Nonlocal Vector Calculus

by

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Abstract

Nonlocal vector calculus, introduced in[4;6], generalizes differential operators' calculus to nonlocal calculus of integral operators. Nonlocal vector calculus has been applied to many fields including peridynamics, nonlocal diffusion, and image analysis[1–3;5;7]. In this report, we present a vector calculus for nonlocal operators such as a nonlocal divergence, a nonlocal gradient, and a nonlocal Laplacian. In Chapter 1, we review the local (differential) divergence, gradient, and Laplacian operators. In addition, we discuss their adjoints, the divergence theorem, Green's identities, and integration by parts. In Chapter 2, we define nonlocal analogues of the divergence and gradient operators, and derive the corresponding adjoint operators. In Chapter 3, we present a nonlocal divergence theorem, nonlocal Green's identities, and integration by parts for nonlocal operators. In Chapter 4, we establish a connection between the local and nonlocal operators. In particular, we show that, for specific integral kernels, the nonlocal operators converge to their local counterparts in the limit of vanishing nonlocality.

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Introduction

In this report, we present a vector calculus for nonlocal operators such as a nonlocal divergence, a nonlocal gradient, and a nonlocal Laplacian. We define nonlocal analogues of divergence and gradient and derive the corresponding adjoint operators; this will be the topic of Chapter 2. In Chapter 3, we present a nonlocal divergence theorem, Green's identities, and integration by parts formula for nonlocal operators. In Chapter 1, we review the local divergence and gradient operators and their adjoints, the local divergence theorem, the local Green's identities and integration by parts. In Chapter 4, we establish a connection between local and nonlocal operators. In particular, we show that, for an appropriate integral kernel, the nonlocal Laplacian operator converges to the local Laplacian operator in the limit of vanishing nonlocality.

Chapter 1

Local Operators

We review some definitions of classical operators in Cartesian coordinates in $\mathbb{R}^n$.

Local divergence operator

Let $\mathbf{v}$ be a differentiable vector field $\mathbf{v} : \mathbb{R}^n \to \mathbb{R}^k$. Then

$$\text{div } \mathbf{v} = \nabla \cdot \mathbf{v} = \frac{dv_1}{dx_1} + \frac{dv_2}{dx_2} + \ldots + \frac{dv_k}{dx_k}.$$

Local gradient operator

$$\text{grad } f = \nabla f = (\frac{df}{dx_1}, \frac{df}{dx_2}, \ldots, \frac{df}{dx_n}), \quad \text{where } f : \mathbb{R}^n \to \mathbb{R} \text{ is differentiable.}$$

Local Laplacian operator

$$\Delta f = \nabla \cdot \nabla f = \nabla \cdot (\frac{df}{dx_1}, \ldots, \frac{df}{dx_n}) = \frac{d^2 f}{dx_1^2} + \ldots + \frac{d^2 f}{dx_n^2},$$

where f, g are twice differentiable.

A product rule

The divergence of the product of a vector field $\mathbf{v}$ and a scalar field u satisfies

$$\nabla \cdot (u\mathbf{v}) = u\nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla u \tag{1.1}$$

1.1 The divergence theorem

Let $\mathbf{v}$ be a differentiable vector field, Ω a bounded domain, and $\mathbf{n}$ a unit normal vector. Then

$$\int_\Omega \nabla \cdot \mathbf{v}\, dx = \int_{\partial\Omega} \mathbf{v} \cdot \mathbf{n}\, ds.$$

In $\mathbb{R}^n$ and for $\mathbf{v}$ compactly supported, the divergence theorem is given by

$$\int_{\mathbb{R}^n} \nabla \cdot \mathbf{v}\, dx = 0.$$

1.2 Adjoints for the divergence and the gradient operators

Let V and U be Hilbert spaces. Then we recall that the adjoint T^* of a linear operator $T : V \to U$ satisfies

$$< Tv, u >_U = < v, T^* u >_V,$$

for all $v \in Dom(T)$ and $u \in \mathrm{Dom}(T^*)$. Next we show the following:

Proposition 1.1. *The adjoints of the gradient and divergence operators are given by*

(I) $(\nabla \cdot)^* = -\nabla$

(II) $(\nabla)^* = -\nabla \cdot$

Proof. Let $\mathbf{v}$ be a differentiable vector field and u be a differentiable compactly supported scalar field.

For (I), we want to show that $< \nabla \cdot \mathbf{v}, u > = < \mathbf{v}, -\nabla u >$.

Using the identity (1.1) along with the divergence theorem

$$\begin{aligned} < \nabla \cdot \mathbf{v}, u > &= \int_{\mathbb{R}^n} (\nabla \cdot \mathbf{v})\; u\; dx = -\int_{\mathbb{R}^n} \mathbf{v} \cdot \nabla u \;\; dx \\ &= \int_{\mathbb{R}^n} \mathbf{v} \cdot (-\nabla u)\; dx = < \mathbf{v}, -\nabla u >, \end{aligned}$$

and hence $(\nabla\cdot)^* = -\nabla$ as required.

For (II), we want to show that $< \nabla u, \mathbf{v} >=< u, -\nabla \cdot \mathbf{v} >$.

Using (1.1) with the divergence theorem,

$$\begin{aligned} < \nabla u, \mathbf{v} > &= \int_{\mathbb{R}^n} \nabla u \cdot \mathbf{v} \; dx = - \int_{\mathbb{R}^n} u \; \nabla \cdot \mathbf{v} \; dx \\ &= \int_{\mathbb{R}^n} u \; (-\nabla \cdot \mathbf{v}) \; dx =< u, -\nabla \cdot \mathbf{v} >, \end{aligned}$$

and hence $(\nabla)^* = -\nabla$. completing the proof. □

1.3 Integration by parts formula

Let $\mathbf{v}$ be a vector field, and u be a scalar field. Then

$$\int_\Omega \mathbf{v} \cdot \nabla u \; dx = \int_{\partial\Omega} u \; \mathbf{v} \cdot \mathbf{n} \; ds - \int_\Omega u \; \nabla \cdot \mathbf{v} \; dx, \quad \Omega \subseteq \mathbb{R}^n \text{ and } \mathbf{n} \text{ unit normal vector.}$$

Proof. By integrating (1.1) on Ω we get,

$$\int_\Omega \nabla \cdot (u \; \mathbf{v}) \; dx = \int_\Omega u \; \nabla \cdot \mathbf{v} \; dx + \int_\Omega \mathbf{v} \cdot \nabla u \; dx,$$

and by divergence theorem we know $\int_\Omega \nabla \cdot (u \; \mathbf{v}) \; dx = \int_{\partial\Omega} u \; \mathbf{v} \cdot \mathbf{n} \; ds$. Then

$$\int_{\partial\Omega} u \; \mathbf{v} \cdot \mathbf{n} \; ds = \int_\Omega u \; \nabla \cdot \mathbf{v} \; dx + \int_\Omega \mathbf{v} \cdot \nabla u \; dx,$$

and hence

$$\int_\Omega \mathbf{v} \cdot \nabla u \; dx = \int_{\partial\Omega} u \; \mathbf{v} \cdot \mathbf{n} \; ds - \int_\Omega u \; \nabla \cdot \mathbf{v} \; dx.$$

