YIELD LINE AND MEMBRANE ACTION ANALYSIS OF CONCRETE PLATES

by

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TABLE OF CONTENTS

List of Figures

List of Tables

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INTRODUCTION

The main object of structural analysis is to ensure a structure will have a suitable factor of safety against failure and in addition, that it will be serviceable when subjected to its design working load. The majority of slab or plate structures in the world have been designed in the past on the elastic theory. However, this theory does not give an accurate indication of the factor of safety against failure because of the condition that the materials may behave plastically. An alternative technique, yield line analysis, is an ultimate load method, in that the load at which a slab will fail is determined. No matter how complex the slab shape or loading configuration is, it is possible to obtain a realistic value of the failure load.

The membrane action analysis, which gives an ultimate load much higher than the ultimate load given by yield line analysis, is another alternative. In this analysis, restraining the edges of a beam or slab results in a higher load carrying capacity when compared with results of yield line analysis.

The object of this report is to describe the techniques of yield line analysis to find the ultimate load of a structural slab, and also, the analysis for tensile membrane action arresting the load when the structure fails.

Chapter I

Yield Line Theory

General Concept: The basic concept of yield line theory for ultimate load design of slabs was developed in detail by Johanson (6)*. He provided, not only introductory theory, but also a variety of practical examples.

In this theory, the strength of the slab is assumed to be controlled by flexure alone. The steel is assumed to be fully yielded along a given line at collapse and the bending and twisting moments are assumed to be uniformly distributed along these "yield" lines. This assumption is justified since most slabs are under-reinforced allowing reinforcement to yield with a corresponding large angle change and only slight increase in moment capacity.

The yield line theory for a one-way slab may, generally, be treated as identical to the limit analysis of a continuous beam. For instance, consider the simply supported one-way slab in Figure 1. As the load approaches its ultimate value, failure of the slab occurs on lines along which the steel has yielded (which are idealized and called yield lines). The curvature of the slab at the yielding section increases sharply, and deflection increases disproportionately. The elastic deformation of the slab is small compared with the deflection of the structure due to plastic deformation along the yield line. Therefore, elastic deformations are neglected and all deformations are assumed to be concentrated at the yield lines. The plastic hinge which forms at the yield line is

*Number in parenthesis refers to references listed in the bibliography.

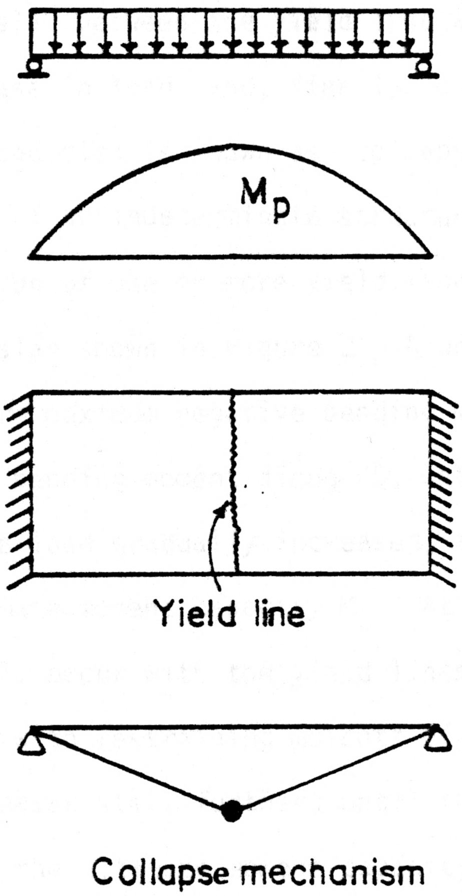
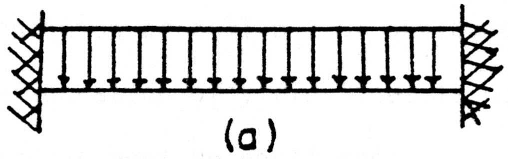


Figure 1. Simply Supported, Uniformly Loaded Slab

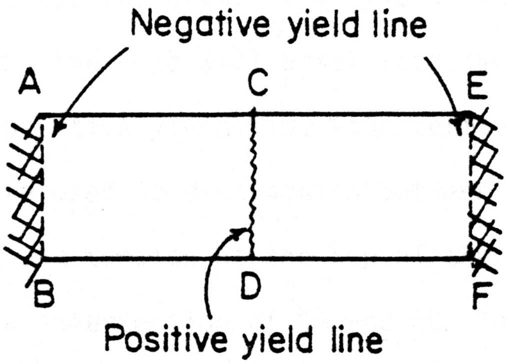
assumed to rotate with a constant moment. This restraining moment at the yield line can be taken equal to the ultimate resisting moment for a determinate slab, such as in the example. The formation of a yield line in this example is identical to failure. A "mechanism" forms and the segments of the slab between the yield line and supports will move without an increase in load, and, finally, collapse of the structure results. This condition is known as "collapse mechanism".

In the case of an indeterminate structure, equilibrium can be maintained after the formation of one or more yield lines. Consider, for example, the fixed-fixed slab shown in Figure 2. A uniform loading on the slab will cause uniform maximum negative bending moment along AB and EF and uniform positive bending moment along CD, which is parallel to the supports. As the load gradually increases, the moments along AB and EF reach their ultimate moment capacity M_p. At this time, rotation of the slab segments will occur with the yield lines AB and EF acting as axes of rotation, but with restraining moments of constant amount M_p. The load can be increased still further, until the moment at the midspan becomes equal to the ultimate moment capacity of the slab, and the third yield line forms along CD then the slab segments between the supports and yield line CD will rotate with no change in resisting moment. This converts the structure into a mechanism, and results in collapse.

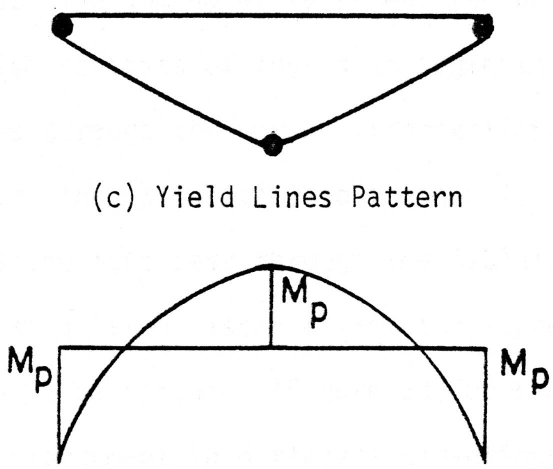
The yield line theory for a two-way slab requires a different treatment from the limit analysis of continuous beams, because, in this case, the yield lines will not, in general, be parallel to each other but instead will form a yield line pattern. The entire slab area will be



(b) Elastic Moment Distribution



(c) Yield Lines Pattern



(d) Plastic Moment Distribution

Fig. 2. Formation of Yield Lines in Uniformly Loaded Slabs.

divided into several segments which can rotate along the yield lines as rigid bodies at the condition of collapse or unstable equilibrium. Frequently, the exact configuration of yield lines is not known and a series of possible patterns must be considered.

Generally, yield lines occur along lines of fixed support, lines of symmetry and along axes passing over a column. Yield lines also pass through the intersection of the axes of rotation of adjacent slab segments. Figure 3 shows several typical examples of yield line patterns. In Figure 3a, at limit condition, the slab will break into two segments and rotate about EA and ED. The positive yield line (the terms positive yield and negative yield line are used to distinguish between those associated with tension at the bottom and tension at the top of the slab, respectively) must go through E, the intersection of EA and ED. The exact location of the positive yield line depends on the amount of reinforcement and its direction, both in the positive and negative moment regions. In Figure 3b, the collapse mechanism consists of four slab segments. The diagonal yield lines must pass through the corner intersection of two yield lines, and the central yield line is parallel to the two long sides. In Figure 3c, the axes of rotation must pass through the isolated columns. For a concentrated load at a large distance from the supported edge, the yield line pattern will be circular (Figure 3d) or eliptical (4) depending on the amount of reinforcement in different directions. If the point load is near the edge or corner of a slab, a partial fan may form, which is shown in Figure 3e. The foregoing can be summarized by the following conditions which assist the prediction of yield lines.

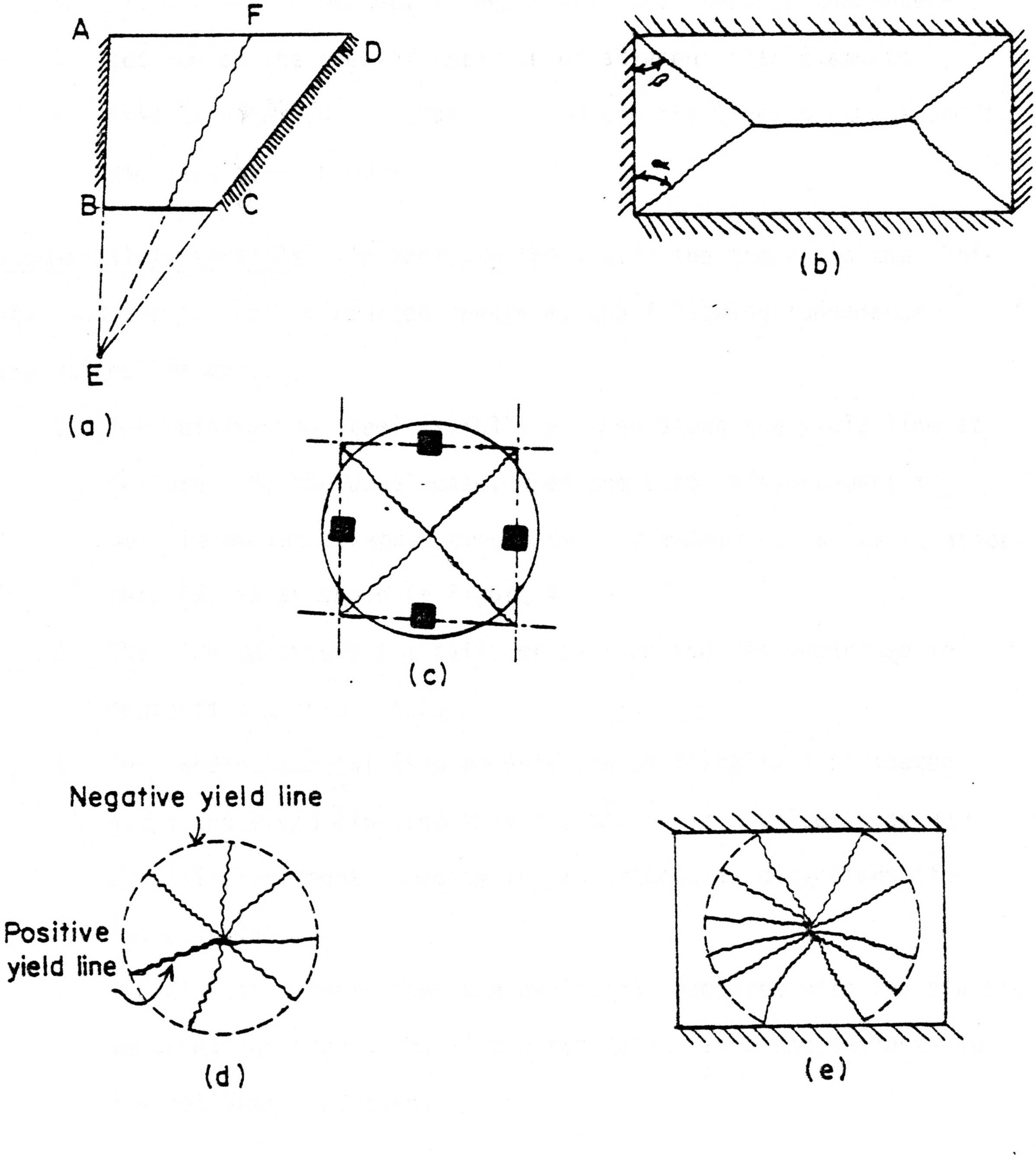


Figure 3. Various Yield Line Patterns

1. Yield lines terminate at a slab boundary.
2. Yield lines are, generally, straight.*
3. Yield lines (produced, if necessary) pass through the intersection of the axes of rotation of adjacent slab elements.
4. Axes of rotation, generally, lie along the lines of the support and pass over columns.

Fundamental Assumptions: In applying the yield line theory to the ultimate load analysis of reinforced concrete, the following fundamental assumptions are made.

1. The reinforcing steel is fully yielded along the yield line at failure. In the usual case, when the slab reinforcement is well below the balanced condition, the moment curvature relationship (7) is as shown in Figure 4.
2. The slab deforms plastically at failure and is separated in segments by yield lines.
3. The bending and twisting moments are uniformally distributed along the yield line and they are the maximum values provided by ultimate moment capacity in two orthogonal directions (for a two-way slab).
4. The elastic deformations are negligible compared with the plastic deformations; thus, the slab parts rotate as plane segments in the collapse condition.

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* It should be noted, however, that for some systems of point loads or circular supports, a multiple system of yield lines can approximate a single curved yield line, e.g. see Figure 3d.

