/A QUANTUM LOGIC APPROACH TO THE ADDITION OF SPIN ANGULAR MOMENTA/

by

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I. INTRODUCTION

Beginning in the early 1900's, new discoveries in physics forced a re-evaluation of long-accepted concepts. The deterministic classical theory was modified by the theories of relativity and quantum mechanics. One intriguing development was that even the rules of logic, previously considered a priori, had to be changed to accommodate quantum theory. Ordinary propositions in the macroworld satisfy a distributive law: for example, if we shake two pennies in a box, and discover that Penny 1 shows heads and Penny 2 shows either heads or tails, then we can be sure that either Penny 1 shows heads and Penny 2 shows heads, or Penny 1 shows heads and Penny 2 shows tails. The equivalence of the two ways of expressing the same compound proposition may not hold if we replace the classical statements with statements about quantum properties of an object. For example, the statement "the spin of the particle in the $\vec{x}$ direction is up and in the $\vec{y}$ direction is either up or down" may be true, but the statement "the spin of the particle in the $\vec{x}$ direction is up and in the $\vec{y}$ direction is up, or the spin of the particle in the $\vec{x}$ direction is up and in the $\vec{y}$ direction is down" is false or meaningless, depending on one's philosophical orientation, but certainly not true.

The first paper on quantum logic was published by Birkhoff and von Neumann in 1936.[1] This pioneering work gave the first description of quantum propositions in terms of non-Boolean lattice theory. Since then, especially since the 1960's, extensive physical, mathematical, philosophical, and popular literature has appeared. An exhaustive bibliography up to 1981 has been compiled by Holdsworth and Hooker.[2] The most frequently cited works on the subject are books by Jauch and by Piron.[3,4]

The purpose of this paper is to use the quantum logic approach to try to solve the well-known (and already solved, through the formalism of the

Hilbert space) problem of the addition of spin angular momenta.

Chapter II is a short discussion of the addition of angular momenta and the Stern-Gerlach experiment for spin-½ and spin-1 particles. An unexpected complication arises in the Stern-Gerlach experiment for spin-1--not every pure state of a spin-1 particle has its spin vector pointing in a definite direction, so a standard Stern-Gerlach apparatus will not filter every pure state. Two theorems about the spin-1 states that <u>can</u> be filtered by the standard Stern-Gerlach apparatus are proven.

In Chapter III we discuss the notions of quantum logic and how to apply them to propositions about systems with spin. The quantum logic for spin-½ is presented. One conjecture about the quantum logic for spin-1, proposed by Hultgren and Shimony, is discussed. Their model for the quantum logic of spin-1 propositions uses only the pure spin-1 states that <u>can</u> be filtered by a standard (i.e., magnetic field only) Stern-Gerlach apparatus. Their challenge to find an experimental procedure to filter any pure spin-1 state was answered successfully by Swift and Wright.

In Chapter IV we discuss the notion of a tensor product of logics in order to combine two logics (for example, those of 2 spin-½ particles) into a new logic in an appropriate way. The goal is to define the tensor product for quantum logics <u>without appealing to the Hilbert space structure of the quantum propositions</u>. Several authors have already considered the problem of defining a tensor product for quantum logic. Their efforts are discussed in light of the goal expressed above.

Chapter V concludes the paper with a discussion of the difficulties and possibilities of further investigation of the logical structure of quantum propositions.

II. ANGULAR MOMENTUM AND THE STERN-GERLACH EXPERIMENT

A. Angular momentum

Angular momentum is one of the most important concepts in classical and quantum physics.[5] The angular momentum vector of a particle with respect to an origin is defined as $\vec{\mathcal{L}} = \vec{r} \times \vec{p}$ where $\vec{r}$ is the position vector and $\vec{p}$ is the momentum vector of the particle.

The time derivative of $\vec{\mathcal{L}}$

$$\frac{d\vec{\mathcal{L}}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

is 0 for a free particle or for a particle in a central potential, so in these cases angular momentum is a constant of the motion.

In terms of the components of $\vec{r}$ and $\vec{p}$, $\vec{\mathcal{L}} = (yp_z - zp_y, zp_x - xp_z, xp_y - yp_x)$. In quantum mechanics, an operator corresponds to each of these quantities. A quantum mechanical operator L is defined as R x P. L is a set of three operators, designated L_x, L_y, L_z, such that, e.g., $L_x = YP_z - ZP_x$.

The commutation relations for these operators are

$$[L_x, L_y] = i\hbar L_z$$

$$[L_y, L_z] = i\hbar L_x$$

$$[L_z, L_x] = i\hbar L_y$$

or more compactly $[L_i, L_j] = i\hbar\varepsilon_{ijk}L_k$ where ε_{ijk} is the Levi-Cevita symbol (completely antisymmetric tensor).

H, L^2, and L_z form a complete set of commuting observables. The eigenvalues of L^2 and L_z associated with the same eigenfunction (denoted by Y_ℓ^m, a spherical harmonic function) are $\ell(\ell + 1)\hbar^2$ and $m\hbar$ where ℓ is a non-negative integer, m is an integer, and m takes on all values from $-\ell$ to ℓ. We see that angular momentum is quantized.

We have noted that the quantum number ℓ is an integer. In the quantum mechanical theory of angular momentum, based on the commutation relations rather than on the quantum analogue to $\vec{r} \times \vec{p}$, it is shown that the quantum number j for an angular momentum $\vec{J}$ can actually be either integral or half-integral. The restriction to position and momentum variables in orbital angular momentum leads to the rejection of half-integral angular momenta. The results of the Stern-Gerlach experiment provide experimental evidence for the existence of a half-integral angular momentum. This half-integral angular momentum is an intrinsic angular momentum or spin.

B. The Stern–Gerlach Experiment

In the Stern–Gerlach experiment, a beam of particles is collimated by slits and directed through a non-uniform magnetic field in a vacuum. This magnetic field may be produced by the north and south poles of a magnet as shown in Fig. 1.

Suppose the beam is composed of silver atoms. The silver atom is electrically neutral. However, it has a magnetic moment $\vec{\mu}$ and therefore has potential energy $W = -\vec{\mu}.\vec{B}$ when placed in the magnetic field $\vec{B}$. $\vec{\mu}$ and the angular momentum $\vec{\mathcal{L}}$ are proportional: $\vec{\mu} = \gamma\vec{\mathcal{L}}$. To see this classically, suppose an electron with charge $-e$ and mass m is orbiting a nucleus at a distance r from the center of the nucleus with momentum $\vec{p}$. Then the orbiting electron can be considered a current loop. The magnetic moment $\vec{\mu}$ is defined as the product of the current i and the area A of the loop:

$$|\mu| = iA = dq/dt\ (\pi r^2) = -e/(2\pi r/v)\ \pi r^2$$

where $2\pi r/v = T$ = time needed to make one revolution; therefore

$$|\mu| = -e/(2\pi rm/p)\ \pi r^2 = -erp/2m = -e\ell/2m$$

where the orbital angular momentum $\ell = rp$.

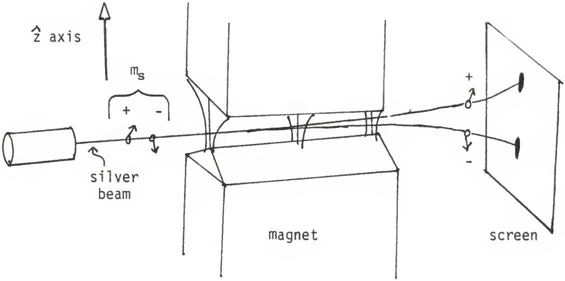
The force exerted on the atom in the magnetic field is

$$\vec{F} = \vec{\nabla}(\vec{\mu}.\vec{B}) = \vec{\nabla}(\mu_z B_z) = \mu_z \vec{\nabla} B_z.$$

(We have taken the direction of $\vec{B}$ to be the z-axis.) Therefore, the force on the atom is proportional to μ_z. Since this force deflects the atom from its initial trajectory, the final position of the particle on the screen is proportional to μ_z.

If the angular momentum of a particle can take on any value, and the beam of particles has randomly distributed angular momenta, we would expect to see particles in a continuously distributed pattern on the screen. This is the classical prediction. The quantum mechanical picture is that, since

Figure 1 The Stern-Gerlach Apparatus



$\hat{z}$ axis
m_s
+
-
silver
beam
magnet
screen
+
-

ℓ (and therefore m_ℓ) is quantized and cannot take on just any value, we should see $(2\ell + 1)$ spots on the screen.

For the silver atom, even this prediction proves false. It implies that that silver atom in any state will produce an odd number of spots on the screen. In fact, silver atoms in the ground state produce not $2(0) + 1 = 1$ spot on the screen, but $2 = 2(\frac{1}{2}) + 1$. Silver atoms possess an intrinsic angular momentum or spin, $j = \frac{1}{2}$, with two possible m_s values, $m_s = \frac{1}{2}$ or $-\frac{1}{2}$. The spin magnetic moment m_s interacts with the gradient of the magnetic field.

We could arrange the Stern-Gerlach apparatus then so that the emerging beams pass through 2 holes on the forward side of the apparatus. Then, if we wish to work with a polarized beam, we can simply close off one hole and retain only one beam.

Summarizing: In addition to its spatial degrees of freedom, every particle also has an intrinsic spin. The spin quantum number of a particle, which may be any non-negative integer or half-integer, is characteristic of the particle in the same way that its mass is characteristic. Given its spin s, the quantum number m_s with $-s \leqslant m_s \leqslant s$ (with m_s in integral increments) gives the possible spin states of the particle. A spin-$\frac{1}{2}$ paricle has $m_s = \frac{1}{2}$ or $-\frac{1}{2}$, with the states usually denoted $|+\rangle$ and $|-\rangle$. A spin-1 particle has $m_s = 1$, 0, or -1, denoted $|1\rangle$, $|0\rangle$, and $|-1\rangle$. If the state of a particle can be written as a linear combination of these basis states, e.g. as $\alpha|1\rangle + \beta|0\rangle + \gamma|-1\rangle$, then that state is called a pure state. α, β, and γ are the amplitudes for the particles to be in the $|1\rangle$, $|0\rangle$, or $|-1\rangle$ states respectively, with the overall phase of the state physically unimportant. If the state of the particle cannot be written this way, it is said to be a mixed state and must be denoted by a density matrix.

(The use of the word "state" demands close attention because "state" as used in Chapter III will not mean the same thing as a state in Hilbert space.)

The Stern-Gerlach apparatus just described constitutes a complete set of experiments for a spin-½ particle. That is, a Stern-Gerlach apparatus with 2 holes for the emerging beams, one closed and one open, may be used to prepare a beam in <u>any</u> pure normalized spin-½ state $(\alpha, \beta) = \alpha|+\rangle_z + \beta|-\rangle_z$.

