u, Jv).$$

The fact that this defines a Riemannian metric follows from the compatibility of J with ω .

Proposition 78. *Let (M, ω) be a symplectic manifold and let J be an ω -compatible almost complex structure. Then*

$$g_J(u, v) = \omega(u, Jv)$$

defines a Riemannian metric on M .

Proof. Using Proposition 75 and Eq. (1.32) allows us to show that g_J is a metric. □

The three structures ω , J , and g_J are therefore closely related. In particular,

$$g_J(Ju, Jv) = g_J(u, v),$$

so J acts as an isometry with respect to g_J shown as follows.

$$\begin{aligned}
g_J(Ju, Jv) &= \omega(Ju, J(Jv)) \\
&= -\omega(Ju, v) \\
&= \omega(u, Jv) \\
&= g_J(u, v).
\end{aligned}$$

Thus, the almost complex structure behaves similar to a rotation by $\pi/2$ in each tangent space while preserving the associated metric.

The associated metric plays an important role in Floer theory. It is used to measure the energy of pseudo-holomorphic curves and Floer trajectories and provides the norm appearing in the corresponding energy functionals. In the later analysis of the twisted cotangent bundle T^*T^2 , we choose an explicit almost complex structure J compatible with the deformed symplectic form and study the resulting metric g_J . In particular, the local flatness of this metric will be used in the analysis of the relevant moduli spaces.

1.2.4 Existence of Compatible Almost Complex Structures

We have seen that an ω -compatible almost complex structure determines a Riemannian metric on a symplectic manifold. It is therefore important to know that such structures always exist. In fact, the space of compatible almost complex structures is much richer than mere existence suggests.

Theorem 79. *Let (M, ω) be a symplectic manifold. Then there exists an ω -compatible almost complex structure on M .*

Proof. Choose an arbitrary Riemannian metric g on M .

Since ω is non-degenerate, there exists a unique bundle endomorphism

$$A : TM \longrightarrow TM$$

such that

$$\omega(u, v) = g(Au, v)$$

for all tangent vectors u, v .

Because ω is skew-symmetric, the endomorphism A is skew-adjoint with respect to g as follows.

$$\begin{aligned} g(Au, v) &= \omega(u, v) \\ &= -\omega(v, u) \\ &= -g(Av, u) \\ &= -g(u, Av). \end{aligned}$$

Hence, $-A^2$ is self-adjoint and positive definite. The positive square root

$$(-A^2)^{1/2}$$

is well-defined. Define,

$$J = A(-A^2)^{-1/2}.$$

Since A commutes with every smooth function of A^2 , we obtain

$$\begin{aligned} J^2 &= A^2(-A^2)^{-1} \\ &= -\text{id}. \end{aligned}$$

Thus, J is an almost complex structure.

Moreover, one checks that J preserves ω and that

$$\omega(v, Jv) > 0$$

for every nonzero tangent vector v . Hence, J is ω -compatible. $\square$

The preceding construction shows that compatible almost complex structures exist on every symplectic manifold. Moreover, there is no canonical choice in general.

Theorem 80. *Let (M, ω) be a symplectic manifold. The space*

$$\mathcal{J}(M, \omega)$$

of ω -compatible almost complex structures is nonempty and contractible.

We will not prove the contractibility statement here. In Floer theory, the almost complex structure is typically chosen as an auxiliary geometric datum. Although the moduli spaces used to define Floer operations depend on this choice, the resulting Floer-theoretic invariants are constructed so as to be independent of the compatible almost complex structure, up to the appropriate notion of equivalence. This is a consequence of Theorem 80.

For the specific twisted cotangent bundle considered later in this thesis, an explicit compatible almost complex structure will be chosen rather than using the abstract existence theorem. This allows the associated metric to be computed directly and its geometric properties to be used in the analysis of pseudo-holomorphic curves.

1.2.5 Almost Complex Structures on Cotangent Bundles

Cotangent bundles provide a natural setting in which compatible almost complex structures can often be described explicitly. Let Q be a smooth manifold and consider T^*Q with its canonical symplectic form

$$\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i.$$

In local coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$, the standard almost complex structure is defined by

$$\begin{aligned} J_0 \left(\frac{\partial}{\partial x_i} \right) &= \frac{\partial}{\partial y_i}, \\ J_0 \left(\frac{\partial}{\partial y_i} \right) &= -\frac{\partial}{\partial x_i}. \end{aligned}$$

It satisfies

$$J_0^2 = -\text{id}$$

and is compatible with ω_0 . The associated metric is the standard Euclidean metric in these coordinates.

More generally, if the symplectic form is deformed to

$$\omega_\theta = \omega_0 + \pi^* \theta,$$

where θ is a closed two-form on the base, the standard almost complex structure need not remain compatible with the new symplectic form. In this case, we choose an almost complex structure adapted to ω_θ .

In the context of this thesis, the cotangent bundle T^*T^2 is equipped with the twisted

symplectic form

$$\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + h dx_1 \wedge dx_2.$$

An explicit ω -compatible almost complex structure will be constructed later and used to define the associated metric and analyze the relevant pseudo-holomorphic curves.

1.2.6 Pseudo-Holomorphic Curves

Let (Σ, j) be a Riemann surface and let (M, J) be an almost complex manifold. A smooth map

$$u : \Sigma \longrightarrow M$$

is called *J-holomorphic* if

$$du \circ j = J \circ du.$$

When Σ is locally identified with an open subset of $\mathbb{C}$ with coordinates (s, t) and standard complex structure j , the J -holomorphic equation takes the form

$$\frac{\partial u}{\partial s} + J(u) \frac{\partial u}{\partial t} = 0. \tag{1.35}$$

If J is compatible with a symplectic form ω , then the associated Riemannian metric g_J can be used to define the energy of a map. For a smooth map $u : \Sigma \rightarrow M$, its energy is given locally by

$$E(u) = \frac{1}{2} \int_{\Sigma} |du|_{g_J}^2. \tag{1.36}$$

For J -holomorphic maps, the energy is closely related to the symplectic area

$$\int_{\Sigma} u^* \omega.$$

In Lagrangian Floer theory one studies J -holomorphic strips with boundary conditions on Lagrangian submanifolds. If $L_0, L_1 \subseteq M$ are Lagrangian and

$$u : \mathbb{R} \times [0, 1] \longrightarrow M,$$

then one imposes

$$u(s, 0) \in L_0,$$

$$u(s, 1) \in L_1.$$

More generally, in the presence of a Hamiltonian perturbation H_t , the relevant equation is the Floer equation

$$\frac{\partial u}{\partial s} + J_t(u) \left(\frac{\partial u}{\partial t} - X_{H_t}(u) \right) = 0. \quad (1.37)$$

Solutions are required to converge at the strip-like ends to Hamiltonian chords. These maps form the moduli spaces used to define the differential and higher products in Lagrangian Floer theory. In the later chapters of this thesis, pseudo-holomorphic strips and polygons with boundary on a cotangent fiber will be used to study the Floer differential and product on the twisted cotangent bundle T^*T^2 . Pseudo-holomorphic curves with different boundary conditions define $\mathcal{A}_{\infty}$ structure on the Lagrangian Floer complex.

1.3 Floer Cohomology

1.3.1 Pseudo-Holomorphic Strips with Lagrangian Boundary Conditions

Let (M, ω) be a symplectic manifold, let J_t be a smooth time-dependent family of ω -compatible almost complex structures, and let $L_0, L_1 \subseteq M$ be Lagrangian submanifolds. Floer theory studies maps from the infinite strip

$$\Sigma = \mathbb{R} \times [0, 1]$$

to M satisfying a perturbed Cauchy–Riemann equation together with boundary conditions on L_0 and L_1 .

Let

$$H : [0, 1] \times M \longrightarrow \mathbb{R}$$

be a smooth time-dependent Hamiltonian, and let X_{H_t} be defined by

$$\iota_{X_{H_t}} \omega = dH_t.$$

A Hamiltonian chord from L_0 to L_1 is a path $x : [0, 1] \rightarrow M$ satisfying

$$\dot{x}(t) = X_{H_t}(x(t)),$$

$$x(0) \in L_0,$$

$$x(1) \in L_1.$$

We denote the set of Hamiltonian chords by

$$\mathcal{X}_H(L_0, L_1).$$

Definition 81. Let $x_-, x_+ \in \mathcal{X}_H(L_0, L_1)$. A *Floer strip* from x_- to x_+ is a smooth map

$$u : \mathbb{R} \times [0, 1] \longrightarrow M$$

satisfying the Floer equation Eq. (1.37), the Lagrangian boundary conditions

$$u(s, 0) \in L_0,$$

$$u(s, 1) \in L_1,$$

and the asymptotic conditions

$$\lim_{s \rightarrow -\infty} u(s, t) = x_-(t),$$

$$\lim_{s \rightarrow +\infty} u(s, t) = x_+(t).$$

The asymptotic convergence is usually required to hold uniformly in t and, under the non-degeneracy assumptions used in Floer theory, is exponential in the s -variable. This guarantees that the energy of a Floer strip is finite and that the asymptotic Hamiltonian chords are well defined. When the Hamiltonian perturbation vanishes, the Floer equation reduces to the ordinary J_t -holomorphic strip equation; Eq. (1.35). The Eq. (1.37) is invariant under translation in the s -variable. If $u(s, t)$ is a solution, then for every $c \in \mathbb{R}$ the map

$$u_c(s, t) = u(s + c, t)$$

is also a solution with the same asymptotic limits. This produces a natural $\mathbb{R}$ -action on the

space of nonconstant Floer strips.

Definition 82. The moduli space of Floer strips from x_- to x_+ is denoted by

$$\mathcal{M}(x_-, x_+; H, J).$$

The quotient by translation in the s -variable is denoted by

$$\widehat{\mathcal{M}}(x_-, x_+; H, J) = \mathcal{M}(x_-, x_+; H, J)/\mathbb{R}.$$

The solutions of the Floer equation interpolate between Hamiltonian chords. In the unperturbed case, they interpolate between intersection points of L_0 and L_1 . In the perturbed case, the correspondence

$$\mathcal{X}_H(L_0, L_1) \cong \varphi_H^1(L_0) \cap L_1 \tag{1.38}$$

shows that Floer strips may still be regarded as connecting intersection points after a Hamiltonian deformation of one Lagrangian.

The Lagrangian boundary conditions are essential to the elliptic nature of the problem. At each boundary component, the tangent space of the image is constrained to lie in a Lagrangian subspace:

$$du(T(\mathbb{R} \times \{0\})) \subseteq TL_0,$$

$$du(T(\mathbb{R} \times \{1\})) \subseteq TL_1.$$

These are totally real boundary conditions with respect to an ω -compatible almost complex structure, and they provide the natural boundary-value problem for the Cauchy–Riemann operator.

The linearization of the Floer equation at a solution u is a first-order Cauchy–Riemann

type operator

$$D_u : W^{1,p}(u^*TM; TL_0, TL_1) \longrightarrow L^p(\Omega^{0,1}(u^*TM)).$$

Under suitable hypotheses, this operator is Fredholm. Its Fredholm index determines the expected dimension of the moduli space $\mathcal{M}(x_-, x_+; H, J)$. The expected dimension is governed by a relative Maslov index. With an appropriate grading convention, one has

$$\dim \mathcal{M}(x_-, x_+; H, J) = |x_-| - |x_+|,$$

while the quotient by the translation action has expected dimension

$$\dim \widehat{\mathcal{M}}(x_-, x_+; H, J) = |x_-| - |x_+| - 1.$$

Rigid Floer strips are those for which,

$$|x_-| - |x_+| = 1.$$

These isolated solutions are counted in the definition of the Floer differential. In the main setting of this thesis, one takes

$$L_0 = L_1 = T_q^*T^2.$$

The asymptotic conditions are Hamiltonian chords

$$\gamma_{ij}, \gamma_{kl},$$

indexed by elements of $\pi_1(T^2, q) \cong \mathbb{Z}^2$. The corresponding Floer strips form moduli spaces whose zero-dimensional components determine the coefficients of the Floer differential. Later,

analogous pseudo-holomorphic polygons with three or more strip-like ends will be used to define the product and the higher A_∞ operations.

