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PULSE RESPONSE OF NONLINEAR NONSTATIONARY VIBRATIONAL SYSTEMS

by

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I. INTRODUCTION

The investigation of nonlinear vibrational systems is of fundamental importance in many areas of physics, engineering and mechanics. Most naturally occuring vibrational systems are inherently nonlinear and technological advancements require increased sophistication in their analyses. A designer, for example, provided with a clear understanding of the behavioral features brought about by nonlinear characteristics, can more easily account for these features and perhaps incorporate them in the design. The understanding of nonlinear systems, however, has been restricted by the lack of techniques available for their analyses. Thus, any new development in this area is a significant contribution.

An important class of nonlinear vibrational systems are those with time dependent parameters, better known as nonstationary systems. The theory of nonlinear, nonstationary systems has many applications and numerous examples can be cited from the areas of quantum mechanics, solid mechanics, fluid mechanics, aeronautics, and engineering design which fall under the above mentioned category. Some recent applications include the studies by Scalia and Torrisi [1] for the motion of a variable mass gyroscope, by Balitskii and Borovkov [2] for the response of systems with limited excitation, by Ilin and Zhigula [3] for the dynamics of viscoelastic cables in load lifting devices, by Tordion and Gauvin [4] for the stability of gear trains with variable meshing stiffness, and by Gopak [5] for the dynamic stability of a rod of variable mass.

A very important class of such problems also occur in the theory of vibrations of parametrically excited nonlinear systems which has numerous applications in mechanics. Some of the recent applications dealing with beams and columns are by Tso [6], Ghoborah and Tso [7], Iwatsubo et al. [8], Elmaraghy and Tabarrok [9], Tesak et al. [10], and Crespo da Silva [11]. Examples of investigation of parametric excitation in other areas of mechanics are studies by Mote [12] and Rhodes [13] for moving strings and bands, by Nishikawa and Willems [14] for satellite attitude stability problems, by Friedmann and Silverthorn [15,16] for helicopter blade vibration problems, by Paidoussis and Sundararajan [17] for pipes conveying fluid with variable flow rate, and by Nayfeh [18] for the problems of re-entry missile dynamics.

The problem one encounters in the analysis of such systems is the inability to obtain an exact solution. Mathematically speaking, the systems lead to the analysis of nonlinear differential equations with time dependent coefficients, which, in general, have no closed form solutions. Thus one must turn to approximate techniques to gain some insight into the problem. It could of course be argued that there are various numerical methods available for the analysis of such systems. However, numerical solutions themselves do not lend much insight into the general behavior of the system as a function of its parameters. Thus any form of analytical solution, exact or approximate, is a valuable asset in the analysis and design of vibrational systems.

The purpose of this study is to investigate the behavior of nonlinear, nonstationary vibrational systems when subjected to an arbitrary pulse type loading. Such analyses are extremely important, for example, if one is interested in determining the transient behavior of structural or mechanical systems, with time dependent parameters, under the action of a shock loading.

However, analyses of this nature pose an additional complication due to the fact that the governing differential equation of a nonlinear, nonstationary system, contains a forcing term, which, in general, is nonperiodic. Many of the available approximate techniques can not be applied directly to such problems. To overcome this obstacle, a transformation of the dependent variable is introduced to yield an equivalent set of tractable differential equations. Approximate solutions are obtained through an application of the technique developed by Krylov, Bogoliubov, and Mitropolsky [19,20].

Presented in this study is a general procedure for obtaining first and second order approximate solutions of nonlinear, nonstationary vibrational systems, subjected to an arbitrary pulse excitation. Specific examples include: step function excitation, blast type loading, exponentially decaying pulse, asymptotic step, and a cosine pulse. In addition, several types of nonlinearities are considered and their effects on the systems' behavior are discussed. Results obtained are also compared with numerical solutions using a fourth order Runge-Kutta integrating scheme.

II. EQUATION OF MOTION AND METHOD OF ANALYSIS

2.1 Introduction

The transient response of nonlinear systems with constant parameters subjected to a pulse excitation has been considered by several authors. Approximate analyses of such systems have been presented by Srirangarajan and Srinivasan [21], Srirangarajan [22], Bapat and Srinivasan [23], Bauer [24-27] , and Ariaratnam [28]. The response of nonlinear, nonstationary systems have been investigated by Nayfeh and Mook [29], Rubenfeld [30], Mitropolsky [20], Evan-Iwanowski [31], and Sinha and Chou [32]. However, these analyses do not deal with arbitrary pulse excitations. In general, when the system parameters are functions of time, it is very difficult to obtain even an approximate solution of a nonlinear system. However, for the case when the parameters vary slowly with time, it is possible to construct an approximate solution for a general class of problems.

In this chapter the pulse response of nonlinear, nonstationary vibrational systems is investigated through an application of the Krylov-Bogoliubov - Mitropolsky method (henceforth called the KBM method). The system parameters are considered to be slowly varying functions of time as compared to the "normal" time period of the system. The general analysis for the single degree of freedom system is presented and solutions for first and second order approximations are obtained.

2.2 Equation of Motion

Many problems of nonlinear vibrations with single degree of freedom and slowly varying parameters subjected to an arbitrary pulse excitation, $g(\tau)$, can be reduced to

$$\frac{d}{dt}\left[m(\tau)\,\frac{dx}{dt}\right]+k(\tau)x + \varepsilon f\left(x\,,\,\frac{dx}{dt}\right) = g(\tau), \tag{2.1}$$

where $f(x\,, dx/dt)$ is a nonlinear function of x and dx/dt, ε is a small non-linearity parameter, and $\tau = \varepsilon t$ is a new time scale called "slowing time" [20]. Here, all mass or inertia type terms have been lumped into one time dependent function $m(\tau)$, and all stiffness or restoring type terms into the function $k(\tau)$. It is assumed that the functions $m(\tau)$, $k(\tau)$, $g(\tau)$, and $f(x,dx/dt)$ all have the desired number of derivatives with respect to τ, x, dx/dt for all their finite values, and that $m(\tau) > 0$ and $k(\tau) > 0$ for all τ within the interval under consideration. Finally, the initial conditions, without any loss of generality, can be taken as

$$x = 0,\ \frac{dx}{dt} = 0 \text{ at } t = 0. \tag{2.2}$$

Many of the available approximate techniques cannot be directly applied to equation (2.1), if $g(\tau)$ is an arbitrary function of time. Approximate solutions are available for the case when $g(\tau)$ is a periodic function of time. The method of multiple scales [29] and the KBM method [19,20] are the two popular techniques commonly employed for such problems. However it is possible to overcome this obstacle through a transformation of the dependent variable. Introducing the transformation of Ariaratram [28], the dependent variable is transformed by

$$x(t) = y(t) + p(t), \tag{2.3}$$

where $p(t)$ is yet unknown, equation (2.1) becomes

$$\frac{d}{dt}\left[m(\tau)\,\frac{dy}{dt}\right] + \frac{d}{dt}\left[m(\tau)\,\frac{dp}{dt}\right] + k(\tau)y$$

$$+ k(\tau)p + \varepsilon f\left(y + p,\ \frac{dy}{dt} + \frac{dp}{dt}\right) = g(\tau). \tag{2.4}$$

Choosing $p(t)$ such that

$$\frac{d}{dt}\left[m(\tau)\,\frac{dp}{dt}\right] + k(\tau)p = g(\tau), \tag{2.5}$$

yields

$$\frac{d}{dt}\left[m(\tau)\,\frac{dy}{dt}\right] + k(\tau)y + \varepsilon f\left(y + p,\ \frac{dy}{dt} + \frac{dp}{dt}\right) = 0, \tag{2.6}$$

where p is the particular solution of equation (2.5). The initial conditions transform as

$$y = -p,\quad \frac{dy}{dt} = -\frac{dp}{dt}\ ,\ \text{at } t = 0. \tag{2.7}$$

Although equation (2.5) is linear, its coefficients are time dependent and in general there are no exact solutions for such equations. However, the structure of equation (2.6) reveals that solutions of first $(O(\varepsilon))$ and second $(O(\varepsilon))$ order approximations require the evaluation of p independent of ε and up to the order of ε, respectively. To realize this, notice that equation (2.5) can be rewritten as

$$\varepsilon^2 m(\tau)\,\frac{d^2p}{d\tau^2} + \varepsilon^2\frac{dm(\tau)}{d\tau}\,\frac{dp}{d\tau} + k(\tau)p = g(\tau), \tag{2.8}$$

so that for first and second order approximations the solution of $p(t)$ is easily given by

$$p(t) = \frac{g(\tau)}{k(\tau)}. \tag{2.9}$$

Once p(t) is known, equation (2.6) takes the standard form

$$\frac{d}{dt}\left[m(\tau)\,\frac{dy}{dt}\right] + k(\tau)y + \varepsilon f\left(\tau,\ y,\ \frac{dy}{dt}\right) = 0, \tag{2.10}$$

a form to which many of the available methods can be directly applied to obtain an approximate solution. The method to be used in this study is that developed by Krylov, Bogoliubov and Mitropolsky [19,20]. This method is outlined in the following section and the procedure for obtaining first and second order approximations for equations of the form (2.10) are presented.

2.3 The Krylov-Bogoliubov-Mitropolsky (KBM) Method

The KBM method is one of the many perturbation techniques available today for obtaining approximate solutions of nonlinear differential equations. The method was first introduced by Krylov and Bogoliubov [43] in 1943. Following this work, Bogoliubov, Mitropolsky and their co-rèsearchers, in a series of papers, further developed the method to include a wide class of problems in nonlinear oscillation. These results are summarized in an excellent monograph by Mitropolsky [20]. Presented below is a brief summary of the method and the procedure for obtaining first and second order approximations for equations of the form (2.10).

Following Mitropolsky [20] the unperturbed equation is obtained by setting ε equal to zero and taking τ as a constant in equation (2.10); i.e.,

$$m(\tau)\,\frac{d^2y}{dt^2} + k(\tau)y = 0, \tag{2.11}$$

where $m(\tau)$ and $k(\tau)$ are now constants.

Motion described by equation (2.11) is obviously purely harmonic and the solution is

$$y = a\cos\psi, \tag{2.12}$$

with a constant amplitude and uniformly rotating phase angle. Or

$$\frac{da}{dt} = 0, \quad \frac{d\psi}{dt} = \omega, \tag{2.13}$$

where the phase angle, ψ, is defined by

$$\psi = \omega t + \theta, \quad \omega = \sqrt{k(\tau)/m(\tau)}, \tag{2.14}$$

and θ, $k(\tau)$ and $m(\tau)$ are constants.

The presence of nonlinear perturbations ($\varepsilon \neq 0$) and the slow variations of some of the parameters ($\tau \neq$ constant), introduce several effects in the solution of equation (2.10), in comparison to those obtained according to equation (2.11). Thus the solution may contain overtones, the instantaneous frequency may cease to be constant and may depend on the amplitude of vibration as well as the slowing time τ, or, a systematic increase or decrease of the amplitude may occur, etc.

Taking all these into account, it is natural to look for a solution of equation (2.10) in the form of an expression

$$x = a\cos\psi + \varepsilon u_1(\tau,a,\psi) + \varepsilon^2 u_2(\tau,a,\psi) + \dots , \tag{2.15}$$

where $u_1(\tau,a,\psi)$, $u_2(\tau,a,\psi),\dots,$ are periodic functions of angle ψ with period 2π, and the quantities a and ψ are functions of time defined by

$$\frac{da}{dt} = \varepsilon A_1(\tau,a) + \varepsilon^2 A_2(\tau,a) + \dots , \tag{2.16}$$

$$\frac{d\psi}{dt} = \omega(\tau) + \varepsilon B_1(\tau,a) + \varepsilon^2 B_2(\tau,a) + \dots . \tag{2.17}$$

Here, $\omega(\tau) = \sqrt{k(\tau)/m(\tau)}$ is the "normal" frequency of the given vibrating system and $\tau = \varepsilon t$.

Proceeding as usual in nonlinear mechanics, the functions $u_1, u_2, \ldots, u_n, A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_n$, are chosen such that equations (2.15) and (2.16) satisfy the differential equation (2.10). In addition, unique definitions of the functions A_n and B_n are ensured by requiring that no u_n contains the first harmonic of the angle ψ. In other words these periodic functions are determined so as to satisfy the equations

$$\int_0^{2\pi} u_1(\tau,a,\psi) \cos\psi \, d\psi = 0, \quad \int_0^{2\pi} u_2(\tau,a,\psi) \cos\psi \, d\psi = 0, \ \ldots \ , \qquad (2.18)$$

$$\int_0^{2\pi} u_1(\tau,a,\psi) \sin\psi \, d\psi = 0, \quad \int_0^{2\pi} u_2(\tau,a,\psi) \sin\psi \, d\psi = 0, \ \ldots \ . \qquad (2.19)$$

These conditions are, from a physical point of view, equivalent to associating the quantity a with the full amplitude of the first fundamental harmonic of vibration.

Under the conditions (2.18) and (2.19) it is possible to construct approximate expressions for the functions A_n, B_n, and u_n, to any order of ε desired. However, as discussed in the previous section, only first and second order approximations are required in the present study. Therefore, it is necessary to determine the functions A_1, B_1, A_2, B_2, and u_1 only. Following the approach presented by Mitropolsky [20], it can be shown that

$$A_1(\tau,a) = - \frac{a}{2m(\tau)\omega(\tau)} \frac{d[m(\tau)\omega(\tau)]}{d\tau} + $$

$$+ \frac{1}{2\pi m(\tau)\omega(\tau)} \int_0^{2\pi} f_0(\tau,a,\psi) \sin\psi \, d\psi, \qquad (2.20)$$

$$B_1(\tau,a) = \frac{1}{2\pi m(\tau)\omega(\tau)a}\int_0^{2\pi} f_0(\tau,a,\psi)\cos\psi\, d\psi, \tag{2.21}$$

$$A_2(\tau,a) = -\frac{1}{2\omega(\tau)}\left[a\frac{\partial B_1}{\partial a}A_1 + 2A_1B_1 + \frac{a}{m(\tau)}\frac{d[m(\tau)B_1]}{d\tau}\right] +$$

$$+\frac{1}{2\pi m(\tau)\omega(\tau)}\int_0^{2\pi} f_1(\tau,a,\psi)\sin\psi\, d\psi, \tag{2.22}$$

$$B_2(\tau,a) = \frac{1}{2\omega(\tau)}\left[\frac{\partial A_1}{\partial a}A_1 - aB_1^2 + \frac{1}{m(\tau)}\frac{d[m(\tau)A_1]}{d\tau}\right] +$$

$$+\frac{1}{2\pi m(\tau)\omega(\tau)a}\int_0^{2\pi} f_1(\tau,a,\psi)\cos\psi\, d\psi, \tag{2.23}$$

$$u_1(\tau,a,\psi) = -\frac{1}{2\pi k(\tau)}\sum_{n\neq\pm1}\frac{e^{in\psi}}{1-n^2}\int_0^{2\pi} f_0(\tau,a,\psi)\, e^{-in\psi}\, d\psi, \tag{2.24}$$

where the following notations have been introduced

$$f_0(\tau,a,\psi) = f(\tau,a\cos\psi,-a\omega\sin\psi), \tag{2.25}$$

$$\begin{aligned} f_1(\tau,a,\psi) = {} & u_1 f'_x(\tau,a\cos\psi,-a\omega\sin\psi) + \\ & + \left[A_1\cos\psi - aB_1\sin\psi + \frac{\partial u_1}{\partial\psi}\omega(\tau)\right] f'_{x'}(\tau,a\cos\psi,-a\omega\sin\psi) - \\ & - 2m(\tau)\omega(\tau)\left[\frac{\partial^2 u_1}{\partial\tau\partial\psi} + \frac{\partial^2 u_1}{\partial a\partial\psi}A_1 + \frac{\partial^2 u_1}{\partial\psi^2}B_1\right] - \frac{\partial u_1}{\partial\psi}\frac{d[m(\tau)\omega(\tau)]}{d\tau}. \end{aligned} \tag{2.26}$$

The symbols f'_x and $f'_{x'}$ denote partial differentiation of f with respect to x and dx/dt respectively.

In summary, solutions of equations of the form (2.1) can be approximated, to the first order of ε, by

$$x = a\cos\psi + \frac{g(\tau)}{k(\tau)}, \tag{2.27}$$

where a and ψ are given by

$$\frac{da}{dt} = \varepsilon A_1(\tau,a), \quad \frac{d\psi}{dt} = \omega(\tau) + \varepsilon B_1(\tau,a). \tag{2.28}$$

A_1 and B_1 can be obtained from equations (2.20) and (2.21) respectively.

Likewise, the solutions of equation (2.1) to the order of ε^2 is given by

$$x = a\cos\psi + \varepsilon u_1(\tau,a,\psi) + \frac{g(\tau)}{k(\tau)}, \tag{2.29}$$

where

$$\begin{aligned} \frac{da}{dt} &= \varepsilon A_1(\tau,a) + \varepsilon^2 A_2(\tau,a), \\ \frac{d\psi}{dt} &= \omega(\tau) + \varepsilon B_1(\tau,a) + \varepsilon^2 B_2(\tau,a). \end{aligned} \tag{2.30}$$

A_1, B_1, A_2, and B_2 are given by equations (2.20), (2.21), (2.22), and (2.23) respectively. u_1 can be obtained from equation (2.24). The following sections deal with some special cases of the first order approximation. In what follows, it is observed that certain types of nonlinearities result in significantly simplified expressions for A_1 and B_1.

2.4 Special Cases of the First Approximation

2.4.1 Nonlinearities of the Displacement Variable Only

Consider a system with slowly varying parameters, subjected to a pulse excitation, and whose nonlinearity arises from the displacement variable only. Such a system could be described by

$$\frac{d}{dt}\left[m(\tau)\,\frac{dx}{dt}\right] + k(\tau)x + \varepsilon f(x) = g(\tau). \tag{2.31}$$

Proceeding as outlined in the previous section, the first order approximation, as given by equation (2.27), is

$$x = a\cos\psi + \frac{g(\tau)}{k(\tau)}, \tag{2.32}$$

where, from equations (2.28), (2.20), and (2.21), a and ψ are determined by

$$\frac{da}{dt} = -\frac{\varepsilon a}{2m(\tau)\omega(\tau)}\frac{d[m(\tau)\omega(\tau)]}{d\tau} + \\ + \frac{\varepsilon}{2\pi m(\tau)\omega(\tau)}\int_0^{2\pi} f(\tau, a\cos\psi)\,\sin\psi\, d\psi, \tag{2.33}$$

$$\frac{d\psi}{dt} = \omega(\tau) + \frac{\varepsilon}{2\pi m(\tau)\omega(\tau)a}\int_0^{2\pi} f(\tau, a\cos\psi)\,\cos\psi\, d\psi. \tag{2.34}$$

Notice that the nonlinear term appearing in equations (2.33) and (2.34) is a function of $\cos\psi$ only, in which case the integral appearing in the amplitude expression will vanish. Thus, the expression for amplitude variations is simply given by

$$\frac{da}{dt} = -\frac{\varepsilon}{2m(\tau)\omega(\tau)}\frac{d[m(\tau)\omega(\tau)]}{d\tau}. \tag{2.35}$$

This equation can be integrated in closed form to yield

$$a = a(0)\left[\frac{\omega(0)m(0)}{\omega(\tau)m(\tau)}\right]^{\frac{1}{2}}, \tag{2.36}$$

where $a(0)$ is the initial amplitude. Or, recalling that $\omega(\tau) = \sqrt{k(\tau)/m(\tau)}$,

$$a = a(0)\left[\frac{k(0)m(0)}{k(\tau)m(\tau)}\right]^{\frac{1}{4}}. \tag{2.37}$$

Once a is known, it can be substituted into equation (2.34), which can then be integrated to yield the expression for phase variation as

$$\psi = \int \left[\omega(\tau) + \frac{\varepsilon}{2\pi m(\tau)\omega(\tau)a} \int_0^{2\pi} f(\tau, a\cos\psi) \cos\psi \, d\psi \right] dt + \Phi_0. \tag{2.38}$$

The constant of integration, Φ_0, and the initial amplitude, a(0), are determined form the initial conditions of a and ψ.

2.4.2 Nonlinearities of the Velocity Term Only

As a second special case, consider a system with slowly varying parameters, subjected to a pulse excitation, whose nonlinearity arises from the velocity term only. Such a system could be described by

$$\frac{d}{dt}\left[m(\tau) \frac{dx}{dt} \right] + k(\tau)x + \varepsilon f(dx/dt) = g(\tau). \tag{2.39}$$

Proceeding as before, the first order approximation is given by

$$x = a\cos\psi + \frac{g(\tau)}{k(\tau)}, \tag{2.27}$$

where, from equations (2.28), (2.20), and (2.21), a and ψ are determined by

$$\frac{da}{dt} = - \frac{\varepsilon a}{2m(\tau)\omega(\tau)} \frac{d[m(\tau)\omega(\tau)]}{d\tau} + \frac{\varepsilon}{2\pi m(\tau)\omega(\tau)} \int_0^{2\pi} f(\tau, -a\omega\sin\psi) \sin\psi \, d\psi, \tag{2.40}$$

$$\frac{d\psi}{dt} = \omega(\tau) + \frac{\varepsilon}{2\pi m(\tau)\omega(\tau)a} \int_0^{2\pi} f(\tau, -a\omega\sin\psi) \cos\psi \, d\psi. \tag{2.41}$$

Notice that the nonlinear term appearing in equations (2.40) and (2.41) is, for this particular case, a function of $\sin\psi$ only.and the integral appearing in the phase equation vanishes. Hence

$$\psi = \int \omega(\tau)\, dt + \Phi_0, \tag{2.42}$$

and

$$a = \int \left[- \frac{\varepsilon a}{2m(\tau)\omega(\tau)} \frac{d[m(\tau)\omega(\tau)]}{d\tau} + \right.$$
$$\left. + \frac{\varepsilon}{2\pi m(\tau)\omega(\tau)} \int_0^{2\pi} f(\tau, -a\omega\sin\psi) \cos\psi \, d\psi \right] dt + A_0. \tag{2.43}$$

The constants of integration, Φ_0 and A_0, are determined from the initial conditions on a and ψ.

2.5 Discussion

Through suitable transformation of the dependent variable, the problem of nonlinear, nonstationary systems, subjected to an arbitrary pulse excitation, can be solved via the approximate methods of Krylov, Bogoliubov and Mitropolsky.

For special cases, where the nonlinearity of the system arises solely from the displacement variable or velocity term, there is a significant simplification in the first order approximations. As seen from equation (2.36), the expression for amplitude variation, in systems whose nonlinearity arises from the displacement variable only, reduces to a simple power law type function. Also, as seen from equation (2.42), the frequency of vibration in systems whose nonlinearity arises from velocity terms only, obviously does not depend, in the first approximation, on the amplitude of vibration, but only on the character of the slowly varying mass and stiffness of the system. The following chapter deals with specific examples of nonlinear, nonstationary systems subjected to step function excitation.

III. STEP RESPONSE OF SOME TYPICAL SYSTEMS

3.1 Introduction

Several authors have investigated the response of nonlinear vibrational systems, subjected to step function excitation. Bapat and Srinivasan [33,34] have obtained exact expressions for the periods and maximum displacements for a class of undamped nonlinear systems subjected to a step function excitation. Bapat and Srinivasan [35] have also applied Ponavko's direct linearization method [36] and Atkinson's superposition method [37] to obtain approximate expressions for the time periods in systems with arbitrary hardening type spring characteristics. Bauer [24] has used the Poincare'-Lighthill perturbation technique and Ergin [38] has proposed a bilinear approximation method for solving undamped problems. Sinha and Srinivasan [39], Liu [40], and Anderson [41] have attempted the problem thru applications of ultraspherical polynomials. These analyses, however, have all been restricted to systems with constant parameters only. Recently, Sinha and Chou [32] have extended the application of ultraspherical polynomials to systems with time dependent parameters, an analysis restricted specifically to step function excitation.

In this chapter the approximate technique outlined in chapter II is applied to some common nonlinear systems with time dependent parameters when subjected to step function excitation. Specific examples are presented for cases where the parameter variations are linear functions of time. Results are obtained in the first order approximation and compared with numerical solutions using a fourth order Runge-Kutta integrating scheme.

