A nonlocal Laplacian in one dimension

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Abstract

An overview on nonlocal Laplace operators acting on real-valued one-dimensional functions is presented. We provide a definition for nonlocal Laplace operators and present some basic examples. In addition, we show that, for vanishing nonlocality, the nonlocal Laplacian of a sufficiently differentiable one-dimensional function approaches the second derivative of the function. Moreover, we compute the Fourier multipliers of the nonlocal Laplacian and show that these multipliers converge to the multipliers of the Laplacian in the limit of vanishing nonlocality. Furthermore, we consider a nonlocal diffusion equation and provide an integral representation for its solution in terms of the Fourier multipliers of the nonlocal Laplacian.

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Introduction

A nonlocal Laplacian is an integral operator of the form

$$L_\gamma u(x) = \int \gamma(x,y)(u(y) - u(x))\, dy, \quad x \in \mathbb{R}^n,$$

where γ is a symmetric kernel $\gamma(x,y) = \gamma(y,x)$[6]. Such operators have been studied in the context of nonlocal diffusion, digital image correlation, and nonlocal wave phenomena among other applications, see for example[3;4;7;9]. Several mathematical and numerical studies have focused on nonlocal Laplace operators including[1;2;5;8].

In this report, we focus on nonlocal Laplace operators acting on real-valued one-dimensional functions. In Chapter 1, we define a nonlocal Laplacian with a compactly supported kernel γ and present some basic examples. In addition, we show a connection between the nonlocal Laplacian in one-dimension and the second derivative. In Chapter 2, we review some basic properties of the Fourier transform and provide examples. In Chapter 3, we present the Fourier multipliers for the nonlocal Laplacian and show that these multipliers converge to the multipliers of the Laplacian in the limit of vanishing nonlocality. Moreover, we consider a nonlocal version of the heat equation and present its solution in terms of the Fourier multipliers and the Fourier transform.

Chapter 1

A Nonlocal Laplace Operator

Definition 1.1. *Let $\delta > 0$ and $\beta < 1$. The nonlocal Laplacian is defined by*

$$L_\delta u(x) = c_\delta \int_{-\delta+x}^{\delta+x} \frac{u(y) - u(x)}{|y-x|^\beta}\, dy,\ x \in \mathbb{R}, \tag{1.1}$$

where $u : \mathbb{R} \to \mathbb{R}$. The constant c_δ is a scaling factor which is given by

$$c_\delta := \frac{3-\beta}{\delta^{3-\beta}}. \tag{1.2}$$

By a change of variables $z = y - x$, the nonlocal Laplacian in (1.1) can be written as,

$$L_\delta u(x) = c_\delta \int_{-\delta}^{\delta} \frac{u(x+z) - u(x)}{|z|^\beta}\, dz. \tag{1.3}$$

We note that for $\beta < 1$, the integral kernel

$$\chi_{[-\delta,\delta]}(z) \frac{1}{|z|^\beta}$$

is integrable. We also note that due to symmetry

$$\int_{-\delta}^{\delta} \frac{g(z)}{|z|^\alpha} = 0, \tag{1.4}$$

for any odd function g.

1.1 Examples

In this section, we compute the nonlocal Laplacian of some basic functions.

1.1.1 Polynomial examples

Let $u(x) = k$, where k is a constant. Then it is straightforward to see that $L_\delta u = 0$.

Next, we consider $u(x)$ be a cubic polynomial. Then $L_\delta u(x) = u''(x)$. To see this, let $u(x) = a_3x^3 + a_2x^2 + a_1x + a_0$. Using (1.3), we note that

$$\begin{aligned}
L_\delta u(x) &= c_\delta \int_{-\delta}^{\delta} \frac{[a_3(x+z)^3 + a_2(x+z)^2 + a_1(x+z) + a_0] - [a_3x^3 + a_2x^2 + a_1x + a_0]}{|z|^\beta}\, dz, \\
&= c_\delta \int_{-\delta}^{\delta} \frac{3a_3xz^2 + 3a_3x^2z + a_3z^3 + 2a_2xz + a_2z^2 + a_1z}{|z|^\beta}\, dz \\
&= c_\delta \int_{-\delta}^{\delta} \left(\frac{3a_3xz^2}{|z|^\beta} + \frac{a_2z^2}{|z|^\beta} \right) dz,
\end{aligned}$$

where in the last equation we used (1.4). Therefore, and by using (1.2), we obtain

$$\begin{aligned}
L_\delta u(x) &= c_\delta \int_{-\delta}^{\delta} (3a_3x + a_2) \frac{z^2}{|z|^\beta}\, dz \\
&= 2\, c_\delta\, (3a_3x + a_2) \int_0^\delta z^{2-\beta}\, dz \\
&= 2\, c_\delta\, (3a_3x + a_2) \frac{\delta^{3-\beta}}{3-\beta} \\
&= 6a_3x + 2a_2 = u''(x).
\end{aligned}$$

