A Künneth theorem for the cyclic homology of A_∞ algebras

by

Henry Chukwunyere Azubuike

B.Tech., FUTO Nigeria, 2015

M.Sc., Kansas State University, 2022

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

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Abstract

The cyclic homology of a $\mathbb{Z}/2\mathbb{Z}$-graded, smooth and proper A_∞ category satisfying the Hodge-to-de-Rham degeneration property carries the structure of a polarized semi-infinite Hodge structure or the so-called EP-structure. Given two A_∞ algebras A and B with the above conditions, we construct a Künneth map from the tensor product of their cyclic homologies to the cyclic homology of the A_∞ tensor product $A \otimes_\infty B$ and show that it respects the EP-structures. As an application, we show that if A and B are equipped with weak Calabi-Yau structures, then $A \otimes_\infty B$ also inherits a weak Calabi-Yau structure. Also, we show that the Künneth quasi-isomorphism respects good splittings of the Hodge filtration on A and B compatible with the weak Calabi-Yau structure. Our explicit calculations rely on the combinatorial (tree) description of the tensor product of A_∞ algebras.

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Approved by:

Major Professor
Lino Amorim

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The cyclic homology of a $\mathbb{Z}/2\mathbb{Z}$-graded, smooth and proper A_∞ category satisfying the Hodge-to-de-Rham degeneration property carries the structure of a polarized semi-infinite Hodge structure or the so-called EP-structure. Given two A_∞ algebras A and B with the above conditions, we construct a Künneth map from the tensor product of their cyclic homologies to the cyclic homology of the A_∞ tensor product $A \otimes_\infty B$ and show that it respects the EP-structures. As an application, we show that if A and B are equipped with weak Calabi-Yau structures, then $A \otimes_\infty B$ also inherits a weak Calabi-Yau structure. Also, we show that the Künneth quasi-isomorphism respects good splittings of the Hodge filtration on A and B compatible with the weak Calabi-Yau structure. Our explicit calculations rely on the combinatorial (tree) description of the tensor product of A_∞ algebras.

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Acknowledgments

I would like to thank my advisor, Lino Amorim, for his patience, teaching, and support.

Dedication

This work is dedicated to my wife, Angela Chinenye for making me a husband, and father of our liitle one, Flora Chukwunyere; and to my parents Samuel and Chinyere Nwachukwu.

Chapter 1

Introduction

The notion of a variation of semi-infinite hodge structure was introduced by Barannikov[1] as a generalization of *variations of hodge structures* on the moduli spaces of non-commutative deformations of complex manifolds. Mirror symmetry predicts the existence of certain mirror pairs of Calabi–Yau manifolds, X and Y, so that the Gromov–Witten invariants of X, the A-side can be extracted from certain hodge-theoretic invariants of Y, the B-side. The mirror relationship between the enumerative invariants of X and Y can be stated as an isomorphism of their variations of semi-infinite hodge structures (VSHS).

Kontsevich proposed the homological mirror symmetry conjecture[2] as a generalization of hodge-theoretic mirror symmetry. It predicts a quasi-equivalence of A_∞ categories - the (derived) Fukaya category of the A-side, and the derived category of coherent sheaves on the B-side. One question is how to get from this very sophisticated statement to the classical one. The expectation is that the cyclic homology of a smooth, proper, Calabi-Yau A_∞-category carries a VSHS. So taking the cyclic homology of the Fukaya category should recover the A-side VSHS and taking cyclic homology of the bounded derived category should recover the B-side VSHS. This was made precise by Barannikov[3] and Katzarkov-Kontsevich-Pantev[4], as well as the works of Gantara-Perutz-Sheridan[5], and[6].

We study the tensor product of two smooth and proper, weak Calabi-Yau A_∞ algebras A and B, and their Hochschild invariants. We construct a Künneth map from the tensor product of their cyclic homologies to the cyclic homology of the A_∞ tensor product $A \otimes_\infty B$ and show that it respects the semi-infinite Hodge structures - namely, the Mukai and higher residue pairings, as well as the u-connections.

Our main result is stated below.

Theorem 1. *Let A and B be two $\mathbb{Z}/2\mathbb{Z}$-graded, smooth, proper, (weak) Calabi-Yau A_∞ algebras over a field* k *of characteristic zero. Consider their negative cyclic homologies $HC_*^-(A)$, $HC_*^-(B)$ with the usual semi-infinite hodge structures $\mathcal{V}^A = (HC_*^-(A), \nabla_{d/du}, \langle , \rangle_{hres})$, and $\mathcal{V}^B$ defined analogously, where the higher residue pairing and u-connection (say, on A) are maps*

$$\langle -, - \rangle_{hres} : HC_*^-(A) \otimes HC_*^-(A) \to k[[u]]$$

$$\nabla_{\frac{d}{du}} : HC_*^-(A) \to u^{-2} HC_*^-(A)$$

defined in (2.2) and (2.5). There is a chain map of the negative cyclic complexes

$$k + uK : C_*^-(A) \otimes_{k[[u]]} C_*^-(B) \to C_*^-(A \otimes_\infty B)$$

defined in (4.3, 4.17), such that

(a) $k + uK$ is a quasi-isomorphism of complexes. [= Proposition (29)]

(b) $k + uK$ respects the higher residue pairings and u-connections, and hence the semi-infinite hodge structures. [= Theorem (30)]

The Künneth maps $k, K : C_*(A) \otimes C_*(B) \to C_*(A \otimes_\infty B)$ are constructed by extending the well-known shuffle and cyclic shuffle maps in the associative algebra case (see[7]). Shklyarov[8] carried out this extension in the dg algebra case and proved that the cyclic Künneth quasi-isomorphism respects the u-connections on periodic cyclic homology. Our Main Theorem 1 generalizes Shklyarov's work to the A_∞ world, and further shows that under the finiteness conditions - smooth and proper, the (cyclic) Künneth map also respects the Mukai and higher residue pairings on negative cyclic homology.

A major challenge to extending results concerning tensor products to A_∞ algebras (or categories) is that the operation of tensor product for A_∞ algebras is not canonical, and may not preserve extra structure. It is well-studied by many authors however. Saneblidze and Umble[9;10], Markl and Shnider[11] using the language of operads. More recently, Amorim[12] generalized the construction to filtered A_∞ algebras, without the language of operads. In this paper, we work with the version of Amorim's construction for classical (unfiltered) A_∞ algebras, with the $\mathfrak{m}_n$ operations given by trees. However, this tensor product framework does not preserve cyclic structures. For a cyclic structure-preserving tensor product, see[13]. To overcome this hurdle, we replace the cyclic structure with a weak Calabi-Yau structure on A. This in turn gives us the Poincaré-type isomorphism.

Proposition 2 (= Proposition (39))**.** *A weak Calabi-Yau structure ϕ determines an isomorphism $D : HH^*(A) \longrightarrow HH_*(A)$ in equation (5.1). This is an isomorphism of $HH^*(A)$-modules, that is,*

$$D(\theta \cup \psi) = \theta \cap D(\psi)$$

for Hochschild cochains $\theta, \psi \in HH^*(A)$

We obtain a weak Calabi-Yau structure on $A \otimes_\infty B$ given weak Calabi-Yau structures on A and B respectively.

Theorem 3. *(= Theorem 42) $\phi_A \otimes \phi_B := \langle \omega^\otimes, \ - \rangle_{Muk}$ where $\omega^\otimes := k(\omega_A \otimes \omega_B)$ is a weak Calabi-Yau structure on $A \otimes B$ if ω_A, ω_B are weak Calabi-Yau structures on A and B respectively.*

We apply the results of Main Theorem 1 to obtain splittings of the Hodge filtration on $A \otimes_\infty B$ compatible with $\omega_{A\otimes B}$.

Theorem 4 (=Theorem 45)**.** *Let s_A, s_B be splittings of the hodge filtration on A, and B respectively. Then $s_{A\otimes B} := (k + uK) \circ (s_A \otimes s_B) \circ k^{-1}$ is a splitting of the Hodge filtration on $A \otimes_\infty B$. Further, if s_A, s_B are ω_A, ω_B-compatible good splittings of the Hodge filtration on A, and B respectively, then $s_{A\otimes B}$ is a $\omega^\otimes$-compatible good splitting of the Hodge filtration of $A \otimes_\infty B$, where $\omega^\otimes = k(\omega_A \otimes \omega_B)$.*

The existence of ω-compatible good splittings is very non-trivial. Amorim-Tu[14] showed that they exist in the case where the Hochschild cohomology ring $HH^*(A)$ is semi-simple. They further showed that in the case of a versal deformation of A, the set of ω-compatible good splittings of the Hodge filtration on the central fiber is in bijection with the set of primitive forms of the corresponding variational semi-infinite Hodge structure. With the data of a primitive form on a variational semi-infinite Hodge structure, one may construct a formal Frobenius manifold structure on the versal deformation of the A_∞-category. This is the categorical construction of Frobenius manifolds from A_∞ categories, analogous to Saito-Takahashi[15]'s theory of primitive forms in singularity theory.

1.1 Notation and Conventions

We shall work with $\mathbb{Z}/2\mathbb{Z}$-graded strictly unital A_∞ algebras A over a field k of characteristic zero. We let $|a|$ denote the degree of $a \in A$, and $|a|' = |a| + 1$ its shifted degree in $A[1]$. All our degrees will be shifted unless explicitly specified. A $\mathbb{Z}/2\mathbb{Z}$-graded A_∞ algebra is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space

$$A = \bigoplus_{p \in \mathbb{Z}/2\mathbb{Z}} A^p$$

with graded homogeneous k-linear maps

$$\mathfrak{m}_n : A[1]^{\otimes n} \longrightarrow A[1], \quad n \geq 1$$

of degree 1 satisfying the following relations:

$$\sum (-1)^{|a_1|' + \ldots + |a_{i-1}|'} \mathfrak{m}_{n-j+1}\left(a_1, ..., a_{i-1}, \mathfrak{m}_j(a_i, ..., a_{i+j-1}), a_{i+j}, ..., a_n\right) = 0.$$

The strict unitality condition means we have $\mathfrak{m}_n\left(\ldots, \mathbf{1}_A, \ldots\right) = 0$ for all $n \neq 2$, and $\mathfrak{m}_2\left(\mathbf{1}_A, a\right) = a = (-1)^{|a|}\mathfrak{m}_2\left(a, \mathbf{1}_A\right)$.

We use bold letters to denote elements of the Hochschild chains $\mathbf{a} := a_0 \otimes a_1 \otimes a_2 \otimes \ldots \otimes a_n = a_0|a_1, a_2, ..., a_n = a_0|a$. Also, denote $\underline{a}_i = a_i \otimes \mathbf{1}_B$, and $\underline{b}_i = \mathbf{1}_A \otimes b_i$ in $A \otimes_\infty B$. The commas, as well as $|$ and $\otimes$ will be used interchangeably to mean the same thing. Furthermore, we use Sweedler notation for the coproduct on the tensor coalgebra, namely:

$$\sum_{(a)} a^{(1)} \otimes a^{(2)} = \sum_{i=0}^{n} (a_1 \otimes \ldots \otimes a_i) \otimes (a_{i+1} \otimes \ldots \otimes a_n) \tag{1.1}$$

Chapter 2

Hochschild Invariants of A_∞ Algebras

We review the basic definitions that we will need, following conventions from[14].

Definition 5. *The space of Hochschild cochains of length n are defined as:*

$$C^*(A)^n := \mathrm{Hom}_{\mathrm{k}}\left(\left(\frac{A}{k \cdot \mathbf{1}_A}[1]\right)^{\otimes n}, A[1]\right)[-1]$$

The Hochschild cochain complex is defined as $C^*(A) = \prod_{n \geq 0} C^*(A)^n$.

On the Hochschild cochain complex one defines the Gerstenhaber product:

$$\varphi \circ \psi(a) = \sum (-1)^{|a^{(1)}|'|\psi|'} \varphi\left(a^{(1)}, \psi\left(a^{(2)}\right), a^{(3)}\right)$$

The corresponding bracket $[\varphi, \psi] := \varphi \circ \psi - (-1)^{|\varphi|'|\psi|'} \psi \circ \varphi$ is a graded Lie bracket, and can be used to define the Hochschild differential on $C^*(A)$ as $\delta(-) := [\mathfrak{m}, -]$, where $\mathfrak{m} = \prod_k \mathfrak{m}_k$ are the A_∞-operations. The cochain complex is closed under the Gerstenhaber product, so the Hochschild differential descends to a differential in the complex since A is strictly unital. The corresponding cohomology is called the Hochschild cohomology of A and is denoted by $HH^*(A)$.

Definition 6. *The Hochschild chain complex of A is defined as the vector space*

$$C_*(A) := \bigoplus_{n=0}^{\infty} A \otimes \left(\frac{A}{\mathrm{k}\cdot \mathbf{1}_A}\right)^{\otimes n}.$$

with differential $\mathfrak{b}$ defined as

$$\mathfrak{b}\left(\mathbf{a}\right) = \mathfrak{b}_0(\mathbf{a}) + \mathfrak{b}_1(\mathbf{a}) = \sum (-1)^{|\mathbf{a}_{<\mathfrak{m}}|'} a_0 \otimes a^{(1)} \otimes \mathfrak{m}\left(a^{(2)}\right) \otimes a^{(3)} + \sum (-1)^{@} \mathfrak{m}\left(a^{(3)}, a_0, a^{(1)}\right) \otimes a^{(2)}$$

where $|\mathbf{a}_{<\mathfrak{m}}|' = |a_0|' + \left|a^{(1)}\right|'$ - the sum of (shifted) degrees of a's to the left of $\mathfrak{m}$, and $@ = \left(|a_0|' + \left|a^{(1)}\right|' + \left|a^{(2)}\right|'\right)\left|a^{(3)}\right|'$.

Definition 7. *The negative cyclic chain complex of A is defined by*

$$C_*^-(A) := \left(C_*\left(A\right)[[u]], \mathfrak{b} + u\mathfrak{B}\right)$$

where the so-called Connes' differential $\mathfrak{B} : C_*(A) \to C_*(A)$ is defined by:

$$\mathfrak{B}(\mathbf{a}) = \sum (-1)^{|a^{(2)}|'\left(|a_0|' + |a^{(1)}|'\right)}\ \mathbf{1}|a^{(2)}, a_0, a^{(1)}$$

and satisfies $\mathfrak{B}^2 = \mathfrak{b}^2 = \mathfrak{B}\mathfrak{b} + \mathfrak{b}\mathfrak{B} = 0$. The cohomology of the negative cyclic chain complex is called the negative cyclic homology denoted $HC_*^-(A)$.

Definition 8. *The Lie derivative, $\mathcal{L} : C^*(A) \otimes C_*(A) \to C_*(A)$ defined in[16] can be written as:*

$$\mathcal{L}_\phi(\mathbf{a}) = \sum (-1)^{|\phi|'\left(|a^{(1)}|' + |a_0|'\right)} a_0 | a^{(1)} \otimes \phi_*\left(a^{(2)}\right) \otimes a^{(3)}$$
$$+ \sum (-1)^{|a^{(3)}|'\left(|a^{(2)}|' + |a^{(1)}|' + |a_0|'\right)} \phi_*\left(a^{(3)} \otimes a_0 \otimes a^{(1)}\right) | a^{(2)}$$

Observe that $\mathcal{L}_\mathfrak{m} = \mathfrak{b}$.

2.1 Mukai and higher residue pairings

We recall that an A_∞ algebra A is called smooth (or homologically smooth) if the diagonal bimodule A_Δ is perfect, that is, split-generated by tensor products of Yoneda modules (see[17]). Also, we say that an A_∞ algebra A is proper if the cohomology $HH^*(A, \mathfrak{m}_1)$ is finite-dimensional.

Definition 9. *Assume A is proper. There is a degree-zero pairing known as the Mukai pairing naturally defined on Hochschild homology:*

$$\langle -, - \rangle_{Muk} : HH_*(A) \otimes HH_*(A) \to \mathrm{k}$$

Shklyarov[18] proves that this pairing is non-degenerate if A is in addition smooth. When A is finite dimensional (which can be achieved after passing to the minimal model of A), Sheridan[6] shows that the Mukai-pairing can be described at the chain level in the following way: for $\mathbf{a} = a_0|a_1, ..., a_n$ and $\mathbf{b} = b_0|b_1, ..., b_m$,

$$\langle \mathbf{a}, \mathbf{b} \rangle_{Muk} = \sum tr[c \mapsto (-1)^{\#} \mathfrak{m}_* \left(a^{(3)}, a_0, a^{(1)} \otimes \mathfrak{m}_*(a^{(2)}, c, b^{(3)}, b_0, b^{(1)}) \otimes b^{(2)}\right)] \tag{2.1}$$

where $\# = 1 + |c||\mathbf{b}| + (|a^{(3)}|' + |a_0|' + |a^{(1)}|') + (|a_0|' + |a^{(1)}|' + |a^{(2)}|')|a^{(3)}|' + (|b_0|' + |b^{(1)}|' + |b^{(2)}|')|b^{(3)}|'$.

On $HC_*^-(A)$ we can define the so-called *higher residue pairing*[6]:

$$\langle -, - \rangle_{\mathrm{hres}} : CC_*^-(A) \otimes CC_*^-(A) \to \mathrm{k}[[u]], \tag{2.2}$$

by extending the Mukai pairing (2.1) sesquilinearly. That is $\langle u\mathbf{a}, \mathbf{b} \rangle_{\mathrm{hres}} = -\langle \mathbf{a}, u\mathbf{b} \rangle_{\mathrm{hres}} = u\langle \mathbf{a}, \mathbf{b} \rangle_{\mathrm{Muk}}$, for Hochschild chains $\mathbf{a}, \mathbf{b}$. It commutes with the cyclic differential $\mathfrak{b} + u\mathfrak{B}$ and so descends to the negative cyclic homology.

2.2 The u-connection on negative cyclic homology

Definition 10. *Let $\mathfrak{m}'$ be the cocycle in $C^*(A)$ defined as $\mathfrak{m}' := \prod_{k\geq 0}(2-k)\mathfrak{m}_k$, and Γ the length operator defined by $\Gamma(a_0|a_1,\ldots,a_n) := -n \cdot a_0|a_1,\ldots,a_n$. Further, let $\iota\{\mathfrak{m}'\} := \mathfrak{b}\{\mathfrak{m}'\} + u\mathfrak{B}\{\mathfrak{m}'\}$ where $\mathfrak{b}\{\mathfrak{m}'\}(-) = \mathfrak{b}^{1,1}(\mathfrak{m}',-)$ and $\mathfrak{B}\{\mathfrak{m}'\}(-) = \mathfrak{B}^{1,1}(\mathfrak{m}',-)$ are bilinear maps $\mathfrak{b}^{1,1}, \mathfrak{B}^{1,1} : C^*(A)\otimes C_*(A) \to C_*(A)$, defined by:*

$$\mathfrak{b}^{1,1}(\phi,\mathbf{a}) = \sum(-1)^{\dagger}\mathfrak{m}_*\left(a^{(3)}\otimes\phi\left(a^{(4)}\right)\otimes a^{(5)}\otimes a_0\otimes a^{(1)}\right)\otimes a^{(2)} \tag{2.3}$$

where $\dagger = \left(|a^{(3)}|'+|a^{(4)}|'+|a^{(5)}|'\right)\left(|a_0|'+|a^{(1)}|'+|a^{(2)}|'\right)+|\phi|'|a^{(3)}|'$, *and*

$$\mathfrak{B}^{1,1}(\phi,\mathbf{a}) = \sum(-1)^{|\phi|'|a^{(2)}|'+\left(|a^{(1)}|'+|a_0|'\right)\left(|a^{(2)}|'+|a^{(3)}|'+|a^{(4)}|'\right)}\mathbf{1}|a^{(2)},\phi\left(a^{(3)}\right),a^{(4)},a_0,a^{(1)} \tag{2.4}$$

The meromorphic u-connection $\nabla_{\frac{d}{du}}$ on $HC^-_*(A)$ was first introduced by Kontsevich-Soibelman and Shklyarov (see[19 4 17 18]), $\nabla_{\frac{d}{du}} : HC^-_*(A) \to u^{-2}HC^-_*(A)$ and is defined by the formula

$$\nabla_{\frac{d}{du}} := \frac{d}{du} + \frac{\Gamma}{2u} + \frac{\iota\{\mathfrak{m}'\}}{2u^2} \tag{2.5}$$

Amorim-Tu[14] showed the higher residue pairing is parallel with respect to the u connection. That is,

$$\frac{d}{du}\langle\mathbf{a},\mathbf{b}\rangle_{\text{hres}} = \left\langle\nabla_{\frac{d}{du}}\mathbf{a},\mathbf{b}\right\rangle_{\text{hres}} - \left\langle\mathbf{a},\nabla_{\frac{d}{du}}\mathbf{b}\right\rangle_{\text{hres}}$$

The DG version was showed earlier by Shklyarov[20].

The Lie derivative $\mathcal{L}$ and contraction ι are related by the Cartan homotopy formula

$$[\mathfrak{b}+u\mathfrak{B},\iota\{\varphi\}] = -u\cdot\mathcal{L}_\varphi - \iota\{[\mathfrak{m},\varphi]\} \tag{2.6}$$

See Getzler[16] for a proof.

The Hochschild cochain complex $C^*(A)$ acts on $C_*(A)$ via the cap product

$$\varphi\cap(-) := (-1)^{|\varphi|}\mathfrak{b}\{\varphi\}(-)$$

and defines a left module structure, with respect to the cup product $\bigcup$ on $HH^*(A)$ defined on the chain-level by

$$\phi \cup \psi = \sum (-1)^{|\phi|'+|\psi|(|a^{(2)}|'+|a^{(3)}|')+|\phi||a^{(1)}|'} \ \mathfrak{m}_*(a^{(1)}, \phi(a^{(2)}), a^{(3)}, \psi(a^{(4)}), a^{(5)})$$

for cochains $\phi, \psi \in C^*(A)$.

2.3 Semi-infinite Hodge structures

Definition 11. *A semi-infinite Hodge structure of dimension $N \in \mathbb{Z}/2\mathbb{Z}$ consists of a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathrm{k}[[u]]$-module $\mathcal{E}$ that is free and of finite rank; a meromorphic connection of even degree*

$$\nabla_{\frac{d}{du}} : \mathcal{E} \to u^{-2}\mathcal{E}$$

with at most a second order pole at $u = 0$; and an even k-linear pairing $\langle -, - \rangle : \mathcal{E} \otimes \mathcal{E} \to \mathrm{k}[[u]]$ subject to the following conditions: for $s_1, s_2 \in \mathcal{E}$

- *the pairing $\langle -, - \rangle$ is $\mathrm{k}[[u]]$-sesquilinear, that is:*

$$\langle f \cdot s_1, g \cdot s_2 \rangle = fg^* \langle s_1, s_2 \rangle; \quad f, g \in \mathrm{k}[[u]], \ \textit{and } g^*(u) = g(-u)$$

- *the pairing is covariantly constant with respect to the u-connection, that is:*

$$\frac{d}{du} \langle s_1, s_2 \rangle = \langle \nabla_{\frac{d}{du}} s_1, s_2 \rangle - \langle s_1, \nabla_{\frac{d}{du}} s_2 \rangle$$

- *the pairing is symmetric in the sense that*

$$\langle s_1, s_2 \rangle = (-1)^{N+|s_1||s_2|} \langle s_2, s_1 \rangle^*$$

A semi-infinite Hodge structure is called polarized if the induced pairing $\mathcal{E}/u\mathcal{E} \otimes \mathcal{E}/u\mathcal{E} \to \mathrm{k}$ is non-degenerate. This is also called an EP-structure in the literature.

A fundamental conjecture of non-commutative Hodge theory, as formulated by Kontsevich-Soibelman[17] - the so-called Hodge-to-de-Rham degeneration property in this case is:

Conjecture 12. *Let A be $\mathbb{Z}/2\mathbb{Z}$-graded smooth and proper A_∞ algebra. Then, the Hodge-to-de-Rham spectral sequence starting with Hochschild homology and converging to the negative-cyclic homology degenerates. So we have the isomorphism $HH_*(A)[[u]] \cong HC^-_*(A)$*

The conjecture has been proved by Kaledin in the case that A is $\mathbb{Z}$-graded (see[21]). In the general case, to the knowledge of the author, the conjecture remains open if A is, for example, $\mathbb{Z}/2\mathbb{Z}$-graded.

Theorem 13 ([6])**.** *Let A be $\mathbb{Z}/2\mathbb{Z}$-graded smooth, proper and weak Calabi-Yau A_∞ algebra such that the Hodge-to-de-Rham spectral sequence degenerates. Then the triple*

$$\left(HC^-_*(A), \nabla_{\frac{d}{du}}, \langle -, - \rangle_{hres}\right)$$

forms a polarized semi-infinite Hodge structure.

Chapter 3

Tensor Product of A-infinity Algebras

3.1 A combinatorial (tree) description of the tensor product

We will now recall the definition of the tensor product of two A_∞-algebras A and B adapting Amorim's work on the filtered A_∞ version[12].

Let G_n denote the set of planar rooted trees with n leaves such that the valency of incoming edges $\mathrm{val}(v) \geqslant 2$ for all vertices $v \in V(T)$. Given a vertex v we will denote by $v(i)$ the i^{th} incident edge at v.

