A STUDY OF EFFECT OF TORSION ON RECTANGULAR REINFORCED CONCRETE BEAMS

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NOTATION

A = cross sectional area.

A_{bl} = area of bottom longitudinal steel.

A_e = area of core enclosed by stirrups.

A_s = cross sectional area of one leg of stirrup.

A_t = total area of longitudinal steel.

A_{tl} = area of top longitudinal steel.

A_{vl} = area of steel located near one vertical side of beam.

a_1 = bottom cover for steel.

a_2 = side cover for steel.

b = width of rectangular beam.

b' = smaller center to center dimension of the rectangular spiral reinforcement.

C = total flexure compression.

D = diameter of circle.

d' = longer center to center dimension of the rectangular spiral reinforcement.

d_c = depth of core area.

d_o, d = depth from extreme compression fiber to centroid of tension reinforcement.

f = yield stress for pure shear.

f_1 = horizontal normal stress (compressive) due to the applied bending moment.

f_b = stress in bottom longitudinal steel.

f_c = uniaxial compressive strength of concrete.

f_{cc} = ultimate compressive strength of 6 in. concrete cube at 28 days;

f_{cm} = stress at microcracking.

f_{cy} = ultimate compressive strength of 6x12 in. cyclinder at 28 days.

f_o = stress in concrete.

f_r = modulus of ruputre of concrete.

f_s = stress in bottom longitudinal steel due to flexure.

f'_s = stress in top longitudinal steel due to flexure.

f_{sl} = stress in longitudinal steel near the vertical face.

f_{sy} = yield strength of web steel.

f_{st} = stress in transverse steel.

f_t = uniaxial tensile strength of concrete.

f_{tc} = tensile strength of concrete with perpendicular compression of equal magnitude.

f_{ut} = ultimate tensile strength of concrete.

f_v = stress in steel A_{vl}.

f_y = yield stress of bottom longitudinal steel.

f'_y = yield stress of top longitudinal steel.

h = height of rectangular beam.

K_t = stiffness.

M = bending moment.

M_1 = moment, due to force in bottom longitudinal steel.

M_{bo} = flexure capacity of a beam under pure torsion.

M_c = moment, due to compressive force in concrete.

M_{hs} = moment, due to force in horizontal stirrups.

M_u = ultimate bending moment in combined bending and torsion.

M_{uo} = ultimate moment in pure bending.

M_{vs} = moment, due to force in vertical stirrups.

m_p = fully plastic moment.

n_1 = number of ties intersected by a potential failure crack on the longer side of the beam.

n_2 = number of ties intersected by a potential failure crack on the shorter side of the beam.

P = perimeter.

p = ratio of force in transverse steel to that of bottom longitudinal steel.

p_1 = percentage of reinforcement.

p_2 = ratio of force in transverse steel to that of A_{v1}.

p_3 = ratio of area of tension steel to the effective area of concrete.

Q_u = angle of twist at failure.

R = lateral force developed by longitudinal bar.

R_1 = ratio of top to bottom longitudinal steel.

r = radius.

r_1 = distance of longitudinal bar from the axis of the beam.

s = spacing of stirrups.

T = torque.

T_1, T_2, T_3 = torsional strength of the section in combined bending and torsion by mode 1,2 and 3 respectively.

T_b = bending component of torque.

T_c = torque carried by concrete based on elastic theory.

T_{c1} = contribution of compressed concrete to torsional strength.

T_{c2} = contribution of non-compressed concrete to torsional strength.

T_{cm} = torque at microcracking.

T_{cu} = torsional strength of corresponding plain concrete section.

T_c^e = torque resisted by concrete given by elastic theory.

T_c^p = torque resisted by concrete given by plastic theory.

T_e = torque resisted by inside core.

T_o = torque resisted by outside ring.

T_p = plastic torque;

T_r = torsional moment carried by concrete.

T_{s1} = torsional strength contributed by transverse steel.

T_{s2} = torsional strength contributed by longitudinal steel.

T_{su} = torsional strength contributed by reinforcement.

T_t = twisting component of torque.

T_u = ultimate torsional strength of concrete section.

T_{uo} = ultimate torque of plain concrete beam in pure torsion.

T_{ub} = ultimate torque in combined bending and torsion.

V = transverse shear.

V_{ch} = shear strength based on cracking in combined bending and torsion.

V_{co} = shear strength based on diagonal cracking in pure shear.

Z = static moment of compression zone about the neutral axis.

z = relationship between transverse and longitudinal steel.

∞ = ratio of height to width of beam.

β_1 = angle of inclination of crack.

β_e = elastic torsional shear stress coefficient corresponding to the middle of the shorter sides of the rectangular beam.

γ = $\sqrt{1 + (\lambda\phi)^2}$

δ = Bv/2T.

ε_t = strain in concrete.

θ_1, θ_2 = angles of inclination of cracks.

λ = $2\tau/f\phi$

σ = compressive stress due to flexure.

$\sigma_1\sigma_2$ = principal compressive and tensile stress respectively.

τ = shear stress.

τ_0 = unit torsional strength of concrete in pure torsion.

τ_1 = maximum torsional stress.

τ_2 = unit torsional strength of compressed concrete.

ϕ,ψ = T/M.

ϕ_{cr} = capacity reduction factor.

ω = $p\ f_y/f_c$.

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INTRODUCTION

STATEMENT OF THE PROBLEM

There is an uneasy awareness among designers that torsion is indeed becoming important in the design of concrete structures. Not only are architects demanding new forms in which torsion is an important consideration, but also the advent of ultimate strength design has resulted in members with greater slenderness, in which secondary torsion effects are no longer negligible. Usually torsion is a secondary effect, but it is very prominent in structural members such as spandrel beams, orthogonal shells, spiral staircases etc.

SCOPE OF THE STUDY

The scope of this report is limited to a review of current literature on the topic of torsion effects in plain and reinforced concrete beams. This topic has been previously studied by Pai.[10*] This report will describe different aspects of this problem which were not contained in this previous report. An example is presented to illustrate the application of two of the design methods described.

*Superscripts denote references listed in Bibliography

1. PLAIN CONCRETE SUBJECTED TO PURE TORSION

INTRODUCTION

In plain concrete subjected to pure torsion, failure occurs when the tensile component of the torsional shear exceeds the tensile strength of concrete. The torsional shear is usually calculated by one of two classical theories; the elastic and the plastic theory. The former assumes concrete to be linearly elastic, and the latter assumes concrete as fully plastic. Because neither is true the elastic theory underestimates and the plastic theory overestimates the torsional strength of a concrete member.

To account for the inelastic behavior of concrete, theories were developed by Miyamoto[7], Turner and Davis[16] and Marshall and Tembe[16] on the basis of different idealized non-linear stress strain curves for concrete. These theories may be categorically called semi-plastic theories. Although all these theories differ in assumptions concerning the behavior of concrete, they all employ the maximum tensile stress theory to predict the failure torque of a beam. In each case, it is assumed that failure occurs when the maximum tensile stress becomes equal to the tensile strength of concrete.

According to T.C. Hsu[13] all these theories are less than satisfactory and indicate that the maximum tensile stress assumption which is the basis of all the theories, may be incorrect. Therefore, the basic mechanism of failure was re-examined. It was found that plain concrete beams subjected to pure torsion actually fail by bending. Hsu developed a new theory based on the contention that failure is reached when the tensile stresses induced by a 45-deg bending component of torque on the wider face reach the modulus of rupture. All of these theories will be examined in this chapter.

PLASTIC THEORY-SAND HEAP ANALOGY[8]

If a concrete beam is severely twisted, certain parts will undergo plastic deformation. The states of stress in the small elements of a prismatic bar subjected to pure torsion are taken as states of simple shear. As long as the shearing strains are quite small the assumption will be made that no components of normal stress act in the sections normal or parallel to the bar axis. Also the resultant of the two components of the shearing stress τ_{xz} and τ_{yz}, which are the only components acting in the cross section perpendicular to the axis of the bar everywhere in the section where the yield point has been reached, has a constant value k (See Fig 1-1). The constant k is equal to $f_t/2$ if the maximum shear stress theory is assumed and equal to $f_t/\sqrt{3}$ if an ideally plastic substance is postulated, where f_t is the yield point for tension.

At a certain point of the cross section at which the shearing stress τ has reached the yield point, both shear-stress components τ_{xz} and τ_{yz} must satisfy the condition of plasticity:

$$\tau_{xz}{}^2 + \tau_{yz}{}^2 = k^2 = \text{constant} \qquad (1\text{-}1)$$

The above equation states that of all shearing stresses occuring at various cross sections of the bar, the largest, which in this case is τ, has at the yield point a constant value of k. These components τ_{xz}

and τ_{yz} must satisfy the condition of equlibrium.

$$\partial\tau_{xz}/\partial x + \partial\tau_{yz}/\partial y = 0 \tag{1-2}$$

This equation is again satisfied if one substitutes

$$\tau_{xz} = \partial F/\partial y, \qquad \tau_{yz} = -\partial F/\partial x \tag{1-3}$$

The function F(x,y) determined by these relations may again be represented as a surface over the cross section. F(x,y) is called the plastic stress function of the cross section. In those parts of the cross section in which plastic flow occurs, the function F must satisfy, according to Eqs. (1-1) and (1-3), the following equation:

$$(\partial F/\partial x)^2 + (\partial F/\partial y)^2 = k^2. \tag{1-4}$$

The differential expression on the left side of this equation is the square of the absolute value of the gradient - grad F (of the maximum slope of the surface F). Everywhere in the cross section where flow occurs the following equation holds true:

$$|\text{grad } F| = k = \text{constant}$$

This property and the further condition

$$-\tau_{yz}dx + \tau_{xz}dy = \partial F/\partial x.dx + \partial F/\partial y.dy = 0 \tag{1-5}$$

implies that the shearing stress τ at each point along the edge of a plastically distorted part of the cross section is directed tangentially to the edge $y = f(x)$ or the condition that along the edge F = constant, the plastic stress function F of the section is determined. Since an additive constant in F does not affect the value of the stresses along the edge, F may be taken equal to zero. From these properties of F it has been seen that the plastic stress function is a surface of constant maximum slope which one may construct over the edge of the cross section.

If the contour of the cross section is thought of as cut out of a piece of stiff paper and covered with sand while lying horizontally, there results a heap whose natural slope gives a picture of the surface F. Its form is independent of the amount of twist. The conditions in which all parts of a twisted bar yield, may be designated as the completely plastic state. The volume of the sand heap is numerically equal to half the plastic torque.[8]

The twisting moment of a circular bar can be calculated by considering the plastic stress surface. The plastic stress surface is a circular cone having the equation:

$$F = f\,(r-a)$$

where f = yield stress for pure shear;

a = radius of the bar.

For $r = 0$, $F = -fa$ and the cone has the height $h = fa$. The volume of the cone is $\pi a^2 h/3$, or $\pi a^3 f/3$. The plastic torque is equal to twice the volume of the cone.

$$T_p = 2 \times \text{volume} = 2\pi a^3 f/3 \qquad (1\text{-}6)$$

For the rectangular cross section, the plastic torque is:[8]

$$T_p = fb^3/3 + fb^2(b-h)/2$$

$$\text{or } T_p = \frac{1}{2}\, b^2(h-b/3)f$$

where h is the height and b is the width of the rectangular cross section.

SEMI-PLASTIC THEORIES[7,16]

Besides elastic and plastic theories, there are theories developed by Miyamoto[7], Turner and Davis[16] and Marshall and Tembe,[16] known as semi-plastic theories. Miyamoto considered the stress-strain curve for concrete in torsion as second degree parabolic. It is interesting to note that all the theories described here can be expressed in a basic form.

$$\tau_1 = T/KAD \qquad \text{or} \qquad T = \tau_1 KAD$$

where,

T = torque;

τ_1 = maximum torsional stress;

A = cross sectional area;

D = diameter of inscribed circle;

K = coefficient depending upon cross sectional shape. (See Table 1-1)

It is clear that the plastic torque and elastic torque are the upper and lower bounds, respectively, for the torque determined by any semi plastic theory. The Turner and Davis[16] theory is not valid for $h/b > 2.5$.

All the theories discussed postulate that the failure mechanism of a plain concrete section under pure torsion is in the form of helical cracks as shown in Fig. 1-3. However latest results show that helical cracks do not fully develop. Instead, when cracks develop on three sides, failure proceeds in the form of skewed bending with the neutral axis parallel to the long side of the section, and inclined to the axis of the twist as shown in Fig. 1-4.

HSU'S THEORY: ULTIMATE STRENGTH ANALYSIS:[13,15]

This theory is based on the contention that failure is reached when the tensile stresses, induced by a 45-deg bending component of torque on the wider face, reach a reduced modulus of rupture.

The failure surface for a rectangular section is actually bounded on its side as shown in Fig. 1-4. The longer edge of the failure surface shown in the foreground is inclined at 45-deg to the axis of the beam, while two shorter edges are curves inclined at various angles to the beam axis. The second longer edge in the background is a straight line connecting the ends of the two short curves. The direction of this straight edge is contrary to the spiral form.

Hsu derived an equation for ultimate torsional strength. Refering to Fig. 1-5, the applied torque can be divided into two components acting on the failure surface - the bending component T_b, and the twisting component T_t. According to the elastic bending theory

$$T_b = T_u \operatorname{Cos} \phi = \frac{b^2 h \operatorname{Csc} \phi}{6} f_r .$$

T_b = bending component of torque;

T_u = ultimate torsinal strength;

ϕ = angle between tensile crack on the wider face and the axis of the beam;

f_r = modulus of rupture of concrete;

b = shorter side of the rectangular beam;

h = longer side of the rectangular beam.

The above may be written as

$$T_u = \frac{b^2 h}{6} f_r \text{ (Sec } \phi \text{ Csc } \phi).$$

Minimum torque resistance occurs when $\phi = 45$ deg.

Hence, $$T_u = \frac{b^2 h}{3} f_r$$

An element 'A' is considered (See Fig. 1-5). Compression is caused by the twisting component of the torque T_t. By using Mohr's straight line failure theory one obtains,

$$f_{tc} = 0.85 f_t$$

where, f_{tc} = the tensile strength of concrete with perpendicular compression of equal magnitude;

f_t = unaxial tensile strength of concrete.

The modulus of rupture should be reduced in the same proportion because of the perpendicular compression.

Therefore $$T_u = \frac{b^2 h}{3} (0.85 f_r) \tag{1-6}$$

In order to simplify the application, Eqn. (1-6) has been transformed into an expression using the compressive strength.

$$T_u = 6(b^2+10)h\ 3\sqrt{f_c} \tag{1-7}$$

$$Q_u = \frac{0.025}{\sqrt{bh}} (1 + \frac{10}{b^2}) \tag{1-8}$$

where, Q_u = angle of twist at failure.

h = 3.5 b should be used when ever h > 3.5b.

$K_t = T_u/Q_u$, where k_t is the stiffness.

V. NAVARATNARAJAH'S THEORY[9]

Results obtained by this author on torsion in beams support the hypothesis that the concrete behaves elastically up to micro-cracking and non-elastically thereafter up to the ultimate failure of the beam. Mambers subjected to pure torsional loads develop diagonal tensile stresses due to the torsional shear stresses. Hence, the study of failure of concrete in pure torsion is primarily dependent on the behaviour of concrete in tension.

Strain at initiation of cracking or microcracking is not a constant value of 100×10^{-6} but varies with strength. It was also observed that the stress-strain relationship for concrete in tension is linear up to this strain at microcracking. Studies of torque-twist curves for plain circular sections, showed deviation from linearity at about 80% of the ultimate torque. The reason for this behavior is initiation of micro-cracking in the outer concrete at this value of the torque. Beyond this torque, the outer concrete continues to crack with an elastic core at the center until the maximum tensile strength of the concrete is reached in the outermost fiber, when the specimen ruptures. Brandtzaeg, et al[9] found from tests on plain concrete cylinders in compresssion that a serious breakdown in continuity of the specimens occured at about 70 to 80 percent of the ultimate load and at failure an outer concrete shell spalled off.

Prompted by these observations, Navaratnarajah estimated the diameter of the inside elastic core to be three-fourths of the diameter of the section. Only one-eighth of the diameter of the outer section is in a state of failure between microcracking and ultimate tensile failure but it is assumed that the outer ring is completely microcracked. Navaratnarajah showed experimentally that

$$T_{cm} = 0.8(T_e + T_o) = 0.8 \text{ (ultimate torque)}$$

where T_e = torque contributed by concrete core;

T_o = torque contributed by outside shell;

T_{cm} = torque at microcracking.

$$T_{cm} = \frac{\pi f_{cm} r^3}{2} \tag{1-9}$$

$$T_e = \frac{\pi f_{cm}}{2} (.75r)^3 \tag{1-10}$$

$$T_o = 0.67\pi f_{cm}\left(r^3 - (0.75r)^3\right) \tag{1-11}$$

r = radius of the cyclinder;

f_{cm} = stress at microcracking.