Now we consider a quartic polynomial, $p_4(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$. Similar to the calculation for the cubic polynomial above, we obtain

$$\begin{aligned}
L_\delta p_4(x) &= 12a_4x^2 + 6a_3x + 2a_2 + 2\frac{\beta-3}{\beta-5}\delta^2 a_4 \\
&= p_4''(x) + 2\frac{\beta-3}{\beta-5}\delta^2 a_4.
\end{aligned}$$

We observe that

$$\lim_{\delta\to 0^+} L_\delta p_4 = p_4''.$$

Another example of polynomials, we consider $p_5(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$. Performing a calculation similar to that for the cubic polynomial above, we obtain,

$$L_\delta p_5(x) = 20a_5x^3 + 12a_4x^2 + 6a_3x + 2a_2 + 10\frac{\beta-3}{\beta-5}\delta^2 a_5 x + 2\frac{\beta-3}{\beta-5}\delta^2 a_4.$$

Again we observe that

$$\lim_{\delta\to 0^+} L_\delta p_5 = p_5''.$$

1.1.2 A trigonometric function

Let $g(x) = \sin(x)$. Using (1.3) we obtain,

$$\begin{aligned} L_\delta g(x) &= c_\delta \int_{-\delta}^{\delta} \frac{\sin(x+z) - \sin(x)}{|z|^\beta}\, dz \\ &= c_\delta \int_{-\delta}^{\delta} \frac{\sin(x)\cos(z) + \cos(x)\sin(z) - \sin(x)}{|z|^\beta}\, dz \\ &= c_\delta \int_{-\delta}^{\delta} \frac{\cos(z)-1}{|z|^\beta}\, dz\ \sin(x) + c_\delta \int_{-\delta}^{\delta} \frac{\sin(z)}{|z|^\beta}\, dz\ \cos(x). \end{aligned}$$

Due to symmetry as shown in (1.4) we obtain

$$c_\delta \int_{-\delta}^{\delta} \frac{\cos(z)-1}{|z|^\beta}\, dz\ \sin(x) + c_\delta \int_{-\delta}^{\delta} \frac{\sin(z)}{|z|^\beta}\, dz\ \cos(x) = \int_{-\delta}^{\delta} \frac{\cos(z)-1}{|z|^\beta}\, dz\ \sin(x). \tag{1.5}$$

Using the series expansion of cosine

$$\cos(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!},$$

we observe that

$$c_\delta \int_{-\delta}^{\delta} \frac{\cos(z)-1}{|z|^\beta}\, dz = \left(-c_\delta \int_{-\delta}^{\delta} \frac{z^2}{2!\,|z|^\beta} dz + c_\delta \int_{-\delta}^{\delta} \frac{z^4}{4!\,|z|^\beta} dz - c_\delta \int_{-\delta}^{\delta} \frac{z^6}{6!\,|z|^\beta} dz + \cdots\right)$$

$$= \left(-1 + \frac{2}{4!}\frac{3-\beta}{5-\beta}\delta^2 - \frac{2}{6!}\frac{3-\beta}{7-\beta}\delta^4 + \cdots\right). \tag{1.6}$$

Using (1.5) and (1.6) we obtain

$$\begin{aligned} L_\delta \sin(x) &= -\sin(x) + \left(\frac{2}{4!}\, \frac{3-\beta}{5-\beta}\delta^2 - \frac{2}{6!}\, \frac{3-\beta}{7-\beta}\delta^4 + \cdots \right) \sin(x) \\ &= \frac{d^2}{dx^2} \sin(x) + \left(\frac{2}{4!}\, \frac{3-\beta}{5-\beta}\delta^2 - \frac{2}{6!}\, \frac{3-\beta}{7-\beta}\delta^4 + \cdots \right) \sin(x). \end{aligned}$$