For each pair $x, y \in V(T) \cup E(T)$ we say x is under y if the unique path P_x in T from x to the root of T contains y. Note that this defines a partial ordering. Suppose x and y are not comparable and denote by w the first (furthest from the root) vertex that P_x and P_y have in common. Then P_x and P_y contain edges $w(i)$ and $w(j)$ respectively. We say that x is to the left of y if $i < j$. Let $E(T)$ denote the set of edges of the tree T. Given $x \in V(T) \cup E(T)$ we denote by p_x (respectively q_x) the left (respectively right) most leaf of T under x. Additionally, given $e \in E(T)$ we define integers J_e, I_e and $\tilde{I}_e$ to be zero when e is a leaf and when e is not a leaf, J_e is the number of leaves under $e, \tilde{I}_e$ is the number of internal edges under e and $I_e = \tilde{I}_e + 1$.

Let G_n^{bin} denote the subset of G_n consisting of trees T with $\mathrm{val}(v) = 2$ for all $v \in V(T)$.

Elements of G_n^{bin} are called binary trees. There is a partial order on the set G_n^{bin}, which was introduced in[11]. It is generated by the relation in Figure (3.1).

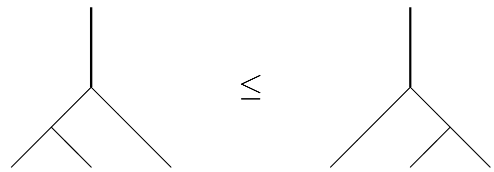

Figure 3.1: *Generator relation for the partial order on binary trees*

There is an absolute minimum and maximum for this order. The minimum (or maximum), which we denote by $\underline{M}_n$ (or $\overline{M}_n$), is the binary tree with strictly right- (or left-) leaning internal edges.

Given $T \in G_n$ we define $T_{\max} \in G_n^{bin}$ as the maximal (with respect to the order above) binary tree that resolves T. That is, T can be obtained by collapsing several edges in $T_{\max}$. An example is shown in Figure 3.2.

Similarly, we define $T_{\min}$ as the minimal binary tree that resolves T. The tree in G_n with exactly one internal vertex, called the n-corolla, is denote by c_n.

We introduce another subset $L_n \subset G_n$. Let L_n denote the subset of G_n obtained by grafting corollas in all possible ways while avoiding the last leaf on any vertex. More formally, $T \in L_n$ if either $T = c_n$ or there exist integers $1 \leqslant i_1 < \ldots < i_l < m$ and trees $T_j \in L_{n_j}$ such that

$$T = c_m \circ_{i_1,\ldots,i_l} (T_1, \ldots, T_l),$$

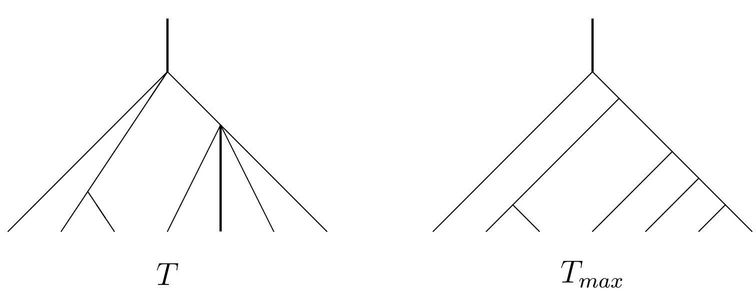

Figure 3.2: *A maximal tree resolving T*

where $\circ_{i_1,\ldots,i_l}$ tells us to graft the root of T_j to the leaf i_j of c_m.

Now suppose that $T \in L_n$ has k internal edges. Let $R(S)$ denote the set of right-leaning internal edges of a binary tree S and let $|R(S)|$ denote its order. We define

$$\alpha(T) := \sum_{\substack{S \in G_n^{bin} \\ S \geqslant V_{\max} \\ |R(S)|=k}} S/R(S),$$

where $S/R(S)$ is the tree obtained by collapsing all the edges of S in $R(S)$. Note that $\alpha(T)$ is non-zero as it always includes $\alpha_0(T) := T_{\max}/R\left(T_{\max}\right)$. For an example see Figure 3.3 below.

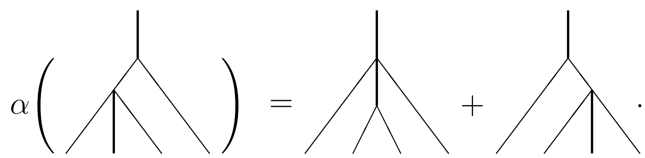

Figure 3.3: *Example of α*

We use $U \in G_n$ as a flow chart to define a map

$$U : A^{\otimes n} \longrightarrow A$$

by assigning to each $v \in V(U)$ the map $\mathfrak{m}^A_{val(v)}$. Similarly we define maps $W : B^{\otimes n} \longrightarrow B$.

Theorem 14 ([12]). *Let A and B be unital A_∞-algebras. Then the tensor product $A \otimes_\infty B$ is quasi-isomorphic to the A_∞-algebra $(A \otimes B, \mathfrak{m}^\otimes)$ with operations given by*

$$\mathfrak{m}_1^\otimes = \mathfrak{m}_1^A \otimes id + id \otimes \mathfrak{m}_1^B$$

$$\mathfrak{m}_n^\otimes \left(a_1 \otimes b_1, \ldots, a_n \otimes b_n\right) = \sum_{T \in L_n} (-1)^\epsilon T^A \left(a_1, \ldots a_n\right) \otimes \alpha(T)^B \left(b_1, \ldots, b_n\right).$$

The sign exponent is defined as $\epsilon = \theta(T) + \lhd + |a|\,|E_{int}\,(T)| + \gamma_a + \gamma_b$*, with*

$$\lhd = \sum_{p>q} |b_q||a_p|$$

$$\gamma_a = \sum_{v\in V(T)} \sum_{i<p_v} |a_i|\,, \quad \gamma_b = \sum_{v\in V(\alpha(T))} \sum_{i<p_v} |b_i| \ \textit{ and}$$

$$\theta(T) = \sum_{v\in V(T)} \left(\sum_{i} \left((\mathrm{val}(v) - i - 1) I_{v(i)} + \tilde{I}_{v(i)} + J_{v(i)} \right) + \sum_{i<j} I_{v(i)} \left(\tilde{I}_{v(j)} + J_{v(j)} \right) \right) \qquad (3.1)$$

We give an illustration of the theorem with explicit formulas for $\mathfrak{m}_3^{\otimes}$, and $\mathfrak{m}_4^{\otimes}$ in Figure 3.4 below with signs determined by $\theta(T)$.

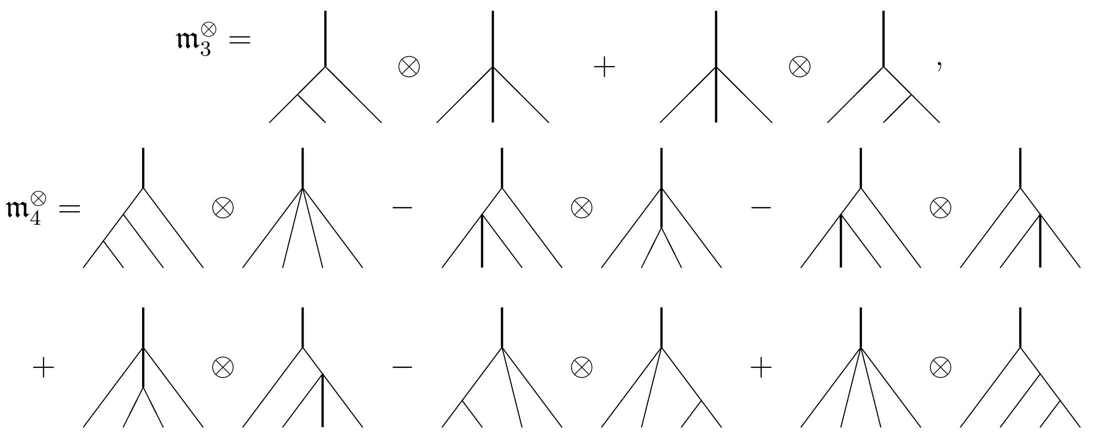

Figure 3.4: *Formulas for* $\mathfrak{m}_3^{\otimes}$ *and* $\mathfrak{m}_4^{\otimes}$*.*

Proposition 15. *There is the following alternative description when* $n \geqslant 3$

$$\mathfrak{m}_n^{\otimes} = \sum_{\substack{|E_{int}\,(U)|+|E_{int}\,(W)|=n-2 \\ U_{\max}\leqslant W_{min}}} \pm U^A \otimes W^B$$

Proof. See[12].

We will not distinguish the notations $A \otimes B$ and $A \otimes_{\infty} B$. They will both mean the same thing.

The following lemma will be used to prove a subsequent result, so we state it here.

Lemma 16. [12] *Consider a tree $U \in L_n$, and $a_1, \ldots a_n \in A$. Let j_a be the number of a_i such that $a_i = \mathbf{1}_A$ and denote by $b(U)$ the number of vertices of U of valency 2 . If $U(a_1, \ldots, a_n) \neq 0$, then $b(U) \geqslant j_a$ unless $j_a = n$, in which case $b(U) = n-1$. Moreover the same is true if $U \in R_n$.*

3.2 Properties of the tensor product

Definition 17. *Let $(A, \mathfrak{m}^A)$ and (C, μ) be strictly unital A_∞-algebras, we say A is a subalgebra of C if $A \subseteq C$, $\mathbf{1}_A = \mathbf{1}_C$ and for all $k > 0$ and $a_1, \ldots a_k \in A$ we have*

$$\mu_k(a_1, \ldots, a_k) = \mathfrak{m}_k^A(a_1, \ldots, a_k)$$

Definition 18. *Let $(A, \mathfrak{m}^A)$ and $(B, \mathfrak{m}^B)$ be A_∞-algebras. Suppose A and B are A_∞-subalgebras of (C, μ). Define a map $K : A \otimes B \longrightarrow C$ by $K(a \otimes b) = (-1)^{|a|}\mu_2(a, b)$. We say A and B are commuting subalgebras if given $c = K(a \otimes b)$ and $c_1, \ldots, c_k \in C$ such that for each i, $c_i = a_i$ or $c_i = b_i$ for some $a_i \in A$ and $b_i \in B$, the following conditions hold.*

(a) For $k > 0$, $\mu_k(c_1, \ldots, c_k) = 0$ unless

- *$k = 2$ and $c_1 \in A, c_2 \in B$ (or vice-versa) in which case,*

$$\mu_2(c_1, c_2) + (-1)^{|c_1|'|c_2|'}\mu_2(c_2, c_1) = 0 \tag{3.2}$$

- *$c_i = a_i$ for all i*
- *$c_i = b_i$ for all i,*

(b) $\mu_{k+1}(c_1, \ldots, c_i, c, c_{i+1}, \ldots, c_k) = 0$ unless

- $c_i = a_i$ *for all* i*, in which case it equals*

$$(-1)^{|b|\sum_{j>i}|a_j|'} K\left(\mathfrak{m}^A_{k+1}\left(a_1 \dots a_i, a, \dots, a_k\right) \otimes b\right) \tag{3.3}$$

- $c_i = b_i$ *for all* i*, in which case it equals*

$$(-1)^{|a|\left(\sum_{j\leqslant i}|b_j|'+1\right)} K\left(a \otimes \mathfrak{m}^B_{k+1}\left(b_1 \dots b_i, b, \dots, b_k\right)\right) \tag{3.4}$$

A more conceptual proof of the following proposition for filtered A_∞ algebras can be found in[12]. For completeness, we give a more combinatorial proof using the planar trees introduced above.

Proposition 19. *The* A_∞ *algebras* $(A, \mathfrak{m}^A)$ *and* $(B, \mathfrak{m}^B)$ *are commuting subalgebras of* $A \otimes B$ *under the inclusions*

$$\begin{aligned} A &\hookrightarrow A \otimes B & \qquad B &\hookrightarrow A \otimes B \\ a &\mapsto a \otimes \mathbf{1}_B & \qquad b &\mapsto \mathbf{1}_A \otimes b \end{aligned}$$

Proof. First, we need to check that A is a subalgebra. That is, for each $n > 0$ and $a_1, \dots, a_n \in A$, we need to show that $\mathfrak{m}^\otimes_n(a_1 \otimes \mathbf{1}_B, \dots, a_n \otimes \mathbf{1}_B) = \mathfrak{m}^A_n(a_1, \dots, a_n) \otimes \mathbf{1}_B$.
By definition,

$$\mathfrak{m}^\otimes_n(a_1 \otimes \mathbf{1}_B, \dots, a_n \otimes \mathbf{1}_B) = \sum_{\substack{U \in L_n,\ W \in R_n \\ |E_{int}(U)|+|E_{int}(W)|=n-2}} \pm\, U(a_1, \dots, a_n) \otimes W(\mathbf{1}_B, \dots, \mathbf{1}_B)$$

But $|E_{int}(U)|+|E_{int}(W)| = n-2$ implies $|V(U)|-1+|V(W)|-1 = n-2$ or $|V(U)|+|V(W)| = n$, and hence, $b(U) + b(W) \leq n$. Now, $j_b = n$ and so by Lemma 16, $W(\mathbf{1}_B, \dots, \mathbf{1}_B) \neq 0$ would mean $b(W) \geq j_b$ unless $j_b = n$, which further means $b(W) = n-1$, that is, W is a binary tree, and $|V(W)| = n-1$. Substituting $|V(W)| = n-1$ we have, $|V(U)| + n - 1 = n$

or $|V(U)| = 1$, hence U is an $n-$corolla. Thus we have

$$\mathfrak{m}_n^{\otimes}(a_1 \otimes \mathbf{1}_B, \ldots, a_n \otimes \mathbf{1}_B) = \mathfrak{m}_n^A(a_1, \ldots, a_n) \otimes \mathbf{1}_B.$$

Similarly,

$$\mathfrak{m}_n^{\otimes}(\mathbf{1}_A \otimes b_1, \ldots, \mathbf{1}_A \otimes b_n) = \mathbf{1}_A \otimes \mathfrak{m}_n^B(b_1, \ldots, b_n).$$

Next we check the commuting relations in Definition 18. Consider $c, c_1, \ldots, c_n \in A \otimes_\infty B$ such that for all i, either $c_i = a_i \otimes \mathbf{1}_B$ or $c_i = \mathbf{1}_A \otimes b_i$ and $c = a \otimes b$. Following the same argument as above, we have

$$\mathfrak{m}_n^{\otimes}(c_1, \ldots, c_n) = \sum_{\substack{U \in L_n,\ W \in R_n \\ |E_{\text{int}}(U)| + |E_{\text{int}}(W)| = n-2}} \pm\, U(a_1, \ldots, a_n) \otimes W(b_1, \ldots, b_n)$$

where $a_i = \mathbf{1}_A$ for some i, and $b_i = \mathbf{1}_B$ for some i. Clearly, $0 < j_a, j_b < n$ and $j_a + j_b = n$. By the same argument, $|V(U)| + |V(W)| = n$ or $b(U) + b(W) \leq n$. Suppose $U_n(a_1, \ldots, a_n) \neq 0$, then by Lemma 16, $b(U) \geq j_a$. Similarly, $b(w) \geq j_b$.

Now, $\mathfrak{m}_n^{\otimes}(c_1, \ldots, c_n) \neq 0$ implies $n \geq b(U) + b(w) \geqslant j_a + j_b = n$, a contradiction except when $b(U) + b(w) = n$. But

$$\begin{aligned} |V(U)| + |V(W)| = b(U) + nb(U) + b(W) + nb(W) = n \\ \Rightarrow nb(U) = nb(W) = 0 \\ \Rightarrow |V(U)| = b(U) \text{ and } |V(W)| = b(W) \end{aligned}$$

That is, U and W are binary trees with $|V(U)| = n - 1 = |V(W)|$, where $nb(U)$ means non-binary vertices of U. Hence, $|V(U)| + |V(W)| = 2n - 2$ contradicting the above, except when $n = 2$, in which case we have $c_1 = a_1 \otimes \mathbf{1}_B$, $c_2 = \mathbf{1}_A \otimes b_2$ and

$$\mathfrak{m}_2^{\otimes}(c_1, c_2) = \mathfrak{m}_2^{\otimes}(a_1 \otimes \mathbf{1}_B, \mathbf{1}_A \otimes b_2) = \mathfrak{m}_2^A(a_1, \mathbf{1}_A) \otimes \mathfrak{m}_2^B(\mathbf{1}_B, b_2) = (-1)^{|a_1|} a_1 \otimes b_2$$

Similarly, $\mathfrak{m}_2^{\otimes}(c_2, c_1) = \mathfrak{m}_2^{\otimes}(\mathbf{1}_A \otimes b_2, a_1 \otimes \mathbf{1}_B) = (-1)^{|a_1||b_2| + |b_2|} a_1 \otimes b_2$. Therefore, we conclude

$$\mathfrak{m}_2^{\otimes}(c_1, c_2) + (-1)^{|c_1|'|c_2|'}\mathfrak{m}_2^{\otimes}(c_2, c_1) = 0$$

Finally we compute

$$\mathfrak{m}_{n+1}^{\otimes}(c_1, \ldots, c_i, c, c_{i+1}, \ldots, c_n) = \sum_{|E_{\text{int}}(U)|+|E_{\text{int}}(W)|=n-1} \pm U \otimes W$$

Clearly, $|E_{\text{int}}(U)|+|E_{\text{int}}(W)| = n-1$ implies $|V(U)|+|V(W)| = n+1$ and so $b(U)+b(W) \leq n+1$, with $j_a + j_b = n$. By Lemma 16 above, nontrivial contributions come from when $b(U) \geq j_a$ and $b(w) \geq j_b$. Therefore we have

$$n = j_a + j_b \leq b(U) + b(W) \leq n + 1$$

giving two possibilities: U is binary and W has one internal vertex, or vice versa. When $n = 0$, both cases contribute and we have:

$$\mathfrak{m}_1^{\otimes}(a \otimes b) = \mathfrak{m}_1^{A}(a) \otimes b + (-1)^{|a|} a \otimes \mathfrak{m}_1^{B}(b)$$

When $n \geq 1$, since A, B are unital, the nontrivial contributions come from when all $c_i = \mathbf{1}_A \otimes b_i$ or $c_i = a \otimes \mathbf{1}_B$. In the first case we have

$$\mathfrak{m}_{n+1}^{\otimes}(c_1, \cdots, c, \ldots, c_n) = (-1)^{|a|(1+\sum_{j\leq i}|b_j|')} a \otimes \mathfrak{m}_{n+1}^{B}(b_1, \ldots, b_i, b, b_{i+1}, \ldots, b_n)$$

and in the second case we have

$$\mathfrak{m}_{n+1}^{\otimes}(c_1, \cdots, c, \ldots, c_n) = (-1)^{|b|\sum_{i+1\leq j}|a_j|'} \mathfrak{m}_{n+1}^{A}(a_1, \ldots, a_i, a, a_{i+1}, \ldots, a_n) \otimes b$$

as required. $\square$

The commuting subalgebras relation will be enough to show that the Künneth map on Hochschild complexes (to be defined in the next chapter) is a chain map. Our first main result (Theorem 20) is an extension of the commuting subalgebras relation. It will turn out to be the key in showing that the Künneth map respects Mukai pairings (and hence, the higher residue pairings on negative cyclic homology.)

Theorem 20. *Let $c_i = a_i \otimes \mathbf{1}_B = \underline{a}_i$ or $c_i = \mathbf{1}_A \otimes b_i = \underline{b}_i$ for each i and let $u_A \otimes u_B,\ v_A \otimes v_B$ be elements of $A \otimes_\infty B$. Then for $n \geq 2$,*

$$\mathfrak{m}^{\otimes}_{n+2}\left(c_1, ..., c_i, u_A \otimes u_B, c_{i+1}, ..., c_j, v_A \otimes v_B, c_{j+1}, ..., c_n\right) = 0$$

except when

- $c_1, ..., c_i = \underline{a}_1, ..., \underline{a}_p, \underline{b}_1, ..., \underline{b}_q =: \underline{a}^{(1)}, \underline{b}^{(1)}$

- $c_{i+1}, ..., c_j = \underline{b}_{q+1}, ..., \underline{b}_r, \underline{a}_{p+1}, ..., \underline{a}_s =: \underline{b}^{(2)}, \underline{a}^{(2)}$

- $c_{j+1}, ..., c_n = \underline{a}_{s+1}, ..., \underline{a}_t, \underline{b}_{r+1}, ..., \underline{b}_w =: \underline{a}^{(3)}, \underline{b}^{(3)}$

in which case we have:

$$\begin{aligned} &\mathfrak{m}^{\otimes}_{n+2}\left(\underline{a}^{(1)}, \underline{b}^{(1)}, u_A \otimes u_B, \underline{b}^{(2)}, \underline{a}^{(2)}, v_A \otimes v_B, \underline{a}^{(3)}, \underline{b}^{(3)}\right) = \\ &\quad (-1)^{\epsilon}\ \mathfrak{m}^{A}_{*}\left(a^{(1)}, u_A, a^{(2)}, v_A, a^{(3)}\right) \otimes \mathfrak{m}^{B}_{*}\left(b^{(1)}, u_B, b^{(2)}, v_B, b^{(3)}\right) \end{aligned} \tag{3.5}$$

with the tree given in Figure 3.11 and ϵ is explicitly computed in equation (4.16).

Proof. By definition,

$$\mathfrak{m}^{\otimes}_{n+2}\left(- \otimes -\right) = \sum_{\substack{U \in L_{n+2},\ W \in R_{n+2} \\ |E_{\text{int}}(U)| + |E_{\text{int}}(W)| = n \\ U_{\max} \leq W_{\min}}} \pm\, U\left(-\right) \otimes W\left(-\right)$$

Now $|E_{\text{int}}(U)| + |E_{\text{int}}(W)| = |V(U)| - 1 + |V(W)| - 1 = n$ which implies

$$|V(U)| + |V(W)| = n + 2 \tag{3.6}$$

Let $b(U)$ and $nb(U)$ (respectively $b(W)$ and $nb(W)$) denote the number of binary and non-binary vertices of U, (respectively W). Equation (3.6) further implies $b(U) + b(W) \leq n + 2$. Clearly, $j_a + j_b = n$, where $j_a, j_b \geq 0$. By Lemma 16, if $U(\cdots) \neq 0$, then $b(U) \geq j_a$ (unless $j_a = n+2$, which is impossible). So we have $b(U) \geq j_a$, and similarly, $b(W) \geq j_b$. Therefore, we have

$$n = j_a + j_b \leq b(U) + b(W) \leq n + 2 \tag{3.7}$$

However, the equation (3.6) can be rewritten as $b(U) + nb(U) + b(W) + nb(W) = n + 2$. Together with the above inequality yields the following three cases:

1. $b(U) + b(W) = n + 2$; and $nb(U) + nb(W) = 0$
2. $b(U) + b(W) = n + 1$; and $nb(U) + nb(W) = 1$
3. $b(U) + b(W) = n$; and $nb(U) + nb(W) = 2$

In case (1): the equation $nb(U) + nb(W) = 0$ implies $nb(U) = nb(W) = 0$, or U and W have no non-binary vertices. Hence they are binary trees, and so $b(U) = b(W) = n + 1$. We obtain

$$n + 2 = |V(U)| + |V(W)| = b(U) + b(W) = n + 1 + n + 1 = 2n + 2$$

if and only if $n = 0$. This corresponds to $\mathfrak{m}_2^{\otimes}(u_A \otimes u_B, v_A \otimes v_B) = (-1)^{|v_A|'|u_B|'}\mathfrak{m}_2^A(u_A, v_A) \otimes \mathfrak{m}_2^B(u_B, v_B)$, as shown in Figure 3.5 below.

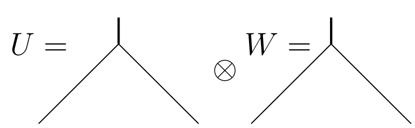

Figure 3.5: *Case 1: U and W have only binary vertices*

In case (2): the equation $nb(U) + nb(W) = 1$ implies $nb(U) = 0$ and $nb(W) = 1$, or $nb(U) = 1$ and $nb(W) = 0$. Suppose $nb(U) = 0$ and $nb(W) = 1$. Then U is a binary tree with $b(U) = n + 1$, so from (3.6) we have $n + 1 + |V(W)| = n + 2$ or $|V(W)| = 1$. Hence, W is the $n + 2$-corolla, $n \geq 1$. The corresponding operation is only non-zero when $c_i = \underline{b}_i$ for all i. Rewrite $\underline{b}^{(1)} = c_1, ..., c_i$; $\underline{b}^{(2)} = c_{i+1}, ..., c_j$; and $\underline{b}^{(3)} = c_{j+1}, ..., c_n$, then this case becomes

$$\mathfrak{m}_{n+2}^{\otimes}(\underline{b}^{(1)}, u_A \otimes u_B, \underline{b}^{(2)}, v_A \otimes v_B, \underline{b}^{(3)}) = (-1)^{\epsilon}\mathfrak{m}_2^A(u_A, v_A) \otimes \mathfrak{m}_{n+2}^B(b^{(1)}, u_B, b^{(2)}, v_B, b^{(3)}) \quad (3.8)$$

as shown in Figure 3.6.