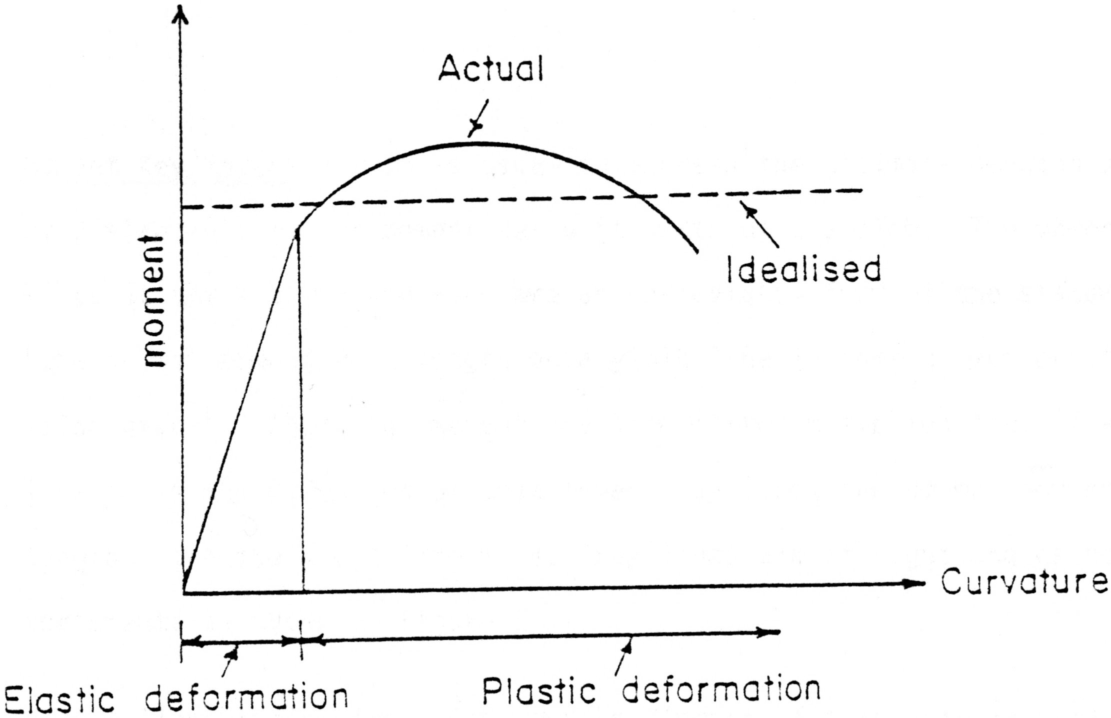


Fig. 4. Typical and Idealized Moment-Curvature Relationship for Reinforced Concrete Slab

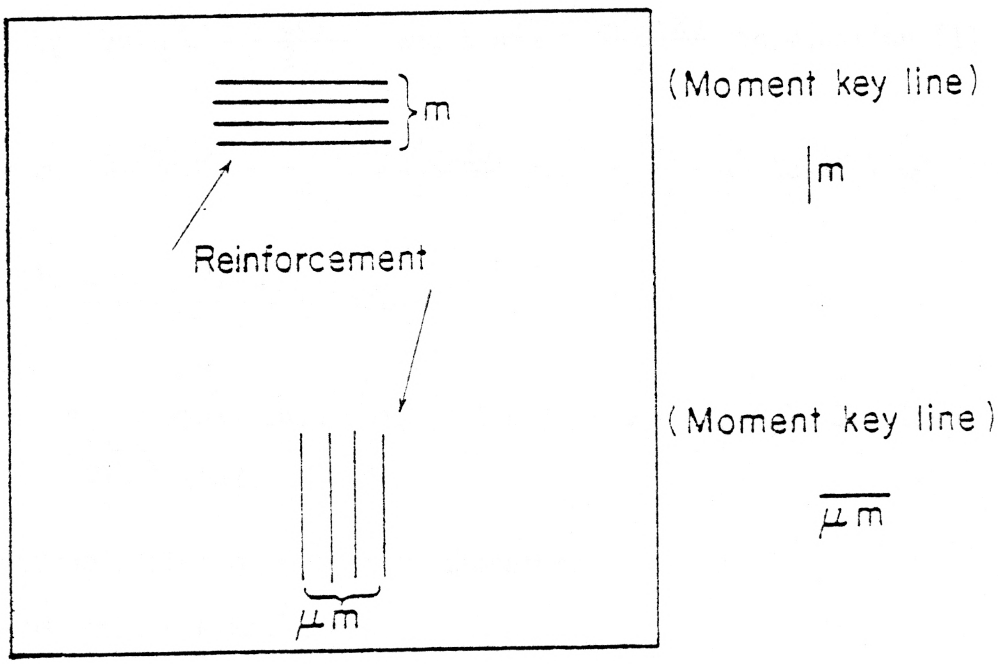


Fig. 5. Moment Key Line Notation

Moment Key Notation: It is usual to express the ultimate bending strength of a slab in terms of moment per unit width of the slab. The moment key lines at the side of the slab are an abbreviated form of the statement "the normal moment/unit length on a yield line in this direction is the value given". Thus, the moment key line marked m implies that if a yield line is in the direction of that moment key line, the normal moment/unit length along the yield line is m. Key lines are at right angles to reinforcements as shown in Figure 5.

Moment Along Yield Line: Consider an element of slab with resisting ultimate moments m_x and m_y, (Figure 6). If m_{nb} and m_{nt} are the normal and twisting moments respectively, both per unit length applied along the yield line which makes an angle α with the x axis. Then, resolving vectors and writing equilibrium equations along the n and t directions, gives:

$$m_{nb}\,\ell = m_y\,(\ell\,\mathrm{Sin}\,\alpha)\,\mathrm{Sin}\,\alpha + m_x\,(\ell\,\mathrm{Cos}\,\alpha)\,\mathrm{Cos}\,\alpha. \qquad (1)$$

Substituting $\mathrm{Cos}^2\alpha = \frac{1 + \mathrm{Cos}\,2\alpha}{2}$ and $\mathrm{Sin}^2\alpha = \frac{1 - \mathrm{Cos}\,2\alpha}{2}$ in equation (1) yields,

$$m_{nb} = m_y\left(\frac{1-\mathrm{Cos}\,2\alpha}{2}\right) + m_x\left(\frac{1 + \mathrm{Cos}\,2\alpha}{2}\right) = \frac{m_y}{2} + \frac{m_x}{2} - \frac{m_y}{2}\,\mathrm{Cos}2\alpha + \frac{m_x}{2}\,\mathrm{Cos}2\alpha \qquad (2)$$

or,

$$m_{nb} = \frac{m_y + m_x}{2} + \frac{m_y - m_x}{2}\,\mathrm{Cos}\,2\alpha \qquad (3)$$

Also,

$$m_{nt}\,\ell = m_y\,(\ell\,\mathrm{Sin}\alpha)\,\mathrm{Cos}\alpha - m_x\,(\ell\,\mathrm{Cos}\alpha)\,\mathrm{Sin}\alpha = (m_y - m_x)\,\ell\,\mathrm{Sin}\alpha\mathrm{Cos}\alpha \qquad (4)$$

$$m_{nt} = \frac{m_y - m_x}{2}\,\mathrm{Sin}2\alpha \qquad (5)$$

For an isotropic slab $m_y = m_x = m$, therefore

$$m_{nb} = m\,\mathrm{Cos}^2\alpha + m\,\mathrm{Sin}^2\alpha = m \qquad (6)$$

$$m_{nt} = \frac{m-m}{2}\,\mathrm{Sin}2\alpha = 0 \qquad (7)$$

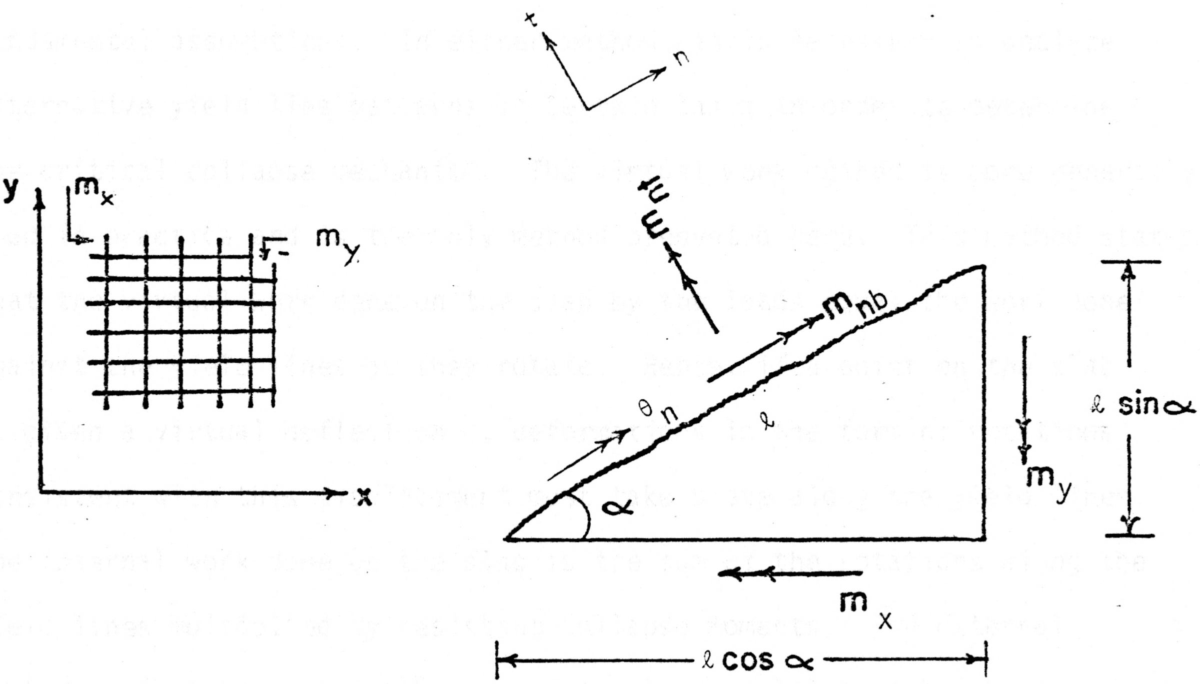


Fig. 6 Bending and Twisting Moment on Yield Line.

In using equations (3) and (5), it is important to note that α is the counterclockwise angle measured from positive x axis.

Method of Analysis: There are two methods of analysis:

1. The virtual work method.
2. The equilibrium method.

Both methods are based on the postulated yield line pattern and fundamental assumptions. In either method, it is necessary to analyze alternative yield line patterns in certain cases in order to determine the critical collapse mechanism. The virtual work method is more generally used in practice and is the only method presented here. This method states that the virtual work done on the slab by the loads equal the work done against the yield lines as they rotate. Hence, if a point on the slab is given a virtual deflection δ, deformations in the form of rotations consistent with this displacement must take place along the yield lines. The internal work done on the slab is the sum of the rotations along the yield lines multiplied by resisting collapse moments. The external work done by the loads is the sum of the loads multiplied by their respective deflections. Equating internal and external work gives the relation between resistance moment in the slab and collapse load. To find the virtual work due to external load, consider a particular rigid region of slab which is in equilibrium under the action of forces. The work done by the external load on the small element of sides dx, dy will be $\iint w\delta dxdy$. If the slab contains many regions, the total virtual work done on the slab will be the summation of $\iint w\delta dxdy$, therefore,

$$\text{total virtual work} = \Sigma\ [\iint w\delta dxdy]_{\text{each region}} \tag{8}$$

where:

w = distributed load on the slab at collapse

δ = virtual displacement

For a uniformly distributed load, equation (8) may be written as

$$w\,\Sigma\iint \delta dxdy = w\Sigma(\text{Volume under each region of slab}). \tag{9}$$

The energy absorbed during rotation along the yield line is the total moment along the yield line multiplied by rotation of the yield line. If θ is the rotation which is a function of δ, then the total internal work done on all yield lines is given by,

$$\text{Internal work done} = \Sigma(m\ell\theta) \tag{10}$$

where:

θ = angle of rotation

ℓ = length of yield line

By equating external work done to internal energy absorbed, we get:

$$\Sigma\iint w\delta dxdy = \Sigma\ (m\ell\theta) \tag{11}$$

From the above equation, which is called "work equation", the ultimate load (or moment) can be calculated. The failure load found by this equation is not necessarily the smallest value of load for a given value of bending moment. If the yield lines are moved to other positions while retaining the same basic shape of the pattern, there exists a certain worst layout where the failure load is minimum for a given resisting moment capacity. This layout can be established by trial and error, or by a differential equation. Since the general pattern of the yield lines is often defined by unknown parameters, the work equation is obtained in terms of these parameters in the form of $m = w\ (\alpha, \beta, --- \phi)$. Only one equation relating m and w is obtained from virtual work. Since m must be maximum (or w minimum),

$$\frac{\partial m}{\partial \alpha} = \frac{\partial m}{\partial \beta} = ---- = \frac{\partial m}{\partial \phi} = 0 \tag{12}$$

By the above equation the ultimate load can be obtained.

To work with the virtual work method it is of great convenience to express equation (10) in a vector form. The total moment vector along the yield line M_n and the rotation vector about the axis of rotation θ_n can be expressed in x and y coordinates. Hence, the internal work

$$E = \vec{M}_n \cdot \vec{\theta}_n \tag{13}$$

may be written as

$$E = m_x \ell_x \theta_x + m_y \ell_y \theta_y \tag{14}$$

where ℓ_x and ℓ_y are the projected length of the yield line in x and y directions, θ_x and θ_y are components of θ_n along the x and y directions, and m_x and m_y are ultimate moment capacities of the reinforced section per unit length, with respect to x and y axes, respectively. (See Appendix for proof of equation 14.)

Example 1: For the isotropically-reinforced square slab find the ultimate load. Assume the load is uniformly distributed and the slab is simply-supported.

Solution: Assume the yield line pattern shown in Figure 7a. Since the slab is equally reinforced, the yield moment along the diagonal yield line will be m. For a small deflection at e the rotations at the yield lines are as shown in Figure 7b which is a view on the diagonal yield line deb. The center of gravity of each of the triangular segments of the slab deflects by 1/3 δ, hence,

External work done = $\Sigma(w\delta)$ = 1/3 $w\delta\ell^2$ (15)

From Figure 7b, the rotation at yield line aec is $\theta = \frac{2\delta\sqrt{2}}{\ell}$,

and hence, internal work done on ac = $m\theta\ell\sqrt{2} = m\,\frac{2\sqrt{2}}{\ell}\,\delta\ell\sqrt{2} = 4m\delta$ (16)

Total internal work = $2 \cdot 4m\delta = 8\delta m$ (17)

Equating equations (15) and (17) gives:

$1/3\ w\delta\ell^2 = 8m\delta$ or, (18)

$w = 24m/\ell^2$ (19)

<u>Example 2</u>: For an orthotropically reinforced rectangular slab, simply supported and uniformly loaded, find the ultimate load.

Solution: Assume the ultimate moment on yield lines parallel to the short side is μm. In the yield line pattern assumed in Figure 8, the angle ϕ is unknown and the correct yield line pattern will be the one corresponding to the value of ϕ which gives the lowest value of the collapse load.