Again, we note that

$$\lim_{\delta \to 0^+} L_\delta \sin(x) = \frac{d^2}{dx^2} \sin(x).$$

1.1.3 A discontinuous function

Consider the Heaviside function

$$H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0\ , \end{cases}$$

which has a jump discontinuity at $x = 0$. Applying (1.3) we obtain

$$\begin{aligned} L_\delta H(x) &= c_\delta \int_{-\delta}^{\delta} \frac{H(x+z) - H(x)}{|z|^\beta}\, dz \\ &= \frac{3-\beta}{\delta^{3-\beta}} \frac{1}{1-\beta\delta^\beta} \begin{cases} \left(\delta + \delta^\beta (-1)^\beta x^{\beta-1}\right), & -\delta \leq x < 0 \\ \left(\delta^\beta x^{1-\beta} - \delta\right), & 0 \leq x \leq \delta \\ 0, & \text{otherwise} \end{cases} . \end{aligned}$$

Figure 1.1 shows the graph of $L_\delta u(x)$ for $\delta = 1$ and $\beta = \frac{1}{2}$.

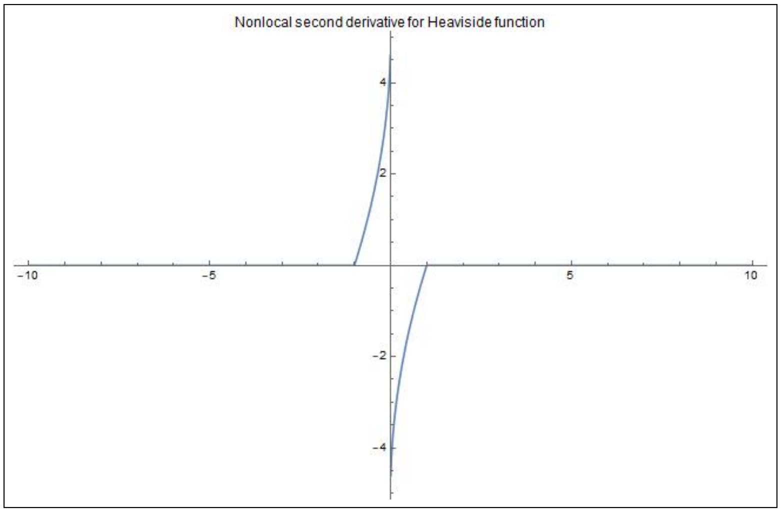

Figure 1.1: Nonlocal second derivative for the Heaviside function

It is interesting to observe that as $\delta \to 0^+$ $L_\delta H$ converges to the distributional second derivative of the Heaviside function $H''(x)$, for example see[2].

1.2 Convergence of the nonlocal Laplacian

In this section, we show that the nonlocal Laplacian of a sufficiently differentiable function converges to the Laplacian of the function in the limit of vanishing nonlocality.

Theorem 1.1. *Let* $\beta < 1, u \in C^3(\mathbb{R}),$ *and* $x \in \mathbb{R}$. *Then*

$$\lim_{\delta \to 0^+} L_\delta u(x) = u''(x).$$

Proof. Using Taylor's theorem, we expand u about $z = x$,

$$u(x+z) = u(x) + u'(x)z + \frac{1}{2}u''(x)z^2 + r(x+z), \tag{1.7}$$

where

$$r(x+z) = \frac{1}{3!}u'''(x+sz)z^3, \tag{1.8}$$

for some $s \in [0,1]$. Substituting (1.7) and (1.8) in (1.3), we obtain

$$L_\delta u(x) = c_\delta \int_{-\delta}^{\delta} \frac{z}{|z|^\beta}\, dz\; u'(x) + c_\delta \int_{-\delta}^{\delta} \frac{z^2}{|z|^\beta}\, dz\; \frac{1}{2}u''(x) + c_\delta \int_{-\delta}^{\delta} \frac{z^3}{|z|^\beta}\, dz\; \frac{1}{3!}u'''(x). \tag{1.9}$$

Using (1.4), the first integral above vanishes and by applying (1.2), we see that

$$c_\delta \int_{-\delta}^{\delta} \frac{z^2}{2|z|^\beta}\, dz = 1. \tag{1.10}$$

Therefore,

$$|Lu(x) - u''(x)| = \left| c_\delta \int_{-\delta}^{\delta} \frac{z^3}{|z|^\beta} \frac{1}{3!} u'''(x)\, dz \right| \leq c_\delta \int_{-\delta}^{\delta} |z|^{3-\beta}\, dz\; H_x,$$

where,

$$H_x = \max_{\substack{|z|\leq\delta \\ s\in[0,1]}} \frac{1}{3!} \left| u'''(x+sz) \right|.$$

