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# Uniqueness of the solution to inverse scattering problem with non-overdetermined data

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**Abstract**

Let $q(x)$ be a real-valued compactly supported sufficiently smooth function. It is proved that the scattering data $A(-\beta,\beta,k)$ $\forall\beta \in S^2$, $\forall k > 0$, determine $q$ uniquely. Under the same assumptions on $q(x)$ it is proved that the scattering data $A(\beta,\alpha_0,k)$ $\forall\beta \in S^2$, $\forall k > 0$, and a fixed $\alpha_0$, the direction of the incident plane wave, determine $q$ uniquely. The above scattering data are non-overdetermined in the sense that the scattering data depends on the same number of variables as the potential $q(x)$, that is, on three variables. earlier there were no uniqueness results for the solution of three-dimensional inverse scattering problems with non-overdetermined scattering data.

MSC: 35P25, 35R30, 81Q05;
*Key words:* inverse scattering, non-overdetermined data, backscattering.

## 1 Introduction

The scattering solution $u(x,\alpha,k)$ solves the scattering problem:

$$[\nabla^2 + k^2 - q(x)]u = 0 \quad in \quad \mathbb{R}^3, \tag{1}$$

$$u = e^{ik\alpha\cdot x} + A(\beta,\alpha,k)\frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad r := |x| \to \infty, \ \beta := \frac{x}{r}. \tag{2}$$

Here $\alpha,\beta \in S^2$ are the unit vectors, $S^2$ is the unit sphere, the coefficient $A(\beta,\alpha,k)$ is called the scattering amplitude, $q(x)$ is a real-valued compactly supported sufficiently smooth function. We want to determine $q(x)$ given the backscattering data $A(-\beta,\beta,k)$, or the data $A(\beta,\alpha_0,k)$ $\forall\beta \in S^2$, $\forall k > 0$, $\alpha_0 \in S^2$ is fixed. These are 3D inverse scattering problems with non-overdetermined data: the data depend on the same number of variables as the $q(x)$, i.e., on three variables. The first uniqueness results for such problems were obtained in [2]- [4].

*Assumption A): We assume that $q$ is compactly supported, i.e., $q(x) = 0$ for $|x| > a$, where $a > 0$ is an arbitrary large fixed number; $q(x)$ is real-valued, i.e., $q = \overline{q}$; and $q(x) \in H_0^\ell(B_a)$, $\ell > 3$.*

Here $B_a$ is the ball centered at the origin and of radius $a$, and $H_0^\ell(B_a)$ is the closure of $C_0^\infty(B_a)$ in the norm of the Sobolev space $H^\ell(B_a)$ of functions whose derivatives up to the order $\ell$ belong to $L^2(B_a)$.

It was proved in [6] that if $q = \overline{q}$ and $q \in L^2(B_a)$ is compactly supported, then the resolvent kernel $G(x, y, k)$ of the Schrödinger operator $-\nabla^2 + q(x) - k^2$ is a meromorphic function of $k$ on the whole complex plane $k$, analytic in $\mathrm{Im}k \geq 0$, except, possibly, of a finitely many simple poles at the points $ik_j$, $k_j > 0$, $1 \leq j \leq n$, where $-k_j^2$ are negative eigenvalues of the selfadjoint operator $-\nabla^2 + q(x)$ in $L^2(\mathbb{R}^3)$. Consequently, the scattering amplitude $A(\beta, \alpha, k)$, corresponding to the above $q$, is a restriction to the positive semiaxis $k \in [0, \infty)$ of a meromorphic on the whole complex $k$-plane function.

It was proved by the author ([7]), that the *fixed-energy scattering data* $A(\beta, \alpha) := A(\beta, \alpha, k_0)$, $k_0 = const > 0$, $\forall \beta \in S_1^2$, $\forall \alpha \in S_2^2$, determine real-valued compactly supported $q \in L^2(B_a)$ uniquely. Here $S_j^2$, $j = 1, 2$, are arbitrary small open subsets of $S^2$ (solid angles).

In [10] (see also monograph [11], Chapter 5, and [8]) an analytical formula is derived for the reconstruction of the potential $q$ from exact fixed-energy scattering data, and from noisy fixed-energy scattering data, and stability estimates and error estimates for the reconstruction method are obtained. To the author's knowledge, these are the only known until now theoretical error estimates for the recovery of the potential from noisy fixed-energy scattering data in the three-dimensional inverse scattering problem.