Chapter 2

Computations on cotangent bundle of torus

2.1 Moduli Spaces on $T^*\mathbb{T}^2$

2.1.1 Notations

Note that the majority of the discussion in this section is independent of the fact that the symplectic form we consider on the cotangent bundle of the torus is non-exact. Consider $L = T_q^*$, a Lagrangian submanifold in $(T^*\mathbb{T}^2, \omega)$. We sometimes use $M = T^*\mathbb{T}^2$. As mentioned above, $\omega = \omega_0 + h\pi^*\theta$, where h is a real parameter, ω_0 is the tautological symplectic form on M and π is the bundle map of the cotangent bundle.

Let x_i, y_i be coordinate functions on M . Here, x_i are base coordinates and y_i are fiber coordinates. In these coordinates, the form ω can be explicitly written as follows.

$$\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + hdx_1 \wedge dx_2$$

As suggested in¹ we pick $H = y_1^2 + y_2^2$ as the Hamiltonian. We start by noting that a cotangent fiber is a Lagrangian submanifold with symplectic form ω . Let $\Psi^H : M \times [0, 1] \rightarrow$

M be the Hamiltonian diffeomorphism generated by H .

Let $\chi(L, L)$ denote the set of Hamiltonian curves $\gamma : I \rightarrow M$ that satisfy $\gamma(0), \gamma(1) \in L$. It is useful to note that $\chi(L, L)$ has a bijective correspondence with the set $L \cap \Psi_H^1(L)$. L is a fiber of the cotangent bundle and therefore carries the structure of a vector space. We make the identification $\mathbb{T}^2 \simeq \mathbb{R}^2/\Lambda$. We have the following lemma for $L \cap \Psi_H^1(L)$ which we prove later in section 2.2.1.

Lemma 83. *There exists an abelian group isomorphism $\Phi : \Lambda \rightarrow L \cap \Psi_H^1(L)$.*

We denote $(i, j) = \Phi(iv_0 + jv_1)$ where v_0, v_1 are generators of the lattice Λ . We use the notation $\gamma_{ij}^H : [0, 1] \rightarrow T^*\mathbb{T}^2$ for the integral flow of the Hamiltonian such that $\gamma_{ij}^H(0) \in L$ and $\gamma_{ij}^H(1) = (i, j) \in \Psi_H^1(L) \cap L$. The curves γ_{ij} and the points (i, j) can be interpreted as generators of $CF(L, L)$.

The Lagrangian Floer complex $(CF(L, L), \mu_1)$ is defined as follows:

$$CF(L, L) = \bigoplus_{i,j} \mathbb{Z}\gamma_{ij}^H$$

$$\mu_1(\gamma_{ij}^H) = \sum_{k,l} \mathcal{M}(\gamma_{ij}^H, \gamma_{kl}^H, [u], \omega) \gamma_{kl}^H$$

where u is a rigid pseudo-holomorphic strip satisfying certain boundary conditions and $[u]$ is the corresponding homotopy class in $\pi_2(M, L \cup \gamma_{ij}^H \cup \gamma_{kl}^H)$. If π is the bundle map from $T^*\mathbb{T}^2$ to the base, then $\pi_*(\gamma_{kl}^H)$ is a loop in the base space and hence an element of $\Omega_q\mathbb{T}^2$. We denote by $\mathcal{M}(\gamma_{ij}^H, \gamma_{kl}^H, [u], \omega)$ the moduli space of such pseudo-holomorphic strips. Furthermore, for each (i, j) the paths $\pi(\gamma_{ij}^H)$ lie in different connected components of the loop space. We have the following result on moduli spaces with three asymptotic boundary conditions. The number of elements in this space form the coefficients of μ_2 .

Lemma 84. *The moduli space of pseudo-holomorphic strips with asymptotic boundary con-*

ditions to be $\gamma_{ij}, \gamma_{kl}, \gamma_{pq}$ denoted as $\mathcal{M}_{ij,kl}^{pq}$ satisfies the following:

$$\mathcal{M}_{ij,kl}^{pq} = \phi \quad \text{if } (p, q) \neq (i + j, k + l)$$

$$\mathcal{M}_{ij,kl}^{pq} = \{*\} \quad \text{if } (p, q) = (i + j, k + l)$$

The proof of lemma 84 requires an analysis of the energy functional E , whose critical points are pseudo-holomorphic strips. We discuss this in the section on the energy functional (2.3).

2.1.2 Compatible structures

In this sub-section, we define compatible structures on (M, ω) . We conclude this section by showing that (M, ω) admits a compatible local flat metric. Let $(M, \omega) = (T^*\mathbb{T}^2, \omega)$ be as above and (x_1, x_2, y_1, y_2) be the coordinate system on M induced by realizing M as a quotient of $T^*\mathbb{R}^2$. The coordinates y_i are the fiber coordinates and x_i are the base coordinates.

Proposition 85. *Given $x \in M$, define $J_x \in \text{End}(T_x M)$ as follows:*

$$\begin{aligned} \partial_{x_1} &\rightarrow \partial_{y_1} \\ \partial_{x_2} &\rightarrow \partial_{y_2} - h\partial_{x_1} \\ \partial_{y_1} &\rightarrow -\partial_{x_1} \\ \partial_{y_2} &\rightarrow -\partial_{x_2} + h\partial_{y_1} \end{aligned} \tag{2.1}$$

Then J is an almost-complex ω -compatible structure such that $\omega(v, Jv) > 0$.

Proof. We use $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + hdx_1 \wedge dx_2$ in local coordinates as above. We proceed to show that $\omega(Ju, Jv) = \omega(u, v)$. Note that it is sufficient to show that the above

equation is true when u, v are elements of the basis.

$$\begin{aligned}
J_x^* \omega(\partial_{x_1}, \partial_{x_2}) &= \omega(\partial_{y_1}, \partial_{y_2}) - h\omega(\partial_{y_1}, \partial_{x_1}) = h = \omega(\partial_{x_1}, \partial_{x_2}) \\
J_x^* \omega(\partial_{x_1}, \partial_{y_1}) &= 1 = \omega(\partial_{x_1}, \partial_{y_1}) \\
J_x^* \omega(\partial_{x_1}, \partial_{y_2}) &= \omega(\partial_{y_1}, -\partial_{x_2}) + h\omega(\partial_{y_1}, \partial_{y_1}) = 0 = \omega(\partial_{x_1}, \partial_{y_2}) \\
J_x^* \omega(\partial_{x_2}, \partial_{y_1}) &= \omega(\partial_{y_2}, -\partial_{x_1}) - h\omega(\partial_{x_1}, -\partial_{x_1}) = 0 = \omega(\partial_{x_2}, \partial_{y_1}) \\
J_x^* \omega(\partial_{y_1}, \partial_{y_2}) &= \omega(-\partial_{x_1}, -\partial_{x_2}) - h\omega(\partial_{x_1}, \partial_{y_1}) = 0 = \omega(\partial_{y_1}, \partial_{y_2}) \\
\\
J_x^* \omega(\partial_{x_2}, \partial_{y_2}) &= \omega(\partial_{y_2}, -\partial_{x_2}) - h\omega(\partial_{x_1}, -\partial_{x_2}) - h^2\omega(\partial_{x_1}, \partial_{y_1}) \\
&= 1 + h^2 - h^2 = 1 = \omega(\partial_{x_2}, \partial_{y_2})
\end{aligned} \tag{2.2}$$

Now we show that $\omega(v, Jv) > 0$ as follows.

$$\begin{aligned}
\omega(\partial_{x_1}, J\partial_{x_1}) &= \omega(\partial_{x_1}, \partial_{y_1}) = 1 \\
\omega(\partial_{x_2}, J\partial_{x_2}) &= \omega(\partial_{x_2}, \partial_{y_2}) + h\omega(\partial_{x_1}, \partial_{x_2}) = 1 + h^2 \\
\omega(\partial_{y_1}, J\partial_{y_1}) &= \omega(\partial_{y_1}, -\partial_{x_1}) = 1 \\
\omega(\partial_{y_2}, J\partial_{y_2}) &= \omega(\partial_{y_2}, -\partial_{x_2}) + h\omega(\partial_{y_2}, \partial_{y_1}) = 1
\end{aligned} \tag{2.4}$$

The above statement can then be linearly extended for any v . This finishes the proof. $\square$

Given J as above, let g_J denote the (ω, J) compatible metric on M . We have the following proposition.

Proposition 86. *The metric g_J on (M, ω) is locally flat.*

Proof. Using Proposition 2.1, we obtain

$$g_J = hdx_2 \otimes dy_1 + dx_1 \otimes dx_1 + (h^2 + 1)dx_2 \otimes dx_2 + dy_1 \otimes dy_2 + dy_2 \otimes dy_2$$

Let $\{\partial_1, \partial_2, \partial_3, \partial_4\}$ denote the orthonormal frame $\{\partial_{x_1}, \partial_{x_2} - h\partial_{y_1}, \partial_{y_1}, \partial_{y_2}\}$. Note the following

properties of the frame.

$$g_J(\partial_i, \partial_j) = \text{constant}$$

$$[\partial_i, \partial_j] = 0$$

Let ∇ denote the Levi-Civita connection of the metric g_J . Then ∇ is given by the following equation ⁽³⁾.

$$2\langle \nabla_X Y, Z \rangle = X\langle Y, Z \rangle + Y\langle Z, X \rangle - Z\langle X, Y \rangle - \langle X, [Y, Z] \rangle + \langle Y, [Z, X] \rangle + \langle Z, [X, Y] \rangle$$

Let Γ_k^{ij} denote the Christoffel symbols of the connection ∇ with respect to the frame $\{\partial_i\}_i$. Then $\Gamma_{ij}^k = \langle \nabla_{\partial_i} \partial_j, \partial_k \rangle$. Using the formula above for the Levi-Civita connection, we see that each individual term is 0 as $g_J(\partial_i, \partial_j) = \text{constant}$ and $[\partial_i, \partial_j] = 0$. This completes the proof. $\square$

2.1.3 Energy functional

Let J be the ω -compatible almost complex structure on (M, ω) as defined above, and let L be the cotangent fiber. Let Σ be the unit disk in $\mathbb{C}$ with three punctured points on the boundary z_i for $i = 1, 2, 3$. Denote by $Z^\pm = \mathbb{R}^\pm \times [0, 1]$. Equip Σ with three strip-like ends $e_i : Z^\pm \rightarrow D_3$. Here, e_1, e_2 are treated as incoming ends and e_3 as an outgoing end. Let $u : \Sigma \rightarrow M$ be a smooth map with the following boundary conditions:

$$\begin{aligned} u|_{\partial\Sigma} &\in L \\ ue_i|_{\partial Z^\pm} &\in L \\ \lim_{s \rightarrow -\infty} u(e_1, \cdot) &= \gamma_{ij}(\cdot) \\ \lim_{s \rightarrow -\infty} u(e_2, \cdot) &= \gamma_{kl}(\cdot) \\ \lim_{s \rightarrow \infty} u(e_3, \cdot) &= \gamma_{i+k, j+l}(\cdot) \end{aligned} \tag{2.5}$$

Note that the convergence at $\pm\infty$ is required to be of exponential speed. This ensures that the energy functional is well-defined. Let $\mathfrak{F}^\infty(\Sigma, M)$ be the space of smooth maps for which E is well-defined. The energy of u , $E(u)$ is defined as follows.

$$E(u) = \frac{1}{2} \int_{\mathbb{R} \times [0,1]} (|\partial_s u|^2 + |\partial_t u - X_t(u)|^2) ds dt$$

Here, $||$ denotes the pairing under the metric g_J . We now focus on our case. The above integral defines $E : \mathfrak{F}^\infty(\Sigma, M) \rightarrow \mathbb{R}$. A J -holomorphic strip is a critical point of the energy functional E . It minimizes the energy functional E .

Suppose ∇ is Levi-Civita on M . Let ∇_u be the induced connection on the bundle u^*TM . A vector field along u is a section of $u^*(TM)$. A vector field v along u defines a variation of u by $u_t(x) = \exp_{u(x)}(tv(x))$. This can be interpreted as a deformation of u along the vector field v . Given two vector fields v, w along u , we choose a variation $u_{s,t}$ such that

$$\begin{aligned} v &= \left. \frac{\partial u_{s,t}}{\partial s} \right|_{(s,t)=0} \\ w &= \left. \frac{\partial u_{s,t}}{\partial t} \right|_{(s,t)=0} \end{aligned} \tag{2.6}$$

The Hessian of u is then defined as the symmetric bilinear form given by

$$H_u(v, w) = \left. \frac{\partial^2 E(u_{s,t})}{\partial s \partial t} \right|_{(s,t)=0}$$

Note the following result from⁴ about the Hessian (H_u) of E .