The procedure is as follows. Suppose a beam of particles defines an axis $\vec{y}$ and the original orientation of the Stern-Gerlach apparatus defines the $\vec{x}$-$\vec{z}$ plane. Since $|\alpha|^2 + |\beta|^2 = 1$, we know an angle $\theta/2$ exists such that $\cos\theta/2 = |\alpha|$, $\sin\theta/2 = |\beta|$. θ is uniquely defined if $0 \leqslant \theta \leqslant \pi$. As noted above, only the difference of the phases of α and β is important in physical predictions, not the overall phase. So define

$$\phi = \mathrm{Arg}\,\beta - \mathrm{Arg}\,\alpha$$

$$\chi = \mathrm{Arg}\,\beta + \mathrm{Arg}\,\alpha$$

Then $\alpha|+\rangle_z + \beta|-\rangle_z = e^{i\chi/2}\{\cos\theta/2\; e^{-i\phi/2}|+\rangle + \sin\theta/2\; e^{i\phi/2}|-\rangle\}$.

The expression in brackets represents a particle certain to have its spin up along the $\hat{u}(\theta,\phi)$ axis. That is, $(\cos\theta/2\; e^{-i\phi/2}, \sin\theta/2\; e^{i\phi/2})$ is an eigenstate of $\vec{S}.\hat{u} = \hbar/2 \begin{pmatrix} \cos\theta & \sin\theta\; e^{-i\phi} \\ \sin\theta\; e^{i\phi} & -\cos\theta \end{pmatrix}$ for $\hat{u}$ defined by the polar angles θ, ϕ. Therefore, any beam of spin-½ particles prepared in a pure spin state can be completely passed by a Stern-Gerlach apparatus aligned along the correct axis $\hat{u} = (\theta,\phi)$ corresponding to that state. The pure states can therefore be modelled by the points on the surface of a sphere of radius 1 in 3-dimensional space.

The corresponding conjecture for spin-1 particles, that the Stern-Gerlach apparatus with 3 holes, 2 closed and 1 open, is a complete set of experiments, is false.[6] There are many pure states (α, β, γ) of a spin-1 particle that will not be passed by such a Stern-Gerlach apparatus oriented in <u>any</u> direction. That is, (α, β, γ) is not an eigenstate of $\vec{S}.\hat{n}$ for any $\hat{n}$. Such a state (α, β, γ) is said not to have a definite spin vector. In order for a vector to be an eigenstate of $\vec{S}.\hat{n}$, it must be obtainable by rotating either $|1\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $|0\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}$, or $|-1\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}$ from the $\hat{z}$-axis to the $\hat{n}$-axis. I.e., $(\alpha, \beta, \gamma) = R\psi(\hat{z}, \lambda)$ (where $\lambda = 0, 1, \text{or } -1$)

where $R = \begin{pmatrix} \cos^2(\frac{1}{2}\theta)e^{-i\phi} & -1/\sqrt{2}\sin\theta\, e^{-i\phi} & \sin^2(\frac{1}{2}\theta)e^{-i\phi} \\ 1/\sqrt{2}\sin\theta & \cos\theta & -1/\sqrt{2}\sin\theta \\ \sin^2(\frac{1}{2}\theta)e^{i\phi} & 1/\sqrt{2}\sin\theta\, e^{i\phi} & \cos^2(\frac{1}{2}\theta)e^{i\phi} \end{pmatrix}$

Therefore (α, β, γ) is an eigenstate for $\vec{S}.\hat{n}$ only if it can be expressed as one of the columns in the matrix R. An example that does not fulfill this requirement is $\begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}$.[7]

The set of spin-1 states that <u>are</u> eigenstates of $\vec{S}.\hat{n}$ for some $\hat{n}$, and therefore <u>can</u> be passed by a Stern-Gerlach apparatus with one hole open and two holes closed, is a set of measure zero and in fact is not even dense in the space of all spin-1 states. This will be proven in IIE.

Although not every spin-1 state has a definite spin vector, we shall prove in IID that any pure spin-1 state (α, β, γ) <u>can</u> be rotated to a state $(\alpha', 0, \gamma')$. (Therefore it will always be possible to rotate the Stern-Gerlach apparatus with the middle hole open, to an orientation such that <u>none</u> of the beam emerges.)

C. Addition of spin angular momenta

A system may be made up of several particles each of which has an angular momentum $\vec{J}_i$. If the particles interact, each individual angular momentum may not be a constant of the motion, but if the system constitutes a free particle or a particle in an external central potential, the total angular momentum $\Sigma \vec{J}_i$ is a constant of the motion, and H, $\vec{J}^{\,2}$, and J_z are a complete set of commuting observables. Therefore, it is an important problem to determine the angular momentum of a system of particles, given their individual angular momenta.

As an example of addition of spins in the Hilbert space formulation, consider two spin-½ particles. Their basis spin states may be written $|+\rangle_1$, $|-\rangle_1$, $|+\rangle_2$, $|-\rangle_2$. The basis states of the two-particle system may be written as the tensor product of a particle 1 state with a particle 2 state: $|+_1+_2\rangle$, $|+_1-_2\rangle$, $|-_1+_2\rangle$, $|-_1-_2\rangle$. The particle "created" by the combination of two particles with spins S_1 and S_2 may have spin $S = |S_1 + S_2|$, ..., $|S_1 - S_2|$ with $-S \leqslant M \leqslant S$. In the case we are considering, S = 0 or 1. If S = 0 then M = 0, and if S = 1 then M = 1, 0, or -1. The relationship between the description using $|S, M\rangle$ and that using $|\pm_1, \pm_2\rangle$ is

$$|0\ 0\rangle = 1/\sqrt{2}\{|+_1-_2\rangle - |-_1+_2\rangle\} \qquad \text{(antisymmetric)}$$

$$\left.\begin{aligned} |1\ 1\rangle &= |+_1+_2\rangle \\ |1\ 0\rangle &= 1/\sqrt{2}\{|+_1-_2\rangle + |-_1+_2\rangle\} \\ |1\ {-1}\rangle &= |-_1-_2\rangle \end{aligned}\right\} \qquad \text{(symmetric)}$$

Suppose two spin-½ particles are known to have spin up along the $\hat{r}_1$ and $\hat{r}_2$ axes respectively. Let the bisector of these axes define the $\hat{z}$-axis and let $\hat{r}_1$ and $\hat{r}_2$ define the $\hat{x}$-$\hat{z}$ plane as shown in Fig. 2. Denote the angle between $\hat{r}_1$ and $\hat{z}$ by θ. In this coordinate system particle 1 is in the state $\psi_1 = \cos\theta/2|+\rangle_1 + \sin\theta/2|-\rangle_1$. Particle 2 is in the state $\psi_2 = -\cos\theta/2|+\rangle_2 + \sin\theta/2|-\rangle_2$. The tensor product of these two states is

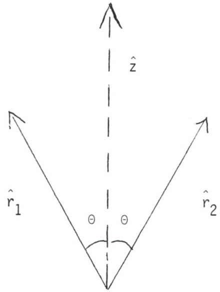
$$\begin{aligned}\psi_1 \times \psi_2 &= \sum_{i,j} a_i b_j |\varepsilon_i\rangle|\varepsilon_j\rangle \\ &= \cos^2\theta/2|{+}{+}\rangle + (-\sin\theta/2\cos\theta/2)|{-}{+}\rangle + \\ &\qquad (\sin\theta/2\cos\theta/2)|{+}{-}\rangle + \sin^2\theta/2|{-}{-}\rangle \\ &= \sqrt{2}\sin\theta/2\cos\theta/2|00\rangle - \cos^2\theta/2|11\rangle + \sin^2\theta/2|1\ {-1}\rangle\end{aligned}$$

The system has amplitude $\sqrt{2}\sin\theta/2\cos\theta/2$ to be spin 0. Notice that the symmetric part, the part producing a particle of spin 1, has 0 amplitude to have $M = 0$ along $\hat{z}$. This is an elementary proof of the earlier assertion that there exists an orientation of a Stern-Gerlach apparatus such that no beam emerges if only the middle hole is open.

This shows how we can combine two spin-½ particles symmetrically to form a spin-1 particle. On the other hand, can we take an arbitrary spin-1 particle and describe it as the combination of two spin-½ particles? (The "particles" may be purely figments of our imagination--they are simply supplying indices.) An angular momentum j can always be written as the totally symmetric sum of 2j kinematically independent spin-½ angular momenta. Each of these individual spin-½ angular momenta may be characterized as a point on the sphere, so a spin-j angular momentum may be characterized by 2j points on a sphere.[8]

In the next section we show how to rotate a general (α, β, γ) state of a spin-1 particle to a $(\alpha', 0, \gamma')$ state. The $(\alpha', 0, \gamma')$ state can be considered the symmetric part of the spin-½ combination.

Figure 2 Spin vectors of 2 spin-$\frac{1}{2}$ particles


$\hat{z}$
$\hat{r}_1$
Θ
Θ
$\hat{r}_2$

D. Rotating a pure state of a spin-1 system to $\begin{pmatrix} \alpha''' \\ 0 \\ \gamma''' \end{pmatrix}$

Every pure state $\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$ of a spin-1 system can be expressed in the form $\begin{pmatrix} \alpha''' \\ 0 \\ \gamma''' \end{pmatrix}$. Refer to Fig. 3.

Step 1. Rephase so that β is real:

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} |\alpha|e^{ia} \\ |\beta|e^{ib} \\ |\gamma|e^{ig} \end{pmatrix} = e^{ib} \begin{pmatrix} |\alpha|e^{i(a-b)} \\ |\beta| \\ |\gamma|e^{i(g-b)} \end{pmatrix} = e^{ib} \begin{pmatrix} \alpha' \\ \beta' \\ \gamma' \end{pmatrix}$$

Step 2. We want the phase of $\alpha - \gamma$ to equal the phase of β, i.e. to be real (the reason for this becomes evident in Step 3). We need to rotate $\begin{pmatrix} \alpha' \\ \beta' \\ \gamma' \end{pmatrix}$ around the z-axis to obtain $\begin{pmatrix} \alpha' e^{i\phi} \\ \beta' \\ \gamma' e^{-i\phi} \end{pmatrix} = \begin{pmatrix} \alpha'' \\ \beta'' \\ \gamma'' \end{pmatrix}$

Geometrically (see Fig. 3) what we need is an angle ϕ that will make the α'' and γ'' components of the state have the same i-component in the complex plane. Then

$$|\alpha|\sin(a - b + \phi) = |\gamma|\sin(g - b - \phi)$$

$$|\alpha|\{\sin(a - b)\cos\phi + \cos(a - b)\sin\phi\} = |\gamma|\{\sin(g - b)\cos\phi - \cos(g - b)\sin\phi\}$$

$$\{|\alpha|\sin(a - b) - |\gamma|\sin(g - b)\}\cos\phi = \{-|\alpha|\cos(a - b) - |\gamma|\cos(g - b)\}\sin\phi \quad .$$