In [9] stability results are obtained for the inverse scattering problem for obstacles.

The scattering data $A(\beta, \alpha)$ depend on four variables (two unit vectors), while the unknown $q(x)$ depends on three variables. In this sense the inverse scattering problem, which consists of finding $q$ from the fixed-energy scattering data $A(\beta, \alpha)$, is overdetermined.

*Historical remark.* In the beginning of the forties of the last century physicists raised the the following question: is it possible to recover the Hamiltonian of a quantum-mechanical system from the observed quantities, such as $S$-matrix? In the non-relativistic quantum mechanics the simplest Hamiltonian $\mathbf{H} = -\nabla^2 + q(x)$ can be uniquely determined if one knows the potential $q(x)$. The $S$-matrix in this case is in one-to-one correspondence with the scattering amplitude $A$: $S = I - \frac{k}{2\pi i} A$, where $I$ is the identity operator in $L^2(S^2)$, $A$ is an integral operator in $L^2(S^2)$ with the kernel $A(\beta, \alpha, k)$, and $k^2 > 0$ is energy. Therefore, the question, raised by the physicists, is reduced to an inverse scattering problem: can one determine the potential $q(x)$ from the knowledge of the scattering amplitude. We have briefly discussed this problem above.

Since the above question was raised, there were no uniqueness theorems for three-dimensional inverse scattering problems with non-overdetermined data. The goal of this paper is to outline a proof of such theorems. The results are formulated in Theorem 1.1:

**Theorem 1.1** *If Assumption A) holds, then the data* $A(-\beta, \beta, k)$ $\forall \beta \in S^2$, $\forall k > 0$, *determine* $q$ *uniquely. Under the same assumptions, the data* $A(\beta, \alpha_0, k)$ $\forall \beta \in S^2$, *a fixed* $\alpha = \alpha_0 \in S^2$, *and* $\forall k > 0$, *determine* $q$ *uniquely.*

**Remark 1.** The conclusion of Theorem 1.1 remains valid if the scattering data $A(-\beta, \beta, k)$, or the scattering data $A(\beta, \alpha_0, k)$ are known $\forall \beta \in S_1^2$ and $k \in (k_0, k_1)$, where $(k_0, k_1) \subset [0, \infty)$ is an arbitrary small interval, $k_1 > k_0$, and $S_1^2$ is an arbitrary small open subset of $S^2$. The assumption $\ell > 3$ can be relaxed to $\ell > 2$.

In Section 2 we formulate some known auxiliary results and introduce some notations. In Section 3 an outline of the proof of Theorem 1.1 is given.

The results, presented in this paper, were reported in [3], [4]. We follow closely the outline of the ideas from these papers.

# 2 Auxiliary results

Let

$$F(g) := \tilde{g}(\xi) = \int_{\mathbb{R}^3} g(x) e^{i\xi \cdot x} dx, \quad g(x) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{-i\xi \cdot x} \tilde{g}(\xi) d\xi. \tag{3}$$

If $f * g := \int_{\mathbb{R}^3} f(x-y) g(y) dy$, then

$$F(f * g) = \tilde{f}(\xi)\tilde{g}(\xi), \quad F(f(x)g(x)) = \frac{1}{(2\pi)^3} \tilde{f} * \tilde{g}. \tag{4}$$

If

$$G(x-y, k) := \frac{e^{ik[|x-y| - \beta \cdot (x-y)]}}{4\pi |x-y|}, \tag{5}$$

then

$$F(G(x,k)) = \frac{1}{\xi^2 - 2k\beta \cdot \xi}, \qquad \xi^2 := \xi \cdot \xi. \tag{6}$$

