METHODS OF MATRIX INVERSION

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INTRODUCTION

A technique that is often used when solving a system of equations involves representing the equations in matrix form. To solve the system we then work with the matrix of coefficients. Closely related to this problem is that of finding the inverse of a matrix. The connection between the two problems will become more apparent when we discuss Gaussian elimination.

In addition to the technique of Gaussian elimination, we will discuss three other techniques for finding the inverse of a matrix. They are partitioning, rank annihilation, and the L-F method. Some variations of these methods will be discussed in the corresponding sections.

With each method, a general outline of the theory behind it will be discussed. This will be followed by a short discussion of the actual computation of the inverse matrix and an example using the particular technique. Throughout the paper, the same example is used in order to tie the methods together.

GAUSSIAN ELIMINATION

One of the methods for finding an inverse of a matrix is based on Gaussian elimination. The theory behind this method of finding an inverse is derived from the solution of a system of equations by elimination.

A system of n equations in n unknowns,

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_b \\
\cdots\cdots\cdots\cdots\cdots\cdots & \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}$$

can be written in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

or AX = B. We augment the matrix A with the column vector B, then reduce $[AB]$ to $[IX]$ by a series of operations on the augmented matrix. These operations, known as elementary row operations, consist of three types: type I, multiplication of a row by a constant; type II, addition of a multiple of one row to another; and type III, interchange of rows.

The aforementioned operations can be carried out by matrix multiplication. The matrices used to perform these operations are called elementary matrices and are denoted as E_i. They are derived from the identity matrix. That is, if we perform these operations (multiplication of a row by a constant, addition of a multiple of one row to another, and interchange of rows) on the identity matrix, we obtain the corresponding elementary matrix E_i

To perform the elimination, it is then necessary to multiply A by a series of these E_i's to obtain the matrix I. By applying these elementary operations to the augmented matrix $[AB]$, we also change the column vector B to the solution vector X.

For the actual mechanics of Gaussian elimination, we use the following procedure. Consider the matrix of coefficients,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} .$$

To change this matrix to the identity matrix we first of all convert the a_{11} to 1 by dividing all elements of the first row by a_{11}. This is a type I operation. Next we perform a series of n - 1 type II operations to make the remaining elements of the first column zeros. As a result, we have a new matrix whose first column consists of a 1 in the first position and zeros elsewhere.

Next we consider the auxiliary system consisting of the principal submatrix formed by deleting the first row and the first column. Applying the previous procedure to this submatrix, we obtain a column vector with 1 in the first position and zeros elsewhere. Continuing this process results in a matrix composed of 1's on the diagonal and 0's below it. This procedure is called the forward solution.

Now, the problem is to change the coefficients above the diagonal to zeros. The procedure used for this is known as the backward solution. Consider the element in the n^{th} row and n^{th}

column. This element will be 1, but all other elements in the column could be nonzero numbers. To change these elements to zeros, we subtract the corresponding multiple of the n^{th} row from the other $n - 1$ rows. This series of type II operations will give all zeros in that column except for the element on the diagonal of the matrix.

Considering the principal submatrix formed by deleting the n^{th} row and n^{th} column, we repeat the procedure. If the matrix is nonsingular, a continuation of this process gives the identity matrix for the final result.

A problem could arise if the leading element (the element in the upper left-hand corner of the matrix or submatrix is 0. If the matrix is nonsingular, this is easily resolved by using a type III operation to interchange this row with another which has a nonzero element in the corresponding column.

Applying this same series of operations to the column vector B gives the solution vector X. Or, if we augment the matrix A with B, we can carry out the operations simultaneously. This completes the solution of a system of equations by Gaussian elimination. Now we shall consider the problem of inverting matrices using Gaussian elimination procedures.

Letting the E_i stand for the needed elementary matrices, we could represent the transformation of A to I by multiplication by a series of E_i's. That is, $E_kE_{k-1} \ . \ . \ . \ E_2E_1A = I$. Multiplication on the right by A^{-1} gives $E_kE_{k-1} \ . \ . \ . \ E_2E_1I = A^{-1}$. But then these same elementary operations applied to the identity matrix will give the inverse matrix A^{-1}.

For computational purposes, as we did in the solution of a system of equations by elimination, we augment the matrix A with the matrix I and perform the operations on both simultaneously. Hence $E_k E_{k-1} \cdot \cdot \cdot E_2 E_1 \left[AI\right] = \left[IA^{-1}\right]$. If A is n x n, then A^{-1} will be the last n columns of the augmented form after the backward solution has been completed.