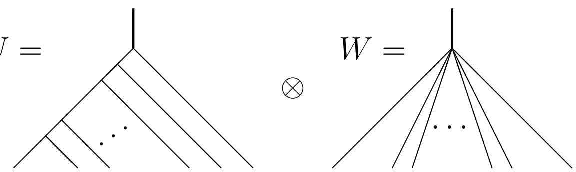

Figure 3.6: *Case 2a: U has only binary vertices and W is a corolla*

Similarly, if $nb(U) = 1$ and $nb(W) = 0$, then W is a binary tree and U is the $n+2$-corolla. This similarly corresponds to $c_i = \underline{a}_i$ for all i, that is,

$$\mathfrak{m}_{n+2}^{\otimes}(\underline{a}^{(1)}, u_A \otimes u_B, \underline{a}^{(2)}, v_A \otimes v_B, \underline{a}^{(3)}) = (-1)^{\epsilon}\mathfrak{m}_{n+2}^A(a^{(1)}, u_A, a^{(2)}, v_A, a^{(3)}) \otimes \mathfrak{m}_2^B(u_B, v_B) \quad (3.9)$$

as shown in Figure 3.7.

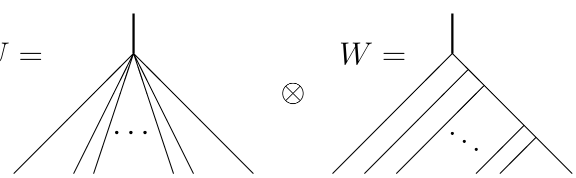

Figure 3.7: *Case 2b: U is the corolla, and W has only binary vertices*

In case (3): the equation $nb(U) + nb(W) = 2$ further yields the following subcases: (a) $nb(U) = 2$ and $nb(W) = 0$; (b) $nb(U) = 0$ and $nb(W) = 2$; (c) $nb(U) = nb(W) = 1$. In subcase (a): $nb(W) = 0$ implies W is a binary tree and so $b(W) = n + 1 = |V(W)|$. On the other hand, U has 2 non-binary vertices. Applying equation (3.6), we obtain $|V(U)| = 1$. But $|V(U)| = b(U) + nb(U)$ which forces $b(U) = -1$, a contradiction. A similar result also follows from subcase (b).

Finally we consider the case $nb(U) = 1 = nb(W)$. By Lemma 16, $U(\cdots) = 0$ unless

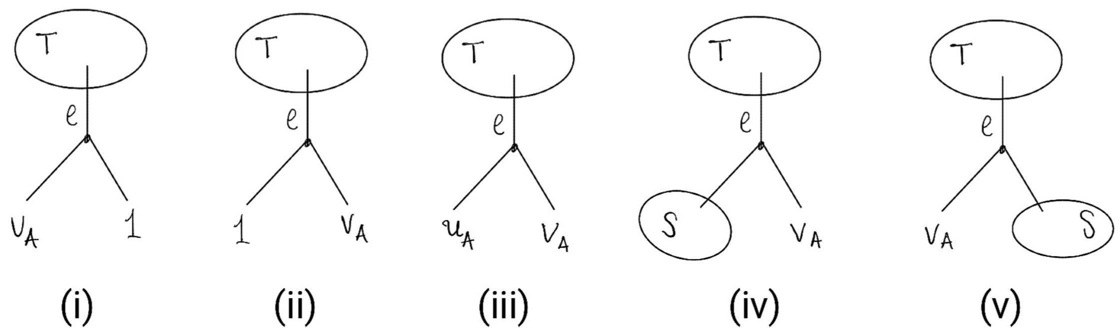

Figure 3.8: *All 5 subcases*

$b(U) \geq j_a$ and similarly, $W(\cdots) = 0$ unless $b(W) \geq j_b$, where $j_a, j_b \geq 0$. Further, from (3.6), we obtain $b(U) + b(W) = n$, or

$$n = j_a + j_b \leq b(U) + b(W) = n \tag{3.10}$$

which implies $j_a + j_b = b(U) + b(W)$ or $b(U) = j_a$ and $b(W) = j_b$, that is, on U the number of binary vertices corresponds to j_a (the number of b's) and on W the number of binary vertices corresponds to j_b (the number of a's).

Note that if $j_a = 0$, $(j_b > 0)$ then U is the $n+2$-corolla but W will have only binary vertices, which is fine, except that it does not belong to this case since $nb(W) = 0$. A similar thing happens when $j_b = 0$, $(j_a > 0)$. If $j_a = j_b = 0$, we are back to Case 1. So we consider $j_a, j_b > 0$.

On U, the a's (coming from $\underline{a}$'s) must be on leaves incident with the unique non-binary vertex since otherwise, the right-leaning internal edges created are collapsed in W forcing the corresponding 1's to be inserted on a non-binary vertex on W, which is zero by unitality. Similarly, on W, the b's (coming from $\underline{b}$'s) must be inserted on the unique non-binary vertex.

Again, on U, we must have v_A inserted on the unique non-binary vertex. Otherwise, we have the following cases shown in Figure 3.8.

In case (i) since there are certainly leaves to the left of v_A, (namely, u_A) the edge e could be the last leaf and we would have attached corollas on it leading to a contradiction.

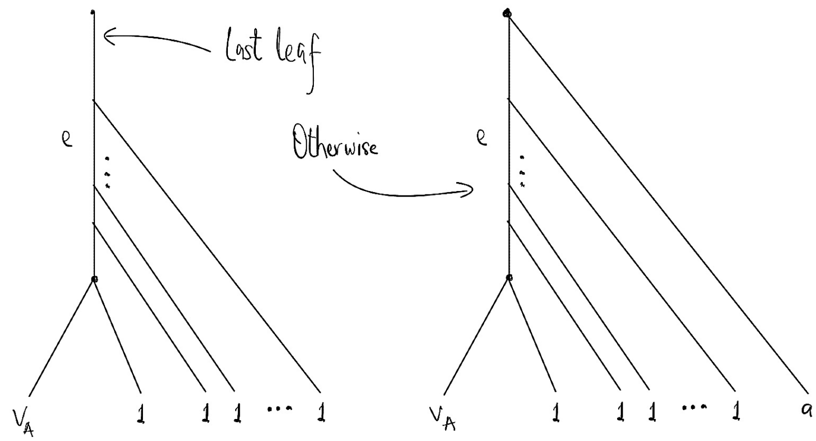

Figure 3.9: *Case (i)*

Otherwise, e must eventually (up to finitely many binary attachments) join the unique non-binary vertex with a attached to the right as shown in Figure 3.9. However, resolving this tree leads to attaching edge e on the leaf containing a. This creates a right-leaning internal edge which is collapsed to obtain the unique non-binary vertex on tree W containing the unit 1. Hence, 0 by unitality. A similar argument applies to case (ii).

In case (iii) you automatically create an additional binary vertex and so $b(U) > j_a$, a contradiction.

In case (iv), if edge e is not the last leaf, then the unique non-binary vertex is contained in T with a attached to the right as in case (i) as shown in Figure 3.10. Once again, resolving this tree yields $W = 0$ by unitality. A similar argument works for case (iv).

So, on tree U, we must have all $a's$ and v_A inserted on the unique non-binary vertex. On the other hand, 1's must be inserted on binary vertices, while u_A is free in general to join a binary vertex or the non-binary vertex. Similarly on tree W, we must have all $b's$ and u_B inserted on the unique non-binary vertex, while v_B is free in general to join a binary vertex or the non-binary vertex.

We thus have a unique non-trivial arrangement, namely, $\underline{c}_1, ..., \underline{c}_i$ must be of the ordered form $(\underline{a}^{(1)}, \underline{b}^{(1)})$, that is, all the $\underline{a}$'s must come before $\underline{b}$'s since any term of the form $\underline{b}, \underline{a}$ forces one to attach corollas on the last leaf or have 1 share a binary vertex with a on U. Similarly,

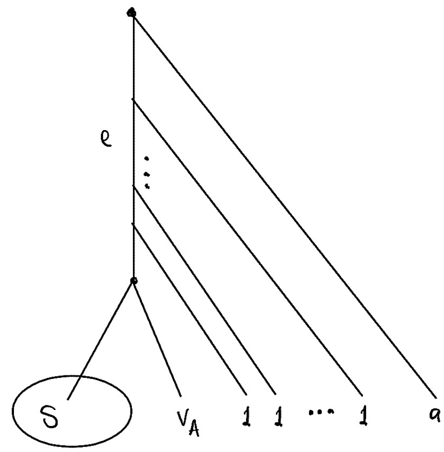

Figure 3.10: *Case (iv)*

$\underline{c}_{i+1}, ..., \underline{c}_j$ must be ordered as $(\underline{b}^{(2)}, \underline{a}^{(2)})$; since any term of the form $\underline{a}, \underline{b}$ forces 1 to share a binary vertex with a or v_A on U. Similarly, we must also have $\underline{c}_{j+1}, ..., \underline{c}_n$ be of the form $(\underline{a}^{(3)}, \underline{b}^{(3)})$.

These restrictions guarantee the existence of unique $U \in L_{n+2}$ and $W \in R_{n+2}$ such that

$$\mathfrak{m}_*^{\otimes}\left(\underline{a}^{(1)}, \underline{b}^{(1)}, u_A \otimes u_B, \underline{b}^{(2)}, \underline{a}^{(2)}, v_A \otimes v_B, \underline{a}^{(3)}, \underline{b}^{(3)}\right) =$$
$$(-1)^{\epsilon}\, U\left(a^{(1)}, u_A, a^{(2)}, v_A, a^{(3)}\right) \otimes W\left(b^{(1)}, u_B, b^{(2)}, v_B, b^{(3)}\right)$$

where the trees U and W are shown below.

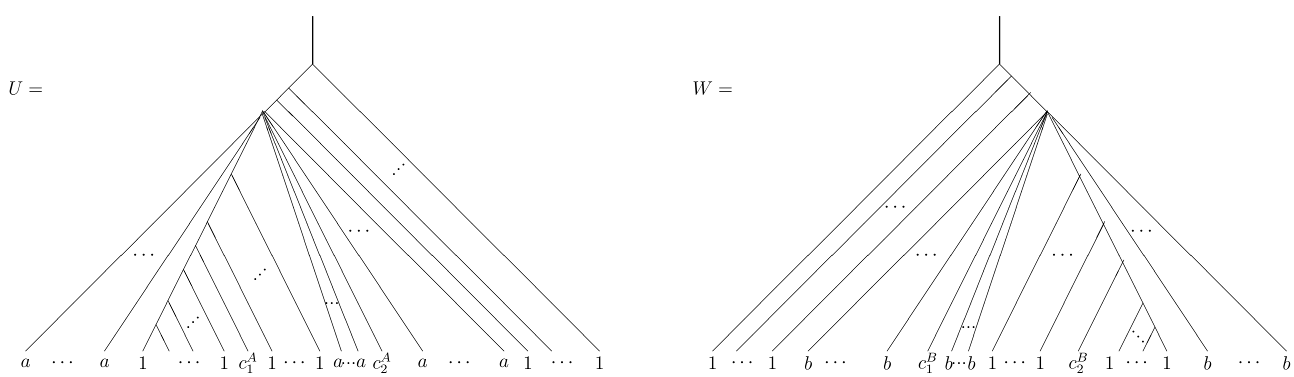

Figure 3.11: *Unique trees U and W*

where

$$
\begin{aligned}
\epsilon =&(\#a^{(2)} + \#a^{(3)})(\#b^{(1)} + \#b^{(2)}) + \#b^{(3)}(\#a^{(1)} + \#a^{(2)} + \#a^{(3)}) \\
|v_B||a^{(3)}|+&(|u_B| + |b^{(2)}|)(|a^{(2)}| + |v_A| + |a^{(3)}|) + |b^{(1)}|(|u_A| + |a^{(2)}| + |v_A| + |a^{(3)}|) \\
&+\#b^{(3)}|a^{(1)}| + (\#b^{(1)} + \#b^{(2)} + \#b^{(3)})(|a^{(2)}| + |a^{(3)}| + |u_A| + |v_A|) \\
&+(\#a^{(2)} + 1)(|b^{(1)}| + |u_B| + |b^{(2)}|) + (\#a^{(3)} - 1)(|b^{(1)}| + |b^{(2)}| + |u_B| + |v_B|)
\end{aligned} \tag{3.11}
$$

and $\#a^{(i)} = m$ if $a^{(i)} = a_1^{(i)} \otimes \cdots \otimes a_m^{(i)}$, similarly defined for $\#b^{(i)}$. $\square$

Chapter 4

The Künneth maps

In this chapter, all our A_∞ algebras will be smooth and proper, strictly unital, and $\mathbb{Z}/2\mathbb{Z}$-graded as defined, unless otherwise stated. We will use the shuffle product to define a Künneth map between tensor product of A_∞ algebras as follows.

4.1 Shuffle and cyclic shuffle products

The (cyclic) shuffle product will play an important role in defining the Künneth maps. We introduce them here.

Definition 21. *Let $m, n \geq 0$ be integers. We say a permutation $\sigma \in S_{n+m}$ is a (n,m)-shuffle if $\sigma(i) < \sigma(j)$ for $i < j \leq n$ or $n+1 \leq i < j$. Consider $a = a_1 \otimes \ldots \otimes a_n \in A^{\otimes n}$ and $b = b_1 \otimes \ldots \otimes b_m \in B^{\otimes m}$ and denote*

$$u_i = \begin{cases} a_i & ,\ 1 \leqslant i \leqslant n, \\ b_{i-n} & , n < 1 \leqslant n+m. \end{cases}$$

Given a (n,m)-shuffle σ, we define $\sigma(a;b) = (-1)^\epsilon u_{\sigma^{-1}(1)} \otimes \ldots \otimes u_{\sigma^{-1}(n+m)}$, where $\epsilon =$

$\sum\limits_{\substack{i\leqslant n<j\\ \sigma(j)<\sigma(i)}} |u_j|'|u_i|'$. *With the same notation, define the shuffle product*

$$\mathrm{sh}(a;b) = \sum_{\sigma} \sigma(a;b) \tag{4.1}$$

where we sum over all (n,m)-shuffles σ.

Some properties of the shuffle product sh that will be useful for us are stated.

Lemma 22. [12] *We have the following:*

$$(a) \sum_{(\mathrm{sh}(a;b))} \mathrm{sh}(a;b)^{(1)} \otimes \mathrm{sh}(a;b)^{(2)} = \sum_{(a)(b)} (-1)^{|a^{(2)}|'|b^{(1)}|'} \,\mathrm{sh}\left(a^{(1)};b^{(1)}\right) \otimes \mathrm{sh}\left(a^{(2)};b^{(2)}\right)$$

(b) $\mathrm{sh}\left(a^{(1)},x,a^{(2)};b\right) = \sum_{(b)} (-1)^{|b^{(1)}|'(|x|'+|a^{(2)}|')} \,\mathrm{sh}\left(a^{(1)};b^{(1)}\right) \otimes x \otimes \mathrm{sh}\left(a^{(2)};b^{(2)}\right)$

(c) $\mathrm{sh}\left(a;b^{(1)},y,b^{(2)}\right) = \sum_{(a)} (-1)^{|a^{(2)}|'(|y|'+|b^{(1)}|')} \,\mathrm{sh}\left(a^{(1)};b^{(1)}\right) \otimes y \otimes \mathrm{sh}\left(a^{(2)};b^{(2)}\right)$

Definition 23. *A (n,m)-cyclic shuffle is a permutation $\{\sigma(1),...,\sigma(n+m)\}$ of $\{1,...,n+m\}$ obtained as follows. Perform a cyclic permutation of any order on the set $\{1,...,n\}$ and a cyclic permutation of any order on the set $\{n+1,...,n+m\}$. Then shuffle the two results to obtain $\{\sigma(1),...,\sigma(n+m)\}$. This is a cyclic shuffle if $\sigma(n+1)$ appears before $\sigma(1)$, denoted as $\sigma(n+1) < \sigma(1)$ in this sequence.*

Note how this compares to the definition in[7] or[8]. In our context with $\mathbf{a} = (a_0|...,a_n)$ and $\mathbf{b} = (b_0|...,b_m)$, the cyclic shuffle of $\{\mathbf{a},\mathbf{b}\}$ will involve performing a cyclic permutation of any order on $\mathbf{a}$ and $\mathbf{b}$ respectively, and then shuffling the result so that b_0 stays to the left of a_0. We denote this operation by Sh.

Let

$$v_i = \begin{cases} a_i & ,\ 1 \leqslant i \leqslant n+1, \\ b_{i-n-1} & ,n+1 < 1 \leqslant n+m+2. \end{cases}$$

Given a $(n+1,m+1)$-cyclic shuffle σ, we define $\sigma(\mathbf{a};\mathbf{b}) = (-1)^{\epsilon} v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(n+m+2)}$, where $\epsilon = \sum_{\substack{i \leqslant n+1 < j \\ \sigma(j) < \sigma(i)}} |v_j|'|v_i|'$. Define the cyclic shuffle product

$$\mathrm{Sh}(\mathbf{a}; \mathbf{b}) = \sum_{\sigma} (-1)^{@_A + @_B} \sigma(\mathbf{a}; \mathbf{b}) \tag{4.2}$$

where we sum over all $(n+1, m+1)$-cyclic shuffles σ.

4.2 Hochschild homology

Definition 24. *Let A, B be two A_∞ algebras, and let $(C_*(A), \mathfrak{b}), (C_*(B), \mathfrak{b})$ be their respective Hochschild chain complexes. Denote $\mathbf{a} = a_0|a$ where $a = a_1 \otimes ... \otimes a_n \in A^{\otimes n}$ and $\mathbf{b} = b_0|b$ where $b = b_1 \otimes ... \otimes b_m \in B^{\otimes m}$. Define $k : C_*(A) \otimes C_*(B) \to C_*(A \otimes B)$ by*

$$k(\mathbf{a} \otimes \mathbf{b}) = \sum_{(n,m)-shuffles} (-1)^{|b_0||a|'} a_0 \otimes b_0 | \, \mathrm{sh}(\underline{a}; \underline{b}) \tag{4.3}$$

where $\underline{a} = a_1 \otimes \mathbf{1}_B, ..., a_n \otimes \mathbf{1}_B$ and $\underline{b} = \mathbf{1}_A \otimes b_1, ..., \mathbf{1}_A \otimes b_m$.

Proposition 25. *The map k defined above is a quasi-isomorphism of complexes.*

Proof. We first show that it is a chain map. That is: $\mathfrak{b} \circ k = k \circ (\mathfrak{b} \otimes id + id \otimes \mathfrak{b})$, where $\mathfrak{b}$ is the Hochschild boundary map.

On the first term of $\mathfrak{b}$ we have

$$\begin{aligned}
\mathfrak{b}_0(k(\mathbf{a},\mathbf{b})) &= \sum (-1)^{\alpha+\gamma} a_0\otimes b_0|\operatorname{sh}(\underline{a};\underline{b})^{(1)}\mathfrak{m}(\operatorname{sh}(\underline{a};\underline{b})^{(2)})\operatorname{sh}(\underline{a};\underline{b})^{(3)} \\
&= \sum (-1)^{\alpha+\beta+\gamma} a_0\otimes b_0|\operatorname{sh}(\underline{a}^{(1)};\underline{b}^{(1)})\otimes \mathfrak{m}\left(\operatorname{sh}(\underline{a}^{(2)};\underline{b}^{(2)})\right)\otimes \operatorname{sh}(\underline{a}^{(3)};\underline{b}^{(3)}) \qquad (4.4)\\
&= \sum (-1)^{\alpha+\beta+\gamma} a_0\otimes b_0|\operatorname{sh}(\underline{a}^{(1)};\underline{b}^{(1)})\otimes \mathfrak{m}(\underline{a}^{(2)})\otimes \operatorname{sh}(\underline{a}^{(3)};\underline{b}^{(3)}) \\
&\quad + \sum (-1)^{\alpha+\beta+\gamma} a_0\otimes b_0|\operatorname{sh}(\underline{a}^{(1)};\underline{b}^{(1)})\otimes \mathfrak{m}(\underline{b}^{(2)})\otimes \operatorname{sh}(\underline{a}^{(3)};\underline{b}^{(3)}) \qquad (4.5)\\
&= \sum (-1)^{\alpha+\beta+\gamma+\delta} a_0\otimes b_0|\operatorname{sh}(\underline{a}^{(1)},\mathfrak{m}(\underline{a}^{(2)}),\underline{a}^{(3)};\underline{b}) \\
&\quad + \sum (-1)^{\alpha+\beta+\gamma+\phi} a_0\otimes b_0|\operatorname{sh}(\underline{a};\underline{b}^{(1)},\mathfrak{m}(\underline{b}^{(2)}),\underline{b}^{(3)}) \\
&= k(\mathfrak{b}_0(\mathbf{a});\mathbf{b}) + (-1)^{|\mathbf{a}|}k(\mathbf{a};\mathfrak{b}_0(\mathbf{b}))
\end{aligned}$$

where equations (4.4) and (4.5) follow from Lemma 22 and Proposition 19 respectively. The signs are given by $\alpha = |b_0||a|'$, $\beta = |\underline{b}^{(1)}|'(|\underline{a}^{(2)}|'+|\underline{a}^{(3)}|') + |\underline{b}^{(2)}|'|\underline{a}^{(3)}|'$, $\gamma = |a_0\otimes b_0|' + |\underline{a}^{(1)}|' + |\underline{b}^{(1)}|'$, $\delta = |\underline{b}^{(1)}|'(|\underline{a}^{(2)}|'+1+|\underline{a}^{(3)}|')$, $\phi = |\underline{a}^{(3)}|'(|\underline{b}^{(2)}|'+1+|\underline{b}^{(1)}|')$

Similarly, on the second term of $\mathfrak{b}$, we have

$$\begin{aligned}
\mathfrak{b}_1(k(\mathbf{a},\mathbf{b})) &= \sum (-1)^{\alpha+\eta}\mathfrak{m}(\operatorname{sh}(\underline{a};\underline{b})^{(3)}, a_0\otimes b_0, \operatorname{sh}(\underline{a};\underline{b})^{(1)}) \;|\; \operatorname{sh}(\underline{a};\underline{b})^{(2)} \\
&= \sum (-1)^{\alpha+\beta+\eta}\mathfrak{m}\left(\operatorname{sh}(\underline{a}^{(3)};\underline{b}^{(3)}), a_0\otimes b_0, \operatorname{sh}(\underline{a}^{(1)};\underline{b}^{(1)})\right)|\operatorname{sh}(\underline{a}^{(2)};\underline{b}^{(2)}) \qquad (4.6)\\
&= \sum (-1)^{\alpha+\beta+\eta}\mathfrak{m}\left(\underline{a}^{(3)}, a_0\otimes b_0, \underline{a}^{(1)}\right)|\operatorname{sh}(\underline{a}^{(2)};\underline{b}^{(2)}) \\
&\quad + \sum (-1)^{\alpha+\beta+\eta}\mathfrak{m}\left(\underline{b}^{(3)}, a_0\otimes b_0, \underline{b}^{(1)}\right)|\operatorname{sh}(\underline{a}^{(2)};\underline{b}^{(2)}) \qquad (4.7)\\
&= \sum (-1)^{\alpha+\beta+\eta+\theta_1}\mathfrak{m}\left(\underline{a}^{(3)}, a_0, \underline{a}^{(1)}\right)\otimes b_0|\operatorname{sh}(\underline{a}^{(2)};\underline{b}^{(2)}) \\
&\quad + \sum (-1)^{\alpha+\beta+\eta+\theta_2} a_0\otimes\mathfrak{m}\left(\underline{b}^{(3)}, b_0, \underline{b}^{(1)}\right)|\operatorname{sh}(\underline{a}^{(2)};\underline{b}^{(2)}) \\
&= k(\mathfrak{b}_1(\mathbf{a});\mathbf{b}) + (-1)^{|\mathbf{a}|}k(\mathbf{a};\mathfrak{b}_1(\mathbf{b})) \qquad (4.8)
\end{aligned}$$

where equations (4.6) and (4.7) follow from Lemma 22 and Proposition 19 respectively.

The signs are given by $\eta = (|\underline{a}^{(3)}|' + |\underline{b}^{(3)}|')(|\underline{a}^{(2)}|' + |\underline{b}^{(2)}|' + |\underline{a}^{(1)}|' + |\underline{b}^{(1)}|' + |a_0|' + |b_0|)$, $\theta_1 = |b_0||\underline{b}^{(1)}|'$, $\theta_2 = |a_0|(|\underline{b}^{(3)}|'+1)$. Hence, k is a chain map.