For a deflection of unity at yield line ef,

volume under region 1 = $1/4\ \ell^2(\alpha - \tan\phi) + 2 \cdot 1/2\,\frac{\ell\tan\phi}{2} \cdot 1/2\ \ell \cdot 1/3$

$= 1/4\ \ell^2\ (\alpha - 2/3\tan\phi)$ (20)

volume under region 2 = $1/2 \cdot 1/2\ \ell\tan\phi \cdot \ell \cdot 1/3 = 1/12\ \ell^2\tan\phi$ (21)

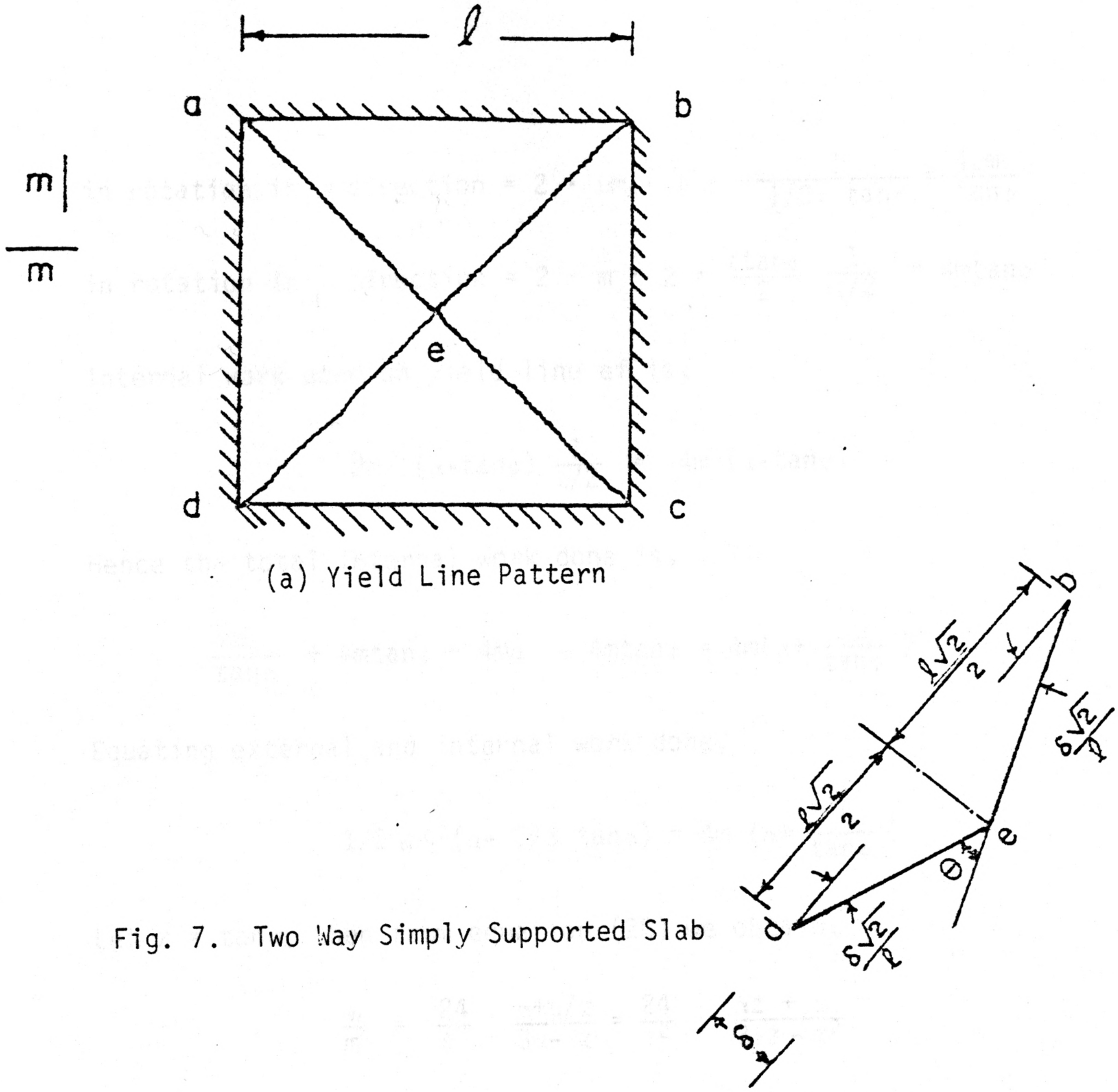
Total volume = $2 \cdot 1/4\ \ell^2(\alpha - 2/3\tan\phi) + 2 \cdot 1/12\ \ell^2\tan\phi$

$= 1/2\ \ell^2(\alpha - 1/3\tan\phi)$ (22)

Using equation (9) we obtain:

Total external work done = $\frac{W}{2}\,\ell^2(\alpha - 1/3\tan\phi)$ (23)

Using equation (14) the internal work done on yield lines ae, bf, cf, and de, may be obtained as follows:



(a) Yield Line Pattern

(b) Rotations

Fig. 7. Two Way Simply Supported Slab

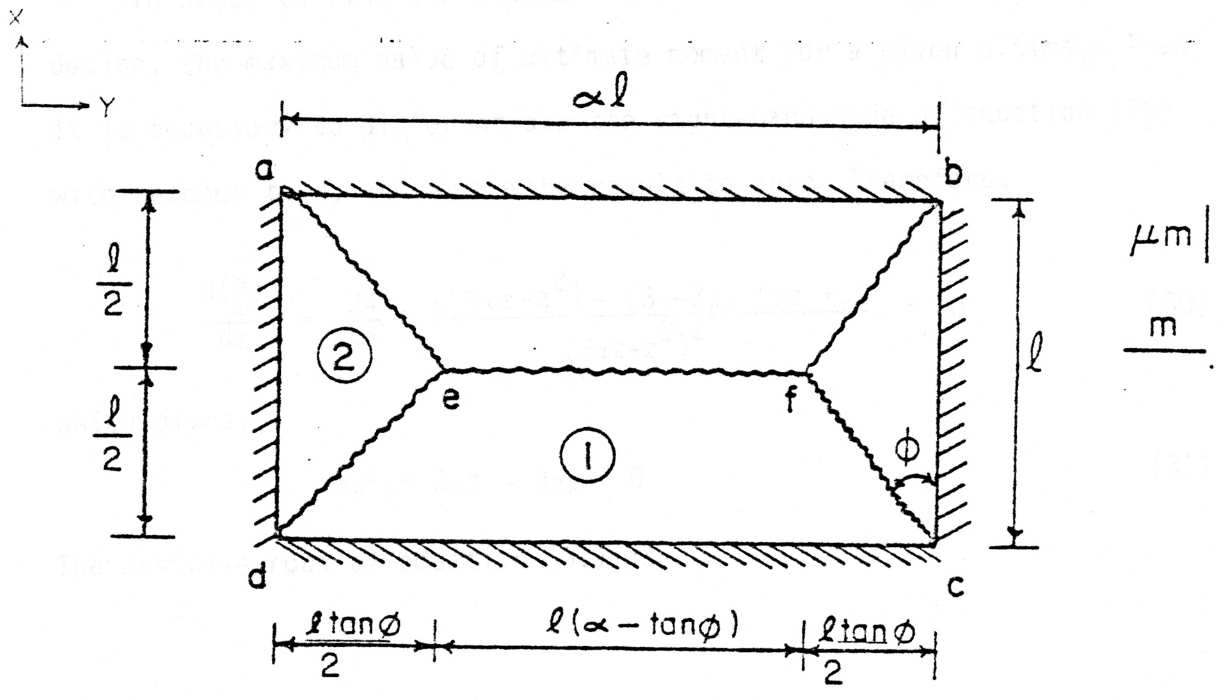


Fig. 8. Yield Line Pattern For Two-Way Rectangular Slab

in rotation in y direction $= 2 \cdot \mu m \cdot \ell \cdot \dfrac{1}{1/2\ell \tan\phi} = \dfrac{4\mu m}{\tan\phi}$ (24)

in rotation in x direction $= 2 \cdot m \cdot 2 \cdot \dfrac{\ell\tan\phi}{2} \cdot \dfrac{1}{\ell/2} = 4m\tan\phi$ (25)

Internal work done on yield line ef is,

$$2m\ell\,(\alpha-\tan\phi)\frac{1}{\ell/2} = 4m\,(\alpha-\tan\phi) \tag{26}$$

Hence the total internal work done is,

$$\frac{4m\mu}{\tan\phi} + 4m\tan\phi + 4m\alpha - 4m\tan\phi = 4m(\alpha+\frac{\mu}{\tan\phi}) \tag{27}$$

Equating external and internal work done,

$$1/2\,w\,\ell^2(\alpha-1/3\tan\phi) = 4m\,(\alpha+\frac{\mu}{\tan\phi}) \tag{28}$$

Let $z = \tan\phi$, then from equation (28) we obtain,

$$\frac{w}{m} = \frac{24}{\ell^2}\cdot\frac{\alpha+\mu/z}{3\alpha-z} = \frac{24}{\ell^2}\cdot\frac{\alpha z+\mu}{3\alpha z-z^2} \tag{29}$$

In order to find the minimum value of the collapse load, or, for design, the maximum value of ultimate moment for a given ultimate load it is necessary to differentiate the right-hand side of equation (29) with respect to z, and equate the result to zero. Therefore,

$$\frac{d(\frac{w}{m})}{dz} = \frac{24}{\ell^2}\cdot\frac{\alpha(3\alpha z-z^2)-(3\alpha-2z)(\alpha z+\mu)}{(3\alpha z-z^2)^2} = 0 \tag{30}$$

which gives,

$$\alpha z^2 + 2\mu z - 3\alpha\mu = 0 \tag{31}$$

The positive root of equation (31) is

$$z = \tan\phi = \sqrt{(3\mu + \frac{\mu^2}{\alpha^2})} - \frac{\mu}{\alpha} \tag{32}$$

If $y = \frac{f(x)}{g(x)}$ we have $y' = \frac{f'(x)\ g(x) - g'(x)\ f(x)}{[g(x)]^2}$. Suppose, x is a root of the equation $y' = 0$. Then $f'(x_o)\ g(x_o) - g'(x_o)\ f(x_o) = 0$ which leads to

$$\frac{f(x_o)}{g(x_o)} = \frac{f'(x_o)}{g'(x_o)} \text{, provided } g'(x_o) \neq 0 \tag{33}$$

Applying this result, to equation (29) gives,

$$\frac{w}{m} = \frac{24}{\ell^2} \cdot \frac{-\mu/z^2}{-1} = \frac{24}{\ell^2} \cdot \frac{\mu}{z^2} \tag{34}$$

Substituting $z = \tan\phi$ in equation (34) leads to,

$$w = \frac{24m}{\ell^2} \cdot \frac{\mu}{\tan^2\phi}$$

where (35)

$$\tan\phi = \sqrt{3\mu + \frac{\mu^2}{\alpha^2}} - \frac{\mu}{\alpha} .$$

Assume the slab is continuous over all four sides. As a result of this continuity, collapse will not take place until the four negative yield lines at the supports have formed. The negative steel in the top of the slab at the supports is assumed to be such that the ultimate moment of yield lines parallel to the long sides is m' and parallel to the short sides is μm'.

Equation (14) can be used to find the work done on negative yield lines. Therefore,

$$\text{total work done on negative yield lines} = 2\left[m'\alpha\ell\frac{2}{\ell} + 2\frac{\mu m'}{\tan\phi}\right] = 4m'(\alpha + \frac{\mu}{\tan\phi}) \tag{36}$$

Adding equation (36) and (28) gives the total internal work done in rotation at yield lines as

$$\text{internal work done} = 4(m + m')\left(\alpha + \frac{\mu}{\tan\phi}\right) \tag{37}$$

It will seem that equation (37) is the same as equation (28) with the moment increased by m'. As a result the collapse load may be written down directly as,

$$w = \frac{24\,(m + m')}{\ell^2} \cdot \frac{\mu}{\tan^2\phi}$$

where (38)

$$\tan\phi = \sqrt{3\mu + \frac{\mu^2}{\alpha^2}} - \frac{\mu}{\alpha} \quad .$$

Example 3: Find the ultimate load for an isotropically reinforced square slab, supported at the corners only, and uniformly loaded. Solution: In this case it is geometrically possible for failure to occur by the formation of diagonal yield lines, and the ultimate load will be the same as the results of example 1, i.e. $w = \frac{24m}{\ell^2}$. However, the axes of rotation may pass over the column heads at any angle, so it is also possible that a yield line pattern as shown in Fig. 9a, may develop. The rotation at yield lines are shown in Fig. 9b for a deflection of unity at center.

$$\text{External work done} = w\ 1/2\ell^2 = 1/2\ w\ell^2 \tag{39}$$

$$\text{Internal work done} = m2\ell x\ \frac{2}{\ell} = 4m \tag{40}$$

Equating external and interal work done, leads to

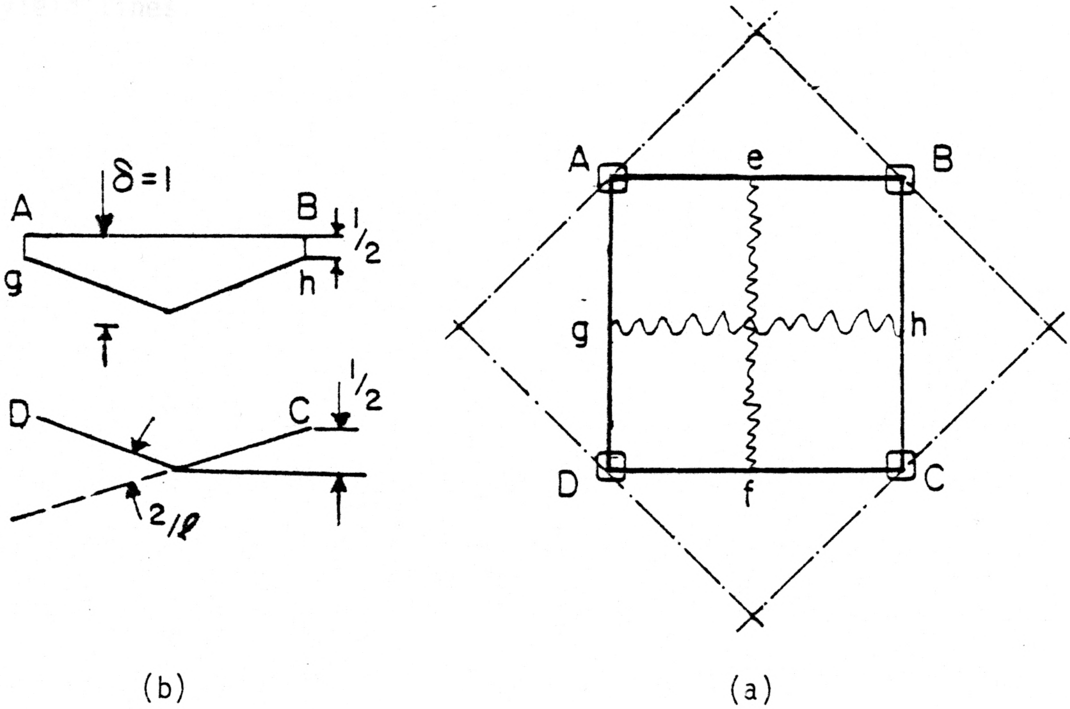


Fig. 9. Uniformly loaded square slab supported at corners with free edges.

$$w = \frac{8m}{\ell^2} \tag{41}$$

It is worth mentioning again that all possible yield line patterns or modes must be analysed. Since each result is an upper bound solution of the collapse load, the least upper bound is the correct solution. In this example the lower upper bound solution is 1/3 of the value for diagonal yield lines.