To illustrate this discussion we can find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 5 \end{bmatrix}.$$

Note how at each step the augmented matrix is multiplied by a suitable elementary matrix until the identity matrix is obtained.

$$\left[AI\right] = \begin{bmatrix} 2 & 1 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 5 & 7 & 5 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[AI\right] = \begin{bmatrix} 1 & 1/2 & 3/2 & 1/2 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 5 & 7 & 5 & 0 & 0 & 1 \end{bmatrix} = \left[AI\right]_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[AI\right]_1 = \begin{bmatrix} 1 & 1/2 & 3/2 & 1/2 & 0 & 0 \\ 0 & \cancel{1}3 & 0 & -2/3 & 1/3 & 0 \\ 5 & 7 & 5 & 0 & 0 & 1 \end{bmatrix} = \left[AI\right]_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix} [AI]_2 = \begin{bmatrix} 1 & 1/2 & 3/2 & 1/2 & 0 & 0 \\ 0 & 3 & 0 & -2 & 1 & 0 \\ 0 & 9/2 & -5/2 & -5/2 & 0 & 1 \end{bmatrix} = [AI]_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} [AI]_3 = \begin{bmatrix} 1 & 1/2 & 3/2 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -2/3 & 1/3 & 0 \\ 0 & 9/2 & -5/2 & -5/2 & 0 & 1 \end{bmatrix} = [AI]_4$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -9/2 & 1 \end{bmatrix} [AI]_4 = \begin{bmatrix} 1 & 1/2 & 3/2 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -2/3 & 1/3 & 0 \\ 0 & 0 & -5/2 & 1/2 & -3/2 & 1 \end{bmatrix} [AI]_5$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2/5 \end{bmatrix} [AI]_5 = \begin{bmatrix} 1 & 1/2 & 3/2 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -2/3 & 1/3 & 0 \\ 0 & 0 & 1 & -1/5 & 3/5 & -2/5 \end{bmatrix} = [AI]_6$$

$$\begin{bmatrix} 1 & 0 & -3/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} [AI]_6 = \begin{bmatrix} 1 & 1/2 & 0 & 4/5 & -9/10 & 3/5 \\ 0 & 1 & 0 & -2/3 & 1/3 & 0 \\ 0 & 0 & 1 & -1/5 & 3/5 & -2/5 \end{bmatrix} = [AI]_7$$

$$\begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} [AI]_7 = \begin{bmatrix} 1 & 0 & 0 & 17/15 & -16/15 & 3/5 \\ 0 & 1 & 0 & -2/3 & 1/3 & 0 \\ 0 & 0 & 1 & -1/5 & 3/5 & -2/5 \end{bmatrix}$$

Thus $A^{-1} = \begin{bmatrix} 17/15 & -16/15 & 3/5 \\ -2/3 & 1/3 & 0 \\ -1/5 & 3/5 & -2/5 \end{bmatrix}$.

COMPACT SCHEMES

A set of methods called compact schemes is derived from Gaussian elimination. In the basic Gaussian elimination we worked with the entire matrix A and finally at the last step came up with the matrix A^{-1}. The compact schemes use formulas to obtain A^{-1} element by element.

Some of the schemes utilize the factorization of the matrix into the product of two triangular matrices. We then work with these triangular matrices to find the inverse of the original matrix.

In the forward solution by Gaussian elimination, we change the matrix A to a matrix U with ones on the diagonal and zeros below it. This can be carried out by a series of elementary row operations E_i. Representing the forward solution in matrix form, we have

$$E_k E_{k-1} \cdot \cdot \cdot E_2 E_1 A = U$$

where U is an upper triangular matrix with ones on the diagonal. The E_i are going to have nonzero elements only on the diagonal or below it (provided no type III operations are used). That is, they are lower triangular matrices. As can be easily verified, the product of the E_i's is also going to be a lower triangular matrix C. Thus we have

$$CA = U$$

so that

$$A = C^{-1}U$$

where C^{-1} is also a lower triangular matrix.

Now with A factored into the product of two triangular matrices, the problem of finding A^{-1} is reduced to finding the inverses of U and C^{-1}. This is easily seen since if

$$A = C^{-1}U ,$$

then

$$A^{-1} = \left[C^{-1}U\right]^{-1} = U^{-1}\left[C^{-1}\right]^{-1} = U^{-1}C .$$

To illustrate the method of obtaining the inverse, let us consider the 3 x 3 case. We have $A = C^{-1}U$, or

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} c_{11} & 0 & 0 \\ c_{21} & c_{22} & 0 \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

where the c_{ij} and u_{ij} are unknown. By multiplying the rows of C^{-1} by the first columns of U, we obtain

$$c_{11} = a_{11}$$
$$c_{21} = a_{21}$$
$$c_{31} = a_{31} .$$

Multiplying the first row of C^{-1} by the second and third column of U, we obtain

$$c_{11}u_{12} = a_{12} \text{ or } u_{12} = \frac{a_{12}}{c_{11}} = \frac{a_{12}}{a_{11}}$$

$$c_{11}u_{13} = a_{13} \text{ or } u_{13} = \frac{a_{13}}{c_{11}} = \frac{a_{13}}{a_{11}}$$

The element a_{11} will not be zero because the type III operations will remove all zero elements from the diagonal.