Finally,

$$-\frac{|\alpha|\sin(a - b) - |\gamma|\sin(g - b)}{|\alpha|\cos(a - b) + |\gamma|\cos(g - b)} = \tan\phi$$

Step 3. Now we want to rotate $\begin{pmatrix} \alpha'' \\ \beta'' \\ \gamma'' \end{pmatrix}$ around the y-axis by the angle θ' to obtain $\begin{pmatrix} \alpha''' \\ 0 \\ \gamma''' \end{pmatrix}$. Is this possible? We must have

$$\begin{pmatrix} \frac{1}{2}(1+\cos\theta') & -1/\sqrt{2}\sin\theta' & \frac{1}{2}(1-\cos\theta') \\ 1/\sqrt{2}\sin\theta' & \cos\theta' & -1/\sqrt{2}\sin\theta' \\ \frac{1}{2}(1-\cos\theta') & 1/\sqrt{2}\sin\theta' & \frac{1}{2}(1+\cos\theta') \end{pmatrix} \begin{pmatrix} \alpha'' \\ \beta'' \\ \gamma'' \end{pmatrix} = \begin{pmatrix} \alpha''' \\ 0 \\ \gamma''' \end{pmatrix}$$

Thus,

$$1/\sqrt{2}\sin\theta'(\alpha'' - \gamma'') + \cos\theta'\beta'' = 0$$

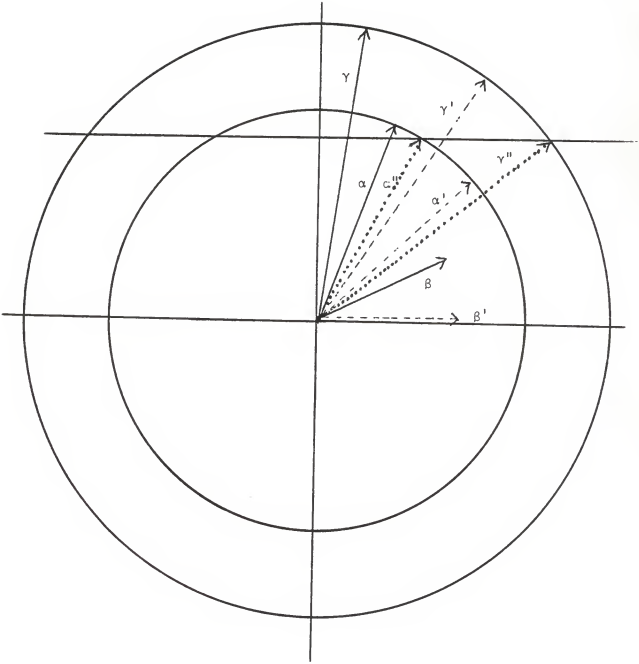
$$1/\sqrt{2}\sin\theta'(\alpha'' - \gamma'') = -\cos\theta'\beta''$$

Hence,

$$\tan\theta' = -\sqrt{2}\ \frac{\beta''}{\alpha'' - \gamma''}$$

which is a real number because β'' and $\alpha'' - \gamma''$ have the same phase.

Figure 3 Rotating $\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}$ to $\begin{pmatrix}\alpha''\\0\\\gamma''\end{pmatrix}$


γ
γ'
γ''
α
α''
α'
β
β'

Example 1. What rotation will give the state $\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$ in the form $\begin{pmatrix} \alpha \\ 0 \\ \gamma \end{pmatrix}$?

$\alpha - \gamma$ already has the same phase as β, and $\alpha - \gamma = 0$. So we need $\cos\theta = 0$, i.e. $\theta = \pi/2$ or $3\pi/2$.

$\theta = \pi/2$:

$$\begin{pmatrix} \frac{1}{2} & -1/\sqrt{2} & \frac{1}{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ \frac{1}{2} & 1/\sqrt{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} - 1/\sqrt{6} \\ 0 \\ 1/\sqrt{3} + 1/\sqrt{6} \end{pmatrix} =$$

$$\begin{pmatrix} \cos\phi/2 \\ 0 \\ \sin\phi/2 \end{pmatrix} \qquad \text{where } \phi/2 = 80.26°.$$

Notice however that this vector is not in the form (standard form from IIC)

$$N \begin{pmatrix} -\cos^2\theta'/2 \\ 0 \\ \sin^2\theta'/2 \end{pmatrix}$$

where N is a normalization factor. So we also need to rotate now around the z-axis by $-\pi/2$:

$$\begin{pmatrix} \alpha''' \\ 0 \\ \gamma''' \end{pmatrix} \to \begin{pmatrix} \alpha''' e^{-i\pi/2} \\ 0 \\ \gamma''' e^{i\pi/2} \end{pmatrix} = i \begin{pmatrix} -\alpha''' \\ 0 \\ \gamma''' \end{pmatrix}$$

and rephase again:

$$e^{-i\pi/2}\, i \begin{pmatrix} -\alpha''' \\ 0 \\ \gamma''' \end{pmatrix} = \begin{pmatrix} -\alpha''' \\ 0 \\ \gamma''' \end{pmatrix}$$

So $\begin{pmatrix} \cos\phi/2 \\ 0 \\ \sin\phi/2 \end{pmatrix} \to i \begin{pmatrix} -\cos\phi/2 \\ 0 \\ \sin\phi/2 \end{pmatrix} \to \begin{pmatrix} -\cos\phi/2 \\ 0 \\ \sin\phi/2 \end{pmatrix} = N \begin{pmatrix} -\cos^2\theta'/2 \\ 0 \\ \sin^2\theta'/2 \end{pmatrix}$

with $N = 2/\sqrt{3}$ and $\theta' = 135°$.

Example 2. $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ could be considered a worst possible case.

Step 1. β is already real.

Step 2. The phase of $\alpha - \gamma = 0$ is already the same as the phase of β.

Let $\theta = \pi/2$ (angle of rotation around the y-axis).

$$\begin{pmatrix} \frac{1}{2} & -1/\sqrt{2} & \frac{1}{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ \frac{1}{2} & 1/\sqrt{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$$

$$= \sqrt{2} \begin{pmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

$$= \sqrt{2} \begin{pmatrix} -\cos^2 45° \\ 0 \\ \sin^2 45° \end{pmatrix}$$

E. Spin-1 States With Definite Spin Vector

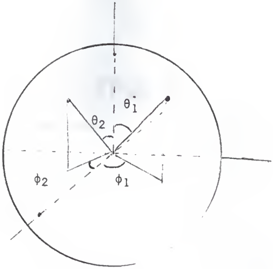
Theorem 1. The set of spin-1 states with definite spin vector is a set of measure 0 in the Hilbert space of spin-1 states.

Proof. A spin-½ state may be represented by a point (θ_1, ϕ_1) on the surface of a sphere of radius 1 in 3-space (i.e. a point of S^2), as shown in section IIB. As we discussed in IIC, a spin-1 state can be represented as a set of two such points, (θ_1, ϕ_1) and (θ_2, ϕ_2), one from each of two spin-½ state spaces (see Fig. 4a). The set of spin-1 states is therefore modelled by two points on a 2-dimensional surface, $S^2 \times S^2$. This is a 4-dimensional space. Although the spin-1 state $\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}$ appears to be 6-dimensional, two dimensions are eliminated by the normalization and the overall phase freedom. The general spin-1 state will be denoted by $(\theta_1, \phi_1, \theta_2, \phi_2)$. The spin-1 states with definite spin vector are those for which $\theta_1 = \theta_2$ and $\phi_1 = \phi_2$ (see Fig. 4b) or $\theta_2 = \pi - \theta_1$ and $\phi_2 = \pi + \phi_1$ (Fig. 4c). Two parameters (θ_1, ϕ_1) suffice to define all such spin-1 states. The set of spin-1 states with definite spin vector is therefore a 2-dimensional set. A 2-dimensional subset of a 4-dimensional space has measure 0.

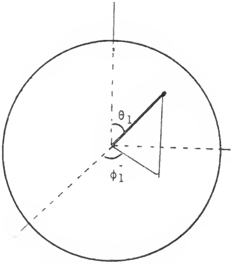
Theorem 2. The set of spin-1 states with definite spin vector is not dense in the space of all spin-1 states. That is, there exists an open set of spin-1 states that contains no spin-1 state with definite spin vector. (By open set we mean open in the topology on the Hilbert space defined using the norm or distance: $d(\psi, \phi) = \|\psi - \phi\|$.)

Proof. Let $\begin{pmatrix}\alpha\\ 0\\ \gamma\end{pmatrix}$ be a state without definite spin vector. This is a rotation of the most general spin 1 state. We have $0 < |\alpha| , |\gamma| < 1$.

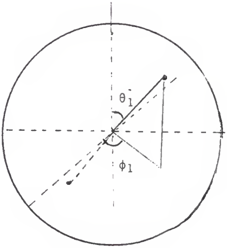
Figure 4 Spin-1 states with definite spin vector



(a)



(b)



(c)

We shall show that there exists an ε-neighborhood of $\begin{pmatrix}\alpha\\0\\\gamma\end{pmatrix}$ containing no spin-1 state with definite spin vector. I.e.,

(a) $\|(\alpha, 0, \gamma) - R(1, 0, 0)\| > 0$

(b) $\|(\alpha, 0, \gamma) - R(0, 1, 0)\| > 0$

(c) $\|(\alpha, 0, \gamma) - R(0, 0, 1)\| > 0$

where R is a rotation matrix and R(1, 0, 0), R(0, 1, 0), and R(0, 0, 1) are the spin-1 states with definite spin vector.

(a) $$\|(\alpha, 0, \gamma) - R(1, 0, 0)\|^2 = |\alpha - \cos^2(\tfrac{1}{2}\theta)e^{-i\phi}|^2 + |1/\sqrt{2}\sin\theta|^2 + |\gamma - \sin^2(\tfrac{1}{2}\theta)e^{i\phi}|^2$$

In order for this expression to equal 0, $\sin^2\theta$ must equal 0. So $\theta = 0$ or π. If $\theta = 0$ then $(\gamma - \sin^2(\tfrac{1}{2}\theta)e^{i\phi})^2 = \gamma^2 > 0$. If $\theta = \pi$ then $(\alpha - \cos^2(\tfrac{1}{2}\theta)e^{-i\phi})^2 = \alpha^2 > 0$. Therefore $\|(\alpha, 0, \gamma) - R(1, 0, 0)\|$ can never equal 0. Case (c) reduces to case (a) because $(0, 0, 1) = R'(1, 0, 0)$ where R' is the rotation by π around the y-axis, so $R(0, 0, 1) = RR'(1, 0, 0)$.