□

1.4 Green's identities for local operators

First Green's identity

Let u, v are twice differentiable and $\mathbf{n}$ be a unit normal vector. Then

$$\int_{\Omega} u \ \Delta v + \nabla u \cdot \nabla v \ dx = \int_{\partial\Omega} u \ (\nabla v \cdot \mathbf{n}) ds, \tag{1.2}$$

Proof. Since $\nabla(uv) = u \ \nabla v + v \nabla u$ and,

$$\nabla \cdot (u \ \nabla v) = u \ \nabla \cdot \nabla v + \nabla u \cdot \nabla v = u \ \Delta v + \nabla u \cdot \nabla v,$$

By integrating on Ω we get,

$$\int_{\Omega} \nabla \cdot (u \ \nabla v) \ dx = \int_{\Omega} u \ \Delta v + \nabla u \cdot \nabla v \ dx,$$

and by the divergence theorem, we obtain

$$\int_{\partial\Omega} u \ \nabla v \cdot \mathbf{n} \ ds = \int_{\Omega} u \ \Delta v + \nabla u \cdot \nabla v \ dx.$$

□

Second Green's identity

Let u, v are twice differentiable and $\mathbf{n}$ be a unit normal vector. Then

$$\int_{\Omega} u \ \Delta v - v \ \Delta u \ dx = \int_{\partial\Omega} (u \ \nabla v - v \ \nabla u) \cdot \mathbf{n} \ ds.$$

Proof. Here we use (1.2) in the proof,

$$
\begin{aligned}
\int_\Omega u\ \Delta v - v\ \Delta u\ dx &= \int_{\partial\Omega} u\ (\nabla v \cdot \mathbf{n})\ ds - \int_\Omega \nabla u \cdot \nabla v\ dx - \int_\Omega v\ \Delta u\ dx, \\
&= \int_{\partial\Omega} u\ (\nabla v \cdot \mathbf{n})\ ds - \int_\Omega \nabla u \cdot \nabla v\ dx - \left[\int_{\partial\Omega} v\ (\nabla u \cdot \mathbf{n})\ ds - \int_\Omega \nabla v \cdot \nabla u\ dx \right], \\
&= \int_{\partial\Omega} u\ (\nabla v \cdot \mathbf{n})\ ds - \int_{\partial\Omega} v\ (\nabla u \cdot \mathbf{n})\ ds, \\
&= \int_{\partial\Omega} (u\ \nabla v - v \nabla u) \cdot \mathbf{n}\ ds.
\end{aligned}
$$

$\square$

Chapter 2

Nonlocal Operators

2.1 Overview

In this chapter and the following chapters, we deal with two different types of functions and two different types of nonlocal operators. A one-point scalar function u refers to a function $u = u(x)$, with $x \in \mathbb{R}^n$, and a two-point scalar function u refers to a function defined for a pair of points $u = u(x, y)$, with $x, y \in \mathbb{R}^n$. Similarly, a one-point vector field $\mathbf{v}$ refers to a vector field $\mathbf{v} = \mathbf{v}(x)$, and a two-point vector field is of the form $\mathbf{v} = \mathbf{v}(x, y)$. One-point operators map two-point functions to one-point functions, and two-point operators map one-point functions to two-point functions. In addition, symmetric and antisymmetric two-point scalar functions satisfy $\psi(x, y) = \psi(y, x)$ and $\psi(x, y) = -\psi(y, x)$, respectively, and similarly for symmetric and antisymmetric two-point vector fields.

For example, let $\boldsymbol{\psi}(x, y) = \frac{y-x}{||y-x||}$, $x \neq y$, with $x, y \in \mathbb{R}^n$. Then $\boldsymbol{\psi}$ is an antisymmetric two-point vector field. An example of a symmetric two-point scalar function is given by $\phi(x, y) = \frac{||y||+||x||}{||y-x||}$, $x \neq y$, with $x, y \in \mathbb{R}^n$.

Theorem 2.1. *The following are equivalent:*

(i) *the antisymmetry of a two-point function* $\psi(x, y) = -\psi(y, x), \qquad \forall x, y \in \mathbb{R}^n.$

(ii) $\displaystyle\int_\Omega \int_\Omega \psi(x, y) dy dx = 0, \qquad \forall\, \Omega \subseteq \mathbb{R}^n.$

(iii) $\displaystyle\int_{\Omega_1} \int_{\Omega_2} \psi(x, y) dy dx + \int_{\Omega_2} \int_{\Omega_1} \psi(x, y) dy dx = 0, \qquad \forall\, \Omega_1, \Omega_2 \subseteq \mathbb{R}^n.$

Proof. $(i) \Rightarrow (ii)$

Let $\psi(x,y) = -\psi(y,x), \qquad \forall\, x, y \in \mathbb{R}^n$ then

$$\int_\Omega \int_\Omega \psi(x,y)dydx = -\int_\Omega \int_\Omega \psi(y,x)dydx.$$

By interchanging the variables $(x \leftrightarrow y)$ in the second integral

$$\int_\Omega \int_\Omega \psi(x,y)dydx = -\int_\Omega \int_\Omega \psi(x,y)dydx,$$

and hence $\int_\Omega \int_\Omega \psi(x,y)dydx = 0$.

$(ii) \Rightarrow (i)$

Assume $\int_\Omega \int_\Omega \psi(x,y)dydx = 0, \qquad \forall\, \Omega \subseteq \mathbb{R}^n$. Then

$$\int_\Omega \int_\Omega \psi(x,y)dydx + \int_\Omega \int_\Omega \psi(x,y)dydx = 0,$$

By interchanging $(x \leftrightarrow y)$ in the second integral, we obtain

$$\int_\Omega \int_\Omega \psi(x,y)dydx + \int_\Omega \int_\Omega \psi(y,x)dydx = 0,$$

$$\int_\Omega \int_\Omega (\psi(x,y) + \psi(y,x))dydx = 0, \qquad \forall x, y \in \mathbb{R}^n,$$

which implies $\psi(x,y) + \psi(y,x) = 0$, and hence $\psi(x,y) = -\psi(y,x)$. $\forall\, x, y \in \mathbb{R}^n$.