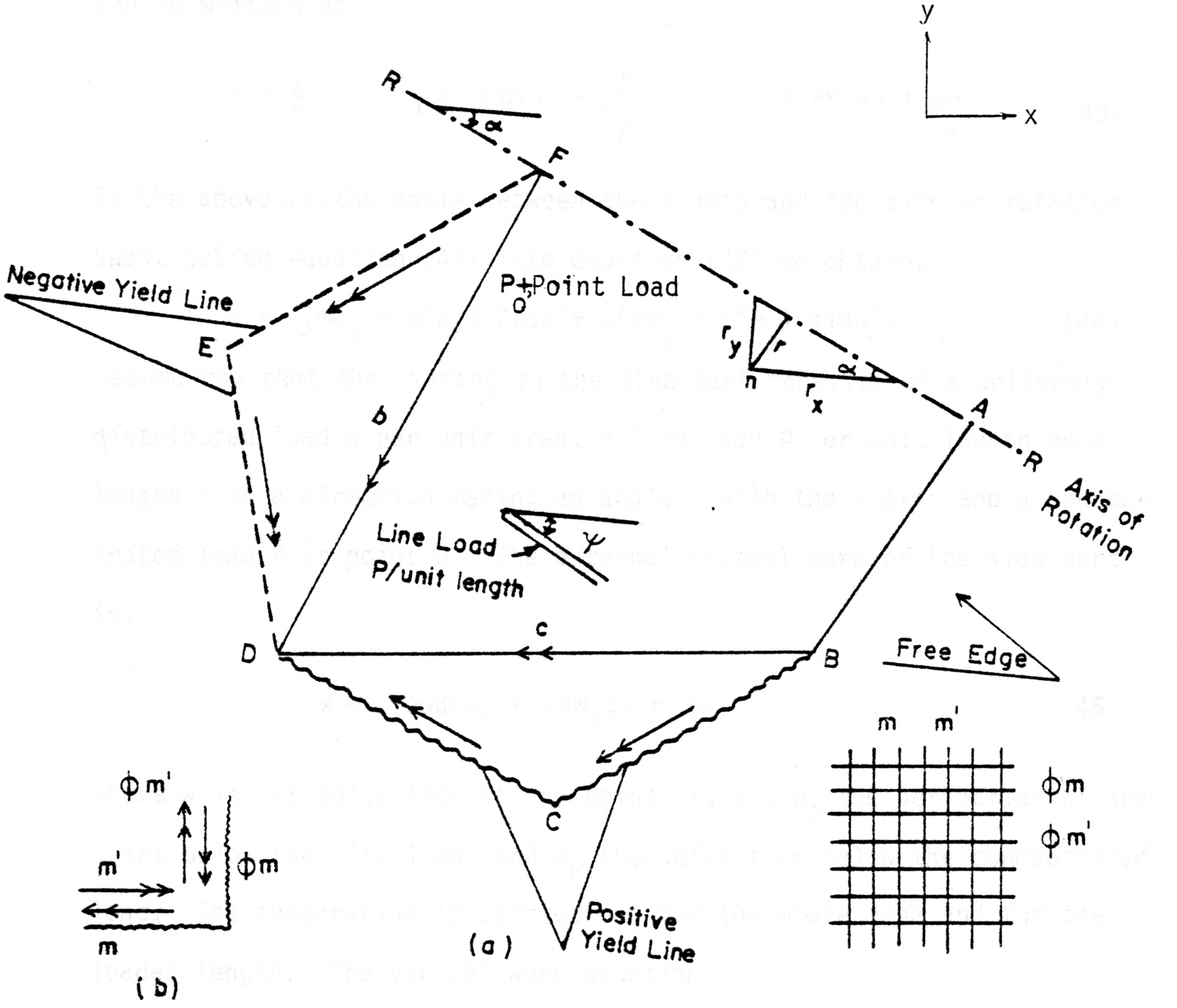
Affinity theorem: An orthotropically reinforced slab is one in which the moment per length of slab is different in two orthogonal directions. This implies a different percentage of steel in the two directions. The analysis of certain types of slab was simplified by Johanson (6) to that of an "affine" slab. An affine slab is an isotropic slab which, for purposes of analysis, may be considered to be equivalent to an orthotropic slab and is obtained by transforming the shape and loading of the orthotropic slab according to certain rules. This equivalent isotropic slab gives the same solution, i.e. value of m, as the orthotropic slab.

Consider a part of a slab ABCDEF shown in Fig. 10a limited by positive and negative yield lines and a free edge, assumed to rotate through a virtual angle θ about an axis of rotation R-R. Assume bottom and top reinforcement are placed in the x and y directions. Let the reinforcement in the y direction provide ultimate moments of m and m', and let the corresponding values in the x direction be ϕm and ϕm'. This means that the ratio of the top to bottom reinforcement is the same in both directions. The ultimate moments in the x and y directions are indicated by vectors in Fig. 10b. From equation (14) we have

$$E = m_x \ell_x \theta_x + m_y \ell_y \theta_y \tag{14}$$

Using this equation, the internal virtual work for this slab part is,

$$U = (mc_x + m'b_x)\theta_x + \phi(mc_y + m'b_y)\,\theta_y \tag{42}$$



Ultimate moments:

m and ϕm positive
m' and ϕm' negative

Moments of resistance
Provided by reinforcement
in x and y direction

Fig. 10. Conversion of an orthotropic slab to an affin slab.

Where c_x, c_y and b_x, b_y are the projections in the x and y directions of the lengths c and b, and θ_x and θ_y are the x and y components of the rotation vector $\vec{\theta}$. Assuming that a unit deflection occurs at a point n, a distance r from the axis R-R, the rotation θ and its components can be written as

$$\theta = \frac{1}{r} \qquad \theta_x = \theta \operatorname{Cos}\alpha = \frac{1}{r_y} \qquad \theta_y = \theta \sin\alpha = \frac{1}{r_x} \tag{43}$$

In the above α is the angle between the x axis and the axis of rotation. Substituting equation (43) into equation (42) we obtain,

$$U = [(mc_x + m'b_x) \operatorname{Cos}\alpha + \phi(mc_y + m'b_y) \sin\alpha]\theta \tag{44}$$

Assume now that the loading on the slab part consists of a uniformly distributed load q per unit area, a line load p per unit length on a length ℓ in a direction making an angle Ψ with the x axis and a concentrated load P at point O. The external virtual work of the slab part is,

$$W = \int\int qwdxdy + \int pw_\ell d\ell + Pw_P \tag{45}$$

where w is the deflection at any point (x, y), w_ℓ the deflection of any point below the line load, and w_P the deflection below the concentrated load. The integration is carried out for the whole area and for the loaded length. The virtual work equation is

$$\Sigma \left[(mc_x + m'b_x) \frac{1}{r_y} + \phi(mc_y + m'b_y) \frac{1}{r_x}\right] = \Sigma \left(\int\int qwdxdy + \int pw_\ell d\ell + Pw_P\right) \tag{48}$$

where the summation is for all slab parts.

Consider an affine slab equally reinforced in the x and y directions so that the ultimate positive and negative moments are m and m' respectively. Suppose this affine slab has all its dimensions in the x direction equal to those of the actual slab multiplied by λ. The pattern of fracture remains similar and the corresponding points can still have the same vertical displacements. The internal virtual work for the part of the affine slab is

$$U' = (m\lambda c_x + m'\lambda b_x)\frac{1}{r_y} + (mc_y + m'b_y)\frac{1}{\lambda r_x} \tag{49}$$

Let the load on the affine slab be a distributed load q' per unit area, a line load p' per unit length, and a concentrated load P'. The external work for this part of affine slab is

$$W' = \iint q'w\lambda dxdy + \int p'w_\ell \sqrt{dy^2 + \lambda^2 dx^2} + P'w_p \tag{50}$$

Dividing both internal and external work by λ will not change the work equation, which then becomes

$$\Sigma(mc_x + m'b_x)\frac{1}{r_y} + \frac{1}{\lambda^2}(mc_y + m'b_y)\frac{1}{r_x} = \Sigma\left(\iint q'wdxdy + \int p'w_\ell\sqrt{\frac{dy^2}{\lambda^2} + dx^2} + \frac{P'}{\lambda}w_p\right) \tag{51}$$

All terms of virtual work equations (48) and (51) are identical, provided that

$$\phi = \frac{1}{\lambda^2} \qquad \text{or } \lambda = \sqrt{\frac{1}{\phi}} \tag{52}$$

$$q' = q \tag{53}$$

$$p'\sqrt{\frac{dy^2}{\lambda^2} + dx^2} = p d\ell$$

or

$$p' = \frac{p}{\sqrt{\phi \text{Sin}^2\Psi + \text{Cos}^2\Psi}} \tag{54}$$

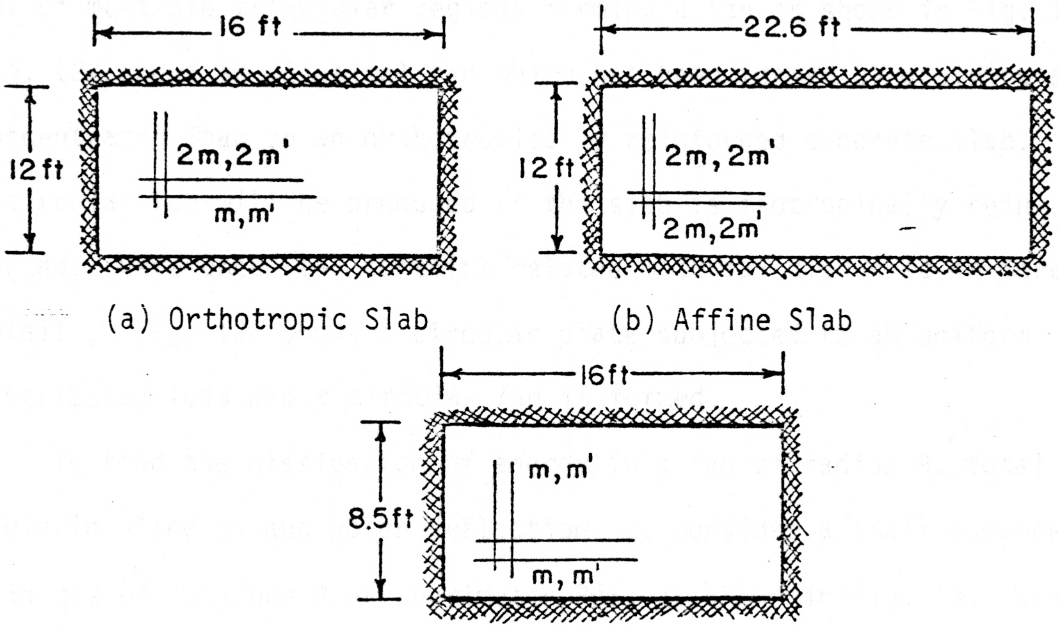
and

$$P' = P\sqrt{\frac{1}{\phi}} \tag{55}$$

This means that an orthotropic slab with positive and negative ultimate moments m and m' in the x direction and ϕm and ϕm' in the y direction, and be analyzed as an isotropic slab with moments m and m' and the linear dimensions in the x multiplied by $\sqrt{\frac{1}{\phi}}$. The intensity of a uniformly distributed load remains the same. A linear load has to be multiplied by $\frac{1}{\sqrt{\phi \text{Sin}^2\Psi + \text{Cos}^2\Psi}}$, Ψ being the angle between the load line and the x axis. The concentrated load has to be multiplied by $\sqrt{\frac{1}{\phi}}$.

Example 4: A rectangular orthotropically reinforced slab is shown in Fig. 11. Find the dimensions of an isotropic affine slab.

Solution: We have $\phi = 1/2$, so that the 16-ft side is to be changed to $16\sqrt{\frac{1}{\phi}} = 22.6$ ft. For an isotropic affine slab with ultimate moment of 2m and 2m' see Fig. 11b. The same orthotropic slab can also be analyzed as an isotropic slab for which the side 16 ft remains unchanged and the 12 ft side is changed to $\frac{12}{\sqrt{2}} = 8.5$ ft, the ultimate moment being m and m' (See Fig. 11c).



(c) Alternative Affine Slab

Fig. 11 Orthotropic and Affine Slabs

Dissipation of Energy in Circular Fans of Yield Lines

Straight yield lines have been studied in the previous part of this report and an accurate value of collapse load was determined. There are certain circumstances in which a "fan mechanism" could be produced in slab. For example in the corner of the slab it is common to see a set of yield lines radiating from a common focus, and forming an elaborate cut-off, whereby a simple corner "lever" is replaced by a set of multiple triangular regions forming a fan as shown in Fig. 12b. Fig. 12a shows an eliptical fan which has been formed by applying a concentrated load on an orthotropically reinforced concrete slab. A circular fan will be produced if the slab is isotropically reinforced. Reference (4) contains related material discussed in more detail . Fig. 12c shows a circular plate subjected to an uniform distributed load and a circular fan is formed.