In a similar manner, we can compute the unknown elements of the second column of C^{-1}. We obtain first of all the equations:

$$c_{21}u_{12} + c_{22} = a_{22} \quad \text{or} \quad c_{22} = a_{22} - c_{21}u_{12}$$

$$c_{31}u_{12} + c_{32} = a_{32} \quad \text{or} \quad c_{32} = a_{32} - c_{31}u_{12}$$

and then for the unknown element of the second row of U we obtain

$$c_{21}u_{13} + c_{22}u_{23} = a_{23} \quad \text{or} \quad u_{23} = \frac{a_{23} - c_{21}u_{13}}{c_{22}} .$$

The element c_{22} cannot be zero since this would make the determinant of C^{-1} equal to zero, and hence A would be singular. The last undetermined element c_{33} is computed from the equation

$$c_{31}u_{13} + c_{32}u_{23} + c_{33} = a_{33}$$

or

$$c_{33} = a_{33} - c_{31}u_{13} - c_{32}u_{23} .$$

General equations can be set up for obtaining these elements. They are

$$c_{ij} = a_{ij} - \sum_{k=1}^{j-1} c_{ik}u_{kj} \qquad i \geqslant j \tag{1}$$

$$u_{ij} = \frac{a_{ij} - \sum_{k=1}^{i-1} c_{ik}u_{kj}}{c_{ij}} \qquad i < j . \tag{2}$$

To have the c_{ij}'s and u_{ij}'s available as needed for the formulas, we follow a certain scheme for obtaining the elements. We compute the elements of the first column of C^{-1}, followed by the elements of the first row of U, then the elements of the second column of C^{-1}, followed by the elements of the second row of U, and so on.

These calculations bring us up to the point where we have the factors C^{-1} and U of A. To obtain A^{-1}, we then have to find the inverses of C^{-1} and U and form their product. That is,

$$A^{-1} = U^{-1}C \ .$$

With the matrices in triangular form we can use Gaussian elimination to produce the U^{-1} and C. Gaussian elimination is particularly easy with a triangular matrix since we need only compute the forward solution on a lower triangular matrix and the backward solution on an upper triangular matrix.

A good example of the compact scheme is the Crout process. To carry out this process, we augment the matrix A with I and proceed with the calculation in two parts called the forward and the backward solutions. Note that since A is n x n, the augmented matrix is n x 2n.

For the forward solution, we calculate the elements p_{ij} of the n x 2n matrix P by using the following formulas.

$$p_{ij} = a_{ij} - \sum_{k=1}^{j-1} p_{kj}p_{ik} \qquad i \geqslant j \tag{3}$$

$$p_{ij} = \frac{a_{ij} - \sum_{k=1}^{i-1} p_{ik}p_{kj}}{p_{ij}} \qquad i < j \ . \tag{4}$$

These formulas look similar to the ones used in computing the c_{ij} and u_{ij}. The first equation is used when $i \geqslant j$. This holds only when p_{ij} is on the diagonal or below it. Formula (3) then gives us the elements corresponding to the c_{ij}. Similarly, formula (4) changes those elements to the right of the diagonal, and if we consider those where $j \leq n$, we have the previously computed u_{ij}'s.

Through the use of the previous formulas, the I part of the augmented matrix $[AI]$ has been changed only in its elements on or below its diagonal. By comparing the following example with that presented during the discussion of Gaussian elimination, it is readily seen that we have changed I to the form obtained by the forward solution in Gaussian elimination. Hence the name forward solution for this part of the Crout process.

To complete the process, we carry out the backward solution and obtain an n x n matrix $D = A^{-1}$. We calculate the d_{ij}'s by rows starting with the last row. The formulas are

$$d_{nj} = p_{n\ n+j} \qquad j = 1 \ldots n$$

$$d_{n-1\ j} = p_{n-1\ n+j} - p_{n-1\ n}d_{nj} \qquad j = 1 \ldots n$$

$$d_{ij} = p_{i\ n+j} - \sum_{k=i+1}^{n} p_{ik}d_{kj} \qquad \text{for } i = n-2, \ldots 2, 1 \quad j = 1 \ldots n .$$

These last formulas actually correspond to computing the backward solution by Gaussian elimination. With the backward solution completed, the I part of the matrix $[AI]$ has been changed to A^{-1}.