$(i) \Rightarrow (iii)$

Assume $\psi(x,y) = -\psi(y,x), \qquad \forall\, x, y \in \mathbb{R}^n$. Then,

$$\int_{\Omega_1} \int_{\Omega_2} \psi(x,y)dydx = -\int_{\Omega_1} \int_{\Omega_2} \psi(y,x)dydx, \qquad \forall\ \Omega_1, \Omega_2 \subseteq \mathbb{R}^n$$

By interchanging $(x \leftrightarrow y)$ in the second integral, we obtain

$$\int_{\Omega_1}\int_{\Omega_2} \psi(x,y)dydx + \int_{\Omega_2}\int_{\Omega_1} \psi(x,y)dydx = 0.$$

$(iii) \Rightarrow (i)$

Assume $\int_{\Omega_1}\int_{\Omega_2} \psi(x,y)dydx + \int_{\Omega_2}\int_{\Omega_1} \psi(x,y)dydx = 0, \qquad \Omega_1, \Omega_2 \subseteq \mathbb{R}^n.$
By interchanging $(x \leftrightarrow y)$ in the second integral, we obtain

$$\int_{\Omega_1}\int_{\Omega_2} \psi(x,y)dydx + \int_{\Omega_1}\int_{\Omega_2} \psi(y,x)dydx = 0,$$

$$\int_{\Omega_1}\int_{\Omega_2} (\psi(x,y) + \psi(y,x))dydx = 0, \qquad \forall x, y \in \mathbb{R}^n$$

and hence $\psi(x,y) = -\psi(y,x), \quad \forall x, y \in \mathbb{R}^n.$ □

2.2 Nonlocal operators

Nonlocal divergence operator

Definition 2.1. *Given a two-point vector field* $\mathbf{v} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *and an antisymmetric two-point vector field* $\boldsymbol{\alpha} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. *Then the action of the nonlocal point divergence operator* $\mathcal{D}$ *on* $\mathbf{v}$ *is defined by*

$$\mathcal{D}(\mathbf{v})(x) \coloneqq \int_{\mathbb{R}^n} (\mathbf{v}(x,y) + \mathbf{v}(y,x)) \cdot \boldsymbol{\alpha}(x,y)dy, \qquad for\ x \in \mathbb{R}^n,$$

where $\mathcal{D}(\mathbf{v}) : \mathbb{R}^n \to \mathbb{R}$.

Note that if we take $\psi(x,y) = (\mathbf{v}(x,y)+\mathbf{v}(y,x))\cdot\boldsymbol{\alpha}(x,y)$, *then* $\psi(x,y)$ *is an antisymmetric two-point scalar function.*

Nonlocal gradient operator

Definition 2.2. *Given a two-point scalar function* $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *and an antisymmetric two-point vector field* $\boldsymbol{\alpha} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. *Then the action of the nonlocal point gradient* $\mathcal{G}$

on η is defined by

$$\mathcal{G}(\eta)(x) := \int_{\mathbb{R}^n} (\eta(x,y) + \eta(y,x))\boldsymbol{\alpha}(x,y)dy, \qquad for\ x \in \mathbb{R}^n,$$

where $\mathcal{G}(\eta)(x) : \mathbb{R}^n \to \mathbb{R}^n$.

Note that if we take $\boldsymbol{\psi}(x,y) = (\eta(x,y) + \eta(y,x))\boldsymbol{\alpha}(x,y)$, then $\boldsymbol{\psi}$ is an antisymmetric two-point vector field.

2.3 Free space nonlocal Calculus theorem

The free-space nonlocal divergence and nonlocal gradient theorems are given by

Theorem 2.2.

(i) $\int_{\mathbb{R}^n} \mathcal{D}(\mathbf{v})(x)dx = 0,$

(ii) $\int_{\mathbb{R}^n} \mathcal{G}(\eta)(x)dx = \mathbf{0}.$

Proof. Let $\mathbf{v}$ be a two-point vector field, $\boldsymbol{\alpha}$ an antisymmetric two-point vector field, and η a two-point scalar function. Then,

(i) since

$$\int_{\mathbb{R}^n} \mathcal{D}(\mathbf{v})(x)dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (\mathbf{v}(x,y) + \mathbf{v}(y,x)) \cdot \boldsymbol{\alpha}(x,y)dydx,$$

and by using Theorem (2.1) with

$$\begin{aligned} \psi(x,y) &= (\mathbf{v}(x,y) + \mathbf{v}(y,x)) \cdot \boldsymbol{\alpha}(x,y) \\ &= (\mathbf{v}(y,x) + \mathbf{v}(x,y)) \cdot -\boldsymbol{\alpha}(y,x) \\ &= -\psi(y,x), \end{aligned}$$

we obtain

$$0 = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \psi(x,y)dydx = \int_{\mathbb{R}^n} \mathcal{D}(\mathbf{v})(x)dx.$$

(ii) Since
$$\int_{\mathbb{R}^n} \mathcal{G}(\eta)(x)dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (\eta(x,y)+\eta(y,x))\boldsymbol{\alpha}(x,y)dydx,$$
then by using Theorem (2.1) with

$$\begin{aligned}\boldsymbol{\phi}(x,y) &= (\eta(x,y)+\eta(y,x))\boldsymbol{\alpha}(x,y)\\ &= (\eta(y,x)+\eta(x,y))(-\boldsymbol{\alpha}(y,x))\\ &= -\boldsymbol{\phi}(y,x),\end{aligned}$$

and hence
$$\mathbf{0} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \boldsymbol{\phi}(x,y)dydx = \int_{\mathbb{R}^n} \mathcal{G}(\eta)(x)dx.$$

□

2.4 Adjoints of nonlocal operators

Adjoint of the nonlocal divergence

Proposition 2.1. *Given a scalar point function $u : \mathbb{R}^n \to \mathbb{R}$ and an antisymmetric two-point vector field $\boldsymbol{\alpha} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. Then the adjoint of $\mathcal{D}$, denoted by $\mathcal{D}^*$, is the two-point operator acting on u which is given by*

$$\mathcal{D}^*(u)(x,y) = -(u(y)-u(x))\boldsymbol{\alpha}(x,y), \quad \textit{for } x,y \in \mathbb{R}^n,$$

with $\mathcal{D}^*(u) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$.