Rectangular Plain Concrete Beams

Initially the section behaves elastically, but as more torque is applied, the behaviour becomes non-elastic. When torque is increased, sections close to the middle of the longer sides behave non-elastically due to microcracking. Non elastic behaviour extends until the middle of the shorter sides develop microcracks.

Torques may be calculated assuming a plastic stress distribution over the entire cross section and assuming first a stress value equal to the stress at microcracking and then a stress value equal to the ultimate tensile strength. The mean of the two torques gives the ultimate torsional strength. The ultimate tensile strength f_{ut} is calculated as

$$f_{ut} = 0.68(f_{cy})^{0.75}$$

This equation is given by Gonnerman and Schuman,[9] f_{cy} being the 6 x 12 inch cyclinder strength obtained from 6 inch cube strength at 28 days, f_{cc}, using L'Hermites[9] formulas.

$$\frac{f_{cy}}{f_{cc}} = 0.76 + 0.20 \log \frac{f_{cc}}{2844} \approx 0.80$$

where f_{cc} = ultimate compressive strength of 6 inch concrete cube at 28 days;

f_{cy} = ultimate compressive strength of 6 x 12 inch concrete cyclinder at 28 days.

For plastic torsion in rectangular sections, the torque is given by the sand heap analogy equation of Nadia.[8]

$$T_p = 0.5\ b^2(h - b/3)f.$$

where T_p = plastic torque;

h = longer side of the section;

b = shorter side of the rectangle;

f = maximum stress for pure shear.

If the section is microcracked and the torque then is designated T_{cm}, one obtains,

$$T_{cm} = 0.5b^2(h-b/3)f_{cm} \tag{1-12}$$

and if the section has a plastic stress distribution, the stress is equal to the ultimate tensile strength f_{ut}. The torque is:

$$T_{ct} = 0.5b^2(h-b/3)f_{ut} \tag{1-13}$$

Hence the ultimate torque

$$T_u = 0.5(T_{cm} + T_{ct}) \tag{1-14}$$

2. REINFORCED CONCRETE SUBJECTED TO PURE TORSION

INTRODUCTION

It is generally agreed that in reinforced concrete subjected to torsion, the reinforcement has no appreciable effect on the stiffness before cracking. Similarly, the longitudinal or transverse reinforcement acting alone provides little additional strength beyond the capacity of plain concrete. However, if the longitudinal and the transverse steels are combined, the torque corresponding to first cracking is usually somewhat increased. After cracking, the stiffness is markedly reduced but a considerable increase in strength and a large amount of plastic detrusion are possible, depending upon the amount and disposition of the reinforcement.

A common premise of most theories is that the torsional strength of a reinforced concrete member is the sum of the strength of plain concrete and the strength of reinforcement, namely,

$$T_u = T_{cu} + T_{su}$$

where T_u = ultimate torque of a reinforced concrete section;

T_{cu} = torsional strength of corresponding plain concrete section based on elastic theory;

T_{su} = torsional strength contributed by reinforcement.

It is important to note that this hypothesis satisfies the condition of equilibrium but not necessarily the condition of compatibility. The principal difference among various theories is the method of calculation of T_{su}.

TURNER - DAVIS AND MARSHALL - TEMBE THEORIES[16]

These authors conducted some experiments on plain and reinforced concrete in torsion and proposed certain formulas, which are basically empirical. The term T_{cu} is based on the elastic theory and the ratio T_{su}/T_{cu} is expressed as a function of the total percentage of steel, p_1. Turner and Davis gave, for equal longitudinal steel and ties $T_{su}/T_{cu} = 0.25p_1$, when $p_1 \leq 1.5$, and for spirals $T_{su}/T_{cu} = 0.5p_1$, when $0.6 \leq p_1 \leq 2$. Marshall and Tembe gave, for equal longitudinal steel and ties, $T_{su}/T_{cu} = 0.33 + 0.1p_1$, when $1.5 \leq p_1 \leq 5$. The fact that these formulas are derived from rather limited test data and that they require equal percentages of longitudinal and transverse steels greatly reduces their practical significance.

RAUCH - ANDERSON - COWAN - ERNST THEORY[2]

In contrast to the empirical formulas, the theories given by these above authors are rational, and are all except the one given by Ernst, based on the elastic concept. It has been shown by Cowan that these theories can be stated in the form

$$T = T_c + T_r = T_c + K\,A_e\,A_s\,f_{sy}/s, \tag{2-1}$$

where

T = total torsional moment;

T_c = torque carried by concrete based on elastic theory;

T_r = torsional moment carried by steel;

A_s = cross sectional area of one leg of steel;

f_{sy} = yield strength of web steel;

A_e = area of core enclosed by stirrups;

s = spacing of stirrups in the direction parallel to the longitudinal axis;

K = a constant given in Table 2-1.

The web reinforcement A_s in equation (2-1) must be augmented by an equal volume of longitudinal reinforcement as determined by

$$A_t = 2\ T_r\ (b_c + d_c)/f_{sy}\ A_e\ K \tag{2-2}$$

where

A_t = total area of longitudinal steel;

b_c = width of the core area;

d_c = depth of the core area.

Although the theories by Rauch, Anderson, and Cowan are nearly indistinguishable in form, they are far apart in some of their basic assumptions. Rauch assumed that both steel and concrete are elastic, the lateral reinforcement is to take the full amount of the principle tension, and all the bars in the section reach their yield stress. Rauch devised a network of bars to represent the action of a reinforced concrete member. In the model the concrete is represented by compression bars and the reinforcement by tension bars. For the case of 45° spiral reinforcement, Rauch used $K = 2\sqrt{2}$ in the torque equation.

For beams with longitudinal and transverse reinforcement, Rauch's theory requires equal volume of longitudinal and transverse reinforcement.

Each part of the reinforcement resists a 45^{o} component of the diagonal force. For this case it was found that K = 2 for all shapes of cross section. Rauch's formula is based on circular section theory of torsion, and when applied to non-circular sections, he assumes that the stress in the reinforcement at any point is directly proportional to the distance from the center of twist. By Saint-Venent's theory for rectangular sections, the reverse is really true in the rectangular section, maximum stress at the middle of the longer sides and minimum at the corners. Since concrete is badly cracked near ultimate load, Rauch assumes that plain concrete is unable to carry torque, so that T_c is assumed to be zero.

The Anderson-Cowan theory is based on Saint-Venant's classical theory. It assumes that plain concrete fails under torsion when the maximum principal tensile stress reaches the tensile strength of concrete. In developing the theory, a circular section with 45^{o} spiral reinforcement was tested. Principal tension and compression are numerically equal on an element which is inclined at 45^{o} to the longitudinal axis. They calculated as follows:

$$T_r = 2\sqrt{2}\, A_e\, A_s\, f_{sy} \,/\, s \qquad (2\text{-}3)$$

Anderson assumed that the reinforcement takes only the part of the principal tension in excess of the allowable stress taken by the concrete. For the analysis of the rectangular section, Anderson made use of an equivalent circular section. His theory overlooks the variation in distance of the steels from the center of twist and, consequently, it fails to account for the difference in their torque resistance.

Cowan's theory has been developed by equating the external work done by the applied torque to the strain energy stored in steel and the concrete. The total strain energy is assumed to be equally divided between the tensile strain energy in the steel and the compressive strain energy in the concrete. It is implied that concrete has no tensile strength. However, in the analysis of a rectangular section, the strain energy is determined on the basis of Saint-Venant's elastic theory, which is applicable to an elastic, homogeneous, and uncracked section. This shows that the tensile strength of concrete is first neglected and then included which is an obvious inconsistency.

For circular sections with 45° reinforcement, Cowan obtained $K = 2\sqrt{2}$. In the rectangular section case, he got $K = 1.59\sqrt{2} - 1.689\sqrt{2}$ within the range of $d'/b' = 1.00$ to 3.00. He suggested $K = 1.6$ for all values of b'/d', in which b' is the smaller center to center dimension of the rectangular spiral reinforcement, and d' is the larger one. Cowan's theory, though developed primarily for working stress design, is believed to be applicable also to ultimate strength. According to Cowan, "the inelastic deformation in the concrete does not immediately produce plastic strains in the steel. The steel, therefore, remains elastic almost up to the point of failure; the beam may sometimes fail before the steel yields." Hence the ultimate strength is obtained by a formula analogous to equation (2-1)

$$T_u = T_{cu} + K\, A_e\, A_s\, f_{sy}/s \tag{2-4}$$

Cowan's procedure is probably acceptable if the mode of failure is dominated by crushing of the concrete. It has been shown by Ernst[4] however that failure may result from excessive yielding of steel. In such a case, the ultimate torque must be resisted by the forces in the closed stirrups. Let b_c and d_c be the width and the depth, respectively, of the core enclosed in the closed stirrup, and assume that the torsional cracks on each side of the rectangular section are inclined at 45^o. The torque resisted by the vertical legs of stirrup crossing the torsional crack is $A_s\ f_{sy}\ (\frac{d_c}{s})\ K_h\ b_c$

where d_c/s = number of stirrups crossing the crack on the long side of the rectangle;

K_h = a coefficient defining the internal moment arm of the force in the vertical leg of stirrups.

Likewise, the torque resisted by the horizontal legs of the stirrups crossing the torsional crack is $A_s\ f_{sy}\ (\frac{b_c}{s})\ K_v\ d_c$. Combining these two torques

$$T_u = A_s\ f_{sy}\ b_c\ d_c\ (K_h + K_v)/s$$

$$= (K_h + K_v)\ A_s\ A_e\ f_{sy}/s \qquad (2\text{-}5)$$

It is clear that the constant $(K_h + K_v)$ defines the internal moment arms of the stirrup forces. Its value will depend on the arrangement of the reinforcement and the shape of the member. However, it must be noted that it is arrived at empirically. At present, there is no sufficient test data for the determination of this quantity.

It is often argued that at ultimate torque the cracked concrete is incapable of producing the torque resistance, and therefore the steel must resist the entire amount of applied torque. A series of tests conducted at Portland Cement Association Structural Laboratory indicated that the core action is not an acceptable explanation of the torque resistance attributed to concrete. Rather, it seems that the strength corresponding to T_{cu} is the shear resistance of the compression zone when skewed bending takes place.

IYENGAR AND RANGAN THEORY[5]

These authors have made an attempt to establish a simple and at the same time a rational solution to the problem. The analysis is based upon the state of stresses existing at the critical point of failure and upon a failure criterion for concrete under compressive-tensile stresses. The following assumptions are made in the theory:

1) Concrete in the tension zone of the beam is neglected in the calculation of bending stresses.

2) The torsional rigidity of the beam is obtained by using a semiplastic concept in the case of beam failure in tension.

3) The transverse steel yields before torsion failure of the beam and hence the stress is assumed to be at the yield point.

4) The contribution of the horizontal legs of the hoops toward the torque capacity of the beam is neglected.

5) The contribution from the dowel action of the longitudinal steel is negligible.

In the beam under pure torsion, torsional shear stresses are produced by the applied torque. Every element in the cross section is subjected to a state of simple shear. This gives rise to two unlike principal stresses, each of which is equal in magnitude to the shear stress. Thus, failure is caused by destruction of concrete when the principal stresses satisfy a failure criterion for concrete under compressive-tensile stresses.

In 1964 Krishnaswamy[5] developed a failure criterion for concrete under compressive-tensile stresses. It can be put in the following form:

$$(\sigma_1/f_c) + (\sigma_2/f_t)^2 = 1 \qquad (2\text{-}6)$$

where

σ_1 = principal compressive stress;

σ_2 = principal tensile stress;

f_c, f_t = usual compressive and tensile strengths of concrete, respectively,

and $\sigma_1 = -\sigma_2 = \tau$ where τ is the shear stress. Substituting these into equation (2-6) we get

$$(\tau/f_c) + (\tau/f_t)^2 = 1 \qquad (2\text{-}7)$$

Letting $f_c/f_t = K$, and putting this in equation (2-7) and then solving for τ one obtains

$$\tau \cong [(2K-1)/2K]f_t \qquad (2\text{-}8)$$

In beams with transverse reinforcement, the torque will be resisted by both the concrete and the steel. The torque capacity of such beams is taken approximately as follows:

$$T = T_c + T_r \tag{2-9}$$

where T = total torque;

T_c = torque resisted by concrete;

T_r = torque resisted by steel.

In T_r, the contribution due to longitudinal steel has been neglected, as the force in the longitudinal steel is neglible.

The above approximation has also been considered by Cowan in the evaluation of the torque capacity of such beams. The contribution of the concrete is obtained by using the semiplastic approach. The elastic theory gives good results at higher ranges of concrete strength and the rigid-plastic-theory gives good results when the concrete strength is very low, because the plasticity ratio is high so as to cause the major portion of the cross section to attain the plastic state of failure. For normal strengths of concrete, only the semi-plastic theory can give the results nearer truth as suggested by Ernst[4].

Hence $$T_c = \frac{1}{2}\,(T_c^e + T_c^p)$$

or $$T_c = b^2\, h\beta_s \tau \tag{2-10}$$

where T_c^e = torque given by elastic theory,

T_c^p = torque given by rigid plastic theory,

β_s = semiplastic coefficient as given in Table 3-1.

b = width of rectangular beam,

h = total depth of rectangular beam.

Experiments[3] have shown that the transverse steel yields at failure of the beam; thus the axial stress in the vertical legs of the transverse hoops can be assumed to be equal to the yield stress of the transverse steel. Ignoring the contribution of horizontal legs, the torque resisted by one hoop is $A_s\ f_{sy}\ b'$, where b' is breadth of the transverse steel cage (center to center distance of vertical legs of the transverse steel cage). The failure crack travels along the periphery like a helix with an inclination of 45^o to the longitudinal axis. Let the depth of the transverse steel cage be d' then the projected length of the crack along the longitudinal axis is $d' \cot 45^o = d'$. The torque resisted by the transverse hoops is

$$T_r = A_s\ f_{sy}\ b'\ d'/s. \tag{2-11}$$

Substitute the values of T_c and T_r in equation (2-9) to obtain

$$T = \tau b^2\ h\beta_s + A_s\ f_{sy}\ b'\ d'\ /\ s \tag{2-12}$$

If failure is to occur, τ in equation (2-12) should satisfy the failure criterion for concrete as discussed earlier.

Substituting the value of τ from equation (2-8), the failure torque in pure torsion is

$$T = [\left((2K-1)/(2K)\right)\beta_s b^2\ h\ f_t + A_s\ f_{sy}\ b'\ d'/s] \tag{2-13}$$

Thus torque can be calculated by using Table 3-1.

3. REINFORCED CONCRETE SUBJECTED TO TORSION AND BENDING

INTRODUCTION

Even though *torsional* stresses are rarely high enough to have control over designs, their influence on the flexural capacity of beams is not small. A number of investigations have been carried out on the strength of reinforced concrete beams under combined loading. Iyenger and Rangan[5], Pandit and Warwaruk[11] and Collin, Walsh, Archer and Hall[2] have proposed solutions to predict the ultimate strength of beams under combined bending and torsion.

PANDIT AND WARWARUK THEORY[11]

This analysis gives the conditions under which the increase in torsional strength may be obtained under combined bending and torsional action and provides means of determining the limit of flexural moment beyond which the torsional strength drops with increase in flexural moment. The torsional strength of a reinforced section consists of the contribution to the strength of both steel and concrete as given here:

$$T_u = T_c + T_r \tag{3-1}$$

where T_u = ultimate torsional strength of reinforced concrete section in combined bending and torsion;

T_c = contribution of concrete to torsional strength;

T_r = contribution of steel to the torsional moment.

and

$$T_c = T_{c1} + T_{c2} \tag{3-2}$$

where T_{c1} = torsional moment resisted by compressed concrete;

T_{c2} = contribution of non compressed concrete to the torsional strength.

The authors consider that only about one fourth of the cross sectional area of a beam is in compression because of the flexural compressive stress and the area of compressed concrete is not significantly changed by torsion. The main effect of torsion is to shift the zone of flexural compression towards the center of the cross section of the beam (See Fig. 3-1). This has the effect of reducing the lever arm, jd, of the internal flexural moment as shown in Fig (3-1). The shaded area represents the uncracked concrete which carries flexural compression. For the case of pure bending, the value of j is 0.9. It is assumed that the value of j for the case of combined bending and torsion depends upon the ratio:

$$k_1 = M_u/M_{uo}$$

where M_u = ultimate bending moment in combined bending & torsion;

M_{uo} = ultimate moment in pure bending.