3.2 Equation of Motion and First Order Approximation

Consider a nonlinear system with slowly varying parameters, subjected to step function excitation. Such a system could be described by

$$\frac{d}{dt}\left[m(\tau)\,\frac{dx}{dt}\right] + k(\tau)x + \varepsilon f\left(x\,,\ \frac{dx}{dt}\right) = F_0 u(\tau), \tag{3.1}$$

where $m(\tau)$, $k(\tau)$, and $f(x,dx/dt)$ satisfy the conditions stated in section 2.2. F_0 is a constant and $u(\tau)$ is the unit step function defined by

$$u(\tau) = \begin{cases} 0 \text{ if } \tau \leq 0^- \\ 1 \text{ if } \tau \geq 0^+ \end{cases}\ . \tag{3.2}$$

The initial conditions, without any loss of generality, can be taken as

$$x = 0,\ \ \frac{dx}{dt} = 0, \ \ \text{at} \ \ t = 0. \tag{3.3}$$

Proceeding as outlined in chapter II, the first order approximation is given, by equation (2.27), as

$$x = a\cos\psi + \frac{F_0 u(\tau)}{k(\tau)}\ . \tag{3.4}$$

The expressions for a and ψ, from equations (2.28), (2.20), and (2.21), are

$$\frac{da}{dt} = -\frac{\varepsilon a}{2m(\tau)\omega(\tau)}\,\frac{d[m(\tau)\omega(\tau)]}{d\tau} + \frac{\varepsilon}{2\pi m(\tau)\omega(\tau)}\,\times$$

$$\times\int_0^{2\pi} f\left[a\cos\psi + \frac{F_0 u(\tau)}{k(\tau)},\ -a\omega\sin\psi + \frac{d}{dt}\left(\frac{F_0 u(\tau)}{k(\tau)}\right)\right]\sin\psi\ d\psi, \tag{3.5}$$

$$\frac{d\psi}{dt} = \omega(\tau) + \frac{\varepsilon}{2\pi m(\tau)\omega(\tau)a}\,\times$$

$$\times\int_0^{2\pi} f\left[a\cos\psi + \frac{F_0 u(\tau)}{k(\tau)},\ -a\ \sin\psi + \frac{d}{dt}\left(\frac{F_0 u(\tau)}{k(\tau)}\right)\right]\cos\psi\ d\psi. \tag{3.6}$$

The initial conditions of a and ψ are obtained from the transformed initial conditions of x given by equation (2.7) as

$$y = -p \, , \quad \frac{dy}{dt} = -\frac{dp}{dt} \, , \quad \text{at} \quad t = 0, \tag{2.7}$$

where, in this case,

$$p = \frac{F_0 u(\tau)}{k(\tau)} \; . \tag{3.7}$$

Substituting for y and p, equation (2.7) yields

$$a(0)\cos\psi(0) = -\frac{F_0}{k(0)} \, , \tag{3.8}$$

$$-\omega(0)a(0)\sin\psi(0) = -\frac{d}{dt}\left[\frac{F_0 u(\tau)}{k(\tau)}\right]_{t=0,} \tag{3.9}$$

which can be solved simultaneously for a(0) and $\psi(0)$ as

$$a(0) = -\frac{F_0}{k(0)\cos\psi(0)} \tag{3.10}$$

$$\psi(0) = \arctan\left[-\frac{k(0)}{\omega(0)F_0}\frac{d}{dt}\left[\frac{F_0 u(\tau)}{k(\tau)}\right]_{t=0}\right] \; . \tag{3.11}$$

Thus the first order approximation of nonlinear systems with slowly varying parameters, subjected to step function excitation of magnitude F_0, is given by equations (3.4), (3.5), (3.6), (3.10), and (3.11). In the following section the response of various nonlinear systems, subjected to step function excitation is investigated.

3.3 Application to Some Typical Systems

The procedure for obtaining first order approximations of systems subjected to step function excitation has been applied to systems with the following types of nonlinearities

$$f\left(x\,,\ \frac{dx}{dt}\right) = \begin{cases} \alpha x^3 \\ \alpha\frac{dx}{dt} \\ \alpha(1 + x^2)\,\frac{dx}{dt} \\ \alpha x^3 + \beta\frac{dx}{dt}\,, \quad \alpha > 0,\ \beta > 0. \end{cases} \tag{3.12}$$

Results are presented for cases where some or all of the parameters vary according to a linear function of time.

3.3.1 Systems with Cubic Nonlinearity

Consider a system whose nonlinearity is of the form, $f(x\,, dx/dt) = \alpha x^3$. This is a special case of a nonlinearity arising from the displacement variable only. As outlined in section 2.4.1, the expression for amplitude variation in systems of this nature simply reduces to

$$a = a(0)\left[\frac{k(0)m(0)}{k(\tau)m(\tau)}\right]^{\frac{1}{4}}. \tag{3.13}$$

The phase expression, given by equation (2.34), takes the form

$$\frac{d\psi}{dt} = \omega(\tau) + \frac{\varepsilon\alpha}{2\pi m(\tau)\omega(\tau)a}\int_0^{2\pi}\left[\ a\cos\psi + \frac{F_0 u(\tau)}{k(\tau)}\right]^3 \cos\psi\ d\psi, \tag{3.14}$$

which, after performing the necessary integration, reduces to

$$\frac{d\psi}{dt} = \omega(\tau) + \frac{3\varepsilon\alpha a^2}{8m(\tau)\omega(\tau)} + \frac{3\varepsilon\alpha F_0^2 u(\tau)^2}{2k(\tau)^2 m(\tau)\omega(\tau)}\,. \tag{3.15}$$

Upon substituting in the expression of a, given by equation (3.13), this equation can be integrated to yield the expression for phase variation

$$\psi = \int\left[\omega(\tau) + \frac{3\varepsilon\alpha a(0)^2}{8m(\tau)\omega(\tau)}\left[\frac{k(0)m(0)}{k(\tau)m(\tau)}\right]^{\frac{1}{2}} + \frac{3\varepsilon\alpha F_0^2 u(\tau)^2}{2k(\tau)^2 m(\tau)\omega(\tau)}\right] dt + \Phi_0\,, \tag{3.16}$$

where the constant of integration, Φ_0, is determined from the initial condition of ψ.

As a specific example, consider a system whose stiffness is constant, i.e., $k(\tau) = k_0$, and whose mass varies according to the linear law, $m(\tau) = m_0 + m_1\tau$, where, k_0, m_0, and m_1 are constants [see figures 3.1 thru 3.4]. For this case the amplitude and phase expressions become

$$a = a(0)\left[\frac{m_0}{m_0 + m_1\tau}\right]^{\frac{1}{4}}, \tag{3.17}$$

$$\psi = \frac{2m(\tau)\omega(\tau)}{m_1\varepsilon} + \frac{3\alpha a(0)^2\sqrt{m_0}}{8m_1\sqrt{k_0}}\ \ln\left[m(\tau)\right] + \frac{3\alpha F_0^2}{m_1 k_0^2\omega(\tau)} + \Phi_0, \tag{3.18}$$

where, from equations (3.10) and (3.11),

$$a(0) = -\frac{F_0}{k_0}, \tag{3.19}$$

$$\psi(0) = 0\ . \tag{3.20}$$

Thus, the constant of integration, Φ_0, is

$$\Phi_0 = -\left[\frac{2m(0)\omega(0)}{m_1\varepsilon} + \frac{3\alpha a(0)^2\sqrt{m_0}}{8m_1\sqrt{k_0}}\ \ln\left[m_0\right] + \frac{3\alpha F_0^2}{m_1 k_0^2\omega(0)}\right]. \tag{3.21}$$

3.3.2 Systems with Linear Damping

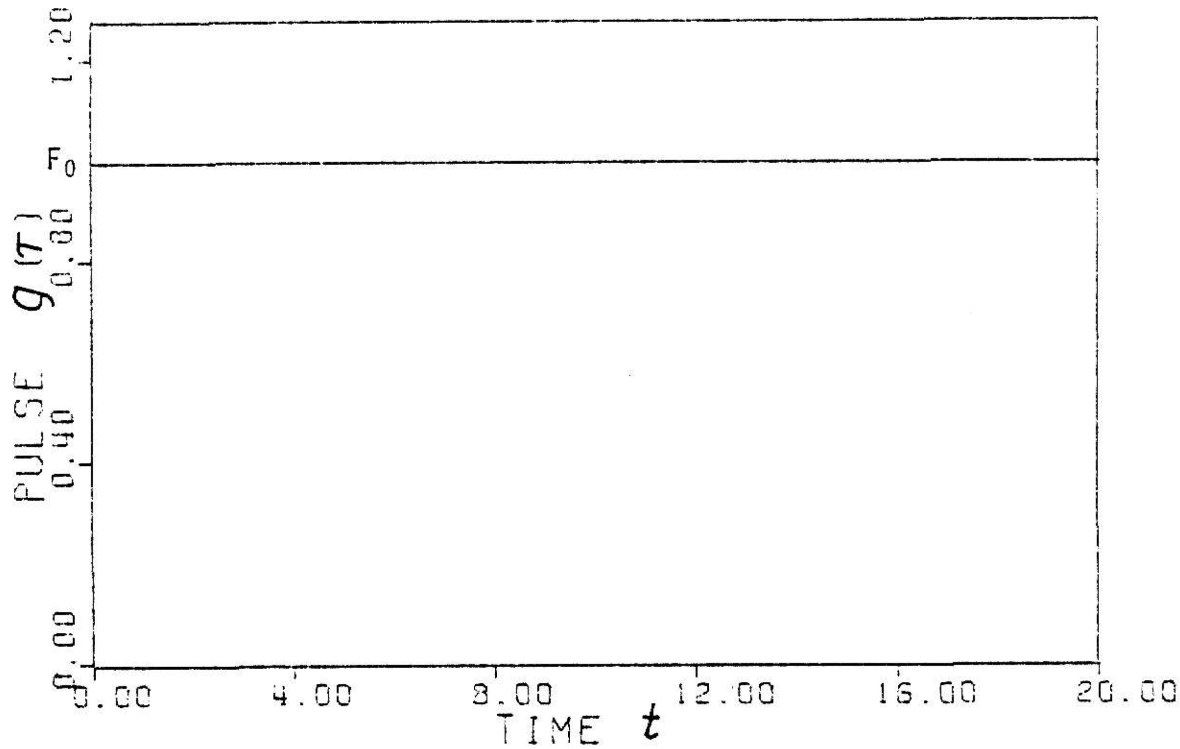
Consider a system where $f(x\,, dx/dt) = \alpha dx/dt$. This is a special case of a nonlinearity arising from the velocity term only. As shown in section 2.4.2, the expression for phase variation reduces to

$$\psi = \int \omega(\tau)\ dt + \Phi_0\ . \tag{2.42}$$

Amplitude variation is given by equation (2.40) as

$$\frac{da}{dt} = -\frac{\varepsilon a}{2m(\tau)\omega(\tau)}\frac{d[m(\tau)\omega(\tau)]}{d\tau} + $$
$$+ \frac{\varepsilon\alpha}{2\pi m(\tau)\omega(\tau)}\int_0^{2\pi}\left[-a\omega\sin\psi + \frac{d}{dt}\left(\frac{F_0 u(\tau)}{k(\tau)}\right)\right]\sin\psi\ d\psi, \tag{3.22}$$

which reduces to



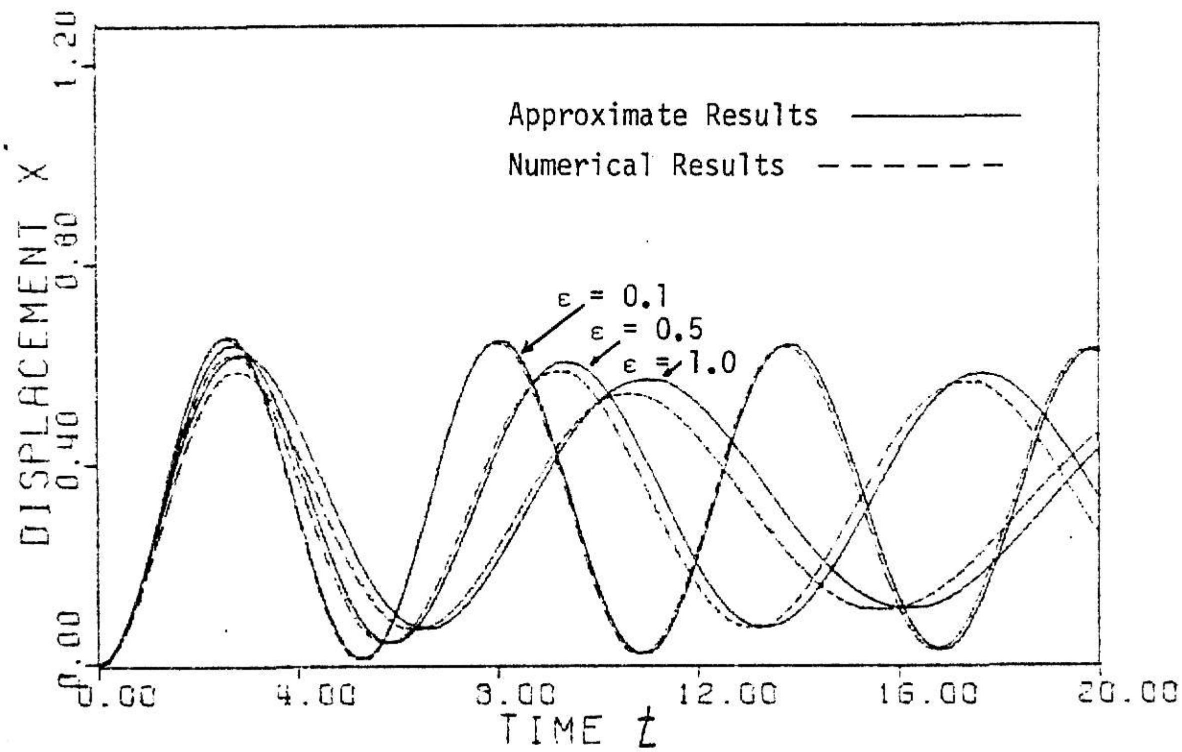


Figure 3.1. Step Function and Time - Displacement results for

$$\frac{d}{dt}\left[(2.0 + 0.5\tau)\frac{dx}{dt}\right] + 3.0x + \varepsilon x^3 = 1.0u(\tau).$$

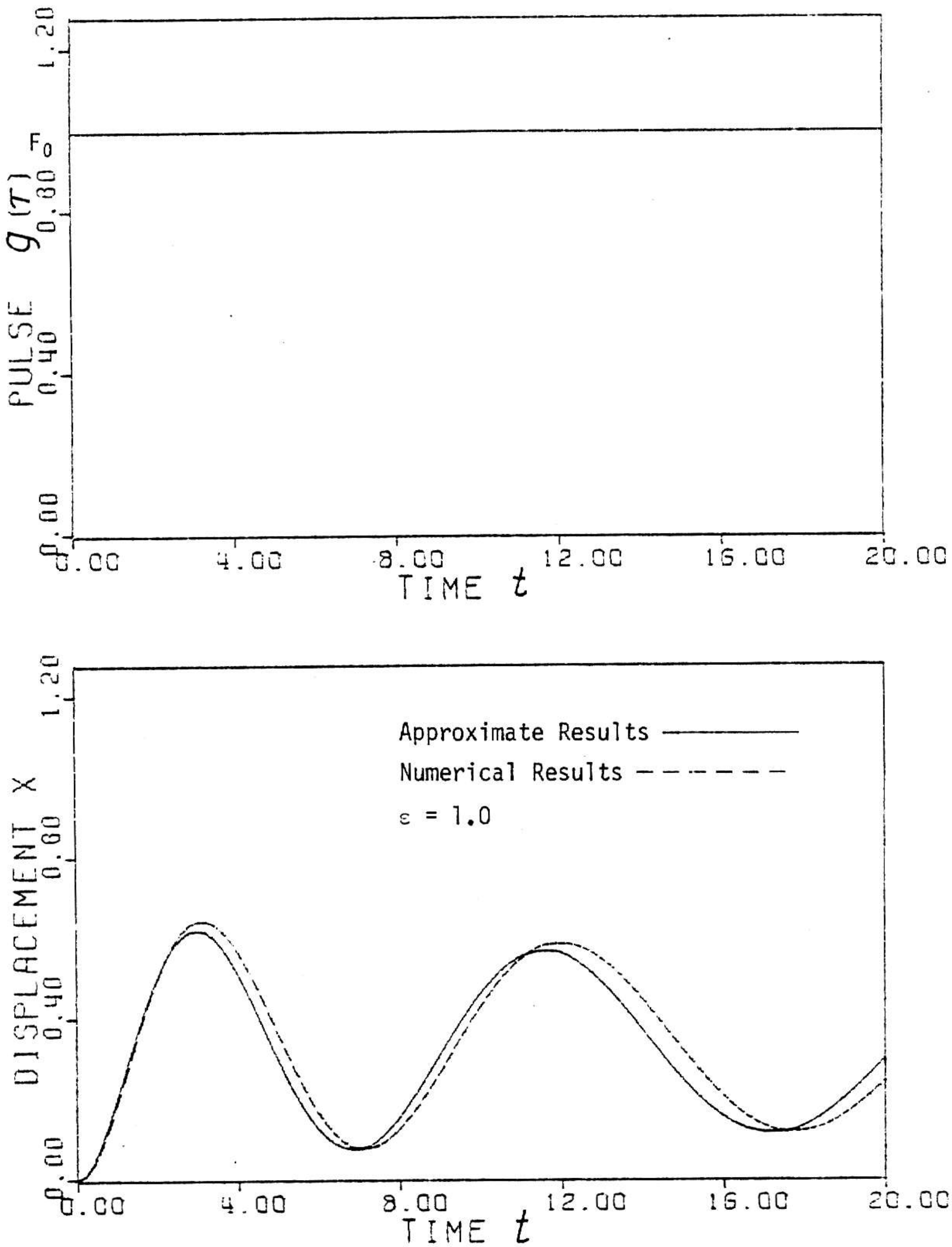
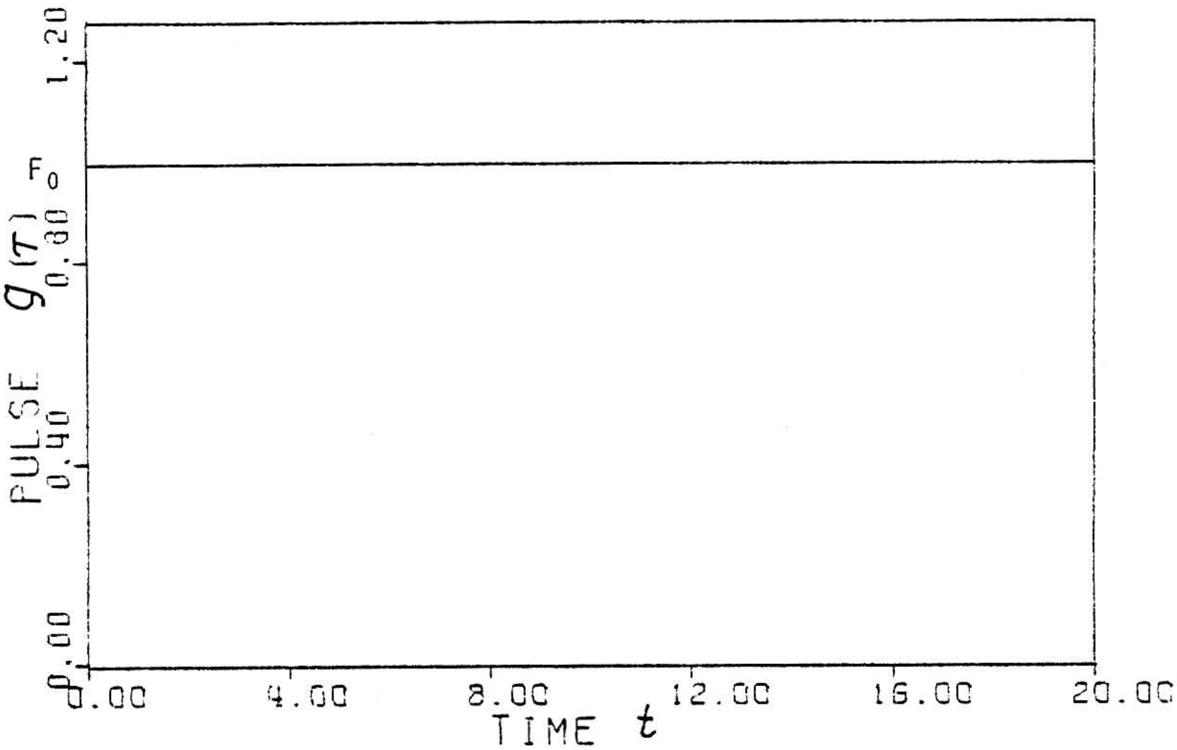


Figure 3.2. Step Function and Time - Displacement results for

$$\frac{d}{dt}\left[(2.0 + 0.5\tau)\,\frac{dx}{dt}\right] + 3.0x - 0.5x^3 = 1.0u(\tau).$$



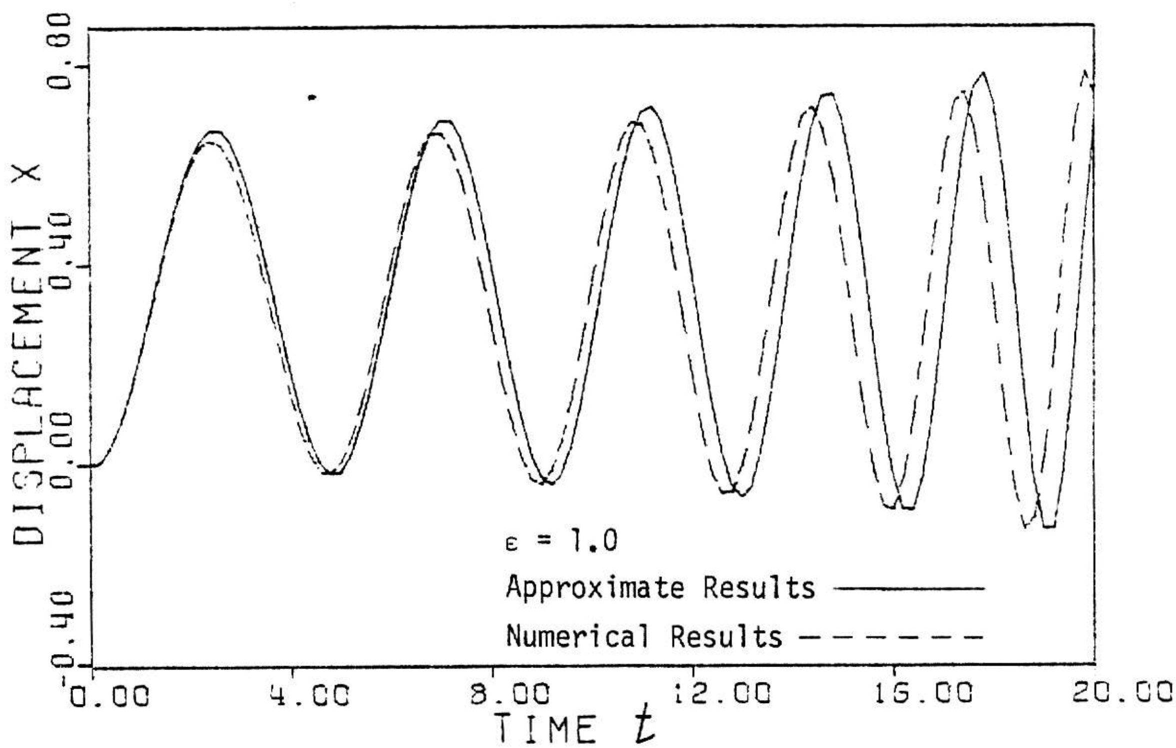


Figure 3.3. Step Function and Time - Displacement results for

$$\frac{d}{dt}\left[(2.0 - 0.08\tau)\,\frac{dx}{dt}\right] + 3.0x + 0.5x^3 = 1.0u(\tau).$$

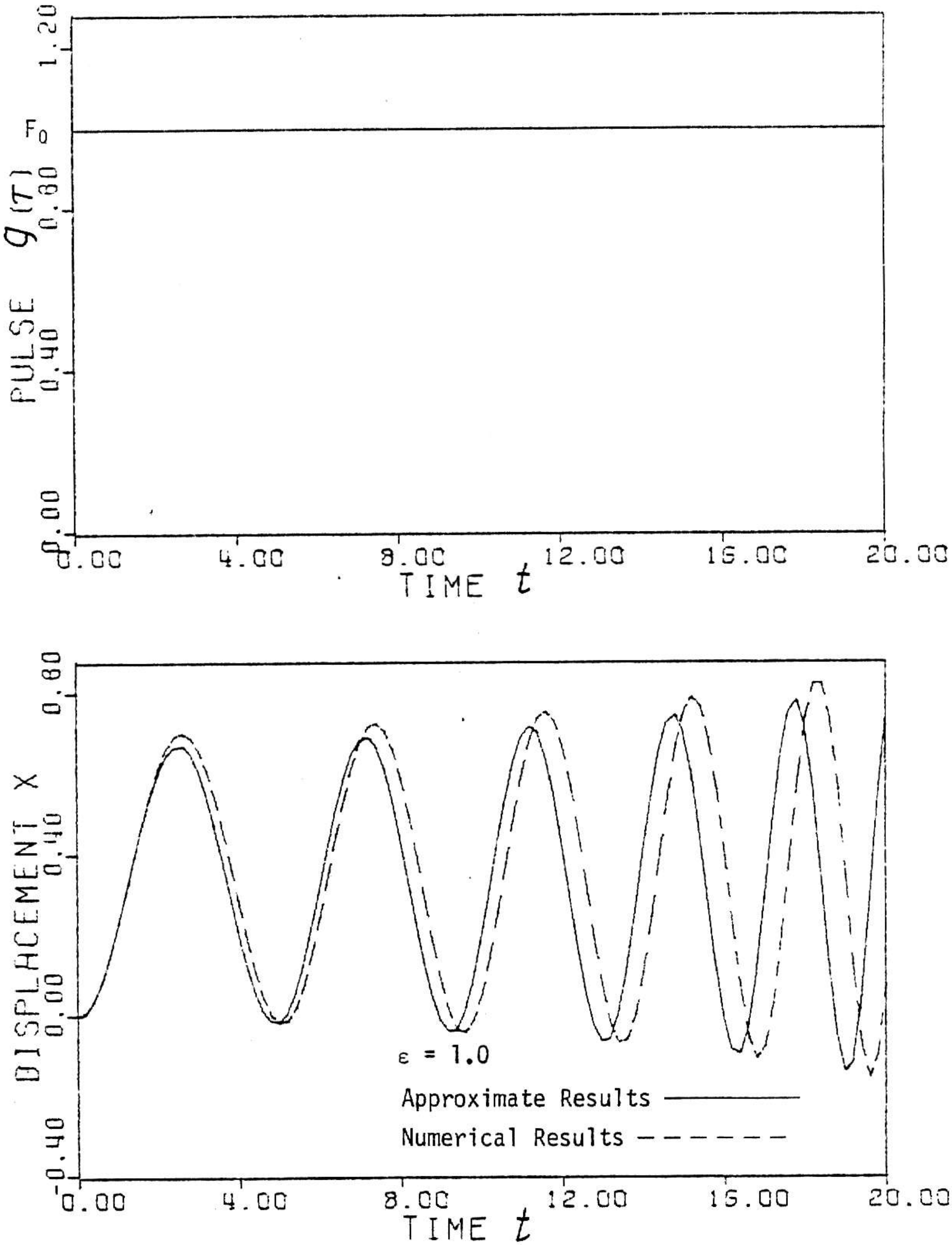


Figure 3.4. Step Function and Time - Displacement results for

$$\frac{d}{dt}\left[(2.0 - 0.08\tau)\,\frac{dx}{dt}\right] + 3.0x - 0.5x^3 = 1.0u(\tau).$$

$$\frac{da}{dt} = -\frac{\varepsilon a}{2m(\tau)\omega(\tau)}\frac{d[m(\tau)\omega(\tau)]}{d\tau} - \frac{\varepsilon\alpha a}{2m(\tau)} \;. \tag{3.23}$$