Proof. Let $\mathbf{v}$ be a two-point vector field and u a scalar point function. Define $\mathbf{w} = u\mathbf{v}$. From Theorem (2.2) we know that $\int_{\mathbb{R}^n} \mathcal{D}(\mathbf{w})(x)dx = 0$, which implies

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (\mathbf{w}(x,y)+\mathbf{w}(y,x)) \cdot \boldsymbol{\alpha}(x,y)dydx = 0,$$

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(u(x)\mathbf{v}(x,y)+u(y)\mathbf{v}(y,x))\cdot\boldsymbol{\alpha}(x,y)dydx=0,$$

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(u(x)\mathbf{v}(x,y)+u(x)\mathbf{v}(y,x)-u(x)\mathbf{v}(y,x)+u(y)\mathbf{v}(y,x))\cdot\boldsymbol{\alpha}(x,y)dydx=0,$$

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}u(x)(\mathbf{v}(x,y)+\mathbf{v}(y,x))\cdot\boldsymbol{\alpha}(x,y)dydx+\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(u(y)-u(x))\mathbf{v}(y,x)\cdot\boldsymbol{\alpha}(x,y)dydx=0,$$

$$\int_{\mathbb{R}^n}\mathcal{D}(\mathbf{v})(x)u(x)dx=-\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(u(y)-u(x))\mathbf{v}(y,x)\cdot\boldsymbol{\alpha}(x,y)dydx.$$

By interchanging $(x\leftrightarrow y)$ in the second integral above, we obtain

$$\int_{\mathbb{R}^n}\mathcal{D}(\mathbf{v})(x)u(x)dx=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}-(u(y)-u(x))\mathbf{v}(x,y)\cdot\boldsymbol{\alpha}(x,y)dydx,$$

$$\int_{\mathbb{R}^n}\mathcal{D}(\mathbf{v})(x)u(x)dx=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(\mathcal{D}^*u)(x,y)\cdot\mathbf{v}(x,y)dydx,$$

and hence $\langle\mathcal{D}\mathbf{v},u\rangle=\langle\mathbf{v},\mathcal{D}^*u\rangle$. □

Adjoint of the nonlocal gradient

Proposition 2.2. *Given a point vector field $\mathbf{v}:\mathbb{R}^n\to\mathbb{R}^n$ and an antisymmetric two-point vector field $\boldsymbol{\alpha}:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$, the adjoint of $\mathcal{G}$ is the two-point operator acting on $\mathbf{v}$, which is given by*

$$\mathcal{G}^*(\mathbf{v})(x,y)=-(\mathbf{v}(y)-\mathbf{v}(x))\cdot\boldsymbol{\alpha}(x,y),\qquad \textit{for } x,y\in\mathbb{R}^n,$$

where $\mathcal{G}^*(\mathbf{v}):\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$.

Proof. Here we want to show $\langle\mathbf{v},\mathcal{G}\eta\rangle=\langle\mathcal{G}^*\mathbf{v},\eta\rangle$, for all two-point scalar functions η and all one-point vector field $\mathbf{v}$,

$$\begin{aligned}
< \mathbf{v}, \mathcal{G}\eta > &= \int_{\mathbb{R}^n} \mathbf{v}(x) \cdot (\mathcal{G}\eta)(x) dx = \int_{\mathbb{R}^n} \mathbf{v}(x) \cdot \int_{\mathbb{R}^n} (\eta(x,y) + \eta(y,x)) \boldsymbol{\alpha}(x,y) dy dx, \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (\eta(x,y) + \eta(y,x)) \mathbf{v}(x) \cdot \boldsymbol{\alpha}(x,y) dy dx, \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \eta(x,y) \mathbf{v}(x) \cdot \boldsymbol{\alpha}(x,y) dy dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \eta(y,x) \mathbf{v}(x) \cdot \boldsymbol{\alpha}(x,y) dy dx.
\end{aligned}$$

Then by interchanging $(x \leftrightarrow y)$ in the second integral above, we obtain

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \eta(x,y) \mathbf{v}(x) \cdot \boldsymbol{\alpha}(x,y) dy dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \eta(x,y) \mathbf{v}(y) \cdot \boldsymbol{\alpha}(y,x) dy dx,$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} -(\mathbf{v}(y) - \mathbf{v}(x)) \cdot \boldsymbol{\alpha}(x,y)) \eta(x,y) dy dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{G}^*(\mathbf{v}) \; \eta(x,y) dy dx = \langle \mathcal{G}^* \mathbf{v}, \eta \rangle.$$

□

Proposition 2.3. *The operators $\mathcal{G}^*$ and $\mathcal{D}^*$ are symmetric two-point operators.*

Proof. Let $\mathbf{v}$ be a point vector field and $\boldsymbol{\alpha}$ an antisymmetric two-point vector field. Then

$$\begin{aligned}
\mathcal{G}^*(\mathbf{v})(y,x) &= -(\mathbf{v}(x) - \mathbf{v}(y)) \cdot \boldsymbol{\alpha}(y,x) \\
&= (\mathbf{v}(y) - \mathbf{v}(x)) \cdot \boldsymbol{\alpha}(y,x), \\
&= -(\mathbf{v}(y) - \mathbf{v}(x)) \cdot \boldsymbol{\alpha}(x,y) = \mathcal{G}^*(\mathbf{v})(x,y),
\end{aligned}$$

for all $x, y \in \mathbb{R}^n$. Similarly, we can show that

$$\mathcal{D}^*(u)(x,y) = \mathcal{D}^*(u)(y,x),$$

for all scalar functions u. □

Chapter 3

Nonlocal Interaction Operators

We introduce nonlocal vector calculus theorems on bounded domains that are analogous to the classical local vector calculus theorems of differential operators. In local vector calculus, interactions between a domain Ω and the outside region occur through contact which is characterized by the flux through the boundary $\partial\Omega$. In contrast, nonlocal interactions occur over a finite distance and not necessarily through contact.

3.1 Interaction domain

Definition 3.1. *Given an open subset $\Omega \subset \mathbb{R}^n$ and an antisymmetric two-point vector field $\boldsymbol{\alpha}$. Then the interaction domain associated with Ω is given by*

$$\Omega_I \coloneqq \{y \in \mathbb{R}^n \setminus \Omega, \quad \textit{such that } \boldsymbol{\alpha}(x,y) \neq 0, \qquad \textit{for some } x \in \Omega\}.$$

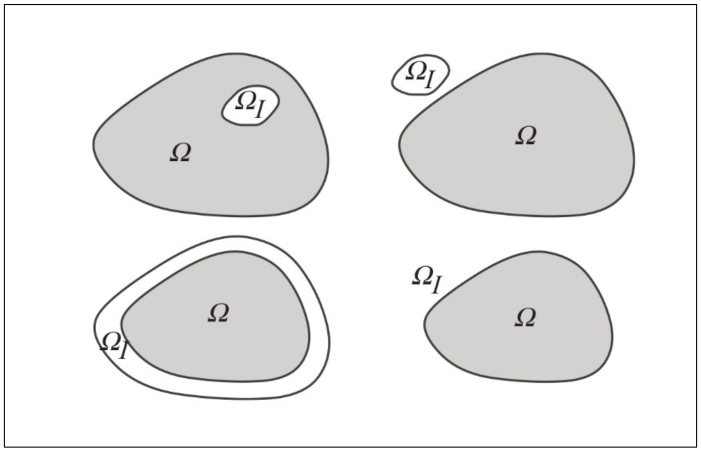

Figure 3.1: *A domain Ω and different possibilities for its interaction domain Ω_I*[4].

Note that Ω_I consists of those points outside of Ω that interact with points in Ω. In addition, points in $\mathbb{R}^n \setminus (\Omega \cup \Omega_I)$ do not interact with any point in Ω, as illustrated in Figure 3.1.