If the ratio of the effective depth to over-all depth of the beam cross-section is 0.9, then

$$\lim_{k_1 \to 0} (j) = \frac{0.5 - 0.1}{0.9} = 0.45$$

Hence, the value of j varies from 0.45 to 0.9 as k_1 varies from 0 to 1.

The exact nature of the variation of j is not known, but for simplicity a linear relationship is assumed.

$$j = 0.45\ (1+k_1)$$

From statics,

$$A_{b1}f_s = C = M_u/jd \tag{3-3}$$

where A_{b1} = area of bottom longitudinal steel;

f_s = stress in bottom longitudinal steel due to flexure;

C = total flexure compression.

It is assumed that one-fourth of the cross sectional area carries flexural compression, hence the compressive stress is

$$\sigma = 4C/hb$$

where h and b are over-all dimensions of the cross-section.

The torsional resistance of a portion of concrete carrying flexural compression is generally higher than that of the equivalent concrete subjected to pure torsion, and the torsional resistance of concrete which carries flexural tension is lower than that of the equivalent concrete subjected to pure torsion. The unit torsional strength of compressed concrete may be determined by using one of the failure theories for concrete under combined stresses. According to Cowan's theory,[11] the compressed concrete may fail by cleavage (tension) failure, or shear (compression) failure, depending upon the magnitude of σ. The critical value of σ, at which the failure changes from one type to another, is given as

$$\sigma_{cr} = f_c - \frac{2\tau_o}{1\text{-sin } 37^o} \tag{3-4}$$

where f_c = compressive strength of concrete;

τ_o = unit torsional strength of concrete in pure torsion.

The unit torsional strength of concrete (compressed) is given as

$$\tau_1 = \sqrt{(\sigma + \tau_o)\tau_o} \tag{3-5}$$

for cleavage failure, or

$$\tau_1 = \sqrt{R^2 - (\sigma/2)^2} \tag{3-6}$$

for shear failure,

$$R = \left(\frac{1}{2} f_c \text{ (cosec } 37^o - 1) + \sigma/2\right)\sin 37^o.$$

The sand-heap analogy, proposed by Nadai for the case of pure torsion, is extended here for the case of combined bending and torsion. The analogy, as used here, is approximate since it involves a stress discontinuity at the junction of the compressed concrete and the non-compressed concrete. Also, the shifting of the zone of compressed concrete due to torsion is ignored, for simplicity, in the derivation of T_c. Modified sand-heap analogy is shown in Fig. 3-1b. Taking the torque equal to twice the volume of the sand heap and replacing the slopes, θ_1 and θ_2, by the analogous unit torsional strengths, τ_1 and τ_2, for the compressed concrete and the non-compressed concrete, respectively, T_c is:

$$T_c = \frac{b^2}{2}(d_1 - \frac{b}{6})\ \tau_1 + \frac{b^2}{2}(d_2 - \frac{b}{6})\ \tau_2$$

when $d_1 \geq b/2$

and $$T_c = \left[\frac{b^2}{2}(d_1 - \frac{b}{6}) + \frac{1}{12}(b - 2d_1)^3 \right] \tau_1 + \frac{b^2}{2}(d_2 - \frac{b}{6})\ \tau_2$$

when $d_1 < b/2$

where d_1 and d_2 are the depths of the compressed and the non-compressed concrete, respectively. The torsional resistance of the non-compressed concrete is reduced by the flexural tension. Hence

$$0 < \tau_2 < \tau_o$$

Therefore, if $K_2 = \tau_2/\tau_o$, then the values of K_2 equal to zero and unity give, respectively, the theoretical lower and upper bounds of the unit torsional strength.

The contribution of steel to the torsional strength is the sum of the contributions of the transverse and longitudinal reinforcement, namely T_{s1} and T_{s2} respectively.

$$T_r = T_{s1} + T_{s2} \qquad (3\text{-}7)$$

Since the twisting moment produces a state of pure shear giving rise to diagonal tension at 45^o to the axis of the beam, it is necessary to have steel in both longitudinal and transverse directions to resist the components of diagonal tension in these directions.

The term T_{s1} is determined from the number of ties intersected by a potential crack on the faces of the beam. Ties intersected by a potential failure crack yield at failure, provided enough longitudinal steel exists both at the top and bottom faces to resist the longitudinal component of diagonal tension. If enough longitudinal steel does not exist, the ties can not be stressed to the yield point and the contribution of transverse steel to torsional strength is assumed to be linearly reduced.

$$T_{s1} = n_1 A_s f_{st} b' + n_2 A_s f_{st} d' \tag{3-8}$$

where n_1 = number of ties intersected by a potential failure crack on the longer side of the beam;

n_2 = number of ties intersected by a potential failure crack on the shorter side of the beam;

A_s = area of cross section of one leg of the tie;

f_{st} = stress in the transverse steel.

Also, $f_{st} = f_{sy}$, the yield stress of transverse steel, only if

$$A_{bl}(f_y - f_s) \geq (n_1 + n_2) A_s f_{sy} \tag{3-9}$$

and

$$A_{tl}(f'_y + f'_s) \geq (n_1 + n_2) A_s f_{sy}. \tag{3-10}$$

In the above, A_{bl} = area of bottom longitudinal steel;

A_{tl} = area of top longitudinal steel;

f_y = yield stress of bottom longitudinal steel;

f'_y = yield stress of top longitudinal steel;

f_s = stress in bottom longitudinal steel due to flexure;

f_s' = stress in top longitudinal steel due to flexure.

In Eqn. 3-10, the (+) sign applies for cleavage failure and the (-) sign for shear failure. However, if either or both of the above two inequalities are violated, then

$$f_{st} = \left[\frac{A_{b1}(f_y - f_s)}{(n_1+n_2)A_s f_{sy}}\right] f_{sy} \qquad (3\text{-}11)$$

or

$$f_{st} = \left[\frac{A_{t1}(f_y' + f_s'}{(n_1+n_2)A_s f_{sy}}\right] f_{sy} \qquad (3\text{-}12)$$

whichever is smaller. Eqns. 3-11 and 3-12 indicate that the stress developed in the transverse steel is governed by yielding of the bottom steel or the top steel, whichever occurs first.

The twisting moment has the tendency to bend the longitudinal bars in the lateral direction, giving rise to lateral forces. The contribution of the longitudinal steel to torsional strength, T_{s2} appears mainly to be a function of the size and position of longitudinal bars and the tie spacing. The term T_{s2} is determined as described:

$$T_{s2} = \sum R r_1 \qquad (3\text{-}13)$$

where R = lateral force developed by longitudinal bar;

r_1 = distance of a longitudinal bar from the axis of the beam.

The summation is taken for all of the longitudinal bars. Assuming that the total lateral force which a bar develops is uniformly distributed and that its magnitude is governed by the full plastic moment of the bar, then

$$R = 2m_p/s \tag{3-14}$$

where m_p = fully plastic moment of a longitudinal bar;

s = spacing of ties.

The value of R is limited by this equation to

$$R \leq \frac{1}{2}\,(A_s f_{sy} \;/\; \sqrt{3}) \tag{3-15}$$

The action of the longitudinal bars in resisting torsion has been idealized, the actual action is more complex. T_{s2} is usually small compared to T_{s1} and T_c, so it is often ignored.

IYENGER AND RANGAN THEORY[5]

These authors have made an attempt to establish a simple, and at the same time rational, solution to the problem. The analysis is based upon the state of stress existing at the critical point of failure and upon a failure citerion for concrete under compressive tensile stresses. The assumptions of the theory are given in the Chapter 2.

In a beam subjected to combined bending and torsion, there are two possible causes of failure.

(a) Flexure failure by crushing of concrete in the compression zone after yielding of longitudinal steel;

(b) Torsion failure by diagonal splitting.

<u>Flexure failure</u>

At the critical point of failure, let f_1 be the horizontal normal stress (compressive) due to the applied bending moment, and τ be the shear stress due to the applied torque at the same point.

$$\sigma_{1,2} = \frac{f_1}{2}\left[\sqrt{1 + \left(\frac{2\tau}{f_1}\right)^2} \pm 1\right] \tag{3-16}$$

Taking ϕ as the ratio of the applied torque to applied bending moment, designating

$$2\,\frac{\tau}{f_1} = \lambda\phi$$

and

$$\sqrt{1 + (\lambda\phi)^2} = \gamma \tag{3-17}$$

and substituting into Eqn. (3-16) one obtains

$$\sigma_{1,2} = \frac{f_1}{2}(\gamma \pm 1) \qquad \begin{array}{l}(+)\ \text{compressive}\\ (-)\ \text{tensile}\end{array}$$

substituting the values of σ_1, and σ_2 into Eqn. (2-6)

$$\frac{f_1}{f_c} = \left(\frac{f_t}{f_c}\right)^2 \frac{\gamma+1}{(\gamma-1)^2}\left[\sqrt{1 + \left[\left(\frac{\gamma-1}{\gamma+1}\right)\frac{2f_c}{f_t}\right]^2} - 1\right] \tag{3-18}$$

The depth of the compression block of a beam subjected to pure bending and the one under combined bending and torsion is almost the same, so that we can equate the ratio.

$$\frac{f_1}{f_c} = \frac{M_u}{M_{uo}}$$

where M_u = ultimate flextural capacity of a beam under combined bending and torsion;

M_{uo} = flexural capacity of same beam under pure bending.

Hence

$$\frac{M_u}{M_{uo}} = \left(\frac{f_t}{f_c}\right)^2 \frac{\gamma+1}{(\gamma-1)^2} \left[\sqrt{1 + \left\{\left(\frac{\gamma-1}{\gamma+1}\right)^2 \frac{f_c}{f_t}\right\}^2} - 1\right] \tag{3-19}$$

Variation of M_u/M_{uo} with $1/\lambda\phi$ has been drawn (See Fig 3-3).

<u>Determination of $\lambda\phi$</u>

The most critical point for a flexure failure is the extreme fiber in the compression zone of the beam (top of beam) where the effect of shear stresses on the bending stresses is maximum. Torsional shear stresses can be calculated using the elastic theory, when the beam fails in flexure by crushing of concrete as the torsional stresses are in the elastic range and the axial stresses in the transverse hoops are very small. Although the maximum shear stress does not occur at the middle of the shorter side, it is still considered here as this is the critical section for a flexure failure. So the shear stress corresponding to the middle of the shorter sides is considered in the analysis. Then

$$T = \beta_e b^2 h\tau \tag{3-20}$$

where T = applied torque;

β_e = elastic torsional shear stress coefficient corresponding to the middle of the shorter sides of the rectangular beam.

Since the axial stress in the transverse hoops is small in a beam failing in flexure, its contribution to the torque, is neglected.

Assuming a nonlinear stress block for the compression zone of the beam, (see Fig. 3-4),

$$M_u = k_1 k_3 bcf'jd$$

or
$$f' = M_u / k_1 k_3 bcjd \tag{3-21}$$

where f' = maximum stress.

Assuming the stress at the extreme fiber as $k'f'$,

$$f_1 = k'f' = k' \; M_u / k_1 k_3 bcjd \tag{3-22}$$

Now, as previously defined,

$$\lambda\phi = 2\tau / f_1$$

Substituting for τ and f_1 from Eqns. 3-20 and 3-22, and observing that $\phi = T/M_u$ and $k_2 c \simeq d - jd$,

$$\lambda = \frac{2k_1 k_3 bd^2 j(1-j)}{k'\beta_e b^2 hk_2} \tag{3-23}$$

Further, putting $h \simeq 1.2d$, $j = 7/8$, and $k' = 0.9$ (based on the ideal stress-strain curve of concrete corresponding to a maximum concrete strain of 0.3%, Eqn. 3-23 reduces to

$$\lambda = (h/b) \frac{1}{6\beta_e(k_2/k_1k_3)}$$

Taking $k_2/k_1k_3 = 0.525$, and observing that the value of λ is a function of h/b ratio, λ has been tabulated for different values of (h/b). These values are given in Table 3-1. Now using the tabulated values of λ, for a given value of f_c/f_t, the calculation of M_u/M_{uo} with the help of Fig. 3-3 can be easily done.

Torsion failure: This occurs in the case of a large value of ϕ. The critical point for failure in torsion is that point where the influence of bending stresses on shear stresses is minimum. So the critical point for torsion failure is at the level of the neutral axis where the bending stresses are zero. The concrete element at the level of the neutral axis is subjected to a state of simple shear. This results in two principal stresses of equal magnitude and, hence, the problem is the same as that of a beam under pure torsion. Therefore, the torque capacity of a beam failing in torsion, under combined bending and torsion, is assumed equal to its torque capacity under pure torsion.

COLLINS, WALSH, ARCHER AND HALL THEORY[2]

From the experiments carried out by the authors, it seems likely that the ratio of top to bottom steel has a marked effect on the interaction behavior. An attempt has been made to develop equations which would have the merit of simplicity without undue loss of accuracy. Just prior to the failure of any reinforced concrete member sustaining torsion and/or bending, a cracked tensile zone develops on one side of a neutral axis, and a zone of compression on the other. With pure bending the compression zone is normal to the axis of the beam, and the tensile crack forms a plane below the neutral axis. In the presence of torsion, the tensile crack forms a warped surface which intersects the three exterior faces of the beam in a rectangular helix; the compression zone on the fourth face joins the two ends of this helix and is consequently inclined to the axis of the beam.

Most recent investigators have alluded to the "failure mechanism" in describing the behavior of the member as failure ensues and relatively large displacements occur. The opening of the tensile crack permits rotation of the member about an axis in the compression zone, generally referred to as the "compression hinge". The failure with the compression hinge on the top surface is referred to as a mode 1 failure, while a mode 2 failure will indicate that the hinge forms on a side surface. Sometimes the hinge develops on the bottom surface, and this is called a mode 3 failure. The appearance of these failure surfaces is shown in an idealized form in Fig. 3-2.

Mode 1

The total moment of the internal forces about the "compression hinge" is equated to the moment of the external forces. For a mode 1 failure, the only internal forces which have a significant moment about this axis are the forces in the bottom longitudinal steel and the forces in the bottom branches of the transverse steel. As the forces in the vertical leg of the stirrups have small effect, they have been ignored. The level of the bottom branches of the stirrups is assumed to be the same as that of the longitudinal steel.

If the angle between the "compression hinge" and the normal cross-section is θ_1 (see Fig. 3-2), the total moment of the external forces about the compression hinge is

$$M_{ext} = M \cos \theta_1 + T \sin \theta_1$$
$$= T (\cos \theta_1/\psi + \sin \theta_1).$$

In the above, M = bending moment in combined bending and torsion (at failure);

T = torsional strength of reinforced section in combined bending and torsion (at failure);

ψ = ratio T/M.

If the bottom branches of the stirrups were distributed as a continuous layer, the area of such steel intercepted by the failure surface would be

$$(A_s/s).b^2 \tan \theta_1/(b+2h)$$

where A_s = cross sectional area of one leg of transverse steel;

b = width of the rectangular section;

h = height of the rectangular section;

s = spacing of the stirrups in the direction parallel to the longitudinal axis.

However, the stirrups occur at discrete intervals and only an integral number are intercepted. For this reason the total area intercepted may be taken as 0.8 of the above value.

The total moment of the internal forces about the compression hinge is then

$$M_{int} = A_{b1}f_b(h-a_1-x_1)\cos\theta_1 + 0.8\,\frac{A_s f_{st}}{s}\,\frac{b^2\tan\theta_1}{b+2h}\,(h-a_1-x_1)\sin\theta_1 \tag{3-26}$$

where A_{b1} = area of bottom longitudinal steel;

f_b = stress in bottom longitudinal steel;

a_1 = distance of bottom longitudinal steel from the bottom surface of the rectangular section;

x_1 = distance between the top surface of the concrete section and the center line in the compression zone;

f_{st} = stress in bottom transverse steel.

The relationship between transverse steel and longitudinal steel may be expressed by a parameter, z, where

$$z = \frac{0.8A_s f_{st}}{s} \cdot \frac{b}{A_{b1}f_b}\,. \tag{3-27}$$

Equating M_{int} to M_{ext}, and substituting $\infty = h/b$ one obtains

$$T = A_{b1}f_b(h-a_1-x_1)\left(\frac{1 + \frac{z}{1+2\infty}\tan\theta_1^2}{1/\psi + \tan\theta_1}\right) \tag{3-28}$$

The inclination, θ_1, of the hinge will be such as to make the failure torque a minimum. If $dT/d\theta_1$, is equated to zero, it is found that T is a minimum when

$$\tan \theta_1 = -\ 1/\psi + \sqrt{(1/\psi)^2 + (1 + 2\infty)/z} \tag{3-29}$$

When this value is substituted into Eqn. 3-28, the failure torque for a mode 1 failure is obtained as

$$T_1 = M_{uo} \frac{2z}{1 + 2\infty} \left[\sqrt{(1/\psi)^2 + (1 + 2\infty)/z} - (1/\psi)\right] \tag{3-30}$$

where M_{uo} = the ultimate capacity of the member in flexure alone;

$= A_{b1} f_b (h - a_1 - x_1)$;

T_1 = torsional strength of the section in combined bending and torsion (by mode 1 failure).