Lemma 87. ⁴ *If (M, g) and (N, h) are two Riemannian manifolds and $u : M \rightarrow N$ is a smooth map and R^N is the curvature tensor of N , then*

$$H_u(v, w) = \int_M [\langle \nabla_u v, \nabla_u w \rangle - \langle \text{Trace} R^N(du, v) du, w \rangle] dx$$

We now provide a proof of the lemma 84.

Proof. Assume that $\mathcal{M}_{ij,kl}^{pq} \neq \emptyset$. Let $M = \Sigma$ and $N = \mathbb{T}^*\mathbb{T}^2$ in lemma 87. Since $(\mathbb{T}^*\mathbb{T}^2, g_J)$ is a flat manifold, $R^N = 0$. We have the following for all maps $u \in \mathfrak{F}^\infty(\Sigma, M)$.

$$H_u(v, v) = \int_{\Sigma} |\nabla_u v|^2 dx \geq 0$$

The energy functional E is positive semi-definite at all points u in $\mathfrak{F}^\infty(\Sigma, M)$, implying the uniqueness of minimizing u . This shows that there is a unique element in $\mathcal{M}_{ij,kl}^{pq}$ when the moduli-space is not empty. $\square$

2.2 Calculations on $\mathbb{T}^*\mathbb{T}^2$ with a deformed symplectic form

As is evident from the work of⁵, deforming the cohomology class of the symplectic form corresponds to deforming the derived category of coherent modules from a commutative to a non-commutative algebra on the mirror side. The non-commutativity is then determined by the deformed symplectic form. Here, we obtain a result in a similar vein for the algebra $CF(\mathbb{T}_q^*\mathbb{T}^2, \mathbb{T}_q^*\mathbb{T}^2)$. As has been shown earlier and elsewhere in the literature, the above is the commutative ring of Laurent polynomials in two variables if the symplectic form is exact. We show that with a perturbed symplectic form the ring is a non-commutative deformation determined by the symplectic form on the base used to perturb. We start by picking

$$\theta = \begin{pmatrix} 0 & h \\ -h & 0 \end{pmatrix}$$

where $h \in \mathbb{R}$. The above matrix represents the symplectic form $\theta = h dx_1 \wedge dx_2$ and x_i are co-ordinate maps on the 2-torus.

2.2.1 Hamiltonian flow lines

In this section we will make a more general computation for the flow on $T^*\mathbb{T}^n$. In this section (M, ω) represents the symplectic manifold $(T^*\mathbb{T}^n, \omega_0 + \pi^*\theta)$, where

$$\theta = \sum_{i < j} h_{ij} dx_i \wedge dx_j$$

and M has local coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$ where x_i are the base coordinates and y_i are fiber coordinates. Computing the Hamiltonian vector field X_H associated with H , one obtains

$$X_H = \sum_i y_i \partial_{x_i} + \sum_{i > k} h_{ki} y_k \partial_{y_k} - \sum_{i < k} h_{ki} y_k \partial_{y_k}$$

We make the identification of $(T^*\mathbb{T}^n, \omega)$ with $(T^*\mathbb{R}^n/\Lambda, \omega)$. Let $\tilde{H}$ denote the lift of H to $T^*\mathbb{R}^n$. Any variable with tilde on it, refers to the pullback of the corresponding variable on $T^*\mathbb{R}^n$, in this subsection. With this in mind, we note that $\tilde{\theta} \in \Lambda^2(\mathbb{R}^n)$. There is a canonical identification $\mathfrak{o}(n) = T_e(O(n)) \simeq \Lambda^2(\mathbb{R}^n)$. The Hamiltonian vector field $\tilde{X}_H$ on $T^*\mathbb{R}^n$ has flow lines as the solution of the following system of equations:

$$\begin{aligned} \tilde{y}'(t) &= -[\theta]\tilde{y}(t) \\ \tilde{x}'(t) &= \tilde{y}(t) \end{aligned} \tag{2.7}$$

Here, we think of $[\theta]$ as the matrix in $\mathfrak{o}(n)$ corresponding to $\theta \in \Lambda^2(n)$. With the identification $\mathfrak{o}(n) \simeq \Lambda^2(\mathbb{R}^n)$, we can think of $\tilde{\theta} \in \mathfrak{o}(n)$. The first equation in equation 2.7 therefore implies that the solution in the $\tilde{y}$ -coordinates is simply the one-parameter subgroup of θ in $O(n)$. Thus, we have the following lemma.

Lemma 88. *The Hamiltonian flow lines of $\tilde{H}$ on $(T^*\mathbb{R}^n, \tilde{\omega}_0 + \tilde{\theta})$ with coordinates $(\tilde{x}_1, \dots, \tilde{x}_n, \tilde{y}_1, \dots, \tilde{y}_n)$ at the point $(\tilde{x}_0, \tilde{y}_0)$ is given as follows. Here, $\exp_{\tilde{\theta}}(t)$ denotes the exponential of $\tilde{\theta}$ in $O(n)$,*

where $\tilde{\theta}$ is viewed as an element in $T(O(n))$.

$$\begin{aligned}\gamma_{\tilde{H}}(T) &= (x(T), y(T)) \\ y(T) &= \exp_{-\tilde{\theta}}(T).y_0 \\ x(T) &= \int_0^T \exp_{-\tilde{\theta}}(t).y_0 dt + x_0\end{aligned}\tag{2.8}$$

Proof. We start by noting that $T^*\mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}^n$. Suppose that the general solution is $\gamma_{\tilde{H}}(T) = (x(T), y(T))$. Equation 2.7 can be viewed as a differential equation in $\{0\} \times \mathbb{R}^n$

$$\tilde{y}'(t) = -[\theta]\tilde{y}(t).$$

Hence, $\tilde{y}$ is precisely the integral flow at time t of the vector field $[\theta] \in \mathfrak{o}(n)$. Since $\tilde{y}(0) = y_0$, we have the following.

$$y(T) = \exp_{-\tilde{\theta}}(T).y_0\tag{2.9}$$

Note that the solution in equation 2.9 is computable as the exponential of the matrix $[\theta]$. Since $\tilde{x}' = \tilde{y}$, we can compute $x(T)$ as follows.

$$x(T) = \int_0^T \exp_{-\tilde{\theta}}(t).y_0 dt + C$$

Since $x(0) = x_0$, we have the following.

$$x(T) = \int_0^T \exp_{-\tilde{\theta}}(t).y_0 dt + x_0$$

□

Applying the lemma 88 to the case $n = 2$, we obtain the following result.

Lemma 89. *Let $\tilde{\Psi}_{\tilde{H}}^t : T^*\mathbb{R}^2 \rightarrow T^*\mathbb{R}^2$ be the Hamiltonian diffeomorphism at time t . Then,*

we have:

$$\begin{aligned}\tilde{\Psi}_{\tilde{H}}^t(q_1, q_2, p_1, p_2) = & \left(p_1 \frac{\sin(ht)}{h} + p_2 \frac{\cos(ht) - 1}{h} + q_1, \right. \\ & - p_1 \frac{\cos(ht) - 1}{h} + p_2 \frac{\sin(ht)}{h} + q_2, \\ & p_1 \cos(ht) - p_2 \sin(ht), \\ & \left. p_1 \sin(ht) + p_2 \cos(ht) \right)\end{aligned}$$

Proof. Let (x_1, x_2, y_1, y_2) be the coordinate functions of $T^*\mathbb{R}^2 \simeq \mathbb{R}^2 \times \mathbb{R}^2$. Equation 2.8 reduces to the following when $n = 2$.

$$\begin{bmatrix} y_1(T) \\ y_2(T) \end{bmatrix} = \exp \begin{pmatrix} 0 & -hT \\ hT & 0 \end{pmatrix} \cdot \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \quad (2.10)$$

We use the following known fact.

$$\exp \begin{pmatrix} 0 & -ht \\ ht & 0 \end{pmatrix} = \begin{bmatrix} \cos(ht) & -\sin(ht) \\ \sin(ht) & \cos(ht) \end{bmatrix} \quad (2.11)$$

From equation 2.11 we obtain the following.

$$y_1(T) = p_1 \cos(hT) - p_2 \sin(hT)$$

$$y_2(T) = p_1 \sin(hT) + p_2 \cos(hT)$$

The above equations give us the following.

$$x_1(T) = \int_0^T y_1(t) dt + q_1$$

$$x_2(T) = \int_0^T y_2(t) dt + q_2$$

The above computation gives us the following.

$$\begin{aligned} x_1(T) &= p_1 \frac{\sin(hT)}{h} + p_2 \frac{\cos(hT) - 1}{h} + q_1 \\ x_2(T) &= -p_1 \frac{\cos(hT) - 1}{h} + p_2 \frac{\sin(hT)}{h} + q_2 \end{aligned}$$

Since $\tilde{\Psi}_H^t(q_1, q_2, p_1, p_2) = (x_1(t), x_2(t), y_1(t), y_2(t))$, the above computations complete the proof. $\square$

We now have all the ingredients to prove the lemma 83.

Proof. (Lemma 83) As before, let L denote the Lagrangian subspace $T_q^*\mathbb{T}^2$ of $(T^*\mathbb{T}^2, \omega)$. Let $\pi : T^*\mathbb{R}^2 \rightarrow \mathbb{R}^2$ denote the bundle map. Without loss of generality, we can assume that the pullback of q to $\mathbb{R}^2$ is the lattice Λ . We have the following.

$$p \in \Psi_H^1(L) \cap L \iff \pi(\tilde{\Psi}_H^1(0, 0, p_1, p_2)) \in \Lambda \quad (2.12)$$

Lemma 89 gives us the following.

$$p \in \Psi_H^1(L) \cap L \iff \begin{bmatrix} \frac{\sin(ht)}{h} & \frac{\cos(ht)-1}{h} \\ \frac{1-\cos(ht)}{h} & \frac{\sin(ht)}{h} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \in \Lambda \quad (2.13)$$

Let $\phi_{t,h}$ denote the following linear map.

$$\phi_{t,h} = \begin{bmatrix} \frac{\sin(ht)}{h} & \frac{\cos(ht)-1}{h} \\ \frac{1-\cos(ht)}{h} & \frac{\sin(ht)}{h} \end{bmatrix} \quad (2.14)$$

Note that $\phi_{t,h}$ is invertible for generic values of h . The above statements give us the following conclusion.

$$(q, p_1, p_2) \in \Psi_H^t(L) \cap L \iff \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \in \phi_{t,h}^{-1}(\Lambda) \quad (2.15)$$

Thus, the following map is a group isomorphism.

$$\Phi : \Lambda \rightarrow L \cap \Psi_H^t(L)$$

$$v \rightarrow \phi_{t,h}^{-1}(v)$$

□

2.2.2 Grading on Floer complex

The Floer complex $CF(L, L)$ can be given a $\mathbb{Z}$ -grading, which turns the Floer complex into an $\mathcal{A}_\infty$ algebra. Any Lagrangian-Floer complex can be given a $\mathbb{Z}$ -grading if the following conditions are satisfied (see⁶):

1. The fiber bundle $\mathcal{L}Gr(M, \omega)$ with fibers $LGr(T_p M, \omega_p)$ that are Lagrangian Grassmannians of the vector space $(T_p M, \omega_p)$ has a fiber-wise $\mathbb{Z}$ -covering space $\widetilde{\mathcal{L}Gr}(M, \omega)$. Such a space exists iff the first Chern class $c_1(TM, J)$ is two torsion.
2. Given a Lagrangian embedding $i : L \rightarrow M$, there is a canonical section $\alpha_L : L \rightarrow \mathcal{L}Gr(M, \omega)$. This section must admit a lift to $\widetilde{\mathcal{L}Gr}(M, \omega)$. A choice of such a lift $(L, \tilde{\alpha}_L)$ is called a graded Lagrangian.

Note that the second condition will be satisfied since L is contractible. Hence $H^1(L) = 0$, implying that the pullback of the global Maslov class is 0.

Proposition 90. *With $(M, \omega) = (T^*\mathbb{T}^2, \omega_0 + \pi^*\theta)$, and J a compatible, almost complex structure $c_1(TM, J) = 0 \in H^2(M; \mathbb{Z})$ and therefore, $CF(L, L)$ admits a $\mathbb{Z}$ -grading.*

Proof.