(b) $$\|(\alpha, 0, \gamma) - R(0, 1, 0)\|^2 = |\alpha + 1/\sqrt{2}\sin\theta e^{-i\phi}|^2 + \cos^2\theta + |\gamma - 1/\sqrt{2}\sin\theta e^{i\phi}|^2$$

For this expression to equal 0, we must have both $\alpha = -1/\sqrt{2}\sin\theta e^{-i\phi}$ and $\gamma = 1/\sqrt{2}\sin\theta e^{i\phi}$; then $\alpha = -\gamma^*$. But then $\begin{pmatrix}\alpha\\0\\\gamma\end{pmatrix} = \begin{pmatrix}-\gamma^*\\0\\\gamma\end{pmatrix}$ with $|-\gamma^*|^2 + |\gamma|^2 = 1 = 2|\gamma|^2$, implying that $|\gamma| = 1/\sqrt{2}$. The vector $\begin{pmatrix}-\gamma^*\\0\\\gamma\end{pmatrix}$ is a spin-1 state with definite spin vector--it is the second column of the R matrix given in section IIB with $\theta = \pi/2$ and ϕ = phase of γ. This is contrary to the assumption that $\begin{pmatrix}\alpha\\0\\\gamma\end{pmatrix}$ is a state without definite spin vector. Thus if $\psi = \begin{pmatrix} \alpha \\ 0 \\ \gamma \end{pmatrix}$ is a state <u>without</u> definite spin vector then there are no states with definite spin vector within $\mathrm{Min}(|\alpha|, |\gamma|, 2 - \sqrt{2}|\ |\alpha| - |\gamma|\ |)$ of ψ.

III. LATTICE THEORY

A. Introduction[9]

A partially ordered set (or poset) is a nonempty set of elements with a partial ordering $\leqslant$ defined on the elements. Two elements a and b of a poset have a least upper bound c if $a \leqslant c$, $b \leqslant c$, and ($\exists$ d such that $a \leqslant d$, $b \leqslant d$ $\Rightarrow$ $c \leqslant d$). We denote this l.u.b. by $a \vee b$, which we read as "a join b." The greatest lower bound of a and b is similarly defined, and denoted by $a \wedge b$, which is read as "a meet b."

A lattice is a poset for which a l.u.b. and g.l.b. exist for every pair of elements in the lattice. A lattice is called complete if a l.u.b. and g.l.b. exist for every nonempty subset of the lattice. The following laws hold for elements a, b, c of a lattice L:

(i)	associative	$a \vee (b \vee c) = (a \vee b) \vee c$
		$a \wedge (b \wedge c) = (a \wedge b) \wedge c$
(ii)	commutative	$a \vee b = b \vee a \qquad a \wedge b = b \wedge a$
(iii)	absorption	$a \vee (a \wedge b) = a$
		$a \wedge (a \vee b) = a$

If a lattice contains an element e such that $x \leqslant e$ for all $x \in L$, e is the greatest element and is denoted by 1. A least element 0 is similarly defined. If 0 and 1 exist in L, L is said to be bounded. An element b is called a complement of a if $a \vee b = 1$ and $a \wedge b = 0$. The atoms of a lattice are defined to be elements $a \in L$ such that $0 < a$ (i.e., $0 \leqslant a$ and $0 \neq a$) and if $\exists$ $x \in L$ such that $0 < x \leqslant a$, then $x = a$. A lattice L is atomic if every $x \neq 0$ in L is greater than or equal to an atom.

An orthocomplementation on a lattice is a function $'$: $L \rightarrow L$ such that $\forall$ $a, b \in L$,

(i) $(a')' = a$

(ii) $a \leqslant b \Rightarrow b' \leqslant a'$

(iii) $\underline{a}'$ is a complement of $\underline{a}$.

$\underline{a}'$ is called the orthocomplement of $\underline{a}$.

The DeMorgan laws hold in a bounded orthocomplemented lattice: if $\{\underline{a}_\alpha \mid \alpha \in I\}$ is an arbitrary subset of L, and $\bigvee_\alpha$ and $\bigwedge_\alpha$ are the join and meet, respectively, of the subset,

(i) if at least one of $\bigvee_\alpha \underline{a}_\alpha$ or $\bigwedge_\alpha \underline{a}'_\alpha$ exists in L, then they both exist and $(\bigvee_\alpha \underline{a}_\alpha)' = \bigwedge_\alpha \underline{a}'_\alpha$.

(ii) if at least one of $\bigwedge_\alpha \underline{a}_\alpha$ or $\bigvee_\alpha \underline{a}'_\alpha$ exists in L, then they both exist and $(\bigwedge_\alpha \underline{a}_\alpha)' = \bigvee_\alpha \underline{a}'_\alpha$.

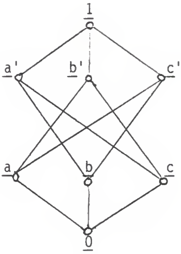
Two elements $\underline{a}$, $\underline{b}$ of an orthocomplemented lattice are said to be orthogonal (to each other) if $\underline{a} \leqslant \underline{b}'$. We denote this by $\underline{a} \perp \underline{b}$. Two elements $\underline{a}$, $\underline{b}$ of an orthocomplemented lattice are said to be compatible or commutative if $(\underline{a} \vee \underline{b}') \wedge \underline{b} = \underline{a} \wedge \underline{b}$. The orthocomplement $\underline{a}'$ of $\underline{a}$ is the complement of $\underline{a}$ that is compatible with $\underline{a}$.

A sublattice of L is a lattice generated by a subset of elements of L, closed under $\vee$ and $\wedge$ but not necessarily under $'$.

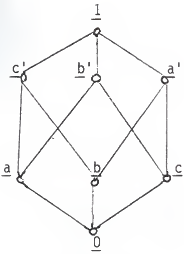
A Hasse diagram of a lattice is a picture in which each element is represented by a small circle, and the circles are arranged such that each element $\underline{a}$ is connected by a line upward to all elements $\underline{b}$ such that $\underline{a} < \underline{b}$ and $\nexists$ $\underline{x} \in L$, $\underline{x} \neq \underline{b}$ such that $\underline{a} < \underline{x} < \underline{b}$. (These elements $\underline{b}$ are said to cover $\underline{a}$.) Figure 5 shows examples of Hasse diagrams. Figs. 5(a) and (b) represent the same lattice. Fig. 5(a) is preferred because the orthocomplement of each atom appears directly above the atom. In Fig. 5(d), $\underline{a}$ and $\underline{d}$ are complements but not orthocomplements.

An orthomodular lattice is an orthocomplemented lattice satisfying the orthomodular law: if $\underline{a} \leqslant \underline{b}$ then $\underline{a} \vee (\underline{a}' \wedge \underline{b}) = \underline{b}$. It can be shown that in an orthomodular lattice the complements of $\underline{a}$ are precisely the elements $(\underline{b} \wedge (\underline{b}' \wedge \underline{a}')) \vee (\underline{b}' \wedge \underline{a}')$ for arbitrary $\underline{b} \in L$.[10]

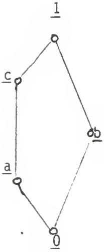
Figure 5 Hasse diagrams for lattices



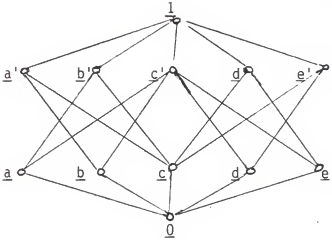
(a)



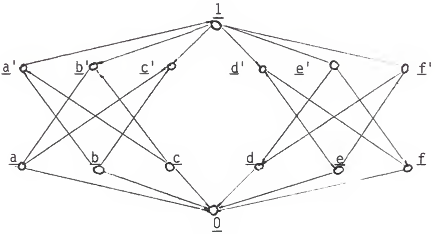
(b)



(c)



(d)



(e)

Two elements $\underline{b}$, $\underline{c}$ of a lattice L are said to be a modular pair if whenever $\underline{a} \leqslant \underline{c}$ then $(\underline{a} \vee \underline{b}) \wedge \underline{c} = \underline{a} \vee (\underline{b} \wedge \underline{c})$. A modular lattice is a lattice in which every pair of elements is a modular pair. An alternative definition of an orthomodular lattice is a lattice in which every orthogonal pair of elements is a modular pair (hence the name orthomodular). An orthomodular lattice is not modular iff it contains a sublattice of the form N_5 (Fig. 5(c)).

A Boolean lattice (or algebra) is an orthocomplemented lattice satisfying the distributive law: $\underline{a} \vee (\underline{b} \wedge \underline{c}) = (\underline{a} \vee \underline{b}) \wedge (\underline{a} \vee \underline{c})$ for all $\underline{a}, \underline{b}, \underline{c}, \in$ L. G_{12} (Fig. 5(d)) is a modular lattice that is not Boolean. 2^3 (Fig 5(a)) is a Boolean lattice. A Boolean lattice may be alternatively defined as a lattice in which each element has a unique complement.

The three types of lattices--Boolean, modular orthocomplemented lattices, and orthomodular lattices--may be compared as follows:

$$\{\text{Boolean lattices}\} \subset \{\text{modular orthocomplemented lattices}\} \subset \{\text{orthocomplemented lattices}\}$$

Boolean:

any $\underline{a}, \underline{b}, \underline{c}$ $\qquad (\underline{a} \vee \underline{b}) \wedge \underline{c} = (\underline{a} \wedge \underline{c}) \vee (\underline{b} \wedge \underline{c})$

Modular:

$\underline{a} \leqslant \underline{c}$ $\qquad (\underline{a} \vee \underline{b}) \wedge \underline{c} = \underline{a} \vee (\underline{b} \wedge \underline{c})$

Orthomodular:

$\underline{a} \leqslant \underline{c}, \underline{b} \leqslant \underline{c}'$ $\qquad (\underline{a} \vee \underline{b}) \wedge \underline{c} = \underline{a} \vee (\underline{b} \wedge \underline{c})$

The lattice $L(\mathcal{H})$ of closed subspaces of a Hilbert space $\mathcal{H}$, when partially ordered by set-theoretic inclusion, is a canonical example of a complete orthomodular lattice. The meet of two closed subspaces is their set-theoretic intersection and the join of two closed subspaces is the subspace they span. $L(\mathcal{H})$ is modular iff $\mathcal{H}$ is finite dimensional. The orthocomplement of a subspace A of Hilbert space is the set of all vectors in $\mathcal{H}$ orthogonal (in $\mathcal{H}$) to every vector in A.

