A METHOD FOR DETERMINING UPPER AND LOWER BOUNDS FOR HEAT CONDUCTION PROBLEMS

by

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NOMENCLATURE

q	internal heat generation per cubic foot
k_t	thermal conductivity
$t(x,y)$	temperature
$T(X,Y)$	transformed temperature
$T_L(X,Y)$	lower bound temperature
$T_U(X,Y)$	upper bound temperature
$T_E(X,Y)$	exact temperature of example problem
T_{mean}	mean value of T_L and T_U
Z_o	temperature function satisfying boundary conditions
Z_{ij}	temperature function having the value zero on the boundaries of region
m,n,i,j,k,p	integers
a_{ij}	unknown coefficients of approximate temperature solution
d	additional heat generation divided by k_t
D	length of plate
B	height of plate
$H = B/D$	dimensionless height of plate
x,y	coordinates of plate
X,Y	dimensionless coordinates of plate

INTRODUCTION

There are many boundary value problems for which exact solutions are either impossible or impractical to obtain using existing methods. With the development of high-speed digital computers, attempts have been made to solve difficult problems by approximate techniques. Many of these techniques result in a good approximate solution; but unless the exact solution is known, the error associated with the approximate solution cannot be precisely determined.

In using most approximate methods an attempt is made to obtain an "approximate solution" for the "exact problem," but the approach used in this report is to determine an "exact solution" to an "approximate problem."

In the present case it has been possible to obtain exact solutions to problems in such a way that these solutions provide upper and lower bounding solutions for the problem of interest. In particular the study concerns the evaluation of a proposed variational procedure for obtaining upper and lower bounding functions for temperature in a steady, two-dimensional heat conduction problem. The convergent properties of these upper and lower bounding functions are of particular interest. If convergent, then the solution may be obtained to any degree of accuracy within the limits of present-day computers.

In this study a variational procedure was used in which the solutions of the first approximate problems were always numerically above the exact solution. Then the problems were varied until the solution was everywhere numerically below the exact solution. By systematically applying this variational procedure, the exact solution was bounded above by a solution which was too large and bounded below by a solution which was too small.

By assuming the best approximation to be the average value of the upper and lower bounding solutions, the maximum error associated with this average solution can easily be calculated from the formula

$$\text{Maximum per cent error} = \left[\frac{S_U - S_L}{2\,S_L}\right]100,$$

where S_U is the upper bounding solution

S_L is the lower bounding solution.

BOUNDS FOR A TWO-DIMENSIONAL HEAT CONDUCTION PROBLEM

Definition of the Problem

The differential equation for temperature t in two-dimensional, steady state heat conduction with internal heat generation q is shown by Schneider [2] to be

$$\nabla^2 t + \frac{q}{k_t} = 0 \tag{1}$$

where ∇^2 indicates the operator $\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$

The boundary conditions for a two-dimensional plate (see Fig. 1) were taken to be

1. $t(o,y) = g_1(y)$
2. $t(D,y) = g_2(y)$
3. $t(x,o) = f_1(x)$
4. $t(x,B) = f_2(x)$

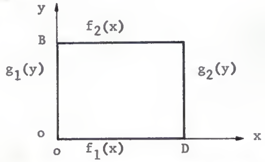

Fig. 1. Two-dimensional plate.

The problem is to determine the temperature t which satisfies equation (1) together with the prescribed boundary conditions.

Variational Procedure for Upper and Lower Bounds

The first step in the variational procedure is to assume an approximate solution of the form

$$t_{mn} = Z_o + \sum_{j=1}^{m} \sum_{i=1}^{n} a_{ij} Z_{ij} \qquad (2)$$

Z_o must satisfy the boundary conditions of the problem, and Z_{ij} must be zero on the boundaries of the region.

Substitute t_{mn} into the differential equation (1) to get

$$\nabla^2 t_{mn} + \frac{q_{mn}}{k_t} = 0 \qquad (3)$$

Now t_{mn} may be considered as the exact solution of a new problem with the specified boundary conditions but with a heat generation term of q_{mn}, which may vary over the region.

If the heat generation of the new problem q_{mn} were always greater than the heat generation of the original problem q, it would seem reasonable to expect the temperature distribution of the new problem t_{mn} to always be greater than the temperature distribution of the original problem t. Likewise, if q_{mn} were always less than q, it would seem reasonable to expect t_{mn} to always be less than t. This is precisely the reasoning upon which this variational procedure is based.

To find an upper bounding temperature, the residual $\left[\frac{q_{mn}}{k_t} - \frac{q}{k_t}\right]$ must be a positive number everywhere within the region of the problem. Thus,

$$\frac{q_{mn}}{k_t} - \frac{q}{k_t} = d \qquad d \geq 0 \qquad (4)$$

Combine equations (3) and (4) to get

$$\nabla^2 t_{mn} + \frac{q}{k_t} + d = 0 \tag{5}$$

The quantity $(d\,k_t)$ may be viewed as an additional heat generation which is added to the heat generation q of the original problem.