□

The grading on $CF(L, L)$ is then chosen to be compatible with the Hamiltonian flow

restricted to Lagrangian L . We have the following map α .

$$\alpha : \Psi_H(L)|_{L \times I} \rightarrow \mathcal{LGr}(M, \omega)$$

$$\Psi_H(p, t) \rightarrow [d\Psi_H^t(T_p L)]$$

Given a lift of $\alpha|_{L \times \{0\}}$, a lift $\tilde{\alpha}$ to $\widetilde{\mathcal{LGr}(M, \omega)}$ can be chosen so that $\tilde{\alpha}$ agrees with the lift chosen of $\alpha|_{L \times \{0\}}$. Furthermore, if $\gamma_{ij}^H : I \rightarrow M$ is a Hamiltonian flow line such that the endpoints $\gamma_{ij}^H(0), \gamma_{ij}^H(1) \in L$, then the degree of the generator γ_{ij}^H is defined as the Maslov index ⁽⁷⁾ of the path $\alpha(\gamma_{ij}^H(t))$ in $\mathcal{LGr}(M, \omega)$.

The crossing form for a pair of paths (λ_0, λ_1) in $\mathcal{LGr}(V)$ is defined as follows ⁽⁸⁾. Choose families of linear maps $\phi_{k,r,s} : \lambda_k(s) \rightarrow \lambda_k(r)$ for $k = 0, 1$ and $|r - s|$ small, such that $\phi_{k,s,s} = Id$. The crossing form is the quadratic form on $\lambda_0(s) \cap \lambda_1(s)$ defined by

$$q_{\lambda_0, \lambda_1}(s) : v \rightarrow - \left(\frac{d}{dr} \right)_{r=s} \omega_V(\phi_{0,r,s}(v), \phi_{1,r,s}(v))$$

We need to choose a path in $\mathcal{P}^-Gr(V)$ (as mentioned in ⁽⁸⁾), which is the space of paths $\lambda : [0, 1] \rightarrow Gr(V)$ with the following properties:

1. $\lambda(0), \lambda(1)$ intersect transversely.
2. The pair $(\lambda, \lambda(1))$, where the second component denotes the constant path with value $\lambda(1)$, has negative definite crossing form at $s = 1$.

Given a generator γ_{ij} , we parallel transport the Lagrangian subspace $T_{\gamma_{ij}(0)}L$ to a Lagrangian subspace (with the help of a symplectic connection) in the fiber over $\gamma_{ij}(1)$ in $\mathcal{LGr}(M, \omega)$. Denote this Lagrangian subspace Λ_0 . Let $T_{\gamma_{ij}(1)}\Psi_H^1(L)$ be denoted as Λ_1 . Note that Λ_0 and Λ_1 are two elements of $\mathcal{LGr}(M, \omega)$ in the same fiber. Choose a path λ in the fiber such that $\lambda(k) = \Lambda_k$ for $k = 0, 1$. Denote the crossing form $q_\lambda = q_{\lambda, \lambda(1)}$. The degree of

the generator γ_{ij} is then given as;

$$\deg(\gamma_{ij}) = I(\lambda) = \sum_{s < 1} \text{sign}(q_\lambda(s))$$

Note, $d\Psi_H^1(T_{(0,0,p_1,p_2)}L)$ is given as $\{A(1)v \oplus v : v \in \Lambda_0\}$ and the matrix $A(t)$ is as follows:

$$A = \begin{pmatrix} \frac{\sin(ht)}{h} & \frac{\cos(ht)-1}{h} \\ \frac{\cos(ht)-1}{h} & \frac{\sin(ht)}{h} \end{pmatrix}$$

Thus, $\lambda(t) = \{A(t)v \oplus v\}$ is a path that joins Λ_0, Λ_1 . We need to check that λ is a path in $\mathcal{P}^-Gr(V_{ij})$ where V_{ij} is the fiber over $\gamma_{ij}(1)$ in (TM, ω) .

Proposition 91. *λ as defined above is a path in $\mathcal{P}^-Gr(V_{ij})$*

Proof. The crossing form at $s = 1$ is given as follows:

$$q_\lambda(1)(Aw \oplus w) = \omega_{\gamma_{ij}(1)}(A(1)w \oplus w, A'(1)w \oplus w)$$

Using the values of $A(1)$ and $A'(1)$ from above, it can be shown that $q_\lambda(1)$ is negative definite. $q_\lambda(1)$ is the quadratic form as defined above with $\lambda_0(t) = \lambda(t)$ and λ_1 is the constant path $\lambda_1(t) = \lambda_1$.

□

Proposition 92. *For h sufficiently small, $\deg(\gamma_{ij}) = 0$.*

Proof. For any t , $\lambda(t) \cap \lambda(1)$ is non-trivial iff the linear map $A(t) - A(1)$ has a non-trivial kernel. $\det(A(t) - A(1)) \neq 0$ when h is sufficiently small except when $t = 1$. Thus, the quadratic form q_λ is trivial.

$$I(\lambda) = \sum_{s < 1} \text{sign}(q_\lambda) = 0$$

□

Proposition 93. *The A_∞ algebra $CF(L, L)$ has $\mu_i = 0$ for $i \neq 2$.*

Proof. The degree of the generators γ_{ij} is 0. Hence, $CF(L, L)$ is an honest algebra. $\square$

2.2.3 Computing $\text{End}(\mathbf{T}_0^*\mathbb{T}^2, \mathbf{T}_0^*\mathbb{T}^2)$

In this section, we prove Proposition 2. For the following discussion, we use the notation $L = \mathbf{T}_0^*\mathbb{T}^2$. To compute the product map $\mu_2 : CF(L, L) \otimes CF(L, L) \rightarrow CF(L, L)$, we have to compute the symplectic area of a triangle as shown in the figure below.

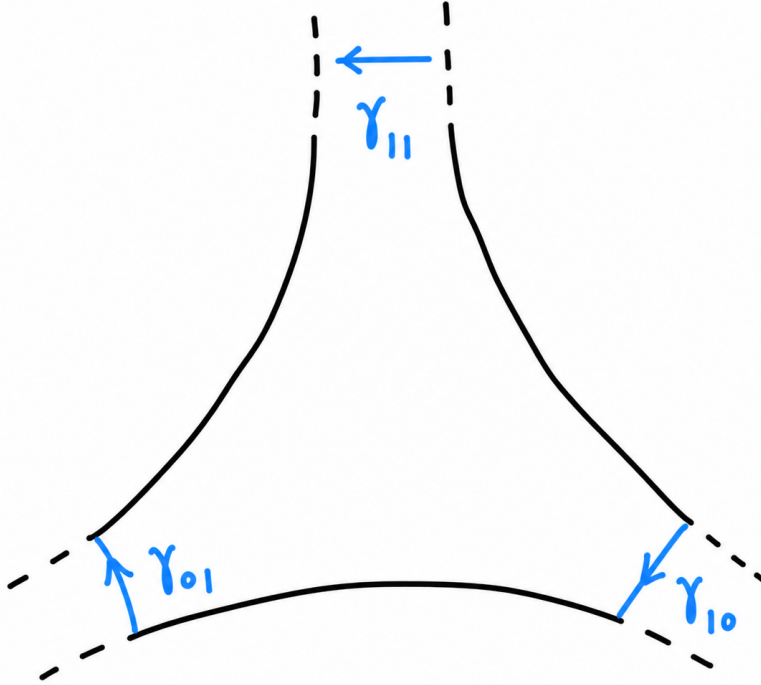


Figure 2.1: Product in the Fukaya category via counting pseudoholomorphic triangles.

Figure 2.1 depicts the triangle we need to compute the following product.

$$\mu_2 : CF(L, L) \otimes CF(L, L) \rightarrow CF(L, L)$$

$$\gamma_{01} \otimes \gamma_{10} \rightarrow \gamma_{11}$$

For a given J-holomorphic strip $u : D_3 \times [0, 1] \rightarrow \mathbf{T}_0^*\mathbb{T}^2$, we have the following commutative

diagram.

$$\begin{array}{ccc}
 & D_3 & \\
 \tilde{u} \swarrow & & \searrow u \\
 T^*\mathbb{R}^2 & \xrightarrow{\pi} & T^*\mathbb{T}^2
 \end{array}$$

Let $\pi^*(L) = \cup_{(i,j)} L_{ij}$, where $L_{ij} = T^*_{(iv_0+jv_1)}\mathbb{R}^2$. Since $T^*\mathbb{R}^2$ is contractible, the 2-form $\pi^*\omega$ is exact. Let $\alpha = -(\tilde{y}_1 d\tilde{x}_1 + \tilde{y}_2 d\tilde{x}_2 + h\tilde{x}_2 d\tilde{x}_1)$ be such that $d\alpha = \omega$. The points $\pi^*(i, j) \in L_{ij}$ and the curves $\tilde{\gamma}_{ij} : [0, 1] \rightarrow T^*\mathbb{R}^2$ that are the lifts of the curves γ_{ij} . (Note that there are $\mathbb{Z}^2$ worth of such lifts and we use the same notation for all the curves, and exactly which lift is being considered can be determined from the context.)

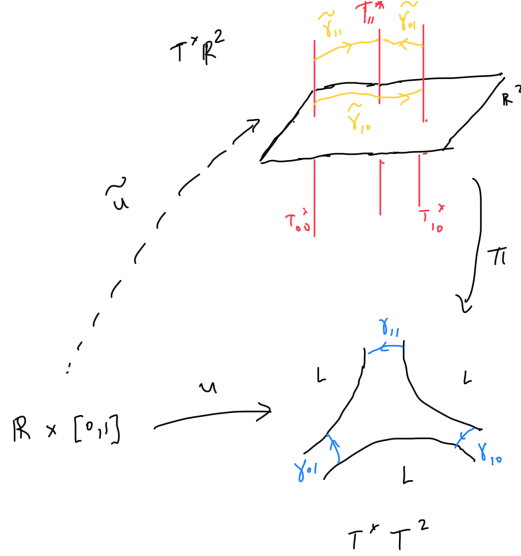


Figure 2.2: *Lifted triangle in $T^*\mathbb{R}^2$ to compute the product*

The Floer product $\mu_2((i, j), (k, l)) = T^{\int u^*\omega}(i + j, k + l)$ where u is an element in Moduli space.

The symplectic area can then be computed as

$$\begin{aligned}
\int u^* \omega &= \int \tilde{u}^* \pi^* \omega \\
&= \int \tilde{u}^* d\alpha \\
&= \int_{\tilde{\gamma}_{kl}} \alpha + \int_{\tilde{\gamma}_{k-i, l-j}} \alpha - \int_{\tilde{\gamma}_{kl}} \alpha + \int_{L_{00}} \alpha + \int_{L_{ij}} \alpha + \int_{L_{kl}} \alpha
\end{aligned}$$

The above integral further simplifies since $\alpha \upharpoonright_{L_{mn}} = 0$. Note the use of Stokes Theorem in the above computation. Using the known form for $\tilde{\Psi}_H^t$, one can compute the integral $\int_{\tilde{\gamma}_{mn}} \alpha$,

$$\begin{aligned}
\int_{\tilde{\gamma}_{mn}} \alpha &= \int_0^1 (\tilde{\Psi}_H^t)^* \alpha \\
&= i_{X_H}(\alpha)|_{\tilde{\gamma}_{mn}(0)}^{\tilde{\gamma}_{mn}(1)} \\
&= - \int_0^1 (\tilde{y}_1^2(\tilde{\Psi}_H^t) + \tilde{y}_2^2(\tilde{\Psi}_H^t) + h\tilde{x}_2(\tilde{\Psi}_H^t)\tilde{y}_1(\tilde{\Psi}_H^t))dt
\end{aligned}$$

where $\tilde{\gamma}_{mn}(t)$ for $t = 0, 1$ can be found simply by solving the following equations:

$$\tilde{\Psi}_H^0(q_1, q_2, p_1, p_2) \in L_{q_1 q_2}$$

$$\tilde{\Psi}_H^1(q_1, q_2, p_1, p_2) = (m, n, y_1, y_2) \in L_{mn}$$

Solving the above two equalities, we get the following solution.

$$\begin{aligned}
p_1 &= \frac{h(m - q_1) \sin(h) - (n - q_2)(\cos(h) - 1)}{2(1 - \cos(h))} \\
y_1 &= h \left(\frac{n - q_2}{2} + \frac{1}{2} \frac{(m - q_1) \sin(h)}{1 - \cos(h)} \right)
\end{aligned}$$

Note that the above solution is invariant under translation by elements of the lattice. Hence, in particular, it shows that the integral along a Hamiltonian flow line depends only on the homotopy class of the projection of γ_{mn} onto the base space. It also shows that there is a unique Hamiltonian flow loop in each homotopy class. In order to show that $(1, 0) \cdot (0, 1) =$

$h(0, 1).(1, 0)$, we need to show the equality below:

$$\int_{\tilde{\gamma}_{01}} \alpha + \int_{\tilde{\gamma}_{10}} \alpha + \int_{\tilde{\gamma}_{10}} \alpha + \int_{\tilde{\gamma}_{01}} \alpha = h$$

The above statement can be deduced from the figure above and the discussion of computing $\int_{\tilde{\gamma}_{mn}} \alpha$. The explicit result in lemma 88 completes the above computations.