As a special case, consider a system where both the mass and stiffness vary according to linear functions of time, i.e.,

$$\left.\begin{aligned} m(\tau) &= m_0 + m_1\tau \\ k(\tau) &= k_0 + k_1\tau \end{aligned}\right\} , \tag{3.24}$$

where m_0, m_1, k_0, and k_1 are constants [see figures 3.5 thru 3.8]. The expressions for amplitude and phase variations can be integrated in closed form to yield

$$a = A_0\left[\frac{1}{\omega(\tau)m(\tau)^{\left[1+\frac{\alpha}{m_1}\right]}}\right]^{\frac{1}{2}} \tag{3.25}$$

$$\psi = \frac{[m(\tau)\omega(\tau)]}{m_1\varepsilon} - \beta^*\left[\frac{m_0k_1 - k_0m_1}{2m_1}\right] + \Phi_0, \tag{3.26}$$

where

$$\beta^* = \begin{cases} \dfrac{2}{\sqrt{m_1k_1\varepsilon^2}}\arctan\left[-\dfrac{k}{m}\dfrac{m(\tau)}{k(\tau)}\right] & \text{if } k_1m_1 < 0 \\ -\dfrac{1}{\sqrt{-m_1k_1\varepsilon^2}}\arcsin\left[\dfrac{2m_1k_1\varepsilon\tau + m_0k_1 + k_0m_1}{k_0m_1 - m_0k_1}\right] & \text{if } \begin{matrix} m_1 > 0 \\ k_1 < 0 \end{matrix} \\ \dfrac{2}{\sqrt{m_1k_1\varepsilon^2}}\ln\left|[m_1k_1\varepsilon^2m(\tau)]^{\frac{1}{2}} + m_1\varepsilon k(\tau)^{\frac{1}{2}}\right| & \text{if } k_1m_1 > 0 \end{cases} \;. \tag{3.26a}$$

From equations (3.10) and (3.11), the initial conditions are

$$\psi(0) = \arctan\left[\frac{k_1\varepsilon}{k_0\omega(0)}\right], \tag{3.27}$$

$$a(0) = -\frac{F_0}{k_0\cos\psi(0)} \;. \tag{3.28}$$

These initial conditions can be used to evaluate A_0 and ψ_0 in equations

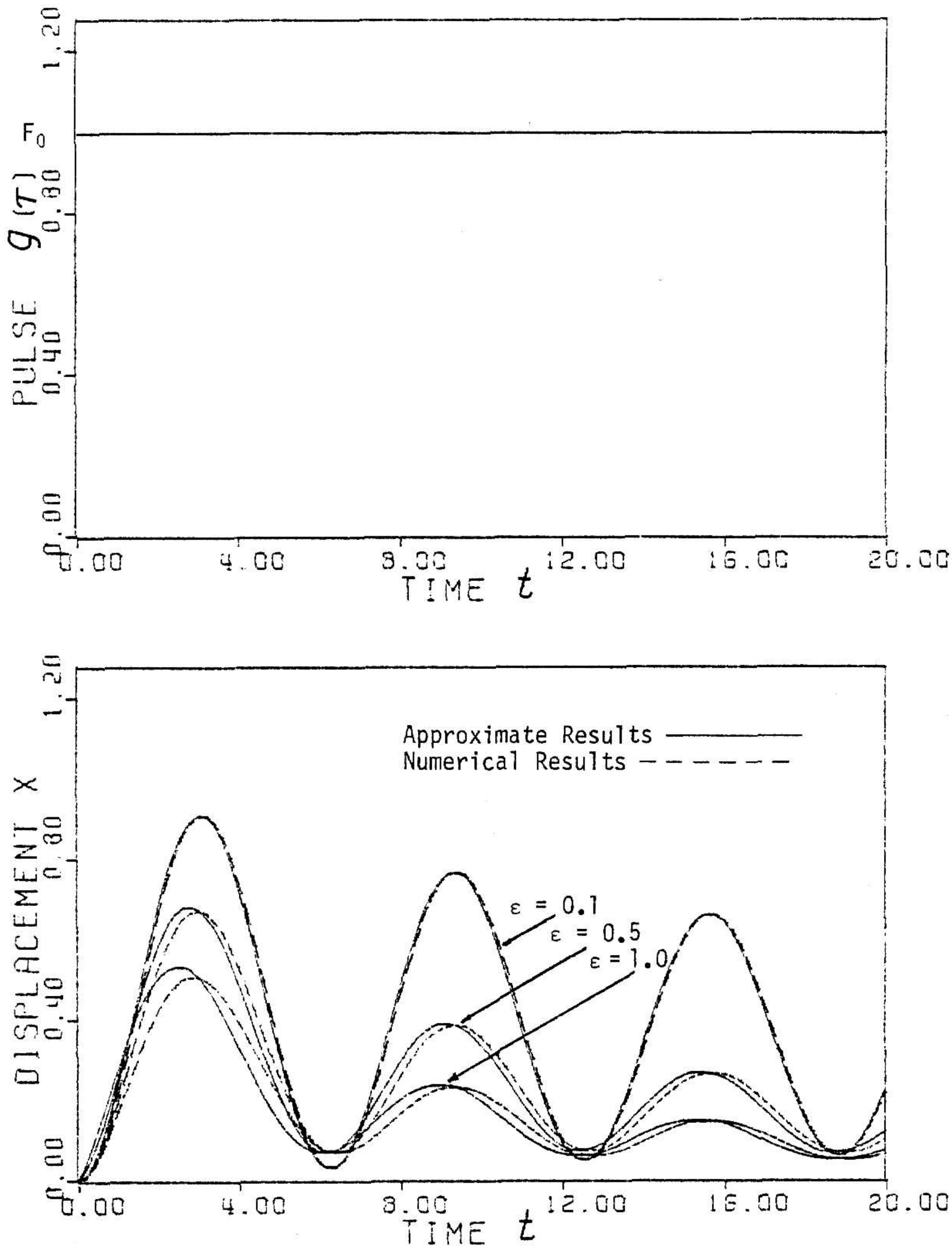
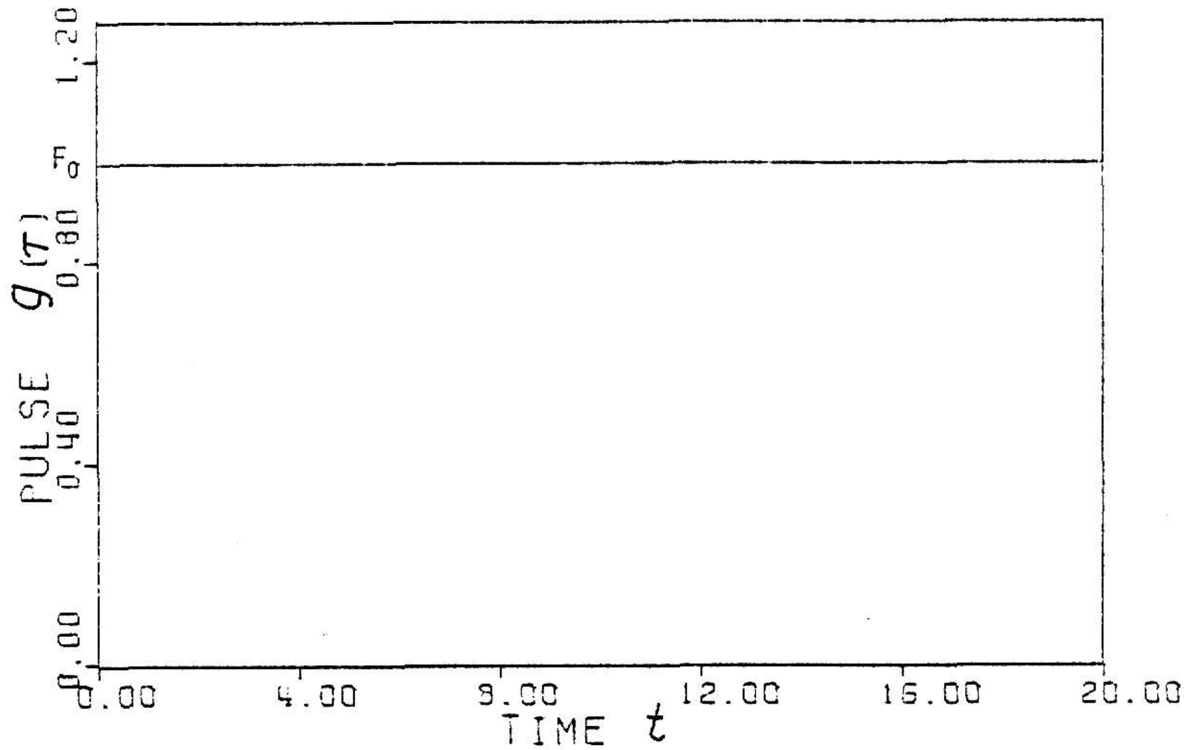


Figure 3.5. Step Function and Time - Displacement results for

$$\frac{d}{dt}\left[(2.0 + 0.5\tau)\frac{dx}{dt}\right] + (2.0 + 0.5\tau)x + \varepsilon 0.2\frac{dx}{dt} = 1.0u(\tau).$$



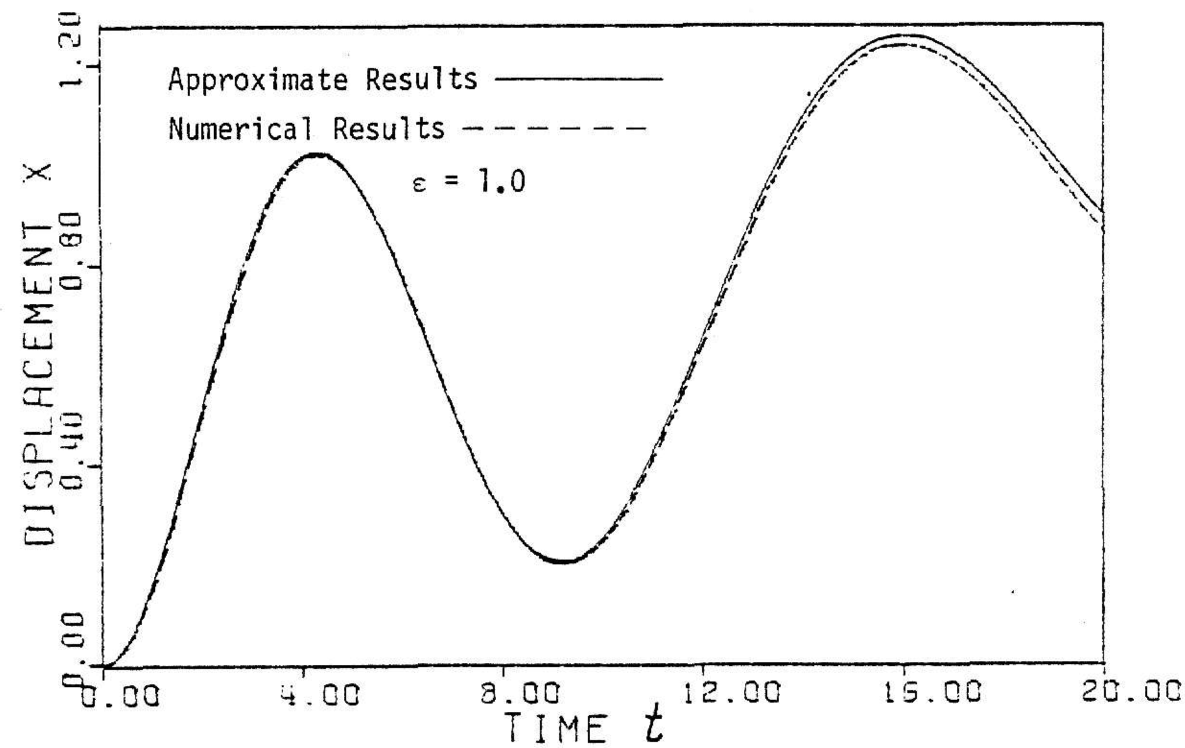


Figure 3.6. Step Function and Time - Displacement results for

$$\frac{d}{dt}\left[(3.0 + 0.2\tau)\frac{dx}{dt}\right] + (2.0 - 0.05\tau)x + 0.2\frac{dx}{dt} = 1.0u(\tau).$$

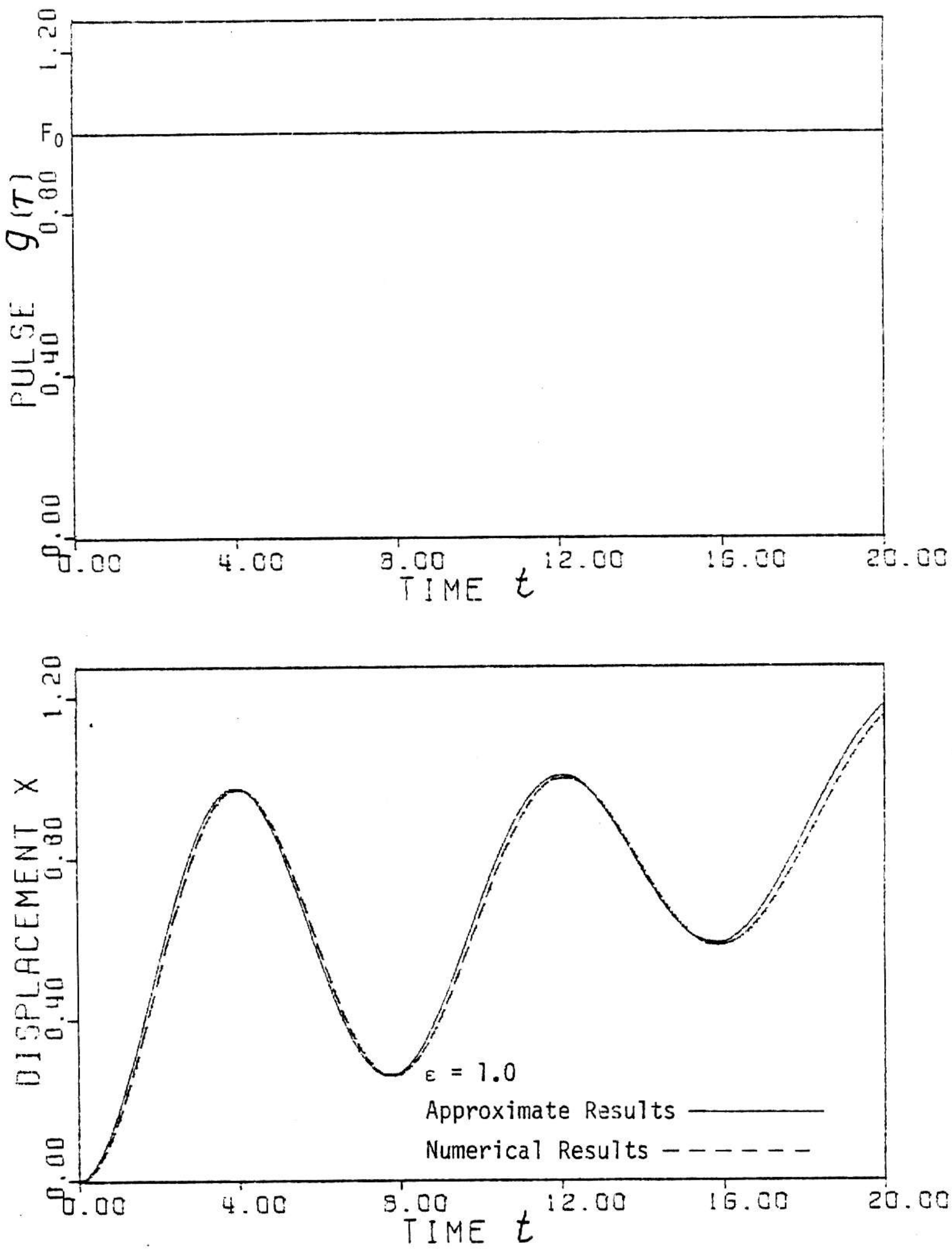
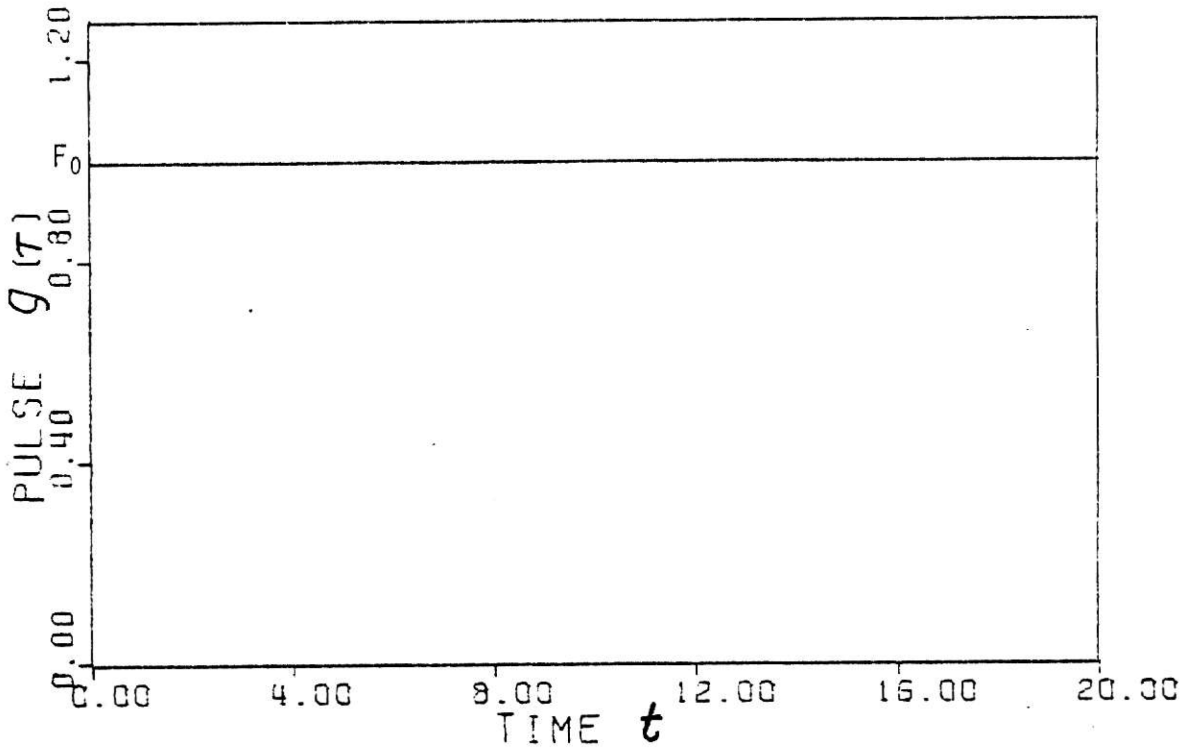


Figure 3.7. Step Function and Time - Displacement results for

$$\frac{d}{dt}\left[(3.0 - 0.05\tau)\frac{dx}{dt}\right] + (2.0 - 0.05\tau)x + 0.2\frac{dx}{dt} = 1.0u(\tau).$$



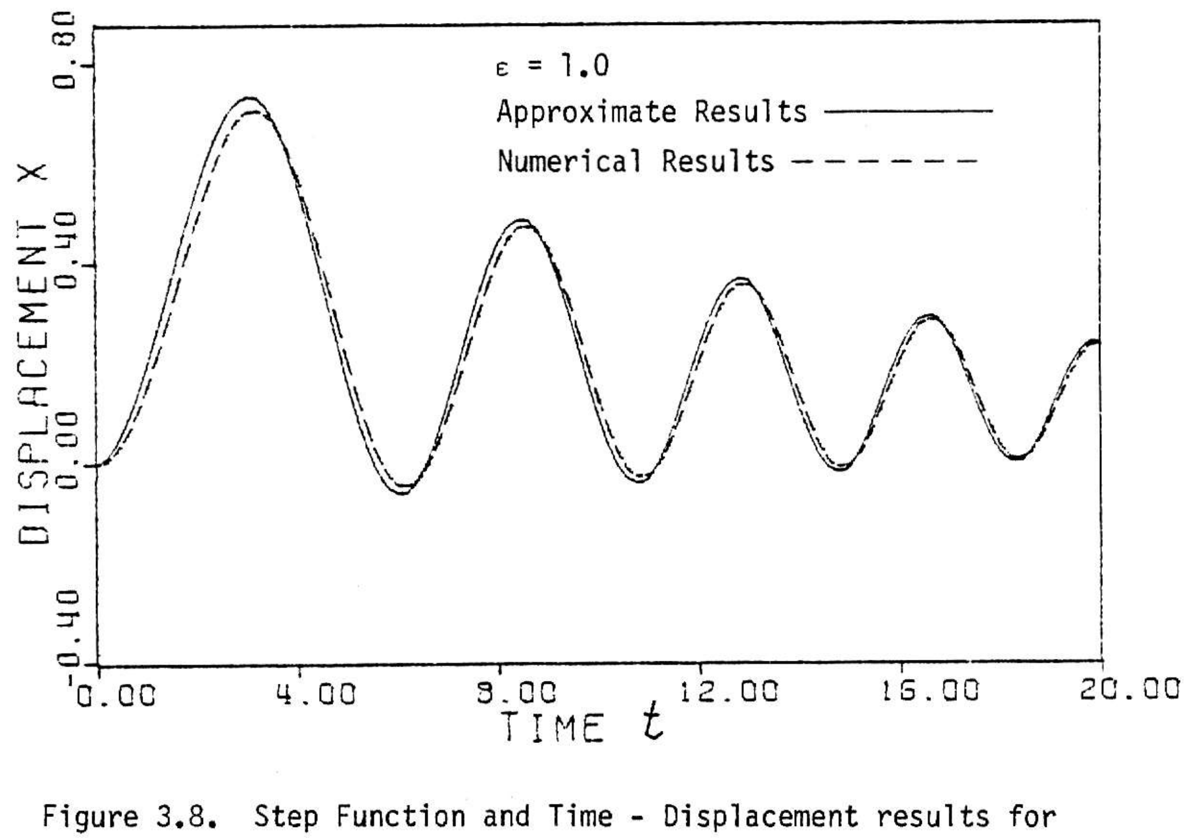


Figure 3.8. Step Function and Time - Displacement results for

$$\frac{d}{dt}\left[(3.0 - 0.05\tau)\frac{dx}{dt}\right] + (2.0 + 0.3\tau)x + 0.2\frac{dx}{dt} = 1.0u(\tau).$$

(3.25) and (3.26). Figures 3.5 thru 3.8 show results for systems with some typical variations in mass and stiffness.