3.2 Nonlocal interaction operators

Here we want to define nonlocal operators on the set Ω_I analogous to boundary operators in the local setting.

The interaction divergence operator

Definition 3.2. *Given a domain $\Omega \subset \mathbb{R}^n$ and a two-point vector field $\mathbf{v}$. Then corresponding to the point divergence operator $\mathcal{D}(\mathbf{v}) : \mathbb{R}^n \to \mathbb{R}$, we define the point interaction divergence operator $\mathcal{N}(\mathbf{v}) : \Omega_I \to \mathbb{R}$ through its action on $\mathbf{v}$ by*

$$\mathcal{N}(\mathbf{v})(x) := -\int_{\Omega \cup \Omega_I} (\mathbf{v}(x,y) + \mathbf{v}(y,x)) \cdot \boldsymbol{\alpha}(x,y) dy, \qquad \textit{for } x \in \Omega_I.$$

The interaction gradient operator

Definition 3.3. *Given a domain $\Omega \subset \mathbb{R}^n$ and a two-point scalar function η. Then corresponding to the point gradient operator $\mathcal{G}(\eta) \colon \mathbb{R}^n \to \mathbb{R}^n$, we define the point interaction* *gradient operator* $\mathcal{S}(\eta)\colon \Omega_I \to \mathbb{R}^n$ *through its action on* η *by*

$$\mathcal{S}(\eta)(x) := -\int_{\Omega\cup\Omega_I} (\eta(x,y)+\eta(y,x))\boldsymbol{\alpha}(x,y)dy, \qquad \textit{for}\, x \in \Omega_I.$$

3.3 A nonlocal integral theorem

Theorem 3.1. *Let* $\mathbf{v}$ *be a two-point vector field and* η *a two-point scalar function. Then*

(i) $\int_\Omega \mathcal{D}(\mathbf{v})dx = \int_{\Omega_I} \mathcal{N}(\mathbf{v})dx.$

(ii) $\int_\Omega \mathcal{G}(\eta)dx = \int_{\Omega_I} \mathcal{S}(\eta)dx.$

We note that Part (i) is a nonlocal divergence theorem.

Proof. Part (i). Define $\psi(x,y) = (\mathbf{v}(x,y)+\mathbf{v}(y,x))\cdot\boldsymbol{\alpha}(y,x) = -\psi(y,x)$, then by Theorem 2.1 we have $\int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}\psi(x,y)dydx = 0$, which implies

$$\int_\Omega\int_{\Omega\cup\Omega_I}(\mathbf{v}(x,y)+\mathbf{v}(y,x))\cdot\boldsymbol{\alpha}(x,y)dydx + \int_{\Omega_I}\int_{\Omega\cup\Omega_I}(\mathbf{v}(x,y)+\mathbf{v}(y,x))\cdot\boldsymbol{\alpha}(x,y)dydx = 0,$$

and hence

$$\int_\Omega \mathcal{D}(\mathbf{v})(x)dx = \int_{\Omega_I}\mathcal{N}(\mathbf{v})(x)dx.$$

Part (ii). From part (i) choose $\mathbf{v}(x,y) = \eta(x,y)\mathbf{b}$, *where* $\mathbf{b}$ is an arbitrary constant vector. Then

$$\int_\Omega\int_{\Omega\cup\Omega_I}(\eta(x,y)+\eta(y,x))\mathbf{b}\cdot\boldsymbol{\alpha}(x,y)dydx = -\int_{\Omega_I}\int_{\Omega\cup\Omega_I}(\eta(x,y)+\eta(y,x))\mathbf{b}\cdot\boldsymbol{\alpha}(x,y)dydx,$$

which implies that

$$\mathbf{b}\cdot\left(\int_\Omega \mathcal{G}(\eta)(x)dx - \int_{\Omega_I}\mathcal{S}(\mu)(x)dx\right) = 0,$$

and since $\mathbf{b}$ is arbitrary, we conclude

$$\int_\Omega \mathcal{G}(\eta)(x)dx = \int_{\Omega_I}\mathcal{S}(\mu)(x)dx.$$

$\square$

3.4 Nonlocal integration by parts formulas

Theorem 3.2. *Given* $u\colon \mathbb{R}^n \to \mathbb{R}$, $\mathbf{w}\colon \mathbb{R}^n \to \mathbb{R}^n$, *a two-point scalar function* $\eta\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, *and a two-point vector field* $\mathbf{v}\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. *Then*

(i)

$$\int_\Omega u(x)\mathcal{D}(\mathbf{v})(x)dx - \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I} \mathcal{D}^*(u)(x,y)\cdot\mathbf{v}(x,y)dydx = \int_{\Omega_I} u(x)\mathcal{N}(\mathbf{v})(x)dx. \tag{3.1}$$

(ii)

$$\int_\Omega \mathbf{w}(x)\cdot\mathcal{G}(\eta)(x)\,dx - \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I} \mathcal{G}^*(\mathbf{w})(x,y)\eta(x,y)dydx = \int_{\Omega_I} \mathbf{w}(x)\cdot\mathcal{S}(\eta)(x)\,dx. \tag{3.2}$$

Proof. Part (i). Let $\boldsymbol{\xi}(x,y) = u(x)\mathbf{v}(x,y)$, then we have

$$\begin{aligned}
\mathcal{D}(\boldsymbol{\xi})(x) =&\mathcal{D}(u\mathbf{v})(x) = \int_{\mathbb{R}^n} (u(x)\mathbf{v}(x,y) + u(y)\mathbf{v}(y,x))\cdot\boldsymbol{\alpha}(x,y)\,dy,\\
&= \int_{\mathbb{R}^n} (u(x)(\mathbf{v}(x,y)+\mathbf{v}(y,x))\cdot\boldsymbol{\alpha}(x,y) + (u(y)-u(x))\mathbf{v}(y,x)\cdot\boldsymbol{\alpha}(x,y))\,dy,\\
&=u(x)\int_{\mathbb{R}^n} (\mathbf{v}(x,y)+\mathbf{v}(y,x))\cdot\boldsymbol{\alpha}(x,y)dy + \int_{\mathbb{R}^n}(u(y)-u(x))\mathbf{v}(y,x)\cdot\boldsymbol{\alpha}(x,y)dy,\\
&=u(x)\mathcal{D}(\mathbf{v})(x) + \int_{\Omega\cup\Omega_I}(u(y)-u(x))\mathbf{v}(y,x)\cdot\boldsymbol{\alpha}(x,y)dy.
\end{aligned}$$