Working with the previously used matrix

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 5 \end{bmatrix},$$

we augment with I and obtain

$$[AI] = \begin{bmatrix} 2 & 1 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 5 & 7 & 5 & 0 & 0 & 1 \end{bmatrix}.$$

Carrying out the forward solution, we obtain

$$\begin{bmatrix} 2 & 1/2 & 3/2 & 1/2 & 0 & 0 \\ 4 & 3 & 0 & -2/3 & 1/3 & 0 \\ 5 & 9/2 & -5/2 & -1/5 & 3/5 & -2/5 \end{bmatrix}.$$

The backward solution then yields

$$A^{-1} = \begin{bmatrix} 17/15 & -16/15 & 3/5 \\ -2/3 & 1/3 & 0 \\ -1/5 & 3/5 & -2/5 \end{bmatrix}.$$

Waugh and Dwyer present a method for obtaining the inverse matrix that is similar to the compact schemes. It depends upon the solution of two systems of equations.

Given that $A = C^{-1}U$, where the elements of the C^{-1} and U are computed by formulas (1) and (2), we obtain $A^{-1} = U^{-1}C$. Multiplying on the right by C^{-1}, we obtain $A^{-1}C^{-1} = U^{-1}$. But U^{-1} is an upper triangular matrix with ones on the diagonal and zeros below. We thus know $\frac{n(n + 1)}{2}$ of its elements; $\frac{n(n - 1)}{2}$

will be zeros and n will be ones. Similarly, multiplying $A^{-1} = U^{-1}C$ on the left by U, we obtain $UA^{-1} = C$. This gives $\frac{n(n-1)}{2}$ equations in the elements above the diagonal.

As an example, take the case where $n = 4$. Then $A^{-1}C^{-1} = U^{-1}$ gives the following ten equations, where the a_{ij}' denote the elements of A^{-1}.

$$
\begin{array}{llcccc}
 & i = & 4 & 3 & 2 & 1 \\
c_{44}a_{i4}' & = & 1 & & & \\
c_{43}a_{i4}' + c_{33}a_{i3}' & = & 0 & 1 & & \\
c_{42}a_{i4}' + c_{32}a_{i3}' + c_{22}a_{i2}' & = & 0 & 0 & 1 & \\
c_{41}a_{i4}' + c_{31}a_{i3}' + c_{21}a_{i2}' + c_{11}a_{i1}' & = & 0 & 0 & 0 & 1
\end{array}
$$

Also from $UA^{-1} = C$, six more equations for the case $n = 4$ are:

$$
\begin{array}{llccc}
 & j = & 4 & 3 & 2 \\
u_{34}a_{4j}' + a_{3j}' & = & 0 & & \\
u_{24}a_{4j}' + u_{23}a_{3j}' + a_{2j}' & = & 0 & 0 & \\
u_{14}a_{4j}' + u_{13}a_{3j}' + u_{12}a_{2j}' + a_{1j}' & = & 0 & 0 & 0
\end{array}
$$

From these 16 equations, we can compute the a_{ij}'. We determine from the first set of equations in succession a_{44}', a_{43}', a_{42}', and a_{41}' (the case $i = 4$). Next we compute a_{34}', a_{24}', and a_{14}' (the case $j = 4$). Continuing analagously, using the first and second groups of equations in turn, we obtain a_{31}', a_{32}', and a_{33}', followed by a_{23}', a_{13}', and so on.

Other compact schemes exist such as the Doolittle process and modernized Gauss. They are variations based on the same theory.

PARTITIONING

Two other methods for inverting matrices arise from the concept known as partitioning. These methods consist of breaking up a large matrix into smaller matrices, the inverses of which can be found more easily.

Starting with a matrix A, we partition it in such a manner that we have two square matrices on the diagonal, at least one of which is nonsingular. Let A be partitioned so that

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

Then if B is the inverse of A, it can be partitioned in the same manner. Thus

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$

Forming the product of A and its inverse B, we have

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} I & Z \\ Z & I \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where Z denotes a matrix whose elements are all zeros.

Multiplying the matrices, we obtain the following sets of equations.

$$A_{11}B_{11} + A_{12}B_{21} = I \qquad\qquad B_{11}A_{11} + B_{12}A_{21} = I$$

$$A_{11}B_{12} + A_{12}B_{22} = Z \qquad\qquad B_{11}A_{12} + B_{12}A_{22} = Z$$

$$A_{21}B_{11} + A_{22}B_{21} = Z \qquad B_{21}A_{11} + B_{22}A_{21} = Z$$

$$A_{21}B_{12} + A_{22}B_{22} = I \qquad B_{21}A_{12} + B_{22}A_{22} = I$$

Suppose that we had partitioned A in such a manner that A_{11}^{-1} existed. Then we could solve the equations for values of the B_{ij}.

$$B_{22} = (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \tag{5}$$

$$B_{12} = -A_{11}^{-1}A_{12}B_{22} \tag{6}$$

$$B_{21} = -B_{22}A_{21}A_{11}^{-1} \tag{7}$$

$$B_{11} = A_{11}^{-1} + A_{11}^{-1}A_{12}B_{22}A_{21}A_{11}^{-1} \tag{8}$$

The solution of these formulas will depend upon the solution for B_{22}. This requires finding an inverse of a matrix of the same dimension as A_{22}. If the dimension of B_{22} is large, we may wish to find its inverse by partitioning also. Hence we can reduce the problem of inversion of a large matrix to the problem of finding inverses of smaller matrices. The inverses of these smaller matrices can be computed by again using partitioning or by the use of other techniques.