The coefficients a_{ij} in the approximate temperature solution t_{mn} may be determined by applying Galerkin's method [1] to equation (5). Multiply equation (5) by a weighting function Z_{kp} and integrate over the region R to obtain

$$\iint_R (\nabla^2 t_{mn} + \frac{q}{k_t} + d) Z_{kp}\, dA = 0 \tag{6}$$

$k = 1, 2, 3, \cdots n$

$p = 1, 2, 3, \cdots m$

A new equation is generated each time k or p varies. Equation (2) is differentiated twice with respect to x and y to get

$$\nabla^2 t_{mn} = \nabla^2 Z_o + \sum_{j=1}^{m} \sum_{i=1}^{n} a_{ij} \nabla^2 Z_{ij} \tag{7}$$

Combining equations (6) and (7) yields

$$-\sum_{j=1}^{m} \sum_{i=1}^{n} \left[\iint_R (\nabla^2 Z_{ij}) Z_{kp}\, dA \right] a_{ij} = \iint_R (\nabla^2 Z_o + \frac{q}{k_t} + d) Z_{kp}\, dA \tag{8}$$

$k = 1, 2, 3, \cdots n$

$p = 1, 2, 3, \cdots m$

The equations (8) are linear algebraic equations and can be solved for the coefficients a_{ij}. $\nabla^2 t_{mn}$ can then be found from equation (7) and $\frac{q_{mn}}{k_t}$ from equation (3). The region R is scanned for the minimum $\frac{q_{mn}}{k_t}$ and the maximum $\frac{q_{mn}}{k_t}$ corresponding to the particular d chosen. Then d is varied until the minimum residual $\left[(\frac{q_{mn}}{k_t})_{min} - \frac{q}{k_t}\right]$ becomes zero at some point in the region. This d is then the smallest positive d corresponding to the least upper bound temperature distribution t_{mn}.

To find the lower bounding temperature, the residual must be a negative number everywhere within the region of the problem. Therefore, d is varied until the maximum residual $\left[(\frac{q_{mn}}{k_t})_{max} - \frac{q}{k_t}\right]$ becomes zero at some point in the region. This d is then the largest negative d corresponding to the greatest lower bound temperature distribution t_{mn}.

A sketch summarizing the procedure for given values of m and n is as shown:

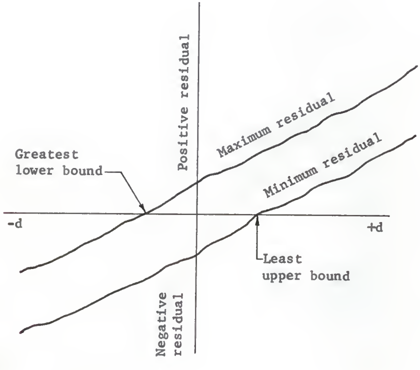

Fig. 2. Maximum and minimum residuals versus d.

EXAMPLE PROBLEM

The example problem is a two-dimensional problem with no internal heat generation. The boundary conditions are shown in Fig. 3.

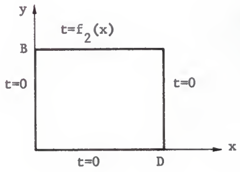

Fig. 3. Two-dimensional plate.

For this problem the general heat conduction equation reduces to

$$\nabla^2 t = 0 \tag{9}$$

The boundary conditions are

1. $t(0, y) = 0$
2. $t(D, y) = 0$
3. $t(x, 0) = 0$
4. $t(x, B) = f_2(x) = 64\, t_{max} (x/D)^3 (1 - x/D)^3$

The plate is transformed into one of dimensionless form by letting $Y = y/D$ and $X = x/D$. The transformed height of the plate becomes B/D and the transformed length becomes D/D or 1. The transformed temperature $T(X, Y)$ equals $t(x/D, y/D)$. The boundary conditions for the transformed problem become, letting $H = B/D$,

1. $T(0, Y) = 0$
2. $T(1, Y) = 0$
3. $T(X, 0) = 0$
4. $T(X, H) = f_2(DX) = F_2(X) = 64\ T_{max}\ X^3(1 - X)^3$

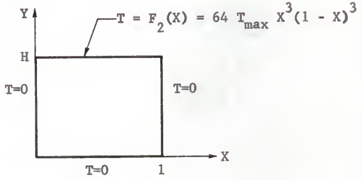

Fig. 4. Transformed two-dimensional plate.

The function Z_o, which must satisfy the boundary conditions, is chosen to be

$$Z_o = \frac{64}{H} T_{max}\ Y\ X^3(1 - X)^3 \tag{10}$$

The functions Z_{ij}, which must be zero on the boundaries of the plate, are chosen to be

$$Z_{ij} = X^i\ Y^j(X - 1)(Y - H) \tag{11}$$

Equations (10) and (11) are differentiated twice with respect to X and twice with respect to Y to get

$$\nabla^2 Z_o = \frac{384}{H} T_{max}\ Y\ X(1 - X)(5X^2 - 5X + 1) \tag{12}$$

and

$$\nabla^2 Z_{ij} = \left[i(i + 1)X^{i-1} - i(i - 1)X^{i-2}\right]\left[Y^{j+1} - H\ Y^j\right] + \left[j(j + 1)Y^{j-1} - j(j - 1)H\ Y^{j-2}\right]\left[X^{i+1} - X^i\right] \tag{13}$$

By using equations (11) and (12) the right-hand side of equation (8) is evaluated as

$$\iint_R (\nabla^2 Z_o + d) Z_{kp}\, dA = \frac{384\, T_{max}}{H} \int_o^1 \int_o^H \Big[X^{k+1} (X - 1)(1 - X)$$

$$(5X^2 - 5X + 1)(Y^{p+2} - H\, Y^{p+1}) + d\, X^k (X - 1) Y^p (Y - H) \Big] dY\, dX$$

$$= \frac{768\, T_{max}\, H^{p+2} (k^2 - 4k)}{(p+2)(p+3)(k+2)(k+3)(k+4)(k+5)(k+6)} + \frac{d\, H^{p+2}}{(k+1)(k+2)(p+1)(p+2)} \qquad (14)$$