Mode 2

The compression zone is located along one side (Fig. 3-2) and the compression hinge is at a distance x_2 from the side face. The external bending moment has no component about this hinge axis, but the shear force does exert a moment about this axis. The total moment of the external forces is

$$M_{ext} = V(b/2 - x_2) \sin \theta_2 + T \sin \theta_2$$

$$= T \left(\delta(1 - 2x_2/b) \sin \theta_2 + \sin \theta_2\right) \tag{3-31}$$

where $\delta = Vb/2T$;

V = shear force (at failure).

The internal moment is mainly provided by the longitudinal steel near the side face remote from the hinge and by the vertical legs of the stirrups on that face. The same assumptions are made as in mode 1 and the internal moment is

$$M_{int} = A_{v1} f_v (b-a_2-x_2) \cos \theta_2 + \frac{0.8 A_s f_{st}}{s} \frac{h^2 \tan \theta_2}{h + 2b} (b-a_2-x_2) \sin \theta_2. \tag{3-32}$$

By the same procedure as used for mode 1, the failure torque for a mode 2 is found to be

$$T_2 \left[1 + \delta\left(1 - \frac{2x_2}{2}\right)\right] = 2A_{v1} f_v (b-a_2-x_2) \sqrt{\frac{0.8 A_s f_{st} h}{A_{v1} f_v s(1-2/\infty)}} \tag{3-33}$$

where A_{v1} = cross sectional area of longitudinal steel located near one vertical side of the beam;

f_v = stress in steel A_{v1};

a_2 = distance of the longitudinal steel from the vertical face of the section;

x_2 = distance between the vertical side of the section and the center line of the compression zone.

The ratio of the forces in the top and bottom longitudinal steels is 'R_1'.

$R_1 = A_{t1} f_{t1} / A_{b1} f_b$

and A_{t1} = area of top longitudinal steel;

f_{t1} = stress in top longitudinal steel.

For most beams, the forces in the side longitudinal steel may be taken as

$$A_{v1}f_v = \frac{1}{2}(A_{b1}f_b + A_{t1}f_{t1})$$

or

$$\frac{A_{v1}f_v}{A_{b1}f_b} = \frac{1 + R_1}{2}$$

Let

$$\frac{h-a_1-x_1}{b-a_2-x_2} \simeq \frac{h-a_1}{b-a_2} = \beta$$

as x_1 is nearly equal to x_2 .

The expression for the failure torque in the second mode takes the form

$$T_2 = M_{uo}\,\frac{1}{1 + \delta} \cdot \frac{\infty}{\beta}\sqrt{\frac{2(1 + R_1)z}{2 + \infty}} \qquad (3\text{-}34)$$

Mode 3

In a mode 3 failure the compression zone forms along the bottom face of the beam (Fig. 3-2), the face on which bending moment alone would cause tension. The analysis is very similiar to that of mode 1, except that the bending moment now opposes the rotation occuring during failure in this mode. Assuming that the cover to the top and bottom longitudinal steel is the same, the following expression for the failure torque is obtained:

$$T_3 = M_{uo}\,\frac{2z}{1 + 2\infty}\left(\sqrt{(1/\psi)^2 + \frac{(1 + 2\infty)R_1}{z}} + (1/\psi)\right) \qquad (3\text{-}35)$$

For a given value of ψ and known beam dimensions, the three torques T_1, T_2 and T_3 can be computed from Eqns. 3-30, 3-34 and 3-35. The smallest of these values gives the twisting moment at failure, for beams in which both the longitudinal and the transverse steel yield.

The effect of compression reinforcement has been ignored in the analysis, as it has little or no effect on the solution. Under certain ratios of load and certain arrangements of steel, failure may occur before both the longitudinal and the transverse steel have yielded. The range of applicability of this theory excludes the above case.

4. REINFORCED CONCRETE SUBJECTED TO COMBINED TORSION, BENDING AND SHEAR

INTRODUCTION

There is a widespread occurance of members subjected to torsion, and in almost every case torsion occurs in combination with bending and shearing resultants. Thus, the problem of combined loading, while being the most difficult to analyze, is, nevertheless, the most important. The importance of the interaction of shear on torsional strength is now reflected in the 1971 ACI Code[1]. A few attempts have been made to introduce a rational theory for concrete members under combined loadings. Although the methods presented are probably not the final solution of this complicated problem, the trends shown may be useful to the design engineer as a guide until improved methods are available. In this chapter the combined torsion, bending and shear analysis by Lessig[6] and the interaction surface as plotted by Hsu[14] are presented.

LESSIG'S THEORY[6]

According to Lessig there are two possible modes of failure for rectangular reinforced concrete beams under combined torsion, bending and shear loads. When torsion and bending predominate, the neutral axis of the failure surface that occurs, intersects both the vertical sides of the beam and the failure is known as the first mode of failure. In the case of the rectangular section subjected to torsion and shear predominantly the neutral axis of the failure surface intersects both horizontal sides of the beam. This is called the second mode of failure.

Lessig made the following assumptions:

1). The tension capacity of the concrete in the failure surface is zero.

2). When the plastic hinge is formed, all the reinforcing steel intersecting the tension part of the failure surface reaches its yield point.

3). The transverse reinforcement is uniformly distributed over the beam.

4). The concrete stress in the compression zone of the failure surface reaches the ultimate strength as in the pure flexure compression.

5). No external loads are applied within the section in which the beam fails.

First Mode

The external and the internal moments with respect to the neutral axis, acting on a plane perpendicular to the neutral axis are equated.

$$M_{ext} = Mb/L + Tc/L$$

where M = bending moment;

T = torsional moment;

b,c and L are the beam dimensions (see Fig. 4-3).

The direct shear V, does not contribute to the external moment because it lies in the plane of the axis of rotation.

$$M_{int} = M_c + M_l + M_{hs} + M_{vs}.$$

M_c = moment, due to normal compression forces in the concrete.

M_l = moment, due to the forces in the bottom longitudinal steel.

M_{hs} = moment, due to the forces in the horizontal stirrups.

M_{vs} = moment, due to the forces in the vertical stirrups.

All these terms are evaluated and summed up to get the internal moment.

1). $M_c = f_c Z$

where f_c = compressive strength of concrete in combined bending and torsion and has been assumed approximately equal to the ultimate strength of concrete in flexure;

Z = static moment of compression zone about the neutral axis.

Referring to Fig. 4-3,

$$Z = \int_A x \, dy \left(\frac{x}{2} \cdot \frac{\sqrt{c^2 + b^2}}{L} \right)$$

and

$$x = x_2 + (x_1 - x_2) \frac{y}{\sqrt{c^2 + b^2}} = \frac{x_2 \sqrt{c^2 + b^2} + (x_1 - x_2) y}{\sqrt{c^2 + b^2}} \qquad (4\text{-}1)$$

Therefore,

$$Z = \frac{1}{2L\sqrt{c^2 + b^2}} \int_0^{\sqrt{c^2+b^2}} \left(x_2 \sqrt{c^2 + b^2} + (x_1 - x_2) y \right)^2 dy$$

$$= \frac{c^2 + b^2}{6L} (x_1^2 + x_1 x_2 + x_2^2)$$

L, c, x_1, x_2 and y are shown in Fig 4-2 and 4-3.

Hence

$$M_c = \frac{f_c(c^2 + b^2)}{6L} (x_1^2 + x_2^2 + x_1 x_2) \qquad (4\text{-}2)$$

2). $$M_l = A_{bl} f_b (d_o - \frac{x_1+x_2}{2}) \frac{b}{L} \qquad (4\text{-}3)$$

where A_{bl} = cross-sectional area of all bottom longitudinal steel;

f_b = tensile yielding stress of the longitudinal steel;

d_o is shown in Fig 4-2.

3). $$M_{hs} = f_{sy} A_s \frac{c\theta_1}{s} (h-d_3- \frac{x_1+x_2}{2}) \frac{c}{L} \qquad (4\text{-}4)$$

where f_{sy} = yield stress of stirrup;

A_s = cross sectional area of one leg of stirrup;

s = spacing of stirrups;

d_3 = bottom cover for steel.

4). Assume the angle β_1 between the cracks on both vertical faces of the beam with respect to the vertical axis are equal,

so $$\cot \beta_1 = \frac{c(1-\theta_1)}{2h - (x_1+x_2)}$$

The lever arm of the total stirrups on the front face is

$$\left[(\frac{h-x_2}{2} \cot \beta_1) \frac{b}{c} - b_1 \right] \frac{c}{L}$$

Then the resisting moment M_{vc}^f, due to the total stirrups on the front face is

$$M^f_{vc} = f_{sy} \frac{A_s}{s} (h-x_2) \cot \beta_1 \left(\frac{h-x_2}{2} \cot \beta_1 - b_1\right) \frac{c}{L}$$

Similarly, the resisting moment, M^r_{vc}, due to the total stirrups on the rear face is found and added to M^f_{vc}, to get M_{vc}.

$$M_{vc} = f_{sy} \frac{A_s}{s} \frac{c^2(1-\theta_1)}{L} \left[\frac{b}{2} (1-\theta_1) \frac{(h-x_1)^2 + (h-x_2)^2}{(2h-x_1-x_2)^2} - b_1\right] \tag{4-5}$$

Summing up equations 4-2 to 4-5 one obtains

$$M.b + T.c = \frac{f_c(c^2 + b^2)}{6} (x_1^2+x_1x_2+x_2^2) + f_bA_{b1}(d_o - \frac{x_1+x_2}{2})b$$

$$+ \frac{f_{sy}A_sc^2(1-\theta_1)}{s} \left[\frac{b}{2} (1-\theta_1) \frac{(h-x_1)^2 + (h-x_2)^2}{(2h-x_1-x_2)} - b_1\right]$$

$$+ f_{sy}A_s \frac{c^2\theta_1}{s} (h-d_3 - \frac{x_1+x_2}{2}) \tag{4-6}$$

The parameters x_1 and x_2 are symmetrical in Eqn. 4-6. Therefore differentiating it first for term x_1 and then for x_2 yields identical results if $x_1=x_2$. The minimum value of torque is obtained when it is assumed that $x_1=x_2$. Substituting x for x_1 and x_2 into Eqn 4-6 and differentiating it for x one obtains

$$f_c(c^2+b^2)x - f_bA_{b1}b - f_{sy}A_s \frac{c^2\theta_1}{s} = 0 \tag{4-7}$$

This is the equation of the projections of all forces acting in the direction normal to the plane of the compression zone. From this equation one can determine x.

$$x = \frac{f_b A_{b1}}{f_c(c^2+b^2)} \left(b+p\theta_1 \frac{c^2}{h}\right) \tag{4-8}$$

in which $p = \frac{f_{sy} A_s h}{f_b A_{b1} s}$.

Let $\phi = T/M$

and substituting x, p and ϕ into Eqn 4-6, noting that $x_1 = x_2 = x$, one obtains

$$M(1 + \frac{c}{b}\phi) = T(1/\phi + c/b) = f_b A_{b1} \left[d_o - \frac{x}{2} + \frac{pc^2}{bh}\left(\theta_1(h-d_1 - \frac{x}{2})\right)\right.$$
$$\left. + \frac{b}{4}(1-\theta_1)(1-\theta_1 - \frac{4b_1}{b})\right] \tag{4-9}$$

Based on tests, Lessig suggests that the failure crack on three sides of beam can have the same angle of inclination, so θ_1 can be

$$\theta_1 = \frac{b}{2h+b} . \tag{4-10}$$

Introducing the designations

$$d_j = d_o - x/2$$

and $$J = \theta_1(h-d_1-x/2) + \frac{b}{4}(1-\theta_1)(1-\theta_1 - \frac{4b_1}{b}) \tag{4-11}$$

and substituting into Eqn. 4-9 one gets a simple equation

$$T = \phi M = A_{b1} f_b \frac{d_j + pJc^2/bh}{1/\phi + c/b} \tag{4-12}$$

Theoretically the least favorable direction of the neutral axis must correspond to minimum value of T or M for a given ratio ϕ. The value c, corresponding to the theoretical minimum value of T or M, can be derived by setting dT/dc or dM/dc equal to zero, while d_j and J, which are affected relatively little by x, are assumed constant. The ratio c to b is given as

$$c/b = -1/\phi + \sqrt{\frac{1}{\phi^2} + \frac{d_j}{pJ}\frac{h}{b}} \qquad (4\text{-}13)$$

The value of c from this formula is usually higher than the actual value, because of the difference between the assumptions used and the actual condition of the failure surface. Through experimental observation, Lessig suggests the following formulas to determine c_{max} and J.

$$c_{max} \leq 2h + b \qquad (4\text{-}14)$$

and $$J = \theta_1(h - d_1 - x/2) \simeq \theta_1 d_j \qquad (4\text{-}15)$$

Substituting Eqn. 4-15 in Eqns. 4-12 and 4-13, one obtains

$$T = \phi M = A_{b1} f_b \frac{1 + p\theta_1 c^2/bh}{1/\phi + c/b}\, d_j \qquad (4\text{-}16)$$

and $$c/b = -\,1/\phi + \sqrt{\frac{1}{\phi^2} + \frac{1}{p\theta_1}\frac{h}{b}} \qquad (4\text{-}17)$$

For solving practical problems Lessig suggests the following procedure.

1). calculate θ_1 and c from Eqns. 4-10 and 4-14 or 4-17.

2). Determine x and d_j from Eqns. 4-8 and 4-11.

3). Compute values of M and T from Eqn. 4-16.

Second Mode

When a rectangular section is subjected to torsion and shear, the neutral axis of the failure surface intersects both horizontal sides of the beam. The final equation for the second mode of failure can be derived by the procedures used in deriving the first mode of failure.

$$T = A_{v1} f_v \frac{b_j h}{c_2} \frac{1 + p_2 \theta_2 c_2^2 / bh}{1 + \psi} \tag{4-18}$$

and $\psi = Vb/2T$;

V = transverse shear ;

A_{v1} = the area of longitudinal steel located near one vertical side of the beam ;

$$b_j = (b - b_2 - x/2) ;$$

$$c_2 = h \sqrt{\frac{1}{p_2 \theta_2} \frac{b}{h}} \leq 2b + h ;$$

$$\theta_2 = h/(2b+h) ;$$

$$p_2 = \frac{A_s f_{sy} b}{A_{v1} f_v s} ;$$

$$x = \frac{A_{v1} f_v (h + p_2 \theta_2 c_2{}^2 / b)}{f_c (c_2^2 + h^2)} .$$

The capacity reduction factor for torsion be taken as 0.85. One can calculate the ultimate torsional capacity of the beam by choosing the lower value of the torque from those obtained, from these two different modes of failure. In the practical design, the yield stress of stirrups are multiplied by 0.8, which takes into consideration that transverse bars are placed at certain intervals, contrary to the assumption of constant intensity of transverse steel used while deriving the torque equation.

INTERACTION SURFACE[14]

The effect of simultaneous application of two different types of forces on the strength of a member can often be expressed by an interaction curve. For example, the interaction of torsion and bending can be represented on a rectangular co-ordinate system. One axis represents torsion and the other bending. To study combinations of three different types of forces, however, the interaction must be expressed in a three dimensional rectangular co-ordinate system. For combined torsion shear and bending, each axis represents one type of traction. The strength of a beam subjected to a certain magnitude of torsion, shear and bending is then represented by a point on an interaction surface.

The torsion-bending, shear-bending and torsion-shear interaction curves are actually special cases of the torsion-shear-bending interaction surface. Two of these three special interaction curves, torsion-bending and shear-bending, can be obtained directly from tests. However, the torsion-shear interaction curve can not be obtained in this way because it is impossible to obtain a constant shear in a finite length of a member, without the simultaneous presence of bending moment.