To show that it induces an isomorphism in homology, consider the length filtration on the chain complex of A defined by

$$\mathcal{F}^l C_*(A) = \bigoplus_{n \le l} A \otimes A^{\otimes n}.$$

Clearly, the Hochschild boundary map $\mathfrak{b}$ restricts to a boundary map on each filtered piece making it a chain complex for each l. Moreover we have the inclusion

$$\ldots \subset \mathcal{F}^{l-1} C_*(A) \subset \mathcal{F}^l C_*(A) \subset \mathcal{F}^{l+1} C_*(A) \subset \ldots \qquad \text{and} \qquad C(A) = \bigcup_l \mathcal{F}^l C(A)$$

Taking homology of the quotient $E^0 = \mathcal{F}^l C_*(A)/\mathcal{F}^{l-1} C_*(A)$, with respect to the induced differential gives the E^1-page as the algebra complex

$$E^1 = \left(\bigoplus_l H^*(A)^{\otimes l}, \mathfrak{b}_{\mathfrak{m}_2} \right)$$

where $H^*(A)$ is the cohomology of A with differential $\mathfrak{m}_1$, and $\mathfrak{b}_{\mathfrak{m}_2}$ is the Hochschild differential induced by $\mathfrak{m}_2$ only. Taking homology now gives the Hochschild homology of the associative algebra above. For two A_∞-algebras A, B we have the map on E^2-pages

$$[k] : HH_*(H^*(A), \mathfrak{m}_2^A) \otimes HH_*(H^*(B), \mathfrak{m}_2^B) \longrightarrow HH_*(H^*(A \otimes B), \mathfrak{m}_2^{\otimes})$$

given by the shuffle map which is known to be an isomorphism for associative algebras (see for example[7;22]). Hence, by the spectral sequence comparison theorem [Theorem 5.2.12][23] , the result follows. $\square$

Theorem 26. *Assume A and B are finite-dimensional, (one can always do this by going to the minimal model since A and B are proper.) The quasi-isomorphism k respects the chain-level Mukai pairings.*

Proof. We show this at the chain-level. Let $\mathbf{a}^i = (a_0^i | a_1^i, ..., a_n^i) \in C_*(A)$, for $i = 1, 2$ and similarly define $\mathbf{b}^j \in C_*(B)$ for $j = 1, 2$.

$$
\begin{aligned}
k(\mathbf{a}^1; \mathbf{b}^1) &= \sum (-1)^{|b_0^1||a^1|'} a_0^1 \otimes b_0^1 | \operatorname{sh}(\underline{a}^1; \underline{b}^1)^{(1)}, \operatorname{sh}(\underline{a}^1; \underline{b}^1)^{(2)}, \operatorname{sh}(\underline{a}^1; \underline{b}^1)^{(3)} \\
&= \sum (-1)^{|b_0^1||a^1|' + |sh^1|} a_0^1 \otimes b_0^1 | \operatorname{sh}(\underline{a}^{1(1)}; \underline{b}^{1(1)}), \operatorname{sh}(\underline{a}^{1(2)}; \underline{b}^{1(2)}), \operatorname{sh}(\underline{a}^{1(3)}; \underline{b}^{1(3)}) \qquad (4.9) \\
k(\mathbf{a}^2; \mathbf{b}^2) &= \sum (-1)^{|b_0^2||a^2|'} a_0^2 \otimes b_0^2 | \operatorname{sh}(\underline{a}^2; \underline{b}^2)^{(1)}, \operatorname{sh}(\underline{a}^2; \underline{b}^2)^{(2)}, \operatorname{sh}(\underline{a}^2; \underline{b}^2)^{(3)} \\
&= \sum (-1)^{|b_0^2||a^2|' + |sh^2|} a_0^2 \otimes b_0^2 | \operatorname{sh}(\underline{a}^{2(1)}; \underline{b}^{2(1)}), \operatorname{sh}(\underline{a}^{2(2)}; \underline{b}^{2(2)}), \operatorname{sh}(\underline{a}^{2(3)}; \underline{b}^{2(3)}) \qquad (4.10)
\end{aligned}
$$

where $|sh^1| = |a^{1(2)}|'|b^{1(1)}|' + |a^{1(3)}|'|b^{1(2)}|' + |a^{1(3)}|'|b^{1(1)}|'$ and
$|sh^2| = |a^{2(2)}|'|b^{2(1)}|' + |a^{2(3)}|'|b^{2(2)}|' + |a^{2(3)}|'|b^{2(1)}|'$
Applying the Mukai pairing, we obtain

$$
\begin{aligned}
&\left\langle k(\mathbf{a}^1; \mathbf{b}^1), k(\mathbf{a}^2; \mathbf{b}^2) \right\rangle_{Muk} = \\
&\sum (-1)^{|b_0^1||a^1|' + |sh^1| + |b_0^2||a^2|' + |sh^2| + \#} \; tr[c_A \otimes c_B \mapsto \mathfrak{m}_*^{\otimes}(\operatorname{sh}(\underline{a}^1; \underline{b}^1)^{(3)}, a_0^1 \otimes b_0^1, \operatorname{sh}(\underline{a}^1; \underline{b}^1)^{(1)}, \\
&\qquad \mathfrak{m}_*^{\otimes}(\operatorname{sh}(\underline{a}^1; \underline{b}^1)^{(2)}, c_A \otimes c_B, \operatorname{sh}(\underline{a}^2; \underline{b}^2)^{(3)}, a_0^2 \otimes b_0^2, \operatorname{sh}(\underline{a}^2; \underline{b}^2)^{(1)}), \operatorname{sh}(\underline{a}^2; \underline{b}^2)^{(2)})]
\end{aligned}
\tag{4.11}
$$

where $\#$ is the corresponding sign of the Mukai pairing as in equation (2.1). Applying Theorem 20 on the inner $\mathfrak{m}_*^{\otimes}$ we obtain

$$
\sum (-1)^{* + \epsilon_{in}} \; tr[c_A \otimes c_B \mapsto \mathfrak{m}_*^{\otimes}(\operatorname{sh}(\underline{a}^1; \underline{b}^1)^{(3)}, a_0^1 \otimes b_0^1, \operatorname{sh}(\underline{a}^1; \underline{b}^1)^{(1)}, \alpha \otimes \beta, \operatorname{sh}(\underline{a}^2; \underline{b}^2)^{(2)})] \tag{4.12}
$$

Further applying Theorem 20, and observing that the trace of a tensor product is the product of the traces, we obtain

$$\sum(-1)^{*+\epsilon_{in}+\epsilon_{out}}\ tr[c_A\otimes c_B\mapsto \mathfrak{m}_*^A(a^{1(3)},a_0^1,a^{1(1)},\alpha,a^{2(2)})\otimes\mathfrak{m}_*^B(b^{1(3)},b_0^1,b^{1(1)},\beta,b^{2(2)})$$

$$\begin{aligned}&=(-1)^{|\mathbf{b}^1|'|\mathbf{a}^2|'}\ \sum(-1)^{\#_A}\ tr[c_A\mapsto\mathfrak{m}_*^A(a^{1(3)},a_0^1,a^{1(1)},\alpha,a^{2(2)})]\\&\qquad\cdot\ \sum(-1)^{\#_B}\ tr[c_B\mapsto\mathfrak{m}_*^B(b^{1(3)},b_0^1,b^{1(1)},\beta,b^{2(2)})] \qquad (4.13)\\&=(-1)^{|\mathbf{b}^1|'|\mathbf{a}^2|'}\langle\mathbf{a}^1,\mathbf{a}^2\rangle_{Muk}\cdot\langle\mathbf{b}^1,\mathbf{b}^2\rangle_{Muk} \qquad (4.14)\end{aligned}$$

where $\#_A$, $\#_B$ are the corresponding signs of the Mukai pairing as in equation (2.1), $*=|b_0^1||a^1|'+|sh^1|+|b_0^2||a^2|'+|sh^2|+\#$, $\alpha=\mathfrak{m}_*^A(a^{1(2)},c_A,a^{2(3)},a_0^2,a^{2(1)})$, $\beta=\mathfrak{m}_*^B(b^{1(2)},c_B,b^{2(3)},b_0^2,b^{2(1)})$;

$$\begin{aligned}\epsilon_{in}=&(\#a^{2(3)}+\#a^{2(1)})(\#b^{1(2)}+\#b^{2(3)})+\#b^{2(1)}(\#a^{1(2)}+\#a^{2(3)}+\#a^{2(1)})\\&|b_0^2||a^{2(1)}|+(|c_B|+|b^{2(3)}|)(|a^{2(3)}|+|a_0^2|+|a^{2(1)}|)+|b^{1(2)}|(|c_A|+|a^{2(3)}|+|a_0^2|+|a^{2(1)}|)\\&+\#b^{2(1)}|a^{1(2)}|+(\#b^{1(2)}+\#b^{2(3)}+\#b^{2(1)})(|a^{2(3)}|+|a^{2(1)}|+|c_A|+|a_0^2|)\\&+(\#a^{2(3)}+1)(|b^{1(2)}|+|c_B|+|b^{2(3)}|)+(\#a^{2(1)}-1)(|b^{1(2)}|+|b^{2(3)}|+|c_B|+|b_0^2|) \qquad (4.15)\end{aligned}$$

and

$$\begin{aligned}\epsilon=&(\#a^{1(1)}+\#a^{2(2)})(\#b^{1(3)}+\#b^{1(1)})+\#b^{2(2)}(\#a^{1(3)}+\#a^{1(1)}+\#a^{2(2)})\\&|\beta||a^{2(2)}|+(|b_0^1|+|b^{1(1)}|)(|a^{1(1)}|+|\alpha|+|a^{2(2)}|)+|b^{1(3)}|(|a_0^1|+|a^{1(1)}|+|\alpha|+|a^{2(2)}|)\\&+\#b^{2(2)}|a^{1(3)}|+(\#b^{1(3)}+\#b^{1(1)}+\#b^{2(2)})(|a^{1(1)}|+|a^{2(2)}|+|a_0^1|+|\alpha|)\\&+(\#a^{1(1)}+1)(|b^{1(3)}|+|b_0^1|+|b^{1(1)}|)+(\#a^{2(2)}-1)(|b^{1(3)}|+|b^{1(1)}|+|b_0^1|+|\beta|) \qquad (4.16)\end{aligned}$$

with $\#a^{i(j)}=m$ if $a^{i(j)}=a_1^{i(j)}\otimes\cdots\otimes a_m^{i(j)}$, and similarly defined for $\#b^{i(j)}$.

□

Corollary 27. *Mukai pairing is non-degenerate on $A\otimes B$.*

4.3 Negative cyclic homology

We use the cyclic shuffle map to define a Künneth map for the negative cyclic complexes as follows.

Definition 28. *Let A, B be two A_∞ algebras, and let $(C_*^-(A), \mathfrak{b}+u\mathfrak{B}), (C_*^-(B), \mathfrak{b}+u\mathfrak{B})$ be their respective negative cyclic chain complexes. Denote $\mathbf{a} = a_0|a$ where $a = a_1, ..., a_n$ and $\mathbf{b} = b_0|b$ where $b = b_1, ..., b_m$. Define $K : C_*(A) \otimes C_*(B) \to C_*(A \otimes B)$ by*

$$K(\mathbf{a} \otimes \mathbf{b}) = \sum \mathbf{1}_A \otimes \mathbf{1}_B | \operatorname{Sh}(\underline{a}_0, \underline{a}_1, \ldots, \underline{a}_n; \underline{b}_0, \underline{b}_1, \ldots, \underline{b}_m) \tag{4.17}$$

where $\underline{a}_i = a_i \otimes \mathbf{1}_B$, and $\underline{b}_j = \mathbf{1}_A \otimes b_j$, and extend u-linearly.

Proposition 29. *The map $k + uK : C_*^-(A) \otimes_{k[[u]]} C_*^-(B) \to C_*^-(A \otimes B)$ is a quasi-isomorphism of complexes.*

Proof. We first show that it is a chain map. That is:

$$(\mathfrak{b}+u\mathfrak{B})(k+uK)(\mathbf{a};\mathbf{b}) = (k+uK)((\mathfrak{b}+u\mathfrak{B})(\mathbf{a});\mathbf{b}) + (-1)^{|\mathbf{a}|}(k+uK)(\mathbf{a};(\mathfrak{b}+u\mathfrak{B})(\mathbf{b})) \tag{4.18}$$

Comparing coefficients in u, this is equivalent to the three equations:

$$u^0 : \mathfrak{b}\ k(\mathbf{a};\mathbf{b}) = k(\mathfrak{b}(\mathbf{a});\mathbf{b}) + (-1)^{|\mathbf{a}|} k(\mathbf{a};\mathfrak{b}(\mathbf{b})) \tag{4.19}$$

$$u : \ \mathfrak{B}\ k(\mathbf{a};\mathbf{b}) + \mathfrak{b}K(\mathbf{a};\mathbf{b}) = k(\mathfrak{B}(\mathbf{a});\mathbf{b}) + K(\mathfrak{b}(\mathbf{a});\mathbf{b}) + (-1)^{|\mathbf{a}|} k(\mathbf{a};\mathfrak{B}(\mathbf{b})) + (-1)^{|\mathbf{a}|} K(\mathbf{a};\mathfrak{b}(\mathbf{b}))$$

$$u^2 : \mathfrak{B}\ K(\mathbf{a};\mathbf{b}) = K(\mathfrak{B}(\mathbf{a});\mathbf{b}) + (-1)^{|\mathbf{a}|} K(\mathbf{a};\mathfrak{B}(\mathbf{b})) \tag{4.20}$$

Equation (4.19) follows from Proposition 25 since k is a chain map. The left hand side of equation (4.20) vanishes since our chains are reduced. The terms on the right hand side also vanish individually for the same reason since

$$
\begin{aligned}
K\left(\mathfrak{B}(\mathbf{a});\mathbf{b}\right) &= \sum \pm K\left(\ 1\ |a^{(2)}, a_0, a^{(1)};\ \underline{b}_0, ..., \underline{b}_m\right) \\
&= \sum \pm\ 1\otimes 1\ |\ \mathrm{Sh}(1\otimes 1, \underline{a}^{(2)}, \underline{a}_0, \underline{a}^{(1)};\ \underline{b}_0, ..., \underline{b}_m) = 0 \\
\text{and } K(\mathbf{a};\mathfrak{B}(\mathbf{b})) &= \sum \pm\ 1\otimes 1\ |\ \mathrm{Sh}(\underline{a}_0, ..., \underline{a}_n;\ 1\otimes 1, \underline{b}^{(2)}, \underline{b}_0, \underline{b}^{(1)}) = 0
\end{aligned}
$$

For the coefficient of u, a calculation is given in Appendix A.1.

To show that it induces an isomorphism in homology, consider the u-filtration on the negative cyclic complex of A defined by

$$
\mathcal{F}^l C_*^-(A) = \bigoplus_{p\geq l} u^p C_*^-(A)
$$

The boundary map $\mathfrak{b} + u\mathfrak{B}$ restricts to a boundary map on each filtered piece and we have the reverse inclusion

$$
\ldots \supset \mathcal{F}^{l-1}C_*^-(A) \supset \mathcal{F}^l C_*^-(A) \supset \mathcal{F}^{l+1}C_*^-(A) \supset \ldots, \qquad \text{and} \quad \bigcap_l \mathcal{F}^l C_*^-(A) = 0
$$

Taking the quotient $E^0 = \mathcal{F}^l C_*^-(A)/\mathcal{F}^{l+1}C_*^-(A)$, we have the induced differential as $\mathfrak{b}$ since the term $u\mathfrak{B}$ increases filtration and hence killed in the quotient. Taking homology we obtain the E^1-page of the spectral sequence as the Hochschild homology of A. For two A_∞-algebras A, B we have the map on E^1-pages

$$
[k] : HH_*(A) \otimes HH_*(B) \longrightarrow HH_*(A\otimes B)
$$

which is an isomorphism from above. Hence, by the spectral sequence comparison theorem [Theorem 5.2.12][23], the result follows. □

We therefore prove our main theorem.

Theorem 30. *The cyclic Kunneth quasi-isomorphism of negative cyclic homology respects* *the semi-infinite Hodge structures. In particular, it respects the higher residue pairings, as well as the u-connection on negative cyclic homology.*

Proof. We first show that the cyclic Künneth map respects the higher residue pairings. Let $\mathbf{a}^i = (a_0^i | a_1^i, ..., a_n^i) \in C_*(A)$, for $i = 1, 2$ and similarly define $\mathbf{b}^j \in C_*(B)$ for $j = 1, 2$.

$$\left\langle (k + uK)(\mathbf{a}^1; \mathbf{b}^1), (k + uK)(\mathbf{a}^2; \mathbf{b}^2) \right\rangle_{hres} = \left\langle k(\mathbf{a}^1; \mathbf{b}^1), k(\mathbf{a}^2; \mathbf{b}^2) \right\rangle_{Muk} \tag{4.21}$$

$$-u \left\langle k(\mathbf{a}^1; \mathbf{b}^1), K(\mathbf{a}^2; \mathbf{b}^2) \right\rangle_{Muk} + u \left\langle K(\mathbf{a}^1; \mathbf{b}^1), k(\mathbf{a}^2; \mathbf{b}^2) \right\rangle_{Muk} \tag{4.22}$$

$$-u^2 \left\langle K(\mathbf{a}^1; \mathbf{b}^1), K(\mathbf{a}^2; \mathbf{b}^2) \right\rangle_{Muk} \tag{4.23}$$

The coefficient of u^2 is readily seen to be zero since

$$\begin{aligned} \left\langle K(\mathbf{a}^1; \mathbf{b}^1), K(\mathbf{a}^2; \mathbf{b}^2) \right\rangle_{Muk} &= \sum \left\langle 1 \otimes 1 | \operatorname{Sh}(\underline{\mathbf{a}}^1; \underline{\mathbf{b}}^1), 1 \otimes 1 | \operatorname{Sh}(\underline{\mathbf{a}}^2; \underline{\mathbf{b}}^2) \right\rangle_{Muk} \\ = \sum \pm\, tr[c_A \otimes c_B &\mapsto \mathfrak{m}_*^{\otimes}(\operatorname{Sh}(\underline{\mathbf{a}}^1; \underline{\mathbf{b}}^1)^{(3)}, 1 \otimes 1, \operatorname{Sh}(\underline{\mathbf{a}}^1; \underline{\mathbf{b}}^1)^{(1)}, \\ \mathfrak{m}_*^{\otimes}(\operatorname{Sh}(\underline{\mathbf{a}}^1; \underline{\mathbf{b}}^1)^{(2)}, &c_A \otimes c_B, \operatorname{Sh}(\underline{\mathbf{a}}^2; \underline{\mathbf{b}}^2)^{(3)}, 1 \otimes 1, \operatorname{Sh}(\underline{\mathbf{a}}^2; \underline{\mathbf{b}}^2)^{(1)}), \operatorname{Sh}(\underline{\mathbf{a}}^2; \underline{\mathbf{b}}^2)^{(2)})] \end{aligned} \tag{4.24}$$

By the commuting subalgebras relation of Definition 18, the inner $\mathfrak{m}_*^{\otimes}$ is only non-zero when $* = 2$, forcing all the inner Sh terms to be empty. The same situation applies to the outer $\mathfrak{m}_*^{\otimes}$ except that all the Sh terms cannot be simultaneously empty. Thus, the outer $\mathfrak{m}_*^{\otimes}$ evaluates to zero, and we are taking the (sum of) traces of the zero map, hence zero.

For the coefficient of u, the terms vanish independently. We show this for the first term, a very similar computation shows this for the second term.

$$\begin{aligned} \left\langle k(\mathbf{a}^1; \mathbf{b}^1), K(\mathbf{a}^2; \mathbf{b}^2) \right\rangle_{Muk} &= \sum \pm\, tr[c_A \otimes c_B \mapsto \mathfrak{m}_*^{\otimes}(\operatorname{sh}(\underline{a}^1; \underline{b}^1)^{(3)}, a_0^1 \otimes b_0^1, \operatorname{sh}(\underline{a}^1; \underline{b}^1)^{(1)}, \\ \mathfrak{m}_*^{\otimes}(\operatorname{sh}(\underline{a}^1; \underline{b}^1)^{(2)}, c_A \otimes c_B, &\operatorname{Sh}(\underline{a}^2; \underline{b}^2)^{(3)}, 1 \otimes 1, \operatorname{Sh}(\underline{a}^2; \underline{b}^2)^{(1)}), \operatorname{Sh}(\underline{a}^2; \underline{b}^2)^{(2)})]. \end{aligned} \tag{4.25}$$

Now the inner $\mathfrak{m}_*^{\otimes}$ is non-zero except when $* = 2$ by Definition 18. So we obtain $\mathfrak{m}_2^{\otimes}(c_A \otimes c_B, 1 \otimes 1) = \pm\, c_A \otimes c_B$ forcing all the inner shuffle and cyclic shuffle terms to be empty.

Equation (4.25) now becomes

$$= \sum \pm\, tr[c_A \otimes c_B \mapsto \mathfrak{m}_*^{\otimes}(\mathrm{sh}(\underline{a}^1;\underline{b}^1)^{(2)}, a_0^1 \otimes b_0^1, \mathrm{sh}(\underline{a}^1;\underline{b}^1)^{(1)}, c_A \otimes c_B, \mathrm{Sh}(\underline{a}^2;\underline{b}^2)]. \tag{4.26}$$

From Theorem 20, the $\mathrm{Sh}(\underline{a}^2;\underline{b}^2)$ term must be ordered such that all $\underline{a}'s$ appear before any $\underline{b}'s$, otherwise we get zero. But by the definition of cyclic shuffles we must have $\underline{b}_0^2$ to the left of $\underline{a}_0^2$. Hence, zero by Theorem 20.

So we have

$$\big\langle (k+uK)(\mathbf{a}^1;\mathbf{b}^1), (k+uK)(\mathbf{a}^2;\mathbf{b}^2)\big\rangle_{hres} = \langle \mathbf{a}^1, \mathbf{a}^2\rangle_{Muk} \cdot \langle \mathbf{b}^1, \mathbf{b}^2\rangle_{Muk} = \langle \mathbf{a}^1, \mathbf{a}^2\rangle_{hres} \cdot \langle \mathbf{b}^1, \mathbf{b}^2\rangle_{hres}$$

By sesqui-linearity, the general case follows.

Next, we need to show that $k + uK : C_*(A)[[u]] \otimes C_*(B)[[u]] \to C_*(A \otimes B)[[u]]$ preserves the u-connection defined as

$$\nabla_{\frac{d}{du}} = \frac{d}{du} + \frac{\mathfrak{b}\{\mathfrak{m}'\}}{2u^2} + \frac{\mathfrak{B}\{\mathfrak{m}'\} - \Gamma}{2u} \tag{4.27}$$

on homology. We compute

$$\begin{aligned}
\nabla_{\frac{d}{du}} \circ (k+uK)(u^p \mathbf{a} \otimes \mathbf{b}) &= \left(\frac{d}{du} + \frac{\mathfrak{b}\{\mathfrak{m}'\}}{2u^2} + \frac{\mathfrak{B}\{\mathfrak{m}'\} - \Gamma}{2u}\right) \circ \left(u^p k(\mathbf{a};\mathbf{b}) + u^{p+1} K(\mathbf{a};\mathbf{b})\right) \\
&= u^p \left((p+1)K(\mathbf{a};\mathbf{b}) + \frac{1}{2}\mathfrak{B}\{\mathfrak{m}'\} \circ K(\mathbf{a};\mathbf{b}) - \frac{1}{2}\Gamma \circ K(\mathbf{a};\mathbf{b})\right) \\
&+ u^{p-1} \left(pk(\mathbf{a};\mathbf{b}) + \frac{1}{2}\mathfrak{b}\{\mathfrak{m}'\} \circ K(\mathbf{a};\mathbf{b}) + \frac{1}{2}\mathfrak{B}\{\mathfrak{m}'\} \circ k(\mathbf{a};\mathbf{b}) - \frac{1}{2}\Gamma \circ k(\mathbf{a};\mathbf{b})\right) \\
&+ u^{p-2} \left(\frac{1}{2}\mathfrak{b}\{\mathfrak{m}'\} \circ k(\mathbf{a};\mathbf{b})\right)
\end{aligned}$$

On the other hand, we have

$$
\begin{aligned}
(k+uK)\circ\left(\nabla_{\frac{d}{du}}\otimes id+id\otimes\nabla_{\frac{d}{du}}\right)(u^p\mathbf{a}\otimes\mathbf{b}) &= (k+uK)(pu^{p-1}\mathbf{a}\otimes\mathbf{b})\\
&+(k+uK)\left(\frac{u^{p-2}}{2}(\mathfrak{b}\{\mathfrak{m}'\}(\mathbf{a})\otimes\mathbf{b}+\mathbf{a}\otimes\mathfrak{b}\{\mathfrak{m}'\}(\mathbf{b}))\right)\\
&+(k+uK)\left(\frac{u^{p-1}}{2}\left(\mathfrak{B}\{\mathfrak{m}'\}(\mathbf{a})\otimes\mathbf{b}+\mathbf{a}\otimes\mathfrak{B}\{\mathfrak{m}'\}(\mathbf{b})-\Gamma(\mathbf{a})\otimes\mathbf{b}-\mathbf{a}\otimes\Gamma(\mathbf{b})\right)\right)\\
&=u^p\left(pK(\mathbf{a};\mathbf{b})+\frac{1}{2}K(\mathfrak{B}\{\mathfrak{m}'\}(\mathbf{a});\mathbf{b})+\frac{1}{2}K(\mathbf{a};\mathfrak{B}\{\mathfrak{m}'\}(\mathbf{b}))-\frac{1}{2}K(\Gamma(\mathbf{a});\mathbf{b})-\frac{1}{2}K(\mathbf{a};\Gamma(\mathbf{b}))\right)\\
&+u^{p-1}\Bigg(pk(\mathbf{a};\mathbf{b})+\frac{1}{2}K(\mathfrak{b}\{\mathfrak{m}'\}(\mathbf{a});\mathbf{b})+\frac{1}{2}K(\mathbf{a};\mathfrak{b}\{\mathfrak{m}'\}(\mathbf{b}))+\frac{1}{2}k(\mathfrak{B}\{\mathfrak{m}'\}(\mathbf{a});\mathbf{b})\\
&\qquad+\frac{1}{2}k(\mathbf{a};\mathfrak{B}\{\mathfrak{m}'\}(\mathbf{b}))-\frac{1}{2}k(\Gamma(\mathbf{a});\mathbf{b})-\frac{1}{2}k(\mathbf{a};\Gamma(\mathbf{b}))\Bigg)\\
&+u^{p-2}\left(\frac{1}{2}k(\mathfrak{b}\{\mathfrak{m}'\}(\mathbf{a});\mathbf{b})+\frac{1}{2}k(\mathbf{a};\mathfrak{b}\{\mathfrak{m}'\}(\mathbf{b}))\right)
\end{aligned}
$$

Comparing coefficients, the commutator $[\nabla_{\frac{d}{du}},k+uK]$ equals

$$u^p\left(K-\frac{1}{2}[\Gamma,K]+\frac{1}{2}[\mathfrak{B}\{\mathfrak{m}'\},K]\right) \tag{4.28}$$

$$+u^{p-1}\left(-\frac{1}{2}[\Gamma,k]+\frac{1}{2}[\mathfrak{B}\{\mathfrak{m}'\},k]+\frac{1}{2}[\mathfrak{b}\{\mathfrak{m}'\},K]\right) \tag{4.29}$$

$$+u^{p-2}\left(\frac{1}{2}[\mathfrak{b}\{\mathfrak{m}'\},k]\right) \tag{4.30}$$

where $\mathfrak{b}\{\mathfrak{m}'\}(-)=\mathfrak{b}^{1,1}(\mathfrak{m}',-)$ and $\mathfrak{B}\{\mathfrak{m}'\}(-)=\mathfrak{B}^{1,1}(\mathfrak{m}',-)$.