2.3 Generalizations to higher genus

We denote Σ_g as a compact and closed surface of genus $g \geq 2$. We note a few facts about Σ_g . All such surfaces have $\mathbb{H}$ (the upper half-plane equipped with the Poincaré metric) as their universal cover. The second homotopy group $\pi_2(\Sigma_g) = 0$. The fundamental group has the following presentation:

$$\pi_1(\Sigma_g) = \langle a_1, b_1, \dots, a_g, b_g | \Pi_i [a_i, b_i] = 1 \rangle$$

The group of deck transformations is a discrete subgroup of $PSL(2, \mathbb{R})$, which is the group of isometries on $\mathbb{H}^2$. Let θ_0 be the unit volume form of Σ_g . Let ω_0 be the tautological form on $T^*\Sigma_g$. We denote by (M, ω) the symplectic manifold $(T^*\Sigma_g, \omega_0 + h\pi^*\theta_0)$, where π is the bundle map of the cotangent bundle $T^*\Sigma_g$. Suppose L is the cotangent fiber $T_q^*\Sigma_g$ which is a lagrangian submanifold of Σ_g . We have the following conjecture.

Proposition 94. *The lagrangian intersection complex $CF(L, L)$ is well defined for sufficiently small h and is A_∞ isomorphic to the following algebra:*

$$\mathbb{C}\langle A_1, B_1, \dots, A_g, B_g | \Pi_i [A_i, B_i] = T^h \rangle$$

Before we outline a sketch for the proof of Proposition 94, we discuss more about the Hamiltonian flow lines in $\mathbb{H}^2$.

2.3.1 Hamiltonian flow in upper half plane

Let p be the covering map $T^*\mathbb{H}^2$ to $T^*\Sigma_g$. Note that the group of deck transformations of the above covering map is given by $\pi_1(\Sigma_g)$. In this section, we denote by (N, α) the symplectic manifold $(T^*\mathbb{H}^2, \pi^*\omega)$. We pick the following Hamiltonian $H : T^*\mathbb{H}^2 \rightarrow \mathbb{R}$.

$$H(x, y, p, q) = \frac{1}{2}y^2(p^2 + q^2)$$

Denote by X_H the Hamiltonian vector field defined by equation $i_{X_H}\alpha = dH$. Let (x, y, p, q) be the local coordinates of N , where x, y are base coordinate functions and p, q are fiber coordinate functions. Let $\gamma_H : I \rightarrow T^*\mathbb{H}^2$ be an integral curve of X_H originating at (x_0, y_0, p_0, q_0) . We have the following lemma.

Lemma 95. *The Hamiltonian flow lines $\gamma_H(t) = (x(t), y(t), p(t), q(t))$ satisfy the following system of differential equations:*

$$\begin{aligned} x' &= y^2 p \\ y' &= y^2 q \\ p' &= -hq \\ q' &= hp - y(p^2 + q^2) \end{aligned}$$

Proof. In local coordinates,

$$\alpha = dx \wedge dp + dy \wedge dq + h \frac{dx \wedge dy}{y^2}$$

Suppose $X_H = X\partial_x + Y\partial_y + P\partial_p + Q\partial_q$, then we can show the following.

$$i_{X_H}\alpha = \left(-P - h\frac{Y}{y^2}\right)dx + \left(h\frac{X}{y^2} - Q\right)dy + Xdp + Ydq$$

Since $dH = y(p^2 + q^2)dy + y^2pdp + y^2qdq$, the equation $i_{X_H}\alpha = dH$ gives us the following

solution for X_H .

$$X = y^2 p$$

$$Y = y^2 q$$

$$P = -hq$$

$$Q = hp - y(p^2 + q^2)$$

Thus, we have the following equation for X_H .

$$X_H = y^2 p \partial_x + y^2 q \partial_y - hq \partial_p + [hp - y(p^2 + q^2)] \partial_q$$

Hence, the integral curves generated by the above vector field should satisfy the differential equation in lemma 95. □

Lemma 96. *The pullback form $\gamma_H^*(pdx + qdy) = c dt$ where $c = y(0)^2(p(0)^2 + q(0)^2)$.*

Proof. This follows from lemma 95.

$$\begin{aligned} \gamma_H^*(pdx + qdy) &= (p(t)x'(t) + y(t)q'(t))dt \\ &= y(t)^2(p(t)^2 + q(t)^2)dt \end{aligned}$$

Since the Hamiltonian H is preserved on a Hamiltonian flow line, the lemma follows. □

Chapter 3

Background on Cyclic Polytopes and Secondary Polytopes

3.1 Cyclic Polytopes

3.1.1 Point Configurations and Cyclic Polytopes

We begin by recalling some basic terminology concerning point configurations and convex polytopes. Let $A = \{a_1, \dots, a_n\} \subset \mathbb{R}^d$ be a finite collection of points. We will generally assume that A affinely spans $\mathbb{R}^d$. The convex hull of A is the polytope

$$\operatorname{conv}(A) = \left\{ \sum_{i=1}^n \lambda_i a_i \mid \lambda_i \geq 0, \sum_{i=1}^n \lambda_i = 1 \right\}.$$

Although a point configuration and its convex hull are closely related, it is useful to distinguish between the two. In particular, constructions such as regular subdivisions and secondary polytopes are naturally associated to the configuration A , and may depend on the particular realization of its points rather than only on the combinatorial type of $\operatorname{conv}(A)$.

A collection of points $a_0, \dots, a_k \in \mathbb{R}^d$ is said to be *affinely independent* if the only relation

$$\sum_{i=0}^k \lambda_i a_i = 0,$$

$$\sum_{i=0}^k \lambda_i = 0$$

is the trivial relation $\lambda_0 = \dots = \lambda_k = 0$. Equivalently, the vectors $a_1 - a_0, \dots, a_k - a_0$ are linearly independent.

A point configuration $A \subset \mathbb{R}^d$ is said to be in *affine general position* if every subset of at most $d + 1$ points is affinely independent.

The d -dimensional moment curve is the map

$$\gamma : \mathbb{R} \longrightarrow \mathbb{R}^d, \tag{3.1}$$

$$t \longmapsto (t, t^2, \dots, t^d). \tag{3.2}$$

Choose real numbers $t_1 < t_2 < \dots < t_n$ and let

$$A(t_1, \dots, t_n) = \{\gamma(t_1), \dots, \gamma(t_n)\}. \tag{3.3}$$

The convex hull of these points is called a *cyclic polytope*. Its combinatorial type is denoted by $C(n, d)$.

Proposition 97. *Every cyclic point configuration $A(t_1, \dots, t_n) \subset \mathbb{R}^d$ is in affine general position.*

Proof. It is enough to show that every collection of $d + 1$ points of $A(t_1, \dots, t_n)$ is affinely independent. Let $t_{i_0} < t_{i_1} < \dots < t_{i_d}$ be distinct parameters. The corresponding points are

affinely independent if and only if

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ t_{i_0} & t_{i_1} & \cdots & t_{i_d} \\ t_{i_0}^2 & t_{i_1}^2 & \cdots & t_{i_d}^2 \\ \vdots & \vdots & \ddots & \vdots \\ t_{i_0}^d & t_{i_1}^d & \cdots & t_{i_d}^d \end{pmatrix}$$

is nonzero. This is the Vandermonde determinant, and therefore

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ t_{i_0} & t_{i_1} & \cdots & t_{i_d} \\ t_{i_0}^2 & t_{i_1}^2 & \cdots & t_{i_d}^2 \\ \vdots & \vdots & \ddots & \vdots \\ t_{i_0}^d & t_{i_1}^d & \cdots & t_{i_d}^d \end{pmatrix} = \prod_{0 \leq r < s \leq d} (t_{i_s} - t_{i_r}).$$

Since the parameters are distinct, every factor on the right-hand side is nonzero. Hence, the determinant is nonzero, and the chosen $d + 1$ points are affinely independent. $\square$

Corollary 98. *Every cyclic polytope $C(n, d)$ is simplicial.*

Proof. A facet of a d -dimensional polytope has dimension $d - 1$. If a facet of $C(n, d)$ contained more than d vertices, then some $d + 1$ of those vertices would lie in a common affine hyperplane. This would contradict the affine general position of the cyclic point configuration. Hence every facet contains exactly d vertices and is therefore a simplex. $\square$

A d -dimensional polytope is called *k -neighborly* if every subset of at most k vertices is the vertex set of a face.

Theorem 99. *The cyclic polytope $C(n, d)$ is $\lfloor d/2 \rfloor$ -neighborly.*

The proof of this theorem follows from the combinatorial description of the faces of cyclic

polytopes. We will give the relevant description in the next subsection using Gale’s evenness criterion.

An important feature of cyclic polytopes is that their combinatorial type depends only on the ordering of the parameters $t_1 < t_2 < \cdots < t_n$, and not on their particular numerical values. Thus, any two choices of strictly increasing parameters determine combinatorially equivalent polytopes. It is therefore customary to label the vertices simply by $1, \dots, n$, with the understanding that this ordering is induced by the moment curve.

The combinatorial rigidity should be distinguished from the rigidity of the underlying point configuration. Different choices of parameters $t_1, \dots, t_n$ determine the same combinatorial type, but the resulting configurations are not necessarily affinely equivalent. This distinction will become important later when we study regular subdivisions, secondary polytopes, and exceptional generality.

3.1.2 Combinatorics of Cyclic Polytopes

The combinatorial structure of a cyclic polytope is completely determined by the ordering of its vertices along the moment curve. A particularly useful characterization of its facets is given by Gale’s evenness criterion. Throughout this subsection, we write the vertices of $C(n, d)$ as $1, 2, \dots, n$, ordered according to increasing parameters on the moment curve.

Theorem 100 (Gale’s evenness criterion). *Let $F \subset \{1, \dots, n\}$ be a subset containing exactly d vertices. Then F is the vertex set of a facet of $C(n, d)$ if and only if, for every pair of vertices $i < j$ not contained in F , the number of elements of F lying strictly between i and j is even.*

Gale’s evenness criterion gives a purely combinatorial description of the facets of a cyclic polytope. In particular, it shows that the face lattice depends only on the linear ordering of the vertices and not on the particular parameters chosen on the moment curve. The criterion becomes particularly transparent in low dimensions. For instance, in dimension three every facet is a triangle. The evenness criterion then allows the facets to be described explicitly.

Proposition 101. *Let $C(n, 3)$ have vertices $1, \dots, n$. A three-element subset $\{i, j, k\}$ with $i < j < k$ is a facet if and only if either*

$$j = i + 1$$

and $k = n$, or

$$i = 1$$

and $k = j + 1$. Equivalently, the facets of $C(n, 3)$ are precisely

$$\{i, i + 1, n\}, \quad 1 \leq i \leq n - 2,$$

and

$$\{1, j, j + 1\}, \quad 2 \leq j \leq n - 1.$$

Proof. Let $F = \{i, j, k\}$ with $i < j < k$. Suppose first that $1 < i$ and $k < n$. Then there is a vertex $p < i$ and a vertex $q > k$ not contained in F . All three vertices of F lie strictly between p and q , so Gale's evenness criterion is violated. Hence every facet must contain either 1 or n .

Suppose that $k = n$. If $j > i + 1$, choose a vertex p with $i < p < j$. Then $p \notin F$, and since $i > 1$ or there is a vertex to the left of i , one may choose a vertex $q < i$ not contained in F whenever $i > 1$. The number of vertices of F lying between q and p is one, namely i , contradicting Gale's evenness criterion. Thus, except for the case where $i = 1$, the vertices i and j must be consecutive. The case $i = 1$ is included in the second family below.

Similarly, suppose that $i = 1$. If $k > j + 1$, then there is a vertex p with $j < p < k$. Comparing p with a vertex $q > k$, when such a vertex exists, shows that the number of

elements of F between them is one, namely k , which violates the evenness condition. Hence $k = j + 1$.