Torsion-Bending Interaction Curve

An extensive study of combined torsion and bending without shear in reinforced concrete beams has been reported by Nylander[14] and Ramakrishnan and Vijayarangan[14]. Ultimate strengths of 53 beams of square and rectangular sections were evaluated. Fig 4-4 shows a nondimensional presentation of these data. The ordinate of this diagram shows the ratio of torque T, to pure torsional strength T_{uo}. On the abscissa, the ratio of the bending moment M to the pure flexural strength M_{uo} is shown. The figure shows that T/T_{uo} increases slightly with increasing values of M/M_{uo} upto about 0.5 M/M_{uo}. This increase is hardly noticeable. Beyond $M/M_{uo} = 0.5$, T/T_{uo} decreases significantly with the increasing M/M_{uo}.

Mathematical expression for these lines are:

$$
\begin{aligned}
T/T_{uo} &= 1 && \text{for} \quad M/M_{uo} \leq 0.5 \\
T/T_{uo} &= 1.7 - 1.4M/M_{uo} && \text{for} \quad M/M_{uo} \leq 1 \\
M/M_{uo} &= 1 && \text{for} \quad T/T_{uo} \leq 0.3
\end{aligned}
\qquad (4\text{-}19)
$$

Values obtained from these equations are believed to be satisfactory for use in design.

Shear-Bending Interaction Curve

The next step in the construction of the interaction surface is to define the interaction curve of combined shear and bending. An interaction equation is available in the Joint Committee Report[14] and has been incorporated into the 1963 ACI Code. It is based on diagonal cracking instead of failure, because the post-cracking capacity has been found unreliable in certain cases. For beams without web reinforcement, the Joint Committee interaction equation is:

$$V = 1.9\sqrt{f_c}\ bd_o + 2500\ \frac{p_3 V b d_o^2}{M} \leq 3.5\sqrt{f_c}\ bd_o \qquad (4\text{-}20)$$

$$\text{for } M \geq Vd_o$$

where V = total shear force at diagonal tension cracking on the section considered;

b = width of cross-section;

d_o = distance of extreme compression fiber to centroid of tension reinforcement;

p_3 = ratio of area of tension reinforcement to effective area of concrete, A_s/bd;

f_c = compressive strength of concrete;

V/M = ratio of shear to moment at section considered.

The Joint Committee Report limits the maximum value of V to $3.5\sqrt{f_c}\ bd_o$. The shear strength based on diagonal cracking in pure shear, V_{co}, can be defined as $3.5\sqrt{f_c}\ bd_o$. Putting p_3 in terms of the dimensions of the cross section in equation 4-20, rearranging the terms and utilizing V_{co} and M_{uo}, Eqn. 4-20 can be changed into the form:

$$\frac{V}{V_{co}} = \frac{0.543\ M/M_{uo}}{M/M_{uo} - \dfrac{2500}{f_y(1-a/2d_o)}} \qquad (4\text{-}21)$$

where f_y = yield strength of tension reinforcement.

With $(1 - a/2d_o) = 7/8$, the interaction curve has been plotted in Fig. 4-5. This interaction curve is also a function of f_y. The shape of the curve for different values of f_y is almost identical.

Torsion-Bending and Shear Interaction Surface

Three mutually perpendicular axis representing parameters, T/T_{uo}, V/V_{co} and M/M_{uo} are shown in Fig. 4-6. In the plane formed by the torsion and bending axis, the interaction curve shown in Fig. 4-4 has been drawn. Similarly, in the plane formed by shear-bending axis, the interaction curve shown in Fig. 4-5 can be plotted.

From Fig 4-6 it is clear that for each value of M/M_{uo} an interaction curve between torsion and shear exists. It is possible to express the series of torsion-shear interaction curves by the equation representing the interaction surface.

$$(T/T_{ub})^m + (V/V_{cb})^n = 1 \tag{4-22}$$

In this equation, m and n are exponents to be determined from the tests. T_{ub} is the ultimate torque in combined torsion and bending. V_{cb} is the shear strength based on cracking in combined shear and bending, T_{ub} and V_{cb} can be derived from the boundary conditions.

If $\quad V = 0 \quad T_{ub} = T_{uo} \quad$ for $\quad M/M_{uo} \leq 0.5$

$$T_{ub} = T_{uo}\ (1.7 - 1.4\ M/M_{uo})$$

$$\text{for } 0.5 < M/M_{uo} \leq 1.0$$

If T = 0, the interaction surface becomes identical with the shear-bending interaction curve. In this case, Eqn 4-22 reduces to either Eqn. 4-20 or 4-21. Since the latter equation is much simpler, it is chosen for design purposes. Comparing Eqn. 4-22 with Eqn. 4-20 results in:

$$V_{cb} = 1.9\sqrt{f_c}\ bd_o + 2500\ p_3 V b d_o^2/M$$

where the previous limits and definations still apply. The surface is plotted in Fig. 4-6. Nylander[14] suggested a conservative interaction surface for design defined by the equation:

$$(T/T_{ub})^2 + (V/V_{cb})^1 = 1 \text{ for } M/M_{uo} \leq 0.5$$

Farmer and Ersoy[14] suggested an interaction surface, which is very useful for design purposes for case where $0.5 \leq M/M_{uo} \leq 1.0$

$$(T/T_{ub})^2 + (V/V_{cb})^2 = 1 \qquad \text{for} \qquad 0.5 \leq M/M_{uo} \leq 1.0.$$

Substituting the values, one obtains

$$\left(\frac{T}{T_{uo}(1.7 - 1.4\ M/M_{uo})}\right)^2 + (V/V_{cb})^2 = 1$$

$$\text{for } 0.5 \leq M/M_{uo} \leq 1.0$$

This surface is plotted in Fig. 4-7.

5. DESIGN EXAMPLE

In this chapter the design of a concrete beam subjected to combined torsion, shear and bending has been described using two different methods, namely, the ACI design approach and Lessig's method. The minimum requirements for ultimate strength design of members subjected to combined torsion and shear have been recently proposed in the ACI Building Code, ACI 318-71.[1] In this method, the steel requirement for bending moment has been separately evaluated by the conventional methods and then added to the steel required to resist the shear and torsion.

Lessig's method is the finest approach to this complicated problem. The ratio of torque to bending moment being used in the design shows that Lessig has actually considered the combined action of these two tractions, whereas, ACI 318-71 still has not provided any interaction equation for torsion and bending. Lessig is the one who has derived the design equations taking into consideration the combined effect of torsion, shear and bending. The writer therefore feels that Lessig's method will give an economical section as compared to the other method.

In the example, a fixed end beam (Fig 5-1), with a $4\frac{1}{2}'$ cantilevered slab supports a live load of 80 psf. The torsional moment, bending moment and shear diagrams are shown in Fig. 5-2. The problem is to design the reinforcement for 15"x24" rectangular section beam. Concrete of $f_c = 4$ ksi and steel of $f_y = 60$ ksi have been used. The beam has been designed at a section d away from the fixed end only. As for shear, the capacity reduction factor for torsion is taken as 0.85.

ACI METHOD OF DESIGN[1]

(a) Allowable torsion stress without considering torsion effect:

$$v_{tu} = 1.5\sqrt{f_c} = 1.5\ \sqrt{4000} = \underline{95\ \text{psi}}$$

(b) Calculate V_{tu}:

$$v_{tu} = 3\ T_u/\phi\sum x^2 y$$

(For slab portion) y = 3 x slab thickness = 3x6 = 18"

T_u = 41.4 ft-kip and $\phi = 0.85$

$\sum x^2 y = (15)^2 \text{x } 24 + (6)^2 \text{ x } 18 = 6048 \text{ in.}^3$

Therefore

$$v_{tu} = \frac{3 \times 41.4 \times 12000}{0.85 \times 6048} = \underline{290\ \text{psi}} > \underline{95\ \text{psi}}$$

So, torsion reinforcement must be considered

(c) Ultimate shear stress, v_u has been calculated.

$$d_o = 24 - 2\frac{1}{2} = 21\frac{1}{2}''.$$

$$v_u = V_u/\phi b d_o = 22000/0.85 \times 15 \times 21.5 = \underline{80.3}\ \text{psi}$$

(d) Maximum v_{tu} allowed, is calculated

$$\text{Max.} \quad v_{tu} = \frac{12\sqrt{f_c}}{\sqrt{1 + \left(\frac{1.2\ v_u}{v_{tu}}\right)^2}}$$

$$= \frac{12\sqrt{4000}}{\sqrt{1 + \left(\frac{1.2 \text{ x } 80.3}{290}\right)^2}} = \underline{720 \text{ psi}} > \underline{290 \text{ psi}}$$

(e) Torsion stress carried by concrete v_{tc} is calculated.

$$v_{tc} = \frac{2.4\sqrt{f_c}}{\sqrt{1 + \left(\frac{1.2 \ v_u}{v_{tu}}\right)^2}} = \underline{144 \text{ psi}} < \underline{290 \text{ psi}}$$

Therefore torsion reinforcement is required.

(f) Shear stress carried by concrete is calculated.

$$v_c = \frac{2\sqrt{f_c}}{\sqrt{1 + \left(\frac{v_{tu}}{1.2 \ v_u}\right)^2}}$$

$$= \frac{2\sqrt{4000}}{\sqrt{1 + \left(\frac{290}{1.2 \text{ x } 80.3}\right)^2}} = 39.9 \text{ psi} < 80.3 \text{ psi}$$

(g) Shear reinforcement

$$V_w = (v_u - v_c) \ bd_o$$

where V_w is the shear taken by the stirrups

$$V_w = (80.3 - 39.9) \text{ x } \frac{15 \text{ x } 21.5}{1000} = 13 \text{ kips}$$

$$A_v/s = V_w/d_o f_y = 13/21.5 \text{ x } 60 = .0101 \text{ sq. in./in.}$$

$$s_{max} = d_o/2 = 21.5/2 = 10.75" \text{ nor } > 24"$$

(h) Torsional web reinforcement:

$$A_k = \frac{(v_{tu} - v_{tc})s\sum x^2y}{3\infty_t x_1 y_1 f_y}$$

= area of web reinforcement

$$x_1 = 15 - (2 \times 1.75) = 11.5"$$

$$y_1 = 24 - (2 \times 2) = 20"$$

$$\infty_t = (0.66 + 0.33\ y_1/x_1) \text{ but not } > 1.5$$

$$= 1.235$$

$$\frac{A_k}{s} = \frac{(290 - 144)\ 6048}{3 \times 1.235 \times 11.5 \times 20 \times 60000} = .01725 \text{ sq. in./in.}$$

$$\text{Max } s = \frac{x_1 + y_1}{4} = \frac{11.5 + 20}{4} = 7.9" < 12"$$

(i) Total web reinforcement:

Closed stirrups have been used for shear and torsional web reinforcement.

Try No. 3 Stirrups:

$$A_k + \frac{1}{2} A_v = 0.11 \text{ sq. in.}$$

$$\frac{A_k}{s} + 0.5\frac{A_v}{s} = 0.01725 + 0.5(0.0101) = .0223 \text{ sq. in.}$$

$s = 0.11/0.0223 = 4.92'' < 7.9''$

Try No. 3 stirrups @ $4\frac{1}{2}''$ throughout.

(j) Check minimum web steel:

$$\text{Closed stirrups} = 2\,A_k + A_v = 50\ b\ s\ /\ f_y$$

$$= \frac{50 \times 15 \times 4.5}{60000} = \underline{0.0563}$$

$2\,A_k + A_v = 2 \times 0.11 = 0.22 > 0.0563$ OK

<u>use No 3 stirrups at 4.5" centers</u>

(k) Torsional longitudinal reinforcement:

$$A_l = 2\,A_k \frac{(x_1 + y_1)}{s}$$

or by:

$$A_l = \left(\frac{400 \times s}{f_y}\,\frac{v_{tu}}{(v_{tu}+v_u)} - 2\,A_k\right)\left(\frac{x_1+y_1}{s}\right)$$

whichever is greater.

$2A_k = 2 \times 0.01725 \times 4.5 = 0.155$ sq. in.

$A_l = 0.155(11.5 + 20)/4.5 = 1.08$ sq. in.

or $$A_l = \left(\frac{400 \times 15 \times 4.5}{60000}\,\frac{290}{290 + 80.3} - 0.155\right)\left(\frac{11.5 + 20}{4.5}\right)$$

$= 1.38$ sq. in.

Maximum spacing of longitudinal torsion steel is 12"

(l) Before choosing bar dimensions, the steel requirement for the bending moment has been calculated, and bars to resist both torsion and bending moment are choosen thereafter

$$M_u / \phi_{cr} f_c b d_o^2$$

$$= 84 \times 12 / 0.9 \times 4 \times 15 \times (21.5)^2 = 0.0404$$

From Table - 2 Section 8-28 PCA workbook.[12]

$\omega = 0.0414$ for $M_u/\phi_{cr} f_c bd_o^2 = 0.0404$.

and $p = \omega f_c/f_y = 0.414 \times 4/60 = .00276$

From Table - 1 Section 8-27 PCA workbook.[12]

$p_{max} = 0.75 p_b = .0214 > .00276$ OK

No compression steel needed

$A_s = 0.00276 \times 15 \times 21.5 = 0.89$ sq. in.

For complete details of the section refer to Fig 5-3.

The area of longitudinal steel at bottom should be

$$\frac{1.38}{3} + 0.89 = 1.32 \text{ sq. in.}$$

The total longitudinal steel area required is 1.38 + .89 = 2.27 in.

Use 2 #7 and 2 #3 bars at bottom and 4 #4 bars one at each top corners and midpoints of the steel cage and 1 #3 bar at top.

The total longitudinal steel provided is 2.33 sq. in.

LESSIG'S METHOD[6]

First Mode

According to Lessig the most economical value of p is:

$$p = \frac{1}{1 + 2\sqrt{\theta_1}\ /\phi}\ \frac{h}{b}$$

and $\theta_1 = b/(2h + b) = 15/(2 \times 24.0 + 15) = .238$

$$\phi = T/M = 41.4/84 = .492$$

$$p = \frac{1}{1 + \frac{2\sqrt{.238}}{.492}}\ \frac{24.0}{15} = .539$$

$$c = b\sqrt{\frac{1}{\phi^2} + \frac{1}{p\theta_1}\ \frac{h}{b}} - \frac{b}{\phi}$$

$$= 15\sqrt{\frac{1}{(.492)^2} + \frac{24.0}{.539 \times .238 \times 15}} - \frac{15}{.492}$$

$$= 30.7 \text{ in.} < (24.0)\ 2 + 15.$$

By trial and error.

Assume $A_{b1} = 1.39$ sq. in.

$$x = \left(\frac{f_b A_{b1}}{f_c(c^2 + b^2)}\right)(b + p\theta_1 c^2/h)$$

$$= \frac{60000 \times 1.39}{0.85 \times 4000 \times (940 + 225)}\ (15 + .539 \times .238 \times 940/24)$$

$$= .422 \text{ in.}$$

$$d_j = 21.5 - .422/2 = 21.29 \text{ in.}$$

$$A_{b1} = \frac{T(1/\phi + c/b)}{\phi_{cr} f_b d_j (1 + p\theta_1 c^2/bh)}$$

$$= \frac{41.4 \times 12 \times 1000\ (1/.492 + 30.7/15)}{0.85 \times 60000 \times 21.29\ (1 + .539 \times .238 \times \frac{940}{24 \times 15})}$$

$$= 1.39 \text{ sq. in. OK}$$

$$\frac{A_s}{s} = \frac{p \times A_{b1}}{0.8 \times h} = \frac{.539 \times 1.39}{0.8 \times 24.0} = .0390 \text{ in.}^2/\text{in.}$$

Try No 4 stirrups:

$$A_s = 0.20 \text{ sq. in.}$$

$$S = A_s/0.0390 = 0.20/0.0390 = 5.14''$$

Try No 4 stirrups @ 5".

Second Mode

$$\theta_2 = h/(2b + h) = 24/(2 \times 15 + 24) = 0.445$$

$$\psi = Vb/2T = \frac{22 \times 15}{2 \times 41.4 \times 12} = 0.332$$

By trial and error.

Assume $A_{v1} = 0.74$ sq. in.

$$p_2 = 0.8 f_{sy} A_s b \ / \ A_{v1} f_v s$$

$$= \frac{0.8 \times 0.0390 \times 15}{0.74} = 0.633$$

$$c_2 = h\sqrt{\frac{b}{p_2 \theta_2\ h}}$$

$$= 24.0\sqrt{\frac{15}{.633 \times .445 \times 24}}$$

$$= 35.8 \text{ in.}$$

$$x = \frac{f_v A_{v1}(h + p_2\theta_2 c_2^2/b)}{f_c\left(c_2^2 + h^2\right)}$$

$$= \frac{60000 \times 0.9 \times (24.0 + .633 \times .445 \times 1280/15)}{0.85 \times 4000 \times (1280 + 576)}$$

$$= .337 \text{ in.}$$

$$b_j = 15 - 1.75 - \frac{.337}{2} = 13.08 \text{ in.}$$

$$A_{v1} = \frac{T}{\phi cr f_v b_j}\,\frac{c_2}{h}\,\frac{1 + \psi}{(1+p_2\theta_2 c_2^2/bh)}$$

$$= \frac{41.4 \times 12000 \quad 35.8 \quad 1 + .332}{.85 \times 60000 \times 13.08 \times 24\left(1 + \frac{.633 \times .445 \times 1280}{15 \times 24}\right)}$$

$$= 0.74 \text{ sq in OK}$$

The details of the section are shown in the Fig. 5-4. Use 2 #7 and 2 #3 bars at bottom and 4 #3 bars one at each top corner and the two midpoints of the steel cage. The total longitudinal steel provided is 1.86 sq. in.