By using equations (11) and (13) the left-hand side of equation (8) is evaluated as

$$-\sum_{j=1}^{m} \sum_{i=1}^{n} a_{ij} \iint_R (\nabla^2 Z_{ij})(Z_{kp})\, dA$$

$$= -\sum_{j=1}^{m} \sum_{i=1}^{n} a_{ij} \int_o^1 \int_o^H \Big[i(i + 1) X^{i-1} - i(i - 1) X^{i-2} \Big] (X^{k+1} - X^k)(Y^{j+1} - H\, Y^j)$$

$$(Y^{p+1} - H\, Y^p)\, dY\, dX - \sum_{j=1}^{m} \sum_{i=1}^{n} a_{ij} \int_o^H \int_o^1 \Big[j(j+1) Y^{j-1} - j(j - 1) H\, Y^{j-2} \Big]$$

$$(Y^{p+1} - H\, Y^p)(X^{i+1} - X^i)(X^{k+1} - X^k)\, dX\, dY$$

$$= \sum_{j=1}^{m} \sum_{i=1}^{n} \Bigg[\frac{4ik\, H^{j+p+3}}{(j+p+1)(j+p+2)(j+p+3)(i+k-1)(i+k)(i+k+1)}$$

$$+ \frac{4jp\, H^{j+p+1}}{(i+k+1)(i+k+2)(i+k+3)(j+p+1)(j+p)(j+p-1)} \Bigg] a_{ij} \qquad (15)$$

Equating equations (14) and (15) yields an equation from which m times n equations are generated by changing k or p. Each of these equations contains m times n unknowns a_{ij}. An IBM 1620 computer was used to solve the equations simultaneously for the coefficients a_{ij}.

Since q is zero, the residual is simply $\frac{q_{mn}}{k_t}$. After the coefficients are calculated, the residuals for any X and Y are found from the equation

$$\frac{q_{mn}}{k_t} = - \nabla^2 T_{mn} = - \nabla^2 Z_o - \sum_{j=1}^{m} \sum_{i=1}^{n} a_{ij} \nabla^2 Z_{ij}$$

$$= \frac{384}{H} T_{max} Y X(X - 1)(5X^2 - 5X + 1)$$

$$- \sum_{j=1}^{m} \sum_{i=1}^{n} a_{ij} \Big[\big[i(i+1)X^{i-1} - i(i - 1)X^{i-2} \big] (Y^{j+1} - H Y^j)$$

$$+ \big[j(j + 1)Y^{j-1} - j(j - 1)H Y^{j-2} \big] (X^{i+1} - X^j) \Big] \tag{16}$$

The computer was used to calculate the residuals at 121 points which were evenly distributed over the plate. For each d, these residuals were examined to find the maximum and minimum residuals on the plete.

When the minimum positive residuel became very close to zero, the coefficients a_{ij} used to calculete this residuel were then used to calculate the upper bounding temperature distribution over the plate. When the maximum negative residual became very close to zero, the coefficients a_{ij} used to calculate this residual were then used to calculate the lower bounding tempereture distribution over the plate.

If the exact temperature distribution is unknown, the mean value of the bounding temperatures T_{mean} is usually taken as the best approximation of the temperature distribution. The mean temperature is found from

$$T_{mean} = \frac{T_U + T_L}{2} \tag{17}$$

The actual per cent error of T_{mean} with respect to the exact temperature T_E is expressed as

$$\text{Actual error (\%)} = \left| \frac{T_{mean} - T_E}{T_E} \right| (100) \tag{18}$$

A numerical study has been made using $H = 0.75$ and $T_{max} = 1.0$. The results are shown in Table 1.

Table 1. Values for the temperature distribution of the example problem. $H = 0.75$, $T_{max} = 1.0$.

X	Y	T(lower)	T(mean)	T(upper)	T(exact)	Error(%)
.1	.075	.000392	.007672	.014952	.010951	29.942
.2	.075	.004731	.015971	.027210	.021001	23.952
.3	.075	.009622	.023152	.036681	.029193	20.695
.4	.075	.013287	.028012	.042736	.034592	19.022
.5	.075	.014638	.029730	.044822	.036481	18.505
.6	.075	.013284	.028008	.042733	.034592	19.031
.7	.075	.009620	.023150	.036680	.029193	20.700
.8	.075	.004734	.015973	.027213	.021001	23.939
.9	.075	.000394	.007673	.014952	.010951	29.937
.1	.150	.005176	.017089	.029002	.022405	23.729
.2	.150	.015396	.034500	.053604	.043047	19.854
.3	.150	.026160	.049529	.072898	.059975	17.417
.4	.150	.034110	.059733	.085357	.071192	16.095
.5	.150	.037025	.063350	.089674	.075131	15.681
.6	.150	.034112	.059732	.085352	.071192	16.097
.7	.150	.026159	.049528	.072897	.059975	17.418
.8	.150	.015393	.034501	.053610	.043047	19.852
.9	.150	.005178	.017089	.029000	.022405	23.727
.1	.225	.013387	.028221	.043056	.034834	18.983
.2	.225	.032023	.056312	.080602	.067173	16.168
.3	.225	.050547	.080567	.110587	.094000	14.290
.4	.225	.064066	.097160	.130254	.111961	13.219
.5	.225	.069013	.103077	.137141	.118306	12.872
.6	.225	.064070	.097165	.130260	.111961	13.215
.7	.225	.050550	.080569	.110588	.094000	14.288
.8	.225	.032020	.056308	.080596	.067173	16.174
.9	.225	.013384	.028221	.043058	.034834	18.985

Table 1 (cont.)