Similarly, we can express $\mathcal{N}(\boldsymbol{\xi})$ as follows

$$\begin{aligned}
\mathcal{N}(\boldsymbol{\xi})(x) =& -\int_{\Omega\cup\Omega_I} (\mathbf{v}(y,x)+\mathbf{v}(y,x))u(x)\cdot\boldsymbol{\alpha}(x,y) + (u(y)-u(x))\mathbf{v}(y,x)\cdot\boldsymbol{\alpha}(x,y)dy,\\
&=u(x)\mathcal{N}(\mathbf{v})(x) - \int_{\Omega\cup\Omega_I}(u(y)-u(x))\mathbf{v}(y,x)\cdot\boldsymbol{\alpha}(x,y)dy.
\end{aligned}$$

Then by using Theorem 3.1, and the above equations, we obtain

$$\begin{aligned}
0 =& \int_\Omega \mathcal{D}(\boldsymbol{\xi})(x)dx - \int_{\Omega_I}\mathcal{N}(\boldsymbol{\xi})(x)dx,\\
=& \int_\Omega u\mathcal{D}(\mathbf{v})(x)dx - \int_{\Omega_I} u\mathcal{N}(\mathbf{v})(x)dx + \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}(u(y)-u(x))\mathbf{v}(y,x)\cdot\boldsymbol{\alpha}(x,y)dy,
\end{aligned}$$

and hence

$$\int_{\Omega} u(x)\mathcal{D}(\mathbf{v})(x)dx - \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I} \mathcal{D}^*(u)(x,y)\cdot\mathbf{v}(x,y)dydx = \int_{\Omega_I} u(x)\mathcal{N}(\mathbf{v})(x)dx.$$

Part (ii). Let $\mu(x,y) = \eta(x,y)\mathbf{b}\cdot\mathbf{w}(x)$, where $\mathbf{b}$ is an arbitrary constant vector.

$$\begin{aligned}\mathcal{G}(\mu)(x) =& \int_{\mathbb{R}^n}((\eta(x,y)+\eta(y,x))\mathbf{b}\cdot\mathbf{w}(x))\boldsymbol{\alpha}(x,y)dy + \int_{\mathbb{R}^n}((\mathbf{w}(y)-\mathbf{w}(x))\cdot\mathbf{b}\;\eta(y,x))\boldsymbol{\alpha}(x,y)dy,\\ =&(\mathbf{w}(x)\cdot\mathcal{G}(\eta)(x))\mathbf{b} + \int_{\Omega\cup\Omega_I}((\mathbf{w}(y)-\mathbf{w}(x))\cdot\mathbf{b}\eta(y,x))\,\boldsymbol{\alpha}(x,y)dy.\end{aligned}$$

$$\begin{aligned}\mathcal{S}(\mu)(x) =& -\int_{\Omega\cup\Omega_I}(\mathbf{w}(x)\cdot(\eta(x,y)-\eta(y,x))\mathbf{b})\;\boldsymbol{\alpha}(x,y)dy + \int_{\Omega\cup\Omega_I}((\mathbf{w}(y)-\mathbf{w}(x))\cdot\eta(y,x)\mathbf{b})\;\boldsymbol{\alpha}(x,y)dy,\\ =&(\mathbf{w}(x)\cdot\mathcal{S}(\eta)(x))\mathbf{b} - \int_{\Omega\cup\Omega_I}((\mathbf{w}(y)-\mathbf{w}(x))\cdot\mathbf{b})\eta(y,x)\;\boldsymbol{\alpha}(x,y)dy.\end{aligned}$$

Then by using Theorem 3.1, and the above equations, we obtain

$$\begin{aligned}\mathbf{0} =& \int_{\Omega}\mathcal{G}(\mu)(x)dx - \int_{\Omega_I}\mathcal{S}(\mu)(x)dx,\\ =&\mathbf{b}\int_{\Omega}\mathbf{w}(x)\cdot\mathcal{G}(\eta)(x)dx - \mathbf{b}\int_{\Omega_I}\mathbf{w}(x)\cdot\mathcal{S}(\eta)(x)dx + \mathbf{b}\int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}(\mathbf{w}(y)-\mathbf{w}(x))\cdot\eta(y,x)\boldsymbol{\alpha}(x,y)dydx,\\ =&\mathbf{b}\left(\int_{\Omega}\mathbf{w}(x)\cdot\mathcal{G}(\eta)(x)dx - \int_{\Omega_I}\mathbf{w}(x)\cdot\mathcal{S}(\eta)(x)dx + \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}(\mathbf{w}(y)-\mathbf{w}(x))\cdot\eta(y,x)\boldsymbol{\alpha}(x,y)dydx\right).\end{aligned}$$

Since $\mathbf{b}$ is arbitrary, we obtain

$$0 = \int_{\Omega}\mathbf{w}(x)\cdot\mathcal{G}(\eta)(x)dx - \int_{\Omega_I}\mathbf{w}(x)\cdot\mathcal{S}(\eta)(x)dx + \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}(\mathbf{w}(y)-\mathbf{w}(x))\eta(y,x)\cdot\boldsymbol{\alpha}(x,y)dydx,$$

and therefore

$$\int_{\Omega}\mathbf{w}(x)\cdot\mathcal{G}(\eta)(x)dx - \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I} -(\mathbf{w}(y)-\mathbf{w}(x))\eta(y,x)\cdot\boldsymbol{\alpha}(x,y)dydx = \int_{\Omega_I}\mathbf{w}(x)\cdot\mathcal{S}(\eta)(x)dx,$$

from which, it follows that

$$\int_{\Omega}\mathbf{w}(x)\cdot\mathcal{G}(\eta)(x)dx - \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}\mathcal{G}^*(\mathbf{w})(x,y)\eta(x,y)dydx = \int_{\Omega_I}\mathbf{w}(x)\cdot\mathcal{S}(\eta)(x)dx.$$

□

3.5 Nonlocal Green's identities

Nonlocal Green's first identities

Given the point scalar functions $u\colon \mathbb{R}^n \to \mathbb{R}$ and $v\colon \mathbb{R}^n \to \mathbb{R}$. Then

$$\int_{\Omega} u\mathcal{D}\left(\mathcal{D}^*(v)\right)dx - \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I} \mathcal{D}^*(u)\cdot\mathcal{D}^*(v)dydx = \int_{\Omega_I} u\mathcal{N}\left(\mathcal{D}^*(v)\right)dx. \tag{3.3}$$

Moreover, given the vector fields $\mathbf{u}\colon \mathbb{R}^n \to \mathbb{R}^n$ and $\mathbf{v}\colon \mathbb{R}^n \to \mathbb{R}^n$. Then

$$\int_{\Omega} \mathbf{v}\cdot\mathcal{G}\left(\mathcal{G}^*(\mathbf{u})\right)dx - \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I} \mathcal{G}^*(\mathbf{v})\mathcal{G}^*(\mathbf{u})dydx = \int_{\Omega_I} \mathbf{v}\cdot\mathcal{S}\left(\mathcal{G}^*(\mathbf{u})\right)dx. \tag{3.4}$$