COMPARISON

1). ACI method

Total longitudinal steel area provided = 2.33 sq. in.

Volume of longitudinal steel per foot = $2.33 \times 12 = 27.96 \text{ in}^3$.

$$\text{Volume of stirrups} = \frac{(20 + 11.5) \times 0.11 \times 12}{4\frac{1}{2}} = 9.25 \text{ in}^3.$$

Total volume of steel = $27.96 + 9.25 = 37.21 \text{ in}^3$.

2). Lessig's method

Total longitudinal steel area provided = 1.86 sq. in.

Volume of longitudinal steel per foot = $1.86 \times 12 = 22.32 \text{ in}^3$.

$$\text{Volume of stirrups per foot} = \frac{31.5 \times 0.2 \times 12}{5} = 15.25 \text{ in}^3.$$

Total volume of steel = $22.32 + 15.25 = 37.67 \text{ in}^3$.

The volume of steel required is almost the same in both the methods of analysis.

6. CONCLUSIONS

1). The classical elastic, plastic and semi-plastic theories employ the maximum tensile stress theory to predict the failure torque of a beam, subjected to pure torsion. Hsu's theory is based on the contention that failure is reached when the tensile stress induced by a 45° bending component of torque on the sider face of the beam reaches the modulus of rupture. Navaratnarajah assumes that the concrete behaves elastically only up to 80% of the ultimate torque, after which microcracks occur.

2). In the case of reinforced concrete beams subjected to pure torsion, the longitudinal or transverse reinforcement acting alone provide little additional strength beyond the capacity of the concrete. The theories of Rauch, Anderson and Cowan are based on the elastic concept, whereas Ernest's theory is based on the ultimate strength analysis. The constant k used for evaluating the contribution of steel to the total torque capacity differs in all the above theories. Iyenger and Rangan's theory assumes the semi-plastic behavior of concrete.

3). The reinforced concrete beam under combined torsion and bending is explained by theories, all of which are based on ultimate strength analysis. Pandit and Warwaruk consider that the main effect of torsion is to shift the zone of flexure compression toward the center of the beam. Collins et al have shown that the ratio of top to bottom steel has a marked effect on the interaction behavior. There may be a slight increase in the torsional strength of the beam under combined bending and torsion action, which drops with increase of flexure moment.

4). Lessig's theory for torsion, shear and flexure is a good method of analysis out of the ones available until now. Lessig has considered the moment vectors along only one axis, which is to some extent inadequate. Moreover Lessig's theory is not applicable to the special condition of pure torsion. The theory put forward by Collins et al can be satisfactorily used for the combined torsion, shear and bending case. The ACI Code does not provide the designer the interaction behavior of torsion and flexure. The interaction surface as plotted by Hsu for the combined tractions provides at a glance how much amount of each traction a particular beam can withstand.

5). The method by Lessig and the ACI method give almost equal volumes of steel. The fact that there are no interaction equations for the combined torsion and bending should have resulted in somewhat higher steel percentages, in the latter method. In the design example presented, the steel requirement in Lessig's method is controlled by the minimum requirements given by the ACI method. The reason for this is that the torque to be resisted by the beam is too low.

Table 1-1

Summary of Torsion Theories for Plain Concrete.

	Type of theory	Assumption on stress strain curve for concrete in torsion	Circular section	Rect section
	Elastic	Linearly elastic	$\tau_1 = \frac{16T}{\pi D^3}$	$\tau_1 = \frac{T}{K_e b^2 h}$
Semi Plastic	Miyamoto[7]	Second degree Parabolic	$\tau_1 = \frac{14T}{\pi D^3}$	Not given
Semi Plastic	Turner & Davis[16]	$f_o = 244\text{-}450\text{X}10^6 (0.000067 - \varepsilon_t)^{1.5}$	$\tau_1 = \frac{13.6T}{\pi D^3}$	$\tau_1 = \frac{0.854(\frac{PD}{4A})^* T}{K_e b^2 h}$
Semi Plastic	Marshall & Tembe[16]	$\varepsilon_t = \frac{(f_o)^m}{E}$, m=1.1 to 1.6	$\tau_1 = \frac{(12+\frac{4}{m})T}{\pi D^3}$	Not given
	Plastic	Fully plastic	$\tau_1 = \frac{12T}{\pi D^3}$	$\tau_1 = \frac{T}{K_p b^2 h}$

Here:

Values of K_e, K_p depend on the ratio $\frac{h}{b}$ - (see Fig. 1-2)

$(\frac{PD}{4A})$ is the shape factor;

*0.854 accounts for the effect of plasticity;

f_o = stress in concrete;

P = Perimenter;

D = inscribed circle diameter;

A = Area of cross section;

ε_t = Strain of concrete.

TABLE 2-1

Values of K.

Theory	Circular section with Spirals	Rectangular section with ties
Rauch	2.83	2.0
Anderson	2.83	1.33
Cowan	2.83	1.60

TABLE 3-1

Values of Coefficient β_s, λ, ϕ_t and ∞ for Rectangular Beams.

h/b	1	1.2	1.5	2.0	2.5	3.0	4.0	5.0	10.0
β_s	0.271	0.285	0.310	0.332	0.346	0.356	0.370	0.379	0.400
λ	1.528	1.640	1.762	2.060	2.460	2.690	3.355	4.060	7.600
ϕ_t	0.865	0.804	0.748	0.640	0.55	0.490	0.393	0.324	0.173
∞	1.186	1.225	1.381	1.820	2.429	3.170	5.049	7.436	26.367

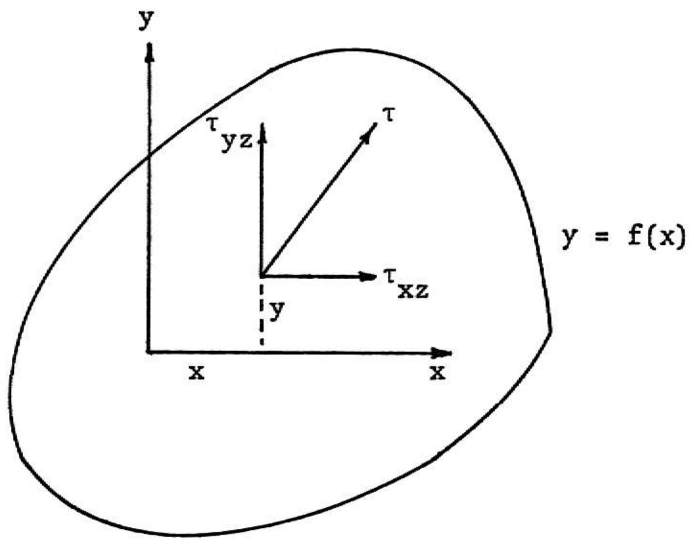


Fig. 1-1. Section through Twisted Bar

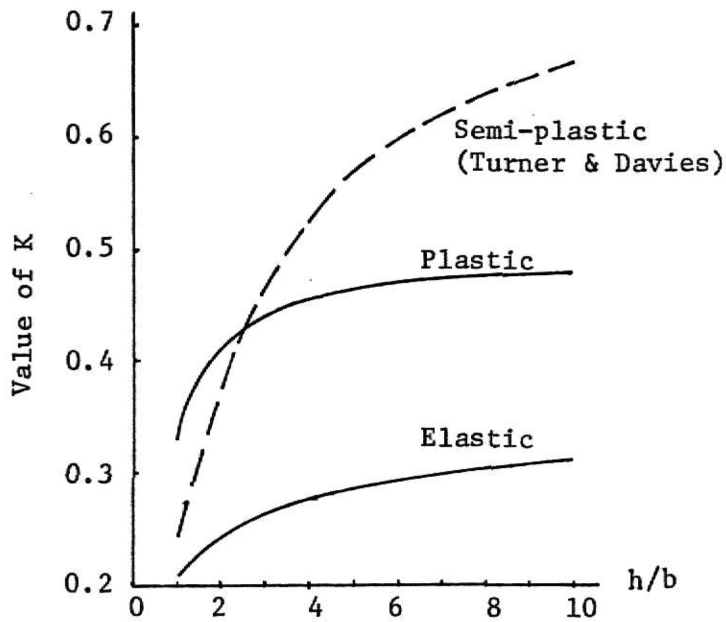


Fig. 1-2. Values of K for Elastic, Semi-Plastic and Plastic Theories of Torsion of Rectangular Section