As a simple example to illustrate the technique, we can again compute the inverse of the matrix

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 5 \end{bmatrix}.$$

Partitioning as follows,

$$A = \left[\begin{array}{cc|c} 2 & 1 & 3 \\ 4 & 5 & 6 \\ \hline 5 & 7 & 5 \end{array}\right]$$

so that

$$A_{11} = \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix},$$

we obtain

$$A_{11}^{-1} = \begin{bmatrix} 5/6 & -1/6 \\ -2/3 & 1/3 \end{bmatrix}.$$

Also

$$A_{12} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \quad A_{21} = [5, \ 7], \text{ and } A_{22} = [5].$$

Computing the B_{ij},

$$B_{22} = \left([5] - [5 \ \ 7] \begin{bmatrix} 5/6 & -1/6 \\ -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right)^{-1} = [-5/2]^{-1} = -\,2/5 ,$$

$$B_{12} = -\begin{bmatrix} 5/6 & -1/6 \\ -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} [-2/5] = \begin{bmatrix} 3/5 \\ 0 \end{bmatrix},$$

$$B_{21} = -\,[-2/5]\,[5 \ \ 7] \begin{bmatrix} 5/6 & -1/6 \\ -2/3 & 1/3 \end{bmatrix} = [-1/5 \ \ 3/5] \ ,$$

$$B_{11} = \begin{bmatrix} 5/6 & -1/6 \\ -2/3 & 1/3 \end{bmatrix} + \begin{bmatrix} 5/6 & -1/6 \\ -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} [-2/5]\,[5, \ 7] \begin{bmatrix} 5/6 & -1/6 \\ -2/3 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 17/15 & -16/15 \\ -2/3 & 1/3 \end{bmatrix} .$$

As a result, we have

$$A^{-1} = B = \begin{bmatrix} B_{11} & B_{12} \\ & \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} 17/15 & -16/15 & 3/5 \\ -2/3 & 1/3 & 0 \\ -1/5 & 3/5 & -2/5 \end{bmatrix}$$

as has been previously verified.

From the method of partitioning, we can develop the method known as bordering. The bordering method starts with a submatrix and successively borders it with another row and column. To find the inverse of a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} ,$$

we first take a_{11} and find its inverse $a_{11}{}^{-1}$. Bordering it with the elements a_{12}, a_{21}, and a_{23}, we can find the inverse of

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} .$$

Having computed this inverse, we can border again and find the inverse of

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} .$$

Continuing the process we eventually find the inverse of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} .$$

The formulas needed at each step of the process can be determined by considering the core of the k x k matrix A. Then let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where A_{11} is a k-1 x k-1 matrix, A_{12} is a k-1 x 1 column vector u_{k-1}, A_{21} is a 1 x k-1 row vector v_{k-1}, and A_{22} is a scalar which we shall denote as $\frac{1}{\alpha_{k-1}}$. The B_{11}, r_{k-1}, q_{k-1}, a_{kk} are respectively the corresponding parts of the inverse matrix B, where B is partitioned in the same manner as A. We have the following equations from $AA^{-1} = I$.

$$A_{11}B_{11} + u_{k-1}\, q_{k-1} = I$$

$$v_{k-1}B_{11} + a_{kk}q_{k-1} = Z$$

$$A_{11}r_{k-1} + \frac{u_{k-1}}{\alpha_{k-1}} = Z$$

$$v_k r_{k-1} + \frac{a_{kk}}{\alpha_{k-1}} = 1 .$$

Solving these equations we obtain

$$\alpha_{k-1} = a_{kk} - v_{k-1}A_{11}^{-1}u_{k-1} \text{ so } \frac{1}{\alpha_{k-1}} = \frac{1}{a_{kk} - v_{k-1}A_{11}^{-1}u_{k-1}} \tag{9}$$

$$q_{k-1} = \frac{-v_{k-1}A_{11}^{-1}}{\alpha_{k-1}} \tag{10}$$

$$r_{k-1} = \frac{-A_{11}^{-1}u_{k-1}}{\alpha_{k-1}} \tag{11}$$

$$B_{11} = A_{11}^{-1} - \frac{A_{11}^{-1}u_{k-1}v_{k-1}A_{11}^{-1}}{\alpha_{k-1}} \quad . \tag{12}$$

Note that these equations are the same as equations (5) - (8), with $B_{22} = \dfrac{1}{\alpha_{k-1}}$, $B_{12} = q_{k-1}$, and $B_{21} = r_{k-1}$. It is evident that to solve these equations it is necessary to find the inverse of A_{11}.