A state on a lattice is defined as a probability measure on the set of elements of the lattice. m is a state on L if for $\underline{a}, \underline{b} \in L$,

$$\text{(i)} \quad 0 \leqslant m(\underline{a}) \leqslant 1$$

$$\text{(ii)} \quad m(\underline{0}) = 0, \quad m(\underline{1}) = 1$$

$$\text{(iii)} \quad \text{if} \quad \underline{a} \perp \underline{b} \quad \text{then } m(\underline{a}) + m(\underline{b}) = m(\underline{a} \vee \underline{b})$$

A set $\mathcal{M}$ of states on a lattice is called full if for $\underline{a}, \underline{b} \in L$

$$\underline{a} \leqslant \underline{b} \iff m(\underline{a}) \leqslant m(\underline{b}) \qquad \forall \quad m \in \mathcal{M}$$

This set of states determines the order on L. A set of states on a lattice is called strong if for $\underline{a}, \underline{b} \in L$

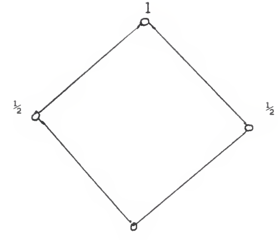
$$\underline{a} \leqslant \underline{b} \iff \left(m(\underline{a}) = 1 \Rightarrow m(\underline{b}) = 1\right) \quad \forall \quad m \in \mathcal{M}$$

Any lattice with a strong set of states also has a full set of states; the reverse implication is false. A pure state is a state m such that $\nexists$ states m_1 and m_2 different from m such that $m = \alpha m_1 + \beta m_2$ with $\alpha + \beta = 1$ and $\alpha, \beta > 0$. The state shown on the lattice in Fig. 6(a) is not a pure state on L because it equals $\frac{1}{2}m_1 + \frac{1}{2}m_2$ where m_1 and m_2 are given in Fig. 6(b). However, m_1 and m_2 are pure states.

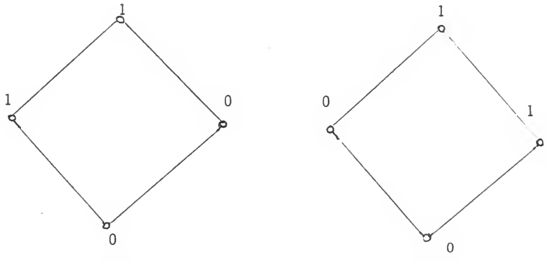
A quantum logic is usually defined as an orthomodular lattice with a full (or sometimes, a strong) set of states. That this notion is non-trivial is shown by Greechie in a paper proving the existence of orthomodular lattices that admit no states.[11] Furthermore, Bennett gives an example of an orthomodular lattice with infinitely many states but no full set of states.[12] $L(\mathcal{H})$ has a full set of states for any Hilbert space $\mathcal{H}$.[13] However, other results of Greechie are that not every orthomodular lattice with a full set of states or a strong set of states is embeddable in a Hilbert space.[14,15]

Figure 6 States on lattices

m:



(a)



m_1 m_2

(b)

B. Orthomodular Lattice Theory Applied to Quantum Mechanics

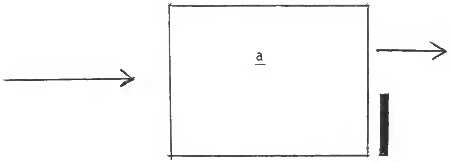
In quantum mechanics we can use the notions of quantum logic to discuss the "propositions" that refer to a physical system. Suppose for example that the physical system is a spin-½ particle. A proposition about the system is a statement such as "the projection of the spin of the particle along the $\vec{a}$ axis is $+\hbar/2$." Denote this proposition by the symbol $\underline{a}$.

Foulis and Randall, Mackey, and Piron identify the propositions with experimental procedures or equivalence classes of experimental procedures designed to confirm the truth of a statement such as $\underline{a}$.[16, 17, 18] If we adopt this view, then in order that a statement such as $\underline{a}$ in fact be a proposition, we need to design an apparatus that will completely pass a beam of which $\underline{a}$ is true. For the spin-½ system, the Stern-Gerlach apparatus with 2 holes, one open and one closed, oriented along the $\vec{a}$-axis, gives us the experimental procedure. We will denote this by the symbol shown in Figure 7. (This symbol will be used to denote <u>any</u> proposition $\underline{a}$ about any sytem, not just the particular spin-½ example cited. If a beam for which proposition $\underline{a}$ is true, or in state $|a\rangle$, enters the apparatus, it is certain to be passed.)

If a beam that passes an apparatus $\underline{a}$ is also certain to pass an apparatus $\underline{b}$, we can define a partial order on the experimental procedures and write $\underline{a} \leqslant \underline{b}$. To extend our use of the logical language and create a lattice of the propositions, we need to define experimental procedures for the meets, joins, orthocomplements, $\underline{0}$, and $\underline{1}$ elements of a lattice.

The join $\underline{a} \vee \underline{b}$ of two experimental procedures is the smallest machine guaranteed to pass all beams that have already been passed by the $\underline{a}$ machine and all beams that have already been passed by the $\underline{b}$ machine. A first guess at $\underline{a} \vee \underline{b}$ is shown in Figure 8(a). However, a beam that has previously passed a $\underline{b}$ machine, and is therefore in the state $|b\rangle$, is not guaranteed

Figure 7 Proposition a


a

to pass the machine of Fig. 8(a). The final emerging beam is given by

$$|+b\rangle\langle +b|-a\rangle\langle -a|b\rangle \; + \; |a\rangle\langle a|b\rangle \quad \neq \quad |+b\rangle.$$

A portion of the beam, $|-b\rangle\langle -b|-a\rangle\langle -a|b\rangle$, is absorbed, since

$$|\; \langle -b|-a\rangle\langle -a|b\rangle \;|^2 \neq 0.$$

This is the probability that any single particle in the beam fails to pass the machine. It can be shown, by iterating the passage of a particle in state $|b\rangle$ through alternating a and b machines as in Fig. 8(b), that a beam in state $|b\rangle$ passes with certainty. The probability that a beam in state $|b\rangle$ fails to pass the Nth stage of the machine is

$$\{ \langle -b \{ \; |-a\rangle\langle -a|-b\rangle\langle -b| \; \}^{N-1} |-a\rangle\langle -a|b\rangle \; \}^2$$

which $\rightarrow 0$ as $N \rightarrow \infty$.

It is very important to note that a particle that passes the $\underline{a} \vee \underline{b}$ machine was not necessarily in state $|a\rangle$ or state $|b\rangle$ before entering the machine. It may well have been in a state $|c\rangle$. The nature of quantum propositions is such that

$$(\underline{a} \vee \underline{b} \text{ is true } \not\Rightarrow \underline{a} \text{ is true or } \underline{b} \text{ is true.})$$

The construction of $\underline{a} \vee \underline{b}$ is based solely on the need for a machine to satisfy the relations

$$\underline{a} \leqslant \underline{a} \vee \underline{b}$$

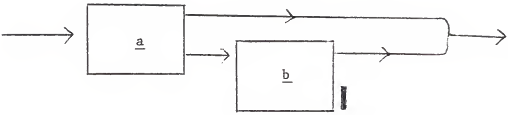
$$\underline{b} \leqslant \underline{a} \vee \underline{b}$$

and $\quad \nexists \quad \underline{c} < \underline{a} \vee \underline{b}$ such that $\underline{a} \leqslant \underline{c}$ and $\underline{b} \leqslant \underline{c}$.

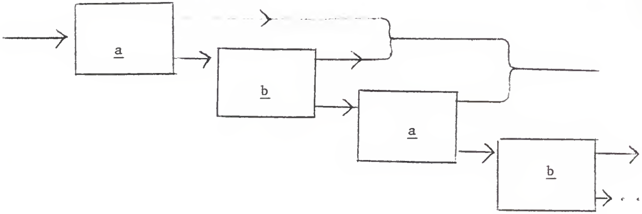
We conjecture that this $\underline{a} \vee \underline{b}$ machine or an equivalence class containing it is indeed the least upper bound for the a and b machines.

The meet $\underline{a} \wedge \underline{b}$ of two propositions should be defined as the largest experimental procedure such that any beam passing $\underline{a} \wedge \underline{b}$ will also pass a and will pass b. If a and b are compatible propositions in the usual QM sense, then a followed by b, or vice versa, or iterations of a followed by b, as in Figure 9, will serve. If a and b are not compatible, then no beam that

Figure 8 $\underline{a} \vee \underline{b}$



(a)



(b)

passes a is guaranteed to also pass b. The machine of Figure 9 then attenuates the intensity of the beam until no beam passes the machine.

To complete the discussion of experimental machines corresponding to propositions, we must design machines for 0, 1, and negations of propositions. The negation of a, denoted by a', should be an apparatus that is guaranteed to pass a beam iff the machine a is guaranteed not to pass the beam. The symbol for this machine is given in Figure 10(a). For the spin-½ beam, a Stern-Gerlach apparatus oriented along the $-\vec{a}$ direction serves the purpose.

The 0 proposition is at the bottom of the lattice: $0 \leqslant a \;\; \forall \; a$. The output of 0 must pass any a machine we choose. No beam will do this, so the output of 0 must be no beam. The symbol for 0 is given in Fig. 10(b). For the spin-½ beam, a Stern-Gerlach apparatus with both holes closed, oriented in any direction, is the appropriate machine.

The 1 machine must pass any beam that has previously been passed by any other machine. The symbol for 1 is given in Fig. 10(c). For the spin-½ beam, the Stern-Gerlach apparatus with both holes open, oriented in any direction, serves as the 1 machine.

The lattice of propositions for a spin-½ system, $L_{\frac{1}{2}}$, is given in Fig. 11. A lattice point corresponds to each direction in 3-space (every possible orientation of the Stern-Gerlach apparatus).

Prior to discussing the more complicated spin-1 system, we discuss the important notion of a state on a spin lattice, using the spin-½ lattice as an example. We will relate the idea of a state on a lattice to a state in Hilbert space. Each lattice point corresponds to one pure state in the Hilbert space of spin-½ states. That is, a corresponds to the state of a beam polarized along the $\vec{a}$ axis. Consider a probability measure (state) on the lattice for which the values on the lattice points represent the probability that a given spin-½ particle in a polarized beam will pass a

Figure 9 $\underline{a} \wedge \underline{b}$

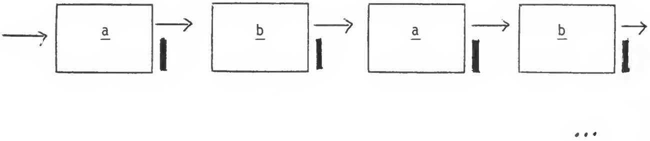
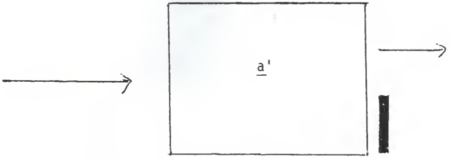
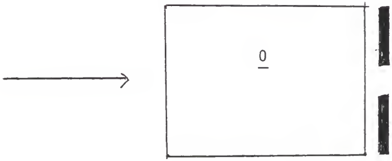

a
b
a
b
...