X	Y	T(lower)	T(mean)	T(upper)	T(exact)	Error(%)
.1	.300	.024778	.041239	.057701	.048620	15.180
.2	.300	.054949	.082183	.109418	.094365	12.909
.3	.300	.084049	.117916	.151782	.133053	11.376
.4	.300	.105207	.142656	.180105	.159395	10.501
.5	.300	.112956	.151538	.190121	.168787	10.219
.6	.300	.105206	.142657	.180109	.159395	10.500
.7	.300	.084048	.117916	.151784	.133053	11.376
.8	.300	.054954	.082184	.109415	.094365	12.908
.9	.300	.024775	.041239	.057702	.048620	15.181
.1	.375	.039324	.056308	.073292	.063906	11.889
.2	.375	.084630	.112815	.141000	.125426	10.054
.3	.375	.128278	.163408	.198537	.179098	8.760
.4	.375	.160324	.199205	.238087	.216581	8.022
.5	.375	.172141	.212204	.252266	.230126	7.788
.6	.375	.160320	.199201	.238083	.216581	8.024
.7	.375	.128275	.163406	.198537	.179098	8.761
.8	.375	.084633	.112818	.141004	.125426	10.051
.9	.375	.039328	.056310	.073291	.063906	11.887
.1	.450	.056483	.072944	.089404	.080265	9.121
.2	.450	.121241	.148469	.175698	.160645	7.579
.3	.450	.185279	.219151	.253024	.234285	6.459
.4	.450	.233409	.270861	.308312	.287600	5.820
.5	.450	.251376	.289954	.328532	.307209	5.616
.6	.450	.233412	.270862	.308312	.287600	5.820
.7	.450	.185278	.219149	.253021	.234285	6.460
.8	.450	.121240	.148469	.175699	.160645	7.579
.9	.450	.056484	.072944	.089405	.080265	9.120
.1	.525	.074482	.089316	.104150	.095943	6.907
.2	.525	.163957	.188246	.212534	.199071	5.437
.3	.525	.257475	.287496	.317517	.300914	4.458
.4	.525	.330324	.363419	.396513	.378216	3.912
.5	.525	.357972	.392036	.426099	.407242	3.733
.6	.525	.330328	.363423	.396517	.378216	3.911
.7	.525	.257480	.287497	.317515	.300914	4.458
.8	.525	.163952	.188240	.212529	.199071	5.440
.9	.525	.074480	.089317	.104155	.095943	6.906
.1	.600	.088882	.100796	.112710	.106166	5.058
.2	.600	.209454	.228565	.247675	.236929	3.530
.3	.600	.347497	.370860	.394223	.381310	2.740
.4	.600	.460011	.485631	.511252	.497139	2.314
.5	.600	.503487	.529816	.556146	.541523	2.161
.6	.600	.460010	.485631	.511252	.497139	2.314
.7	.600	.347496	.370861	.394225	.381310	2.740
.8	.600	.209465	.228569	.247674	.236929	3.528
.9	.600	.088882	.100795	.112709	.106167	5.059
.1	.675	.088617	.095897	.103178	.099190	3.318
.2	.675	.248870	.260107	.271345	.264339	1.601
.3	.675	.458112	.471643	.485173	.477725	1.273
.4	.675	.636803	.651527	.666252	.658413	1.045

Table 1 (cont.)

X	Y	T(lower)	T(mean)	T(upper)	T(exact)	Error(%)
.5	.675	.706890	.721981	.737072	.728615	.910
.6	.675	.636803	.651528	.666254	.658413	1.045
.7	.675	.458111	.471641	.485171	.477725	1.273
.8	.675	.248874	.260113	.271351	.265028	1.854
.9	.675	.088621	.095900	.103180	.099190	3.316

DISCUSSION OF RESULTS

The values of the temperature distribution and the actual error associated with T_{mean} for the example problem are tabulated in Table 1. The percent error is seen to increase as the temperatures at the lower side of the plate become near the zero boundary temperatures. The boundary temperatures are exact, of course, since the approximate temperature solution is composed of terms which satisfy the boundary conditions.

One of the difficulties encountered in this study of two-dimensional problems was that of choosing Z_o and Z_{ij} such that the residuals could be made positive or negative at the corners of the rectangular plate. An example problem was tried in which there was constant internal heat generation, but the above difficulty was not overcome until the internal heat generation was made zero.

Figures 7, 8, and 9 show typical values of the bounding and exact temperatures along vertical rows within the plate. Because of symmetry the temperatures along vertical rows at $X = 0.7$ and $X = 0.9$ would be the same as the temperatures at $X = 0.3$ and $X = 0.1$, respectively.

Figures 5 and 6 show that the maximum and minimum residuals converged toward the value of d as more terms were used in the series solution. As the extreme residuals approached d, then the value of d could be chosen closer to zero with the residuals still all positive or all negative. In Fig. 5 the residuals seemed to diverge when 36 terms were used in the series solution. By increasing the number of places of arithmetic used by the computer, the residuals again became closer to d showing the divergence to be the result of round-off error. As more terms were used, the residuals should have become closer to d, but at the same time the round-off error became worse. The final points of Fig. 5 and Fig. 6 were found by using 49 terms and 13 places of arithmetic as a compromise between using more terms or more places of arithmetic. The approximate temperature distribution and the residuals were both dependent upon the coefficients a_{ij}, which in turn were dependent upon d. Thus, the bounding temperature distribution converged toward the exact temperature distribution as the maximum and minimum residuals converged toward the value of d, which was made closer and closer to zero.

In looking for methods to force the residuals to converge more quickly to the value of d, one might try replacing the weighting function Z_{kp} of equation (6) by its first or second derivative. This should force the slope or curvature, respectively, of the residual surface to approach d. Although this method was not tried in this study, it would seem worth-while to try this technique in future studies.

CONCLUSIONS

The results of the example problem indicate that the variational procedure will provide upper and lower bounding functions for temperature in a steady, two-dimensional heat conduction problem. The procedure will extend to any other problem which is described by a differential equation of the nature of equation (1) and which has specified boundary conditions.

With more efficient and faster computers the variational method has the potential of becoming a powerful tool in solving many problems to any degree of accuracy desired. The method is relatively simple in that nothing more is used than algebra and simple differentiation and integration. The integration could be eliminated by using the collocation method of solving the differential equation (1) instead of the Galerkin method. The collocation method consists of setting the differential equation equal to zero at a finite number of points. Then the coefficients a_{ij} of the assumed series solution are calculated as in the Galerkin method.