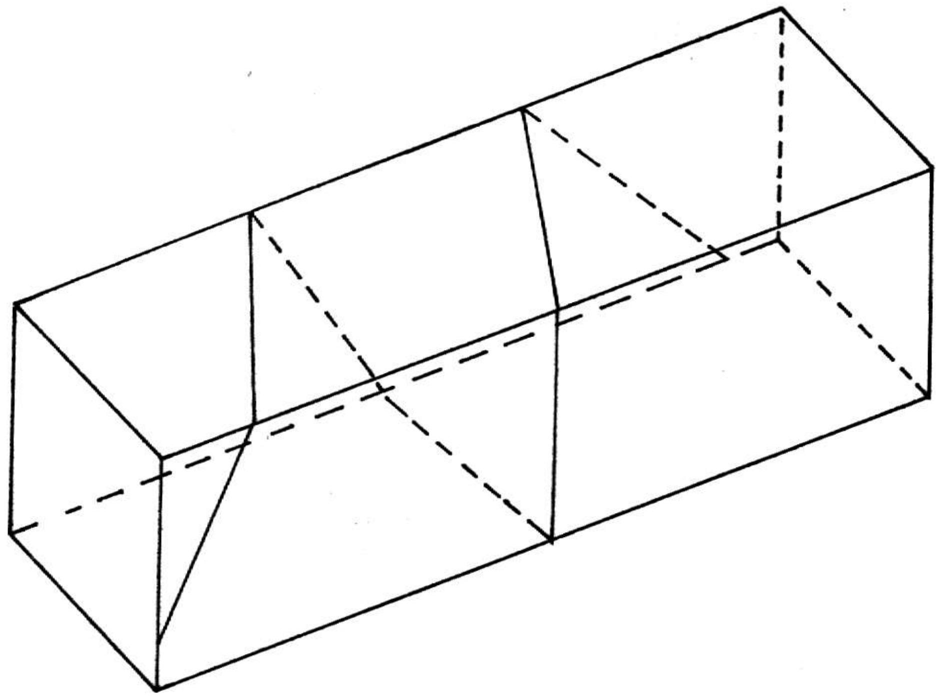


Fig. 1-3. Helical Cracks of Plain Concrete Section under Pure Torsion

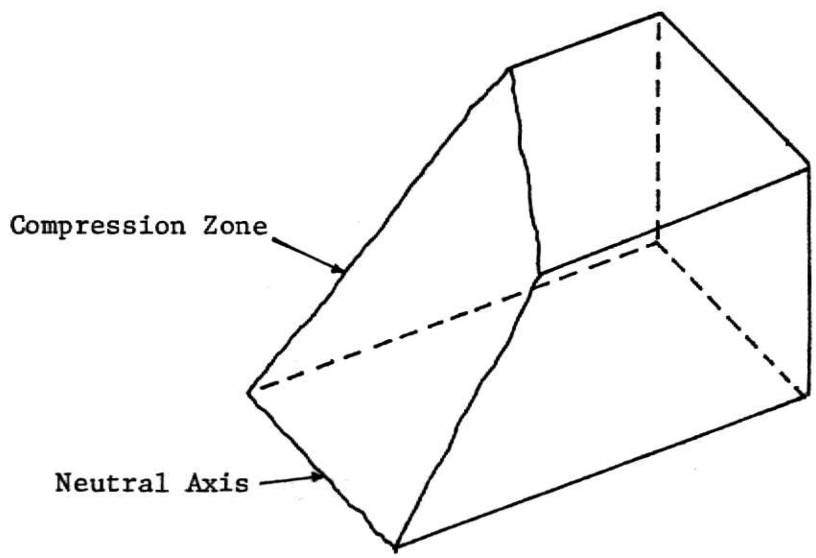


Fig. 1-4. Skewed Bending Failure of Rectangular Section Subject to Pure Torsion

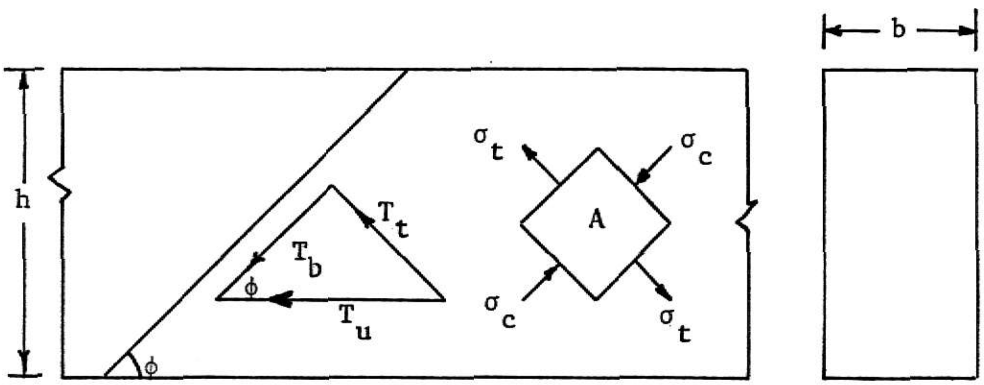


Fig. 1.5. Components of Applied Torque and Stresses on Element in Face of Beam

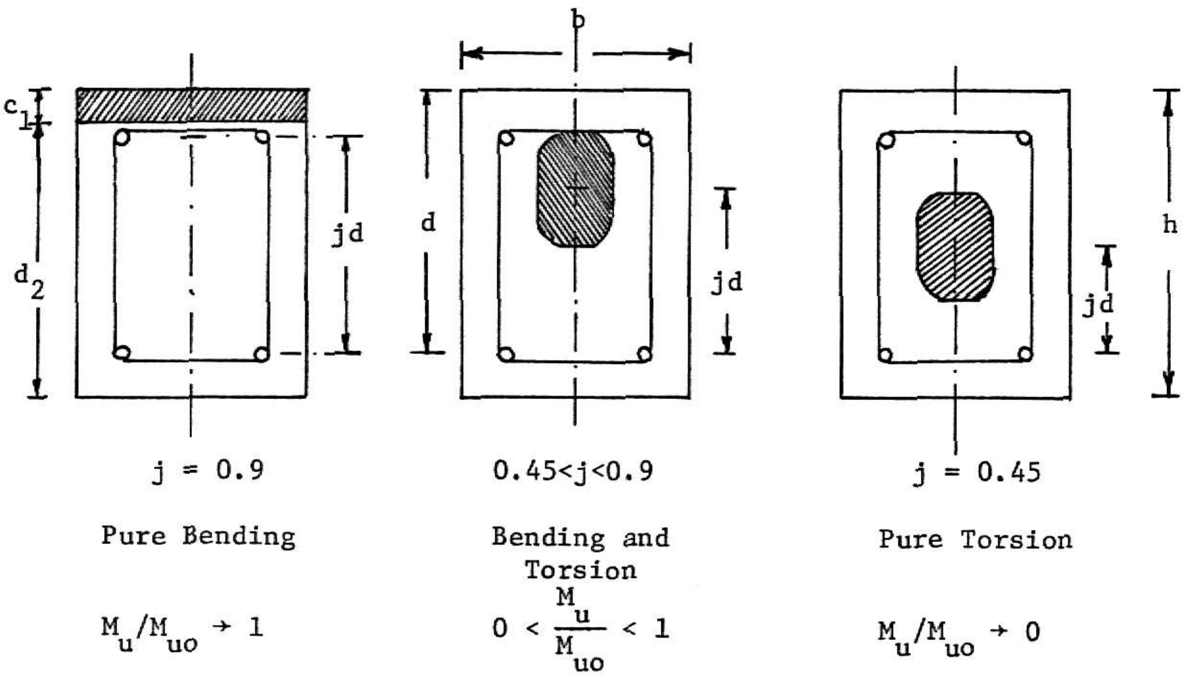


Fig. 3.1a. Effect of Torsion on Internal Lever Arm

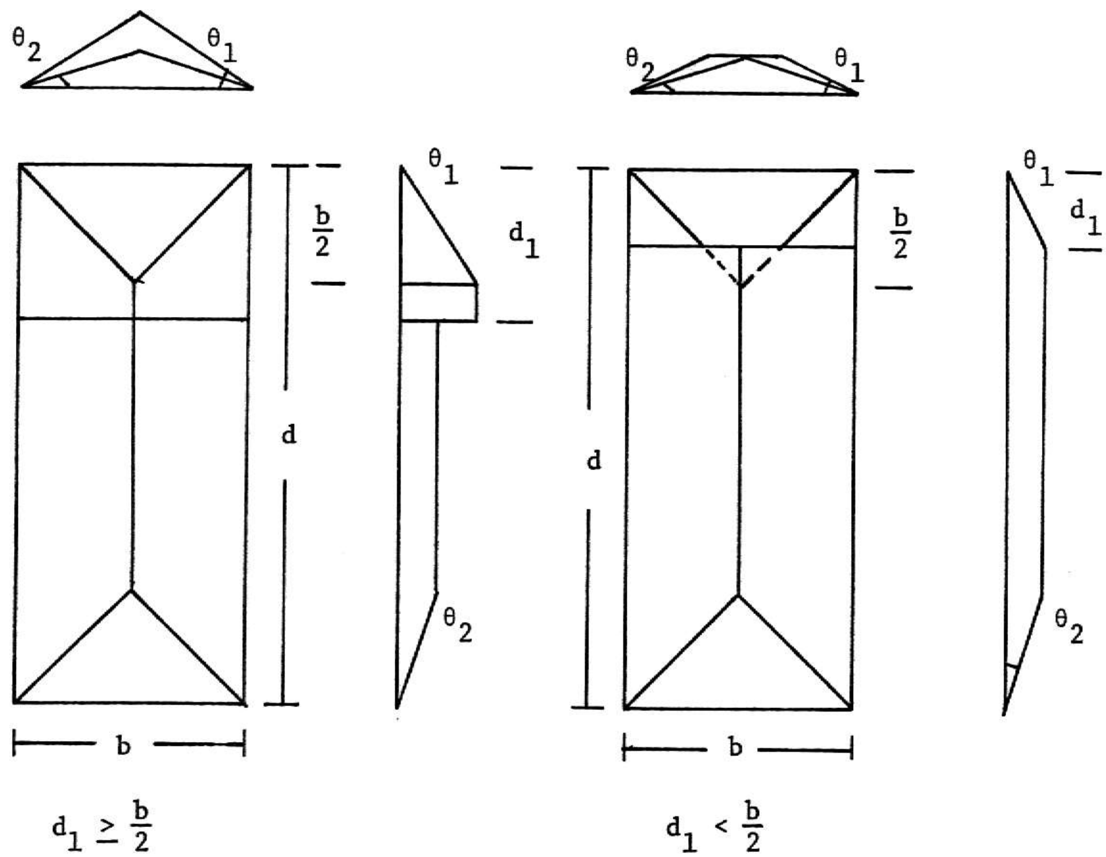


Fig. 3-1b. Modified Sand-Heap Analogy

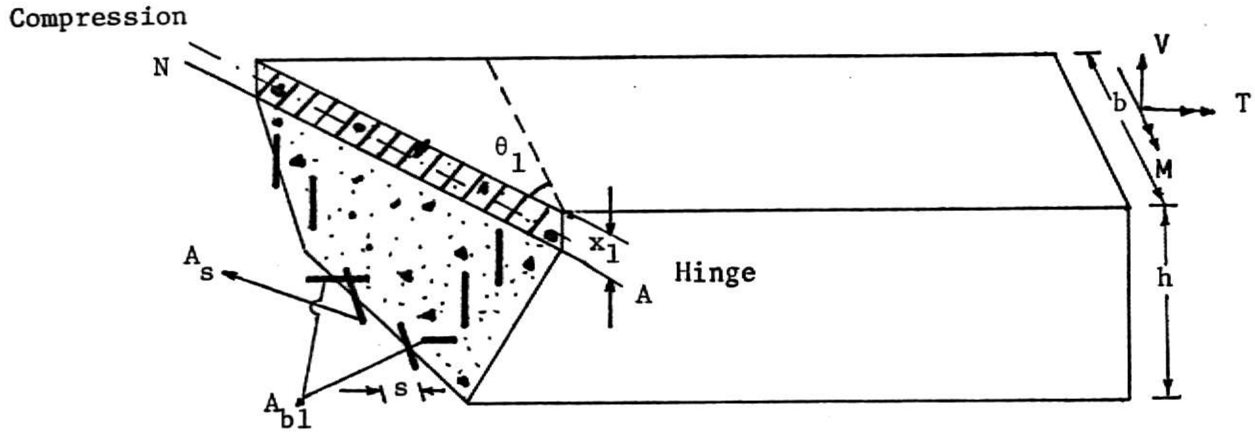


Fig. 3-2a. Mode 1 Failure

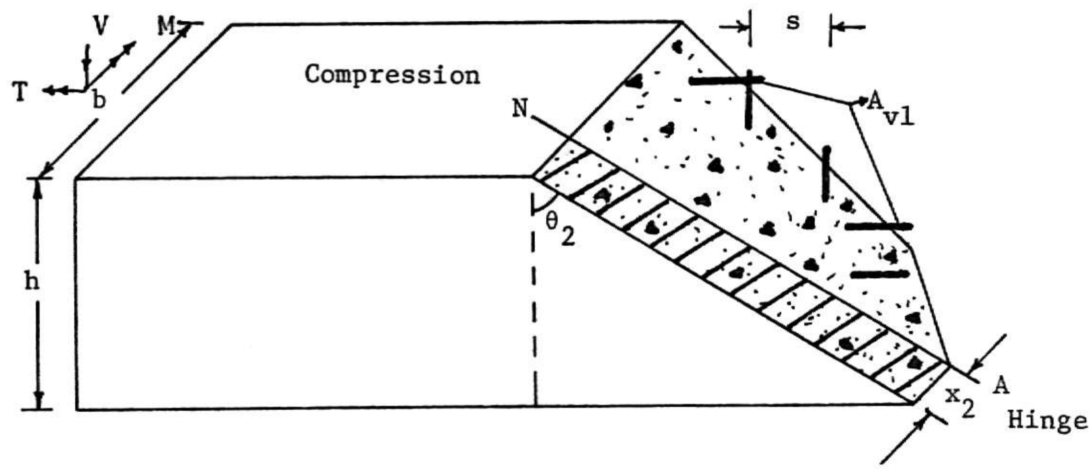


Fig. 3-2b. Mode 2 Failure

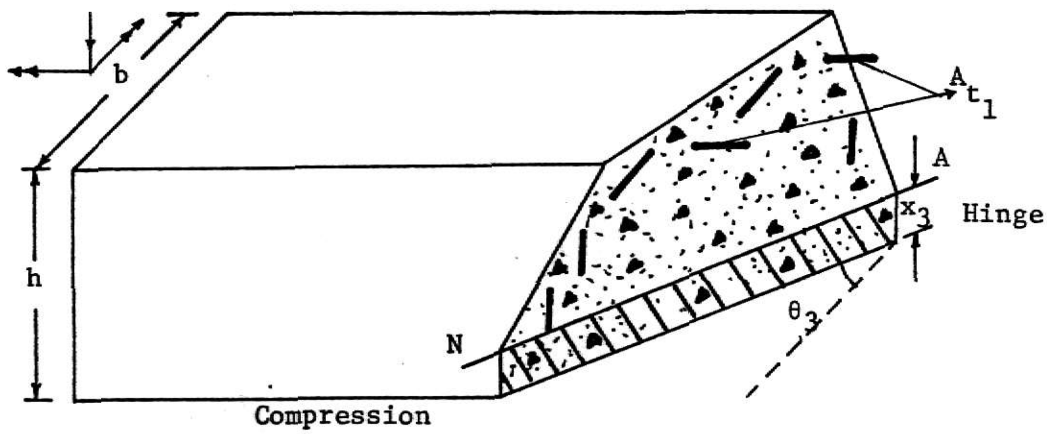


Fig. 3-2c. Mode 3 Failure

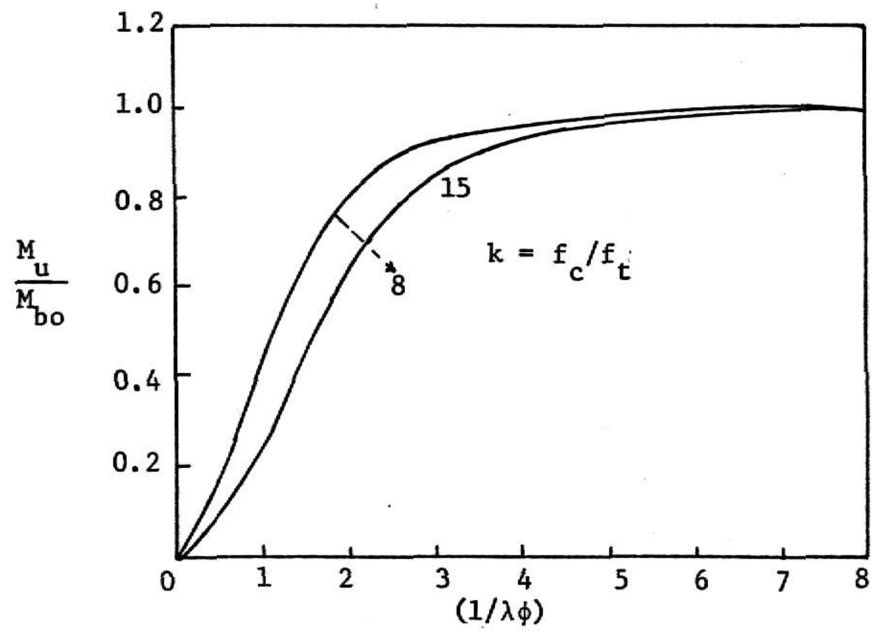


Fig. 3-3. Variation of M_u/M_{bo} with $1/\lambda\phi$

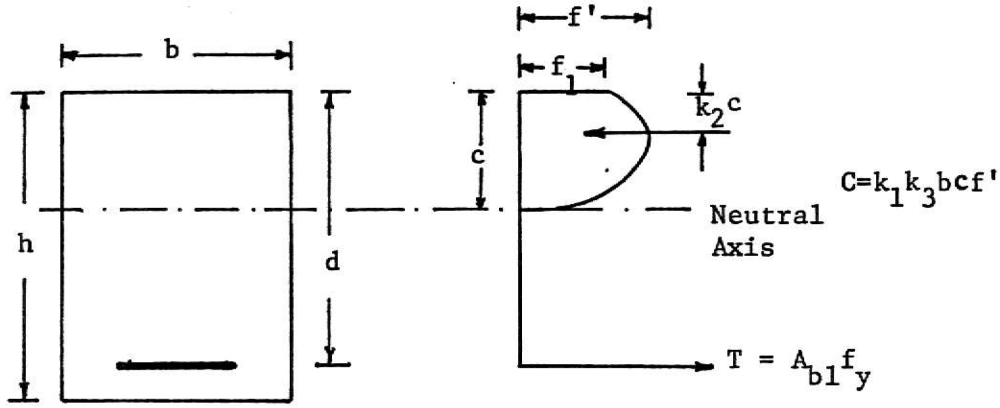


Fig. 3-4. Rectangular Beam under combined Bending and Torsion.

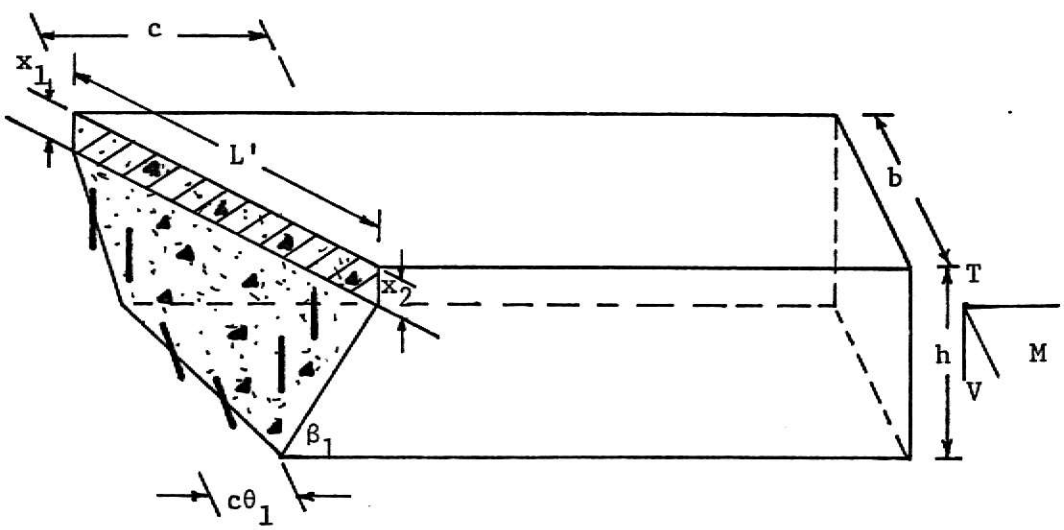


Fig. 4-1. Section of a Reinforced Concrete Beam at First Mode of Failure.