If we let A be the 3 x 3 matrix used previously, then the inverse of a_{11} is 1/2. By bordering, we work with the matrix

$$\begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix} .$$

Thus

$$\alpha_1 = 5 - 4\ (1/2)\ (1) = 3 \ ,$$

$$B_{11} = 1/2 + \frac{(1/2)\ (1)\ (4)\ (1/2)}{3} = 5/6 \ ,$$

$$r_1 = \frac{-(1/2)\ (1)}{3} = -1/6 \ ,$$

$$q_1 = \frac{-4(1/2)}{3} = -2/3 \ .$$

Then the inverse of

$$\begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}$$

is

$$\begin{bmatrix} 5/6 & -1/6 \\ -2/3 & 1/3 \end{bmatrix} .$$

Bordering with the third row and third column, we compute

$$\alpha_2 = 5 - \begin{bmatrix} 5 & 7 \end{bmatrix} \begin{bmatrix} 5/6 & -1/6 \\ -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} = -5/2 ,$$

$$B_{11} = \begin{bmatrix} 5/6 & -1/6 \\ -2/3 & 1/3 \end{bmatrix} + \frac{\begin{bmatrix} 5/6 & -1/6 \\ -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} \begin{bmatrix} 5 & 7 \end{bmatrix} \begin{bmatrix} 5/6 & -1/6 \\ -2/3 & 1/3 \end{bmatrix}}{-5/2}$$

$$= \begin{bmatrix} 17/15 & -16/15 \\ -2/3 & 1/3 \end{bmatrix},$$

$$r_2 = \frac{-\begin{bmatrix} 5/6 & -1/6 \\ -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix}}{-5/2} = \begin{bmatrix} 3/5 \\ 0 \end{bmatrix} ,$$

$$q_2 = \frac{-\begin{bmatrix} 5 & 7 \end{bmatrix} \begin{bmatrix} 5/6 & -1/6 \\ -2/3 & 1/3 \end{bmatrix}}{-5/2} = \begin{bmatrix} -1/5 & 3/5 \end{bmatrix} .$$

Thus

$$A^{-1} = B = \begin{bmatrix} 17/15 & -16/15 & 3/5 \\ -2/3 & 1/3 & 0 \\ -1/5 & 3/5 & -2/5 \end{bmatrix} .$$

Observe that the last step in this example of bordering is the same as the example used for partitioning.

RANK ANNIHILATION

The methods presented thus far will work for most non-singular matrices. For large matrices, however, hand computation is impractical if not impossible. For this reason, the computations of the various methods have to be applicable for use on a computer. A method which in its development was intended for a computer is the method of rank annihilation.

Effectively, what rank annihilation does is to relate the matrix whose inverse is not known to a matrix whose inverse is known. For our purposes we will be relating it to the identity matrix.

From Householder's <u>Principals of Numerical Analysis</u> (p. 79) we have

$$(A + USV^t)^{-1} = A^{-1} - A^{-1}US(S + SV^tA^{-1}US)^{-1}SV^tA^{-1}$$

provided that the indicated inverses exist and that the dimensions are properly matched. The t as a superscript denotes the transpose of the matrix. The A and S are square matrices and U and V are rectangular. To verify that the above formula gives the inverse we must have that $(A + USV^t)(A + USV^t)^{-1} =$

$(A + USV^t) = I$. We shall verify that $(A + USV^t)(A + USV^t)^{-1} = I$. A similar procedure shows that $(A + USV^t)^{-1}(A + USV^t) = I$.

$$(A + USV^t)\ A^{-1} - A^{-1}US(S + SV^tA^{-1}US)^{-1}SV^tA^{-1}$$

$$= I + USV^tA^{-1} - US(S + SV^tA^{-1}US)^{-1}SV^tA^{-1} - USV^tA^{-1}US(S + SV^tA^{-1}US)^{-1}SV^tA^{-1}$$

$$= I + USV^tA^{-1} - U(S + SV^tA^{-1}US)(S + SV^tA^{-1}US)^{-1}SV^tA^{-1}$$

$$= I + USV^tA^{-1} - UISV^tA^{-1} = I + USV^tA^{-1} - USV^tA^{-1}$$

$$= I\ .$$

If we let U and V be column vectors and S be the scalar 1, we obtain

$$(A + UV\ )^{-1} = A^{-1} - (A^{-1}U)(1 + V^tA^{-1}U)^{-1}V^tA^{-1}\ .$$

Householder states that if $1 + V^tA^{-1}U = 0$, then $(A + UV^t)^{-1}$ does not exist. The dimension of $V^tA^{-1}U$ is 1 x 1 since U and V are column vectors. Hence $(1 + V^tA^{-1}U)$ is a nonzero scalar so that the inverse of it is $\dfrac{1}{(1 + V^tA^{-1}U)}$. Now, we can write

$$(A + UV^t)^{-1} = A^{-1} - \frac{(A^{-1}U)(V^tA^{-1})}{(1 + V^tA^{-1}U)}\ .$$

If A is an n x n matrix, we write $A = D + \sum_{i=1}^{n} U_iV_i{}^t$, where D is a matrix whose inverse is known. The U_i and V_i also can be determined to satisfy the particular problem.