3.3.3 Systems with Mixed Damping

As a third case, consider a system whose nonlinearity is of the form $f(x\,,\,dx/dt) = \alpha(1 + x^2)\,dx/dt$. By equations (3.5) and (3.6), the first order approximations of a and ψ are given by

$$\frac{da}{dt} = -\frac{\varepsilon a}{2m(\tau)\omega(\tau)}\frac{d[m(\tau)\omega(\tau)]}{d\tau} + \frac{\varepsilon\alpha}{2\pi m(\tau)\omega(\tau)} \times$$

$$\times \int_0^{2\pi}\left[1 + \left(a\cos\psi + \frac{F_0}{k(\tau)}\right)^2\right]\left[-a\omega\sin\psi + \frac{d}{dt}\left(\frac{F_0}{k(\tau)}\right)\right]\sin\psi\,d\psi, \tag{3.29}$$

$$\frac{d\psi}{dt} = \omega(\tau) + \frac{\varepsilon\alpha}{2\pi m(\tau)\omega(\tau)a}\int_0^{2\pi}\left[1 + \left(a\cos\psi + \frac{F_0}{k(\tau)}\right)^2\right] \times$$

$$\times\left[-a\omega\sin\psi + \frac{d}{dt}\left(\frac{F_0}{k(\tau)}\right)\right]\cos\psi\,d\psi. \tag{3.30}$$

Performing the necessary integrations, these expressions reduce to

$$\frac{da}{dt} = -\left[\frac{\varepsilon}{2m(\tau)\omega(\tau)}\frac{d[m(\tau)\omega(\tau)]}{d\tau} + \frac{\varepsilon\alpha F_0^2}{2m(\tau)k(\tau)^2} + \frac{\varepsilon\alpha}{2m(\tau)}\right]a - \frac{\varepsilon\alpha}{8m(\tau)}a^3, \tag{3.31}$$

$$\frac{d\psi}{dt} = \omega(\tau). \tag{3.32}$$

Notice that equation (3.31) is a special case of the general Bernoulli equation [43]. This can be rewritten as

$$\frac{da}{dt} = \beta_1(t)a + \beta_2(t)a^3, \tag{3.33}$$

where

$$\beta_1(t) = -\left[\frac{\varepsilon}{2m(\tau)\omega(\tau)}\frac{d[m(\tau)\omega(\tau)]}{d\tau} + \frac{\varepsilon\alpha F_0^2}{2m(\tau)k(\tau)^2} + \frac{\varepsilon\alpha}{2m(\tau)}\right], \tag{3.34}$$

$$\beta_2(t) = -\frac{\varepsilon\alpha}{8m(\tau)}. \tag{3.35}$$

Introducing the transformation

$$a = [a^*]^{-\frac{1}{2}}, \tag{3.36}$$

equation (3.33) takes the form

$$\frac{da^*}{dt} + 2\beta_1(t)a^* = -2\beta_2(t). \tag{3.37}$$

Equation (3.37) has the general solution

$$a^* = \exp[-2\textstyle\int\beta_1(t)dt]\displaystyle\int\left[-2\beta_2(t)\exp[2\textstyle\int\beta_1(t)dt]\right]dt + C_0\exp[-2\textstyle\int\beta_2(t)dt], \tag{3.38}$$

where C_0 is a constant.

As a specific example, consider a system whose stiffness coefficient, $k(\tau) = k_0$, is a constant, and whose mass varies as the linear function $m(\tau) = m_0 + m_1\tau$, where k_0, m_0, and m_1 are constants [see figures 3.9 thru 3.10]. The expression for a and ψ are

$$a = A_0\left[\frac{1}{\omega(\tau)m(\tau)^{\left[1+\frac{\alpha}{m_1}+\frac{\alpha F_0^2}{m_1k_0^2}\right]}}\right]^{\frac{1}{2}} \tag{3.39}$$

$$\psi = \frac{2m(\tau)\omega(\tau)}{m_1\varepsilon} + \Phi_0, \tag{3.40}$$

where

$$A_0 = -\frac{F_0}{k_0}\left[\omega(0)m(0)^{\left[1+\frac{\alpha}{m_1}+\frac{\alpha F_0^2}{m_1k_0^2}\right]}\right]^{\frac{1}{2}} \tag{3.41}$$

$$\Phi_0 = -\frac{2m(0)\omega(0)}{m_1\varepsilon}. \tag{3.42}$$

Results for some typical parameters are shown in figures 3.9 and 3.10.

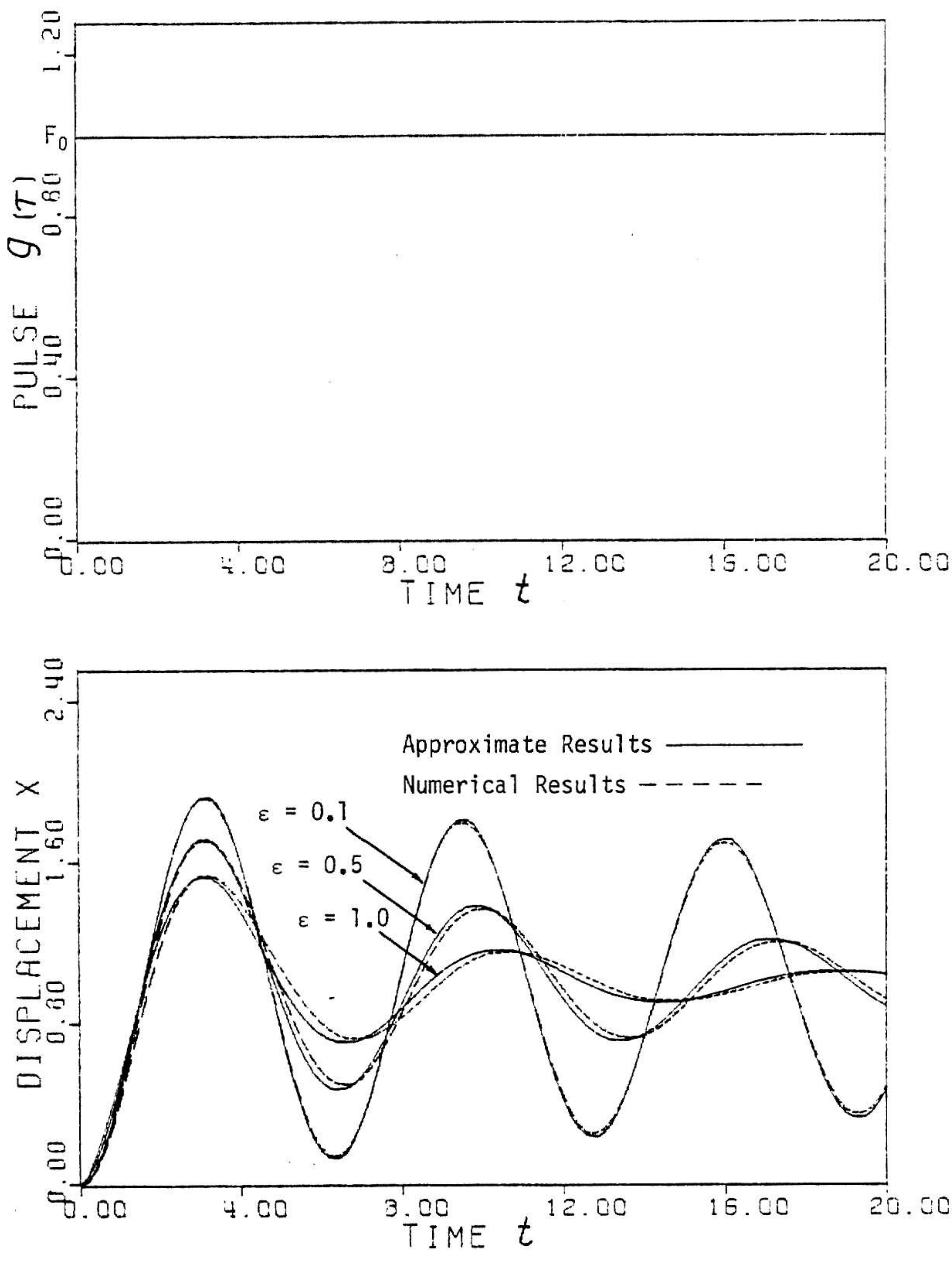


Figure 3.9. Step Function and Time - Displacement results for

$$\frac{d}{dt}\left[(1.0 + 0.05\tau)\frac{dx}{dt}\right] + 1.0x + 0.2\varepsilon(1 + x^2)\frac{dx}{dt} = 1.0u(\tau).$$

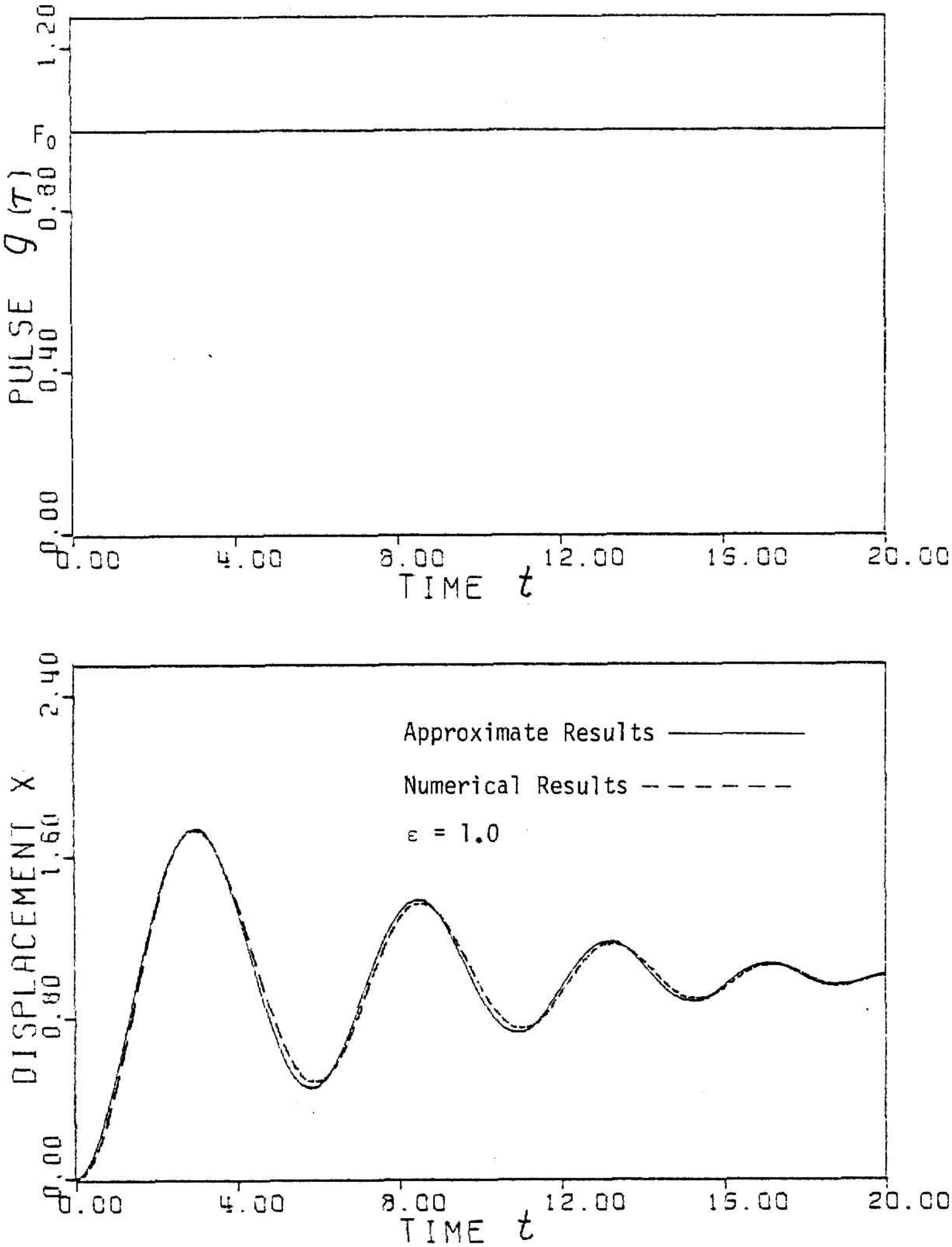


Figure 3.10. Step Function and Time - Displacement results for

$$\frac{d}{dt}\left[(1.0 - 0.04\tau)\,\frac{dx}{dt}\right] + 1.0x + 0.2(1.0 + x^2)\,\frac{dx}{dt} = 1.0u(\tau).$$

3.3.4 Cubic Nonlinearity with Linear Damping

As a final case, consider a system whose nonlinearity is of the form

$$f\left(x\,,\ \frac{dx}{dt}\right) = \alpha x^3 + \beta\frac{dx}{dt}, \tag{3.43}$$

where $\alpha > 0$ and $\beta > 0$. By equations (3.5) and (3.6), the first order approximations for a and ψ are given by

$$\frac{da}{dt} = -\frac{\varepsilon a}{2m(\tau)\omega(\tau)}\frac{d[m(\tau)\omega(\tau)]}{d\tau} + \frac{\varepsilon}{2\pi m(\tau)\omega(\tau)} \times$$

$$\times \int_0^{2\pi}\left[\alpha\left(a\cos\psi + \frac{F_0u(\tau)}{k(\tau)}\right)^3 + \beta\left(-a\omega\sin\psi + \frac{d}{dt}\left(\frac{F_0u(\tau)}{k(\tau)}\right)\right)\right]\sin\psi\, d\psi, \tag{3.44}$$

$$\frac{d\psi}{dt} = \omega(\tau) + \frac{\varepsilon}{2\pi m(\tau)\omega(\tau)a} \times$$

$$\times \int_0^{2\pi}\left[\alpha\left(a\cos\psi + \frac{F_0u(\tau)}{K(\tau)}\right)^3 + \beta\left(-a\omega\sin\psi + \frac{d}{dt}\left(\frac{F_0u(\tau)}{k(\tau)}\right)\right)\right]\cos\psi\, d\psi. \tag{3.45}$$

After performing the necessary integrations these expressions reduce to

$$\frac{da}{dt} = -\frac{\varepsilon\alpha}{2m(\tau)\omega(\tau)}\frac{d[m(\tau)\omega(\tau)]}{d\tau} - \frac{\varepsilon\beta a}{2m(\tau)}\,, \tag{3.46}$$

$$\frac{d\psi}{dt} = \omega(\tau) + \frac{3\varepsilon\alpha a^2}{8m(\tau)\omega(\tau)} + \frac{3\varepsilon\alpha F_0^2}{k(\tau)^2}\,. \tag{3.47}$$

Equations (3.46) and (3.47) can be integrated, depending on the complexity of $m(\tau)$ and $k(\tau)$, in closed form or using a simple numerical scheme.

As an illustrative example, consider a system where the mass and stiffness vary according to the linear laws

$$\left.\begin{aligned} m(\tau) &= m_0 + m_1\tau \\ k(\tau) &= k_0 + k_1\tau \end{aligned}\right\}, \tag{3.48}$$

where k_0, k_1, m_0, and m_1 are constants [see figures 3.11 thru 3.14]. The equation for amplitude can be integrated in closed form to give

$$a = A_0 \left[\frac{1}{\omega(\tau) m(\tau)^{\left[1+\frac{\beta}{m_1}\right]}} \right]^{\frac{1}{2}} . \tag{3.49}$$

This can be substituted in the equation for ψ, to yield

$$\psi = \int \left\{ \omega(\tau) + \frac{3\varepsilon\alpha A_0^2}{8m(\tau)\omega(\tau)} \left[\frac{1}{\omega(\tau) m(\tau)^{\left[1+\frac{\beta}{m_1}\right]}} \right] + \frac{3\varepsilon\alpha F_0}{k(\tau)} \right\} dt + \Phi_0 . \tag{3.50}$$

The constants of integration can be determined from the initial conditions on a and ψ which are given by equations (3.10) and (3.11) as

$$\psi(0) = \arctan \left[\frac{\varepsilon k_1}{k_0 \omega(0)} \right] , \tag{3.51}$$

$$a(0) = - \frac{F_0}{k_0 \cos\psi(0)} . \tag{3.52}$$

Figures 3.11 thru 3.14 show results for some typical parameters.

3.4 Discussion

In this chapter the step response of nonlinear systems with slowly varying parameters was investigated using the method outlined in chapter II. Results were obtained for systems with a variety of nonlinearities. Specific examples were given for systems where the parameters varied according to linear functions of time. For most cases the expressions for amplitude and phase could be obtained in closed form. In all cases the expressions for the amplitude variation were found to be power law type functions. This is in contrast to systems with constant parameters where amplitude variations are in terms of exponential functions.

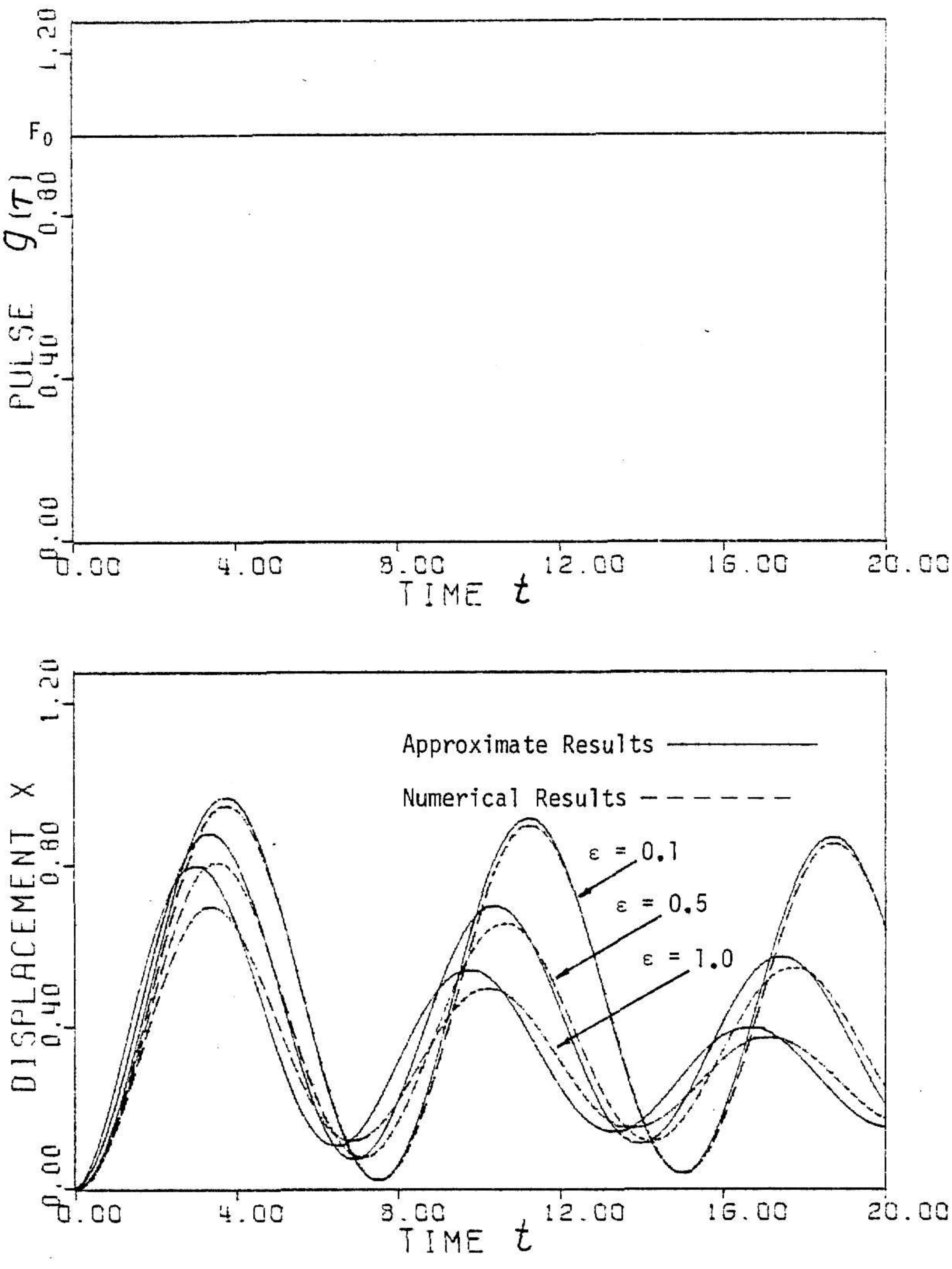


Figure 3.11. Step Function and Time - Displacement results for

$$\frac{d}{dt}\left[(3.0 + 0.1\tau)\frac{dx}{dt}\right] + (2.0 + 0.1\tau)x + \varepsilon x^3 + 0.5\varepsilon\frac{dx}{dt} = 1.0u(\tau).$$

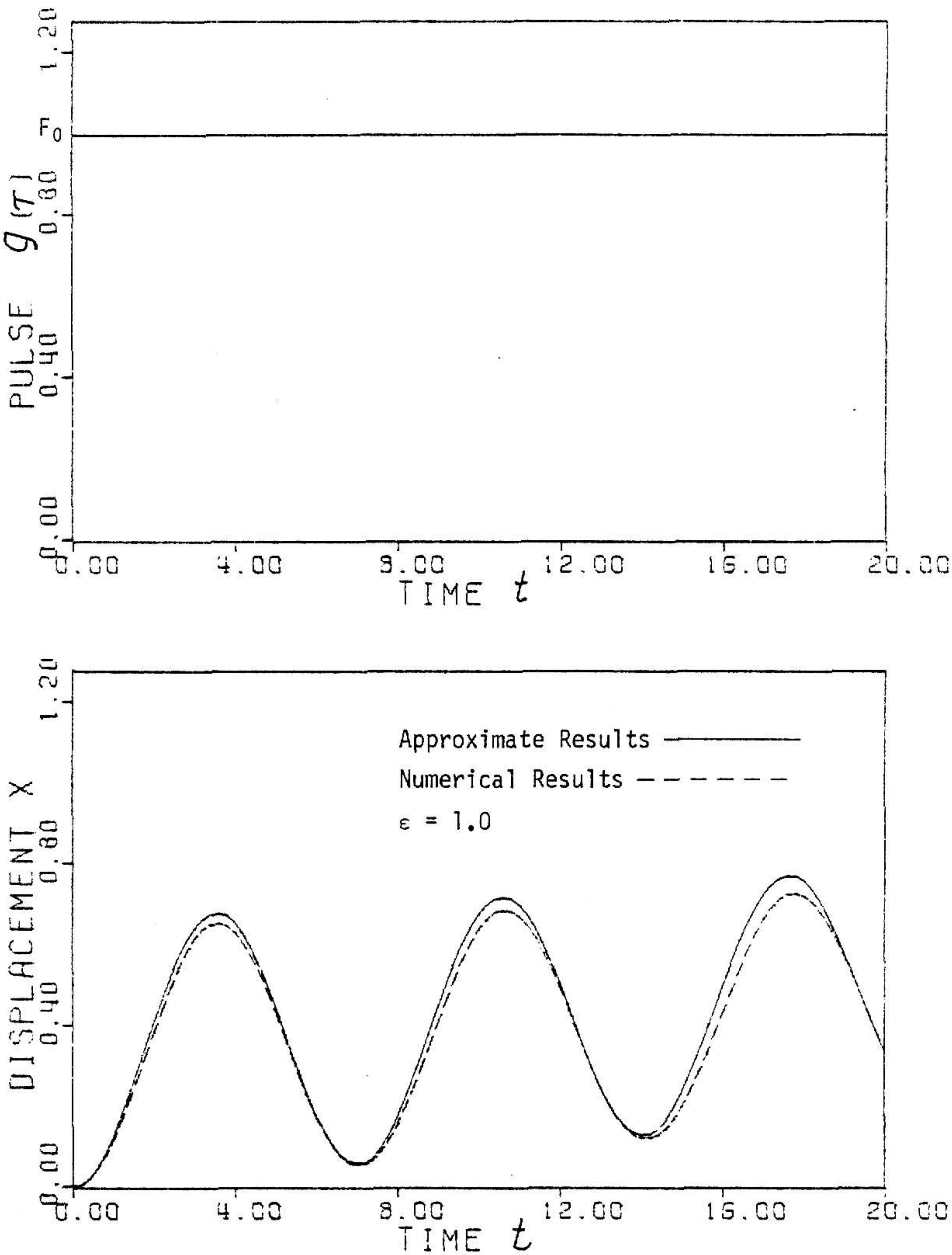


Figure 3.12. Step Function and Time - Displacement results for

$$\frac{d}{dt}\left[(4.0 - 0.05\tau)\,\frac{dx}{dt}\right] + (3.0 - 0.05\tau)x + 0.5x^3 + 0.1\,\frac{dx}{dt} = 1.0u(\tau).$$