Since

$$\int_{-\delta}^{\delta} |z|^{3-\beta}\, dz = 2\frac{\delta^{4-\beta}}{4-\beta},$$

we obtain

$$|L_\delta u(x) - u''(x)| \leq \frac{2(3-\beta)}{4-\beta} \delta\; H_x. \tag{1.11}$$

By taking the limit in (1.11) as $\delta \to 0^+$, the result follows. □

Chapter 2

Fourier Transform

In this chapter, we review some basic properties of the Fourier transform. The Fourier transform is given by

$$\hat{f}(w) = \int_{-\infty}^{+\infty} f(x)e^{-i\omega x}\, dx,$$

and the inverse Fourier Transform is given by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(\omega)e^{i\omega x}\, d\omega.$$

2.1 Properties of Fourier transform

(i) Linearity

$$\mathcal{F}(af + bg)(\omega) = a\ \hat{f}(\omega) + b\ \hat{g}(\omega).$$

(ii) Shift in phase domain

$$\mathcal{F}(f(x-a))(\omega) = e^{-ia\omega}\hat{f}(\omega).$$

(iii) Shift in frequency domain

$$\mathcal{F}(f(x)\ e^{iax})(\omega) = \hat{f}(\omega - a).$$

(iv) Phase Scaling

$$\mathcal{F}(f(ax))(\omega) = \frac{1}{|a|}\hat{f}\left(\frac{\omega}{a}\right).$$

(v) Frequency derivative

$$\mathcal{F}(x^n f)(\omega) = i^n \frac{d^n}{d\omega^n}\hat{f}(\omega).$$

(vi) Convolution property

$$\mathcal{F}(f * g)(\omega) = \hat{f}(\omega)\hat{g}(\omega).$$

(vii) Modulation property

$$\mathcal{F}(f\ g)(\omega) = \frac{1}{2\pi}(\hat{f} * \hat{g})(\omega).$$

Proof. (i)

$$\begin{aligned}
\mathcal{F}(f)(\omega) = \hat{f}(\omega) &= \int_{-\infty}^{+\infty} f(x)e^{-i\omega x}\, dx \\
\mathcal{F}(af + bg)(\omega) &= \int_{-\infty}^{+\infty} \left[af(x) + bg(x)\right] e^{-i\omega x}\, dx \\
&= a\int_{-\infty}^{+\infty} f(x)e^{-i\omega x}\, dx + b\int_{-\infty}^{+\infty} g(x)e^{-i\omega x}\, dx \\
&= a\hat{f}(\omega) + b\hat{g}(\omega).
\end{aligned}$$

(ii)

$$\begin{aligned}
\mathcal{F}(f)(\omega) = \hat{f}(\omega) &= \int_{-\infty}^{+\infty} f(x)e^{-i\omega x}\, dx \\
\mathcal{F}[f(x-a)] &= \int_{-\infty}^{+\infty} f(x-a)e^{-i\omega x}\, dx
\end{aligned}$$

Let $x' = x - a \to dx = dx'$. Therefore,

$$\begin{aligned}
\mathcal{F}[f(x-a)] &= \int_{-\infty}^{+\infty} f(x')e^{-i\omega(x'+a)}\ dx' \\
&= \int_{-\infty}^{+\infty} f(x')e^{-i\omega x'}e^{-i\omega a}\ dx' \\
&= e^{-i\omega a}\int_{-\infty}^{+\infty} f(x')e^{=i\omega x'}\ dx' \\
&= e^{-i\omega a}\hat{f}(\omega).
\end{aligned}$$

(iii)

$$\begin{aligned}
\mathcal{F}(f)(\omega) = \hat{f}(\omega) &= \int_{-\infty}^{+\infty} f(x)e^{-i\omega x}\ dx \\
\mathcal{F}[f(x)\ e^{-i\omega x}] &= \int_{-\infty}^{+\infty} f(x)e^{i\omega x}e^{-i\omega x}\ dx \\
&= \int_{-\infty}^{+\infty} f(x)e^{-i(\omega-a)x}\ dx \\
&= \hat{f}(\omega - a).
\end{aligned}$$

(iv) Consider

$$\mathcal{F}[f(ax)] = \int_{-\infty}^{+\infty} f(ax)e^{-i\omega x}\ dx.$$

Let $x' = ax \to x = \frac{x'}{a} \to dx = \frac{1}{a}\ dx'$. Then

$$\begin{aligned}
\mathcal{F}[f(ax)] &= \int_{-\infty}^{+\infty} f(x')e^{-i\omega(\frac{x'}{a})}\frac{1}{|a|}\ dx' \\
&= \frac{1}{|a|}\int_{-\infty}^{+\infty} f(x')e^{-i\frac{\omega}{a}x'}\ dx' \\
&= \frac{1}{|a|}\ \hat{f}\left(\frac{\omega}{a}\right).
\end{aligned}$$