Clearly, $2K=[K,\Gamma]$ since by definition of the operators, we have

$$K(\Gamma\mathbf{a};\mathbf{b})=-n\sum 1\otimes 1|\operatorname{Sh}(\underline{\mathbf{a}};\underline{\mathbf{b}}) \tag{4.31}$$

$$K(\mathbf{a};\Gamma\mathbf{b})=-m\sum 1\otimes 1|\operatorname{Sh}(\underline{\mathbf{a}};\underline{\mathbf{b}}) \tag{4.32}$$

$$\Gamma K(\mathbf{a};\mathbf{b})=-(n+m+1+1)\sum 1\otimes 1|\operatorname{Sh}(\underline{\mathbf{a}};\underline{\mathbf{b}}) \tag{4.33}$$

The coefficient of u^p is therefore zero by the above calculation. Also $[\mathfrak{B}\{\mathfrak{m}'\},K]=0$ since every term of the composition $\mathfrak{B}\{\mathfrak{m}'\}\circ K$ or $K\circ\mathfrak{B}\{\mathfrak{m}'\}$ is zero in the reduced complex. We show that the coefficient of u^{p-2} is 0.

The first term in the commutator is

$$\begin{aligned}
k\circ(\mathfrak{b}\{\mathfrak{m}'\}\otimes id + id\otimes \mathfrak{b}\{\mathfrak{m}'\})(\mathbf{a}\otimes\mathbf{b}) &= k\Big((\mathfrak{b}\{\mathfrak{m}'\}(\mathbf{a});\mathbf{b}) + (-1)^{|\mathbf{a}|}(\mathbf{a};\mathfrak{b}\{\mathfrak{m}'\}(\mathbf{b}))\Big)\\
&= k\Big(\sum(-1)^{s_A}\mathfrak{m}_*^A(a^{(3)},\mathfrak{m}'_A(a^{(4)}),a^{(5)},a_0,a^{(1)})|a^{(2)};b_0|b^{(1)}\Big)\\
&\qquad + (-1)^{|\mathbf{a}|}k\Big(\sum(-1)^{s_B}a_0|a^{(1)};\mathfrak{m}_*^B(b^{(3)},\mathfrak{m}'_B(b^{(4)}),b^{(5)},b_0,b^{(1)})|b^{(2)}\Big)\\
&= \sum(-1)^{s_A+|b_0||\mathbf{a}|'}\mathfrak{m}_*^A(a^{(3)},\mathfrak{m}'_A(a^{(4)}),a^{(5)},a_0,a^{(1)})\otimes b_0|\,\mathrm{sh}(\underline{a}^{(2)};\underline{b})\\
&\qquad + (-1)^{|\mathbf{a}|}\sum(-1)^{s_B+t|\mathbf{a}|'}a_0\otimes\mathfrak{m}_*^B(b^{(3)},\mathfrak{m}'_B(b^{(4)}),b^{(5)},b_0,b^{(1)})|sh(\underline{a};\underline{b}^{(2)})
\end{aligned}\tag{4.34}$$

where s_A, s_B are the corresponding signs of $\mathfrak{b}\{\mathfrak{m}'\}$ defined in equation (2.3) applied to $\mathbf{a},\mathbf{b}$ respectively; and $t = |\mathfrak{m}_*^B(b^{(3)},\mathfrak{m}'_B(b^{(4)}),b^{(5)},b_0,b^{(1)})|'$.

In a similar fashion, the second term in the commutator gives

$$\begin{aligned}
&\mathfrak{b}\{\mathfrak{m}'\}\circ k(\mathbf{a};\mathbf{b}) = \mathfrak{b}\{\mathfrak{m}'\}\Big(\sum(-1)^{|b_0||\mathbf{a}|'}a_0\otimes b_0|\,\mathrm{sh}(\underline{a};\underline{b})\Big)\\
&= \sum(-1)^{|b_0||\mathbf{a}|'+s_{A\otimes B}}\mathfrak{m}_*^{\otimes}\left(\mathrm{sh}(\underline{a};\underline{b})^{(3)},\mathfrak{m}'_{\otimes}(\mathrm{sh}(\underline{a};\underline{b})^{(4)}),\mathrm{sh}(\underline{a};\underline{b})^{(5)},a_0\otimes b_0,\mathrm{sh}(\underline{a};\underline{b})^{(1)}\right)|\,\mathrm{sh}(\underline{a};\underline{b})^{(2)}
\end{aligned}$$

Since $\mathfrak{m}'_2=0$, it follows from the commuting subalgebras relations of Definition 18 that the argument in $\mathfrak{m}'_{\otimes}$, namely, $\mathrm{sh}(\underline{a};\underline{b})^{(4)}$ must be strictly $\underline{a}'s$ or strictly $\underline{b}'s$ to be non-trivial; and similarly in $\mathfrak{m}_*^{\otimes}$ (but with $a_0\otimes b_0$ fixed). The last sum above thus decomposes as

$$\begin{aligned}
&\sum(-1)^{|b_0||\mathbf{a}|'+s_{A\otimes B}}\mathfrak{m}_*^{\otimes}\left(\underline{a}^{(3)},\mathfrak{m}'_A(\underline{a}^{(4)}),\underline{a}^{(5)},a_0\otimes b_0,\underline{a}^{(1)}\right)|\,\mathrm{sh}(\underline{a};\underline{b})^{(2)}\\
&\quad+\sum(-1)^{|b_0||\mathbf{a}|'+s_{A\otimes B}}\mathfrak{m}_*^{\otimes}\left(\underline{b}^{(3)},\mathfrak{m}'_B(\underline{b}^{(4)}),\underline{b}^{(5)},a_0\otimes b_0,\underline{b}^{(1)}\right)|\,\mathrm{sh}(\underline{a};\underline{b})^{(2)}\\
&=\sum(-1)^{|b_0||\mathbf{a}|'+s_{A\otimes B}+|b_0||a^{(1)}|'}\mathfrak{m}_*^{A}\left(a^{(3)},\mathfrak{m}'_A(a^{(4)}),a^{(5)},a_0,a^{(1)}\right)\otimes b_0|\,\mathrm{sh}(\underline{a}^{(2)};\underline{b})\\
&\qquad+\sum(-1)^{|b_0||\mathbf{a}|'+s_{A\otimes B}+|a_0||b_{<0}|'}a_0\otimes\mathfrak{m}_*^{B}\left(b^{(3)},\mathfrak{m}'_B(b^{(4)}),b^{(5)},b_0,b^{(1)}\right)|\,\mathrm{sh}(\underline{a};\underline{b}^{(2)})
\end{aligned}\tag{4.35}$$

The last equation follows from the commuting subalgebra relations, where $|b_{<0}|$ denotes the sum of degrees of $\underline{b}'s$ to the left of b_0. The (4.34) and (4.35) terms cancel each other, and we have $[k, \mathfrak{b}\{\mathfrak{m}'\}] = 0$.

Lastly, for the coefficient of u^{p-1}, clearly $[k, \Gamma] = 0$. We will show that $[k, \mathfrak{B}\{\mathfrak{m}'\}] + [K, \mathfrak{b}\{\mathfrak{m}'\}]$ is homotopic to 0 via the homotopy $N = H + H' : C_*(A) \otimes C_*(B) \to C_*(A \otimes B)$, where H is defined as (Cf.[8])

$$H(\mathbf{a}, \mathbf{b}) = \sum_{\underline{b}_0 < \underline{a}_0} (-1)^{@_A + @_B + |a_{<\mathfrak{m}'}|'} \, 1_A \otimes 1_B \mid \mathrm{sh}_0 \left(\underline{a}^{(2)}, \mathfrak{m}'_A(\underline{a}^{(3)}), \underline{a}^{(4)}, \underline{a}_0, \underline{a}^{(1)}; \underline{b}^{(2)}, \underline{b}_0, \underline{b}^{(1)} \right)$$
$$+ \sum_{\underline{b}_0 < \underline{a}_0} (-1)^{@_A + @_B + |\underline{a}|' + |b_{<\mathfrak{m}'}|'} \, 1 \otimes 1 \mid \mathrm{sh}_0 \left(\underline{a}^{(2)}, \underline{a}_0, \underline{a}^{(1)}; \underline{b}^{(2)}, \mathfrak{m}'_B(\underline{b}^{(3)}), \underline{b}^{(4)}, \underline{b}_0, \underline{b}^{(1)} \right) \tag{4.36}$$

The condition $\underline{b}_0 < \underline{a}_0$ means $\underline{b}_0$ stays to the left of $\underline{a}_0$, and sh_0 denotes a sub-collection of the shuffles satisfying this condition. We define H' as

$$H' := u^{-1} \, (k + uK) \circ (\iota\{\mathfrak{m}'_A\} \otimes \mathfrak{B})$$

So we need to show that $[\mathfrak{B}\{\mathfrak{m}'\}, k] + [\mathfrak{b}\{\mathfrak{m}'\}, K] = [\mathfrak{b} + u\mathfrak{B}, H + H']$. The RHS equals

$$[\mathfrak{b} + u\mathfrak{B}, H] + [\mathfrak{b} + u\mathfrak{B}, H'] = [\mathfrak{b}, H] + [\mathfrak{b} + u\mathfrak{B}, H']$$

since $[\mathfrak{B}, H] = 0$ in the reduced complex. We will show that $[\mathfrak{b} + u\mathfrak{B}, H'] = -k(\mathcal{L}_{\mathfrak{m}'} \otimes \mathfrak{B})$

$$(\mathfrak{b} + u\mathfrak{B}) \circ H' = (\mathfrak{b} + u\mathfrak{B}) u^{-1} \, (k + uK) \circ (\iota\{\mathfrak{m}'_A\} \otimes \mathfrak{B}) \tag{4.37}$$
$$= u^{-1}(k + uK) \left((\mathfrak{b} + u\mathfrak{B})(\iota\{\mathfrak{m}'_A\}) \otimes \mathfrak{B} + \iota\{\mathfrak{m}'_A\} \otimes (\mathfrak{b} + u\mathfrak{B})\mathfrak{B} \right) \tag{4.38}$$
$$= u^{-1}(k + uK) \circ (\mathfrak{b} + u\mathfrak{B})(\iota\{\mathfrak{m}'_A\} \otimes id) \circ (id \otimes \mathfrak{B}) \tag{4.39}$$
$$+ u^{-1}(k + uK)(\iota\{\mathfrak{m}'_A\} \otimes id)(id \otimes \mathfrak{b}\mathfrak{B}) \tag{4.40}$$

$$H' \circ ((\mathfrak{b} + u\mathfrak{B}) \otimes id + id \otimes (\mathfrak{b} + u\mathfrak{B}))$$

$$= -u^{-1}(k + uK)(\iota\{\mathfrak{m}'_A\}(\mathfrak{b} + u\mathfrak{B}) \otimes id)(id \otimes \mathfrak{B}) \tag{4.41}$$

$$+ u^{-1}(k + uK)(\iota\{\mathfrak{m}'_A\} \otimes id)(id \otimes \mathfrak{B}\mathfrak{b}) \tag{4.42}$$

Now, terms in (4.40) and (4.42) cancel out since $\mathfrak{B}\mathfrak{b} + \mathfrak{b}\mathfrak{B} = 0$, and we are left with

$$[\mathfrak{b} + u\mathfrak{B}, H'] = u^{-1}(k + uK) \circ ([\mathfrak{b} + u\mathfrak{B},\ \iota\{\mathfrak{m}'_A\}] \otimes id)(id \otimes \mathfrak{B}) \tag{4.43}$$

$$= -u^{-1}(k + uK)(u\mathcal{L}_{\mathfrak{m}'} \otimes id)(id \otimes \mathfrak{B}) \tag{4.44}$$

$$= -(k + uK)(\mathcal{L}_{\mathfrak{m}'} \otimes \mathfrak{B}) \tag{4.45}$$

$$= -k(\mathcal{L}_{\mathfrak{m}'} \otimes \mathfrak{B}) \tag{4.46}$$

where equation (4.44) follows from the Cartan homotopy formula (2.6), and since $[\mathfrak{m}, \mathfrak{m}'] = 0$

Putting it all together, we need to prove the lemma below.

Lemma 31. $[\mathfrak{B}\{\mathfrak{m}'\}, k] + [\mathfrak{b}\{\mathfrak{m}'\}, K] = [\mathfrak{b}, H] - k(\mathcal{L}_{\mathfrak{m}'} \otimes \mathfrak{B})$

See Appendix B.1 for a proof.

The result follows. □

Chapter 5

Applications to splittings of the hodge filtration

5.1 Weak Calabi-Yau structure and duality isomorphism

We will generalize Amorim-Tu[14]'s duality isomorphism $D : HH^*(A) \to HH_*(A)$ for cyclic A_∞ structures to weak Calabi-Yau structures. We recall some well-known facts about A_∞-bimodules, following[24].

Definition 32. *An A_∞-bimodule over A (or $A - A$ bimodule) consists of a graded vector space P and a family of maps*

$$\mathfrak{n}_{r,s} : A[1]^{\otimes r} \otimes P \otimes A[1]^{\otimes s} \to P[1]$$

satisfying

$$\begin{aligned}
&\sum (-1)^\star \mathfrak{n}_{r,s}\left(\mathbf{x}^{(1)}, \mathfrak{m}_i\left(\mathbf{x}^{(2)}\right), \mathbf{x}^{(3)}, \underline{p}, \mathbf{y}\right) \\
&+ \sum (-1)^\star \mathfrak{n}_{r,s}\left(\mathbf{x}, \underline{p}, \mathbf{y}^{(1)}, \mathfrak{m}_i\left(\mathbf{y}^{(2)}\right), \mathbf{y}^{(3)}\right) \\
&+ \sum (-1)^\star \mathfrak{n}_{r,s}\left(\mathbf{x}^{(1)}, \mathfrak{n}_{r,s}\left(\mathbf{x}^{(2)}, \underline{p}, \mathbf{y}^{(1)}\right), \mathbf{y}^{(2)}\right) = 0
\end{aligned}$$

where $\star$ is the sum of the degrees to the left of the second multilinear map. For example, in the second line above, $\star = |x_1|' + \ldots + |x_r|' + |y_1|' + \ldots + |y_{j-1}|' + |p|$.

A_∞-bimodules form a dg-category $[A, A]$. Given A_∞-bimodules P and Q an element

$\rho \in \hom^d_{[A,A]}(P,Q)$ of degree d, called a pre-homomorphism consists of maps

$$\rho_{r,s} : A[1]^{\otimes r} \otimes P \otimes A[1]^{\otimes s} \to Q[d]$$

The differential in $hom^\bullet_{[A,A]}(P,Q)$ is defined as

$$\begin{aligned}(\partial\rho)_{r,s}(\mathbf{x},\underline{p},\mathbf{y}) = &\sum(-1)^{|\rho|\star}\mathfrak{n}_{r_1,s_1}\left(\mathbf{x}^{(1)},\underline{\rho_{r_2,s_2}\left(\mathbf{x}^{(2)},\underline{p},\mathbf{y}^{(1)}\right)},\mathbf{y}^{(2)}\right)\\ &+\sum(-1)^{1+|\rho|+\star}\rho_{r_1,s_1}\left(\mathbf{x}^{(1)},\underline{\mathfrak{n}_{r_2,s_2}\left(\mathbf{x}^{(2)},\underline{p},\mathbf{y}^{(1)}\right)},\mathbf{y}^{(2)}\right)\\ &+\sum(-1)^{1+|\rho|+\star}\rho_{r_1,s}\left(\mathbf{x}^{(1)}\mathfrak{m}_i\left(\mathbf{x}^{(2)}\right)\mathbf{x}^{(3)},\underline{p},\mathbf{y}\right)\\ &+\sum(-1)^{1+|\rho|+\star}\rho_{r,s_1}\left(\mathbf{x},\underline{p},\mathbf{y}^{(1)}\mathfrak{m}_i\left(\mathbf{y}^{(2)}\right),\mathbf{y}^{(3)}\right)\end{aligned}$$

Definition 33. *Given A_∞-bimodules P and Q, a pre-homomorphism $\rho \in hom^\bullet_{[A,A]}(P,Q)$ with $\partial\rho = 0$ is called a homomorphism. A homomorphism is called a quasi-isomorphism if $\rho_{0,0} : P \to Q$ induces an isomorphism on cohomology.*

We define two important bimodules for our our work.

Definition 34. *Let A_Δ be the bimodule defined as $A[1]$, with operations*

$$\mathfrak{n}_{r,s}(\mathbf{x},\underline{p},\mathbf{y}) := \mathfrak{m}_{r+s+1}(\mathbf{x},p,\mathbf{y})$$

We will refer to A_Δ as the (shifted) diagonal bimodule.

Definition 35. *Let $A^\vee_\Delta$ be the bimodule defined as $A^\vee[1]$, where $\vee$ stands for the* k*-linear dual. We define the operations*

$$\mathfrak{n}^\vee_{r,s}(\mathbf{x},\underline{\pi},\mathbf{y})(p) := (-1)^{|\mathbf{x}|'+|\mathbf{y}|'+|p|'}\pi\left(\mathfrak{m}_{r+s+1}(\mathbf{y},p,\mathbf{x})\right)$$

We will refer to $A^\vee_\Delta$ as the (shifted) dual diagonal bimodule.

In order to define Calabi-Yau structure we need the following lemma.

Lemma 36 (cf.[24] Lemma 2.11)**.** *There is a quasi-isomorphism of chain complexes*

$$\Psi : C_*(A)^\vee \to \hom^*_{[A,A]}\left(A_\Delta, A^\vee_\Delta\right)$$

given explicitly as

$$\Psi(\varphi)(\boldsymbol{x}, \underline{v}, \boldsymbol{y})(w) = \sum (-1)^{@} \varphi\left(\mathfrak{m}\left(\boldsymbol{y}^{(3)}, w, \boldsymbol{x}, v, \boldsymbol{y}^{(1)}\right), \boldsymbol{y}^{(2)}\right)$$

We are now able to define a weak Calabi-Yau structure on A.

Definition 37. *An element $\phi \in HH_*(A)^\vee$ is called non-degenerate if $\Psi(\phi)$ is a quasi-isomorphism.*

A weak Calabi-Yau structure on A is a non-degenerate element $\phi \in HH_*(A)^\vee$.

Notice that the non-degeneracy of the Mukai pairing implies that the map $\langle -, - \rangle_{Muk} : HH_*(A) \to HH_*(A)^\vee$ defined by $\alpha \mapsto \langle \alpha, - \rangle_{Muk}$ is an isomorphism. Hence, we can write $\phi = \langle \omega, - \rangle_{Muk}$ for some unique $\omega \in HH_*(A)$.

Let $\phi \in HH_*(A)^\vee$ be a weak Calabi-Yau structure on A. That is, the map $\Psi(\phi) : A_\Delta \to A_\Delta^\vee$ is an isomorphism (in the minimal model of A). This means for every $u \in A_\Delta$, there exists $v \in A_\Delta$ such that

$$\Psi(\phi)_{0,0}(u)(v) = (-1)^{|v|'|u|'} \phi(\mathfrak{m}_2(v, u)) = (-1)^{|v|'|u|'} \langle \omega, \mathfrak{m}_2(v, u) \rangle_{Muk} \neq 0$$

Let $\rho \in hom_{[A,A]}(A_\Delta, A_\Delta^\vee[d])$ denote a chain-level representative of $\Psi(\phi)$.

As in[24] we define the contraction map $\iota_- \rho : C^*(A) \to C_{*+d}(A)^\vee$ as follows

$$\iota_Z \rho\left(a_0 \otimes a_1 \cdots \otimes a_n\right) = \sum (-1)^{@+\dagger} \rho\left(a_1, \ldots \underline{Z\left(a_i, \ldots, a_j\right)} \ldots a_n\right)\left(a_0\right).$$

$\dagger := |Z|' \left(|\rho| + |a_1|' + \ldots + |a_{i-1}|'\right)$ and $Z \in C^*(A)$ is a Hochschild cochain.

Further, we have the Mukai pairing defined in (2.1). Since A is smooth and proper, the Mukai pairing is non-degenerate, hence it induces an isomorphism $HH_*(A) \cong HH_*(A)^\vee$. Putting together, we obtain a chain of isomorphisms:

$$HH^*(A) \xrightarrow{\iota_- \rho} HH_*(A)^\vee \xleftarrow{\langle -, - \rangle_{Muk}} HH_*(A) \tag{5.1}$$

Lemma 38 (cf.[14] Proposition A.1)**.** *Given a Hochschild cochain class $\theta \in HH^*(A)$ and* *chains $\alpha, \beta \in HH_*(A)$, we have the identity*

$$\langle \theta \cap \alpha, \beta \rangle_{Muk} = (-1)^{|\phi||\alpha|} \langle \alpha, \theta \cap \beta \rangle_{Muk}$$

We state our main proposition. This is a generalization of[14] which shows this for cyclic structures as opposed to Calabi-Yau structures.