To find the dissipation of energy in a fan of radius R, total angle in plane ϕ, and polar deflection Δ, consider a small component triangle of latitude R and perimeter R$d\phi$, as shown in Fig. 13. Using directions r and ϕ as in polar coordinates, then the vector method for finding the dissipation of energy requires that, for every small triangular rigid element of the fan the concept of dissipation of energy must be referred to axes r and ϕ which are locally at right angles namely.

$$E_D = \Sigma\,(m_r \ell_r \theta_r + m_\phi \ell_\phi \theta_\phi) \qquad (56)$$

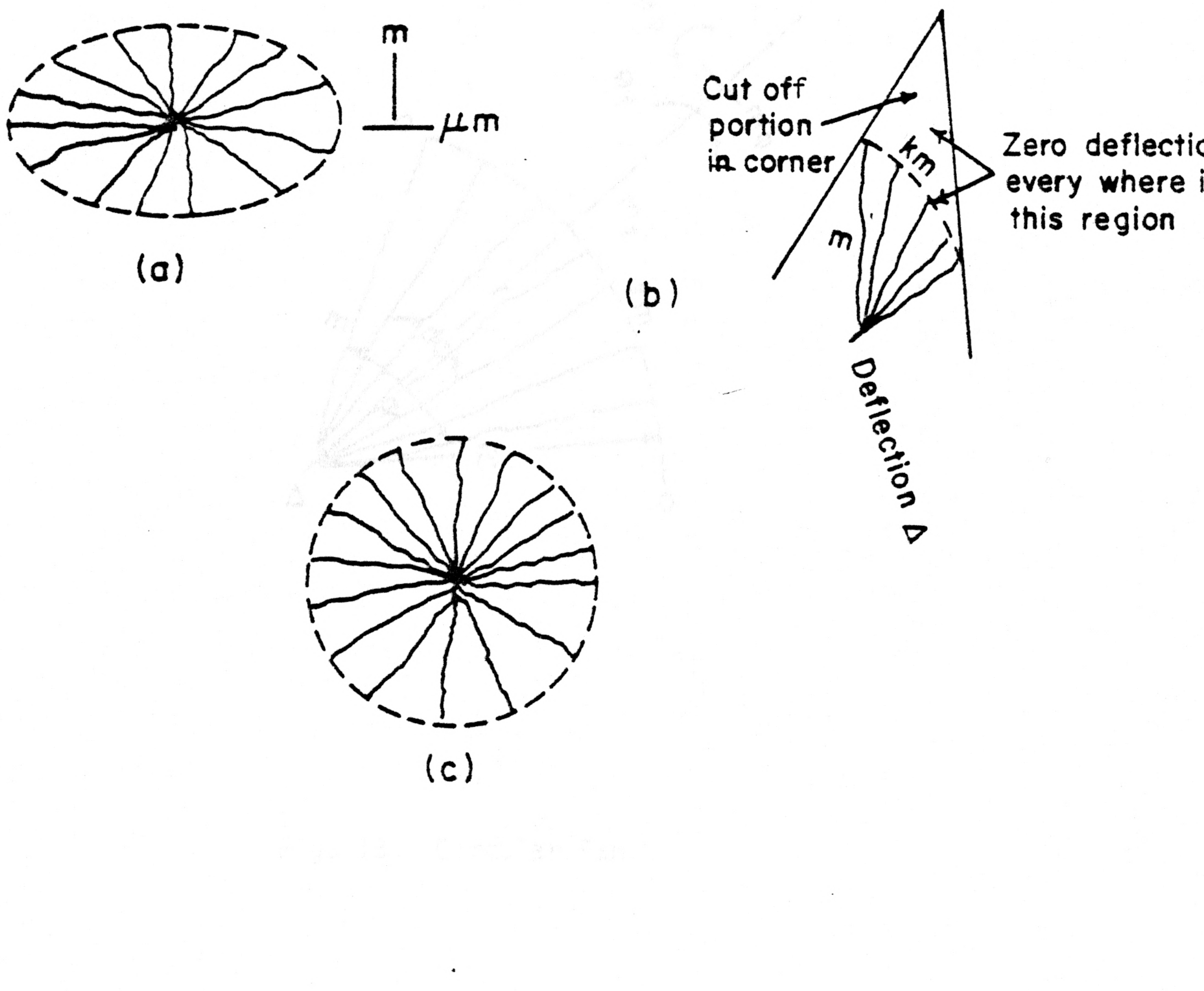


Fig. 12. Various Types of Fan Yield Lines.

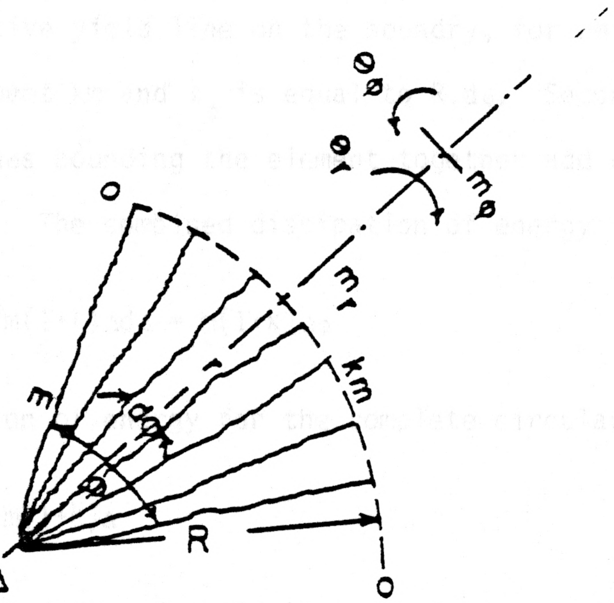


Fig. 13. Circular Fan.

The fact that there is zero displacement around the boundary of the fan makes the rotation of the rigid element about the radius, equal to zero. This leaves only m_ϕ, ℓ_ϕ and θ_ϕ to discuss. θ_ϕ is the rotation of the element about circumferential axis ϕ, so that $\theta_\phi = \Delta/R$. Moreover the quantity $m_\phi \ell_\phi$ is split into two parts. Firstly there is the energy dissipated by the negative yield line on the boundry, for which m_ϕ is the circumferential moment km and ℓ_ϕ is equal to $R.d\phi$. Secondly the two positive radial lines bounding the element together add up to the same vector length $Rd\phi$. The combined dissipation of energy is

$$E_D = \int_0^\phi m(1+k)\Delta d\phi = m(1+k)\Delta\phi \qquad (57)$$

Therefore the dissipation of energy for the complete circular fan is

$$E_D = 2\pi m(1+k)\Delta \qquad (58)$$

Sometimes an inverted fan of yield lines is formed in the slab around the column support (e.g. in flat slab). In this case the radial yield lines now have negative yield moment km, and the boundary of the fan has positive yield moment m, but there is no change in the expression (58) for dissipation of energy. Note that the angle ϕ must be in radians.

Example 5: Circular slab with distributed load p/unit area, and point load P at the center.

Solution: Consider the slab shown in Fig. 12 c, where the negative yield line has a radius R. The expenditure of energy by the external load is equal to

$$E = P\Delta + 1/3\ \pi R^2 \Delta p \tag{59}$$

From equation (58) the dissipation of energy is

$$E_D = 2\pi m\ (1 + k)\Delta \tag{58}$$

Equating equations (58) and (59) we obtain

$$P\Delta + 1/3\pi R^2 \Delta p = 2\pi m\ (1 + k)\Delta \tag{60}$$

Thus it gives

$$m = \frac{P + 1/3\pi R^2 p}{2\pi(1 + k)} \tag{61}$$

If there is no point load, then

$$m = \frac{pR^2}{6(1 + k)} \tag{62}$$

If the distributed load is negligible compared with the point load, then

$$m = \frac{P}{2\pi(1 + k)} \tag{63}$$

In equation (61) and (62) m increases with increase of the radius R of the fan, which means that the radius of the fan in Fig. 13 would extend right up to a circular boundry. Equation (63) is independent of radius R, which means with only a point load a complete fan of any radius could form with no change of collapse load $P = 2\pi m(1 + k)$.

Example 6: Isotropically reinforced circular foundation slab. For this example the result of previous example may be used. If the floor slab is continous with the walls the internal work done can be calculated by use of equation (58) which is equal to $2\pi m(1+k)\Delta$, but if the walls are free at the foot then $k = 0$ and the internal work done will be $2\pi m\Delta$. If it is assumed that the total load w on the soil at failure of the slab is applied through the walls, (See Fig. 14a), then the external work done by the contact pressure, and the value of the ultimate load, depend on the way in which the contact pressure between soil and the slab is distributed. If pressure is assumed to be uniform, the contact pressure will be, $p = \frac{w}{\pi r^2}$ and external work done will be

$$\text{External work done} = 1/3\pi r^2 p\Delta = 1/3\ w\Delta \tag{64}$$

Equating external and internal work done, leads to:

$$w = 6\pi m(1 + k) = 18.8\ m(1 + k) \tag{65}$$

and for the free edge (k = 0):

$$w = 18.8m \tag{66}$$

In reality the distribution of contact pressure depends upon the characteristics of the soil, and of the rigidity of the foundation. Theoretically if the soil is cohesionless, and if the base can be considered rigid, the pressure varies from zero at the edge to a maximum at the center of the base. If it is assumed, for the circular slab under study, that the pressure distribution may be represented by a paraboloid, as shown in Fig. 14b, then the volume of the paraboloid is equal to the total load on the soil, w, therefore

$$\text{Maximum pressure, } p_m = \frac{w}{1/2\pi r^2} \tag{67}$$

$$\text{Pressure at radius } r_A,\ p_A = p_m\left(1 - \left(\frac{r_A}{r}\right)^2\right) \tag{68}$$

If the slab fails by conical collapse modes (forming fan yield lines), then for a centeral upward deflection of unity, the deflection at radius A may be obtained, from triangle ocd (Fig. 14c)

$$\frac{1 - \delta_A}{1} = \frac{r_A}{r} \quad \text{or} \quad \delta_A = \frac{r - r_A}{r} \tag{69}$$

The external work done by the contact pressure on an annular strip dr_A in width at radius r_A is:

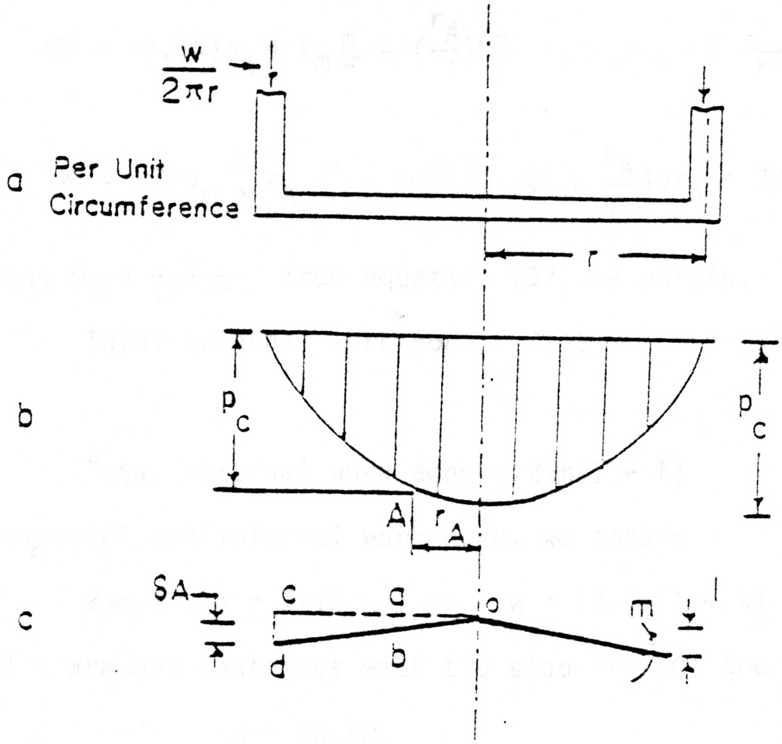


Fig. 14. Circular Foundation of Granular Soil

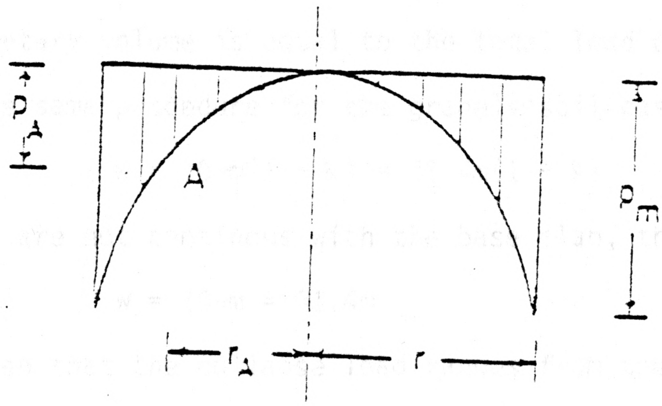


Fig. 15. Circular Foundation on Cohesive Frictionless Soil

$$dW = (P_A dA)\delta_A = p_m\left[1 - \left(\frac{r_A}{r}\right)^2\right] (2\pi r_A dr_A) \left(\frac{r - r_A}{r}\right) \qquad (70)$$

or

$$W = 2\pi p_m \int_0^r r_A \left[1 - \left(\frac{r_A}{r}\right)^2\right] \left(1 - \frac{r_A}{r}\right) dr_A = 7/30\ \pi r^2 p_m \qquad (71)$$

Substituting $p_m = \frac{w}{1/2\pi r^2}$ from equation (67) we obtain,

$$\text{Total external work done} = 7/15\ w \qquad (72)$$

and

$$\text{Total internal work done} = 2\pi m(1 + k) \qquad (73)$$

Equating external and internal work done, we obtain

$$2\pi m(1 + k) = 7/15\ w \quad \text{or} \quad w = 13.4m(1 + k) \qquad (74)$$

If the walls are not continous with the slab ($k = 0$) the collapse load is,

$$w = 13.4m \qquad (75)$$

In the other limit case, that of cohesive soil with no internal friction, the contact pressure is theoretically maximum at the edges, decreasing to the center. If it is assumed that the distribution may be represented by the complimentary volume given by the difference between the paraboloid and cylinder which contains it as shown in Fig. 15, then the complimentary volume is equal to the total load on the soil, w. Following the same procedure for the granular soil case, leads to

$$w = 10\pi m(1 + k) = 31.4m(1 + k) \qquad (76)$$

If the walls are not continous with the base slab, then

$$w = 10\pi m = 31.4m \qquad (77)$$

It can be seen that the collapse load ranges from the upper limit of 31.3m(1 + k) for cohesive soil to the lower limit, for granular soil, of 13.4m(1 + k). An intermediate value for uniformly distributed contact pressure is, 18.8m(1 + k).

In practice, however, the total load will not be applied entirely through the walls, and the external work done must be reduced by the amount of "negative" work done against the action of the weight of the floor slab, or in the case of a tank, from tank contents.

Example 7: Point load on the edge of a slab of infinite extent, (isotropically reinforced).