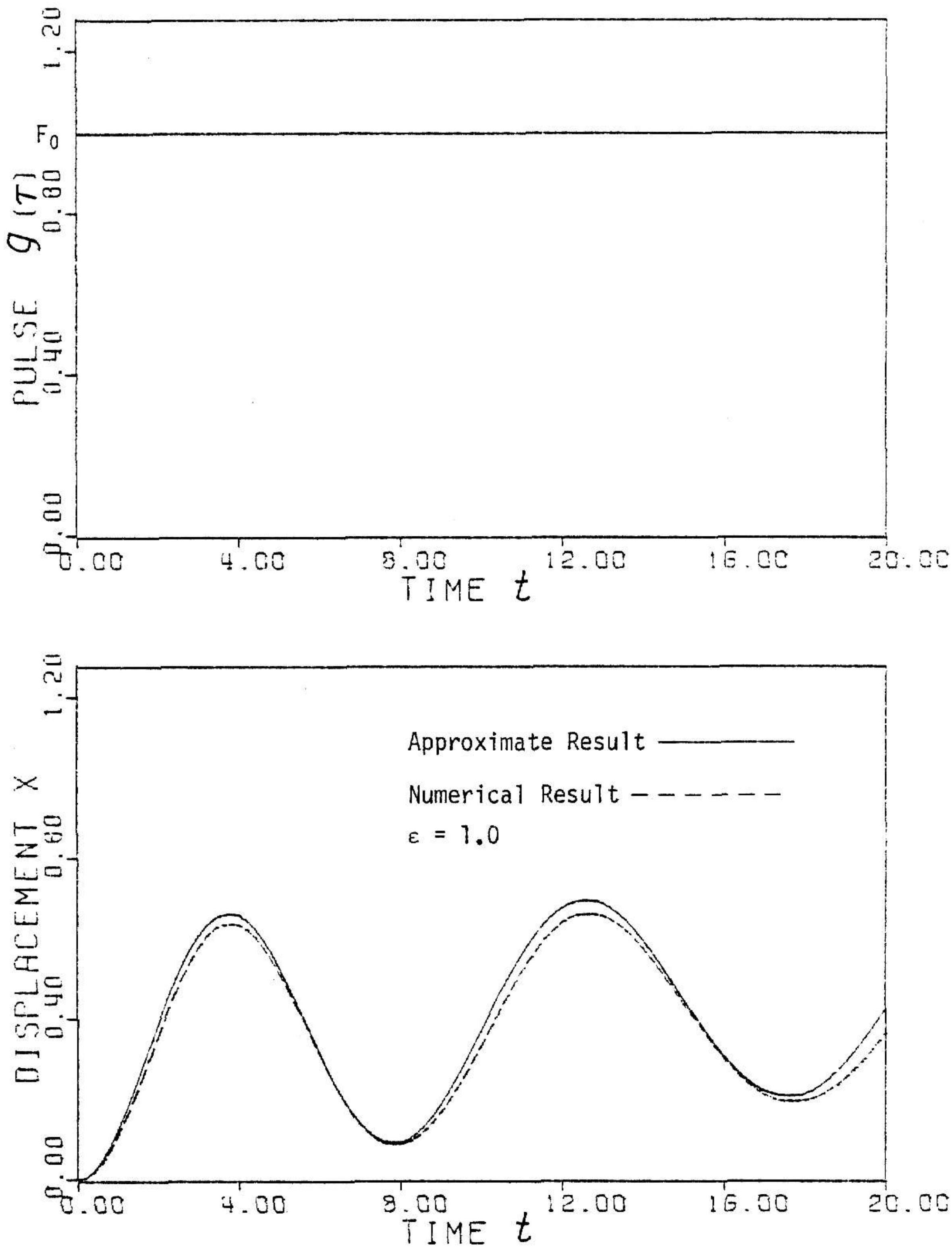
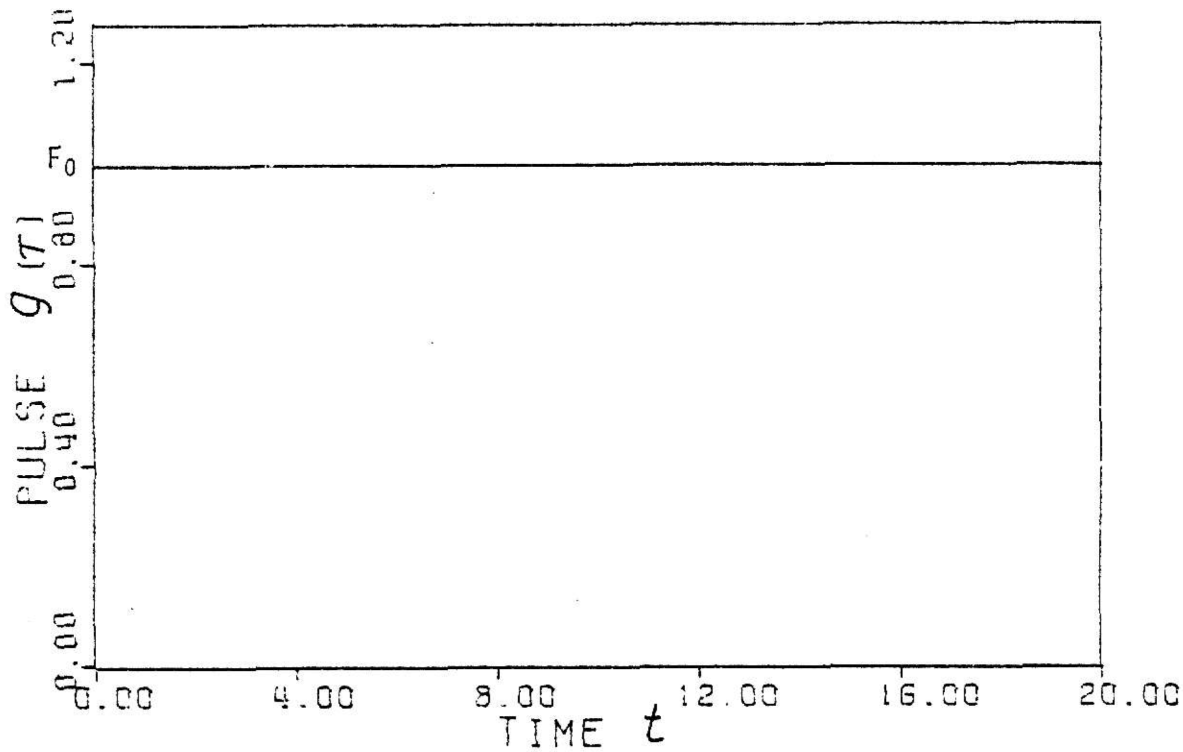


Figure 3.13. Step Function and Time - Displacement results for

$$\frac{d}{dt}\left[(4.0 + 0.05\tau)\frac{dx}{dt}\right] + (3.0 - 0.05\tau)x + 0.5x^3 + 0.1\frac{dx}{dt} = 1.0u(\tau).$$



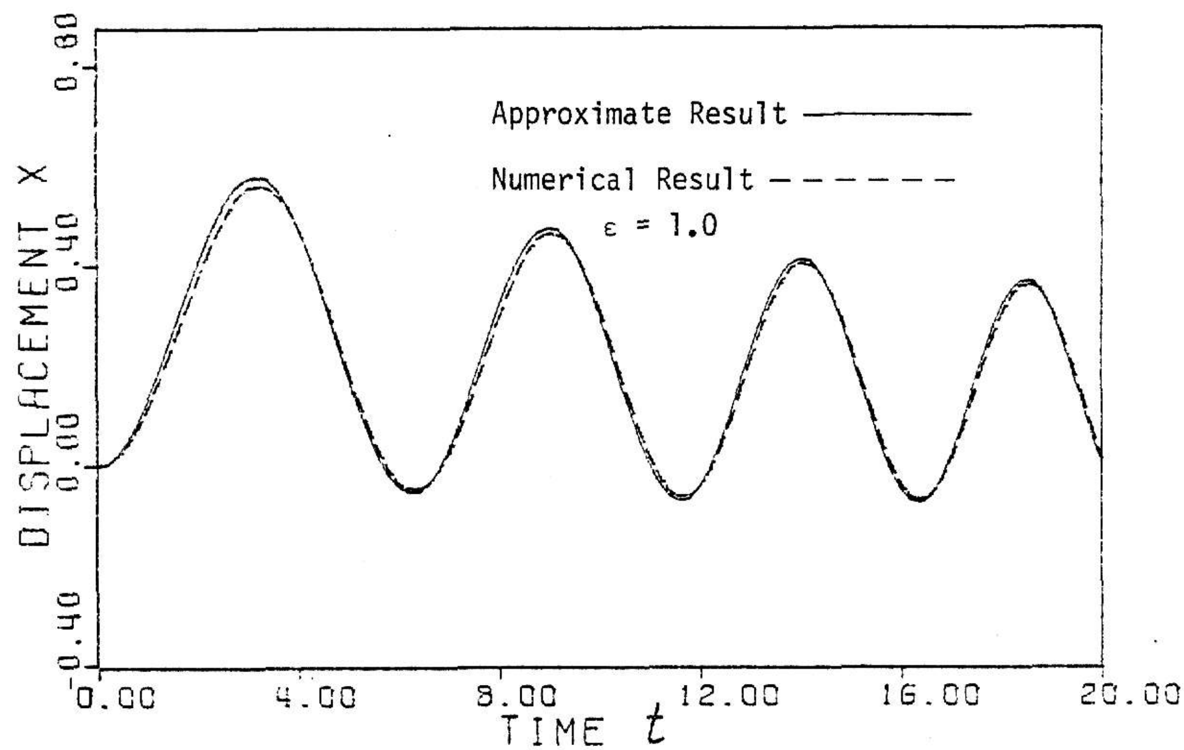


Figure 3.14. Step Function and Time - Displacement results for

$$\frac{d}{dt}\left[(4.0 - 0.05\tau)\frac{dx}{dt}\right] + (3.0 + 0.05\tau)x + 0.5x^3 + 0.1\frac{dx}{dt} = 1.0u(\tau).$$

IV. PULSE EXCITATIONS

4.1 Introduction

The response of systems subjected to pulse excitations form an important class of problems in engineering and mechanics. The response of nonlinear systems subjected to pulse excitations has been studied by Evaldson, Ayre and Jacobson [44] and Jacobson [45], using graphical methods. Ergin [38] and Bapat and Srinivasan [23] used linear approximations to study the response. Ariaratnam [28] and Bauer [24,25-27] have used the perturbation method of Poincare' as modified by Lighthill. Recently, Srirangarajan and Srinivasan [21] and Srirangarajan [22] have applied the generalized averaging technique [39-41] and Krylov-Bogoliubov methods [19] respectively to study the problem. These studies, however, have been restricted to systems with constant parameters only. Recently, Sinha and Chou [32] considered the response of nonlinear, nonstationary systems through an application of a generalized averaging method based on ultraspherical polynomial expansion. However, the analysis was restricted to step function excitation only.

In this chapter, the approximate technique outlined in chapter II is applied to nonlinear, nonstationary systems subjected to arbitrary pulse excitation. The KBM method is used to obtain first and second order approximations. Specific results are obtained for the problem of a pendulum of variable length. Numerous practical problems, such as vibrations of cables in load lifting devices, elevators, etc., reduce to this case and its simple form serves as a good illustration. Once the general solution is obtained, specific results are given for five typical pulses, viz, step function, exponentially decaying pulse, asymptotic step, blast loading, and cosine pulse. Results for a particular set of parameters are compared with numerical solutions obtained using a fourth order Runge Kutta integrating scheme.

4.2 Pendulum of Variable Length Subjected to an Arbitrary Pulse

Consider the vibrations of a pendulum of constant mass when the length varies slowly [20]. Denoting the angle of deviation of the pendulum from the vertical by x, the acceleration due to gravity by g, the mass of the pendulum by m, the slowly varying length by $l = l(\tau)$, and the arbitrary pulse loading by $g(\tau)$, the governing differential equation takes the form

$$\frac{d}{dt}\left[ml^2(\tau)\frac{dx}{dt}\right] + mgl(\tau)\sin x = g(\tau). \tag{4.1}$$

For small amplitude vibrations, $\sin x$ may be replaced by the first two terms of its power series expansion

$$\sin x \simeq x - \frac{x^3}{6}. \tag{4.2}$$

Equation (4.1) may then be written in the form

$$\frac{d}{dt}\left[ml^2(\tau)\frac{dx}{dt}\right] + mgl(\tau)x + \varepsilon f\left(x\,,\,\frac{dx}{dt}\right) = g(\tau), \tag{4.3}$$

$$\text{where } \varepsilon f(x\,,\,dx/dt) = -mgl(\tau)x^3/6\,. \tag{4.4}$$

Following the procedure outlined in chapter II the dependent variable is transformed by

$$x(t) = y(t) + p(t). \tag{2.3}$$

For the first and the second order approximations, p(t) is given by equation (2.9) as

$$p(t) = \frac{g(\tau)}{k(\tau)} = \frac{g(\tau)}{mgl(\tau)} \tag{4.5}$$

Once p(t) is known, the governing equation for y takes the form [c.f. equation (2.10)]

$$\frac{d}{dt}\left[mgl^2(\tau)\,\frac{dy}{dt}\right] + mgl(\tau)y - \frac{mgl(\tau)}{6}\left[a\cos\psi + \frac{g(\tau)}{mgl(\tau)}\right] = 0. \tag{4.6}$$

4.2.1 First Order Approximation

Applying the KBM method as outlined in section 2.3, the first order approximation of equation (4.6) is given by

$$y = a\cos\psi. \tag{4.7}$$

The expressions for a and ψ are given by equations (2.28) as

$$\frac{da}{dt} = \varepsilon A_1, \quad \frac{d\psi}{dt} = \omega(\tau) + \varepsilon B_1. \tag{2.28}$$

The expressions for A_1 and B_1 can be obtained from equation (2.20) and (2.21) as

$$A_1 = \frac{a}{2m(\tau)\omega(\tau)}\,\frac{d[m(\tau)\omega(\tau)]}{d\tau} - $$

$$- \frac{k(\tau)}{12\pi m(\tau)\omega(\tau)}\int_0^{2\pi}\left[a\cos\psi + \frac{g(\tau)}{k(\tau)}\right]^3 \sin\psi\, d\psi, \tag{4.8}$$

$$B_1 = \omega(\tau) - \frac{k(\tau)}{12\pi m(\tau)\omega(\tau)a}\int_0^{2\pi}\left[a\cos\psi + \frac{g(\tau)}{k(\tau)}\right]^3 \cos\psi\, d\psi, \tag{4.9}$$

where $m(\tau) = ml^2(\tau)$, $k(\tau) = mgl(\tau)$, and $\omega(\tau) = \sqrt{g/l(\tau)}$.

After performing the necessary integrations in equations (4.8) and (4.9) and substituting A_1 and B_1 in equation (2.28) the expressions for a and ψ become

$$\frac{da}{dt} = -\varepsilon \frac{3l'(\tau)}{4l(\tau)} a, \tag{4.10}$$

$$\frac{d\psi}{dt} = \omega(\tau)\left[1 - \varepsilon\frac{a^2}{16} - \varepsilon\frac{g(\tau)^2}{4k(\tau)^2}\right], \tag{4.11}$$

where the prime denotes differentiation with respect to τ.

Equation (4.10) can be integrated as

$$a = a(0)\left[\frac{l(0)}{l(\tau)}\right]^{\frac{3}{4}}, \tag{4.12}$$

where $a(0)$ is the initial value of amplitude and is determined from the initial conditions. The expression for ψ, after substituting the value of a into equation (4.11), is given by

$$\psi = \int \omega(\tau)\left[1 - \varepsilon\frac{a(0)^2}{16}\left(\frac{l(0)}{l(\tau)}\right)^{\frac{3}{2}} - \varepsilon\frac{g(\tau)^2}{4k(\tau)^2}\right] dt + \Phi_0, \tag{4.13}$$

where Φ_0 is a constant of integration to be determined from the initial value of ψ.

The initial values of a and ψ are determined from the transformed initial conditions of x given by equation (2.7) as

$$y = -p, \quad \frac{dy}{dt} = -\frac{dp}{dt} \quad \text{at} \quad t = 0. \tag{2.7}$$

Substituting for $y(t)$ and $p(t)$ from equations (4.7) and (4.5) respectively, equation (2.7) yields

$$\begin{aligned} a(0)\cos\psi(0) &= -\frac{g(0)}{k(0)}, \\ -\omega(0)a(0)\sin\psi(0) &= -\frac{d}{dt}\left[\frac{g(\tau)}{k(\tau)}\right]_{t=0}, \end{aligned} \tag{4.14}$$

and can be solved simultaneously for $a(0)$ and $\psi(0)$.

Thus, the total solution of equation (4.3), in the first approximation, is

$$x = a\cos\psi + \frac{g(\tau)}{k(\tau)}\,, \tag{4.15}$$

where a and ψ are given by equations (4.10) and (4.11) respectively and subject to initial conditions obtained from equations (4.14).

4.2.2 Second Order Approximation

The KBM method outlined in section 2.3 yields the second order approximation of equation (4.6) as

$$y = a\cos\psi + \varepsilon u_1(\tau,a,\psi). \tag{4.16}$$

The functions a and ψ are given by

$$\left.\begin{aligned} \frac{da}{dt} &= \varepsilon A_1 + \varepsilon^2 A_2, \\ \frac{d\psi}{dt} &= \omega(\tau) + \varepsilon B_1 + \varepsilon^2 B_2, \end{aligned}\right\} \tag{4.17}$$

where A_1 and B_1 are obtained form equations (4.8) and (4.9), respectively. A_2, B_2, and u_1 are given by equations (2.22), (2.23), and (2.24) respectively as

$$A_2 = -\frac{1}{2\omega(\tau)}\left[a\frac{\partial B_1}{\partial a}A_1 + 2A_1B_1 + \frac{a}{m(\tau)}\frac{d[m(\tau)B_1]}{d\tau}\right] + \\ + \frac{1}{2\pi m(\tau)\omega(\tau)}\int_0^{2\pi} f_1(\tau,a,\psi)\sin\psi\, d\psi, \tag{4.18}$$

$$B_2 = \frac{1}{2\omega(\tau)a}\left[\frac{\partial A_1}{\partial a}A_1 - aB_1^2 + \frac{1}{m(\tau)}\frac{d[m(\tau)A_1]}{d\tau}\right] +$$

(contd.)

$$+ \frac{1}{2\pi m(\tau)\omega(\tau)a} \int_0^{2\pi} f_1(\tau,a,\psi) \cos\psi \, d\psi, \tag{4.19}$$

and

$$u_1(\tau,a,\psi) = \frac{1}{2\pi k(\tau)} \sum_{n\neq\pm 1} \frac{k(\tau)e^{in\psi}}{6(1-n^2)} \int_0^{2\pi} \left[a\cos\psi + \frac{g(\tau)}{k(\tau)} \right]^3 e^{-in\psi} \, d\psi, \tag{4.20}$$

where $k(\tau)$, $m(\tau)$ and $\omega(\tau)$ are as defined in the previous section. Also, from equation (2.26), one obtains

$$f_1(\tau,a,\psi) = - \frac{u_1 k(\tau)}{2} \left[a\cos\psi + \frac{g(\tau)}{k(\tau)} \right]^2$$

$$- 2m(\tau)\omega(\tau) \left[\frac{\partial^2 u_1}{\partial\tau\partial\psi} + \frac{\partial^2 u_1}{\partial a\partial\psi} A_1 + \frac{\partial^2 u_1}{\partial\psi^2} B_1 \right] - \frac{\partial u_1}{\partial\psi} \frac{d[m(\tau)\omega(\tau)]}{d\tau} . \tag{4.21}$$

After performing the necessary integrations in equations (4.18), (4.19), and (4.20), the expressions for A_2, B_2, and u_1 become

$$A_2 = \frac{g(\tau)g'(\tau)a}{4k(\tau)^2} - \frac{l'(\tau)g(\tau)^2 a}{l(\tau)k(\tau)^2} - \frac{3\,l'(\tau)\,a}{64\,l(\tau)} \tag{4.22}$$

$$B_2 = \frac{1}{2\omega(\tau)} \left[- \frac{3\,l'(\tau)}{16\,l(\tau)} - \frac{3\,l(\tau)l''(\tau)}{4\,l(\tau)} - \frac{\omega(\tau)^2 g(\tau)^4}{32k(\tau)^4} + \frac{\omega(\tau)^2 g(\tau)^3 a}{2k(\tau)^3} \right.$$

$$\left. - \frac{\omega(\tau)^2 g(\tau)^2 a^2}{32k(\tau)^2} - \frac{5\omega(\tau)a^4}{1536} \right], \tag{4.23}$$

$$u_1 = - \frac{g(\tau)a^2}{36k(\tau)} \cos 2\psi - \frac{a^3}{192} \cos 3\psi, \tag{4.24}$$

where the prime denotes differentiation with respect to τ. Thus the expressions for da/dt and $d\psi/dt$ take the form [c.f. equations 4.17]

$$\frac{da}{dt} = - \varepsilon \left[\frac{3l'(\tau)}{4l(\tau)} + \frac{\varepsilon l'(\tau)g(\tau)^2}{8l(\tau)k(\tau)^2} - \frac{\varepsilon g(\tau)g'(\tau)}{4k(\tau)^2} \right] a - \frac{3\varepsilon^2 l'(\tau)}{64l(\tau)} a^3, \tag{4.25}$$

$$\frac{d\psi}{dt} = \omega(\tau) \left[1 - \frac{\varepsilon a^2}{16} \right] - \frac{\varepsilon^2}{2\omega(\tau)} \left[\frac{3l'(\tau)^2}{16l(\tau)^2} + \frac{3l(\tau)l''(\tau)}{4l(\tau)^2} + \frac{\omega(\tau)^2 g(\tau)^4}{32k(\tau)} - \right.$$

(contd.)

$$- \frac{\omega(\tau)^2 g(\tau)^3 a}{72k(\tau)^3} + \frac{\omega(\tau)^2 g(\tau)^2 a^2}{32k(\tau)} + \frac{5\omega(\tau)^2 a^4}{1536}\Bigg] . \tag{4.26}$$

Equation (4.25) is a special case of Bernoulli equation [43]. This can be rewritten as

$$\frac{da}{dt} = \alpha(t)a + \beta(t)a^3, \tag{4.27}$$

where

$$\alpha(t) = -\varepsilon\left[\frac{3l'(\tau)}{4l(\tau)} + \frac{\varepsilon l'(\tau)g(\tau)^2}{8l(\tau)k(\tau)} - \frac{\varepsilon g(\tau)g'(\tau)}{4k(\tau)^2}\right], \tag{4.28}$$

$$\beta(t) = -\frac{3\varepsilon^2 l'(\tau)}{64l(\tau)} . \tag{4.29}$$

Introducing the transformation

$$a = [a^*]^{-\frac{1}{2}}, \tag{4.30}$$

equation (4.27) takes the form

$$\frac{da^*}{dt} + 2\alpha(t)a^* = -2\beta(t). \tag{4.31}$$

Equation (4.31) has a general solution

$$a^* = \exp[-2\int\alpha(t)dt]\int\Big[-2\beta(t)\exp[2\int\alpha(t)dt]\Big]\,dt + C_0\exp[-2\int\beta(t)dt], \tag{4.32}$$

where C_0 is a constant.

Hence the total solution of equation (4.3), in the second order approximation, can be obtained from equations (4.16), (4.24), and (4.5) as

$$x = a\cos\psi - \varepsilon\left[\frac{g(\tau)a^2}{12mgl(\tau)}\cos2\psi - \frac{a^3}{192}\cos3\psi\right] + \frac{g(\tau)}{mgl(\tau)} , \tag{4.33}$$

where a and ψ are given by equations (4.25) and (4.26) respectively. It should be pointed out that for the second order approximation, the expressions for a and ψ can be quite complex, to the point that they cannot be expressed in closed form. In such cases it may be necessary to integrate these expressions numerically, using a numerical scheme. In the following sections, explicit results are obtained for various pulses.

4.3 Step Function Excitation

Consider the pendulum described by equation (4.3), subjected to a step function excitation. If the length $l(\tau)$ varies as a linear function of time, the equation of motion can be written as

$$\frac{d}{dt}\left[m[l_0+l_1(\tau)]^2\frac{dx}{dt}\right] + mg[l_0+l_1(\tau)]x - \frac{mg[l_0+l_1(\tau)]}{6}x^3 = F_0u(\tau), \qquad (4.34)$$

where l_0, l_1, and F_0 are constants, and $u(\tau)$ is the unit step function defined by

$$u(\tau) = \begin{cases} 0 \text{ if } \tau \leq 0^- \\ 1 \text{ if } \tau \geq 0^+ . \end{cases} \qquad (4.35)$$

The first order approximation, from equation (4.15) with $g(\tau)$ replaced by $F_0u(\tau)$, is easily given as

$$x = a\cos\psi + \frac{F_0u(\tau)}{mg(l_0+l_1\tau)} . \qquad (4.36)$$

The expressions for amplitude a and phase ψ are obtained from equations (3.10) and (3.11) as

$$a = a(0)\left[\frac{l_0}{l_0+l_1\tau}\right]^{\frac{3}{4}} \qquad (4.37)$$

$$\psi = \int \omega(\tau)\left[1 - \frac{\varepsilon a(0)^2}{16}\left[\frac{l_0}{l_0+l_1\tau}\right]^{\frac{3}{2}} - \frac{F_0^2u(\tau)^2}{4m^2g^2(l_0+l_1\tau)^2}\right]dt + \Phi_0, \qquad (4.38)$$

$$= \frac{\omega(\tau)l(\tau)}{l_1}\left[\frac{2}{\varepsilon} + \frac{a(0)^2}{16}\left[\frac{l_0}{l(\tau)}\right]^{\frac{3}{2}} + \frac{F_0^2u(\tau)^2}{6k(\tau)^2}\right] + \Phi_0, \qquad (4.38a)$$

where $l(\tau) = l_0+l_1\tau$, $k(\tau) = mgl(\tau)$, and $\omega(\tau) = \sqrt{g/l(\tau)}$. The initial amplitude $a(0)$ and constant of integration Φ_0 are determined from the initial conditions given by equations (4.14) as

$$\left.\begin{aligned} a(0)\cos\psi(0) &= -\frac{F_0u(0)}{mgl_0}\,, \\ -\omega(0)a(0)\sin\psi(0) &= \frac{F_0u(0)l_1\varepsilon}{\omega(0)l_0} \end{aligned}\right\}. \tag{4.39}$$

Solving equations (4.39) for $a(0)$ and $\psi(0)$ yields

$$\left.\begin{aligned} \psi(0) &= \arctan\left[\frac{l_1\varepsilon}{\omega(0)l_0}\right], \\ a(0) &= -\frac{F_0u(0)}{mgl_0\cos\psi(0)}\,, \end{aligned}\right\} \tag{4.40}$$

and the constant of integration, Φ_0, of equation (4.38) is

$$\Phi_0 = \arctan\left[\frac{l_1\varepsilon}{\omega(0)l_0}\right] - \frac{\omega(0)l_0}{l_1}\left[\frac{2}{\varepsilon} + \frac{a(0)^2}{16} + \frac{F_0^2u(0)^2}{6k(0)^2}\right].$$

The second order approximation from equation (4.33)

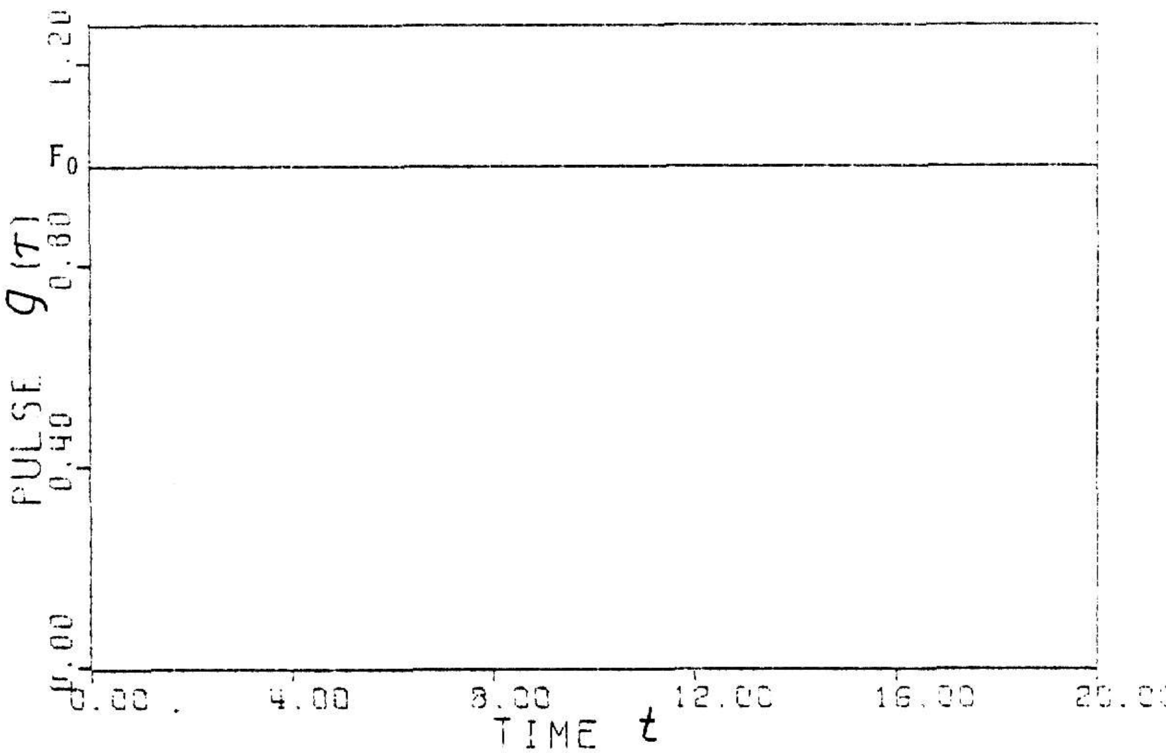
$$x = a\cos\psi - \varepsilon\left[\frac{F_0u(\tau)a^2}{12mgl(\tau)}\cos2\psi - \frac{a^3}{192}\cos3\psi\right] + \frac{F_0u(\tau)}{mgl(\tau)}, \tag{4.41}$$

where a and ψ are given by equations (4.25) and (4.26) respectively.