Proposition 39. *The isomorphism $D : HH^*(A) \longrightarrow HH_*(A)$ in equation (5.1) is an isomorphism of modules. That is,*

$$D(\phi \cup \psi) = \phi \cap D(\psi)$$

for Hochschild cochains $\phi, \psi \in HH^*(A)$

Proof. Since $\langle -, - \rangle_{Muk}$ is non-degenerate, it suffices to show that

$$\langle D(\phi \cup \psi), \mathbf{a} \rangle_{Muk} = \langle \phi \cap D(\psi), \mathbf{a} \rangle_{Muk}, \qquad \text{for any } \mathbf{a} \in HH_*(A)$$

The LHS by definition of D and $\iota_{-}\rho$ is equal to $(\iota_{\phi \cup \psi}\rho)(\mathbf{a})$ or $(-1)^{|\phi||\psi|}(\iota_{\psi \cup \phi}\rho)(\mathbf{a})$ by the commutativity of $\cup$. On the other hand, the right hand side is equal to $(-1)^{|\phi||D(\psi)|}\langle D(\psi), \phi \cap \mathbf{a} \rangle_{Muk}$ by Lemma 38, which is further equal to $(-1)^{|\phi||D(\psi)|}(\iota_{\psi}\rho)(\phi \cap \mathbf{a})$ by definition of D. So we need to show

$$(-1)^{|\phi||\psi|}(\iota_{\psi \cup \phi}\rho)(\mathbf{a}) = (-1)^{|\phi||D(\psi)|}(\iota_{\psi}\rho)(\phi \cap \mathbf{a}) \tag{5.2}$$

Write $\mathbf{a} = a_0|a_1, ..., a_n$, then the LHS of equation (5.2) becomes

$$(\iota_{\psi \cup \phi}\rho)(\mathbf{a}) = \sum (-1)^{@+\dagger} \rho \left(a^{(1)}, \underline{\psi \cup \phi(a^{(2)}, a^{(3)}, a^{(4)}, a^{(5)}, a^{(6)})}, a^{(7)} \right) (a_0) \tag{5.3}$$

$$= \sum (-1)^{@+\dagger+|\cup|} \rho \left(a^{(1)}, \underline{\mathfrak{m}(a^{(2)}, \psi(a^{(3)}), a^{(4)}, \phi(a^{(5)}), a^{(6)})}, a^{(7)} \right) (a_0) \tag{5.4}$$

Similarly, we compute the RHS as

$$(\iota_\psi \rho)(\phi \cap \mathbf{a}) = \sum (-1)^{@+\dagger+|\cap|} \rho \left(a^{(2)}, \underline{\psi(a^{(3)})}, a^{(4)} \right) \left(\mathfrak{m}(a^{(5)}, \phi(a^{(6)}), a^{(7)}, a_0, a^{(1)}) \right) \tag{5.5}$$

where $|\cup| = |\phi|(|a^{(2)}| + |a^{(3)}| + |a^{(4)}|) + |\psi||a^{(2)}|$, $|\cap| = |\phi|'(|a^{(2)}| + |a^{(3)}| + |a^{(4)}| + |a^{(2)}|)$ and $\dagger$ is the sign from the definition of $\iota_- \rho$, for example, in equation (5.5), $\dagger := |\psi|' \left(|\rho| + \left| a^{(2)} \right|' \right)$

Define

$$\iota_{\psi,\phi}(\theta)(\mathbf{a}) := \sum (-1)^{@+\dagger'} \ \theta \left(a^{(1)}, \underline{\psi(a^{(2)})}, a^{(3)}, \phi(a^{(4)}), a^{(5)} \right) (a_0) \tag{5.6}$$

where $\dagger' = |\phi|'(|\theta| + |a^{(1)}| + |a^{(2)}| + |a^{(3)}|) + |\psi|'(|\theta| + |a^{(1)}|)$. The computation below shows that

$$\iota_{\psi,\phi}(\partial \rho)(\mathbf{a}) = \iota_{\psi \cup \phi}(\rho)(\mathbf{a}) + (-1)^{|\rho||\phi|} \iota_\psi \rho(\phi \cap \mathbf{a}) + \iota_{\psi,\phi}(\rho)(\mathfrak{b}(\mathbf{a})) \tag{5.7}$$

$$\iota_{\psi,\phi}(\partial\rho)(\mathbf{a}) = \sum(-1)^{@+\dagger'}\ \partial\rho\left(a^{(1)},\underline{\psi(a^{(2)})},a^{(3)},\phi(a^{(4)}),a^{(5)}\right)(a_0) \tag{5.8}$$

$$= \sum(-1)^{@+\dagger'+1+|\rho|+|\mathfrak{m}_<|}\ \rho\left(a^{(2)},\underline{\psi(a^{(3)})},a^{(4)},\phi(a^{(5)}),a^{(6)}\right)\left(\mathfrak{m}(a^{(7)},a_0,a^{(1)})\right) \tag{5.9}$$

$$+\sum(-1)^{@+\dagger'+1+|\rho|+|\mathfrak{m}_<|}\ \rho\left(a^{(2)},\underline{\psi(a^{(3)})},a^{(4)}\right)\left(\mathfrak{m}(a^{(5)},\phi(a^{(6)}),a^{(7)},a_0,a^{(1)})\right) \tag{5.10}$$

$$+\sum(-1)^{@+\dagger'+1+|\rho|+|\mathfrak{m}_<|}\ \rho\left(a^{(1)},\underline{\mathfrak{m}(a^{(2)},\psi(a^{(3)}),a^{(4)})},a^{(5)},\phi(a^{(6)}),a^{(7)}\right)(a_0) \tag{5.11}$$

$$+\sum(-1)^{@+\dagger'+1+|\rho|+|\mathfrak{m}_<|}\ \rho\left(a^{(1)},\underline{\mathfrak{m}(a^{(2)},\psi(a^{(3)}),a^{(4)}),\phi(a^{(5)}),a^{(6)})},a^{(7)}\right)(a_0) \tag{5.12}$$

$$+\sum(-1)^{@+\dagger'+1+|\rho|+|\mathfrak{m}_<|}\ \rho\left(a^{(1)},\mathfrak{m}(a^{(2)}),a^{(3)},\underline{\psi(a^{(4)})},a^{(5)},\phi(a^{(6)}),a^{(7)}\right)(a_0) \tag{5.13}$$

$$+\sum(-1)^{@+\dagger'+1+|\rho|+|\mathfrak{m}_<|}\ \rho\left(a^{(1)},\underline{\psi(a^{(2)})},a^{(3)},\mathfrak{m}(a^{(4)}),a^{(5)},\phi(a^{(6)}),a^{(7)}\right)(a_0) \tag{5.14}$$

$$+\sum(-1)^{@+\dagger'+1+|\rho|+|\mathfrak{m}_<|}\ \rho\left(a^{(1)},\underline{\psi(a^{(2)})},a^{(3)},\mathfrak{m}(a^{(4)},\phi(a^{(5)}),a^{(6)}),a^{(7)}\right)(a_0) \tag{5.15}$$

$$+\sum(-1)^{@+\dagger'+1+|\rho|+|\mathfrak{m}_<|}\ \rho\left(a^{(1)},\underline{\psi(a^{(2)})},a^{(3)},\phi(a^{(4)}),a^{(5)},\mathfrak{m}(a^{(6)}),a^{(7)}\right)(a_0) \tag{5.16}$$

where $\dagger'$ is the appropriate Koszul sign as in equation (5.6), and $|\mathfrak{m}_<|$ is the sum of (shifted) degrees of terms to the left of $\mathfrak{m}$. Clearly, equations (5.10) and (5.12) correspond to $\iota_\psi\rho(\phi\cap\mathbf{a})$ and $\iota_{\psi\cup\phi}(\rho)(\mathbf{a})$ respectively. Since ψ and ϕ are closed cochains, the terms in 5.11 and 5.15 can be rewritten as $(-1)^{|\psi|+|a^{(2)}|'(|\psi|+1)}\rho\left(a^{(1)},\underline{\psi(a^{(2)},\mathfrak{m}(a^{(3)}),a^{(4)})},a^{(5)},\phi(a^{(6)}),a^{(7)}\right)(a_0)$ and $(-1)^{|\phi|+|a^{(4)}|'(|\phi|+1)}\rho\left(a^{(1)},\underline{\psi(a^{(2)})},a^{(3)},\phi(a^{(4)},\mathfrak{m}(a^{(5)}),a^{(6)}),a^{(7)}\right)(a_0)$ respectively. Combined with the rest of the terms, these now correspond to $\iota_{\psi,\phi}(\rho)(\mathfrak{b}(\mathbf{a}))$. Since $\mathbf{a}$ and ρ are closed, we have $\iota_{\psi\cup\phi}(\rho)(\mathbf{a}) = -(-1)^{|\rho||\phi|}\iota_\psi\rho(\phi\cap\mathbf{a})$. $\square$

Lemma 40. $D(1)=\omega$

Proof.

$$\begin{aligned}
\iota_1\rho(a_0|a_1,...,a_n) &= \sum(-1)^{@_\rho}\rho^{\otimes}(a^{(1)},\underline{1},a^{(2)},a^{(3)},a^{(4)})(a_0)\\
&= \sum(-1)^{@_\rho+@_\phi}\phi(\mathfrak{m}_*(a^{(4)},a_0,a^{(1)},1,a^{(2)}),a^{(3)}), \quad \text{(since } \rho=\Psi(\phi))\\
&= (-1)^{|\rho|+|a_0|(|a_1|'+...+|a_n|')}\phi(a_0|a_1,...,a_n)\\
&= (-1)^{|\rho|+|a_0|(|a_1|'+...+|a_n|')}\langle\omega,a_0|a_1,...,a_n\rangle_{Muk}
\end{aligned}$$

where $@_\rho = |\rho|+|a^{(1)}|'+|a_0|'(|a^{(1)}|'+|a^{(2)}|'+|a^{(3)}|'+|a^{(4)}|')$ and $@\rho = (|a_0|'+|a^{(4)}|')(|a^{(1)}|'+|a^{(2)}|'+1)$. But by construction of D, we have $(\iota_1\rho)(a_0|a_1,...,a_n) = \langle D(1),a_0|a_1,...,a_n\rangle_{Muk}$. Hence, $\langle\omega,a_0|a_1,...,a_n\rangle_{Muk} = \langle D(1),a_0|a_1,...,a_n\rangle_{Muk}$ or $D(1)=\omega$. □

Corollary 41. *$\omega = D(1)$ generates $HH_*(A)$ as a $HH^*(A)$-module.*

Proof. Since $D(\psi\cup 1)=\psi\cap D(1)$, let $\theta\in HH_*(A)$. Then

$$\theta = D(D^{-1}(\theta)\cup 1) = D^{-1}(\theta)\cap D(1)$$

□

5.2 Weak Calabi-Yau structure on the tensor product

We will obtain a weak Calabi-Yau structure on $A\otimes B$ given weak Calabi-Yau structures on A and B respectively. We will abuse notation and write ω_A is a weak Calabi-Yau structure on A if $\phi_A := \langle\omega_A,-\rangle_{Muk}$ is a non-degenerate element of $HH_*(A)$.

Theorem 42. *$\phi_A\otimes\phi_B := \langle\omega^{\otimes},\ -\rangle_{Muk}$ where $\omega^{\otimes} := k(\omega_A\otimes\omega_B)$ is a weak Calabi-Yau structure on $A\otimes B$ if ω_A,ω_B are weak Calabi-Yau structures on A and B respectively.*

Proof. We need to show that $\phi_A\otimes\phi_B := \langle\omega^{\otimes},\ -\rangle_{Muk}$ is a non-degenerate element of $HH_*(A\otimes$

$B)^\vee$. By Lemma 36, we have

$$\Psi(\phi_A \otimes \phi_B)(u_A \otimes u_B)(v_A \otimes v_B) = (-1)^{@} \phi_A \otimes \phi_B(\mathfrak{m}_2^{\otimes}(v_A \otimes v_B, u_A \otimes u_B)) \tag{5.17}$$

$$= (-1)^{@} \langle k(\omega_A \otimes \omega_B), k(\mathfrak{m}_2^A(v_A, u_A) \otimes \mathfrak{m}_2^B(v_B, u_B)) \rangle_{Muk} \tag{5.18}$$

$$= (-1)^{@} \langle \omega_A, \mathfrak{m}_2^A(v_A, u_A) \rangle_{Muk} \cdot \langle \omega_B, \mathfrak{m}_2^B(v_B, u_B) \rangle_{Muk} \quad \text{(by Theorem 26)} \tag{5.19}$$

Since ω_A, ω_B are weak Calabi-Yau structures on A and B respectively, the result follows. □

Corollary 43. *$\omega^{\otimes}$ generates $HH_*(A \otimes B)$ as $HH^*(A \otimes B)$-module.*

Proof. By Theorem 42, $\omega^{\otimes}$ is a weak Calabi-Yau structure on $A \otimes B$. Hence, by Corollary 41, $\omega^{\otimes}$ is a generator of $HH_*(A \otimes B)$ as $HH^*(A \otimes B)$-module. □

5.3 Splittings

In particular, we want *good* splittings of the Hodge filtration that are compatible with ω_A, in the sense of Definition 44. The existence of ω-compatible good splittings is very non-trivial. Amorim-Tu[14] showed that they exist in the case where the Hochschild cohomology ring $HH^*(A)$ is semi-simple, and further classify them in terms of grading operators on Hochschild homology.

We start with the following definition, formulated in terms of the semi-infinite Hodge structure (see[6]) of A given by the triple $\left(HC_\bullet^-(A), \nabla_{\frac{d}{du}}, \langle -, - \rangle_{hres}\right)$.

Definition 44. *Let ω_A be a generator of $HH_*(A)$. A* k*-linear map $s = s_A : HH_*(A) \to HC_*^-(A)$ is called a splitting of the Hodge filtration of A if it satisfies*

S1 (Splitting condition) s splits the canonical map $\pi : HC_*^-(A) \to HH_*(A)$, defined as

$$\pi\left(\sum_{n \geq 0} \alpha_n u^n\right) = \alpha_0$$

S2 (Lagrangian condition) $\langle s(\alpha), s(\beta) \rangle_{hres} = \langle \alpha, \beta \rangle_{Muk}, \forall \alpha, \beta \in HH_*(A)$.

A splitting s is called a good splitting if it satisfies

S3. (Homogeneity) $L^s := \bigoplus_{l\in\mathbb{N}} u^{-l}\cdot \mathrm{Im}(s)$ *is stable under the u-connection* $\nabla_{u\frac{d}{du}}$.

This is equivalent to requiring $\nabla_{u\frac{d}{du}} s(\alpha) \in u^{-1}\,\mathrm{Im}(s) + \mathrm{Im}(s), \forall\alpha$.

A splitting s is called ω-compatible if

S4. (ω-Compatibility) $\nabla_{u\frac{d}{du}} s(\omega) \in r\cdot s(\omega) + u^{-1}\cdot \mathrm{Im}(s)$ *for some* $r \in \mathrm{k}$.

In the semi-simple case, Amorim-Tu[14] denotes by ξ the diagonal operator acting on $HH_*(\mathcal{C})$ by $\lambda_j \cdot id$ for A_∞ category $\mathcal{C}$. They further show that the splitting s satisfies

$$\nabla_{u\frac{d}{du}} s(\alpha) = s(\mu(\alpha)) + u^{-1}(\xi(\alpha)) \tag{5.20}$$

which implies homogeneity and ω-compatibility, where μ is a grading operator described as the regular part of $\nabla_{u\frac{d}{du}} s(\alpha)$ pulled back under the isomorphism $HH_*(A) \cong Im(s)$.

We apply the previous results to obtain a $\omega^{\otimes}$-compatible good splitting of the Hodge filtration on $A\otimes B$ in terms of the splittings on A and B.

Theorem 45. *Let s_A, s_B be splittings of the hodge filtration on A, and B respectively. Then $s_{A\otimes B} := (k+uK)\circ(s_A\otimes s_B)\circ k^{-1}$ is a splitting of the Hodge filtration on $A\otimes_\infty B$. Further, if s_A, s_B are ω_A, ω_B-compatible good splittings of the Hodge filtration on A, and B respectively, then $s_{A\otimes B}$ is a $\omega^{\otimes}$-compatible good splitting of the Hodge filtration of $A\otimes_\infty B$, where $\omega^{\otimes} = k(\omega_A\otimes\omega_B)$.*

$$\begin{tikzcd}
HC_*^-(A)\otimes_{\mathrm{k}[[u]]} HC_*^-(B) \arrow[r, "k+uK", "\simeq"'] \arrow[d, shift left, "\pi_A\otimes\pi_B"] & HC_*^-(A\otimes B) \arrow[d, shift right, "\pi^{\otimes}"'] \\
HC_*(A)\otimes HC_*(B) \arrow[r, "k", "\simeq"'] \arrow[u, bend left, "s_A\otimes s_B"] & HH_*(A\otimes B) \arrow[u, bend right, "s_{A\otimes B}"']
\end{tikzcd}$$

Proof. Clearly, $s_{A\otimes B}$ satisfies the splitting condition since we have

$$
\begin{aligned}
\pi^{\otimes} \circ s_{A\otimes B} &= (\pi^{\otimes} \circ (k+uK)) \circ (s_A \otimes s_B) \circ k^{-1} && (5.21)\\
&= k \circ (\pi_A \otimes \pi_B) \circ (s_A \otimes s_B) \circ k^{-1} && (5.22)\\
&= k \circ (\mathrm{id}_{HH_*(A)} \otimes \mathrm{id}_{HH_*(B)}) \circ k^{-1} && (5.23)\\
&= \mathrm{id}_{HH_*(A\otimes B)} && (5.24)
\end{aligned}
$$

For the Lagrangian condition, let $\alpha, \beta \in HH_*(A \otimes B)$. Denote $k^{-1}(\alpha) = \sum_{(\alpha)} \alpha^{(1)} \otimes \alpha^{(2)}$, and $k^{-1}(\beta) = \sum_{(\beta)} \beta^{(1)} \otimes \beta^{(2)}$.

Then

$$
\begin{aligned}
&\langle s_{A\otimes B}(\alpha),\ s_{A\otimes B}(\beta)\rangle_{hres} && (5.25)\\
&\quad = \left\langle (k+uK)\sum_{(\alpha)} s_A(\alpha^{(1)}) \otimes s_B(\alpha^{(2)}), (k+uK)\sum_{(\beta)} s_A(\beta^{(1)}) \otimes s_B(\beta^{(2)}) \right\rangle_{hres} && (5.26)\\
&\quad = \sum \langle s_A(\alpha^{(1)}),\ s_A(\beta^{(1)})\rangle_{hres} \cdot \sum \langle s_B(\alpha^{(2)}),\ s_B(\beta^{(2)})\rangle_{hres} && (5.27)\\
&\quad = \sum \langle \alpha^{(1)},\ \beta^{(1)}\rangle_{Muk} \cdot \sum \langle \alpha^{(2)},\ \beta^{(2)}\rangle_{Muk} && (5.28)\\
&\quad = \left\langle k\left(\sum \alpha^{(1)} \otimes \alpha^{(2)}\right),\ k\left(\sum \beta^{(1)} \otimes \beta^{(2)}\right)\right\rangle_{Muk} && (5.29)\\
&= \langle \alpha,\ \beta\rangle_{Muk} && (5.30)
\end{aligned}
$$

We will use the ξ, μ equivalent formulation in equation (5.20) to prove homogeneity and $\omega^{\otimes}$-compatibility. Take $\alpha, \beta \in HH_*(A \otimes B)$ as in above. We will show that $s_{A\otimes B}$ satisfies the homogeneity condition.

$$\nabla_{u\frac{d}{du}}(s_{A\otimes B}\circ k(\alpha\otimes\beta)) = \nabla_{u\frac{d}{du}}(k+uK)(s_A(\alpha)\otimes s_B(\beta)) \tag{5.31}$$

$$= (k+uK)(\nabla_{u\frac{d}{du}}\otimes id + id\otimes\nabla_{u\frac{d}{du}})(s_A(\alpha)\otimes s_B(\beta)) \tag{5.32}$$

$$= (k+uK)(\nabla_{u\frac{d}{du}}(s_A(\alpha))\otimes s_B(\beta) + s_A(\alpha)\otimes\nabla_{u\frac{d}{du}}(s_B(\beta))) \tag{5.33}$$

$$\begin{aligned} &= (k+uK)\left((u^{-1}s_A(\xi_A(\alpha)) + s_A(\mu_A(\alpha)))\otimes s_B(\beta)\right) \\ &\quad + (k+uK)\left(s_A(\alpha)\otimes(u^{-1}s_B(\xi_B(\beta)) + s_B(\mu_B(\beta)))\right) \end{aligned} \tag{5.34}$$

$$\begin{aligned} &= u^{-1}(k+uK)\,(s_A\otimes s_B)(\xi_A(\alpha)\otimes\beta + \alpha\otimes\xi_B(\beta)) \\ &\quad + (k+uK)\,(s_A\otimes s_B)(\mu_A(\alpha)\otimes\beta + \alpha\otimes\mu_B(\beta)) \end{aligned} \tag{5.35}$$

$$= s_{A\otimes B}\circ k\left(u^{-1}(\xi_A\otimes id + id\otimes\xi_B) + (\mu_A\otimes id + id\otimes\mu_B)\right)(\alpha\otimes\beta) \tag{5.36}$$

$$\nabla_{u\frac{d}{du}}(s_{A\otimes B}) = s_{A\otimes B}\circ k\left(u^{-1}(\xi_A\otimes id + id\otimes\xi_B) + (\mu_A\otimes id + id\otimes\mu_B)\right)\circ k^{-1} \tag{5.37}$$

$$= s_{A\otimes B}\left(u^{-1}\xi_\otimes + \mu_\otimes\right) = u^{-1}s_{A\otimes B}(\xi_\otimes) + s_{A\otimes B}(\mu_\otimes) \tag{5.38}$$

where $\xi_\otimes = k\circ(\xi_A\otimes id + id\otimes\xi_B)\circ k^{-1}$ and $\mu_\otimes = k\circ(\mu_A\otimes id + id\otimes\mu_B)\circ k^{-1}$

To show that $s_{A\otimes B}$ is $\omega^\otimes$-compatible, where $\omega^\otimes = k(\omega_A\otimes\omega_B)$,

$$\nabla_{u\frac{d}{du}}(s_{A\otimes B}\circ\omega^\otimes) = (k+uK)(\nabla_{u\frac{d}{du}}\otimes id + id\otimes\nabla_{u\frac{d}{du}})(s_A(\omega_A)\otimes s_B(\omega_B)) \tag{5.39}$$

$$\begin{aligned} &= (k+uK)\left((u^{-1}s_A(\xi_A(\omega_A)) + r_A s_A(\omega_A))\otimes s_B(\omega_B)\right) \\ &\quad + (k+uK)\left(s_A(\omega_A)\otimes(u^{-1}s_B(\xi_B(\omega_B)) + r_B s_B(\omega_B))\right) \end{aligned} \tag{5.40}$$

$$= s_{A\otimes B}\circ k\left(u^{-1}(\xi_A\otimes id + id\otimes\xi_B) + (r_A\otimes id + id\otimes r_B)\right)(\omega_A\otimes\omega_B) \tag{5.41}$$

$$\nabla_{u\frac{d}{du}}\left(s_{A\otimes B}(\omega^\otimes)\right) = s_{A\otimes B}\left(u^{-1}\xi_\otimes + r_\otimes\right) = u^{-1}s_{A\otimes B}(\xi_\otimes(\omega^\otimes)) + r_\otimes s_{A\otimes B}(\omega^\otimes) \tag{5.42}$$

where $\xi_\otimes = k\circ(\xi_A\otimes id + id\otimes\xi_B)\circ k^{-1}$ and $r_\otimes = r_A + r_B$ $\square$

Appendix A

Proof of Proposition 29

In the calculations that follow, we will let $N(\mathbf{a}) = \sum(-1)^{@}\ a^{(2)} \otimes a_0 \otimes a^{(1)}$ denote the sum of all cyclic permutations on $\mathbf{a}$, where @ denotes the corresponding sign. Then by definition of the cyclic shuffle, we have

$$\mathrm{Sh}(\underline{\mathbf{a}}, \underline{\mathbf{b}}) = \sum_{\underline{b}_0 < \underline{a}_0} \mathrm{sh}_0(N\underline{\mathbf{a}}, N\underline{\mathbf{b}})$$

where sh_0 denotes the shuffles satisfying $\underline{b}_0 < \underline{a}_0$.

A.1 Proposition 29

$\mathfrak{B}\ k(\mathbf{a};\mathbf{b}) + \mathfrak{b}K(\mathbf{a};\mathbf{b}) = k\left(\mathfrak{B}(\mathbf{a});\mathbf{b}\right) + K\left(\mathfrak{b}(\mathbf{a});\mathbf{b}\right) + (-1)^{|\mathbf{a}|}k\left(\mathbf{a};\mathfrak{B}(\mathbf{b})\right) + (-1)^{|\mathbf{a}|}K\left(\mathbf{a};\mathfrak{b}(\mathbf{b})\right)$

Proof. We compute term by term. The left hand side gives

$$\begin{aligned}\mathfrak{B} \circ k(\mathbf{a};\mathbf{b}) &= \sum(-1)^{|b_0||a|'+@_{A\otimes B}}\ 1\otimes 1 \mid \mathrm{sh}(\underline{a};\underline{b})^{(2)}, a_0 \otimes b_0, \mathrm{sh}(\underline{a};\underline{b})^{(1)} \\ &= \sum(-1)^{|b_0||a|'+|a^{(2)}|'|b^{(1)}|'+@_{A\otimes B}}\ 1\otimes 1 \mid \mathrm{sh}(\underline{a}^{(2)};\underline{b}^{(2)}), a_0\otimes b_0, \mathrm{sh}(\underline{a}^{(1)};\underline{b}^{(1)}) \quad \text{(A.1)}\end{aligned}$$

where $@_{A\otimes B} = (|a^{(2)}|' + |b^{(2)}|')(|a^{(1)}|' + |b^{(1)}|' + |a_0| + |b_0|)$

$$\mathfrak{b}\circ K(\mathbf{a};\mathbf{b}) = \sum_{\underline{b}_0<\underline{a}_0}(-1)^{<+\dagger}\ 1\otimes 1|\ \mathrm{Sh}(\underline{\mathbf{a}};\underline{\mathbf{b}})^{(1)},\mathfrak{m}_*^{\otimes}(\mathrm{Sh}(\underline{\mathbf{a}};\underline{\mathbf{b}})^{(2)}),\mathrm{Sh}(\underline{\mathbf{a}};\underline{\mathbf{b}})^{(3)}$$

$$+\sum_{\underline{b}_0<\underline{a}_0}(-1)^{\dagger+\kappa}\ \mathfrak{m}_*^{\otimes}\left(\mathrm{Sh}(\underline{\mathbf{a}};\underline{\mathbf{b}})^{(3)},1\otimes 1,\mathrm{Sh}(\underline{\mathbf{a}};\underline{\mathbf{b}})^{(1)}\right)\ |\ \mathrm{Sh}(\underline{\mathbf{a}};\underline{\mathbf{b}})^{(2)} \tag{A.2}$$

where $< = 1+|a^{(1)}|'+|b^{(1)}|',\ \dagger = |a^{(2)}|'|b^{(1)}|'+|a^{(3)}|'|b^{(1)}|'+|a^{(3)}|'|b^{(2)}|'$,

and $\kappa = (|a^{(3)}|'+|b^{(3)}|')(|a^{(2)}|'+|b^{(2)}|'+|a^{(1)}|'+|b^{(1)}|')$

$$=\sum(-1)^{<+\dagger+|a^{(2)}|'(|a^{(1)}|'+|a_0|)+|b^{(2)}|'(|b^{(1)}|'+|b_0|)}1\otimes 1|\,\mathrm{sh}(\underline{a}^{(2)};\underline{b}^{(2)}),\mathfrak{m}_2^{\otimes}(\underline{b}_0,\underline{a}_0),\mathrm{sh}(\underline{a}^{(1)};\underline{b}^{(1)}) \tag{A.3}$$

$$+\sum(-1)^{<+\dagger}\ 1\otimes 1|\ \mathrm{sh}(\underline{a}^{(3)};\underline{b}^{(4)},\underline{b}_0,\underline{b}^{(1)}),\mathfrak{m}_*^{\otimes}\left(\mathrm{sh}(\underline{a}^{(4)},\underline{a}_0,\underline{a}^{(1)};\underline{b}^{(2)})\right),\mathrm{sh}(\underline{a}^{(2)};\underline{b}^{(3)}) \tag{A.4}$$

$$+\sum(-1)^{<+\dagger}\ 1\otimes 1|\ \mathrm{sh}(\underline{a}^{(2)};\underline{b}^{(3)}),\mathfrak{m}_*^{\otimes}\left(\mathrm{sh}(\underline{a}^{(3)};\underline{b}^{(4)},\underline{b}_0,\underline{b}^{(1)})\right),\mathrm{sh}(\underline{a}^{(4)},\underline{a}_0,\underline{a}^{(1)};\underline{b}^{(2)}) \tag{A.5}$$

$$+\sum_{\underline{b}_0<\underline{a}_0}(-1)^{<+\dagger}\ 1\otimes 1|\ \mathrm{sh}_0(\underline{a}^{(4)},\underline{a}_0,\underline{a}^{(1)};\underline{b}^{(4)},\underline{b}_0,\underline{b}^{(1)}),\mathfrak{m}_*^{\otimes}\left(\mathrm{sh}(\underline{a}^{(2)};\underline{b}^{(2)})\right),\mathrm{sh}(\underline{a}^{(3)};\underline{b}^{(3)}) \tag{A.6}$$

$$+\sum_{\underline{b}_0<\underline{a}_0}(-1)^{<+\dagger}\ 1\otimes 1|\ \mathrm{sh}(\underline{a}^{(2)};\underline{b}^{(2)}),\mathfrak{m}_*^{\otimes}\left(\mathrm{sh}(\underline{a}^{(3)};\underline{b}^{(3)})\right),\mathrm{sh}_0(\underline{a}^{(4)},\underline{a}_0,\underline{a}^{(1)};\underline{b}^{(4)},\underline{b}_0,\underline{b}^{(1)}) \tag{A.7}$$

$$+\sum(-1)^{<+\dagger}\ 1\otimes 1|\ \mathrm{sh}(\underline{a}^{(2)};\underline{b}^{(4)},\underline{b}_0,\underline{b}^{(1)}),\mathfrak{m}_*^{\otimes}\left(\mathrm{sh}(\underline{a}^{(3)};\underline{b}^{(2)})\right),\mathrm{sh}(\underline{a}^{(4)},\underline{a}_0,\underline{a}^{(1)};\underline{b}^{(3)}) \tag{A.8}$$

$$+\sum(-1)^{\dagger+\kappa}\ \mathfrak{m}_2^{\otimes}(a_0\otimes 1,1\otimes 1)\ |\ \mathrm{sh}(\underline{a};N\underline{\mathbf{b}})\ \ \text{(by strict unitality on A.2)} \tag{A.9}$$

$$+\sum(-1)^{\dagger+\kappa}\ \mathfrak{m}_2^{\otimes}(1\otimes 1,1\otimes b_0)\ |\ \mathrm{sh}(N\underline{\mathbf{a}};\underline{b})\ \ \text{(by strict unitality on A.2)} \tag{A.10}$$

Equation (A.3) cancels out with equation (A.1), with $\dagger$ in (A.3) adjusted to correspond with the renumbering. For example, in this case $\dagger = |a_0||b^{(2)}|'+|a^{(1)}|'|b^{(2)}|'+|a^{(1)}|'|b_0|)$.