For this case the yield line pattern is shown in Fig. 16. It is convenient to take the unknown parameters as ϕ, the fan angle, and the angle β. Let deflection of the slab under load P be unity. The perpendicular distance h is given by $h = R\mathrm{Sin}\beta$. Using the Sine law for triangle Pab, we have

$$\frac{ab}{\mathrm{Sin}(\pi/2 - \phi/2)} = \frac{R}{\mathrm{Sin}[(\pi/2 - (\beta - \phi/2)]} \quad \text{or} \quad \frac{ab}{\mathrm{Cos}(\phi/2)} = \frac{R}{\mathrm{Cos}(\beta - \phi/2)} \tag{78}$$

which gives,

$$ab = \frac{R\ \mathrm{Cos}(\phi/2)}{\mathrm{Cos}(\beta - \phi/2)} \tag{79}$$

The angle of rotation about axis ab is $1/h$ or $1/R\mathrm{Sin}\beta$. To find the dissipation of energy for region 1, with the axis of rotation ab, equation (14) can be used. Therefore

$$\text{for region 1, } E_D = km.\ \frac{R\ \mathrm{Cos}(\phi/2)}{\mathrm{Cos}(\beta - \phi/2)} \cdot \frac{1}{R\mathrm{Sin}\beta} + (m)\ (R\mathrm{Cos}\beta) \cdot \frac{1}{R\mathrm{Sin}\beta} \tag{80}$$

Therefore for region 1 and 2

$$E_D = \frac{2km\ \mathrm{Cos}(\phi/2)}{\mathrm{Cos}\ (\beta - \phi/2)\ \mathrm{Sin}\beta} + 2m\ \mathrm{Cotg}\beta \tag{81}$$

From equation (47) dissipation of energy for circular fan is,

$$E_D = m(1 + k)\phi \tag{82}$$

Therefore

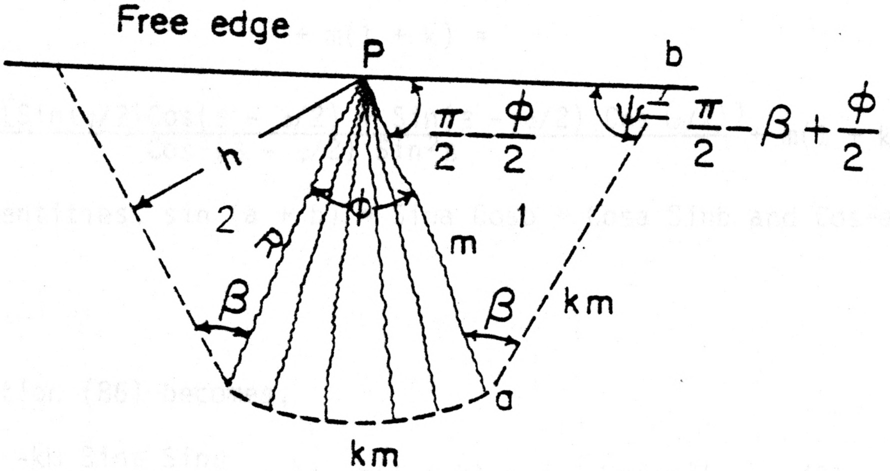


Fig. 16. Yield line Pattern for a Point Load on the Edge of a Slab of Infinite Extent.

The total dissipation of energy $= \dfrac{2km \, Cos\,(\phi/2)}{Cos(\beta - \phi/2)\, Sin\beta} + 2m\, Cot\beta$

$$+ m(1 + k)\phi \qquad (83)$$

External work done is equal to

$$\text{External work done} = P \times 1 = P \qquad (84)$$

Equating equations (83) and (84) we get.

$$P = \frac{2km\, Cos\,(\phi/2)}{Cos(\beta - \phi/2)\, Sin\beta} + 2m\, Cot\beta + m(1 + k)\phi \qquad (85)$$

m is maximum when $\frac{\partial P}{\partial \phi} = 0$, $\frac{\partial P}{\partial \beta} = 0$,

$$\frac{\partial P}{\partial \phi} = \frac{-km\, Sin\,(\phi/2)\, Cos(\beta - \phi/2)\, Sin\beta - km\, Cos\,(\phi/2) Sin(\beta - \phi/2)\, Sin\beta}{Cos^2(\beta - \phi/2) Sin^2\beta}$$

$$+ m(1 + k) =$$

$$\frac{-km\, Sin\beta[Sin(\phi/2) Cos(\beta - \phi/2) + Sin(\beta - \phi/2)\, Cos\,(\phi/2)]}{Cos^2(\beta - \phi/2)\, Sin^2\beta} + m(1 + k) \qquad (86)$$

Using identities, $sin\,(a + b) = Sina\, Cosb + Cosa\, Sinb$ and $Cos^2 a = \dfrac{1}{1 + tan^2 a}$,

equation (86) becomes,

$$\frac{\partial P}{\partial \phi} = \frac{-km\, Sin\beta\, Sin\beta}{Cos^2(\beta - \phi/2)\, Sin^2\beta} + m(1 + k) = m - km tan^2(\beta - \phi/2) \qquad (87)$$

If $\frac{\partial P}{\partial \phi} = 0$, then

$$tan(\beta - \phi/2) = \sqrt{1/k} \qquad (89)$$

Putting $\frac{\partial P}{\partial \beta} = 0$ gives

$$\frac{\partial P}{\partial \beta} = \frac{-2km\, Cos(\phi/2)[Cos(\beta - \phi/2)\, Cos\beta - Sin\beta\, Sin(\beta - \phi/2)]}{Cos^2(\beta - \phi/2)\, Sin^2\beta} - \frac{2m}{Sin^2\beta} = 0 \qquad (90)$$

After adding two fractions and ignoring the denominator, we get

$$-2km\, Cos\,(\phi/2)\, Cos\,(2\beta - \phi/2) - 2m\, Cos^2(\beta - \phi/2) = 0 \qquad (91)$$

Using the identify Cosa Cosb = 1/2 [Cos (a + b) + Cos (a - b)], the above equation gives,

$$-k\ \mathrm{Cos}2\beta - k\ \mathrm{Cos}2(\beta - \phi/2) - 2\ \mathrm{Cos}^2(\beta - \phi/2) = 0 \tag{92}$$

By use of identities $\mathrm{Cos}\ 2a = \frac{1 - \tan^2 a}{1 + \tan^2 a}$, $\mathrm{Cos}^2 a = \frac{1}{1 + \tan^2 a}$ and equation (89), equation (92) reduces to

$$-\ k\ \mathrm{Cos}2\beta - k\frac{k - 1}{k + 1} - 2\frac{k}{k + 1} = 0 \tag{93}$$

After simplification, the above equation leads to:

$$\mathrm{Cos}2\beta = -1 \quad \text{or} \quad \beta = 90^0 \tag{94}$$

Clearly from equation (89) we now have

$$\tan\ (90 - \phi/2) = \mathrm{Cot}\phi = \sqrt{1/k} \tag{95}$$

and by referring to equation (85):

$$P = m(1 + k)\phi + 2m\sqrt{k} \tag{96}$$

It can be seen that radius R of the fan is not part of the solution, so that the mechanism can be reproduced on any scale without affecting the collapse load.

CHAPTER II

Membrane Action Analysis of Concrete Plates

General Concept: Concrete is one of the groups of materials with different tensile and compressive strengths. When a plate of such material is subjected to a flexure , its moment capacity may be governed by either lower or higher rupture strength depending on boundary conditions. If the plate is simply supported, the lower rupture strength is normally critical. If the plate is laterally restrained, the higher strength may generally be utilized in resisting loads. In a plain concrete plate the tensile strength is much the weaker, and transverse load capacity is correspondingly low when the plates are simply supported. When restrained against elongation at the edges, higher compressive stress and a resultingly higher load capacity may be obtained. This behavior is termed compressive membrane action.

In an under reinforced concrete plate, two actions may occur at different stages of deformation. At deformations which are small compared to plate dimensions the compressive strength of the concrete governs, and if the plate is restrained agains elongation, arching occurs as before. At large deformations, however, the concrete may crush leaving only the tensile strength of the reinforcement to resist loading, and if the edges of plate are restrained against inward displacement the full strength of reinforcement may be developed as a tensile net.

<u>Tensile Membrane Action:</u> The load - central deflection relationship (5) for a slab restrained against in - plate motion at its boundry is shown in Fig. 17. As the load increases and inelastic action attempts to develop to the slab, the outward in plane displacements that should accompany that yielding are restrained and compressive forces generated that strengthen and stiffen the slab allowing it to develop the maximum load represented by point A. Prior to point A there is compressive membrane action. The failure at point A is a brittle punching shear failure or a brittle flexural compression failure. After that failure the slab snaps through. The resulting large deflections set-up a tensile membrane with only the reinforcement resisting the load as a hanging net. That tensile membrane if properly proportioned and detailed offers a secondly support system capable of arresting a progressive collapse. Exact solutions to the equilibrium and compatibility expression for tensile membrane can be complex (8, 2). For flat slabs, simplified expressions yielding reasonable agreement with test result are obtained if it is assumed that the membrane takes up a circular, deformed shape. The use of a catenary or parabolic shape could be realistic. However, there is little gained in either understaing or reliability with the resultant more complex expressions. Fig. 18 shows a rectangular slab, L_x by L_y, where L_y is the long direction. The slab is subjected to a uniform loading w, resisted by edge tensions per unit length of T_x in the x direction and T_y in the y-direction. The deformed shaped assumed for the slab in the x direction is shown in Fig. 19. Vertical equilibrium gives

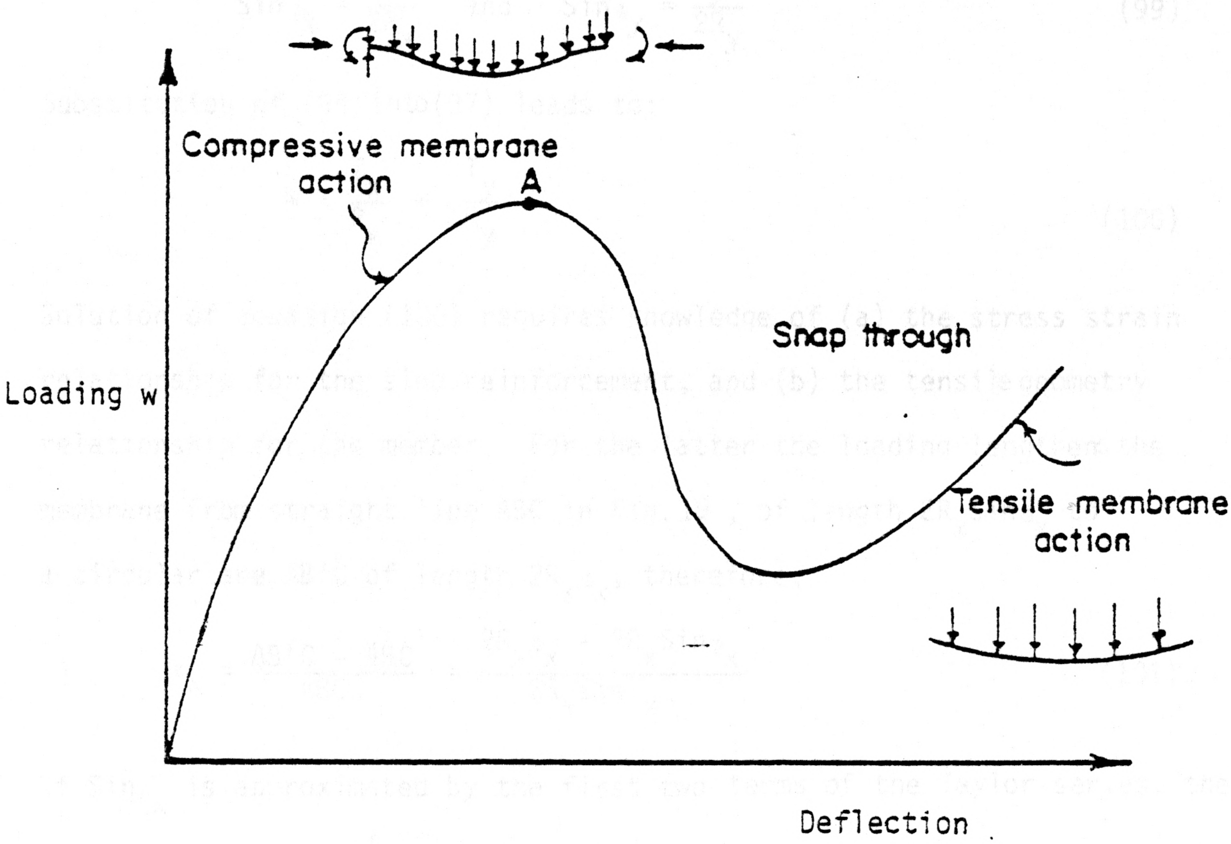


Fig. 17. Development of Tensile Membrane Action.

$$wL_xL_y = 2T_xL_y \, \mathrm{Sin}\,\phi_x + 2T_yL_x \, \mathrm{Sin}\,\phi_y \tag{97}$$

where w is the load per unit area of the slab.