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APPENDIX A

EXACT TEMPERATURE DISTRIBUTION

For the steady-state, two-dimensional, rectangular plate heat conduction problem shown in Fig. 3, Schneider [2] has shown the exact temperature distribution to be

$$t(x,y) = \frac{2}{D} \sum_{n=1}^{\infty} \frac{\sinh \frac{(n\pi y)}{D}}{\sinh \frac{(n\pi B)}{D}} \sin\left(\frac{n\pi x}{D}\right) \int_{o}^{D} f(x) \sin\left(\frac{n\pi x}{D}\right) dx \qquad (19)$$

By transforming the problem into the dimensionless problem shown in Fig. 4, the exact temperature distribution becomes

$$T(X,Y) = 2 \sum_{n=1}^{\infty} \frac{\sinh (n\pi Y)}{\sinh (n\pi H)} \sin (n\pi X) \int_{o}^{1} F(X) \sin (n\pi X)\, dX \qquad (20)$$

Substituting $F(X) = 64\ T_{max}\ X^3\ (1 - X)^3$ into equation (20), integrating and reducing, the exact temperature distribution over the plate becomes

$$T(X,Y) = \frac{18432\ T_{max}}{\pi^5} \sum_{n=1}^{\infty} \frac{\sinh (n\pi Y)}{\sinh (n\pi H)} \sin (n\pi X) \frac{1}{n^5} \left[\frac{10}{n^2\ \pi^2} - 1\right] \qquad (21)$$

$$n = 1, 3, 5, 7, \cdots$$

The exact temperature distribution was calculated using the computer. The number of terms used in the series solution to calculate the temperature at each of the 121 points of the plate was variable. Terms of the series solution were summed as n increased until the last term used was 10^{-8} of the temperature at the particular point of the plate.

APPENDIX B

EXPLANATION OF PLATE I

Fig. 5. Convergence properties of maximum and minimum residuals associated with lower bounding temperatures, $d = -1.0$.

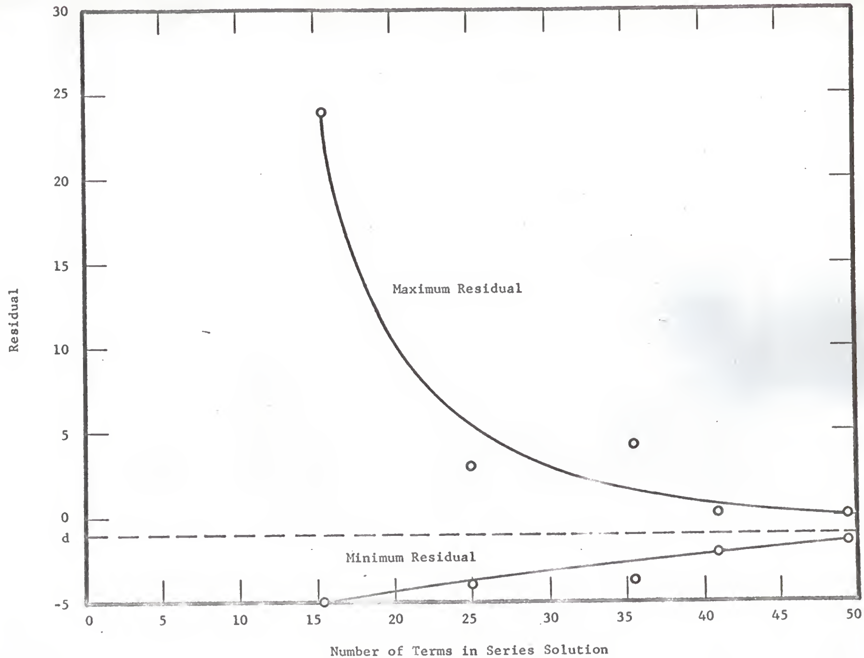
Residual
30
25
20
15
10
5
0
d
-5
Maximum Residual
Minimum Residual
0
5
10
15
20
25
30
35
40
45
50
Number of Terms in Series Solution

Fig. 5

EXPLANATION OF PLATE II

Fig. 6. Convergence properties of maximum and minimum residuals associated with upper bounding temperatures, $d = 1.0$.

PLATE II

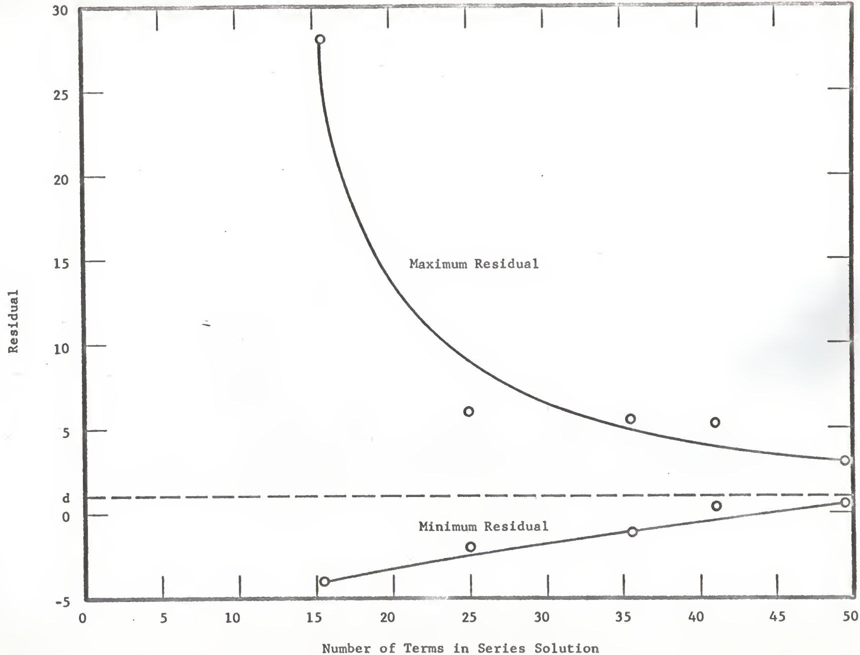

Fig. 6

EXPLANATION OF PLATE III

Fig. 7. Plot of exact and bounding temperatures versus Y/H, X = 0.1.