The strict unitality condition on equation (A.2) yields (A.9 and A.10), but also terms of the form $\pm\,\mathfrak{m}_2^{\otimes}(a_i\otimes 1,1\otimes 1)\,|\,\mathrm{sh}(N\underline{\mathbf{a}};N\underline{\mathbf{b}})$ and $\pm\,\mathfrak{m}_2^{\otimes}(1\otimes b_j,1\otimes 1)\,|\,\mathrm{sh}(N\underline{\mathbf{a}};N\underline{\mathbf{b}})$ $i,j>0$.

These however come in commuting pairs, namely, $\pm\ \mathfrak{m}_2^{\otimes}\,(1\otimes 1, a_i\otimes 1)\,|\,\mathrm{sh}(N\underline{\mathbf{a}}; N\underline{\mathbf{b}})$ and $\pm\ \mathfrak{m}_2^{\otimes}\,(1\otimes 1, 1\otimes b_j)\,|\,\mathrm{sh}(N\underline{\mathbf{a}}; N\underline{\mathbf{b}})$ respectively, and thus cancel each other.

Also, note that from the commuting subalgebras criterion (Definition 18), the entries in $\mathfrak{m}_*^{\otimes}$ in equation (A.5) have to be all $\underline{b}'s$ except for when $* = 2$ in which case $\mathfrak{m}_2^{\otimes}(\underline{b}_0, \underline{a}_i)$, $i > 0$, shows up in a commuting pair with $\mathfrak{m}_2^{\otimes}(\underline{a}_i, \underline{b}_0)$, and thus cancel each other. Similarly, the entries in $\mathfrak{m}_*^{\otimes}$ in equation (A.4) have to be all $\underline{a}$'s except for when $* = 2$ in which case $\mathfrak{m}_2^{\otimes}(\underline{a}_0, \underline{b}_i)$, $i > 0$, shows up in a commuting pair with $\mathfrak{m}_2^{\otimes}(\underline{b}_i, \underline{a}_0)$, and thus cancel each other.

Applying a similar argument in equations (A.6, A.7, A.8), we find that the entries in $\mathfrak{m}_*^{\otimes}$ have to be all $\underline{a}$'s or all $\underline{b}$'s except when $* = 2$ in which case $\mathfrak{m}_2^{\otimes}(\underline{a}_i, \underline{b}_j)$ shows up in commuting pairs with $\mathfrak{m}_2^{\otimes}(\underline{b}_j, \underline{a}_i)$, $i, j > 0$, and thus cancel each other. Hence, each of equations (A.6, A.7, A.8) splits into two sums: one with $\mathfrak{m}_*^{\otimes}(...)$ replaced by $\mathfrak{m}_*^{A}(...)$, and the other by $\mathfrak{m}_*^{B}(...)$ respectively according to Definition (18). The left hand side now becomes

$$\mathfrak{B}\circ k(\mathbf{a};\mathbf{b}) + \mathfrak{b}\circ K(\mathbf{a};\mathbf{b}) =$$

$$\sum(-1)^{<+\dagger}\ 1\otimes 1|\ \ \mathrm{sh}(\underline{a}^{(3)}; \underline{b}^{(3)}, \underline{b}_0, \underline{b}^{(1)}), \mathfrak{m}_*^{A}\left(\underline{a}^{(4)}, \underline{a}_0, \underline{a}^{(1)}\right), \mathrm{sh}(\underline{a}^{(2)}; \underline{b}^{(2)}) \tag{A.11}$$

$$+\sum(-1)^{<+\dagger}\ 1\otimes 1|\ \ \mathrm{sh}(\underline{a}^{(2)}; \underline{b}^{(3)}), \mathfrak{m}_*^{B}\left(\underline{b}^{(4)}, \underline{b}_0, \underline{b}^{(1)}\right), \mathrm{sh}(\underline{a}^{(3)}, \underline{a}_0, \underline{a}^{(1)}; \underline{b}^{(2)}) \tag{A.12}$$

$$+\sum_{\underline{b}_0<\underline{a}_0}(-1)^{<+\dagger}\ 1\otimes 1|\ \ \mathrm{sh}_0(\underline{a}^{(4)}, \underline{a}_0, \underline{a}^{(1)}; \underline{b}^{(3)}, \underline{b}_0, \underline{b}^{(1)}), \mathfrak{m}_*^{A}\left(\underline{a}^{(2)}\right), \mathrm{sh}(\underline{a}^{(3)}; \underline{b}^{(2)}) \tag{A.13}$$

$$+\sum_{\underline{b}_0<\underline{a}_0}(-1)^{<+\dagger}\ 1\otimes 1|\ \ \mathrm{sh}_0(\underline{a}^{(3)}, \underline{a}_0, \underline{a}^{(1)}; \underline{b}^{(4)}, \underline{b}_0, \underline{b}^{(1)}), \mathfrak{m}_*^{B}\left(\underline{b}^{(2)}\right), \mathrm{sh}(\underline{a}^{(2)}; \underline{b}^{(3)}) \tag{A.14}$$

$$+\sum_{\underline{b}_0<\underline{a}_0}(-1)^{<+\dagger}\ 1\otimes 1|\ \ \mathrm{sh}(\underline{a}^{(2)}; \underline{b}^{(2)}), \mathfrak{m}_*^{A}\left(\underline{a}^{(3)}\right), \mathrm{sh}_0(\underline{a}^{(4)}, \underline{a}_0, \underline{a}^{(1)}; \underline{b}^{(3)}, \underline{b}_0, \underline{b}^{(1)}) \tag{A.15}$$

$$+\sum_{\underline{b}_0<\underline{a}_0}(-1)^{<+\dagger}\ 1\otimes 1|\ \ \mathrm{sh}(\underline{a}^{(2)}; \underline{b}^{(2)}), \mathfrak{m}_*^{B}\left(\underline{b}^{(3)}\right), \mathrm{sh}_0(\underline{a}^{(3)}, \underline{a}_0, \underline{a}^{(1)}; \underline{b}^{(4)}, \underline{b}_0, \underline{b}^{(1)}) \tag{A.16}$$

$$+\sum(-1)^{<+\dagger}\ 1\otimes 1|\ \ \mathrm{sh}(\underline{a}^{(2)}; \underline{b}^{(3)}, \underline{b}_0, \underline{b}^{(1)}), \mathfrak{m}_*^{A}\left(\underline{a}^{(3)}\right), \mathrm{sh}(\underline{a}^{(4)}, \underline{a}_0, \underline{a}^{(1)}; \underline{b}^{(2)}) \tag{A.17}$$

$$+\sum(-1)^{<+\dagger}\ 1\otimes 1|\ \ \mathrm{sh}(\underline{a}^{(2)}; \underline{b}^{(4)}, \underline{b}_0, \underline{b}^{(1)}), \mathfrak{m}_*^{B}\left(\underline{b}^{(2)}\right), \mathrm{sh}(\underline{a}^{(3)}, \underline{a}_0, \underline{a}^{(1)}; \underline{b}^{(3)}) \tag{A.18}$$

$$+\sum(-1)^{\dagger+\kappa}\ a_0\otimes 1\ |\ \ \mathrm{sh}(\underline{a}; N\underline{\mathbf{b}}) \tag{A.19}$$

$$+\sum(-1)^{\dagger+\kappa+|b_0|}\ 1\otimes b_0\ |\ \ \mathrm{sh}(N\underline{\mathbf{a}}; \underline{b}) \tag{A.20}$$

We now compute the right hand side term by term and show how they cancel out with the left hand side computed above.

$$\begin{aligned}
k(\mathfrak{B}\otimes id)(\mathbf{a};\mathbf{b}) &= k\left((\sum(-1)^{@_A}\ 1 \mid a^{(2)}, a_0, a^{(1)}); b_0 \mid b\right)\\
&= \sum(-1)^{@_A+|b_0|(|a_0|+|a|')}\ 1\otimes b_0 \mid\ \mathrm{sh}(\underline{a}^{(2)}, \underline{a}_0, \underline{a}^{(1)}; \underline{b})\\
&= \sum(-1)^{@_A+|b_0|(|a_0|+|a|')} 1\otimes b_0 \mid\ \mathrm{sh}(N\underline{\mathbf{a}}; \underline{b}) \qquad \text{(A.21)}\\
(-1)^{|\mathbf{a}|}k(id\otimes\mathfrak{B})(\mathbf{a};\mathbf{b}) &= k\left(a_0 \mid a; (\sum(-1)^{@_B+|\mathbf{a}|}\ 1 \mid b^{(2)}, b_0, b^{(1)})\right)\\
&= \sum(-1)^{@_B+|\mathbf{a}|}\ a_0\otimes 1 \mid\ \mathrm{sh}(\underline{a}; \underline{b}^{(2)}, \underline{b}_0, \underline{b}^{(1)})\\
&= \sum(-1)^{@_B+|\mathbf{a}|}\ a_0\otimes 1 \mid\ \mathrm{sh}(\underline{a}; N\underline{\mathbf{b}}) \qquad \text{(A.22)}
\end{aligned}$$

Equations (A.21) and (A.22) cancel out with (A.20) and (A.19) respectively, where $@_A = |a^{(2)}|(|a_0| + |a^{(1)}|')$ and $@_B = |b^{(2)}|(|b_0| + |b^{(1)}|')$.

$$K(\mathfrak{b}\otimes id)(\mathbf{a};\mathbf{b}) = \sum_{\underline{b}_0<\underline{a}_0}(-1)^{|a_0|+|a^{(1)}|'}\ 1\otimes 1|\,\mathrm{sh}(\underline{a}^{(4)}, \underline{a}_0, \underline{a}^{(1)}, \mathfrak{m}_*^A(\underline{a}^{(2)}), \underline{a}^{(3)}; N\underline{\mathbf{b}}) \qquad \text{(A.23)}$$

$$+ \sum_{\underline{b}_0<\mathfrak{m}_*^A(\underline{a}^{(4)},\underline{a}_0,\underline{a}^{(1)})}(-1)^{|a^{(3)}|'(|a_0|+|a^{(1)}|'+|a^{(2)}|')}\ 1\otimes 1|\,\mathrm{sh}(\underline{a}^{(3)}, \mathfrak{m}_*^A(\underline{a}^{(4)}, \underline{a}_0, \underline{a}^{(1)}), \underline{a}^{(2)}; N\underline{\mathbf{b}}) \qquad \text{(A.24)}$$

We break up the shuffle in (A.23) according to whether $\mathfrak{m}_*^A(\underline{a}^{(2)})$ is to the right, left, or middle of $\underline{b}_0$ and $\underline{a}_0$. Similarly, we break up the shuffle in (A.24) rewriting it more explicitly with $\underline{b}_0$ to the left of $\underline{a}_0$ (and adjusting the signs appropriately) to obtain

$$K(\mathfrak{b}\otimes id)(\mathbf{a};\mathbf{b}) = \sum_{\underline{b}_0<\underline{a}_0} (-1)^{|a_0|+|a^{(1)}|'} 1\otimes 1|\ \mathrm{sh}_0(\underline{a}^{(4)},\underline{a}_0,\underline{a}^{(1)};\underline{b}^{(3)},\underline{b}_0,\underline{b}^{(1)}),\mathfrak{m}_*^A\left(\underline{a}^{(2)}\right),\mathrm{sh}(\underline{a}^{(3)};\underline{b}^{(2)}) \tag{A.25}$$

$$+\sum_{\underline{b}_0<\underline{a}_0} (-1)^{|a_0|+|a^{(1)}|'} 1\otimes 1|\ \mathrm{sh}(\underline{a}^{(2)};\underline{b}^{(2)}),\mathfrak{m}_*^A\left(\underline{a}^{(3)}\right),\mathrm{sh}_0(\underline{a}^{(4)},\underline{a}_0,\underline{a}^{(1)};\underline{b}^{(3)},\underline{b}_0,\underline{b}^{(1)}) \tag{A.26}$$

$$+\sum (-1)^{|a_0|+|a^{(1)}|'} 1\otimes 1|\ \mathrm{sh}(\underline{a}^{(2)};\underline{b}^{(3)},\underline{b}_0,\underline{b}^{(1)}),\mathfrak{m}_*^A\left(\underline{a}^{(3)}\right),\mathrm{sh}(\underline{a}^{(4)},\underline{a}_0,\underline{a}^{(1)};\underline{b}^{(2)}) \tag{A.27}$$

$$+\sum (-1)^{|a^{(3)}|'(|a_0|+|a^{(1)}|'+|a^{(2)}|')} 1\otimes 1|\ \mathrm{sh}(\underline{a}^{(3)};\underline{b}^{(3)},\underline{b}_0,\underline{b}^{(1)}),\mathfrak{m}_*^A\left(\underline{a}^{(4)},\underline{a}_0,\underline{a}^{(1)}\right),\mathrm{sh}(\underline{a}^{(2)};\underline{b}^{(2)}) \tag{A.28}$$

which cancel out respectively with (A.13), (A.15), (A.17), and (A.11).

Lastly, we compute

$$(-1)^{|\mathbf{a}|}K(id\otimes\mathfrak{b})(\mathbf{a};\mathbf{b}) = \sum_{\underline{b}_0<\underline{a}_0} (-1)^{|b_0|+|b^{(1)}|'+|\mathbf{a}|}\ 1\otimes 1|\,\mathrm{sh}_0(N\underline{\mathbf{a}};\underline{b}^{(4)},\underline{b}_0,\underline{b}^{(1)},\mathfrak{m}_*^B(\underline{b}^{(2)}),\underline{b}^{(3)}) \tag{A.29}$$

$$+\sum_{\mathfrak{m}_*^B(\underline{b}^{(4)},\underline{b}_0,\underline{b}^{(1)})<\underline{a}_0} (-1)^{|b^{(3)}|'(|a_0|+|a^{(1)}|'+|a^{(2)}|')+|\mathbf{a}|}\ 1\otimes 1|\,\mathrm{sh}_0(N\underline{\mathbf{a}};\underline{b}^{(3)},\mathfrak{m}_*^B(\underline{b}^{(4)},\underline{b}_0,\underline{b}^{(1)}),\underline{b}^{(2)}) \tag{A.30}$$

Again, we break up the shuffle in (A.29) according to whether $\mathfrak{m}_*^B(\underline{a}^{(2)})$ is to the right, left, or middle of $\underline{b}_0$ and $\underline{a}_0$. Similarly, we break up the shuffle in (A.30) rewriting it with $\underline{b}_0$ to the left of $\underline{a}_0$ (and adjusting the signs appropriately) to obtain

$$\sum_{\underline{b}_0<\underline{a}_0}(-1)^{|b_0|+|b^{(1)}|'+|\mathbf{a}|}1\otimes 1|\,\mathrm{sh}_0(\underline{a}^{(3)},\underline{a}_0,\underline{a}^{(1)};\underline{b}^{(4)},\underline{b}_0,\underline{b}^{(1)}),\mathfrak{m}_*^B\left(\underline{b}^{(2)}\right),\mathrm{sh}(\underline{a}^{(2)};\underline{b}^{(3)}) \tag{A.31}$$

$$+\sum_{\underline{b}_0<\underline{a}_0}(-1)^{|b_0|+|b^{(1)}|'+|\mathbf{a}|}\ 1\otimes 1|\,\mathrm{sh}(\underline{a}^{(2)};\underline{b}^{(2)}),\mathfrak{m}_*^B\left(\underline{b}^{(3)}\right),\mathrm{sh}_0(\underline{a}^{(3)},\underline{a}_0,\underline{a}^{(1)};\underline{b}^{(4)},\underline{b}_0,\underline{b}^{(1)}) \tag{A.32}$$

$$+\sum(-1)^{|b_0|+|b^{(1)}|'+|\mathbf{a}|}1\otimes 1|\ \ \mathrm{sh}(\underline{a}^{(2)};\underline{b}^{(4)},\underline{b}_0,\underline{b}^{(1)}),\mathfrak{m}_*^B\left(\underline{b}^{(2)}\right),\mathrm{sh}(\underline{a}^{(3)},\underline{a}_0,\underline{a}^{(1)};\underline{b}^{(3)}) \tag{A.33}$$

$$+\sum(-1)^{|b^{(3)}|'(|a_0|+|a^{(1)}|'+|a^{(2)}|')+|\mathbf{a}|}\ 1\otimes 1|\,\mathrm{sh}(\underline{a}^{(2)};\underline{b}^{(3)}),\mathfrak{m}_*^B\left(\underline{b}^{(4)},\underline{b}_0,\underline{b}^{(1)}\right),\mathrm{sh}(\underline{a}^{(3)},\underline{a}_0,\underline{a}^{(1)};\underline{b}^{(2)}) \tag{A.34}$$

which cancel out respectively with (A.14), (A.16), (A.18), and (A.12). □

Appendix B

Proof of Lemma 31

B.1 Lemma 31

$$[\mathfrak{B}\{\mathfrak{m}'\}, k] + [\mathfrak{b}\{\mathfrak{m}'\}, K] = [\mathfrak{b}, H] - k(\mathcal{L}_{\mathfrak{m}'} \otimes \mathfrak{B})$$

Proof. We compute the LHS.

$$\mathfrak{B}\{\mathfrak{m}'\} \circ k(\mathbf{a}; \mathbf{b}) = \mathfrak{B}\{\mathfrak{m}'\} \left(\sum (-1)^{\bullet} \, a_0 \otimes b_0 \mid \bigotimes_{i=1}^{4} \mathrm{sh}(\underline{a}^{(i)}; \underline{b}^{(i)}) \right)$$

$$= \sum (-1)^{\bullet + @ + <} \, 1 \otimes 1 | \, \mathrm{sh}(\underline{a}^{(2)}; \underline{b}^{(2)}), \mathfrak{m}'_{\otimes} \left(\mathrm{sh}(\underline{a}^{(3)}; \underline{b}^{(3)}) \right), \mathrm{sh}(\underline{a}^{(4)}; \underline{b}^{(4)}), a_0 \otimes b_0, \mathrm{sh}(\underline{a}^{(1)}; \underline{b}^{(1)}) \tag{B.1}$$

$$\text{where} \quad \bullet = |b_0||\mathbf{a}|' + \left(\sum_{i=1,\ j>i}^{3} |b^{(i)}|' |a^{(j)}|' \right), \ < \, = |a^{(2)}|' |b^{(2)}|'$$

$$\text{and} \quad @ = (|a_0| + |b_0| + |a^{(1)}|' + |b^{(1)}|')\left(\sum_{i=2}^{4} |a^{(i)}|' + |b^{(i)})|'\right)$$

Since $\mathfrak{m}'_2 = 0$ and by the commuting subalgebra relations (18), we must have $\mathrm{sh}(\underline{a}^{(3)}; \underline{b}^{(3)}) = \underline{a}^{(3)}$ or $\underline{b}^{(3)}$ but not both, in equation (B.1), so we split it as

$$\mathfrak{B}\{\mathfrak{m}'\} \circ k(\mathbf{a}; \mathbf{b}) =$$

$$\sum (-1)^{\bullet + @ + <} \, 1 \otimes 1 | \, \mathrm{sh}(\underline{a}^{(2)}; \underline{b}^{(2)}), \mathfrak{m}'_A \left(\underline{a}^{(3)} \right), \mathrm{sh}(\underline{a}^{(4)}; \underline{b}^{(3)}), a_0 \otimes b_0, \mathrm{sh}(\underline{a}^{(1)}; \underline{b}^{(1)}) \tag{B.2}$$

$$\sum (-1)^{\bullet + @ + <} \, 1 \otimes 1 | \, \mathrm{sh}(\underline{a}^{(2)}; \underline{b}^{(2)}), \mathfrak{m}'_B \left(\underline{b}^{(3)} \right), \mathrm{sh}(\underline{a}^{(3)}; \underline{b}^{(4)}), a_0 \otimes b_0, \mathrm{sh}(\underline{a}^{(1)}; \underline{b}^{(1)}) \tag{B.3}$$

$$k \circ (\mathfrak{B}\{\mathfrak{m}'\}\mathbf{a} \otimes \mathbf{b}) = \sum (-1)^{@_A + <_A + |b_0||a|' + 1}\ 1 \otimes b_0 | \operatorname{sh}\left(\underline{a}^{(2)}, \mathfrak{m}'_A(\underline{a}^{(3)}), \underline{a}^{(4)}, a_0, \underline{a}^{(1)}; \underline{b}\right) \tag{B.4}$$

$$(-1)^{|\mathbf{a}|} k(\mathbf{a}; \mathfrak{B}\{\mathfrak{m}'\}\mathbf{b}) = \sum (-1)^{@_B + <_B + |\mathbf{a}|}\ a_0 \otimes 1 | \operatorname{sh}\left(\underline{a}; \underline{b}^{(2)}, \mathfrak{m}'_B(\underline{b}^{(3)}), \underline{b}^{(4)}, b_0, \underline{b}^{(1)}\right) \tag{B.5}$$

where $<_A = |a^{(2)}|'$; $<_B = |b^{(2)}|'$

$$\mathfrak{b}\{\mathfrak{m}'\} \circ K(\mathbf{a}; \mathbf{b}) = \sum_{\underline{b}_0 < \underline{a}_0} (-1)^{\bullet + @ + <}$$

$$\mathfrak{m}^{\otimes}_{*}(\operatorname{Sh}(\mathbf{a}; \mathbf{b})^{(3)}, \mathfrak{m}'_{\otimes}(\operatorname{Sh}(\mathbf{a}; \mathbf{b})^{(4)}), \operatorname{Sh}(\mathbf{a}; \mathbf{b})^{(5)}, 1 \otimes 1, \operatorname{Sh}(\mathbf{a}; \mathbf{b})^{(1)}) | \operatorname{Sh}(\mathbf{a}; \mathbf{b})^{(2)} \tag{B.6}$$

where $\bullet = (\sum_{i=1,\ j>i}^{4} |b^{(i)}|'|a^{(j)}|')$, $< = |a^{(3)}|' + |b^{(3)}|'$