Results for two particular cases are shown in figures 4.1 and 4.2. In the first, the pendulum length increases with time and in the second its length decreases. Notice that there is no significant improvement in the second order approximation over the first.

4.4 Exponentially Decaying Pulse

Consider the pendulum of section 4.2 subjected to a pulse whose magnitude decays according to the exponential law



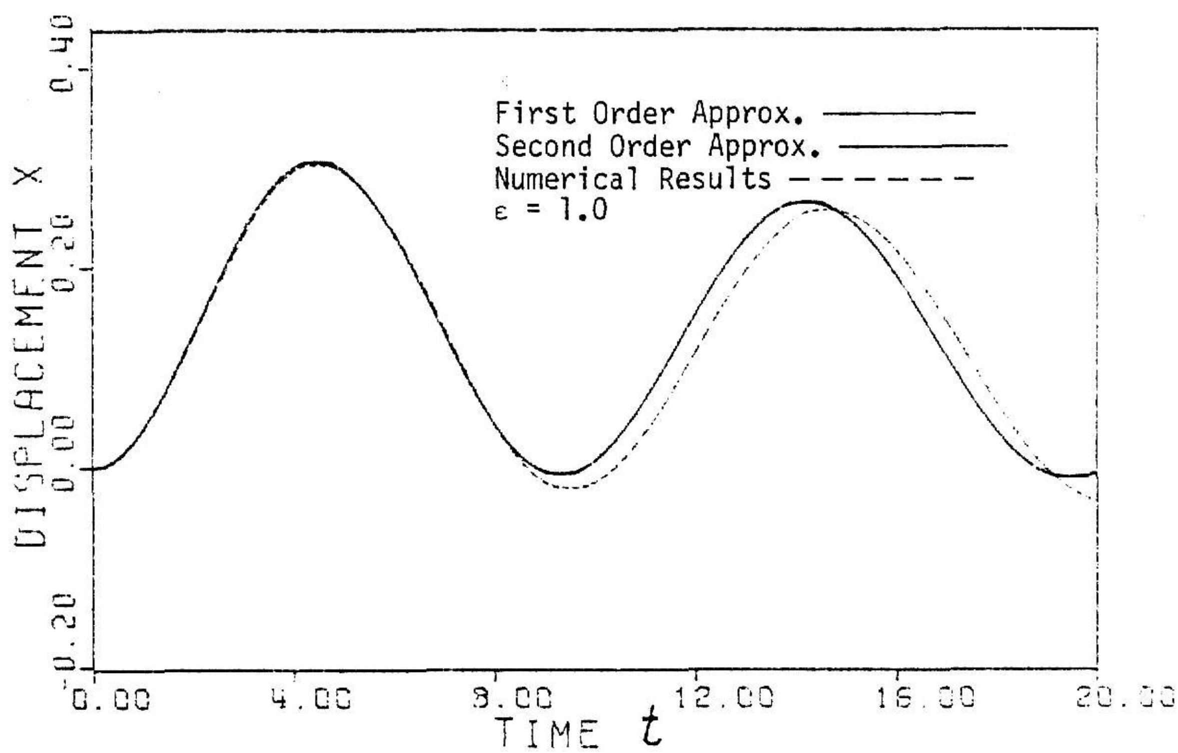
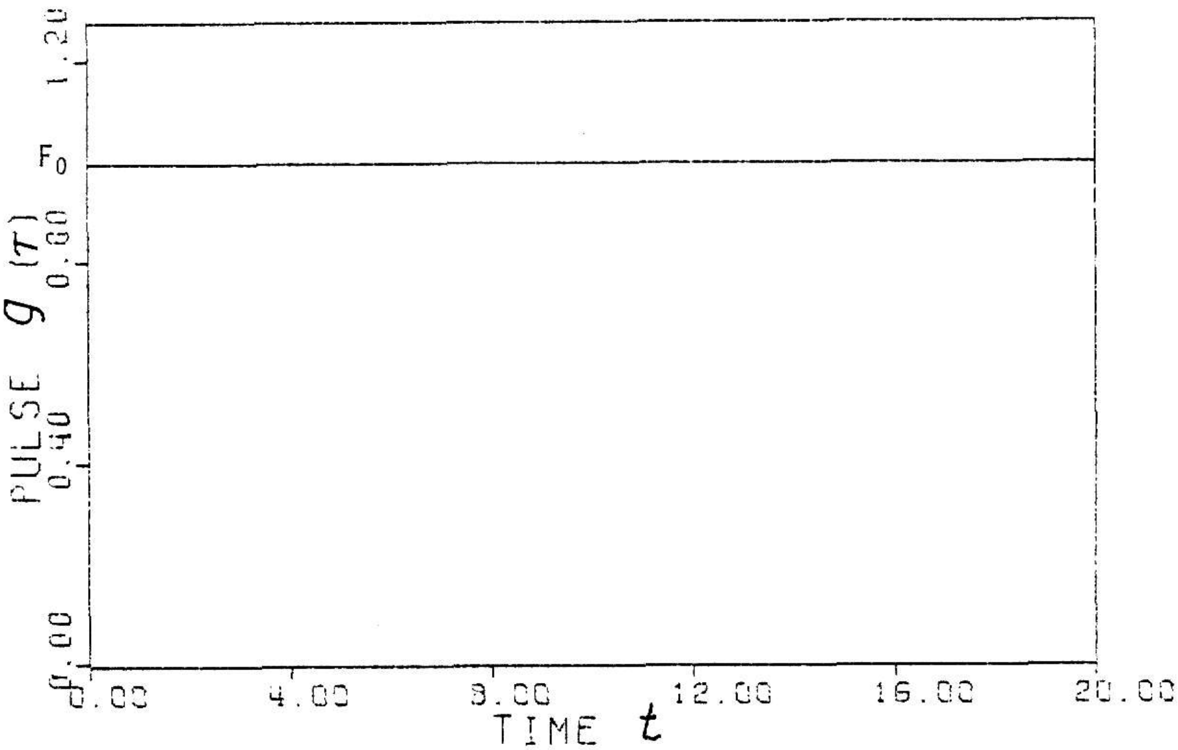


Figure 4.1. Step Function and Time - Displacement results for

$$\frac{d}{dt}\left[3.0(2.0 + 0.04\tau)^2 \frac{dx}{dt}\right] + 3.0(2.0 + 0.04\tau)\left[x - \frac{x^3}{6}\right] = 1.0u(\tau).$$



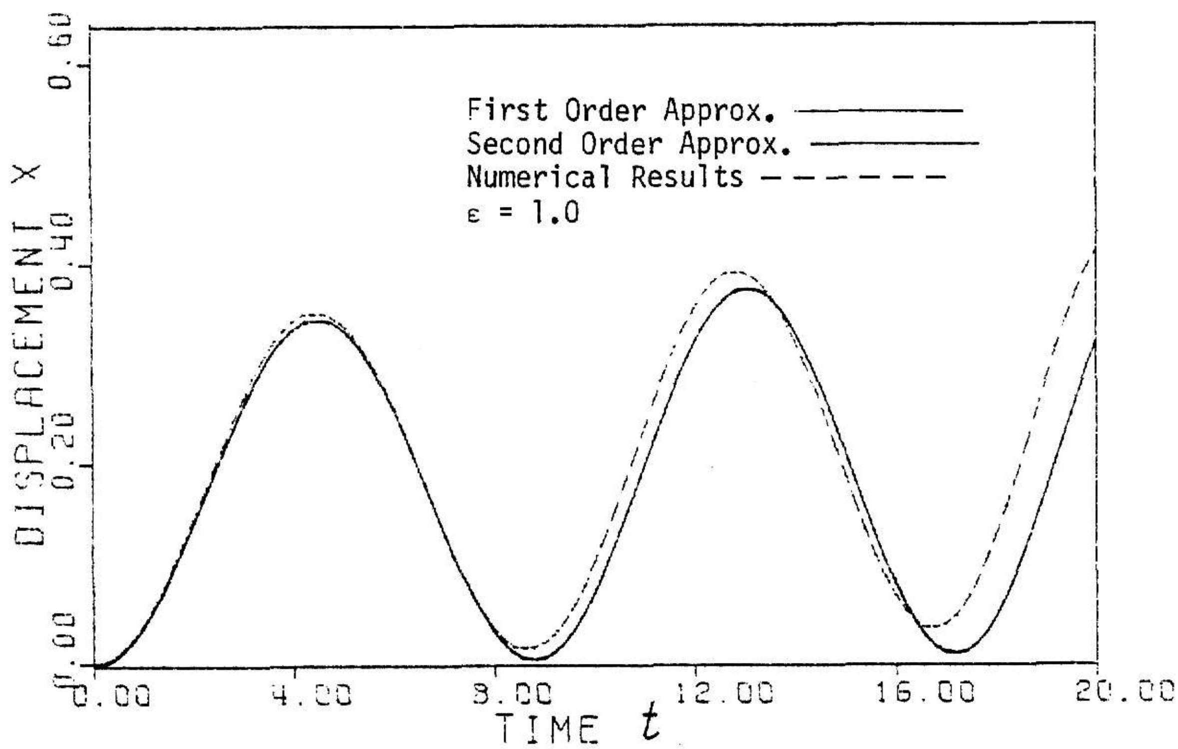


Figure 4.2. Step Function and Time - Displacement results for

$$\frac{d}{dt}\left[3.0(2.0 - 0.02\tau)^2 \frac{dx}{dt}\right] + 3.0(2.0 - 0.02\tau)\left[x - \frac{x^3}{6}\right] = 1.0u(\tau).$$

$$g(\tau) = F_0 e^{-\alpha\tau}; \; \alpha > 0 \, , \tag{4.42}$$

where F_0 and α are constants [see figures 4.3 and 4.4]. The first order approximation, as given by equation (4.15) with $g(\tau)$ replaced by $F_0 e^{-\alpha\tau}$, is easily given as

$$x = a\cos\psi + \frac{F_0 e^{-\alpha\tau}}{mgl(\tau)} \, , \tag{4.43}$$

where $l(\tau) = l_0 + l_1\tau$ and, from equations (3.10) and (3.11), a and ψ are

$$a = a(0)\left[\frac{l_0}{l(\tau)}\right]^{\frac{3}{4}} \tag{4.44}$$

$$\psi = \int \omega(\tau)\left[1 - \frac{\varepsilon a(0)^2}{16}\left[\frac{l_0}{l(\tau)}\right]^{\frac{3}{2}} - \frac{\varepsilon F_0 e^{-2\alpha\tau}}{4m^2g^2l(\tau)^2}\right] dt + \Phi_0 . \tag{4.45}$$

The initial amplitude $a(0)$ and the constant of integration Φ_0 are determined from the initial conditions on a and ψ, obtained from equations (4.14) as

$$\left.\begin{aligned} a(0) &= -\frac{F_0}{mgl_0\cos\psi(0)} \, , \\ \psi(0) &= \arctan\left[\frac{\varepsilon(\alpha l_0 + l_1)}{\psi(0)l_0}\right] \end{aligned}\right\} . \tag{4.46}$$

The second order approximation from equation (4.33) is

$$x = a\cos\psi - \varepsilon\left[\frac{F_0 e^{-\alpha\tau} a^2}{12mgl(\tau)}\cos 2\psi - \frac{a^3}{192}\cos 3\psi\right] + \frac{F_0 e^{-\alpha\tau}}{mgl(\tau)} \, , \tag{4.47}$$

where a and ψ are once again given by equations (4.25) and (4.26) respectively. Figures 4.3 and 4.4 show results for two particular values of α, where the length of the pendulum is increasing with time.

4.5 Blast Type Loading

A common type of pulse loading occurs due to a sudden impact or ex-

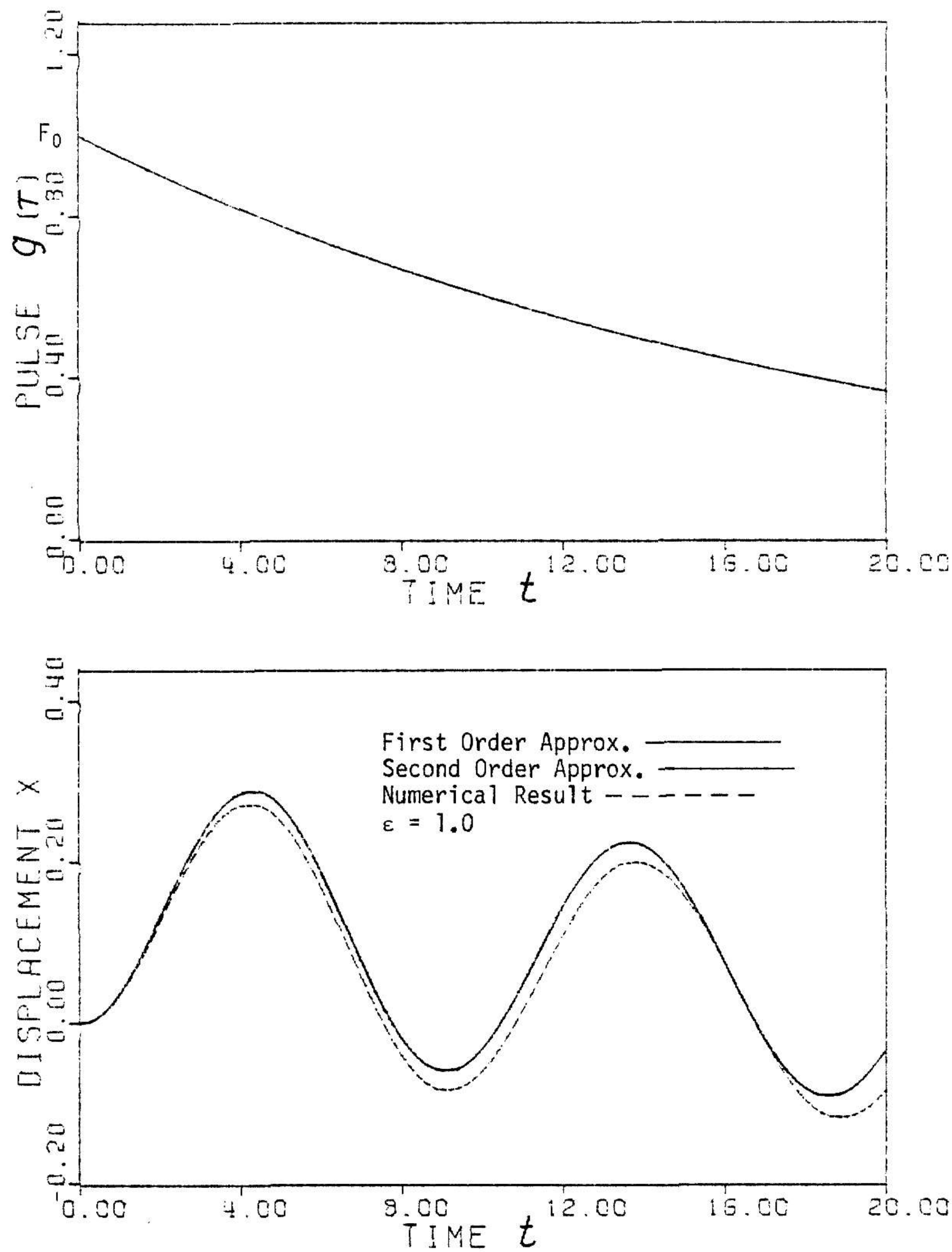


Figure 4.3. Exponentially Decaying Pulse and Time - Displacement results for

$$\frac{d}{dt}\left[3.0(2.0 + 0.02\tau)^2 \frac{dx}{dt}\right] + 3.0(2.0 + 0.02\tau)\left[x - \frac{x^3}{6}\right] = 1.0e^{-0.05\tau}$$

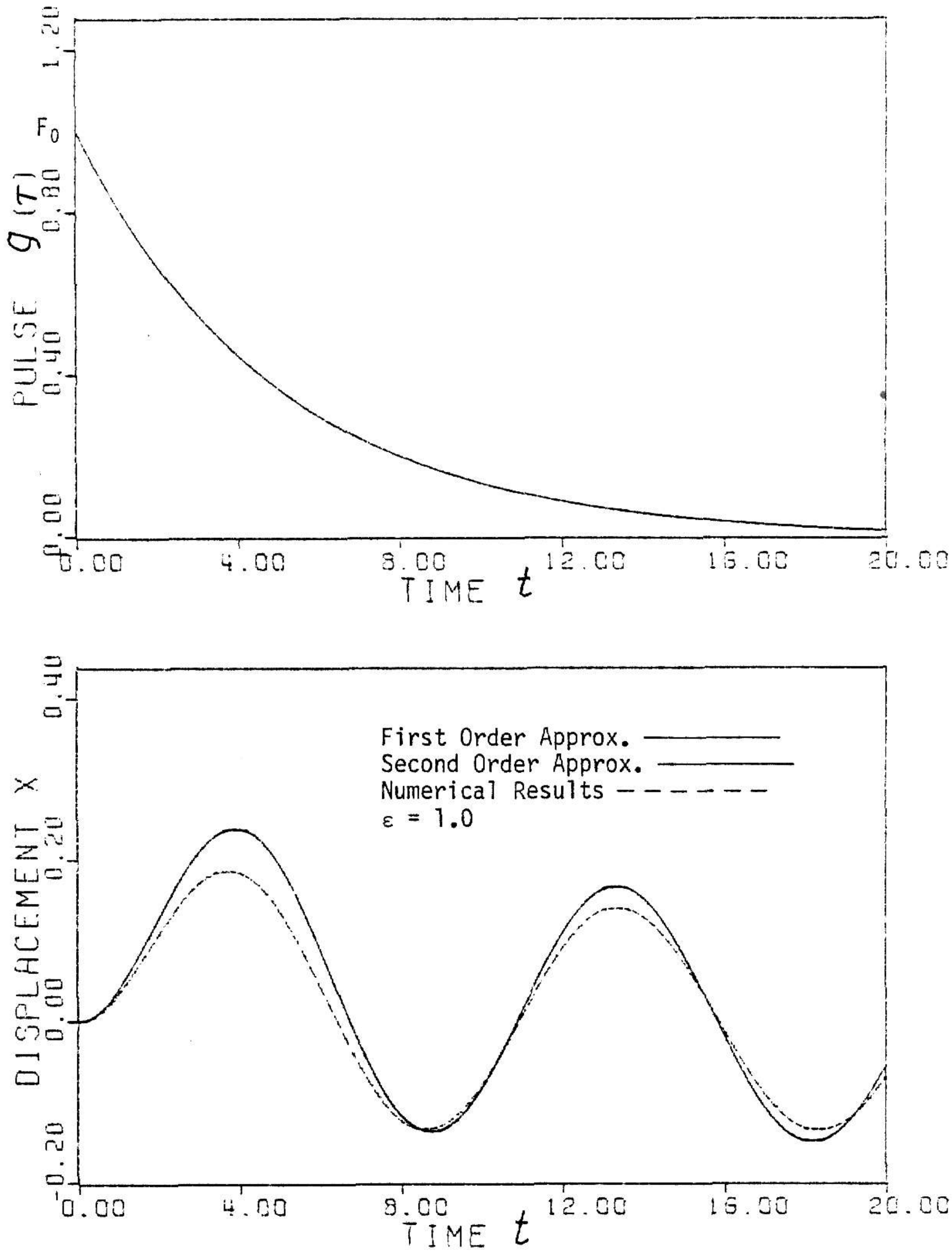


Figure 4.4. Exponentially Decaying Pulse and Time - Displacement results for

$$\frac{d}{dt}\left[3.0(2.0 + 0.02\tau)^2 \frac{dx}{dt}\right] + 3.0(2.0 + 0.02\tau)\left[x - \frac{x^3}{6}\right] = 1.0e^{-0.2\tau}$$

plosion. One method of modelling such a loading is by the exponential law

$$g(\tau) = F_0(e^{-\alpha\tau} - e^{-\beta\tau}); \quad \alpha,\beta > 0, \tag{4.48}$$

where F_0, α, β are constants and $\alpha < \beta$. [see figures 4.5 and 4.6]. The first order approximation for the response of the pendulum of section 4.2, whose length varies according to $l(\tau) = l_0+l_1\tau$, is given by

$$x = a\cos\psi + \frac{F_0(e^{-\alpha\tau} - e^{-\beta\tau})}{mgl(\tau)}, \tag{4.49}$$

where, from equations (4.12) and (4.13),

$$a = a(0)\left[\frac{l_0}{l(\tau)}\right]^{\frac{3}{4}} \tag{4.12}$$

$$\psi = \int \omega(\tau)\left[1 - \varepsilon\frac{a(0)^2}{16}\left[\frac{l_0}{l(\tau)}\right]^{\frac{3}{2}} - \frac{F_0(e^{-\alpha\tau} - e^{-\beta\tau})}{4m^2g^2l(\tau)^2}\right]dt + \Phi_0. \tag{4.50}$$

The initial conditions on a and ψ are, from equations (4.14),

$$\left.\begin{aligned} a(0) &= -\frac{\varepsilon F_0(\alpha-\beta)}{\omega(0)mgl_0}, \\ \psi(0) &= \frac{\pi}{2}. \end{aligned}\right\} \tag{4.51}$$

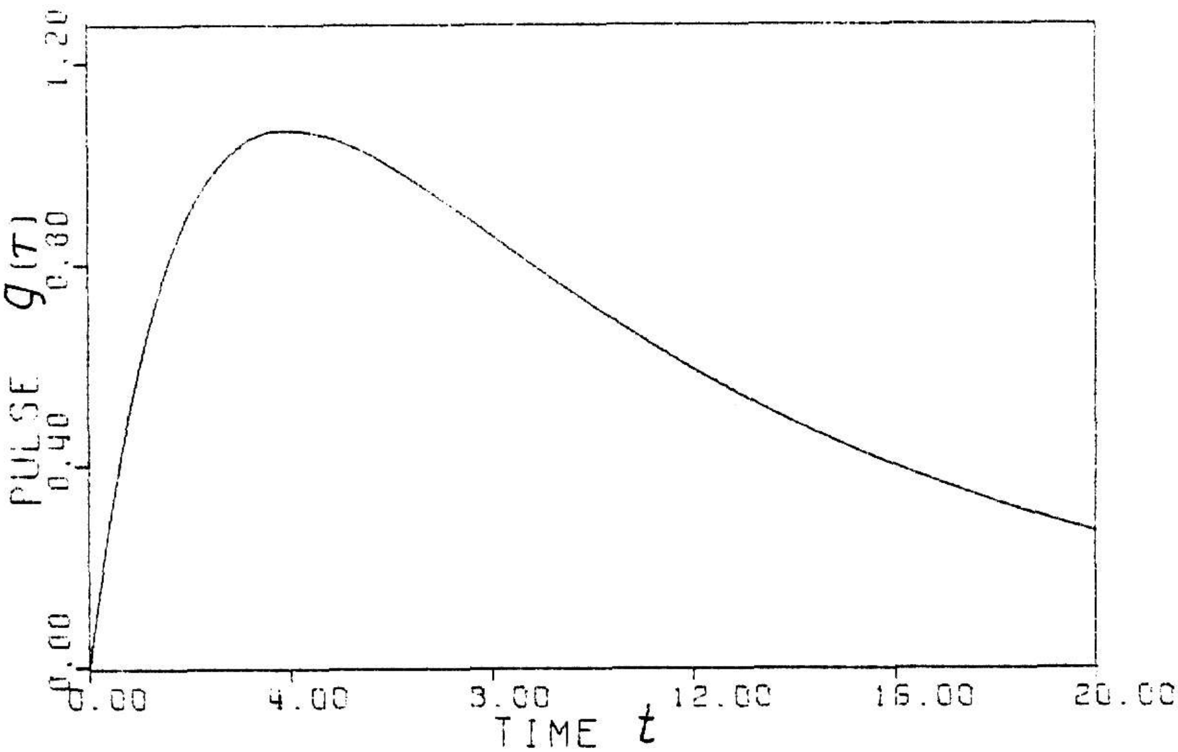
The second order approximation is given by equation (4.33) as

$$x = a\cos\psi - \varepsilon\left[\frac{g(\tau)a^2}{12mgl(\tau)}\cos2\psi - \frac{a^3}{192}\cos3\psi\right] + \frac{F_0(e^{-\alpha\tau} - e^{-\beta\tau})}{mgl(\tau)}, \tag{4.52}$$

where again a and ψ are given by equations (4.25) and (4.26) respectively, with $g(\tau)$ replaced with the exponentially decaying pulse represented by equation (4.48). Results for particular values of α and β are shown in figures 4.5 and 4.6, where the length of the pendulum increases with time.