Conversely, consider a set of the form $\{i, i + 1, n\}$. If $p < q$ are vertices not belonging to this set, then the pair $i, i + 1$ either lies entirely between p and q or entirely outside that interval. Thus the number of vertices of the facet lying between p and q is either zero or two. Hence Gale's evenness criterion is satisfied. The same argument applies to a set of the form $\{1, j, j + 1\}$. Therefore these and only these triples are facets. $\square$

In higher dimensions, Gale's criterion can also be expressed in terms of consecutive pairs of vertices. This description is particularly simple in even dimension.

Proposition 102. *Let $d = 2m$. A subset $F \subset \{1, \dots, n\}$ of cardinality $2m$ is the vertex set of a facet of $C(n, 2m)$ if and only if its vertices can be partitioned into m pairs of consecutive vertices in the cyclic ordering determined by the moment curve, with the usual convention that the endpoints 1 and n may occur as boundary vertices.*

Proof. This is an equivalent reformulation of Gale's evenness criterion. Indeed, if the vertices of F occur in consecutive pairs, then any interval between two vertices outside F contains an even number of elements of F . Conversely, if some block of vertices of F in the ordered set has odd cardinality and is separated on both sides by vertices not in F , then those two exterior vertices violate the evenness criterion. Hence all interior blocks have even cardinality, and the vertices can be grouped into consecutive pairs. $\square$

The high degree of neighborliness of cyclic polytopes follows naturally from the structure of the moment curve.

Theorem 103. *The cyclic polytope $C(n, d)$ is $\lfloor d/2 \rfloor$ -neighborly.*

Proof. Let $k \leq \lfloor d/2 \rfloor$, and let

$$I = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$$

be any collection of k vertices. We show that I is contained in the vertex set of a facet.

Write $d = 2m$ or $d = 2m + 1$. Since $k \leq m$, one can enlarge I to a d -element subset F satisfying Gale's evenness criterion by adding vertices so that the elements of F occur in consecutive pairs, together with an endpoint when necessary in odd dimension. By Gale's evenness criterion, F is a facet of $C(n, d)$. Therefore I is contained in a facet.

Since every subset of the vertex set of a simplex is itself the vertex set of a face, it follows that I determines a face of $C(n, d)$. Thus every collection of at most $\lfloor d/2 \rfloor$ vertices determines a face, and hence $C(n, d)$ is $\lfloor d/2 \rfloor$ -neighborly. $\square$

In particular, when $d = 3$, every individual vertex is a face, while the polytope is not necessarily 2-neighborly. In contrast, when $d = 4$, every pair of vertices forms an edge. The neighborliness of cyclic polytopes is extremal in the sense of the Upper Bound Theorem: among all d -dimensional polytopes with a fixed number of vertices, cyclic polytopes attain the maximal possible number of faces in every dimension. The ordering of the vertices also determines the affine dependence relations among sufficiently small subsets.

3.1.3 Polyhedral and Coherent Subdivisions

Let A be a finite subset of $\mathbb{R}^{k-1}$, and let

$$Q = \text{Conv}(A).$$

We will regard (Q, A) as a polytope, where we do not require every element of A to be a vertex of Q .

A *subdivision* S of (Q, A) is a decomposition of (Q, A) into subpolytopes

$$\{(Q_i, A_i)\}_{i \in I},$$

where each A_i is a subset of A . We write $S[j]$ for the collection of j -dimensional subpolytopes

appearing in the subdivision. A subdivision is called a *triangulation* if every subpolytope Q_i is a simplex.

Given a subdivision S of (Q, A) , a continuous function

$$g : Q \longrightarrow \mathbb{R}$$

is called *S-piecewise-linear* if its restriction to every subpolytope Q_i is affine-linear.

A continuous function $g : Q \rightarrow \mathbb{R}$ is called *concave* if, for every $x, y \in Q$ and every $0 \leq t \leq 1$,

$$g(tx + (1 - t)y) \geq tg(x) + (1 - t)g(y).$$

A *domain of linearity* of a concave function g is a maximal subset of Q on which the restriction of g is affine-linear.

Definition 104. A subdivision S of (Q, A) is called *coherent* if there exists a concave S -piecewise-linear function on Q whose domains of linearity are precisely the maximal subpolytopes of S .

Coherent subdivisions may equivalently be constructed from height functions. Let $\eta \in \mathbb{R}^A$. Thus η assigns a real number $\eta(\omega)$ to each point $\omega \in A$. Consider the subset of $\mathbb{R}^k = \mathbb{R}^{k-1} \times \mathbb{R}$ given by the union of vertical half-lines

$$\{(\omega, y) \mid y \leq \eta(\omega), \omega \in A, y \in \mathbb{R}\}.$$

Let G_η denote the convex hull of this union.

The upper boundary of G_η projects bijectively onto Q . The projections of its faces therefore determine a subdivision of (Q, A) , which we denote by S_η .

Proposition 105. *For every $\eta \in \mathbb{R}^A$, the subdivision S_η is coherent.*

Proof. Define a continuous piecewise-linear function

$$g_{\eta, S_\eta} : Q \longrightarrow \mathbb{R}$$

by assigning

$$g_{\eta, S_\eta}(\omega) = \eta(\omega)$$

for every $\omega \in A$ which occurs as a vertex of a subpolytope of S_η , and extending the function affine-linearly on each subpolytope of the subdivision.

By construction, the graph of g_{η, S_η} is the upper boundary of G_η . Since G_η is convex, its upper boundary is the graph of a concave function. Hence g_{η, S_η} is concave. Moreover, the maximal domains on which this function is affine-linear are exactly the maximal subpolytopes of S_η . Therefore, S_η is coherent. $\square$

Thus, every height function $\eta \in \mathbb{R}^A$ determines a coherent subdivision of (Q, A) . Conversely, every coherent subdivision arises from such a height function.

Proposition 106. *Let S be a coherent subdivision of (Q, A) . Then there exists $\eta \in \mathbb{R}^A$ such that*

$$S = S_\eta.$$

Proof. Since S is coherent, there exists a concave S -piecewise-linear function

$$g : Q \longrightarrow \mathbb{R}$$

whose domains of linearity are precisely the maximal subpolytopes of S . Define $\eta \in \mathbb{R}^A$ by

$$\eta(\omega) = g(\omega)$$

for every $\omega \in A$.

The graph of g is then the upper boundary of the convex hull G_η , and its domains of linearity are precisely the maximal subpolytopes of S . Hence, the subdivision obtained by projecting the upper boundary of G_η is exactly S . Therefore

$$S = S_\eta.$$

□

Given a coherent subdivision S of (Q, A) and an element $\eta \in \mathbb{R}^A$, we can similarly define a S -piecewise-linear function $g_{\eta,S}$, by affine interpolation on the subpolytopes of S , whenever the values of η are compatible with the affine-linear structure on those subpolytopes. In particular, the assignment

$$\eta \longmapsto g_{\eta,S}$$

relates height functions on A to piecewise affine-linear functions on the subdivision. This point of view will become useful later when we introduce the deformation complex associated with a subdivision.

If S_1 and S_0 are subdivisions of (Q, A) , we say that S_1 is a *refinement* of S_0 if every subpolytope of S_1 is contained in a subpolytope of S_0 . Equivalently, S_0 is a *coarsening* of S_1 . The refinement relation gives a natural partial ordering on the set of subdivisions of (Q, A) .

For a coherent subdivision S of (Q, A) , we define

$$C(S) = \{\eta \in \mathbb{R}^A \mid S_\eta = S\}.$$

The set $C(S)$ is a relatively open polyhedral cone in $\mathbb{R}^A$. As S varies over all coherent subdivisions of (Q, A) , the closures of these cones fit together to form the secondary fan. We discuss this construction in the following subsection.

3.1.4 The Secondary Fan

Let (Q, A) be a polytope, with $A \subset \mathbb{R}^{k-1}$ finite and

$$Q = \text{Conv}(A).$$

Recall that every $\eta \in \mathbb{R}^A$ determines a coherent subdivision S_η of (Q, A) . For a fixed coherent subdivision S , we consider the set

$$C(S) = \{\eta \in \mathbb{R}^A \mid S_\eta = S\}.$$

Thus, $C(S)$ consists of height functions which induce the same coherent subdivision.

The subdivision determined by a height function is unchanged when one perturbs the height function sufficiently little without changing the domains of linearity of the associated concave piecewise-linear function. Consequently, the set of height functions inducing a fixed subdivision forms a relatively open polyhedral cone.

Proposition 107. *Let S be a coherent subdivision of (Q, A) . Then $C(S)$ is a relatively open polyhedral cone in $\mathbb{R}^A$.*

Proof. Let the maximal subpolytopes of S be (Q_i, A_i) , for $i \in I$. A height function $\eta \in \mathbb{R}^A$ induces S precisely when, for every maximal subpolytope (Q_i, A_i) , there exists an affine-linear function

$$f_i : \mathbb{R}^{k-1} \longrightarrow \mathbb{R}$$

such that

$$f_i(\omega) = \eta(\omega)$$

for every $\omega \in A_i$, while the graph of f_i lies strictly above the lifted points corresponding

to vertices not contained in A_i whenever those points do not belong to the same domain of linearity.

These conditions are expressed by finitely many linear equalities and strict linear inequalities in the coordinates of η . Hence the set of height functions satisfying them is a relatively open polyhedral cone. $\square$

The closure of $C(S)$ will be denoted by $\overline{C(S)}$. Passing from a subdivision to a coarsening corresponds to allowing some of the strict inequalities defining $C(S)$ to become equalities. Thus the refinement relation between coherent subdivisions is reflected in the face structure of these cones.

Proposition 108. *Let S_1 and S_0 be coherent subdivisions of (Q, A) . If S_1 is a refinement of S_0 , then*

$$C(S_0) \subseteq \overline{C(S_1)}.$$

Proof. Suppose S_1 refines S_0 . The piecewise-linear function defining S_1 has at least as many domains of linearity as the function defining S_0 . A coarsening is obtained when some adjacent domains of linearity become part of a single affine-linear domain. At the level of height functions, this occurs when some of the strict linear inequalities defining $C(S_1)$ become equalities.

Therefore, every height function inducing S_0 can be obtained as a limit of height functions inducing S_1 . Hence

$$C(S_0) \subseteq \overline{C(S_1)}.$$

$\square$

The cones associated with coherent subdivisions fit together to form a complete polyhedral fan.

Theorem 109. *The collection*

$$\{\overline{C(S)} \mid S \text{ is a coherent subdivision of } (Q, A)\}$$

forms a complete polyhedral fan in $\mathbb{R}^A$.

Proof. Every element $\eta \in \mathbb{R}^A$ determines a coherent subdivision S_η . Hence

$$\mathbb{R}^A = \bigcup_S \overline{C(S)},$$

where the union ranges over all coherent subdivisions of (Q, A) . Thus, the collection of cones is complete.

Moreover, if two cones $\overline{C(S_1)}$ and $\overline{C(S_2)}$ intersect, a height function in their intersection induces a common coarsening of S_1 and S_2 . The intersection is therefore a face of each cone, corresponding to such a common coarsening. Hence, the collection satisfies the defining properties of a polyhedral fan. $\square$

Definition 110. The fan

$$\Sigma_{\text{fan}}(A) = \{\overline{C(S)} \mid S \text{ is a coherent subdivision of } (Q, A)\}$$

is called the *secondary fan* of A .

In particular, the cones of the secondary fan encode the coherent subdivisions of (Q, A) . The face relations between cones encode the refinement relations among these subdivisions. The maximal cones correspond to coherent triangulations, whereas lower-dimensional cones correspond to coarser coherent subdivisions.

At this stage, the secondary fan is obtained directly from the space of height functions and does not a priori arise as the normal fan of a polytope. A fundamental result of⁹ shows that it is in fact the normal fan of a naturally associated polytope, called the secondary polytope. We introduce this polytope in the next subsection.

3.1.5 The Secondary Polytope

Let $A \subset \mathbb{R}^{k-1}$ be a finite point configuration, and let

$$Q = \text{Conv}(A).$$

We assume throughout that $\dim(Q) = k - 1$. Fix a translation-invariant volume form Vol on $\mathbb{R}^{k-1}$.