(v)

$$
\begin{aligned}
\mathcal{F}(f)(\omega)=\hat{f}(\omega) &= \int_{-\infty}^{+\infty} f(x) \quad e^{-i\omega x}\, dx \\
\frac{d}{d\omega}\hat{f}(\omega) &= \int_{-\infty}^{+\infty} \quad \left[\frac{d}{d\omega}e^{-i\omega x}\right]\, dx \\
\frac{d}{d\omega}\hat{f}(\omega) &= \int_{-\infty}^{+\infty} f(x)(-ix)e^{-i\omega x}\, dx \\
\frac{d}{d\omega}\hat{f}(\omega) &= -i\int_{-\infty}^{+\infty} xf(x)e^{-i\omega x}\, dx \\
i\,\frac{d}{d\omega}\hat{f}(\omega) &= \int_{-\infty}^{+\infty} xf(x)e^{-i\omega x}\, dx
\end{aligned}
$$

Since $\mathcal{F}(f)(\omega)=\hat{f}(\omega)$ and $\mathcal{F}(xf)(\omega)=i\frac{d}{d\omega}\hat{f}(\omega)$, then $\mathcal{F}(x^n f)(\omega)=i^n\frac{d^n}{d\omega^n}\hat{f}(\omega)$.

(vi) Since the convolution of two functions f and g is

$$
(f*g)(x)=\int_{-\infty}^{+\infty} f(p)g(x-p)\, dp,
$$

then by applying the Fourier transform, we obtain

$$
\begin{aligned}
\mathcal{F}[(f*g)](\omega) &= \int_{-\infty}^{+\infty}(f*g)(x)e^{-i\omega x}\, dx \\
&= \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} f(p)g(x-p)\, dp\right] e^{-i\omega x}\, dx
\end{aligned}
$$

Switching the order of integration, we obtain

$$
\mathcal{F}[(f*g)](\omega)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \quad f(p)g(x-p)e^{-i\omega x}\, dx\, dp \tag{2.1}
$$

Let $q=x-p\rightarrow dq=dx$ and $x=p+q$, thus (2.1) becomes

$$
\begin{aligned}
\mathcal{F}[f*g)(x)] &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(p)g(q)e^{-i\omega(p+q)}\, dq\, dp \\
&= \int_{-\infty}^{+\infty} f(p)e^{-i\omega p}\, dp\int_{-\infty}^{+\infty} g(q)e^{-i\omega q}\, dq \\
&= \hat{f}(\omega)\ \hat{g}(\omega).
\end{aligned}
$$

(vii) From

$$\mathcal{F}[f\ g](\omega) \;=\; \int_{-\infty}^{+\infty} f(x)g(x)e^{-i\omega x}\,dx,$$

and the inverse Fourier transform,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(\omega)e^{i\omega x}\,d\omega,$$

we obtain

$$\begin{aligned}\mathcal{F}[f\ g](\omega) &= \int_{-\infty}^{+\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{f}(\lambda)e^{i\lambda x}d\lambda\right]g(x)e^{-i\omega x}\,dx\\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{f}(\lambda)\left[\int_{-\infty}^{+\infty} g(x)e^{-i(\omega-\lambda)x}\,dx\right]\,d\lambda.\end{aligned}$$

From the definition of Fourier transform, we have

$$\int_{-\infty}^{+\infty} g(x)\,e^{-i(\omega-\lambda)x}\,dx \;=\; \hat{g}(\omega-\lambda),$$

and thus

$$\begin{aligned}\mathcal{F}[f\ g](\omega) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{f}(\lambda)\hat{g}(\omega-\lambda)\,d\lambda\\ &= \frac{1}{2\pi}\quad(\hat{f}*\hat{g})(\omega).\end{aligned}$$

□

2.2 Examples of Fourier transform

In this section, we consider some basic examples of Fourier transform.