The scattering solution $u = u(x, \alpha, k)$ solves (uniquely) the integral equation

$$u(x, \alpha, k) = e^{ik\alpha \cdot x} - \int_{B_a} g(x, y, k) q(y) u(y, \alpha, k) dy, \tag{7}$$

where

$$g(x, y, k) := \frac{e^{ik|x-y|}}{4\pi |x-y|}. \tag{8}$$

If

$$v = e^{-ik\alpha \cdot x} u(x, \alpha, k), \tag{9}$$

then

$$v = 1 - \int_{B_a} G(x-y, k) q(y) v(y, \alpha, k) dy, \tag{10}$$

where $G$ is defined in (5). Define $\epsilon$ by the formula

$$v = 1 + \epsilon. \tag{11}$$

Then (10) can be rewritten as

$$\epsilon(x, \alpha, k) = -\int_{\mathbb{R}^3} G(x-y, k) q(y) dy - T\epsilon, \tag{12}$$

where $T\epsilon := \int_{B_a} G(x-y,k)q(y)\epsilon(y,\alpha,k)dy$. Fourier transform of (12) yields (see (4),(6)):

$$\tilde{\epsilon}(\xi, \alpha, k) = -\frac{\tilde{q}(\xi)}{\xi^2 - 2k\alpha\cdot\xi} - \frac{1}{(2\pi)^3}\frac{1}{\xi^2 - 2k\alpha\cdot\xi}\tilde{q} * \tilde{\epsilon}. \tag{13}$$

An essential ingredient of our proof in Section 3 is the following lemma, proved by the author in [11], p.262, and in [10].

**Lemma 2.1** *If $A_j(\beta, \alpha, k)$ is the scattering amplitude corresponding to potential $q_j$, $j = 1, 2$, then*

$$-4\pi[A_1(\beta,\alpha,k) - A_2(\beta,\alpha,k)] = \int_{B_1} [q_1(x) - q_2(x)] u_1(x,\alpha,k) u_2(x,-\beta,k) dx, \tag{14}$$

*where $u_j$ is the scattering solution corresponding to $q_j$.*

Consider an algebraic variety $\mathcal{M}$ in $\mathbb{C}^3$ defined by the equation

$$\mathcal{M} := \{\theta\cdot\theta = 1, \quad \theta\cdot\theta := \theta_1^2 + \theta_2^2 + \theta_3^2, \quad \theta_j \in \mathbb{C}, \ 1 \le j \le 3.\} \tag{15}$$

This is a non-compact variety, intersecting $\mathbb{R}^3$ over the unit sphere $S^2$.

Let $R_+ = [0, \infty)$. The following result is proved in [12], p.62.

**Lemma 2.2** *If Assumption A) holds, then the scattering amplitude $A(\beta, \alpha, k)$ is a restriction to $S^2 \times S^2 \times R_+$ of a function $A(\theta', \theta, k)$ on $\mathcal{M} \times \mathcal{M} \times \mathbb{C}$, analytic on $\mathcal{M} \times \mathcal{M}$ and meromorphic on $\mathbb{C}$, $\theta'$, $\theta \in \mathcal{M}$, $k \in \mathbb{C}$.*

The scattering solution $u(x, \alpha, k)$ is a meromorphic function of $k$ in $\mathbb{C}$, analytic in $\mathrm{Im}k \ge 0$, except, possibly, at the points $k = ik_j$, $1 \le j \le n$, $k_j > 0$, where $-k_j^2$ are negative eigenvalues of the selfadjoint Schrödinger operator, defined by the potential $q$ in $L^2(\mathbb{R}^3)$. These eigenvalues can be absent, for example, if $q \ge 0$.

We need the notion of the Radon transform: $\hat{f}(\beta, \lambda) := \int_{\beta\cdot x=\lambda} f(x)d\sigma$, where $d\sigma$ is the element of the area of the plane $\beta \cdot x = \lambda$, $\beta \in S^2$, $\lambda$ is a real number. The following properties of the Radon transfor will be used: $\int_{B_a} f(x)dx = \int_{-a}^{a} \hat{f}(\beta,\lambda)d\lambda$,$\int_{B_a} e^{ik\beta\cdot x} f(x)dx = \int_{-a}^{a} e^{ik\lambda}\hat{f}(\beta,\lambda)d\lambda$,$\hat{f}(\beta,\lambda) = \hat{f}(-\beta,-\lambda)$.These properties are proved, e.g., in [13], pp. 12, 15. We also need the following Phragmen-Lindelöf lemma, which is proved, e.g., in [1], p.69.