Proof. Let $\mathbf{v} = \mathcal{D}^*(v)$ in (3.1), we obtain

$$\int_{\Omega} u\mathcal{D}\left(\mathcal{D}^*(v)\right)dx - \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I} \mathcal{D}^*(u)\cdot\mathcal{D}^*(v)dydx = \int_{\Omega_I} u\mathcal{N}\left(\mathcal{D}^*(v)\right)dx.$$

In addition, let $\eta = \mathcal{G}^*(\mathbf{u})$ in (3.2), we obtain

$$\int_{\Omega} \mathbf{v}\cdot\mathcal{G}\left(\mathcal{G}^*(\mathbf{u})\right)dx - \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I} \mathcal{G}^*(\mathbf{v})\mathcal{G}^*(\mathbf{u})dydx = \int_{\Omega_I} \mathbf{v}\cdot\mathcal{S}\left(\mathcal{G}^*(\mathbf{u})\right)dx.$$

□

Nonlocal Green's second identity for nonlocal divergence

Given the point scalar functions $u, v\colon \mathbb{R}^n \to \mathbb{R}$. Then

$$\int_{\Omega} u\mathcal{D}\left(\mathcal{D}^*(v)\right)dx - \int_{\Omega} v\mathcal{D}\left(\mathcal{D}^*(u)\right)dx = \int_{\Omega_I} u\mathcal{N}\left(\mathcal{D}^*(v)\right)dx - \int_{\Omega_I} v\mathcal{N}\left(\mathcal{D}^*(u)\right)dx.$$

Proof. By using (3.3), we obtain

$$
\begin{aligned}
\int_{\Omega} u\mathcal{D}\left(\mathcal{D}^*(v)\right) dx - \int_{\Omega} v\mathcal{D}\left(\mathcal{D}^*(u)\right) dx = &\int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I} \mathcal{D}^*(u)\cdot\mathcal{D}^*(v)dydx + \int_{\Omega_I} u\mathcal{N}\left(\mathcal{D}^*(v)\right) dx \\
&-\int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I} \mathcal{D}^*(u)\cdot\mathcal{D}^*(v)dydx - \int_{\Omega_I} v\mathcal{N}\left(\mathcal{D}^*(u)\right) dx, \\
= &\int_{\Omega_I} u\mathcal{N}\left(\mathcal{D}^*(v)\right) dx - \int_{\Omega_I} v\mathcal{N}\left(\mathcal{D}^*(u)\right) dx.
\end{aligned}
$$

□

Nonlocal Green's second identity for nonlocal gradient

Given the vector fields $\mathbf{u}$, $\mathbf{v}\colon \mathbb{R}^n \to \mathbb{R}^n$. Then

$$
\int_{\Omega} \mathbf{v}\cdot\mathcal{G}\left(\mathcal{G}^*(\mathbf{u})\right) dx - \int_{\Omega} \mathbf{u}\cdot\mathcal{G}\left(\mathcal{G}^*(\mathbf{v})\right) dx = \int_{\Omega_I} \mathbf{v}\cdot\mathcal{S}\left(\mathcal{G}^*(\mathbf{u})\right) dx - \int_{\Omega_I} \mathbf{u}\cdot\mathcal{S}\left(\mathcal{G}^*(\mathbf{v})\right) dx.
$$

Proof. By using (3.4), we have

$$
\begin{aligned}
\int_{\Omega} \mathbf{v}\cdot\mathcal{G}\left(\mathcal{G}^*(\mathbf{u})\right) dx - \int_{\Omega} \mathbf{u}\cdot\mathcal{G}\left(\mathcal{G}^*(\mathbf{v})\right) dx = &\int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I} \mathcal{G}^*(\mathbf{v})\mathcal{G}^*(\mathbf{u})dydx + \int_{\Omega_I} \mathbf{v}\cdot\mathcal{S}\left(\mathcal{G}^*(\mathbf{u})\right) dx \\
&-\int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I} \mathcal{G}^*(\mathbf{v})\mathcal{G}^*(\mathbf{u})dydx - \int_{\Omega_I} \mathbf{u}\cdot\mathcal{S}\left(\mathcal{G}^*(\mathbf{v})\right) dx, \\
= &\int_{\Omega_I} \mathbf{v}\cdot\mathcal{S}\left(\mathcal{G}^*(\mathbf{u})\right) dx - \int_{\Omega_I} \mathbf{u}\cdot\mathcal{S}\left(\mathcal{G}^*(\mathbf{v})\right) dx.
\end{aligned}
$$

□

Chapter 4

A Connection Between Local and Nonlocal Operators

Our goal in this chapter is to show that there is a connection between local and nonlocal operators. We define the nonlocal Laplacian, which we denote by $\mathcal{L}$, and then show that for a specific integral kernel, the nonlocal Laplacian converges to the classical Laplacian operator.

4.1 Nonlocal Laplacian operator

Define a two-point vector field $\boldsymbol{\alpha}$ by

$$\boldsymbol{\alpha}(x, y) = \frac{\sqrt{c_\delta}}{\sqrt{2}} \frac{y - x}{||y - x||^{\frac{\beta+2}{2}}} \chi_{B_\delta(x)}(y)$$

where the exponent $\beta > 0$ and $c_\delta > 0$ is a scaling factor. Note that $\boldsymbol{\alpha}$ is antisymmetric. Let Ω be an open subset of $\mathbb{R}^n$ and $u\colon \Omega \cup \Omega_I \to \mathbb{R}$. Then the nonlocal Laplacian operator

$\mathcal{L}$ acting on u is defined by

$$\begin{aligned}
\mathcal{L}(u)(x) &\coloneqq -\mathcal{D}\left(\mathcal{D}^*(u)\right)(x) = -\int_{\Omega\cup\Omega_I} \left((\mathcal{D}^*u)(y,x) + (\mathcal{D}^*u)(x,y)\right)\cdot \boldsymbol{\alpha}(x,y)dy,\\
&= -\int_{\Omega\cup\Omega_I} \left[-\left(u(x)-u(y)\right)\boldsymbol{\alpha}(y,x) - \left(u(y)-u(x)\right)\boldsymbol{\alpha}(x,y)\right]\cdot \boldsymbol{\alpha}(x,y)dy,\\
&= -\int_{\Omega\cup\Omega_I} 2\left(u(x)-u(y)\right)\boldsymbol{\alpha}(x,y)\cdot \boldsymbol{\alpha}(x,y)dy,\\
&= \int_{\Omega\cup\Omega_I} \left(u(y)-u(x)\right)2\boldsymbol{\alpha}(x,y)\cdot \boldsymbol{\alpha}(x,y)dy,\\
&= \int_{\Omega\cup\Omega_I} \left(u(y)-u(x)\right)\gamma(x,y)dy,
\end{aligned}$$

where $\gamma(x,y) = 2\boldsymbol{\alpha}(x,y)\cdot\boldsymbol{\alpha}(x,y)$, and hence

$$\mathcal{L}(u)(x) = -\mathcal{D}\left(\mathcal{D}^*(u)\right)(x) = \int_{\Omega\cup\Omega_I} \left(u(y)-u(x)\right)\cdot\gamma(x,y)dy, \qquad \text{for } x\in\Omega.$$