PLATE III

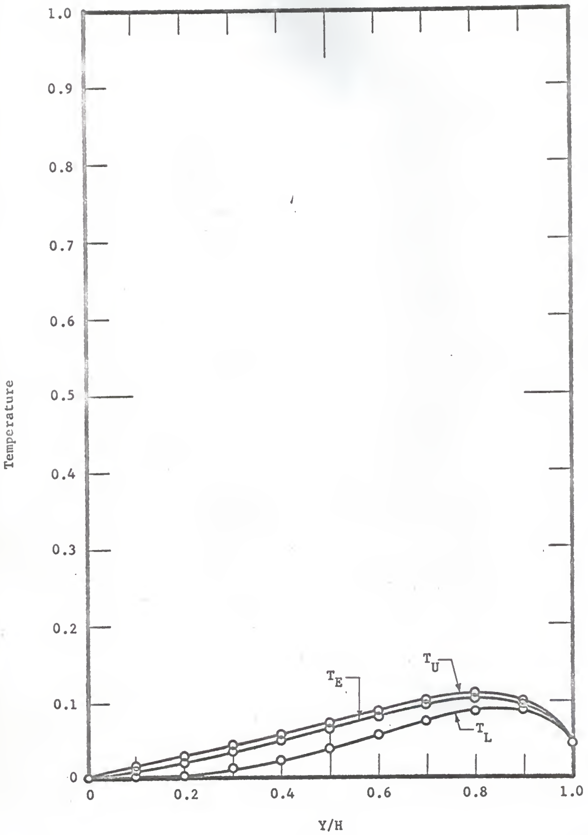

Fig. 7

EXPLANATION OF PLATE IV

Fig. 8. Plot of exact and bounding temperatures versus Y/H, X = 0.3.

PLATE IV

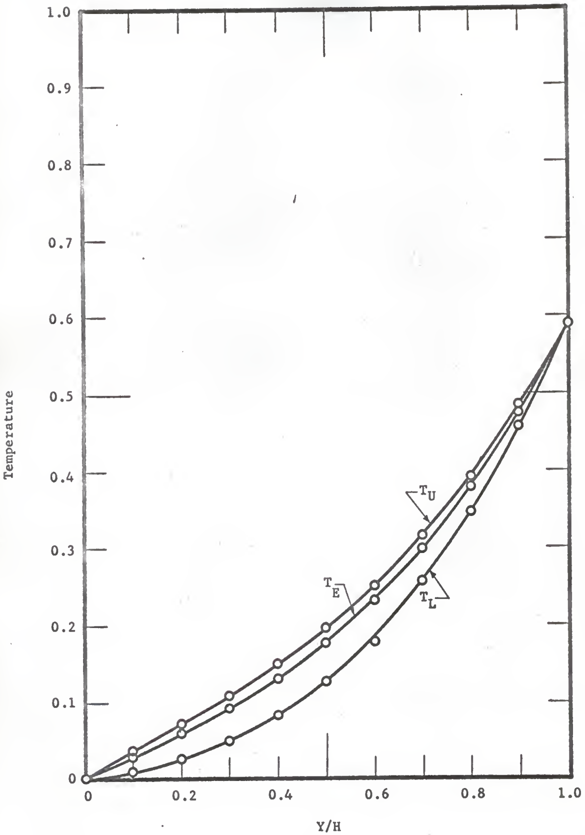

Fig. 8

EXPLANATION OF PLATE V

Fig. 9. Plot of exact and bounding temperatures versus Y/H, X = 0.5.

PLATE V

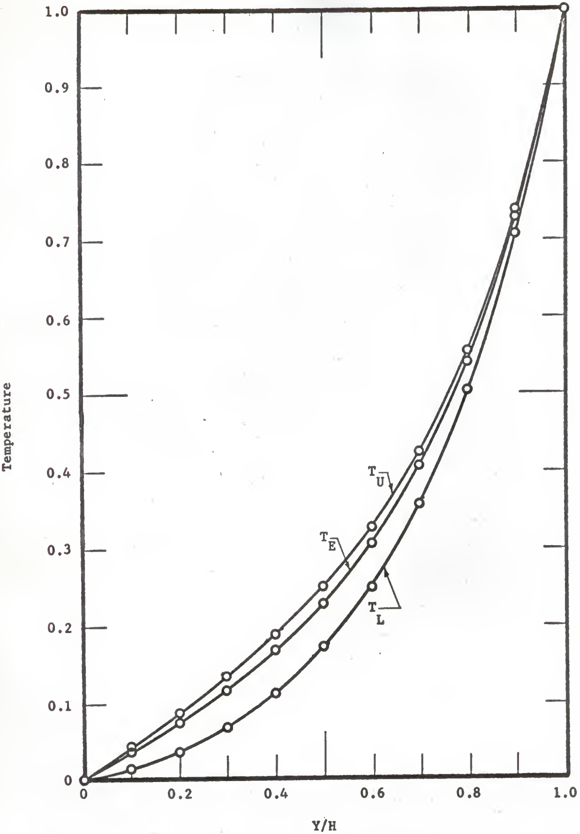
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Temperature
T_U
T_E
T_L
0
0.2
0.4
0.6
0.8
1.0
Y/H