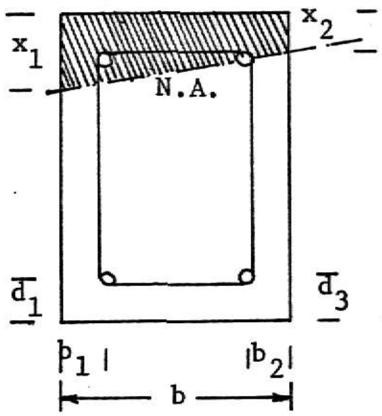


Fig. 4-2. Position of the Compression Zone in the Cross Section

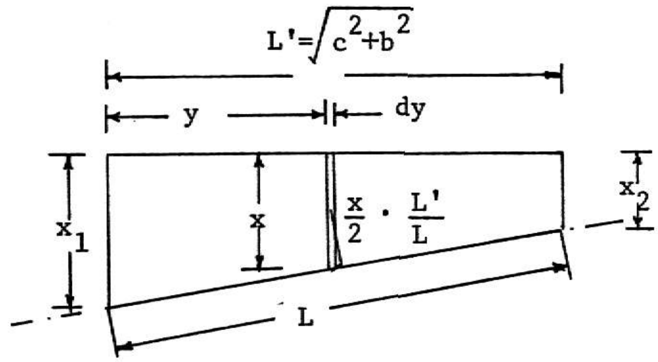


Fig. 4-3. Determination of Static Moment of the Area of the Compression Zone

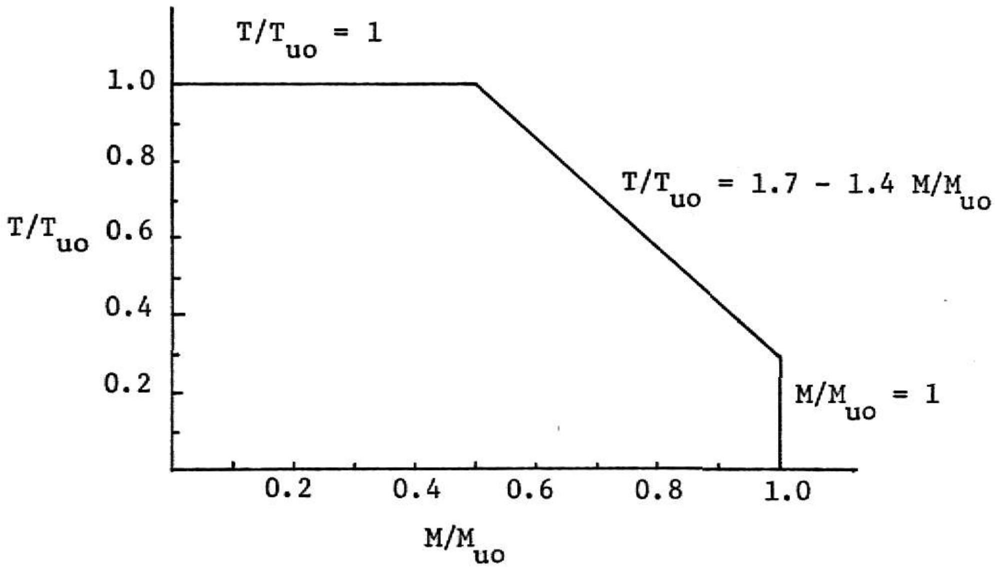


Fig. 4-4. Non-dimensional Interaction Diagram between Torsion and Bending

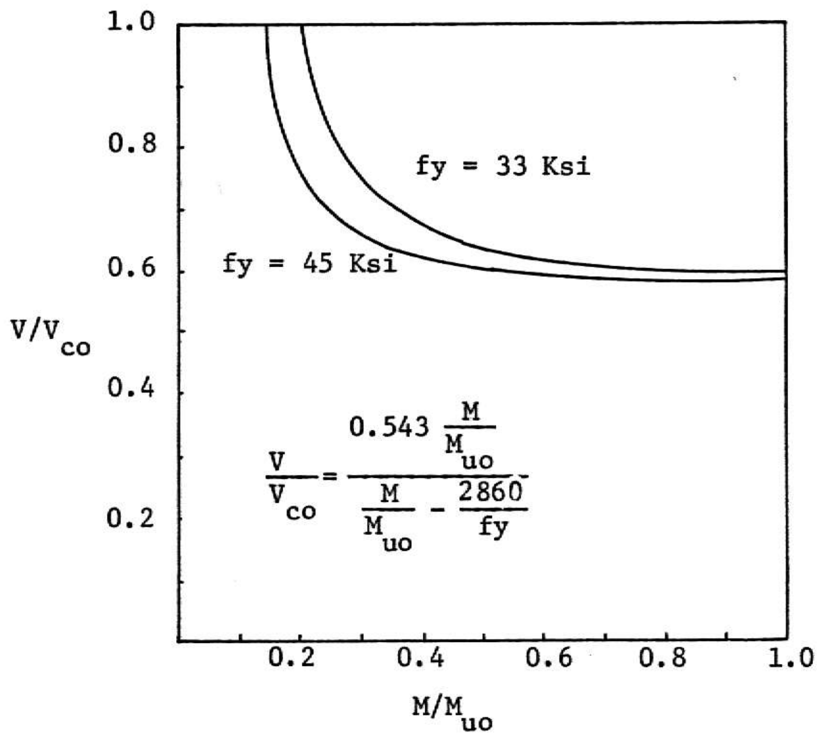


Fig. 4-5. Non-dimensional Interaction Curves between Shear and Bending

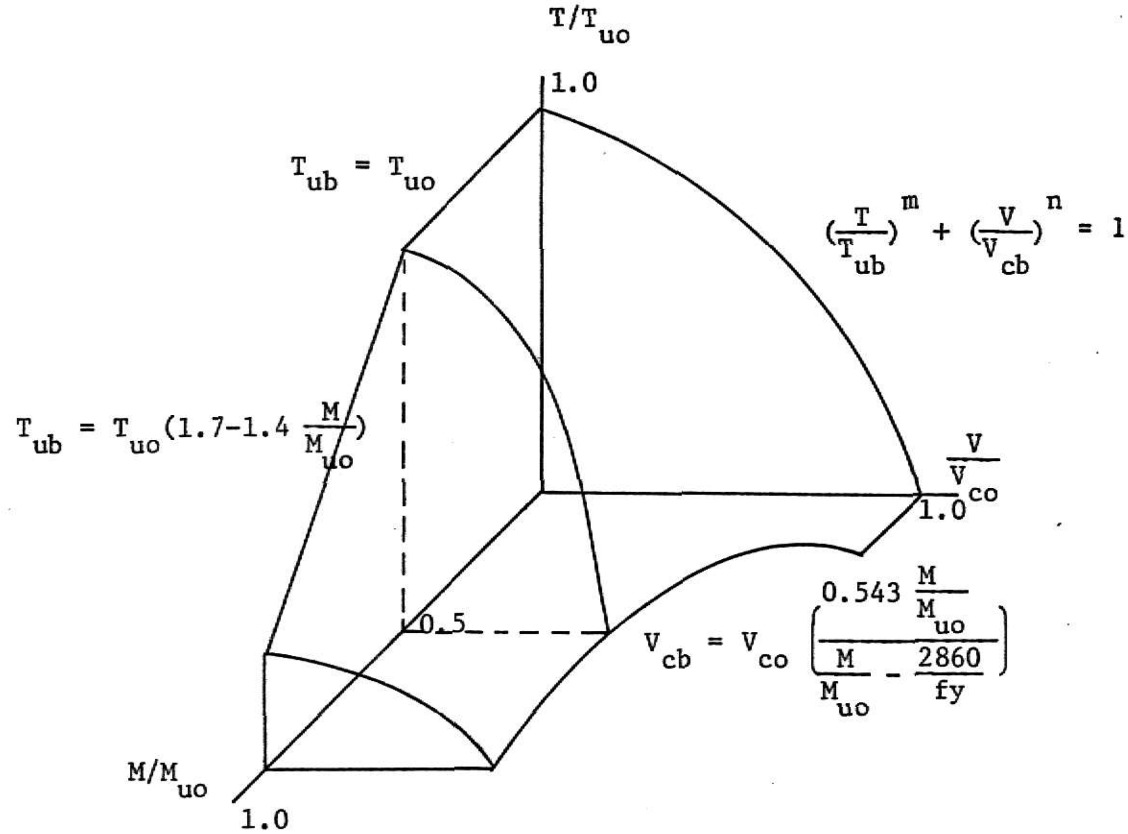


Fig. 4-6. Interaction Surface for combined Torsion, Bending and Shear, (General Form)

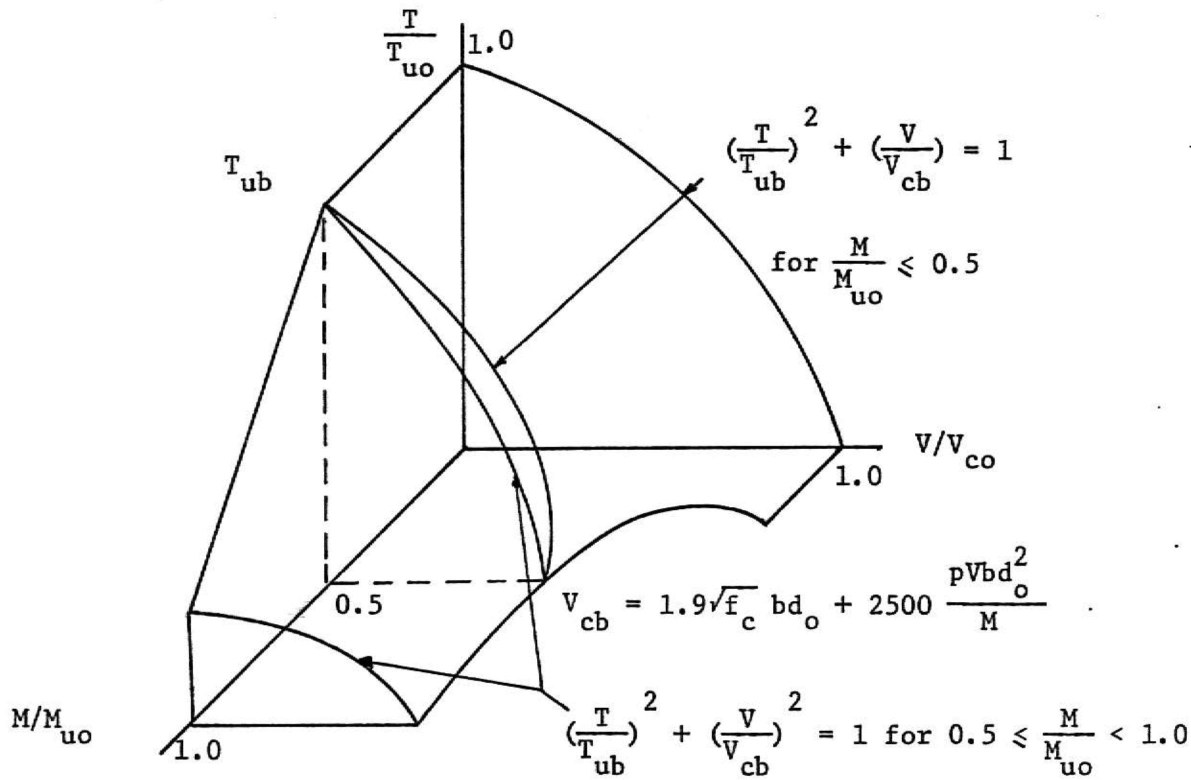


Fig. 4-7. Interaction Surface for combined Bending, Shear and Torsion (Simplified Form for Design)

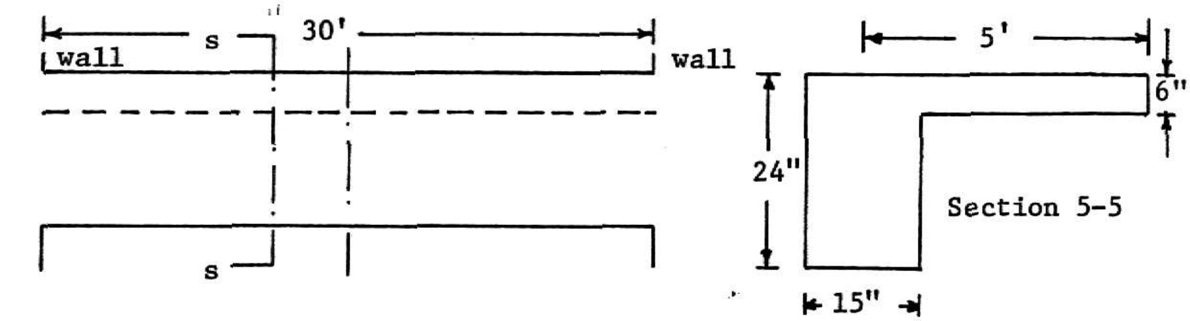
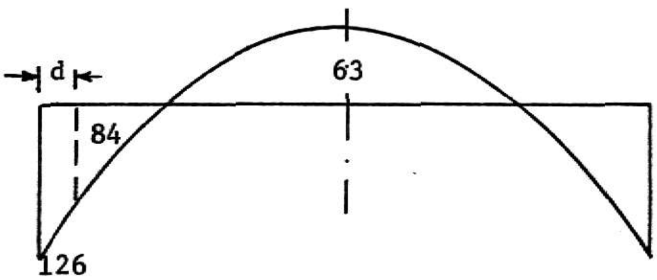
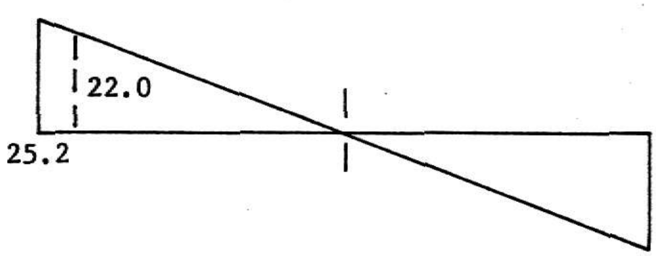


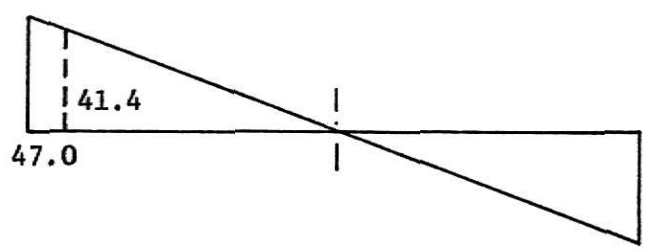
Fig. 5-1. Beam Considered in Design Example



Ultimate Design Bending Moment (k - ft)



Ultimate Design Shear (k)



Ultimate Design Torsion Moment (k - ft)

Fig. 5-2. Moments and Shears Used in Design Example

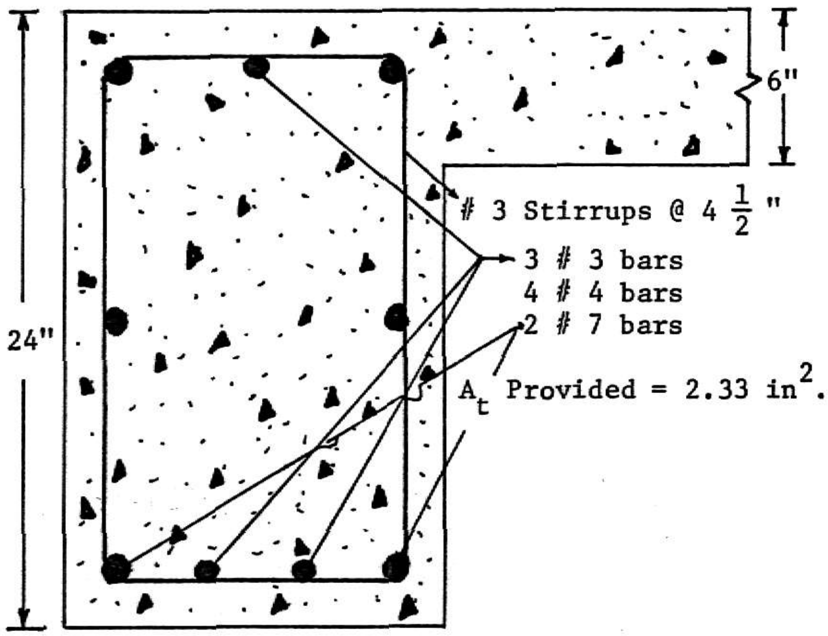


Fig. 5-3. Section showing Reinforcement by ACI Method

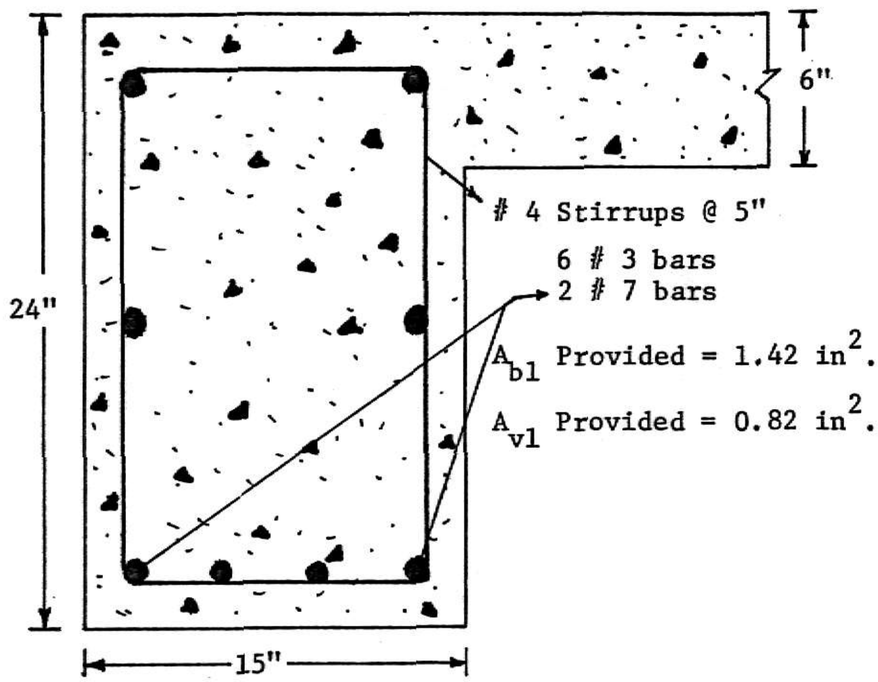


Fig. 5-4. Section showing Reinforcement by Lessig's Method

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ABSTRACT

This report presents an investigation of torsional effects on rectangular reinforced concrete beams. As torsion hardly acts alone in civil engineering structures, its effect combined with those of flexure and shear has been studied, to enable the designer and engineer to understand the torsional effects more clearly. The advent of ultimate strength design has resulted in members with greater slenderness in which the so called secondary torsion effects are no longer negligible. The report summarizes the following:

1). Plain concrete subjected to pure torsion —— a review of the classical plastic theory; semiplastic theories by Miyamoto, Turner and Davis and Marshall and Tembe; ultimate strength analysis by Hsu and Navaratnarajah's method.

2). Reinforced concrete subjected to pure torsion —— a review of the theories of Turner and Davis, and Marshall and Tembe, and a review of the theories of Rauch-Anderson-Cowant-Ernst, and Iyenger and Rangan.

3). Reinforced concrete subjected to torsion and bending —— a review of the theories of Pandit and Warwaruk; Collins, Walsh, Archer and Hall; and Iyenger and Rangan.

4). Reinforced concrete subjected to combined torsion, bending and shear —— a review of the theory of Lessig and interaction surface by Hsu.

5). A design example is presented to illustrate the theory given by Lessig and to compare it with the current ACI method.