4.6 Asymptotic Step

Consider the pendulum of section 3.2, subjected to an asymptotic step



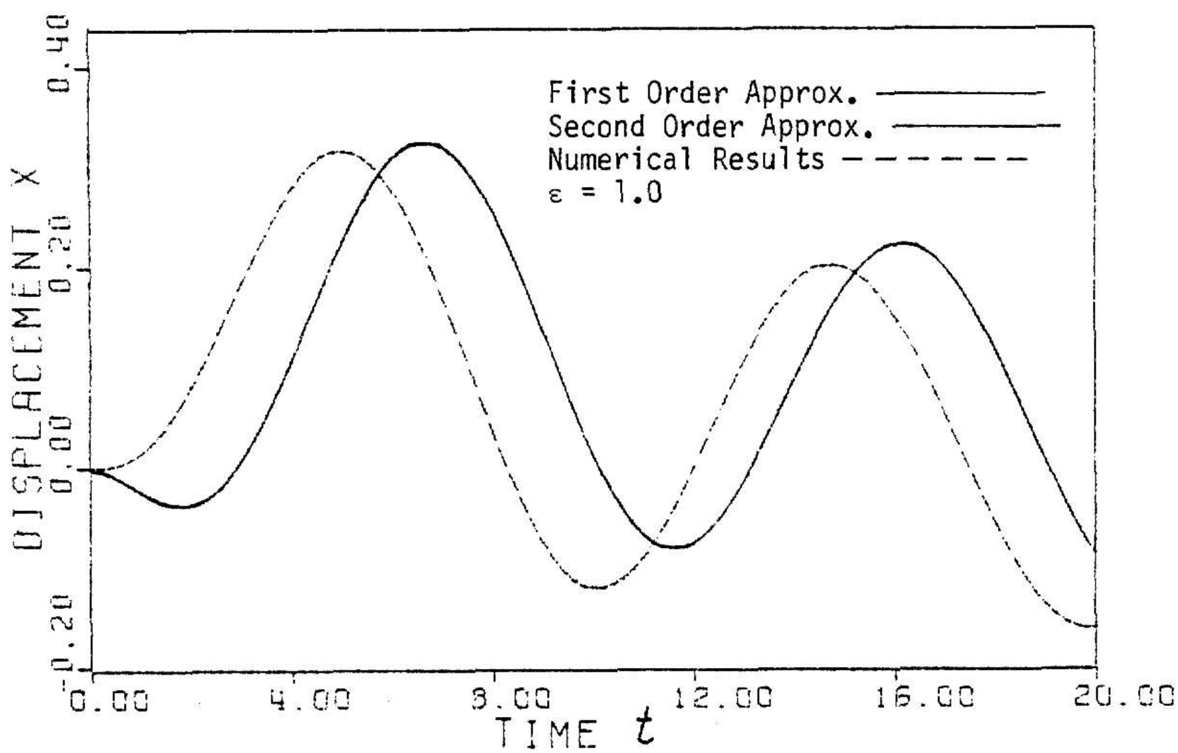
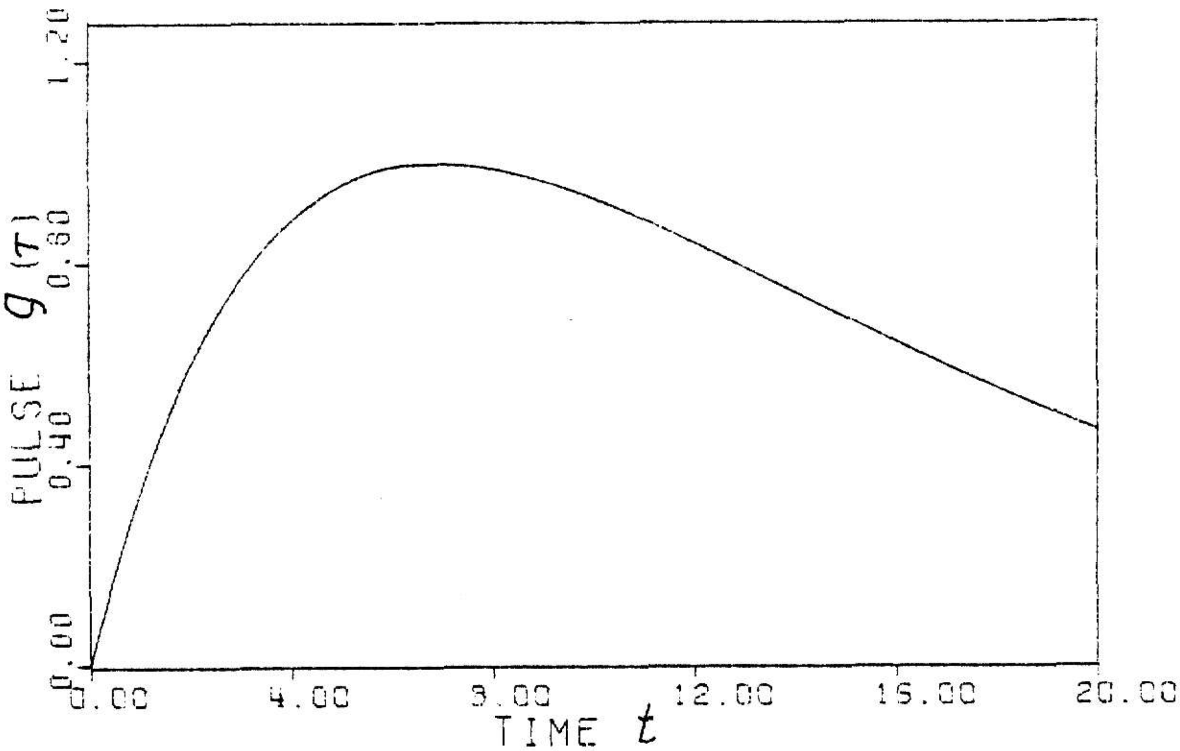


Figure 4.5. Blast Loading and Time - Displacement results for

$$\frac{d}{dt}\left[3.0(2.0+0.02\tau)^2\frac{dx}{dt}\right]+3.0(2.0+0.02\tau)\left[x-\frac{x^3}{6}\right]=2.0(e^{-0.1\tau}-e^{-0.5\tau})$$



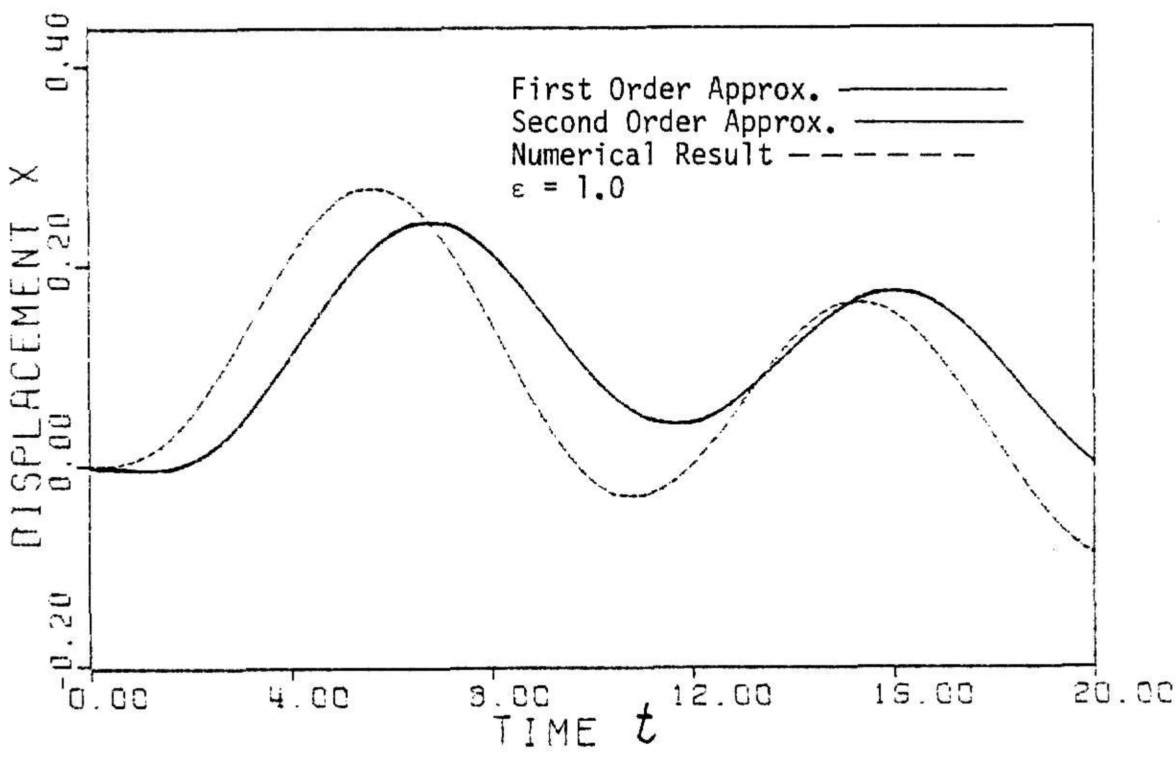


Figure 4.6. Blast Loading and Time - Displacement results for

$$\frac{d}{dt}\left[3.0(2.0 + 0.02\tau)^2 \frac{dx}{dt}\right] + 3.0(2.0 + 0.02\tau)\left[x - \frac{x^3}{6}\right] = 4.0(e^{-0.1\tau} - e^{-0.2\tau})$$

function, given by

$$g(\tau) = F_0(1 - e^{-\alpha\tau}) \; ; \quad \alpha > 0, \tag{4.53}$$

where F_0 and α are constants [see figures 4.7 and 4.8]. As before, the first order approximation is given by equation (4.15) as

$$x = a\cos\psi + \frac{F_0(1 - e^{-\alpha\tau})}{mgl(\tau)} \tag{4.54}$$

where

$$a = a(0)\left[\frac{l_0}{l(\tau)}\right]^{\frac{3}{4}}, \tag{4.12}$$

$$\psi = \int \omega(\tau)\left[1 - \varepsilon\frac{a(\tau)^2}{16}\left[\frac{l_0}{l(\tau)}\right]^{\frac{3}{2}} - \frac{F_0(1 - e^{-\alpha\tau})}{4m^2g^2l(\tau)}\right]dt + \Phi_0, \tag{4.55}$$

with initial contitions

$$\left.\begin{aligned} a(0) &= \frac{\varepsilon\alpha F_0}{\omega(0)mgl_0}\,, \\ \psi(0) &= \frac{\pi}{2}\,. \end{aligned}\right\} \tag{4.56}$$

The second order approximation is

$$x = a\cos\psi - \varepsilon\left[\frac{g(\tau)a^2}{12mgl(\tau)}\cos2\psi - \frac{a^3}{192}\cos3\psi\right] + \frac{F_0(1 - e^{-\alpha\tau})}{mgl(\tau)}\,, \tag{4.57}$$

where a and ψ are given by equations (4.25) and (4.26) respectively with $g(\tau)$ replaced with the asymptotic step given in equation (4.53). Results for a particular set of parameters are shown in figures 4.7 and 4.8.

4.7 Cosine Pulse

As a final example of nonlinear, nonstationary systems subjected to pulse excitation, consider the pendulum described in section 3.2, with

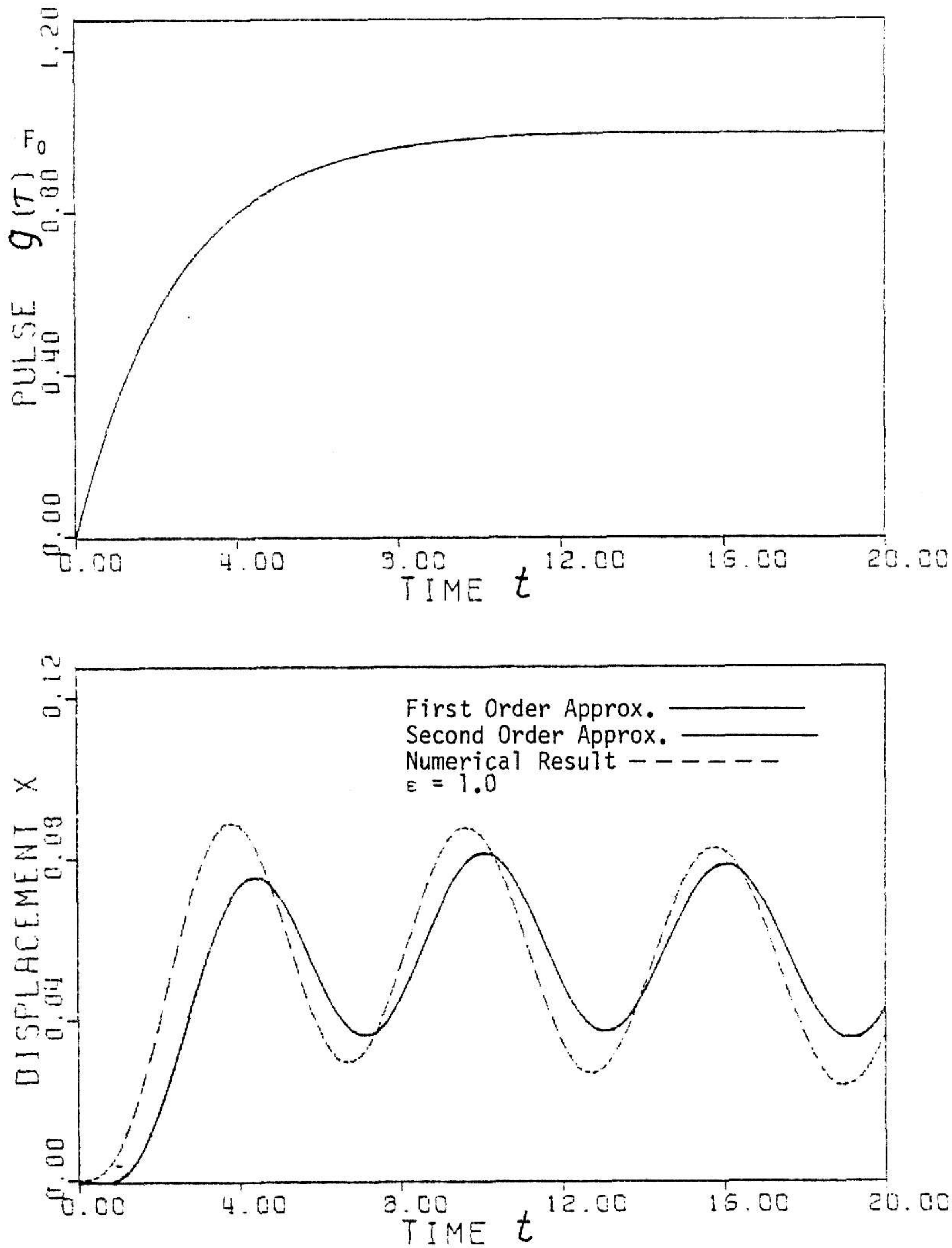
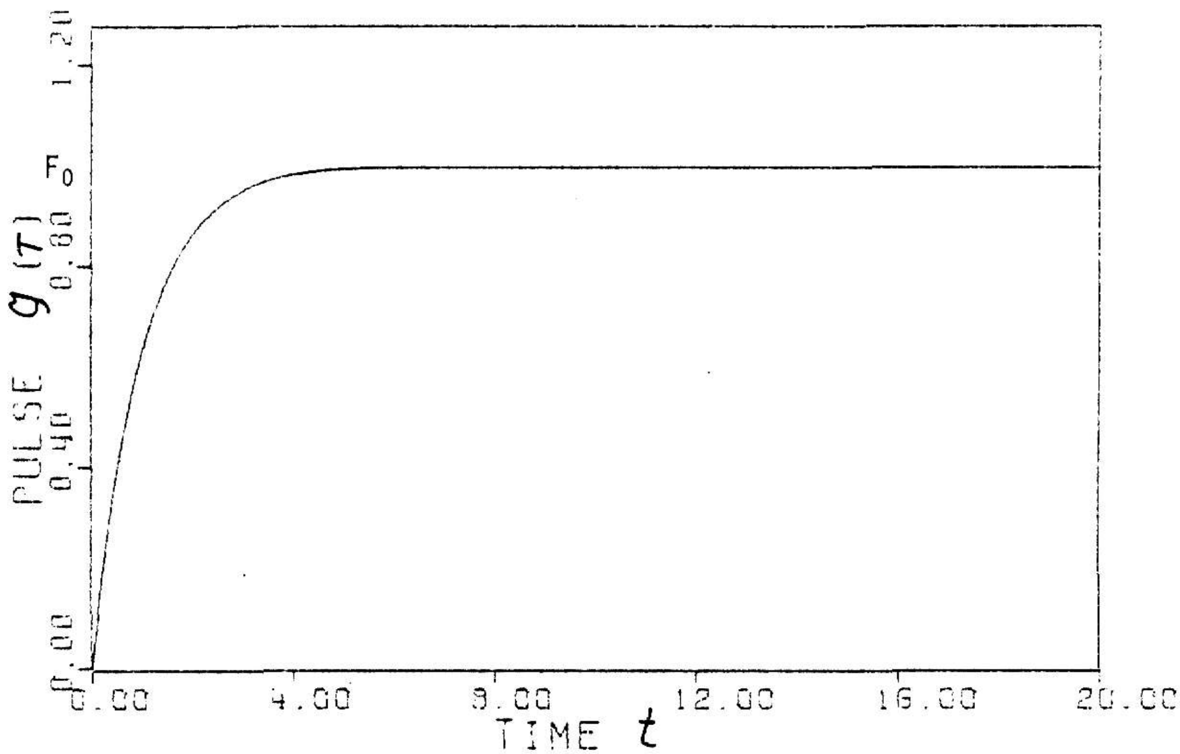


Figure 4.7. Asymptotic Step and Time - Displacement results for

$$\frac{d}{dt}\left[3.0(2.0 + 0.02\tau)^2 \frac{dx}{dt}\right] + 3.0(2.0 + 0.02\tau)\left[x - \frac{x^3}{6}\right] = 1.0(1.0 - e^{-0.4\tau})$$



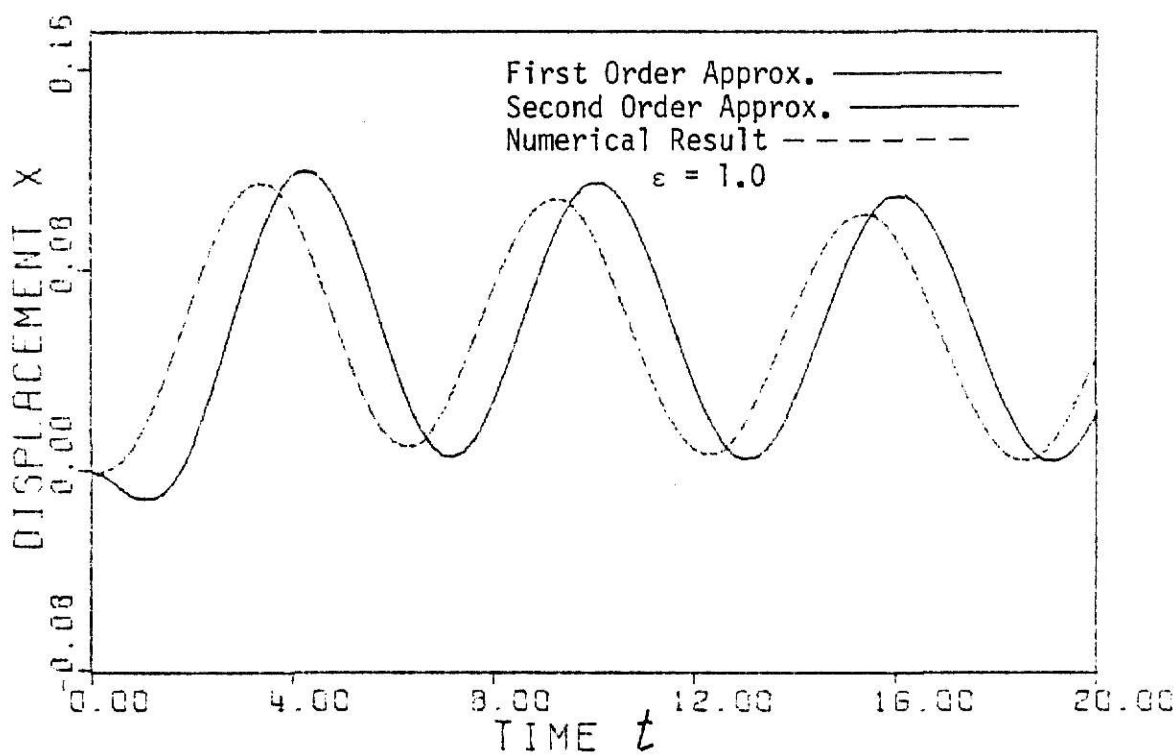


Figure 4.8. Asymptotic Step and Time - Displacement results for

$$\frac{d}{dt}\left[3.0(2.0 + 0.02\tau)^2 \frac{dx}{dt}\right] + 3.0(2.0 + 0.02\tau)\left[x - \frac{x^3}{6}\right] = 1.0(1.0 - e^{-1.0\tau})$$

the length obeying the linear law $l(\tau) = l_0 + l_1\tau$, and subjected to a pulse which is described by

$$g(\tau) = F_0 \cos\alpha\tau, \tag{4.58}$$

where F_0 and ψ are constants [see figures 4.9 and 4.10]. The first order approximation is

$$x = a\cos\psi + \frac{F_0\cos\psi}{mgl(\tau)}, \tag{4.59}$$

where, by equations (4.12) and (4.13), a and ψ are

$$a = a(0)\left[\frac{l_0}{l(\tau)}\right]^{\frac{3}{4}} \tag{4.12}$$

$$\psi = \int \omega(\tau)\left[1 - \varepsilon\frac{a(0)^2}{16}\left[\frac{l_0}{l(\tau)}\right]^{\frac{3}{2}} - \varepsilon\frac{F_0^2\cos^2\alpha\tau}{4m^2g^2l(\tau)^2}\right] dt + \Phi_0. \tag{4.60}$$

The initial conditions on a and ψ are

$$\left.\begin{aligned} a(0) &= -\frac{F_0}{mgl_0\cos\psi(0)}, \\ \psi(0) &= \arctan\left[\frac{\varepsilon l_1}{\omega(0)l_0}\right]. \end{aligned}\right\} \tag{4.61}$$

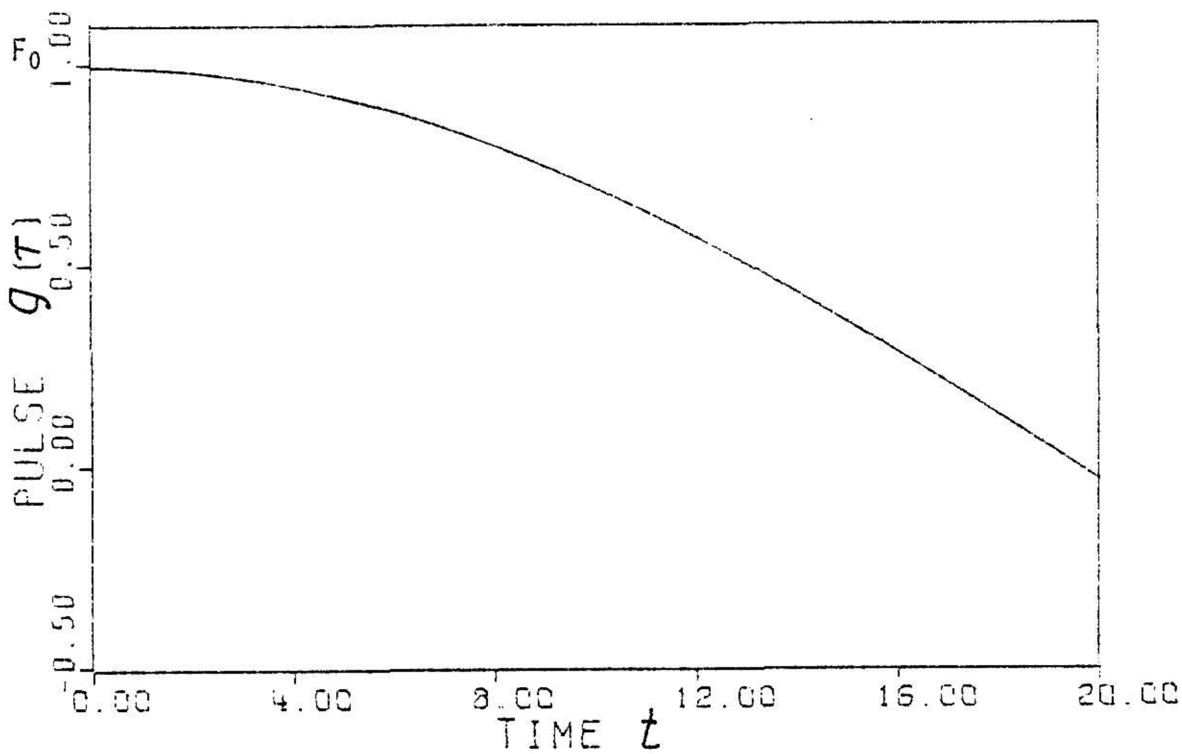
The second order approximation is

$$x = a\cos\psi - \varepsilon\left[\frac{g(\tau)a^2}{12mgl(\tau)}\cos 2\psi - \frac{a^3}{192}\cos 3\psi\right] + \frac{F_0\cos\alpha\tau}{mgl(\tau)}, \tag{4.62}$$

where, again a and ψ are given by equations (4.25) and (4.26) respectively, with $g(\tau)$ replaced by $F_0\cos\alpha\tau$. Results for increasing and decreasing lengths are shown in firgues 4.9 and 4.10 respectively.