Fig. 9

APPENDIX C

The following fortran statements were used on the IBM 1620 computer to solve for the coefficients a_{ij}.

```
C     UPPER AND LOWER BOUNDS FOR HEAT CONDUCTION PROBLEM
      DIMENSION B(49,50)
  158 FORMAT (E14.8,E14.8,E14.8,E14.8)
  160 FORMAT (I3,I3)
  182 READ 158,D,DELD,ZAC,ZCORR
      READ 158,H,TMAX
      READ 160,M,N
      LN=M*N
      I1=1
      DO 250 K=1,N
      DO 250 L=1,M
      J1=1
      DO 248 I=1,N
      DO 248 J=1,M
      A=I
      T=J
      C=K
      E=L
      C1=(4.*A*C)/((T+E+1.)*(T+E+2.)*(T+E+3.)*(A+C-1.)*(A+C))
      C1=C1/(A+C+1.)
      C1=C1*(H**(J+L+3))
      C2=(4.*T*E)/((A+C+1.)*(A+C+2.)*(A+C+3.)*(T+E-1.)*(T+E))
      C2=C2/(T+E+1.)
      C2=C2*(H**(J+L+1))
      B(I1,J1)=C1+C2
      IF(SENSE SWITCH 1) 251,248
  251 PUNCH 2,B(I1,J1),I1,J1
  248 J1=J1+1
  250 I1=I1+1
  252 CONTINUE
      I1=1
      J1=LN+1
      DO 274 K=1,N
      DO 274 L=1,M
      A=K
      T=L
      B(I1,J1)=(768.*TMAX*(H**(L+2)))/((T+2.)*(T+3.)*(A+2.)*(A+3.))
      B(I1,J1)=B(I1,J1)/((A+4.)*(A+5.)*(A+6.))
      B(I1,J1)=B(I1,J1)*((A**2)-(4.*A))
      B(I1,J1)=B(I1,J1)+(D*(H**(L+2)))/((A+1.)*(A+2.)*(T+1.)*(T+2.))
      IF(SENSE SWITCH 1) 272,274
  272 PUNCH 2,B(I1,J1),I1,J1
  274 I1=I1+1
  275 CONTINUE
C     MATRIX SOLUTION 4/9/62 RE
    1 FORMAT(E14.8,E14.8,E14.8,E14.8,E14.8,E14.8)
    2 FORMAT(E14.8,I3,I3)
    4 FORMAT (E22.16,I3)
```

```
      ZK=9.0E10
    5 LN1=LN+1
   14 ZTRY=0.0
  141 ZFA=0.0
  142 ZFAC=1.0
  143 ZZFAC=1.0
  144 DO 146 I=1,LN
  145 ZLL=I
  146 ZZFAC=ZZFAC*ZLL
   15 ZZ=0.0
   16 DO 64 LI=1,LN
   17 LIR=LN+1-LI
   18 IF(B(LIR,LIR)) 23,21,21
   21 ZY=B(LIR,LIR)-ZAC
   22 GO TO 24
   23 ZY=-1.*B(LIR,LIR)-ZAC
   24 IF(ZY) 25,25,64
   25 IF(LN-LI) 26,26,29
   26 LI1=2
   27 ZZ=1.0
   28 GO TO 31
   29 LI1=1I+1
   31 CONTINUE
   32 DO 45 LK=LI1,LN
   33 IF(ZZ) 34,34,36
   34 LK1=LN+1-LK
   35 GO TO 37
   36 LK1=LK
   37 IF(B(LK1,LIR)) 38,40,40
   38 ZY=-1.*B(LK1,LIR)-ZAC
   39 GO TO 41
   40 ZY=B(LK1,LIR)-ZAC
   41 IF(ZY) 42,42,56
   42 IF(ZK-ZY)45,45,43
   43 ZK=ZY
   45 CONTINUE
   46 IF(ZZ)47,47,51
   47 LI1=LIR+1
   48 ZZ=1.
   49 GO TO 31
   50 FORMAT(23H ALL ELEMENTS IN COLUMN,I3,21H ARE SMALLER THAN ZAC)
   51 ZK=ZK+ZAC
   52 PRINT50,LIR
   53 PRINT1,ZK,ZAC
   54 ACCEPT1,ZAC
   55 GO TO 15
C     READY TO INTERCHANGE TWO ROWS
   56 DO 61 LJ=1,LN1
   57 ZTE=B(LIR,LJ)
   61 B(LK1,LJ)=ZTE
  612 ZFA=ZFA+1.0
  613 ZFAC=ZFAC*ZFA
  614 IF(ZZFAC-ZFAC)616,616,62
```

```
  615 FØRMAT(44H REARRANGE EQ ØR INCREASE ZAC AND START ØVER)
  616 PRINT615
  617 PAUSE
   62 IF(ZZ) 64,64,63
   63 GØ TØ 15
   64 CØNTINUE
   65 IF(SENSE SWITCH 1) 66,71
   66 DØ 68 LI=1,LN
   68 PUNCH2,B(LI,LJ),LI,LJ
   71 CØNTINUE
C     MACHINE IS DØNE REARRANGING AND READY TØ SØLVE THE MATRIX
   75 DØ 86 LI=1,LN
   76 ZELD=B(LI,LI)
   77 DØ 78 LJ=LI,LN1
   78 B(LI,LJ)=B(LI,LJ)/ZELD
   81 DØ 86 LK=1,LN
   82 IF(LK-LI) 83,86,83
   83 ZELS=B(LK,LI)
   84 DØ 85 LJ=1,LN1
   85 B(LK,LJ)=B(LK,LJ)-ZELS*B(LI,LJ)
   86 CØNTINUE
   87 ZTRY=ZTRY+1.0
C     MATRIX IS SØLVED
   88 IF(SENSE SWITCH2) 91,94
   91 PRINT1,ZTRY
   92 DØ 93 LI=1,LN
   93 PUNCH2,B(LI,LN1),LI
   94 IF(SENSE SWITCH 3) 95,98
   95 PRINT4,ZTRY
      DØ 96 LI=1,LN
   96 PUNCH 4,B(LI,LN1),LI
   98 CØNTINUE
      GØ TØ 182
      END
```