Again, since $\mathfrak{m}'_2 = 0$ and by the commuting subalgebra relations, the cyclic shuffle term above $\operatorname{Sh}(\mathbf{a}; \mathbf{b})^{(4)} = \mathbf{a}^{(4)}$ or $\mathbf{b}^{(4)}$ but not both. Also, by strict unitality, $* = 2$. We also break up the sum (B.6) into whether $\underline{a}_0$ is an input of $\mathfrak{m}'$, or not. Observe that the restriction $\underline{b}_0 < \underline{a}_0$ means $\underline{b}_0$ is not an input of $\mathfrak{m}'$. So we obtain

$$\mathfrak{b}\{\mathfrak{m}'\} \circ K(\mathbf{a}; \mathbf{b}) = \sum_{\underline{b}_0 < \underline{a}_0} (-1)^{\bullet + @} \mathfrak{m}^{\otimes}_2 \left(\mathfrak{m}'_A(\underline{a}^{(2)}), 1 \otimes 1\right) | \operatorname{sh}_0(\underline{a}^{(3)}, \underline{a}_0, \underline{a}^{(1)}, N\underline{\mathbf{b}}) \tag{B.7}$$

$$+ \sum_{\underline{b}_0 < \underline{a}_0} (-1)^{\bullet + @} \mathfrak{m}^{\otimes}_2 \left(\mathfrak{m}'_B(\underline{b}^{(2)}), 1 \otimes 1\right) | \operatorname{sh}_0(N\underline{\mathbf{a}}, \underline{b}^{(3)}, \underline{b}_0, \underline{b}^{(1)}) \tag{B.8}$$

$$+ \sum (-1)^{\bullet + @} \mathfrak{m}^{\otimes}_2 \left(\mathfrak{m}'_A(\underline{a}^{(3)}, \underline{a}_0, \underline{a}^{(1)}), 1 \otimes 1\right) | \operatorname{sh}(\underline{a}^{(2)}, N\underline{\mathbf{b}}) \tag{B.9}$$

$$K \circ (\mathfrak{b}\{\mathfrak{m}'\}\mathbf{a}; \mathbf{b}) = \sum_{\underline{b}_0 < \mathfrak{m}^A_*(\underline{a}^{(3)}, \mathfrak{m}'_A(\underline{a}^{(4)}), \underline{a}^{(5)}, \underline{a}_0, \underline{a}^{(1)})} (-1)^{@_A + <_A}\ 1 \otimes 1 | \operatorname{sh}_0\left(\mathfrak{m}^A_*(\underline{a}^{(3)}, \mathfrak{m}'_A(\underline{a}^{(4)}), \underline{a}^{(5)}, \underline{a}_0, \underline{a}^{(1)}) | \underline{a}^{(2)}; N\mathbf{b}\right) \tag{B.10}$$

$$(-1)^{|\mathbf{a}|} K(\mathbf{a}; \mathfrak{b}\{\mathfrak{m}'\}\mathbf{b}) = \sum_{\mathfrak{m}^B_*(\underline{b}^{(3)}, \mathfrak{m}'_B(\underline{b}^{(4)}), \underline{b}^{(5)}, \underline{b}_0, \underline{b}^{(1)} < \underline{a}_0} (-1)^{@_A + @_B + |\mathbf{a}|}\ 1 \otimes 1 | \operatorname{sh}_0\left(N\mathbf{a}; \mathfrak{m}^B_*(\underline{b}^{(3)}, \mathfrak{m}'_B(\underline{b}^{(4)}), \underline{b}^{(5)}, \underline{b}_0, \underline{b}^{(1)}) | \underline{b}^{(2)}\right) \tag{B.11}$$

$$-k(\mathcal{L}_{\mathfrak{m}'}\otimes\mathfrak{B})(\mathbf{a}\otimes\mathbf{b})=\sum(-1)^{1+<_A+@_B}\ a_0\otimes 1|\operatorname{sh}(\underline{a}^{(1)},\mathfrak{m}'_A(\underline{a}^{(2)}),\underline{a}^{(3)};\underline{b}^{(2)},\underline{b}_0,\underline{b}^{(1)}) \tag{B.12}$$

$$+\sum(-1)^{1+@_A+@_B}\ \mathfrak{m}'_A(a^{(3)},a_0,a^{(1)})\otimes 1|\operatorname{sh}(\underline{a}^{(2)};\underline{b}^{(2)},\underline{b}_0,\underline{b}^{(1)}) \tag{B.13}$$

Since the shuffle term in equation (B.13) cyclically permutes $\underline{\mathbf{b}}$, it is exactly the shuffle term in (B.9). A computation of the signs show that they cancel each other. We will show that the rest of the above terms are canceled by the terms in $\mathfrak{b}\circ H$.

$$\mathfrak{b}\circ H(\mathbf{a};\mathbf{b})=\mathfrak{b}\left(\sum_{\underline{b}_0<\underline{a}_0}(-1)^{@_A+@_B+|a^{(2)}|'}\ 1\otimes 1|\operatorname{sh}(\underline{a}^{(2)},\mathfrak{m}'_A(\underline{a}^{(3)}),\underline{a}^{(4)},\underline{a}_0,\underline{a}^{(1)};\underline{b}^{(2)},\underline{b}_0,\underline{b}^{(1)})\right) \tag{B.14}$$

$$+\mathfrak{b}\left(\sum_{\underline{b}_0<\underline{a}_0}(-1)^{@_A+@_B+|\mathbf{a}|+|b^{(2)}|'}\ 1\otimes 1|\operatorname{sh}(\underline{a}^{(2)},\underline{a}_0,\underline{a}^{(1)};\underline{b}^{(2)},\mathfrak{m}'_B(\underline{b}^{(3)}),\underline{b}^{(4)},\underline{b}_0,\underline{b}^{(1)})\right) \tag{B.15}$$

Equations (B.2) and (B.3) are canceled in (B.14) and (B.15) when $\mathfrak{b}$ acts by applying $\mathfrak{m}_2^{\otimes}$ to $(\underline{b}_0,\underline{a}_0)$ (with the correct signs). The restriction $\underline{b}_0<\underline{a}_0$ guarantees these do not come in commuting pairs.

Equation (B.4) is canceled in (B.14) when $\mathfrak{b}$ acts by applying $\mathfrak{m}_2^{\otimes}$ before "|" to $(1\otimes 1, 1\otimes \underline{b}_0)$. Equations (B.5) and (B.12) are canceled in (B.15) and (B.14) respectively, when $\mathfrak{b}$ acts by applying $\mathfrak{m}_2^{\otimes}$ before "|" to $(\underline{a}_0\otimes 1, 1\otimes 1)$.

Similarly, (B.7) and (B.8) are canceled in (B.14) and (B.15) when $\mathfrak{b}$ acts by applying $\mathfrak{m}_2^{\otimes}$ to $(1\otimes 1,\mathfrak{m}'_A(a^{(2)}))$, and to $(1\otimes 1,\mathfrak{m}'_B(b^{(2)}))$ respectively. By strict unitality, and the restriction $\underline{b}_0<\underline{a}_0$, these are the only possibilities for applying $\mathfrak{m}_2^{\otimes}$ on the "outside" before "|", with $1\otimes 1$. All the other cases $\mathfrak{m}_2^{\otimes}(a_i\otimes 1, 1\otimes 1)$ and $\mathfrak{m}_2^{\otimes}(1\otimes 1, 1\otimes b_i)$ come in commuting pairs and thus cancel each other.

Equations (B.10) and (B.11) are canceled in (B.14) and (B.15) when $\mathfrak{b}$ acts by applying

$\mathfrak{m}_*^{\otimes}$ to $(\underline{a}^{(3)}, \mathfrak{m}'_A(\underline{a}^{(4)}), \underline{a}^{(5)}, \underline{a}_0, \underline{a}^{(1)})$, and on $(\underline{b}^{(3)}, \mathfrak{m}'_B(\underline{b}^{(4)}), \underline{b}^{(5)}, \underline{b}_0, \underline{b}^{(1)})$.

The remaining terms in $\mathfrak{b} \circ H$ unaccounted for are:

$$\sum_{\underline{b}_0 < \underline{a}_0} (-1)^{@_A + @_B + |\mathbf{a}|' + |b_0| + |b^{(1)}|' + |b^{(4)}|'} \ 1 \otimes 1 |\, \mathrm{sh}(\underline{a}^{(2)}, \mathfrak{m}'_A(\underline{a}^{(3)}), \underline{a}^{(4)}, \underline{a}_0, \underline{a}^{(1)}; \underline{b}^{(4)}, \underline{b}_0, \underline{b}^{(1)}, \mathfrak{m}_*^B(\underline{b}^{(2)}), \underline{b}^{(3)}) \tag{B.16}$$

$$+ \sum_{\underline{b}_0 < \underline{a}_0} (-1)^{@_A + @_B + |b^{(2)}|' + |a^{(2)}|' + |a^{(3)}|'} 1 \otimes 1 |\, \mathrm{sh}(\underline{a}^{(4)}, \underline{a}_0, \underline{a}^{(1)}, \mathfrak{m}_*^A(\underline{a}^{(2)}), \underline{a}^{(3)}; \underline{b}^{(2)}, \mathfrak{m}'_B(\underline{b}^{(3)}), \underline{b}^{(4)}, \underline{b}_0, \underline{b}^{(1)}) \tag{B.17}$$

$$+ \sum_{\underline{b}_0 < \underline{a}_0} (-1)^{@_A + @_B + |\mathbf{a}|'} 1 \otimes 1 |\, \mathrm{sh}(\underline{a}^{(3)}, \mathfrak{m}'_A(\underline{a}^{(4)}), \underline{a}^{(5)}, \mathfrak{m}_*(\underline{a}^{(6)}, \underline{a}_0, \underline{a}^{(1)}), \underline{a}^{(2)}; \underline{b}^{(2)}, \underline{b}_0, \underline{b}^{(1)}) \tag{B.18}$$

$$+ \sum_{\underline{b}_0 < \underline{a}_0} (-1)^{@_A + @_B + |\mathbf{a}|' + |b^{(3)}|' + |a^{(2)}|'} \ 1 \otimes 1 |\, \mathrm{sh}(\underline{a}^{(3)}, \mathfrak{m}_*^A(\underline{a}^{(4)}, \underline{a}_0, \underline{a}^{(1)}), \underline{a}^{(2)}; \underline{b}^{(2)}, \mathfrak{m}'_B(\underline{b}^{(3)}), \underline{b}^{(4)}, \underline{b}_0, \underline{b}^{(1)}) \tag{B.19}$$

$$+ \sum_{\underline{b}_0 < \underline{a}_0} (-1)^{@_A + @_B + |\mathbf{a}|' + |a^{(2)}|' + |b^{(3)}|'} \ 1 \otimes 1 |\, \mathrm{sh}(\underline{a}^{(2)}, \mathfrak{m}'_A(\underline{a}^{(3)}), \underline{a}^{(4)}, \underline{a}_0, \underline{a}^{(1)}; \underline{b}^{(3)}, \mathfrak{m}_*^B(\underline{b}^{(4)}, \underline{b}_0, \underline{b}^{(1)}), \underline{b}^{(2)}) \tag{B.20}$$

$$+ \sum_{\underline{b}_0 < \underline{a}_0} (-1)^{@_A + @_B + |b^{(2)}|' + |b^{(3)}|'} \ 1 \otimes 1 |\, \mathrm{sh}(\underline{a}^{(2)}, \underline{a}_0, \underline{a}^{(1)}; \underline{b}^{(3)}, \mathfrak{m}'_B(\underline{b}^{(4)}), \underline{b}^{(5)}, \mathfrak{m}_*^{\otimes}(\underline{b}^{(6)}, \underline{b}_0, \underline{b}^{(1)}), \underline{b}^{(2)}) \tag{B.21}$$

$$+ \sum_{\underline{b}_0 < \underline{a}_0} (-1)^{@_A + @_B} \ 1 \otimes 1 |\, \mathrm{sh}(\underline{a}^{(2)}, \mathfrak{m}_*^A(\underline{a}^{(3)}, \mathfrak{m}'_A(\underline{a}^{(4)}), \underline{a}^{(5)}), \underline{a}^{(6)}, \underline{a}_0, \underline{a}^{(1)}; \underline{b}^{(2)}, \underline{b}_0, \underline{b}^{(1)}) \tag{B.22}$$

$$+ \sum_{\underline{b}_0 < \underline{a}_0} (-1)^{@_A + @_B} \ 1 \otimes 1 |\, \mathrm{sh}(\underline{a}^{(2)}, \underline{a}_0, \underline{a}^{(1)}; \underline{b}^{(2)}, \mathfrak{m}_*^B(\underline{b}^{(3)}, \mathfrak{m}'_B(\underline{b}^{(4)}), \underline{b}^{(5)}), \underline{b}^{(6)}, \underline{b}_0, \underline{b}^{(1)}) \tag{B.23}$$

We will show that these are all canceled by terms in $H \circ \mathfrak{b}$.

Now $H \circ \mathfrak{b}(\mathbf{a};\mathbf{b}) = H(\mathfrak{b}(\mathbf{a});\mathbf{b}) + (-1)^{|\mathbf{a}|} H(\mathbf{a};\mathfrak{b}(\mathbf{b}))$, so we compute term by term.

$$H(\mathfrak{b}(\mathbf{a});\mathbf{b}) =$$

$$\sum_{\underline{b}_0<\underline{a}_0} (-1)^{@_A+@_B+|a^{(2)}|'+<_A}\ 1\otimes 1|\,\mathrm{sh}(\underline{a}^{(2)}, \mathfrak{m}'_A(\underline{a}^{(3)}), \underline{a}^{(4)}, \mathfrak{m}_*(\underline{a}^{(5)}), \underline{a}^{(6)}, \underline{a}_0, \underline{a}^{(1)}; \underline{b}^{(2)}, \underline{b}_0, \underline{b}^{(1)}) \tag{B.24}$$

$$+ \sum_{\underline{b}_0<\underline{a}_0} (-1)^{@_A+@_B+|a^{(2)}|'+<_A}\ 1\otimes 1|\,\mathrm{sh}(\underline{a}^{(2)}, \mathfrak{m}'_A(\underline{a}^{(3)}, \mathfrak{m}_*(\underline{a}^{(4)}), \underline{a}^{(5)}), \underline{a}^{(6)}, \underline{a}_0, \underline{a}^{(1)}; \underline{b}^{(2)}, \underline{b}_0, \underline{b}^{(1)}) \tag{B.25}$$

$$+ \sum_{\underline{b}_0<\underline{a}_0} (-1)^{@_A+@_B+|a^{(2)}|'+<_A}\ 1\otimes 1|\,\mathrm{sh}(\underline{a}^{(2)}, \mathfrak{m}_*(\underline{a}^{(3)}), \underline{a}^{(4)}, \mathfrak{m}'_A(\underline{a}^{(5)}), \underline{a}^{(6)}, \underline{a}_0, \underline{a}^{(1)}; \underline{b}^{(2)}, \underline{b}_0, \underline{b}^{(1)}) \tag{B.26}$$

$$+ \sum_{\underline{b}_0<\underline{a}_0} (-1)^{@_A+@_B+<_A+|\mathbf{a}|'+|b^{(2)}|'}\ 1\otimes 1|\,\mathrm{sh}(\underline{a}^{(4)}, \underline{a}_0, \underline{a}^{(1)}, \mathfrak{m}^A_*(\underline{a}^{(2)}), \underline{a}^{(3)}; \underline{b}^{(2)}, \mathfrak{m}'_B(\underline{b}^{(3)}), \underline{b}^{(4)}, \underline{b}_0, \underline{b}^{(1)}) \tag{B.27}$$

$$+ \sum_{\underline{b}_0<\underline{a}_0} (-1)^{@_A+|a^{(3)}|'}\ 1\otimes 1|\,\mathrm{sh}(\underline{a}^{(3)}, \mathfrak{m}'_A(\underline{a}^{(4)}), \underline{a}^{(5)}, \mathfrak{m}_*(\underline{a}^{(6)}, \underline{a}_0, \underline{a}^{(1)}), \underline{a}^{(2)}; \underline{b}^{(2)}, \underline{b}_0, \underline{b}^{(1)}) \tag{B.28}$$

$$+ \sum_{\underline{b}_0<\underline{a}_0} (-1)^{@_A+|\mathbf{a}|+|b^{(2)}|'}\ 1\otimes 1|\,\mathrm{sh}(\underline{a}^{(3)}, \mathfrak{m}^A_*(\underline{a}^{(4)}, \underline{a}_0, \underline{a}^{(1)}), \underline{a}^{(2)}; \underline{b}^{(2)}, \mathfrak{m}'_B(\underline{b}^{(3)}), \underline{b}^{(4)}, \underline{b}_0, \underline{b}^{(1)}) \tag{B.29}$$

$$(-1)^{|\mathbf{a}|} H(\mathbf{a}; \mathfrak{b}(\mathbf{b})) =$$

$$\sum_{\underline{b}_0<\underline{a}_0} (-1)^{|\mathbf{a}|+@_A+@_B+|a^{(2)}|'+<_B} \; 1\otimes 1|\,\mathrm{sh}(\underline{a}^{(2)}, \mathfrak{m}'_A(\underline{a}^{(3)}), \underline{a}^{(4)}, \underline{a}_0, \underline{a}^{(1)}; \underline{b}^{(4)}, \underline{b}_0, \underline{b}^{(1)}, \mathfrak{m}^B_*(\underline{b}^{(2)}), \underline{b}^{(3)}) \tag{B.30}$$

$$+\sum_{\underline{b}_0<\underline{a}_0} (-1)^{|\mathbf{a}|+@_A+@_B+|b^{(2)}|'+<_B} \; 1\otimes 1|\,\mathrm{sh}(\underline{a}^{(2)}, \underline{a}_0, \underline{a}^{(1)}; \underline{b}^{(2)}, \mathfrak{m}'_B(\underline{b}^{(3)}), \underline{b}^{(4)}, \mathfrak{m}^B_*(\underline{b}^{(5)}), \underline{b}^{(6)}, \underline{b}_0, \underline{b}^{(1)}) \tag{B.31}$$

$$+\sum_{\underline{b}_0<\underline{a}_0} (-1)^{|\mathbf{a}|+@_A+@_B+|b^{(2)}|'+<_B} \; 1\otimes 1|\,\mathrm{sh}(\underline{a}^{(2)}, \underline{a}_0, \underline{a}^{(1)}; \underline{b}^{(2)}, \mathfrak{m}'_B(\underline{b}^{(3)}, \mathfrak{m}^B_*(\underline{b}^{(4)}), \underline{b}^{(5)}), \underline{b}^{(6)}, \underline{b}_0, \underline{b}^{(1)}) \tag{B.32}$$

$$+\sum_{\underline{b}_0<\underline{a}_0} (-1)^{|\mathbf{a}|+@_A+@_B+|b^{(2)}|'+<_B} \; 1\otimes 1|\,\mathrm{sh}(\underline{a}^{(2)}, \underline{a}_0, \underline{a}^{(1)}; \underline{b}^{(2)}, \mathfrak{m}^B_*(\underline{b}^{(3)}), \underline{b}^{(4)}, \mathfrak{m}'_B(\underline{b}^{(5)}), \underline{b}^{(6)}, \underline{b}_0, \underline{b}^{(1)}) \tag{B.33}$$

$$+\sum_{\underline{b}_0<\underline{a}_0} (-1)^{@_A+@_B+|a^{(2)}|'+|\mathbf{a}|} \; 1\otimes 1|\,\mathrm{sh}(\underline{a}^{(2)}, \underline{a}_0, \underline{a}^{(1)}; \underline{b}^{(3)}, \mathfrak{m}'_B(\underline{b}^{(4)}), \underline{b}^{(5)}, \mathfrak{m}^{\otimes}_*(\underline{b}^{(6)}, \underline{b}_0, \underline{b}^{(1)}), \underline{b}^{(2)}) \tag{B.34}$$

$$+\sum_{\underline{b}_0<\underline{a}_0} (-1)^{@_A+@_B+|b^{(3)}|'+|\mathbf{a}|} \; 1\otimes 1|\,\mathrm{sh}(\underline{a}^{(2)}, \mathfrak{m}'_A(\underline{a}^{(3)}), \underline{a}^{(4)}, \underline{a}_0, \underline{a}^{(1)}; \underline{b}^{(3)}, \mathfrak{m}^B_*(\underline{b}^{(4)}, \underline{b}_0, \underline{b}^{(1)}), \underline{b}^{(2)}) \tag{B.35}$$

We find that (B.16) cancels with (B.30); (B.17) cancels with (B.27); (B.18) cancels with (B.28); (B.19) cancels with (B.29); (B.20) cancels with (B.35); (B.21) cancels with (B.34). Further, (B.31) and (B.33) cancel each other, as well as (B.24) and (B.26). Lastly, (B.22) and (B.23) cancel with (B.25) and (B.32) respectively since $[\mathfrak{m}_*, \mathfrak{m}'] = 0$.

□

Bibliography

[1] Serguei Barannikov. Quantum periods, I: semi-infinite variations of Hodge structures. *International Mathematics Research Notices*, 2001(23):1243–1264, 2001.

[2] Maxim Kontsevich. Homological algebra of mirror symmetry. In *Proceedings of the International Congress of Mathematicians: August 3–11, 1994 Zürich, Switzerland*, pages 120–139. Springer, 1995.

[3] Serguei Barannikov. Non-commutative periods and mirror symmetry in higher dimensions. *Communications in mathematical physics*, 228(2):281–325, 2002.

[4] L. Katzarkov, M. Kontsevich, and T. Pantev. Hodge theoretic aspects of mirror symmetry. In *From Hodge theory to integrability and TQFT tt*-geometry*, volume 78 of *Proc. Sympos. Pure Math.*, pages 87–174. Amer. Math. Soc., Providence, RI, 2008. ISBN 978-0-8218-4430-4. doi: 10.1090/pspum/078/2483750. URL https://doi.org/10.1090/pspum/078/2483750.

[5] Sheel Ganatra, Timothy Perutz, and Nick Sheridan. Mirror symmetry: from categories to curve counts. *arXiv preprint arXiv:1510.03839*, 2015.

[6] Nick Sheridan. Formulae in noncommutative Hodge theory. *Journal of Homotopy and Related Structures*, 15(1):249–299, 2020.

[7] Jean-Louis Loday. *Cyclic homology*, volume 301. Springer Science & Business Media, 2013.

[8] Dmytro Shklyarov. On a Hodge theoretic property of the Künneth map in periodic cyclic homology. *Journal of Algebra*, 446:132–153, 2016.

[9] Samson Saneblidze and Ronald Umble. A diagonal on the associahedra. *arXiv preprint math/0011065*, 2000.

[10] Samson Saneblidze and Ronald Umble. Diagonals on the permutahedra, multiplihedra and associahedra. *Homology, Homotopy and Applications, Homology Homotopy Appl.*, 6(1):363–411, 2004.

[11] Martin Markl and Steve Shnider. Associahedra, cellular W-construction and products of A_∞-algebras. *Transactions of the American Mathematical Society*, 358(6):2353–2372, 2006.

[12] Lino Amorim. Tensor product of filtered A_∞-algebras. *Journal of Pure and Applied Algebra*, 220(12):3984–4016, 2016.

[13] Lino Amorim and Junwu Tu. Tensor product of cyclic A_∞-algebras and their Kontsevich classes. *Transactions of the American Mathematical Society*, 371(2):1029–1061, 2019.

[14] Lino Amorim and Junwu Tu. Categorical primitive forms of Calabi-Yau A_∞-categories with semi-simple cohomology. *Selecta Mathematica*, 28(3):54, 2022.

[15] Kyoji Saito and Atsushi Takahashi. *From primitive forms to Frobenius manifolds*. Kyoto University, Research Institute for Mathematical Sciences, 2008.

[16] Ezra Getzler. Cartan homotopy formulas and the Gauss-Manin connection in cyclic homology. In *Israel Math. Conf. Proc*, volume 7, pages 65–78, 1993.

[17] Maxim Kontsevich and Yan Soibelman. Notes on A-infinity algebras, A-infinity categories and non-commutative geometry. I. *arXiv preprint math/0606241*, 2006.

[18] Dmytro Shklyarov. Hirzebruch–Riemann–Roch-type formula for DG algebras. *Proceedings of the London Mathematical Society*, 106(1):1–32, 2013.

[19] Andrei Căldăraru, Si Li, and Junwu Tu. Categorical primitive forms and Gromov-Witten invariants of A_n singularities. *International Mathematics Research Notices*, 2021 (24):18489–18519, 2021.

[20] Dmytro Shklyarov. Matrix factorizations and higher residue pairings. *Advances in Mathematics*, 292:181–209, 2016.

[21] Dmitry Kaledin. Spectral sequences for cyclic homology. *Algebra, Geometry, and Physics in the 21st Century: Kontsevich Festschrift*, pages 99–129, 2017.

[22] Boris Tsygan. Cyclic homology. *Cyclic homology in non-commutative geometry*, pages 73–113, 2004.

[23] Charles A Weibel. *An introduction to homological algebra*. Number 38. Cambridge university press, 1994.

[24] Lino Amorim and Junwu Tu. On the Morita invariance of categorical enumerative invariants. *arXiv preprint arXiv:2209.02744*, 2022.