Set up then a sequence of matrices $\{C_i\}$ so that

$$C_0 = D$$

$$C_1 = D + U_1V_1^t = C_0 + U_1V_1^t$$

$$C_2 = D + U_1V_1^t + U_2V_2^t = C_1 + U_2V_2^t$$

$$\vdots$$

$$C_n = D + \sum U_iV_i^t = C_{n-1} + U_nV_n^t = A \ .$$

By using the formula

$$(A + UV^t)^{-1} = A^{-1} - \frac{(A^{-1}U)(V^tA^{-1})}{(1 + V^tA^{-1}U)} \quad ,$$

we can compute at each step the inverse of C_i. This gives a sequence of matrices $\{C_i^{-1}\}$ with $C_n^{-1} = A^{-1}$.

The expansion $A = D + \sum_{i=1}^{n} U_iV_i^t$ is clearly not unique. A simple form of the expression would be to let $D = I$. If we then let $U = A - I$, this U can be partitioned by columns so that $U = \left[U_1U_2 \ . \ . \ . \ U_n\right]$. Similarly, let V_i be the i^{th} column vector of the identity matrix. Then $A = I + \sum_{i=1}^{n} U_iV_i^t$.

This form actually involves adding (at the i^{th} step) the i^{th} column of the matrix $A - I$.

Again working with our example matrix:

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 5 \end{bmatrix} .$$

Then

$$U = A - I = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 4 & 4 & 6 \\ 5 & 7 & 4 \end{bmatrix}$$

so that

$$U_1 = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \text{ and } U_3 = \begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix} .$$

Thus

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} .$$

Then

$$C_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ so that } C_0^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} .$$

So

$$C_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}{1 + \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}}$$

$$= \begin{bmatrix} 1/2 & 0 & 0 \\ -2 & 1 & 0 \\ -5/2 & 0 & 1 \end{bmatrix} .$$

Similarly,

$$C_2 = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 5 & 0 \\ 5 & 7 & 1 \end{bmatrix} .$$

$$C_2^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ -2 & 1 & 0 \\ -5/2 & 0 & 1 \end{bmatrix} - \frac{\begin{bmatrix} 1/2 & 0 & 0 \\ -2 & 1 & 0 \\ -5/2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1/2 & 0 & 0 \\ -2 & 1 & 0 \\ -5/2 & 0 & 1 \end{bmatrix}}{1 + \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1/2 & 0 & 0 \\ -2 & 1 & 0 \\ -5/2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}}$$

$$= \begin{bmatrix} 5/6 & -1/6 & 0 \\ -2/3 & 1/3 & 0 \\ 1/2 & -3/2 & 1 \end{bmatrix} .$$

Finally we will get

$$C_3 = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 5 \end{bmatrix} = A$$

and

$$C_3^{-1} = \begin{bmatrix} 17/15 & -16/15 & 3/5 \\ -2/3 & 1/3 & 0 \\ -1/5 & 3/5 & -2/5 \end{bmatrix} = A^{-1} .$$

The disadvantage in this method is that it fails if any C_i is singular. This occurs if and only if one of the principal submatrices of A is singular.

L-F METHOD

The last means of inverting matrices to be discussed is the keverrier-Faddeev method.[1] The L-F method is a byproduct of a technique of finding the coefficients of the characteristic polynomial of a matrix. The method is very convenient for finding inverses of matrices of small order. For $n > 4$, computation becomes quite lengthy.

To present a development of the method requires an approach involving the characteristic polynomial of a matrix. Consequently we shall just present the technique. In the discussion we will use trA to denote the trace of the matrix A (the sum of the elements on the diagonal).

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[1]Faddeev, D. K. and Faddeeva, V. N., Computational Methods of Linear Algebra, San Francisco, W. H. Freeman and Company, pp. 260-265.