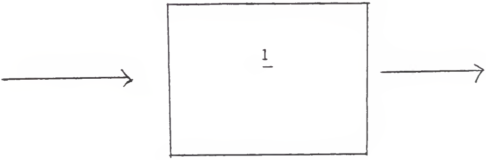
Figure 10 $\underline{a}'$, $\underline{0}$, and $\underline{1}$



(a)

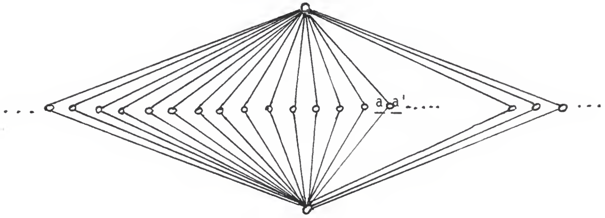


(b)



(c)

Figure 11 The lattice of propositions for a spin-$\frac{1}{2}$ system, $L_{\frac{1}{2}}$

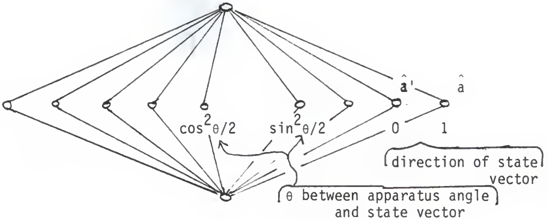

a
a'

Stern-Gerlach apparatus oriented in a specified direction. Such a probability measure can be denoted by the picture in Fig. 12(a). The formal definition of a state on a lattice would allow such states as that shown in Fig. 12(b). Obviously this state does not represent a physical situation. Pulmannova restricts her discussion of the set of lattice states $\mathcal{M}$ under consideration to a set of pure states such that for each atom a in the lattice there is one and only one $m \in \mathcal{M}$ for which $m(a) = 1$.[19] Even this restriction on the states does not guarantee physical states, since it does not specify the value of the state on the other atoms. For example, the state given in Fig. 12(c) would be possible for any a. Only the states m_a as defined above and as shown in Fig. 12(a) can represent physically possible states of a spin-½ system. Each such m_a corresponds to one atom a of $L_{\frac{1}{2}}$, and to one pure state of the Hilbert space for the spin-½ lattice.

We turn now to the case of the spin-1 particle. Another level of complexity is introduced. A beam of spin-1 particles may be in a pure state $\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$ of the Hilbert space of spin-1 states, yet not have its spin vector oriented along any axis (e.g., $\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$ mentioned previously). The Stern-Gerlach apparatus with 3 holes, 1 open and 2 closed, picks out only those pure states with definite spin vector.

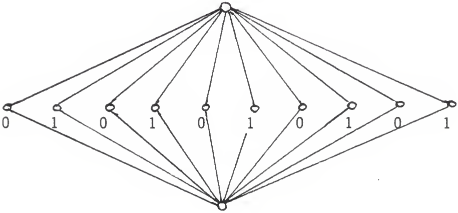
Hultgren and Shimony propose calling these states with definite spin vector "verifiable" states, with the "lattice of verifiable propositions L_V" shown in Fig. 13.[20] The lattice point $a(\hat{n}, \lambda)$ corresponds to the statement "the projection of the spin vector along the $\vec{n}$ direction is λ", where λ = -1, 0, or 1. The lattice point $b(\hat{n}, \lambda)$ corresponds to the statement "the projection of the spin vector along the $\vec{n}$ direction is not λ"--a 2-dimensional subspace of the Hilbert space, a Stern-Gerlach apparatus with two holes open,

Figure 12 States on $L_{\frac{1}{2}}$



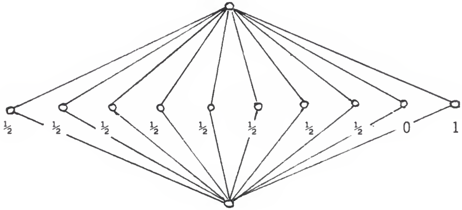
$\hat{a}'$
$\hat{a}$
$\cos^2\theta/2$
$\sin^2\theta/2$
0
1
direction of state vector
θ between apparatus angle and state vector

(a)



0 1 0 1 0 1 0 1 0 1

(b)

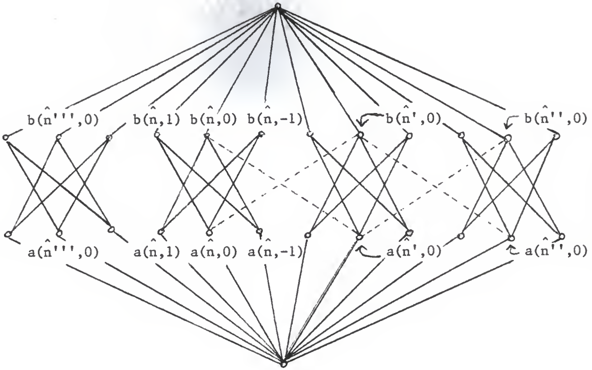


½ ½ ½ ½ ½ ½ ½ ½ 0 1

(c)

Figure 13

The lattice of "verifiable" propositions of a spin-1 system, L_V



A representative sample of propositions of L_V and their relations--

$\hat{n}'$ is orthogonal to both $\hat{n}$ and $\hat{n}''$, but $\hat{n}$ is not parallel, antiparallel, or orthogonal to $\hat{n}''$; and $\hat{n}'''$ is not parallel, antiparallel, or orthogonal to any of the other directions.

one closed. The point $b(\hat{n}, \lambda)$ is the negation of the point $a(\hat{n}, \lambda)$.

Hultgren and Shimony's verifiable propositions are not sufficient for our purposes, since we can build _any_ pure spin-1 state from the symmetric combination of 2 pure spin-½ states, not only those with definite spin vector. Hultgren and Shimony presented a challenge to devise an experimental procedure to filer any pure spin-1 state. Swift and Wright answered this challenge by specifying electric fields to be added to the standard Stern-Gerlach apparatus.[21] Since a spin-1 particle has an electric quadrupole moment as well as a magnetic dipole moment, the particle will interact with the electric field.

C. Generalized Stern-Gerlach Apparatus for the Spin-1 Particle

The lattice L_V of Hultgren and Shimony is based on the assumption that the only "verifiable" states of a spin-1 system are those with definite spin vector. Using the usual Stern-Gerlach apparatus, these are the only states guaranteed to be passed. The challenge to find an apparatus to filter any pure spin state was answered by Swift and Wright. Their program consists of showing how to apply an electric field to the Stern-Gerlach apparatus which, along with the magnetic field, can be adjusted to filter any spin-1 pure state.

Swift and Wright begin with the motivating simpler case of the spin-½ particle. Any 2 x 2 Hermitian operator A (in particular, the projection operators for a spin-½ particle) can be written in the form $A = aI + b_iS_i$ (using the summation convention for repeated indices). I is the identity matrix and the S_i's are the usual spin matrices. In a magnetic field $\lambda(\vec{r})\vec{B}_0 = \lambda(\vec{r})(B_1, B_2, B_3)$, the Hamiltonian $H = H_0 + \mu_0 B_i S_i$ describes the particle's behavior, where H_0 contains the kinetic energy of the particle and μ_0 is its total magnetic moment. If the vector $\vec{B}$ is chosen parallel to the vector $\vec{b}$, then the operators H and A have the same eigenstates. If A is a projection operator, i.e. the operation of determining the spin component in a certain direction, then the application of the given magnetic field spatially separates the particles of the beam which are in different energy eigenstates, as described in section IIA. Swift and Wright show that, with the perturbation necessary to make this magnetic field conform to Maxwell's equations, these eigenstates are correct to first order.

We extend this idea to the spin-1 case by applying the same type of analysis to a projection operator on a 3-dimensional Hilbert space. Any Hermitian 3 x 3 matrix can be written as a combination of matrices thus:

$$A = \sum_{k=0}^{2} a^{(k)}_{i_1 \ldots i_k} T^{(k)}_{i_1 \ldots i_k} =$$

$$a_0 I + b_x S_x + b_y S_y + b_z S_z + a_{xy} T_{xy} + a_{yz} T_{yz} + a_{xz} T_{xz} +$$

$$a_{xx} T_{xx} + a_{yy} T_{yy} + a_{zz} T_{zz}$$

where $a^{(0)} = a_0$, $T^{(0)} = I$, $a^{(1)}_i = b_i$, $T^{(1)}_i = S_i$, $a^{(2)}_{ij} = a_{ij}$ and $T^{(2)}_{ij}$ is obtained from the product $S_i S_j$ by symmetrizing and subtracting off the trace. (This formula is generalized to any spin-s system in Swift and Wright's paper.) The form of these matrices is listed in the Appendix.

Recall that the most general state of a spin-1 particle can be written in some coordinate system as $\psi = (\alpha, 0, \gamma)$. The projection operator onto this state is $\psi\psi^* =$

$$\begin{pmatrix} \alpha\alpha^* & 0 & \alpha\gamma^* \\ 0 & 0 & 0 \\ \alpha^*\gamma & 0 & \gamma\gamma^* \end{pmatrix}$$

or, if we set $\alpha = a + bi$, $\gamma = c + di$,

$$\begin{pmatrix} a^2 + b^2 & 0 & (ac + bd) - (ad - bc)i \\ 0 & 0 & 0 \\ (ac + bd) + (ad - bc)i & 0 & c^2 + d^2 \end{pmatrix}$$

Let this be the operator A. (The necessary coefficients of the Swift-Wright matrices to obtain this matrix are given in the Appendix.) Swift and Wright construct a Hamiltonian with the same eigenstates as A, but whose spatial variation causes different forces on particles in energy eigenstates with different eigenvalues. They show that the operators $T^{(k)}_{i_1 \ldots i_k}$ are proportional to operators measuring the 2k-pole electromagnetic moments.

First define $\Phi^E(\vec{r})$ and $\Phi^M(\vec{r})$ as the scalar potentials of the electric and magnetic fields, respectively. The fields are written as scalar potentials in order to satisfy Laplace's equation:

$$\nabla^2\Phi^E(\vec{r}) = 0 = \nabla^2\Phi^M(\vec{r}).$$

Then Maxwell's equations for static fields are immediately satisfied, and the need to add a perturbation term to the field, as Swift and Wright did in their spin-½ example, is eliminated.

If $^E(r)$ and $^M(r)$ are expanded in a Taylor series about $z = (z_1, z_2, z_3)$ (the particle's position), then we have

$$\Phi^E(\vec{r}) = \sum_{k=0}^{\infty} \frac{1}{k!}\Phi^E{}_{i_1\ldots i_k}(\vec{z})y_{i_1}\ldots y_{i_k}$$

$$\Phi^M(\vec{r}) = \sum_{k=0}^{\infty} \frac{1}{k!}\Phi^M{}_{i_1\ldots i_k}(\vec{r})y_{i_1}\ldots y_{i_k}$$

where $\vec{r} = \vec{z} + \vec{y}$

and

$$\Phi_{i_1 i_2\ldots i_k}(\vec{z}) = \frac{\partial}{\partial x_{i_k}} \frac{\partial}{\partial x_{i_{k-1}}} \cdots \frac{\partial}{\partial x_{i_1}} \Phi(\vec{r}) \Big|_{\vec{r} = \vec{z}}$$

Then we can write the Hamiltonian as

$$H(\vec{z}) = \sum_{k=0}^{2} \Phi^{(k)}{}_{i_1\ldots i_k}(z)T^{(k)}{}_{i_1\ldots i_k}$$

where

$$\Phi^{(k)}{}_{i_1\ldots i_k}(\vec{z}) = \begin{cases} (Q_k/k!)\Phi^E{}_{i_1\ldots i_k}(\vec{z}) & \text{if } k = 0 \text{ or } 2 \\ (M_k/k!)\Phi^M{}_{i_1\ldots i_k}(\vec{z}) & \text{if } k = 1 \end{cases}$$

Q_k is the 2k-pole electric multipole moment and M_k is the 2k-pole magnetic multipole moment.