Geometry gives

$$L_x = 2R_x \mathrm{Sin}\,\phi_x \quad \text{and} \quad L_y = 2R_y \, \mathrm{Sin}\,\phi_y \tag{98}$$

or

$$\mathrm{Sin}\,\phi_x = \frac{L_x}{2R_x} \quad \text{and} \quad \mathrm{Sin}\,\phi_y = \frac{L_y}{2R_y} \tag{99}$$

Substitution of (99) into (97) leads to:

$$w = \frac{T_x}{R_x} + \frac{T_y}{R_y} \tag{100}$$

Solution of equation (100) requires knowledge of (a) the stress strain relationship for the slab reinforcement, and (b) the tensile geometry relationship for the member. For the latter the loading lengthens the membrane from straight line ABC in Fig. 19, of length $2R_x\mathrm{Sin}\,\phi_x$ to a circular are AB'C of length $2R_x\phi_x$, therefore,

$$\varepsilon_x = \frac{AB'C - ABC}{ABC} = \frac{2R_x\phi_x - 2R_x\mathrm{Sin}\,\phi_x}{2R_x\mathrm{Sin}\,\phi_x} \tag{101}$$

If $\mathrm{Sin}\,\phi_x$ is approximated by the first two terms of the Taylor series, then

$$\varepsilon_x = \frac{\phi_x - (\phi_x - \phi_x{}^3/3! + ---)}{\phi_x - \phi_x{}^3/3! + ---} = \frac{\phi_x{}^3/3! + ---}{\phi_x - \phi_x{}^3/3! + ---} \tag{102}$$

If $\phi_x{}^3 << \phi_x$, (102) gives,

$$\varepsilon_x = \phi_x^2/6 \tag{103}$$

similarly

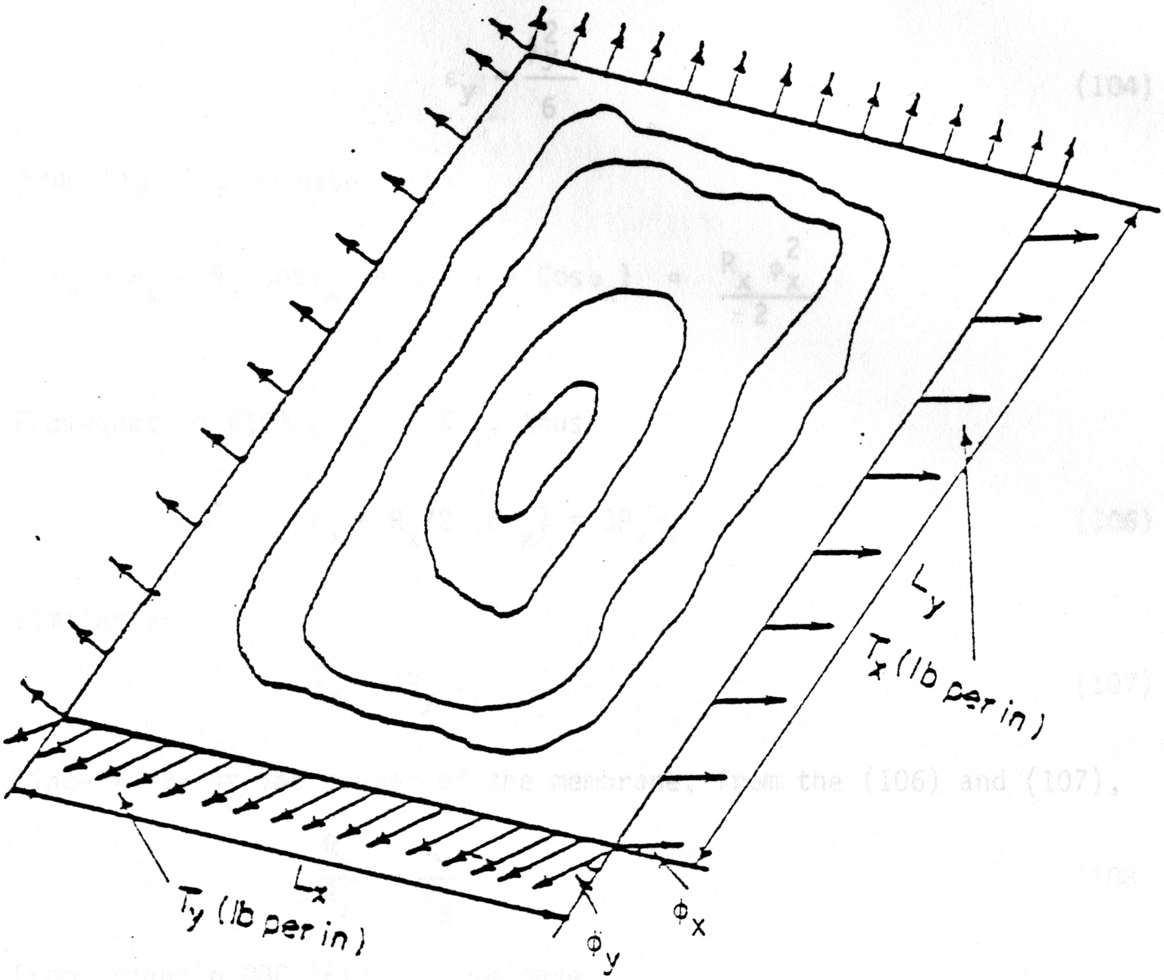


Fig. 18. Equilibrium of Rectangular Tensile Membrane.

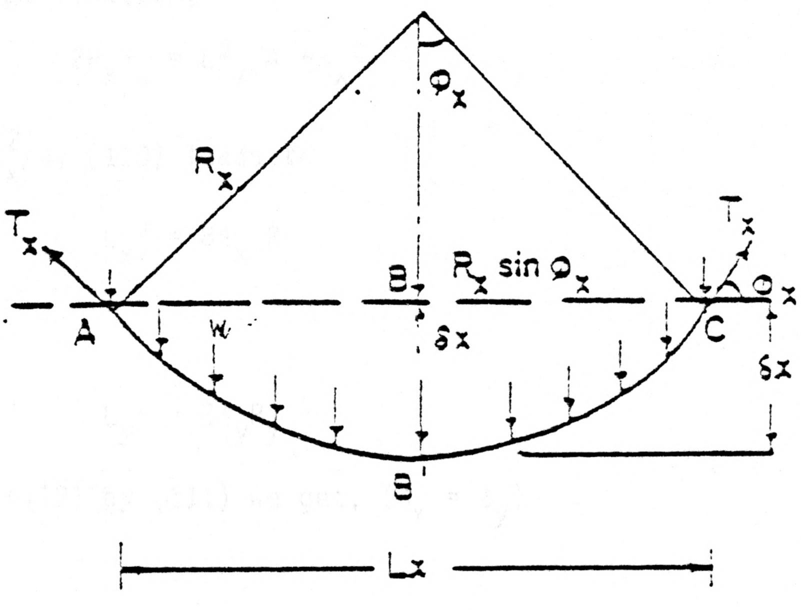


Fig. 19. Deflection in x direction for Unit Width Strip.

$$\varepsilon_y = \frac{\phi_y^2}{6} \tag{104}$$

From Fig. 18, we have

$$\delta_x = R_x - R_x \cos\phi_x = R_x (1 - \cos\phi_x) = \frac{R_x \phi_x^2}{2}$$

From equation (103), $\phi_x{}^2 = 6\varepsilon_x$, thus

$$\delta_x = R_x/2 \; (6\varepsilon_x) = 3R_x \varepsilon_x \tag{106}$$

similarly

$$\delta_y = 3R_y \; \varepsilon_y \tag{107}$$

Since $\delta x = \delta y$ at the center of the membrane, from the (106) and (107),

$$\frac{R_y}{R_x} = \frac{\varepsilon_x}{\varepsilon_y} \tag{108}$$

From triangle OBC (Fig. 19) we have

$$R_x{}^2 = (R_x - \delta_x)^2 + (L_x/2)^2 \tag{109}$$

After simplification,

$$2R_x \delta_x = L^2{}_x/4 + \delta_x{}^2 \tag{110}$$

If $\delta_x{}^2 << L_x^2/4$, (110) leads to

$$L_x{}^2 = 8\delta_x \; R_x \tag{111}$$

similarly

$$L_y{}^2 = 8\delta_y R_y \tag{112}$$

Dividing (112) by (111) we get, $(\delta_x = \delta_y)$:

$$\left(\frac{L_y}{L_x}\right)^2 = \frac{R_y}{R_x} \tag{113}$$

Equations (108) and (113) give

$$\varepsilon_x = \varepsilon_y \left(\frac{L_y}{L_x}\right)^2 \tag{114}$$

Dividing equation (97) by L_xL_y, gives

$$w = \frac{2T_x}{L_x} \text{ Sin}\phi_x + \frac{2T_y}{L_y} \text{ Sin}\phi_y \tag{115}$$

Using $\phi_x = \sqrt{6\varepsilon_x}$, and equation (114), yields

$$w = \frac{2T_x \text{Sin}\sqrt{6\varepsilon_x}}{L_x} + \frac{2T_y \text{Sin } (L_x/L_y\sqrt{6\varepsilon_x})}{L_y} \tag{116}$$

Thus for a square matrix with isotropic reinforcement ($L_x = L_y$, $\varepsilon_x = \varepsilon_y$ and $T_x = T_y$).

$$w = \frac{4T_y \text{Sin}\sqrt{6\varepsilon_x}}{L_y} = \frac{4T_x \text{Sin}\sqrt{6\varepsilon_x}}{L_x} \tag{117}$$

If the stress strain relationship for reinforcement is known the full tensile membrane behavior can be predicted by: choosing a value of ε_x; finding ε_y from equation (114), finding T_x and T_y from the reinforcement stress strain curve; calculating W from equation (116); calculating δ from equations (106) or (107); returning to step one incrementing ε_x and repeating the cycle. An alternative procedure is: calculating R_x from the equation (110), finding ε_x from the equation (106); calculating ε_y from the equation (114); finding T_x and T_y from the stress-strain curve: find w from the equation (117). The comparison of results obtained from using the two methods is shown in the Table on page (53). For most flat plate structures the midspan reinforcement is considerably less than the edge reinforcement so that the midspan reinforcement controls the capacity of the tensile membrane. The force at the midspan of the membrane is $\cos\phi$ times the force at the edges so that equations (116) and (117) can be re-expressed as equations (118) and (119) respectively. From equation (116) we have:

$$w = \frac{2T_x \sin\sqrt{6\varepsilon_x}\ \cos\sqrt{6\varepsilon_x}}{L_x \cos\sqrt{6\varepsilon_x}} + \frac{2T_y \sin(l_x/L_y\sqrt{6\varepsilon_x}) \cdot \cos(Lx/Ly\sqrt{6\varepsilon_x})}{L_y \cos(Lx/Ly\sqrt{6\varepsilon_x})} \qquad (118)$$

or

$$w = \frac{2T'_x}{L_x} \tan\sqrt{6\varepsilon_x} + \frac{2T'_y}{L_y} \tan\left(\left(\frac{L_x}{L_y}\right)\sqrt{6\varepsilon_x}\right) \qquad (119)$$

For a small angle ϕ_x

$$w = 2\sqrt{6\varepsilon_x}\left(\frac{T'_x}{L_x} + T'_y \frac{L_x}{L^2_y}\right) \qquad (120)$$

For a square panel with isotropic reinforcement,

$$w = 4\sqrt{6\varepsilon_x}\ \frac{T'_x}{L_x} = 4\sqrt{6\varepsilon_y}\ \frac{T'_y}{L_y} \qquad (121)$$

where, T'_x and T'_y are midspan force per unit width in the x and y direction.

Comparison with test data: Strengths predicted by this tensile membrane model are compared with experimental results. In Table (1), Powell's test specimens (8, 9), contained both top and bottom steel mats over the whole area of the rectangular slab. For these tests equation (116) was used by him to predict his results. Park's specimens (8) contained continuous bottom reinforcement only. The top steel was discontinued short distances from the edge and therefore equation (120), was used to predict his results. For Park's specimens A_1 and A_2 pure membrane action did not develop. Brotche and Holley (2) tested square slabs with bottom reinforcement only. Therefore equation (117) was used to predict their results. The stress strain curves for steel shown in Figs. 20 and 21 were used for the predictions listed in Table (1). There is reasonable agreement between theoretical and experimental results. In the last column of Table (1), ultimate loads for slabs using yield line theory are shown. Calculations were done based on information given in that table, from appropriate references, and the slabs were assumed to be simply supported. Failure was assumed to occur when the middle deflection δ_x was equal to 0.1 L_x.

Example 8: The following calculations show the alternative procedure to calculate the loading w. The result is compared with value given in table (1).

Let us take Powell's slab Number 47 (See table 1), and calculate the load w.

$\delta_x = 0.1 L_x = 0.1 \times 20.57 = 2.06$ in (5.32 cm)

$$R_x = \frac{\delta_x}{2} + \frac{L_x^2}{8\delta_x} = 2.06/2 + (20.57)^2/\ 8 \times 2.06 = 26.74 \text{ in. } (67.91 \text{ cm})$$

$$\varepsilon_x = \delta_x/3R_x = 2.06/3 \times 26.74 = 0.02564$$

$$\varepsilon_y = \varepsilon_x\,(L_x/L_y)^2 = 0.02564\ (20.57/36)^2 = 0.00837$$

From the reinforcement stress-strain curve (Fig. 21), we get

$\sigma_y = 31.66$ ksi $(218.45 \times 10^3$ kPa$)$

$\sigma_x = 37.80$ ksi $(260.82 \times 10^3$ kPa$)$

$A_s = 0.0039 \times 1.286 = 0.00501$ in^2/in (0.127 cm)

$T = A\sigma$

$T_x = 0.00501 \times 37.8 \times 10^3 = 189.57$ lb/in $(3.32 \times 10^4$ N/m$)$

$T_y = 0.00501 \times 31.66 \times 10^3 = 158.78$ lb/in $(2.78 \times 10^4$ N/m$)$

$$w = \frac{2T_x \operatorname{Sin}\sqrt{6\varepsilon_x}}{L_x} + \frac{2T_y \operatorname{Sin}\,(L_x/L_y)\,\sqrt{6\varepsilon_x}}{L_y} \qquad \text{(Eq. 116)}$$

$$w = \frac{2 \times 189.5 \times \operatorname{Sin}\sqrt{6 \times .002564}}{20.57} + \frac{2 \times 158.78 \times \operatorname{Sin}(20.57/36)\sqrt{6 \times 0.02564}}{36} =$$

$$7.035 + 1.988 = 9.025 \text{ psi } (62.27 \text{ kPa})$$

The theoretical value from Table (1), given by Powell, is 9.2 psi and his experimental value is 9.0 psi, which are very close to the above answer.

Example 9: Calculate the ultimate load w by yield line analysis for slabs listed in table (1). To show a sample calculation, let us take Powell's slab Number 47 from Table (1). We have the following information:

$d = 1.286$ in (3.26 cm)

$L_x = 20.57$ in (52.25 cm)

$L_y = 36$ in (97.44 cm)

$\rho_x = \rho_y = 0.0039$

$f_y = 30 \times 10^3$ psi

$f_c' = 4000$ psi

$m_u = A_s f_y (d - a/2)$

$a = A_s f_y / 0.85\, f_c'\, b$

$A_s = 0.0039 \times (1.286) = 0.00501\ \text{in}^2/\text{in}\ (0.127\ \text{cm}^2/\text{cm})$

$a = 0.00501 \times 30 \times 10^3 / 0.85 \times 4000 = 0.044$ in (1.12 cm)

$m_{ux} = 0.00501 \times 30 \times 10^3 (1.286 - 0.044/2) = 190$ lb-in/in (844 N-m/m)

$m_{uy} = 190$ lb-in/in (844 N-m/m)

From equation (35), Page (18), we have

$w = 24m/\ell^2 \cdot \mu/\tan^2\phi$

$\tan\phi = \sqrt{3\mu + \mu^2/\alpha^2} - \mu/\alpha$

$\alpha = 36/20.57 = 1.75$

$\mu = 1$ (isotropic slab)

$\tan\phi = \sqrt{3 + 1/(1.75)^2} - 1/1.75 = 1.25$

$w = 24 \times 190/(20.57)^2 \cdot 1/(1.25)^2 = 6.87$ psi (47.36 kPa)

Note that this is somewhat lower than obtained from the membrane net solution.