Let T be a triangulation of (Q, A) . For every $\omega \in A$, we define the *characteristic function* of T by

$$\varphi_T(\omega) = \sum_{\substack{\sigma \in T[k-1] \\ \omega \in \text{Vert}(\sigma)}} \text{Vol}(\sigma),$$

where the sum is taken over all maximal simplices σ of T having ω as a vertex. Thus the characteristic function determines an element

$$\varphi_T \in \mathbb{R}^A.$$

The vector φ_T records how the volume of Q is distributed among the vertices of A by triangulation T . Note that the definition makes sense for every triangulation, whether or not the triangulation is coherent.

Definition 111. The *secondary polytope* of A is the convex polytope

$$\Sigma(A) = \text{Conv}\{\varphi_T \mid T \text{ is a triangulation of } (Q, A)\} \subset \mathbb{R}^A$$

Although the definition of $\Sigma(A)$ involves all triangulations of (Q, A) , its vertices are determined by the coherent triangulations. The following theorem is the fundamental result relating secondary polytopes to the secondary fan.

Theorem 112 ⁽⁹⁾. *Let $A \subset \mathbb{R}^{k-1}$ be a finite point configuration with $|A| = n$ and $\dim(Q) = k - 1$. Then the secondary polytope $\Sigma(A)$ satisfies the following properties:*

1.

$$\dim \Sigma(A) = n - k.$$

2. *The vertices of $\Sigma(A)$ are precisely the characteristic functions φ_T corresponding to coherent triangulations T of (Q, A) .*

Moreover, if T is coherent, then

$$\varphi_T \neq \varphi_{T'}$$

for every triangulation $T' \neq T$.

3. *If T is a coherent triangulation, then the normal cone of $\Sigma(A)$ at φ_T is*

$$N_{\varphi_T} \Sigma(A) = \overline{C(T)}.$$

In particular, the normal fan of the secondary polytope coincides with the secondary fan of A . Thus, the polyhedral geometry of $\Sigma(A)$ encodes the coherent subdivisions of (Q, A) .

More generally, coherent subdivisions correspond to the faces of the secondary polytope.

Proposition 113. *Let S be a coherent subdivision of (Q, A) . Then there exists a face F_S of $\Sigma(A)$ whose normal cone is $\overline{C(S)}$.*

Proof. By the preceding theorem, the normal fan of $\Sigma(A)$ is the secondary fan of A . Since $\overline{C(S)}$ is a cone in the secondary fan, it is the normal cone of a unique face F_S of $\Sigma(A)$. Hence

$$N_{F_S} \Sigma(A) = \overline{C(S)}.$$

□

The correspondence between coherent subdivisions and faces reverses refinement.

Proposition 114. *Let S_1 and S_0 be coherent subdivisions of (Q, A) , and let F_{S_1} and F_{S_0} be their corresponding faces in $\Sigma(A)$. If S_1 is a refinement of S_0 , then*

$$F_{S_1} \subseteq F_{S_0}.$$

Proof. If S_1 refines S_0 , then

$$C(S_0) \subseteq \overline{C(S_1)}.$$

Under the correspondence between faces of a polytope and cones in its normal fan, inclusion of normal cones reverses inclusion of faces. Therefore

$$F_{S_1} \subseteq F_{S_0}.$$

□

In particular, coherent triangulations correspond to vertices of $\Sigma(A)$, while coarser coherent subdivisions correspond to positive-dimensional faces. The secondary polytope therefore provides a geometric organization of the coherent subdivisions of (Q, A) together with their refinement relations.

The geometry of a face F_S can depend on more than the combinatorial structure of the subdivision S . In particular, one may ask whether its dimension agrees with the dimension predicted by the combinatorics of the subdivision. This question is closely related the deformation complex of the subdivision.

3.1.6 Exceptional Generality in Dimension Two

We now specialize to two-dimensional point configurations. Following the convention of[?], we identify $\mathbb{R}^2$ with $\mathbb{C}$. Let $A \subset \mathbb{C}$ be a finite point configuration, and let P be a polygonal subdivision of (Q, A) , where $Q = \text{Conv}(A)$.

For such a subdivision, the deformation complex has three terms:

$$\text{Def } \mathcal{P} = \left\{ \bigoplus_{\sigma \in \mathcal{P}^2} \text{Aff}(\sigma) \otimes \text{or}(\sigma) \xrightarrow{d} \bigoplus_{\sigma \in \mathcal{P}^1} \text{Aff}(\sigma) \otimes \text{or}(\sigma) \xrightarrow{d} \bigoplus_{\sigma \in \mathcal{P}^0} \text{Aff}(\sigma) \otimes \text{or}(\sigma) \right\},$$

where only the cells which do not lie in the boundary of Q are included.

The first cohomology of this complex measures the failure of the affine compatibility conditions imposed by the subdivision to be independent.

Definition 115. The *exceptionality* of a polygonal subdivision $\mathcal{P}$ is defined by

$$\text{exc}(\mathcal{P}) = \dim H^1(\text{Def}(\mathcal{P})).$$

The following vanishing result is proved in[?].

Proposition 116. *For every polygonal subdivision $\mathcal{P}$,*

$$H^2(\text{Def}(\mathcal{P})) = 0.$$

The exceptionality of a subdivision enters directly into the dimension of the corresponding face of the secondary polytope.

Proposition 117. *Let P be a coherent polygonal subdivision of (Q, A) , and let F_P denote the corresponding face of the secondary polytope $\Sigma(A)$. Then*

$$\text{codim}(F_P) = 3|\mathcal{P}^2| - 2|\mathcal{P}^1| + |\mathcal{P}^0| - 3 + \text{exc}(\mathcal{P}).$$

Proof. Since $H^2(\text{Def}(P)) = 0$, the Euler characteristic of the deformation complex gives

$$\dim H^0(\text{Def}(P)) - \dim H^1(\text{Def}(P)) = 3|P^2| - 2|P^1| + |P^0|.$$

Here each two-dimensional cell contributes a three-dimensional space of affine-linear functions, each one-dimensional cell contributes a two-dimensional space, and each zero-dimensional cell contributes a one-dimensional space.

By definition,

$$\dim H^1(\text{Def}(P)) = \text{exc}(P).$$

Moreover, $H^0(\text{Def}(P))$ is identified with the space of P -piecewise affine-linear functions. Modulo the three-dimensional space of globally affine-linear functions on $\mathbb{R}^2$, this space determines the normal cone to the face F_P . Therefore

$$\text{codim}(F_P) = \dim H^0(\text{Def}(P)) - 3.$$

Combining the above equalities gives

$$\text{codim}(F_P) = 3|P^2| - 2|P^1| + |P^0| - 3 + \text{exc}(P).$$

□

Thus, when $\text{exc}(P) = 0$, the codimension of the face F_P is determined entirely by the combinatorics of the subdivision. In this case the face has the dimension predicted by the naive count of affine compatibility conditions. If $\text{exc}(P) > 0$, then the subdivision contains additional linear dependencies among these conditions, and the corresponding face has larger codimension than expected.

We now define exceptional generality.

Definition 118. Let $\mathbb{C}_{\text{gen}}^N$ denote the set of configurations of N points in $\mathbb{C}$ which are in generic position. A point configuration

$$A \in \mathbb{C}_{\text{gen}}^N$$

is called *exceptionally general* if

$$\text{exc}(P) = 0$$

for every subdivision P of $(\text{Conv}(A), A)$.

We denote the set of exceptionally general configurations by

$$\mathbb{C}_{\text{Xgen}}^N.$$

The complement

$$D = \mathbb{C}_{\text{gen}}^N \setminus \mathbb{C}_{\text{Xgen}}^N$$

therefore consists of generic configurations admitting at least one subdivision with positive exceptionality.

The following result describes the basic geometric properties of the exceptionally general locus.

Proposition 119 ⁽¹⁰⁾. *The set $\mathbb{C}_{\text{Xgen}}^N$ satisfies the following properties:*

1. $\mathbb{C}_{\text{Xgen}}^N$ is an $\mathbb{R}$ -Zariski open subset of $\mathbb{C}_{\text{gen}}^N$.
2. *The complement*

$$D = \mathbb{C}_{\text{gen}}^N \setminus \mathbb{C}_{\text{Xgen}}^N$$

has codimension at least one.

3. If E is a connected component of $\mathbb{C}_{\text{Xgen}}^N$, then the combinatorial equivalence class of $\Sigma(A)$ is constant for $A \in E$.

4. If $A \in \mathbb{C}_{\text{Xgen}}^N$ and

$$P = \{(Q_i, A_i)\}_{i \in I}$$

is a coherent subdivision, then the corresponding face F_P of $\Sigma(A)$ is combinatorially equivalent to

$$\prod_{i \in I} \Sigma(A_i).$$

The last statement is especially important for the factorization properties of secondary polytopes. For an exceptionally general configuration, the face associated to a coherent subdivision decomposes combinatorially as the product of the secondary polytopes of its maximal cells.

The condition of exceptional generality is therefore substantially stronger than ordinary genericity of a point configuration. A configuration may be in affine general position while still admitting a subdivision with nonzero $H^1(\text{Def}(P))$. In the later chapters we will examine this distinction for cyclic point configurations and show that, even in dimension three, cyclicity does not by itself guarantee exceptional generality.

Chapter 4

A non-exceptional 3-dimensional cyclic point configuration

4.1 Introduction

The preceding chapter introduced point configurations, cyclic polytopes, regular subdivisions, secondary fans, and secondary polytopes. The theory of secondary polytopes was developed systematically by Gel'fand, Kapranov, and Zelevinsky⁹. The relationship between triangulations, regular subdivisions, and their associated polyhedral structures is discussed in detail in¹¹. Further constructions related to secondary polytopes and their complexity appear in¹², while fiber polytopes provide a related geometric framework for studying projections and subdivisions¹³.

A central notion in this chapter is that of exceptional generality. Given a subdivision $\mathcal{S}$, the deformation complex $\text{Def}(\mathcal{S})$ records the affine compatibility conditions imposed by the cells of the subdivision. Following the perspective of Kapranov, Soibelman, and Soukhanov¹⁰, the exceptionality of a subdivision is measured by

$$\text{exc}(\mathcal{S}) = \dim H^1(\text{Def}(\mathcal{S})).$$

A point configuration is called exceptionally general if every subdivision has vanishing exceptionality. As explained in Chapter 3, this condition is related to the expected dimensions of faces of the secondary polytope and to the factorization properties of those faces.

The purpose of this chapter is to construct an explicit three-dimensional cyclic point configuration that is not exceptionally general. We begin by studying the vertex figure of a three-dimensional cyclic polytope at the vertex $\gamma(0)$. Proposition 121 identifies this vertex figure with a marked planar point configuration lying on the graph of the reciprocal function

$$f(x) = \frac{1}{x}.$$

This identification allows us to study a two-dimensional configuration instead of working directly with the original three-dimensional cyclic polytope.

We then relate regular subdivisions of the vertex figure to regular subdivisions of the original cyclic configuration. Proposition 122 shows that a regular subdivision of the vertex figure can be realized as the subdivision induced by a regular subdivision of the cyclic polytope. Proposition 123 further shows that the secondary fan of the cyclic configuration is a refinement of the secondary fan of the vertex figure.

Finally, we consider parameters a and b satisfying $b > a > 1$ and define

$$A = \{\pm a^{\pm 1}, \pm b^{\pm 1}, 0\}.$$

The corresponding planar vertex-figure configuration is

$$\overline{A} = \left\{ \left(x, \frac{1}{x} \right) \mid x \in A \setminus \{0\} \right\}.$$

We construct a regular subdivision of $(\text{Conv}(\overline{A}), \overline{A})$ with nonzero exceptionality. This subdivision lifts to a regular subdivision of the three-dimensional cyclic point configuration $\gamma(A)$, yielding the main result of the chapter.

4.2 Vertex figures

We define vertex figure of a d -dimensional polytope.

Definition 120. Let $P \subset \mathbb{R}^d$ be a d -dimensional polytope and let v be a vertex of P . Let H be an affine hyperplane that separates v from all other vertices of P and intersects every edge of P incident to v . The polytope

$$P_v := P \cap H$$

is called the *vertex figure of P at v* . Its combinatorial type is independent of the choice of such a hyperplane H .

Let

$$t_1 < t_2 < \dots t_{i-1} < 0 < t_{i+1} < \dots t_n$$

be a collection of increasing points in $\mathbb{R}$. We denote the moment curve by $\gamma(t)$ as described in Eq. (3.1) with $d = 3$. The cyclic polytope

$$A(t_1, \dots, t_n)$$

was described in Eq. (3.3). For this section, we will use P for the polytope

$$A(t_1, \dots, 0, \dots t_n)$$

with $d = 3$ in Eq. (3.3). The vertex figure of the cyclic polytope

$$A(t_1, \dots, t_n)$$

is a marked polytope in $\mathbb{R}^2$ with $n - 1$ points. The description of the vertex figure P_0 in $\mathbb{R}^2$

is crucial to understand our example. We state this in the following proposition.