We observe that

$$\begin{aligned}
\gamma(x,y) =& 2\boldsymbol{\alpha}(x,y)\cdot\boldsymbol{\alpha}(x,y),\\
=& c_\delta \frac{||y-x||^2}{||y-x||^{\beta+2}}\chi_{B_\delta(x)}(y),\\
=& c_\delta \frac{1}{||y-x||^{\beta}}\chi_{B_\delta(x)}(y).
\end{aligned}$$

Since $(\Omega\cup\Omega_I)\cap B_\delta(x) = B_\delta(x)$, and $x\in\Omega$,

$$\begin{aligned}
\mathcal{L}(u)(x) &= \int_{\Omega\cup\Omega_I} \left(u(y)-u(x)\right)c_\delta \frac{1}{||y-x||^{\beta}}\chi_{B\delta(x)}(y)dy,\\
&= c_\delta \int_{\mathcal{B}_\delta(x)} \frac{u(y)-u(x)}{||y-x||^{\beta}}dy. \qquad (4.1)
\end{aligned}$$

In the one-dimensional case, let $u : [a-\delta, b+\delta] \to \mathbb{R}$, where $\Omega = (a,b)$ and $\Omega_I = [a-\delta, a]\cup [b, b+\delta]$. Then from (4.1), it follows that the nonlocal Laplacian $\mathcal{L}_\delta$, parametrized by δ, is given by

$$\mathcal{L}_\delta(u)(x) = c_\delta \int_{x-\delta}^{x+\delta} \frac{u(y)-u(x)}{\mid y-x\mid^{\beta}}dy.$$

The exponent β, in the one-dimensional case, satisfies $0 < \beta < 1$, and the scaling factor is given by

$$c_\delta = \left(\int_{-\delta}^{\delta} \frac{z^2}{2|z|^\beta} dz \right)^{-1} = \frac{3-\beta}{\delta^{3-\beta}}.$$

Observe that by a change of variables ($z = y - x$), the nonlocal Laplacian can be written as

$$\mathcal{L}_\delta(u)(x) = c_\delta \int_{-\delta}^{\delta} \frac{u(x+z) - u(x)}{|z|^\beta} dz. \tag{4.2}$$

The convergence of the nonlocal Laplacian in (4.2) to the classical Laplacian (the second derivative in this case) is provided by the following result.

Theorem 4.1. *Let $u \in \mathcal{C}^\infty([a-\delta, b+\delta])$. Then*

$$\lim_{\delta \to 0} \mathcal{L}_\delta u(x) = \Delta u(x) = u''(x), \qquad \textit{for all } x \in (a,b).$$

Proof. The Taylor expansion of $u(z)$ about $z = x$ is given by

$$u(x+z) = u(x) + u'(x)z + \frac{u''(x)}{2!} z^2 + \frac{u'''(x)}{3!} z^3 + \cdots$$

By substituting this expansion in (4.2), we obtain

$$\begin{aligned}
\mathcal{L}_\delta u(x) =& c_\delta \int_{-\delta}^{\delta} \frac{u'(x)z + \dfrac{u''(x)}{2!} z^2 + \cdots}{|\, z \,|^\beta} dz, \\
=& c_\delta \int_{-\delta}^{\delta} \frac{z}{|\, z \,|^\beta} dz \, u'(x) + c_\delta \int_{-\delta}^{\delta} \frac{z^2}{|\, z \,|^\beta} dz \, \frac{u''(x)}{2!} \\
&+ c_\delta \int_{-\delta}^{\delta} \frac{z^3}{|\, z \,|^\beta} dz \, \frac{u'''(x)}{3!} + c_\delta \int_{-\delta}^{\delta} \frac{z^4}{|\, z \,|^\beta} dz \, \frac{u^{(4)}(x)}{4!} + \cdots
\end{aligned}$$

Since $\dfrac{z^k}{|\, z \,|^\beta}$ is an odd function whenever k is odd, it follows that

$$\int_{-\delta}^{\delta} \frac{z^k}{|\, z \,|^\beta} dz = 0,$$

and hence,

$$\mathcal{L}_\delta u(x) = c_\delta \int_{-\delta}^{\delta} \frac{z^2}{\mid z \mid^\beta} dz \, \frac{u''(x)}{2!} + c_\delta \int_{-\delta}^{\delta} \frac{z^4}{\mid z \mid^\beta} dz \, \frac{u^{(4)}(x)}{4!} + \cdots$$

Calculating term by term in the above equation; the first term is

$$\begin{aligned}
\frac{1}{2!} c_\delta \int_{-\delta}^{\delta} \frac{z^2}{\mid z \mid^\beta} dz =& c_\delta \int_0^\delta z^{2-\beta} dz, \qquad , 0 < \beta < 1 \\
=& c_\delta \frac{\delta^{3-\beta}}{3-\beta} = \frac{3-\beta}{\delta^{3-\beta}} \frac{\delta^{3-\beta}}{3-\beta} = 1,
\end{aligned}$$

and the second term is given by

$$\begin{aligned}
\frac{1}{4!} c_\delta \int_{-\delta}^{\delta} \frac{z^4}{\mid z \mid^\beta} dz =& \frac{(3-\beta)}{4!} \frac{1}{\delta^{3-\beta}} 2 \int_0^\delta z^{4-\beta} dz, \\
=& \frac{2(3-\beta)}{4!} \frac{1}{5-\beta} \frac{1}{\delta^{3-\beta}} \delta^{5-\beta}, \\
=& \frac{2(3-\beta)}{4!(5-\beta)} \delta^2.
\end{aligned}$$

In general, it can be shown that,

$$c_\delta \int_{-\delta}^{\delta} \frac{z^{2k}}{|z|^\beta} dz = O(\delta^{2k}).$$

It follows that

$$\mathcal{L}_\delta u(x) = u''(x) + \frac{2(3-\beta)}{4!(5-\beta)} \delta^2 u^{(4)}(x) + \frac{2(3-\beta)}{6!(7-\beta)} \delta^4 u^{(6)}(x) + \cdots$$

and therefore,

$$\lim_{\delta \to 0} \mathcal{L}_\delta u(x) = u''(x), \qquad \forall\, x \in (a, b).$$

□

In the same manner, by choosing appropriate kernels, we can show that

$$\lim_{\delta \to 0} \mathcal{D}(\mathbf{v})(x) = \nabla \cdot \mathbf{v}(x), \quad \forall\, x \in \mathbb{R}^n,$$

and

$$\lim_{\delta \to 0} \mathcal{G}(u)(x) = \nabla u(x), \quad \forall\, x \in \mathbb{R}^n.$$

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