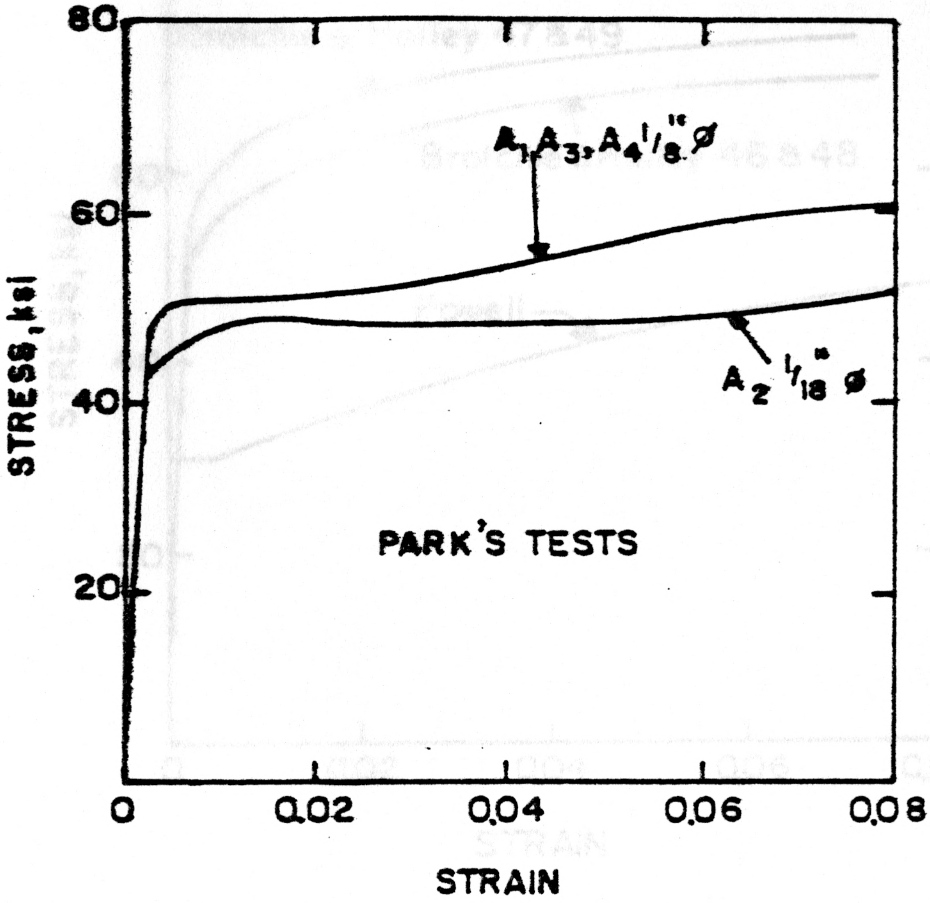


Fig. 20 Reinforcement Stress-Strain Curves for Predicting Results in Park's Test

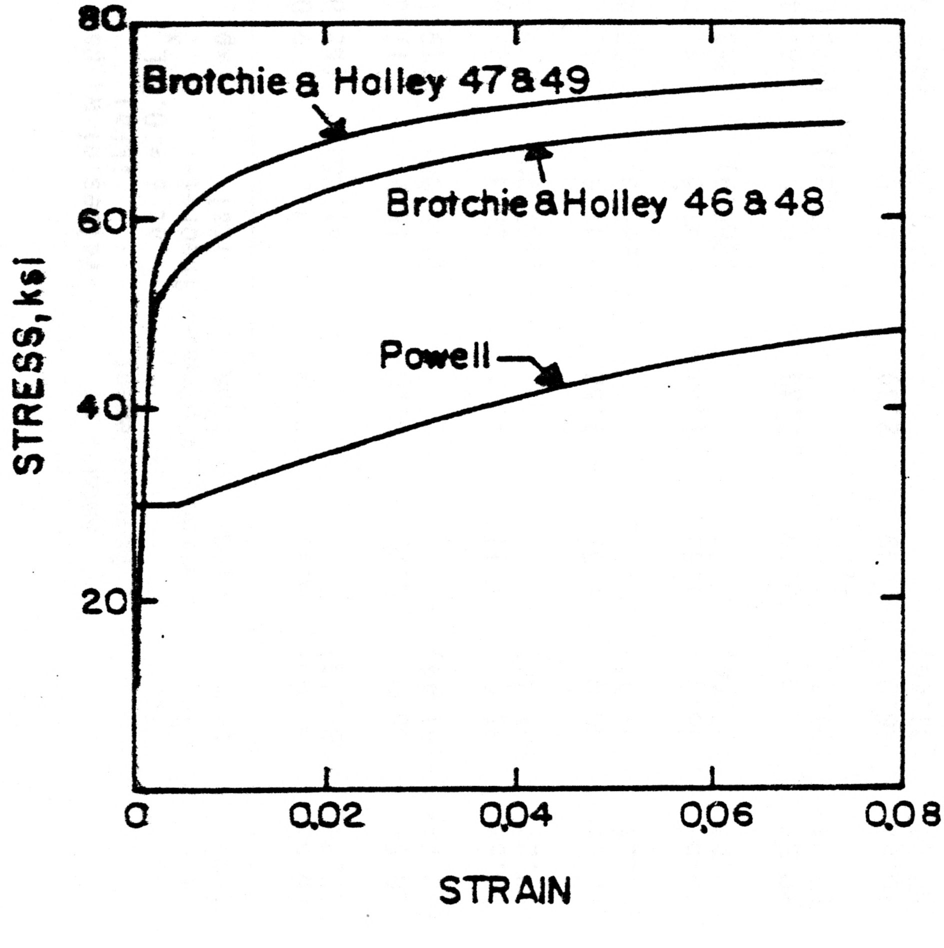


Fig. 21 Reinforcement Stress-Strain Curves for Predicting Results in Powell's and Brotchie & Holley's Test

Table 1. COMPARISON OF MEASURED AND PREDICTED STRENGTHS FOR TENSILE MEMBRANE ACTION

Source	Slab Designation	d in. (cm)	L_x in. (cm)	L_y in. (cm)	Amount of Reinforcement % x direction	Amount of Reinforcement % y direction	Values of w, psi (kPa) at $\delta = 0.1 L_x$ Theoretical	Values of w, psi (kPa) at $\delta = 0.1 L_x$ Exp.	Experimental, w / Theoretical, w	Ultimate Load w by Yield Line Theory, psi (kPa)
	S47	1.286 (3.26)	20.57 (52.24)	36.0 (91.44)	0.39	0.39	9.2 (63.38)	9.0 (62.01)	0.98	6.87 (44.40)
	S50	1.286 (3.26)	20.57 (52.24)	36.0 (91.44)	0.70	0.70	16.7 (115.06)	17.3 (119.19)	1.04	12.22 (84.25)
	S54	1.286 (3.26)	20.57 (52.24)	36.0 (91.44)	1.11	1.11	26.3 (181.20)	25.9 (178.45)	0.98	19.01 (131.06)
Powell (23)	S58	1.286 (3.26)	20.57 (52.24)	36.0 (91.44)	1.52	1.52	41.5 (285.9)	43.6 (300.4)	1.05	25.5 (175.81)
	S59	1.286 (3.26)	20.57 (52.24)	36.0 (91.44)	1.52	1.52	41.5 (285.9)	41.2 (283.86)	0.99	25.5 (175.8)
	S62	1.286 (3.26)	20.57 (52.24)	36.0 (91.44)	2.39	2.39	65.5 (451.29)	61.4 (423.06)	0.94	38.5 (265.44)
	S63	1.286 (3.26)	20.57 (52.24)	36.0 (91.44)	2.39	2.39	65.5 (451.29)	65.6 (452.0)	1.00	38.5 (265.44)
	A1	2.0 (5.08)	40 (101.6)	60 (152.4)	0.16	0.16	4.7 (32.38)	9.2 (62.01)	1.96*	3.40 (23.42)
Park (20)	A2	2.0 (5.08)	40 (101.6)	60 (152.4)	0.35	0.16	7.7 (53.05)	15.2 (104.72)	1.97*	5.22 (35.96)

*Full tensile membrane action did not develop.

Table 1 (continued)

Source	Slab Designation	d in. (cm)	L_x in. (cm)	L_y in. (cm)	Amount of Reinforcement % x direction	y direction	Values of w, psi (kPa) at $\delta = 0.1 L_x$ Theoretical	Exp.	$\frac{\text{Experimental, w}}{\text{Theoretical, w}}$	Ultimate Load w by Yield Line Theory, psi (kPa)
	A3	2.0 (5.08)	40 (101.6)	60 (152.4)	0.59	0.16	14.1 (97.14)	20.8 (143.30)	1.48	8.57 (59.10)
	A4	2.0 (5.08)	40 (101.6)	60 (152.4)	0.96	0.16	20.0 (137.79)	25.6 (196.37)	1.28	12.48 (85.98)
Brotchie and Holley (21)	46	0.56 (1.42)	15 (38.1)	15 (38.1)	1.00	1.00	39.9 (274.9)	35.3 (243.20)	0.88	16.91 (116.59)
	47	1.22 (3.09)	15	15	1.00	1.00	81.1 (558.7)	94.5 (651.10)	1.17	86.8 (597.10)
	48	0.56 (1.42)	15	15	2.00	2.00	79.7 (549.1)	65.6 (454.03)	0.82	30.84 (212.65)
	49	1.22 (3.09)	15	15	2.00	2.00	162.2 (1117.51)	145.5 (1002.45)	0.90	156.89 (1081.75)

Chapter III Conclusions

Yield line analysis provides answers to problems of slab design which cannot be handled by other means. The work method theory gives an upper-bound solution. In practice the actual load of a reinforced concrete slab, may be above the calculated value by this method because of the occurrence of various secondary effects. By means of yield line analysis, a rational solution for the failure load may be found for slabs of any shape supported in a variety of ways and for concentrated loads as well as distributed and partially distributed loads. In most of the cases the work method gives a quick solution to the problems.

There is no necessity for considering the fan mechanism when the slab is carrying distributed load only, however, when heavy concentrated loads are involved, it is important to analyze the slab considering a fan mechanism. A fracture pattern involving fan modes will generally give a lower collapse load than that obtained from corresponding straight line pattern. For analysis of an orthotropic slab a library of solution collected for the isotropic slab is very useful.

When horizontal movement of the slab is not allowed by restraining the boundaries, at large deflections of the slab, it is possible finally to develop tensile membrane action when cracks go right through the slab so that the load is supported on the net of reinforcement. The load carried by reinforcement is usually greater than the ultimate load calculated by yield line analysis.

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APPENDIX

Consider the yield line shown in Fig. 6. The internal work done along the yield line may be written as:

$$E = (\vec{m}_{nb}\ell + \vec{m}_{nt}\ell) \cdot \vec{\theta}_n = m_{nb}\ell\,\theta_n \operatorname{Cos} 0 + m_{nt}\ell\,\theta_n \operatorname{Cos} 90 \qquad (1a)$$

or

$$E = m_{nb}\ell\,\theta_n \qquad (2a)$$

where $\vec{m}_{nb}$ and $\vec{m}_{nt}$ are bending and twisting moments per unit length along the yield line, and $\vec{\theta}_n$ is rotational vector along the yield line. Substituting m_{nb} in terms of the ultimate moment capacity of reinforced section per unit length with respect to x and y axes, as given by equation (1), page (10), into equation (2a), gives

$$E = [m_x\,(\ell \operatorname{Cos}\alpha) \operatorname{Cos}\alpha + m_y\,(\ell \operatorname{Sin}\alpha) \operatorname{Sin}\alpha]\,\theta_n \qquad (3a)$$

or

$$E = m_x\,(\ell \operatorname{Cos}\alpha)(\theta_n \operatorname{Cos}\alpha) + m_y\,(\ell \operatorname{Sin}\alpha)(\theta_n \operatorname{Sin}\alpha) \qquad (4a)$$

substituting $\ell_x = \ell \operatorname{Cos}\alpha$, $\ell_y = \ell \operatorname{Sin}\alpha$, $\theta_x = \theta_n \operatorname{Cos}\alpha$ and $\theta_y = \theta_n \operatorname{Sin}\alpha$ into equation (4a), leads to

$$E = m_x\ell_x\theta_x + m_y\ell_y\theta_y \qquad (5a)$$

which is similar to equation (14).

YIELD LINE AND MEMBRANE ACTION ANALYSIS OF CONCRETE PLATES

by

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Yield line theory and tensile membrane action are presented. The yield line analysis is based on the formation of the yield pattern, with its location dependent on loading and boundary conditions. There are guidelines for establishing yield line patterns. The complexity of analysis of yield line will be reduced by making some basic assumptions. It has been assumed that the collapse load can be arrived at by considering bending action only. It has been also assumed that the elastic deformations are negligible compared to plastic deformations, therefore the segments between the yield lines remain rigid. Once the general crack pattern is known, the work method or the equilibrium method may be used to determine the collapse load. In the work method, the external work done by the loads to cause a small arbitrary virtual deflection must equal the internal work done as the slab rotates at the yield lines to adapt this deflection. Only the work method is discussed.

A fan mechanism will form when a slab is subjected to a heavy concentrated load. There is no need for considering the fan mechanism when the slab is carrying a distributed load only.

An orthotropic slab may be transferred into an equivalent, or affine, isotropic slab. This transformation allows an easy solution compared with the analysis of the orthotropic slab.

When boundaries of any slab are restrained from horizontal movement (as might occur in the interior panels of a continuous slab), the formation of the collapse mechanism develops high compressive forces in the plane of the slab with consequent increase in carring capacity. At large deflections of the slab, it is possible finally to develop tensile membrane action where cracks go right through the slab so that the load is supported on the net of reinforcement.