Proposition 121. *Let f be the function,*

$$\begin{aligned} f : \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto \frac{1}{x}. \end{aligned}$$

Define the set A , as

$$A := \{(t_j, f(t_j)) \mid 1 \leq j \leq n, j \neq i\} \subset \mathbb{R}^2.$$

Then, the 2-dimensional polytope

$$(Q, A) = (\text{Conv}(A), A) \tag{4.1}$$

is affine equivalent to P_0 .

Proof. Since $d = 3$, the equation for the moment curve is

$$\gamma : \mathbb{R} \longrightarrow \mathbb{R}^3 \tag{4.2}$$

$$t \longmapsto (t, t^2, t^3). \tag{4.3}$$

We start by noting that any point on the moment curve has positive y -coordinate. Hence,

$$\text{image}(\gamma) \subseteq \mathbb{R} \times \mathbb{R}_{\geq 0} \times \mathbb{R}$$

Since the i -vertex, $\gamma(t_i) = 0$, all other vertices of P lie in the subset

$$\mathbb{R} \times \mathbb{R}_{>0} \times \mathbb{R}.$$

This is because the only element of the moment curve that lies in the hyperplane $y = 0$ is

$\gamma(0)$. We now show that for $\epsilon > 0$ small enough, the hyperplane $y = \epsilon$ separates $\gamma(0)$ from every other vertex of P and that every edge incident to $\gamma(0)$ intersects the hyperplane $y = \epsilon$. Pick

$$\epsilon = \frac{1}{2} \min\{\|\gamma(t_j) \mid j \neq i\}.$$

ϵ as defined above guaranties that the vertices

$$\begin{aligned} &\{\gamma(t_j) \mid j \neq i\} \\ &\{\gamma(0)\} \end{aligned}$$

lie on different sides of the hyperplane $y = \epsilon$. Thus, every edge incident to $\gamma(0)$ has to intersect the hyperplane $y = \epsilon$.

Now, we compute the projection of the moment curve on the hyperplane $y = \epsilon$. The line joining $\gamma(0)$ and $\gamma(t)$ is given by the equation

$$\lambda(t, t^2, t^3), \quad \lambda, t > 0. \tag{4.4}$$

To lie in the hyperplane $y = \epsilon$,

$$\begin{aligned} \lambda t^2 &= \epsilon \\ \lambda &= \frac{\epsilon}{t^2} \end{aligned}$$

Thus, the projection of points on the moment curve are given by

$$\left\{ \epsilon \left(\frac{1}{t}, 1, t \right) : t > 0 \right\}.$$

Since the map

$$\begin{aligned}\pi : \mathbb{R} \times \{\epsilon\} \times \mathbb{R} &\longrightarrow \mathbb{R}^2 \\ (x, \epsilon, z) &\longmapsto \frac{1}{\epsilon}(x, z)\end{aligned}$$

is an invertible affine map, the proposition is proved. $\square$

4.2.1 Subdivisions of vertex figures

To infer facts about the regular subdivisions of $C(n, d)$, we will work with subdivisions of the vertex figure. Any subdivision S of $C(n, d)$ and a vertex ν of $C(n, d)$ induces a subdivision S_ν of the vertex figure $C_\nu(n, d)$. As in Chapter 3, we will denote $\Sigma_{\text{fan}}(Q, A)$ as the secondary fan of the polytope (Q, A) . The following proposition relates the secondary fan of $(P, (\gamma(t_1), \dots, \gamma(t_n)))$ to that of the secondary fan of the vertex figure (Q, A) .

Proposition 122. $\Sigma_{\text{fan}}(P, (\gamma(t_1), \dots, \gamma(t_n)))$ is a refinement of $\Sigma_{\text{fan}}(Q, A)$.

Proof. $\square$

4.3 A non-exceptional cyclic point configuration of the vertex figure

In this section, we describe a regular subdivision of the vertex figure of a particular 3-dimensional cyclic point configuration. Fix a, b that satisfy $b > a > 1$ and define

$$A := \{\pm a^{\pm 1}, \pm b^{\pm 1}, 0\} \tag{4.5}$$

to be a subset of $\mathbb{R}$. We denote by $\bar{A}$ the subset of $\text{Graph}(f)$ defined as

$$\bar{A} := \{(x, f(x)) \mid x \in A \setminus \{0\}\}. \tag{4.6}$$

We denote by $\gamma(A)$, the set of points on the moment curve γ (Eq. (3.1)) in 3-dimensions,

$$\gamma(A) := \{\gamma(x) \mid x \in A\}. \quad (4.7)$$

Let $(P, \gamma(A))$ denote the polytope $(\text{Conv}(\gamma(A)), \gamma(A))$ in 3-dimensions. Let $(Q, \bar{A})$ denote the polytope $(\text{Conv}(\bar{A}), \bar{A})$ in 2-dimensions. Proposition 121 implies that (P, A) is the vertex figure of $(Q, \bar{A})$. We consider the following regular subdivision of $(Q, \bar{A})$. Define

$$\bar{\eta} : \bar{A} \longrightarrow \mathbb{R} \quad (4.8)$$

$$(\pm b^{\pm 1}, f(\pm b^{\pm 1})) \longmapsto -1 \quad (4.9)$$

$$(\pm a^{\pm 1}, f(\pm a^{\pm 1})) \longmapsto 0. \quad (4.10)$$

$\bar{\eta}$ defined as above induces a subdivision $S_{\bar{\eta}}$ of $(Q, \bar{A})$ (105). We have the following proposition about $S_{\bar{\eta}}$.

Proposition 123.

$$\text{exc}(S_{\bar{\eta}}) \neq 0$$

, i.e., $\bar{A}$ is a non-exceptionally general point configuration in $\mathbb{R}^2$.

Proof. Note that if $\text{exc}(S_{\bar{\eta}}) = 0$, then the secondary polytope satisfies a factorization theorem (?? 119). We will denote the quadrilateral formed by the vertices

$$\{(x_i, f(x_i)) \mid i = 1, \dots, 4\}$$

by the notation $[x_1, x_2, x_3, x_4]$. We claim that $S_{\bar{\eta}}$ consists of the following quadrilaterals.

$$\begin{aligned} & [-b, -a, -\frac{1}{a}, -\frac{1}{b}] \\ & [-\frac{1}{a}, -\frac{1}{b}, a, b] \\ & [b, a, \frac{1}{a}, \frac{1}{b}] \\ & [\frac{1}{a}, \frac{1}{b}, -a, -b] \\ & [\frac{1}{a}, -\frac{1}{a}, -a, a] \end{aligned}$$

Consider the following piecewise-affine linear function:

$$\begin{aligned} & Cy + Cx + C \left(a + \frac{1}{a} \right) & (x, y) \in [-b, -a, -\frac{1}{a}, -\frac{1}{b}] \\ & Dy - Dx + D \left(-a - \frac{1}{a} \right) & (x, y) \in [-\frac{1}{a}, -\frac{1}{b}, a, b] \\ & -Cy - Cx - C \left(a + \frac{1}{a} \right) & (x, y) \in [b, a, \frac{1}{a}, \frac{1}{b}] \\ & -Dy + Dx - D \left(-a - \frac{1}{a} \right) & (x, y) \in [\frac{1}{a}, \frac{1}{b}, -a, -b] \\ & 0 & (x, y) \in [\frac{1}{a}, -\frac{1}{a}, -a, a]. \end{aligned}$$

We can check that the above function gives rise to the subdivision $S_{\bar{\eta}}$. Finally, note that since $S_{\bar{\eta}}$ is a regular proper subdivision, if factorization were to hold true, it should correspond to a face of dimension 5 in the secondary polytope. However, the dimension of the secondary polytope of the above point configuration is also 5. Therefore, the secondary polytope of the above point configuration does not satisfy the factorization property as in ?? 119. Hence, $\text{exc}(S_{\bar{\eta}}) \neq 0$. □

Proposition 122 gives us a subdivision S_η of (P, A) with η defined as

$$\eta : \gamma(A) \longrightarrow \mathbb{R} \quad (4.11)$$

$$\gamma(\pm a^{\pm 1}) \longmapsto 0 \quad (4.12)$$

$$\gamma(\pm b^{\pm 1}) \longmapsto -b^{\pm 2} \quad (4.13)$$

$$\gamma(0) \longmapsto 0. \quad (4.14)$$

To simplify notation, we denote a cyclic polytope with vertices $\gamma(t_{i_1}), \gamma(t_{i_2}), \dots, \gamma(t_{i_n})$ by $[t_{i_1}, t_{i_2}, \dots, t_{i_n}]$.

Proposition 124. *Let*

$$f : A(t_1, \dots, t_n) \longrightarrow \mathbb{R}$$

be an affine linear function on a cyclic polytope of dimension d $A(t_1, \dots, t_n)$ with vertices $\gamma(t_1), \dots, \gamma(t_n)$. Then there exists a unique polynomial p of degree d that satisfies for all $1 \leq j \leq n$,

$$f(\gamma(t_j)) = p(t_j).$$

Proof.

□

Proposition 125. *The subdivision S_η contains, among other cyclic subpolytopes, the fol-*

lowing cyclic subpolytopes:

$$\begin{aligned} &[-b, -a, -\frac{1}{a}, -\frac{1}{b}, 0] \\ &[-\frac{1}{a}, -\frac{1}{b}, a, b, 0] \\ &[b, a, \frac{1}{a}, \frac{1}{b}, 0] \\ &[\frac{1}{a}, \frac{1}{b}, -a, -b, 0] \\ &[\frac{1}{a}, -\frac{1}{a}, -a, a, 0] \end{aligned}$$

Proof.

□

This brings us to the following theorem.

Theorem 126. *The cyclic point configuration*

$$\{\gamma(\pm a^{\pm 1}), \gamma(\pm b^{\pm 1}), \gamma(0)\}$$

is non-exceptionally general.

Proof.

□

4.4 Conclusion

In this chapter, we constructed an explicit family of non-exceptionally general three-dimensional cyclic point configurations. The construction began with the parameter set

$$A = \{\pm a^{\pm 1}, \pm b^{\pm 1}, 0\}, \quad b > a > 1,$$

and the associated cyclic point configuration

$$\gamma(A) = \{\gamma(x) \mid x \in A\}.$$

The main idea was to study the vertex figure at $\gamma(0)$. Proposition 121 identifies this vertex figure with the marked planar configuration

$$\overline{A} = \left\{ \left(x, \frac{1}{x} \right) \mid x \in A \setminus \{0\} \right\}.$$

On this planar configuration, we considered the height function $\overline{\eta}$ defined by

$$\begin{aligned} \overline{\eta}(\pm b^{\pm 1}, f(\pm b^{\pm 1})) &= -1, \\ \overline{\eta}(\pm a^{\pm 1}, f(\pm a^{\pm 1})) &= 0. \end{aligned}$$

The resulting regular subdivision $\mathcal{S}_{\overline{\eta}}$ satisfies

$$\text{exc}(\mathcal{S}_{\overline{\eta}}) \neq 0.$$

Therefore, $\overline{A}$ is not exceptionally general.

By Proposition 122, this regular subdivision of the vertex figure lifts to a regular subdivision of the cyclic point configuration $\gamma(A)$. Proposition 125 identifies several cyclic subpolytopes occurring in the lifted subdivision. These subpolytopes provide the geometric structure needed for the final non-exceptionality statement.

We conclude that the cyclic point configuration

$$\{\gamma(\pm a^{\pm 1}), \gamma(\pm b^{\pm 1}), \gamma(0)\}$$

is non-exceptionally general. Thus, cyclicity alone does not imply exceptional generality in dimension three.

This example also illustrates that the geometry of regular subdivisions and secondary polytopes depends on more than the combinatorial type of the underlying cyclic polytope. Although cyclic polytopes possess a highly rigid combinatorial structure, the affine compati-

bility conditions associated with a regular subdivision can still have nontrivial dependencies. In the language of the deformation complex, these dependencies are detected by the nonvanishing of

$$H^1(\text{Def}(\mathcal{S})).$$

Consequently, the factorization behavior of secondary-polytope faces discussed in [9;10](#) cannot be extended to all three-dimensional cyclic point configurations without imposing an additional exceptional-generality hypothesis.

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