If we expand the $\Phi^{(k)}_{i_1 \ldots i_k}$'s in a MacLaurin series, we get for the Hamiltonian:

$$H(\vec{z}) = \sum_{k=0}^{2} \left(\sum_{n=0}^{\infty} \frac{1}{n!} \Phi^{(k)}_{i_1 \ldots i_k j_1 \ldots j_n}(\vec{0}) z_{j_1} \ldots z_{j_n} \right) T^{(k)}_{i_1 \ldots i_k}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} z_{j_1} \ldots z_{j_n} \sum_{k=0}^{2} \Phi^{(k)}_{i_1 \ldots i_k j_1 \ldots j_n}(\vec{0}) T^{(k)}_{i_1 \ldots i_k}$$

$$= \sum_{k=0}^{2} \Phi^{(k)}_{i_1 \ldots i_k}(\vec{0}) T^{(k)}_{i_1 \ldots i_k}$$

$$+ \; z_j \sum_{k=0}^{2} \Phi^{(k)}_{i_1 \ldots i_k j}(\vec{0}) T^{(k)}_{i_1 \ldots i_k}$$

+ terms of second and higher order in z.

Comparing this with the form of the operator A, we see that if we choose

$$\Phi^{(k)}_{i_1 \ldots i_k}(\vec{0}) = a^{(k)}_{i_1 \ldots i_k}$$

then the 0-order part of the Hamiltonian equals A. The energy eigenstates of a particle in the center of the apparatus are the eigenstates of A. If we also choose

$$\Phi^{(k)}_{i_1 \ldots i_k 1}(\vec{0}) = a^{(k)}_{i_1 \ldots i_k}$$

then the Hamiltonian to first order in a is

$$H_1 = A + z_1 A + z_2 A_2 + z_3 A_3$$

and the energy E_i of a particle in eigenstate $|a_i\rangle$ is

$$E_i = a_i + z_1 a_i + z_2 \langle a_i|A_2|a_i\rangle + z_3 \langle a_i|A_3|a_i\rangle$$

The force in the z_1 direction at the center of the chamber is

$$F_1(\vec{0}) = -\frac{\partial}{\partial z_1} E_i = -a_i \; .$$

Therefore particles in energy eigenstates corresponding to different a_i's will experience different forces in the z_1 direction and be physically separated.

Swift and Wright have shown that if a "verifiable" proposition is one for which an experimental procedure does exist (at least in principle) then we must consider all propositions of a spin system, not just those of L_V, to be verifiable. In the case of a spin-1 system, L_1 ought to be the lattice of closed subspaces of a 3-dimensional Hilbert space.

IV. TENSOR PRODUCTS

If we want to combine the lattices of two spin-½ particles to obtain the lattice for a spin-1 particle, then we need to define some kind of product of lattices. Several combinations of lattices have been defined in the literature: e.g., the horizontal sum and the direct product. The horizontal sum $L = L_1 \circ L_2$ of two lattices is defined:

$$a \in L \iff a \in \{L_1 \setminus \{0_1, 1_1\}\} \cup \{L_2 \setminus \{0_2, 1_2\}\} \cup \{0, 1\}$$

with the relation

$$a \leq_L b \iff \begin{array}{c} a, b \in L_1 \text{ and } a \leq_{L_1} b \\ \text{or} \\ a, b \in L_2 \text{ and } a \leq_{L_2} b \\ \text{or} \\ b = 1 \quad \text{or} \quad a = 0 \end{array}$$

The lattices 2^2 and 2^3 and their horizontal sum are given in Fig. 14. The horizontal sum is not suitable for our purposes because $L_{1/2}^{(1)} \circ L_{1/2}^{(2)} \neq L_1$.

The direct product $L = L_1 \times L_2$ is defined as $L = \{(a_1, a_2) \mid a_1 \in L_1, a_2 \in L_2\}$ with the relation $\leq_L$ defined as:

$$(a_1, a_2) \leq_L (b_1, b_2) \iff a_1 \leq_{L_1} b_1 \text{ and } a_2 \leq_{L_2} b_2.$$

The meets, joins, and orthocomplements can be shown to be

$$(a_1, a_2) \vee (b_1, b_2) = (a_1 \vee b_1, a_2 \vee b_2)$$

$$(a_1, a_2) \wedge (b_1, b_2) = (a_1 \wedge b_1, a_2 \wedge b_2)$$

$$(a_1, a_2)^{\perp} = (a_1^{\perp}, a_2^{\perp}).$$

A Hasse diagram for a direct product lattice can be obtained by the following steps, illustrated in Fig. 15a-d for the lattice $2^1 \times 2^2 = 2^3$ and in Fig. 15e for the lattice $MO(2) \times 2^2$:

(i) Draw L_1 with a large circle in place of each point.

(ii) Inside each circle draw a copy of L_2.

(iii) Connect each point a in a copy of L_2 in row n of the large L_1 to a in each copy of L_2 in row n - 1 below row n.

The direct product does not serve our purposes for a tensor product. It includes elements such as (a, 0) and (0, b) that correspond physically to not doing an experiment on one of the particles, hence to not doing a spin-1 measurement. Several authors, however, have used the direct product as a beginning point for defining a tensor product. Aerts, Foulis and Randall, Pulmannova, and Zecca have worked on the problem.[22, 23, 24, 25, 26]

Pulmannova and Zecca develop a tensor product $\mathcal{L}$ of lattices L_1 and L_2 by defining an appropriate function $\circ : L_1 \times L_2 \to \mathcal{L}$ such that $(a, 0) \to 0_{\mathcal{L}}$ and $(0, b) \to 0_{\mathcal{L}}$ or by defining two functions $h_1 : L_1 \to \mathcal{L}$ and $h_2 : L_2 \to \mathcal{L}$ such that $h_1(0_1) = 0_{\mathcal{L}} = h_2(0_2)$. Following Zecca's paper, the conditions for $\circ$ are:

1. $0_1 \neq x_1 \in L_1 \Rightarrow (x_1 \circ x_2 \leqslant x_1 \circ y_2 \Leftrightarrow x_2 \leqslant y_2)$, $x_2, y_2 \in L_2$

   $0_2 \neq x_2 \in L_2 \Rightarrow (x_1 \circ x_2 \leqslant y_1 \circ x_2 \Leftrightarrow x_1 \leqslant y_1)$, $x_1, y_1 \in L_1$

2. $1_1 \circ 1_2 = 1_{\mathcal{L}}$, $0_1 \circ x_2 = x_1 \circ 0_2 = 0_{\mathcal{L}} \ \forall \ x_1 \in L_1, x_2 \in L_2$

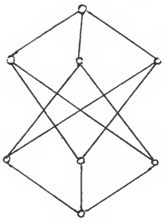
3. $(x_1 \circ 1_2) \wedge (1_1 \circ x_2) = x_1 \circ x_2 \ \forall \ x_1 \in L_1, x_2 \in L_2$

4. $(x_1 \circ 1_2)^{\perp} = x_1^{\perp} \circ 1_2, \quad (1_1 \circ x_2)^{\perp} = 1_1 \circ x_2^{\perp} \ \forall \ x_1 \in L_1, x_2 \in L_2$

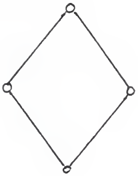
5. $A(L_1) \circ A(L_2) \subset A(\mathcal{L})$ where $A(L)$ denotes the set of the atoms of the logic L.

6. $e(x_1) \circ e(x_2) = e(x_1 \circ x_2) \ \forall \ x_1 \in A(L_1), x_2 \in A(L_2)$, where $e(x)$ denotes the central cover of x.

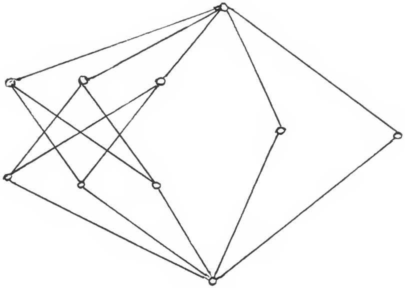
Figure 14 Horizontal sum of 2^2 and 2^3



2^3

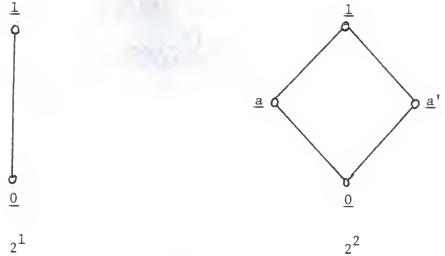


2^2



$2^3 \circ 2^2$

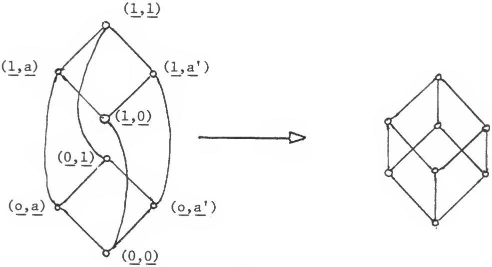
Figure 15 Two examples of direct products of lattices



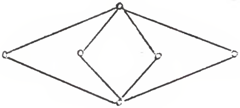
(a)

(b)

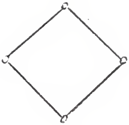
(c)



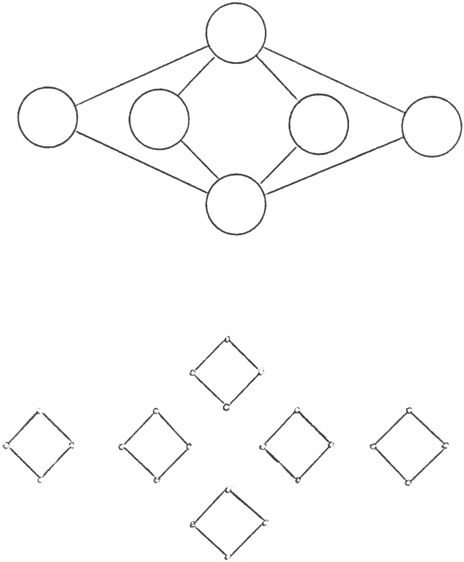
(d)



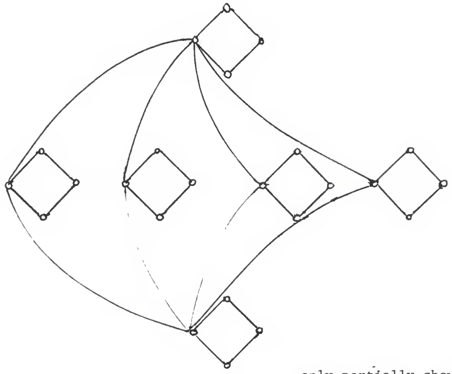
MO(2)



2^2



(e)



only partially shown to avoid confusion of lines

(e) (cont'd)

*. $x_1 \circ L_2$ and $L_1 \circ x_2$ are sublattices of $\mathcal{L}$.