**Lemma 2.3** *Let $f(z)$ be holomorphic inside an angle $\mathcal{A}$ of opening $< \pi$; $|f(z)| \le c_1 e^{c_2|z|}$, $z \in \mathcal{A}$, $c_1, c_2 > 0$ are constants; $|f(z)| \le M$ on the boundary of $\mathcal{A}$; and $f$ is continuous up to the boundary of $\mathcal{A}$. Then $|f(z)| \le M$, $\forall z \in \mathcal{A}$.*

# 3 Outline of Proof of Theorem 1.1

The scattering data in Remark 1 determine uniquely the scattering data in Theorem 1.1 by Lemma 2.2. Assume that potentials $q_j$, $j = 1, 2$, generate the same scattering data: $A_1(-\beta, \beta, k) = A_2(-\beta, \beta, k) \quad \forall\beta \in S^2, \quad \forall k > 0$, and let $p(x) := q_1(x) - q_2(x)$. Then by Lemma 2.1, see equation (14), one gets

$$0 = \int_{B_a} p(x) u_1(x, \beta, k) u_2(x, \beta, k) dx, \qquad \forall\beta \in S^2, \ \forall k > 0. \tag{16}$$

By (9) and (11) one can rewrite (16) as

$$\int_{B_a} e^{2ik\beta\cdot x}[1 + \epsilon(x, k)]p(x)dx = 0 \quad \forall\beta \in S^2, \ \forall k > 0, \tag{17}$$

where $\epsilon(x, k) := \epsilon := \epsilon_1(x, k) + \epsilon_2(x, k) + \epsilon_1(x, k)\epsilon_2(x, k)$. By Lemma 2.2 the relations (16) and (17) hold for complex $k$, $k = \frac{\kappa+i\eta}{2}, \qquad \kappa+i\eta \neq 2ik_j, \quad \eta \geq 0$. Using formulas (3)-(4), one derives from (17) the relation

$$\tilde{p}((\kappa + i\eta)\beta) + \frac{1}{(2\pi)^3}(\tilde{\epsilon} * \tilde{p})((\kappa + i\eta)\beta) = 0 \qquad \forall\beta \in S^2, \ \forall\kappa \in \mathbb{R}, \tag{18}$$

where the notation $(f * g)(z)$ means that the convolution $f * g$ is calculated at the argument $z = (\kappa + i\eta)\beta$.

One has

$$\sup_{\beta\in S^2} |\tilde{\epsilon} * \tilde{p}| := \sup_{\beta\in S^2} \left| \int_{\mathbb{R}^3} \tilde{\epsilon}((\kappa + i\eta)\beta - s)\tilde{p}(s)ds \right| \leq \nu(\kappa, \eta) \sup_{s\in\mathbb{R}^3} |\tilde{p}(s)|, \tag{19}$$

where $\nu(\kappa, \eta) := \sup_{\beta\in S^2} \int_{\mathbb{R}^3} |\tilde{\epsilon}((\kappa + i\eta)\beta - s)|ds$. We prove that if $\eta = \eta(\kappa) = O(\ln\kappa)$ is suitably chosen (see [4]), then the following inequality holds:

$$0 < \nu(\kappa, \eta(\kappa)) < 1, \qquad \kappa \to \infty. \tag{20}$$

We also prove that

$$\sup_{\beta\in S^2} |\tilde{p}((\kappa + i\eta(\kappa))\beta)| \geq \sup_{s\in\mathbb{R}^3} |\tilde{p}(s)|, \quad \kappa \to \infty, \tag{21}$$

and then it follows from (18)-(21) that $\tilde{p}(s) = 0$, so $p(x) = 0$, and Theorem 1.1 is proved. Indeed, it follows from (18) and (21) that, for sufficiently large $\kappa$ and a suitable $\eta(k) = O(\ln k)$, one has $\sup_{s\in\mathbb{R}^3} |\tilde{p}(s)| \leq \frac{1}{(2\pi)^3}\nu(\kappa, \eta(\kappa)) \sup_{s\in\mathbb{R}^3} |\tilde{p}(s)|$. If (20) holds, then the above equation implies that $\tilde{p} = 0$. This and the injectivity of the Fourier transform imply that $p = 0$. A detailed proof of estimates (20) and (21), that completes the proof, is given in [4]. *This completes the outline of the proof of Theorem 1.1.*

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