APPENDIX D

The following fortran statements were used on the IBM 1620 computer to solve for the residuals over the rectangular plate.

```
C     RESIDUALS FOR HEAT COND. PROB.
    3 DIMENSION AL(10,10)
    2 FORMAT(E14.8)
  158 FORMAT (E14.8,E14.8,E14.8,E14.8)
  160 FORMAT (I3,I3)
  182 READ 160,M,N
      DO 10 I=1,N
      DO 10 J=1,M
   10 READ 2,AL(I,J)
  184 READ 158,XO,YO,DELX
  188 READ 158,H,TMAX
  303 D1=(1./DELX)+1.
  307 L1=D1
  309 D2=H*DELX
  302 Y=YO
  313 DO 332 L=1,L1
  304 X=XO
  315 DO 330 K=1,L1
  305 QK=384.*TMAX*Y*X*(X-1.)*((5.*X**2)-(5.*X)+1.)/H
  306 DO 324 I=1,N
  308 DO 324 J=1,M
  310 A=I
  312 T=J
  334 IF(X) 314,336,314
  336 IF(I-1) 314,342,338
  338 IF(I-2) 314,340,314
  340 C1=-2.
      GO TO 316
  342 C1=2.
      GO TO 316
  314 C1=(A*(A+1.)*(X**(I-1)))-(A*(A-1.)*(X**(I-2)))
  316 C2=(Y**(J+1))-(H*(Y**J))
  344 IF(Y) 318,346,318
  346 IF(J-1) 318,352,348
  348 IF(J-2) 318,350,318
  350 C3=-2.
      GO TO 320
  352 C3=2.
      GO TO 320
  318 C3=(T*(T+1.)*(Y**(J-1)))-(T*(T-1.)*H*(Y**(J-2)))
  320 C4=(X**(I+1))-(X**(I))
  322 C5=(AL(I,J)*C1*C2)+(AL(I,J)*C3*C4)
  324 QK=QK-C5
  328 PUNCH 158,QK,Y,X
  330 X=X+DELX
  332 Y=Y+D2
      GO TO 182
      END
```

APPENDIX E

The following fortran statements were used on the IBM 1620 computer to solve for the temperature distribution over the rectangular plate.

```
C      TEMP. BØUNDS FØR HEAT CØND. PRØB.
     2 DIMENSIØN A(10,10)
     3 FØRMAT (I3,I3)
     4 FØRMAT(E14.8)
     5 FORMAT(E14.8,E14.8,E14.8)
     1 READ 3,M,N
     6 DØ 10 I=1,N
     8 DØ 10 J=1,M
    10 READ 4,A(I,J)
    12 READ 5,H,TMAX
    14 READ 5,XØ,YØ,DELX
    16 D1=(1./DELX)+1.
    17 L1=D1
    18 D2=H*DELX
    19 Y=YØ
    20 DØ 40 L=1,L1
    22 X=XØ
    23 DØ 38 K=1,L1
    24 PHI=TMAX*64.*(X**3)*((1.-X)**3)*Y/H
    26 DØ 34 I=1,N
    28 DØ 34 J=1,M
    34 PHI=PHI+(A(I,J)*(X**I)*(Y**J)*(X-1.)*(Y-H))
    36 PUNCH 5,PHI,Y,X
    38 X=X+DELX
    40 Y=Y+D2
       GØ TØ 1
    42 END
```

APPENDIX F

The following fortran statements were used on the IBM 1620 computer to solve for the exact temperature distribution over the rectangular plate.

```
C     EXACT SOLUTION TO H. C. PROB.
    2 FORMAT (E14.8,E14.8,E14.8)
    4 FORMAT (E14.8,I3)
    6 READ 2, X0,Y0, DELX
    8 READ 2,H,TMAX
   10 READ 4,E,K
   12 LN=10
   14 L1=9
   16 D2=H*DELX
   18 Y=Y0
   20 DO 66 L=1,LN
   22 X=X0
   24 DO 64 J=1,L1
   26 N=1
   28 SUM=0.
   30 A=N
   32 C1=A*3.1415926*Y
   34 C2=A*3.1415926*H
   36 C3=A*3.1415926*X
   38 SINHY=EXPF(C1)-(1./EXPF(C1))
   40 SINHH=EXPF(C2)-(1./EXPF(X2))
   42 C4=(10./(9.8696044*A*A))-1.
   44 SUM1=SINHY*C4*SINF(C3)/(SINHH*A*A*A*A*A)
   46 SUM=SUM+SUM1
   48 IF(N-K) 54,50,50
   50 RATIO=SUM1/SUM
   51 IF(RATIO) 53,52,52
   53 RATIO=-RATIO
   52 IF(RATIO-E) 56,54,54
   54 N=N+2
      GO TO 30
   56 T=18432.*TMAX*SUM/306.01968
   58 IF(SENSE SWITCH 1) 60,62
   60 PRINT 2,T,Y,X
   62 PUNCH 2,T,Y,X
   64 X=X+DELX
   66 Y=Y+D2
      GO TO 6
   68 END
```

A METHOD FOR DETERMINING UPPER AND LOWER BOUNDS FOR HEAT CONDUCTION PROBLEMS

by

KEITH MELVIN HOSTETLER

B. S., Kansas State University, 1962

AN ABSTRACT OF
A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mechanical Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1963

A method is presented for finding upper and lower bounds for the temperature distribution of a two-dimensional, steady state heat conduction problem with internal heat generation. The method is systematic and well-adapted to analysis on high-speed computers.

The method is applied to a rectangular plate which has specified boundary conditions and zero internal heat generation. The results of the example compare fairly well with the exact solution which is known. From the example, the upper and lower bounding solutions seem to converge to the exact solution as the number of terms used in the series solution is increased.