Zecca derives the following theorem: Let H_1, H_2, and $\mathcal{H}$ be complex Hilbert spaces with dim $H_1 \geqslant 2$, dim $H_2 \geqslant 2$, and let $L(H_1)$, $L(H_2)$, $L(\mathcal{H})$ be the associated standard Hilbert logics. Suppose $L(\mathcal{H}) = L(H_1) \circ L(H_2)$. Then $L(\mathcal{H})$ is canonically isomorphic to $L(H_1 \times H_2)$ or to $L(H_1 \times H_2^*)$, where H_2^* is the dual of H_2.[27] This tells us that the result we want is what we will get, but we would prefer not to appeal to Hilbert space structure at all.

Pulmannova further defines a mapping β from the direct product of the sets of states on L_1 and L_2 into a set of states on $\mathcal{L}$:

$$\beta: \; [M_1 \times M_2] \rightarrow M$$

such that $\beta(m_1, m_2)(a_1 \circ b_2) = m_1(a_1)m_2(b_2)$.[28]

What state should $(a_1 \circ b_2)$ represent? As in our discussion of spin lattices, we want it to represent an experimental procedure, so we might define it to correspond to measuring particle 1 along axis $\vec{a}_1$ and particle 2 along axis $\vec{b}_2$. If the $(a_1 \circ b_2)$'s correspond to $|a_1\rangle|b_2\rangle$ in the Hilbert space tensor product, they have some non-zero probability of producing a spin-0 particle. If particle 1's spin is certain to be $\hbar/2$ along the $\vec{a}$ axis and particle 2's spin along the $\vec{b}$ axis, then

$$m_1(a_1) = 1 \text{ and } m_2(b_2) = 1 \text{ and } (m_1, m_2)(a_1 \circ b_2) = 1.$$

This implies that the 2-particle system is certain to be in a state called $(a_1 \circ b_2)$. What state is the 2-particle system certain to be in?--A state with some non-zero probability of having spin 0, or spin 1 with $m_s = 1, 0,$ or -1.

Another important question is to find the atoms in this lattice that correspond to superpositions of the $|a\rangle|b\rangle$'s. In particular, how do we generate $\phi = \dfrac{|a\rangle|b\rangle + |b\rangle|a\rangle}{\sqrt{2}}$, a legitimate state of the 2-particle system, a state which does not equal $|c\rangle|d\rangle$ for any of the $|c\rangle|d\rangle$'s?

ϕ should be an atoms of the lattice, below $(a \circ b) \vee (b \circ a)$, but it is only one of many atoms below $(a \circ b) \vee (b \circ a)$. If all the points $(c \circ d) \vee (f \circ g)$ above state ϕ could be given, then we could express ϕ as their meet. But the most reasonable way to do this is to find two planes in Hilbert space that intersect at ϕ; as mentioned above, we had hoped to be able to write ϕ without falling back on the Hilbert space structure we already know, and use the Hilbert space structure only as a check on our result. If $a \circ b$ is to correspond to $|a\rangle|b\rangle$, then an operation in lattice theory corresponding to "+" in Hilbert space still needs to be defined.

Suppose we limit our discussion then to lattice points corresponding to $|a\rangle|a\rangle$. Referring to Fig. 2, suppose $m_1(\vec{r}_1) = 1$, $m_2(\vec{r}_2) = 1$. Let $\vec{x}$ be an axis perpendicular to the plane of the page. Then $m_1(\vec{x}) = .5$, $m_2(\vec{x}) = .5$; if we would turn the apparati measuring particles 1 and 2, half of each beam would emerge in each case. Then also $\beta(m_1, m_2)(x \circ x) = .25$. What does this .25 represent? The probability that the spin-1 system passes what apparatus?

Suppose then alternatively that we decide to consider only spin-1 systems. Then we try the possibility that

$$(a \circ b) = \frac{|a\rangle|b\rangle + |b\rangle|a\rangle}{\sqrt{2}}$$

What apparatus corresponds to measuring this state, if we consider it the combination of two subsystems? Along what axes do we measure the spins of the subsystems? We do not know, but think that we should. We do know, in terms of the Swift-Wright apparatus, but this is not specified in terms of $\vec{a}$, $\vec{b}$--but it should be possible to do so!

A better treatment of the tensor product of lattices, in the spirit of the experimental procedures discussed earlier, would involve a combination of two spin-½ machines that interact independently with each of the spin-½ particles. The anti-symmetric combination, yielding a probability for a spin-0 particle, would then be dealt with together with the symmetric combinations, as in Hilbert space tensor products. An investigation into the connection between this machine and the Swift-Wright non-standard Stern-Gerlach experiment might be useful.

V. CONCLUSIONS

Quantum logic began historically with the observation that the distributive law does not hold for propositions about quantum properties of systems. This intriguing fact has invited comparison with the introduction of non-standard geometries and their eventual incorporation into physical theory.[29] The possibility that quantum logic might entail a similarly fundamental change in our concepts of physical reality is exciting indeed. The analogy is not quite complete, however, in the sense that quantum logic provides a mathematical contruct for experimental results already known, whereas the theory of relativity provided an application for a mathematical construct.

In classical mechanics the formalisms of Newton's laws and the Lagrangian are equally valid. Quantum logic could stand in just such a relationship to Hilbert space, if descriptions of coupled quantum systems, transition probabilities, Planck's constant, and other quantum concepts could be given in quantum logical terms.

One writer, P. Fevrier, has been criticized for supposedly attempting to formalize quantum theory based on a non-standard logic.[30] This task would be equivalent to writing a non-standard _Principia Mathematica_. Bas van Fraassen defends Fevrier against this particular criticism by saying that this is perhaps _not_, in fact, what she attempts to do:

> ...from our point of view a logic of quantum mechanics is simply an attempt to give a systematic account of the semantic relations among the elementary statements of that theory. And these semantic relations are to be deduced from the quantum theory--_that_ is the sense in which this logic is a quantum logic. It is not meant to be the basis for a formalization of the theory, or for a new, non-standard _Principia_.[31]

From a physicist's point of view, however, the status of quantum logic would be enhanced if the very goal for which Fevrier is criticized could be carried out.

Coupled systems are a basic part of physics. The development of a suitable tensor product in any formalism of quantum mechanics is therefore imperative. Gudder remarked on the need for a reasonable lattice theoretical tensor product as late as 1978.[32] As we discussed in section IV, several writers have taken on the challenge of defining a quantum logical tensor product, but have not succeeded in divorcing their constructions from a pre-existing Hilbert space. The questions raised about the physical meaning of the lattice points and their state values, resulting from these definitions of the tensor product, indicate that there is a need for perhaps more structure in the quantum logical formulation of quantum mechanics, and for more clarification of the relationship between the symbols used and objects, experiments, and processes in the physical world. A closer collaboration between those physicists, mathematicians, and logicians interested in the foundations of natural science might promise a more satisfactory insight into this fascinating idea--the non-standard nature of the logic of behavior in the quantum world.

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APPENDIX

Swift-Wright matrices for 3 x 3 Hermitian matrix:

$$S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad S_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad T_{xy} = \frac{\hbar^2}{2} \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$T_{xz} = \frac{\hbar^2}{2\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \qquad T_{yz} = \frac{\hbar^2}{2\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & i \\ 0 & -i & 0 \end{pmatrix}$$

$$T_{xx} = \frac{\hbar^2}{6} \begin{pmatrix} -1 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{pmatrix} \qquad T_{yy} = \frac{\hbar^2}{6} \begin{pmatrix} -1 & 0 & -3 \\ 0 & 2 & 0 \\ -3 & 0 & -1 \end{pmatrix}$$

$$T_{zz} = \frac{\hbar^2}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Coefficients for operator A, A written in terms of Swift-Wright matrices:

If $\alpha = a + bi, \quad \gamma = c + di$, then

$$A = \begin{pmatrix} \alpha \\ 0 \\ \gamma \end{pmatrix} (\alpha^*, 0, \gamma^*) = \begin{pmatrix} \alpha\alpha^* & 0 & \alpha\gamma^* \\ 0 & 0 & 0 \\ \alpha^*\gamma & 0 & \gamma\gamma^* \end{pmatrix}$$

$$= \frac{1}{3}\hbar^2 (a^2 + b^2 + c^2 + d^2) I + \frac{1}{2}\hbar (a^2 + b^2 - c^2 - d^2) S_z$$

$$+ \quad 2 (ad - bc) T_{xy}$$

$$+ \quad \{ 2 (ac + bd) - \frac{1}{2} (a^2 + b^2 + c^2 + d^2) \} T_{xx}$$

$$- \quad \frac{1}{2} (a^2 + b^2 + c^2 + d^2) T_{yy} + (ac + bd) T_{zz}$$

A QUANTUM LOGIC APPROACH TO THE ADDITION OF SPIN ANGULAR MOMENTA

by

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AN ABSTRACT OF A MASTER'S THESIS

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The introduction of quantum theory in the early 1900's, to explain observed phenomena that could not be explained by classical physics, caused a re-evaluation of philosophical concepts of the physical world. One interesting observation, first expressed by Birkhoff and von Newmann in 1936, was that propositions (testable statements of fact) about quantum mechanical properties of objects do not follow the same rules that propositions about classical objects follow. The mathematical framework of lattice theory has provided a formalism for the subject. In the last 25 years a burst of research activity, primarily by mathematicians, has produced a wealth of literature.

Physicists welcome a variety of mathematical models for the same phenomena, so if the quantum logic approach proves fruitful in solving problems in quantum mechanics, its addition to the Hilbert space repertoire of the physicist would enrich our understanding and vocabulary for quantum phenomena. This paper uses the quantum logic approach to try to solve the problem of addition of spins. A discussion of spin and addition of spins is presented. A basic difference between spin-1/2 and spin-1 systems is discussed--i.e. every pure spin-1/2 state has its spin vector pointing in a definite direction, whereas not every pure spin-1 state does. Two theorems are stated and proved concerning those spin-1 states that do have definite spin vector. Lattice theory terms are defined and experimental procedures for propositions are designed. Several types of products, necessary for the addition problem, are presented and their suitability for the tensor product of lattices is discussed. A purely lattice theoretical tensor product appropriate for the addition of two spin-1/2 systems has not yet been developed.