We shall first compute a series of matrices A_k and B_k where

$$
\begin{array}{lll}
A_1 = A & q_1 = \mathrm{tr}A_1 & B_1 = A_1 - q_1 I \\
A_2 = AB_1 & 2q_2 = \mathrm{tr}A_2 & B_2 = A_2 - q_2 I \\
\vdots & \vdots & \vdots \\
A_k = AB_{k-1} & kq_k = \mathrm{tr}A_k & B_k = A_k - q_k I \\
\vdots & \vdots & \vdots \\
A_n = AB_{n-1} & nq_n = \mathrm{tr}A_n & B_n = A_n - q_n I \ .
\end{array}
$$

Note that k ranges from 1 to n where n is the rank of the matrix A. By use of the Cayley-Hamilton theorem it can be shown that $B_n = Z$ so that $A_n - q_n I = Z$, or $\frac{1}{q_n} A_n = I$. Since $A_n = AB_{n-1}$, we have

$$\frac{1}{q_n} AB_{n-1} = I, \text{ and thus } A^{-1} = \frac{1}{q_n} B_{n-1} \ .$$

For $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 5 \end{bmatrix}$, $q_1 = 12$

so

$$B_1 = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 5 \end{bmatrix} - 12 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -10 & 1 & 3 \\ 4 & -7 & 6 \\ 5 & 7 & -7 \end{bmatrix}$$

Then

$$A_2 = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 5 \end{bmatrix} \begin{bmatrix} -10 & 1 & 3 \\ 4 & -7 & 6 \\ 5 & 7 & -7 \end{bmatrix} = \begin{bmatrix} -1 & 16 & -9 \\ 10 & 11 & 0 \\ 3 & -9 & 22 \end{bmatrix}$$

$$q_2 = \frac{32}{2} = 16$$

$$B_2 = \begin{bmatrix} -1 & 16 & -9 \\ 10 & 11 & 0 \\ 3 & -9 & 22 \end{bmatrix} - 16 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -17 & 16 & -9 \\ 10 & -5 & 0 \\ 3 & -9 & 6 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 5 \end{bmatrix} \begin{bmatrix} -17 & 16 & -9 \\ 10 & -5 & 0 \\ 3 & -9 & 6 \end{bmatrix} = \begin{bmatrix} -15 & 0 & 0 \\ 0 & -15 & 0 \\ 0 & 0 & -15 \end{bmatrix}$$

$$q_3 = \frac{-45}{3} = -15$$

Note that for A_n it is only necessary to compute the elements on the diagonal. The other elements can be computed as a check. Now

$$A^{-1} = \frac{1}{q_3} B_2 = \frac{1}{-15} \begin{bmatrix} -17 & 16 & -9 \\ 10 & -5 & 0 \\ 3 & -9 & 6 \end{bmatrix} = \begin{bmatrix} 17/15 & -16/15 & 3/5 \\ -2/3 & 1/3 & 0 \\ -1/5 & 3/5 & -2/5 \end{bmatrix} .$$

CONCLUSION

Of the various methods of matrix inversion presented, none of them is particularly applicable for all cases. For hand computation each becomes rather lengthy when we have a matrix of dimension greater than four. For matrices of small dimension, Gaussian elimination and the L-F method make hand computation of the inverses possible and efficient. If one has access to a calculator, then the compact schemes are particularly useful and allow computation of inverses of matrices of somewhat larger order. The method of rank annihilation was developed for use on a computer, and thus is applicable to matrices of large dimension.

Partitioning gives us a scheme or technique to use in breaking large matrices into matrices of a more manageable size. This can make it possible to use hand computation on matrices of higher degree than one would normally care to handle in this manner.

This paper did not present all the methods of matrix inversion. There are, for example, many others that fall in the category of compact schemes. Methods have also been developed for particular types of matrices such as the symmetric matrix. Basically, though, these other methods will be found to be variations of the four types presented.

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METHODS OF MATRIX INVERSION

by

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In this paper various methods of inverting matrices are considered. For our purposes we grouped the many methods of matrix inversion into four basic types: Gaussian elimination, partitioning, rank annihilation, and the L-F method.

In considering the first type, Gaussian elimination, we showed the relation of matrix inversion to the solution of a system of equations. Then we proceeded to show how the Gaussian elimination technique could be applied to the inversion problem. From the elimination technique it was possible to derive various other matrix inversion schemes known as compact schemes.

The second basic type, partitioning, was then presented. This type consists of breaking up a large matrix into smaller matrices, the inverses of which can be more readily found. The theory behind the method was discussed and a special case of partitioning, known as bordering, was presented.

The method of rank annihilation was then discussed. Effectively what rank annihilation does is to relate a matrix whose inverse is not known to a matrix whose inverse is known. This is done through a series of changes to the matrix whose inverse we know. The theory behind the process is discussed and then the method for inverting the matrix is presented.

Last of all, we discussed briefly the L-F method. The method involves computing a series of matrices and then using a relation of the matrices in this series to produce the inverse. Because the theory is based on the concept of the characteristic polynomial of a matrix, it was not possible to give a development of the theory. Instead, just the technique used in the L-F method was presented.

In conclusion, we compared the merits of the methods and gave an idea of when each was particularly applicable.