4.8 Discussion

In this chapter the approximate method developed in chapter II, for nonlinear systems with slowly varying parameters subjected to pulse exci-



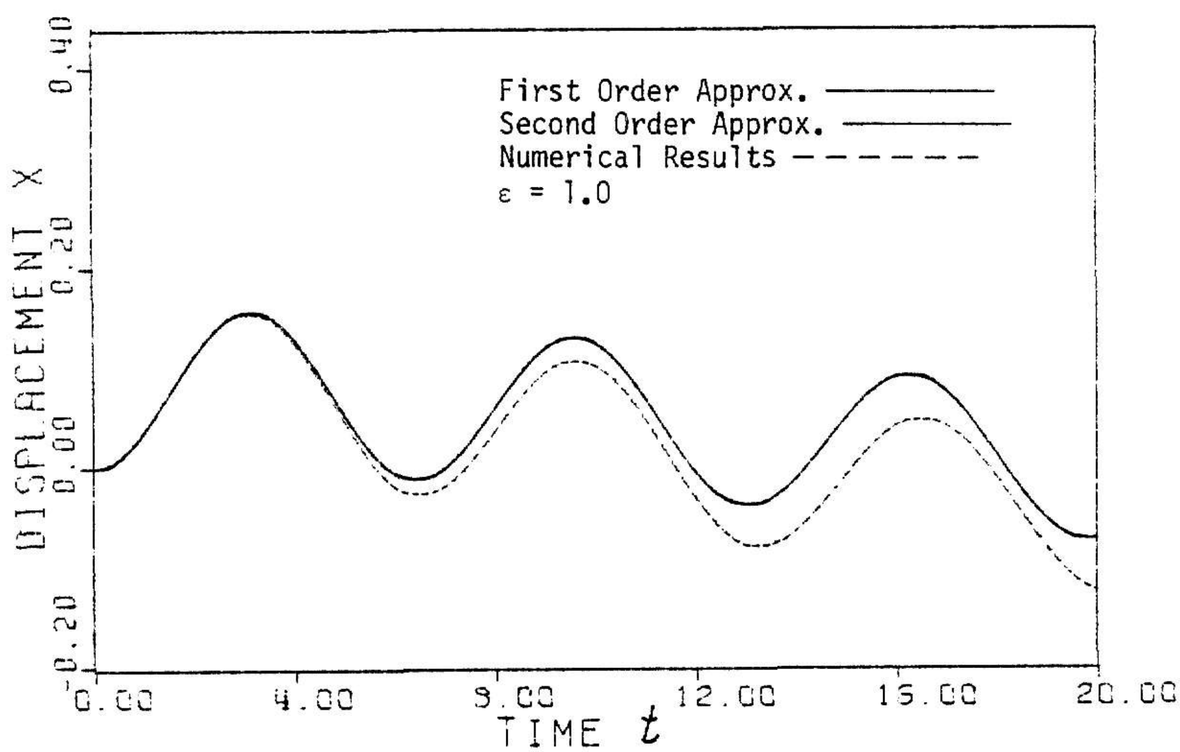
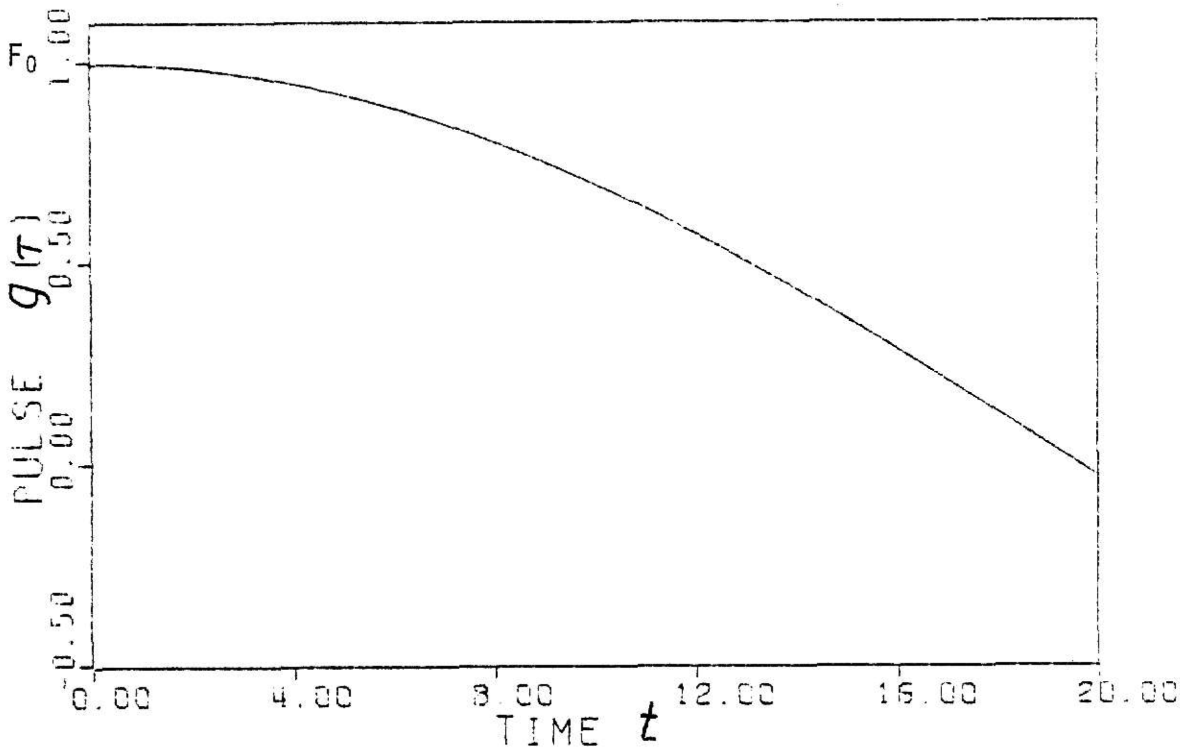


Figure 4.9. Cosine Pulse and Time - Displacement results for

$$\frac{d}{dt}\left[3.0(2.0 - 0.02\tau)^2 \frac{dx}{dt}\right] + 3.0(2.0 - 0.02\tau)\left[x - \frac{x^3}{6}\right] = 1.0\cos(0.08\tau).$$



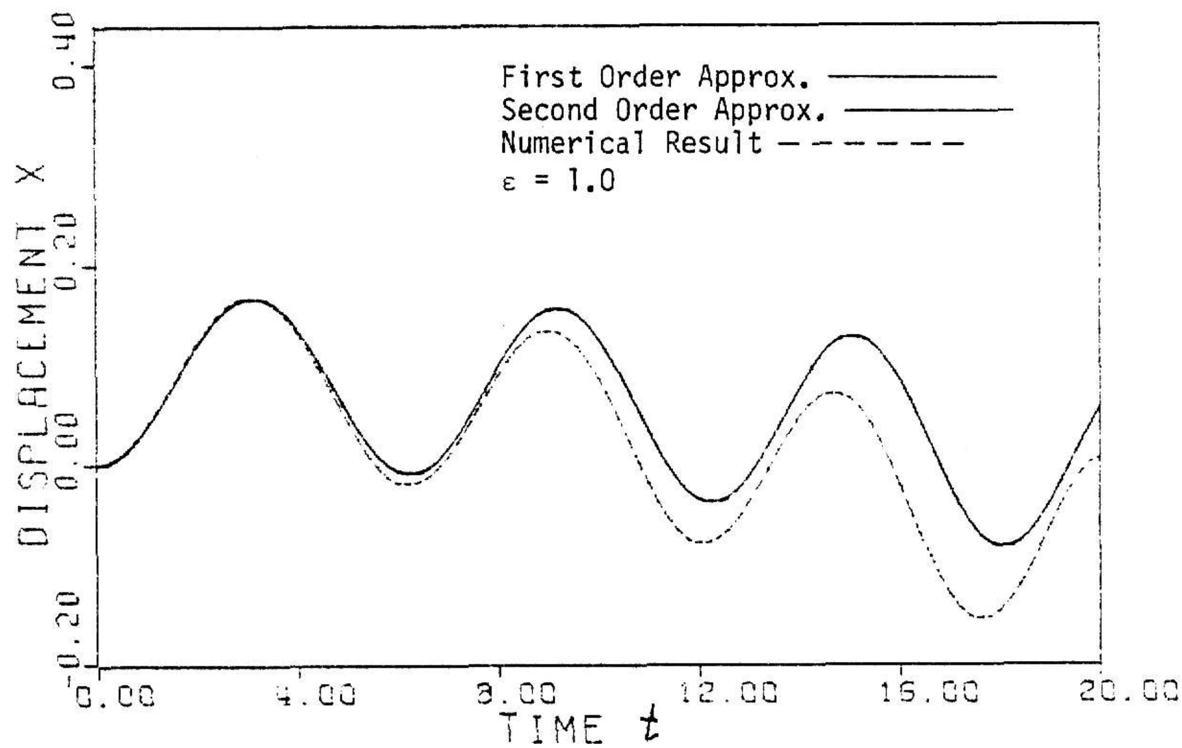


Figure 4.10. Cosine Pulse and Time - Displacement results for

$$\frac{d}{dt}\left[3.0(2.0 + 0.02\tau)^2 \frac{dx}{dt}\right] + 3.0(2.0 + 0.02\tau)\left[x - \frac{x^3}{6}\right] = 1.0 \cos (0.08\tau).$$

tation, was applied to the problem of a variable length pendulum. First and second order approximations were obtained and results were presented for the pendulum subjected to several types of pulse loadings, viz., step function, exponentially decaying pulse, blast loading, asymptotic step, and cosine pulse. Specific results were obtained for the case when the pendulums' length varied according to a linear function of time and were compared with numerical results.

As was seen from the analysis , the first order approximation of amplitude variation reduced to a power law type function. This function was found to be independent of the nonlinear term and depended only on the character of the slowly varying length of the pendulum. The analysis also reveals that the accuracy of the method decreases for the case when the pulse changes rapidly. This could be seen in the cases when the pendulum was excited by an asymptotic step and blast loading. It was also shown that the resulting improvement of the second order approximation was insignificant.

V. DISCUSSION AND CONCLUSION

The purpose of this study was to investigate the response of non-linear vibrational systems with slowly varying parameters, subjected to arbitrary pulse excitations. A transformation of the dependent variable led to an equivalent set of governing differential equations, at which point the method of Krylov, Bogoliubov and Mitropolsky was applied to obtain approximate solutions of first and second order.

In chapter II it was shown that for particular types of nonlinearities, there was a significant simplification, in the first order approximation, of the expressions for amplitude and phase of vibration. In particular, in systems whose nonlinearity arises from the velocity term only, the frequency of vibration does not depend on the amplitude of vibration, but only on the character of the slowly varying mass and stiffness of the system. Likewise it was found that, in systems whose nonlinearities arise from the displacement variable only, the amplitude variations were power law type functions and independent of the nonlinear term.

In chapter III, the approximate method outlined in chapter II was successfully applied to a variety of nonlinear systems with slowly varying parameters subjected to step function excitation. Specific examples showed that expressions for amplitude and phase, in systems where some or all of the parameters varied according to linear functions of time, could be obtained, for most cases, in closed form. Results obtained were compared with numerical solutions and a close agreement was found.

The proposed method was also applied to the problem of a variable length pendulum subjected to different types of pulse loadings. Both first and second order approximations were derived for several types of pulse excitations. These included: step function, exponentially decaying pulse, blast loading, asymptotic step and a cosine pulse. Specific results were presented for the case when the length of the pendulum varied according to a linear function of time. As seen from these results, the contribution of the higher order terms of the second order approximation was insignificant. In most cases the approximate analytical results agreed well with the numerical solutions. The results showed a decrease in the accuracy of the method for systems subjected to a rapidly varying pulse, such as the asymptotic step and the blast loading.

In conclusion, the present investigation provides a general approach for determining the response of nonlinear, nonstationary systems, subjected to arbitrary pulse excitations. The first order approximate solutions were found to be sufficiently accurate for most systems whose parameters vary slowly with time. The method is conceptually simple and in most cases, the expressions for amplitude can be obtained in closed form. The phase variation requires the use of a simple numerical integration scheme such as Simpson's rule or other similar techniques.

It is anticipated that the method presented here would be of some value to the field of nonlinear analysis. A major area for additional investigation is the extension of the method to the problem of nonlinear systems with multiple degrees of freedom, and slowly varying parameters. In addition, any analysis removing the restriction of slowly varying parameters would be an enormous contribution.

REFERENCES

1. Scalia, A., and Torrisi, M., "On the motion of a variable mass gyroscope under the action of a system of forces of zero power," Atti dell Instituo Veneto di Scienze, Vol. 135, pp. 191-202, 1976/77 (in Italian).

2. Babitskii, V.I., and Borovkov, B.A., "Vibration-impact modes in systems with limited excitation," Mechanics of Solids, Vol. 12(1), pp. 46-52, 1977.

3. Ilin, R.F., and Zhigula, T.I., "Dynamics of viscoelastic filaments (cables) in multicable lift of a load," Soviet Applied Mechanics, Vol. 12(8), pp. 801-805, May 1977.

4. Tordion, G.V., and Gauvin, R., "Dynamic Stability of a two stage gear train under the influence of variable meshing stiffness," Journal of Engineering for Industry, TRANS ASME Series B, Vol. 99(3), pp. 785-791, August 1977.

5. Gopak, K.N., "Dynamic Stability of a free rod of variable mass," Soviet Applied Mechanics, Vol. 12(4), pp. 425-428, March 1977.

6. Tso, W.K., "Parametric torsional stability of a bar under axial excitation," Journal of Applied Mechanics, Vol. 35, pp. 13-19, 1968.

7. Ghobarah, A.A., and Tso, W.K., "Parametric stability of thin-walled beams of open section," Journal of Applied Mechanics, Vol. 39, pp. 201-206, 1972.

8. Iwatsubo, T., Sugiyama, Y., and Ogino, S., "Simple and combination resonances of columns under periodic axial loads," Journal of Sound and Vibration, Vol. 33, pp. 211-221, 1974.

9. Elmaraghy, R., and Tabarrok, B., "On the dynamic stability of an axially oscillating beam," Journal of Franklin Institute, Vol. 300, pp. 25-39, 1975.

10. Tesak, E.G., Mook, D.T., and Nayfeh, A.H., "Nonlinear Analysis of the lateral response of columns to periodic loads," ASME Paper No. 77-DET-69, 1977.

11. Crespo da Silva, M.R.M., "Flexural-flexural oscillations of Beck's column subjected to a planer harmonic excitation," Journal of Sound and Vibration, Vol. 60, pp. 133-144, 1978.

12. Mote, C.D., Jr., "Dynamic stability of an axially moving band," Journal of Franklin Institute, Vol. 285, pp. 329-346, 1968.

13. Rhodes, J.E., Jr., "Parametric self-excitation of a belt into transverse vibration," Journal of Applied Mechanics, Vol. 37, pp. 1055-1060, 1971.

14. Nishikawa, Y., and Willems, P.Y., "A method for stability investigation of a periodic dynamic system with many degrees of freedom," Journal of Franklin Institute, Vol. 287, pp. 143-157, 1969.

15. Friedmann, P., and Silverthorn, L.J., "Aeroelastic stability of periodic systems with application to rotor blade flutter," AIAA Journal, Vol. 12, pp. 1559-1565, 1974.

16. Friedmann, P., and Silverthorn, L.J., "Aeroelastic stability of coupled flag-lag motion of hingeless helicopter blades at arbitrary advance ratios," Journal of Sound and Vibration, Vol. 39, pp. 409-428, 1975.

17. Paidoussis, M.P., and Sundararajan, C., "Parametric and combination resonances of a pipe conveying pulsating fluid," Journal of Applied Mechanics, Vol, 42, pp. 780-784, 1975.

18. Nayfeh, A.H., "A multiple time scaling analysis of re-entry vehicle roll dynamics," AIAA Journal, Vol. 7, pp. 2155-2157, 1969.

19. Bogoliubov, N.N., and Mitropolsky, Yu. A., "Asymptotic Methods in the Theory of Nonlinear Oscillations," Gordon and Breach, New York, 1961.

20. Mitropolsky, Yu. A., "Problems of the Asymptotic Theory of Nonstationary Vibrations," Daniel Darey and Co., Inc., New York, 1965.

21. Srirangarajan, H.R., and Srinivasan, P., "The pulse response of nonlinear systems," Journal of Sound and Vibration, Vol. 44(3), pp. 369-377, 1976.

22. Srirangarajan, H.R., "Pulse response of nonlinear systems," Journal of Sound and Vibration, Vol. 63(4), pp. 593-596, 1979.

23. Bapat, V.A., and Srinivasan, P., "An equivalent linearization for the solution of certain class of nonlinear vibration problems," Journal of the India Institute of Science, Vol. 50, pp. 135-149, 1968.

24. Bauer, H.F., "The response of a nonlinear system to pulse excitation." International Journal of Nonlinear Mechanics, Vol. 1, pp. 267-282, 1966.

25. Bauer, H.F., "The response of a nonlinear n-degrees-of-freedom system to pulse excitation," International Journal of Nonlinear Mechanics, Vol. 3, pp. 157-171, 1968.

26. Bauer, H.F., "Nonlinear response of elastic plates to pulse excitation," Journal of Applied Mechanics, Vol. 35, pp. 47-52, 1968.

27. Bauer, H.F., "Vibrational behavior of nonlinear systems to pulse excitations of finite duration," International Journal of Nonlinear Mechanics, Vol. 6, pp. 529-543, 1971.

28. Ariaratnam, S.T., "Response of a nonlinear system to pulse excitation," Journal of Mechanical Engineering Science, Vol. 6, pp. 26-31, 1964.

29. Nayfeh, A.H., and Mook, D.T., "Nonlinear Oscillations," John Wiley and Sons, New York, 1965.

30. Rubenfeld, L.A., "The passage of weakly coupled nonlinear oscillators through internal resonance," Studies in Applied Mathematics, Vol. 57, pp. 77-92, 1977.

31. Evan-Iwanowski, R.M., "Resonance Oscillations in Mechanical Systems," Elsevier Scientific Publishing Co., Amsterdam, 1976.

32. Sinha, S.C., and Chou, C.C., "On nonlinear oscillations with slowly varying system parameters," Journal of Sound and Vibration, Vol. 61, pp. 293-301, 1978.

33. Bapat, V.A., and Srinivasan, P., "Response of undamped nonlinear spring mass system subjected to constant force excitation," Journal of Sound and Vibration, Vol. 9, pp. 53-58, 1969.

34. Bapat, V.A., and Srinivasan, P., "Response of undamped nonlinear spring mass system subjected to constant force excitation," Journal of Sound and Vibration, Vol. 9, pp. 438-446, 1969.

35. Bapat, V.A., and Srinivasan, P., "Approximate methods for step function response of undamped nonlinear spring mass systems with arbitrary hardening type spring characteristics," Journal of Sound and Vibration, Vol. 10, pp. 430-443, 1969.

36. Panovko, Ya. G., "A review of application of the method of direct linearization," Proceedings of the 11th International Congress of Applied Mechanics, pp. 167-171, 1964.

37. Atkinson, C.P., "On a superposition method for determining frequencies of nonlinear systems," Proceedings of the Fourth U.S. National Congress of Applied Mechanics, Vol. 1, pp. 57-62, 1962.

38. Ergin, E.I., "Transient response of a nonlinear system by a bilinear approximation method," Journal of Applied Mechanics, Vol. 23, pp. 635-641, 1965.

39. Sinha, S.C., and Srinivasan, P., "An approximate analysis of nonlinear nonconservative systems subjected to step function excitation," Journal of Sound and Vibration, Vol. 22(2), pp. 211-219, 1972.

40. Liu, Y.K., "The dynamic response of nonlinear systems studied by the method of ultraspherical polynomials approximation," Ph.D. Dissertation, Wayne State University, 1963.

41. Anderson, G.L., "Application of ultraspherical polynomials to nonlinear, nonconservative systems subjected to step function excitation," Journal of Sound and Vibration, Vol. 32, pp. 101-108, 1974.

42. Krylov, N., and Bogoliubov, N.N., "Introduction to Nonlinear Mechanics," Princeton University Press, Princeton, New Jersey, 1943.

43. Davis, H.T., "Introduction to Nonlinear Differential and Integral Equations," Dover Publications, Inc., New York, 1962.

44. Evaldson, R.L., Ayre, R.S., Jacobson, L.S., "Response of an elastically nonlinear system to transient disturbances," Journal of Franklin Institute, Vol. 248, pp. 473-494, 1949.

45. Jacobson, L.S., "on a graphical method of solving second order ordinary differential equations by phase-plane displacements," Journal of Applied Mechanics, Vol. 19, pp. 543-553, 1952.

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PULSE RESPONSE OF NONLINEAR
NONSTATIONARY VIBRATIONAL SYSTEMS

by

DANIEL JOSEPH OLBERDING

B. S., Kansas State University, 1978

AN ABSTRACT OF A MASTER'S THESIS

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requirements for the degree

MASTER OF SCIENCE

Department of Mechanical Engineering
Manhattan, Kansas

1980

ABSTRACT

The present study involves the analysis of nonlinear, nonstationary vibrational systems with single degree of freedom, subjected to an arbitrary pulse excitation. The nonstationary system parameters are considered to be slowly varying functions of time and may include masses, restoring forces, material properties or damping. A transformation of the dependent variable yields an equivalent set of differential equations; at which point a general procedure is presented for obtaining first and second order approximate solutions using the technique of Krylov, Bogoliubov, and Mitropolsky.

Included in this work is an investigation of several types of nonlinear systems with slowly varying parameters subjected to step function excitation. Explicit results for systems whose parameters vary according to linear functions of time are derived and compared with numerical solutions. Also included is an analysis of a pendulum of variable length subjected to a variety of pulse excitations, viz., step function, exponentially decaying pulse, blast loading, asymptotic step, and cosine pulse. First and second order approximations are presented and specific results are compared with numerical solutions.

V. DISCUSSION AND CONCLUSION

The purpose of this study was to investigate the response of nonlinear vibrational systems with slowly varying parameters, subjected to arbitrary pulse excitations. A transformation of the dependent variable led to an equivalent set of governing differential equations, at which point the method of Krylov, Bogoliubov and Mitropolsky was applied to obtain approximate solutions of first and second order.

In chapter II it was shown that for particular types of nonlinearities, there was a significant simplification, in the first order approximation, of the expressions for amplitude and phase of vibration. In particular, in systems whose nonlinearity arises from the velocity term only, the frequency of vibration does not depend on the amplitude of vibration, but only on the character of the slowly varying mass and stiffness of the system. Likewise it was found that, in systems whose nonlinearities arise from the displacement variable only, the amplitude variations were power law type functions and independent of the nonlinear term.

In chapter III, the approximate method outlined in chapter II was successfully applied to a variety of nonlinear systems with slowly varying parameters subjected to step function excitation. Specific examples showed that expressions for amplitude and phase, in systems where some or all of the parameters varied according to linear functions of time, could be obtained, for most cases, in closed form. Results obtained were compared with numerical solutions and a close agreement was found.

The proposed method was also applied to the problem of a variable