2.2.1 Rectangular function

The rectangular and sinc functions are defined as

$$\operatorname{rect}(x) = \begin{cases} 1, & -0.5 < x < 0.5 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

The Fourier transform of the rectangular function is

$$\mathcal{F}(\operatorname{rect}(ax))(\omega) = \frac{1}{|a|}\operatorname{sinc}\left(\frac{\omega}{2\pi a}\right). \tag{2.2}$$

Proof. (2.2) Let $f(x) = \operatorname{rect}(x)$. Then,

$$\begin{aligned}
\mathcal{F}[f(x)] &= \int_{-\infty}^{+\infty} f(x)e^{-i\omega x}\,dx \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} 1e^{-i\omega x}\,dx \\
&= \left.\frac{(e^{-i\omega x})}{-i\,\omega}\right|_{-\frac{1}{2}}^{+\frac{1}{2}} \\
&= -\frac{1}{i\,\omega}\left(e^{-i\frac{\omega}{2}} - e^{i\frac{\omega}{2}}\right) \\
&= \frac{2}{\omega}\left(\frac{e^{i\frac{\omega}{2}} - e^{-i\frac{\omega}{2}}}{2i}\right) \\
&= \frac{2}{\omega}\left(\sin\left(\frac{\omega}{2}\right)\right) \\
&= \operatorname{sinc}\left(\frac{\omega}{2\pi}\right)
\end{aligned}$$

Using property (iv), $\mathcal{F}(f(ax))(\omega) = \frac{1}{|a|}\hat{f}(\frac{\omega}{a})$, thus

$$\mathcal{F}(\operatorname{rect}(ax))(\omega) = \frac{1}{|a|}\operatorname{sinc}\left(\frac{\omega}{2\pi a}\right).$$

□

2.2.2 Gaussian function

The Fourier transform of a Gaussian function is a Gaussian function as

$$\mathcal{F}(e^{-\alpha x^2})(\omega) \quad = \sqrt{\frac{\pi}{\alpha}}.\, e^{\frac{-\omega^2}{4\alpha}}. \tag{2.3}$$

Proof. (2.3)

$$\begin{aligned}
\hat{f}(\omega) &= \int_{-\infty}^{+\infty} f(x)e^{-i\omega x}\, dx \\
&= \int_{-\infty}^{+\infty} e^{-i\omega x}e^{-\alpha x^2}\, dx \\
&= \int_{-\infty}^{+\infty} e^{-(\alpha x^2+i\omega x)}\, dx \\
&= \int_{-\infty}^{+\infty} e^{-\alpha(x^2+\frac{i\omega x}{\alpha})} dx \\
&= \int_{-\infty}^{+\infty} e^{-\alpha[(x^2+\frac{i\omega}{2\alpha})^2+\frac{\omega^2}{4\alpha^2}]}\, dx \\
&= \int_{-\infty}^{+\infty} e^{-\alpha(x+\frac{i\omega}{2\alpha})^2} e^{\frac{-\omega^2}{4\alpha}}\, dx.
\end{aligned}$$

Changing variables $y = x + \frac{i\omega}{2\alpha}$, we obtain

$$\begin{aligned}
\hat{f}(\omega) &= e^{\frac{-\omega^2}{4\alpha}} \int_{-\infty}^{+\infty} e^{-\alpha(x+\frac{i\omega}{2\alpha})^2}\, dx \\
&= e^{\frac{-\omega}{4\alpha}} \int_{-\infty}^{+\infty} e^{-\alpha y^2}\, dy \\
&= \sqrt{\frac{\pi}{\alpha}}.\, e^{\frac{-\omega^2}{4\alpha}},
\end{aligned}$$

where in the last equation we used the fact that

$$\int_{-\infty}^{+\infty} e^{-\alpha y^2}\, dy = \sqrt{\frac{\pi}{\alpha}}.$$

□

Chapter 3

Fourier Multipliers and Nonlocal Equations

In this chapter, we follow the work in[2] that presented the Fourier multipliers approach for studying nonlocal equations.

3.1 Fourier multipliers

Definition 3.1. *For a linear operator L, the Fourier multiplier is a function m such that*

$$\mathcal{F}(Lu) = m\mathcal{F}(u),$$

for any function u in the domain of L.

For example, let $L = \frac{d^2}{dx^2}$. Then $m(\xi) = -|\xi|^2$, for any $\xi \in \mathbb{R}$.

Lemma 3.1. *The Fourier multipliers for the nonlocal Laplacian in (1.3) is given by*

$$m_\delta(\xi) = c_\delta \int_{-\delta}^{\delta} \frac{\cos(\xi z) - 1}{|z|^\beta}\, dz, \quad \xi \in \mathbb{R}. \tag{3.1}$$

Proof. We compute the Fourier transform of $L_\delta u$,

$$\begin{aligned}\mathcal{F}(L_\delta u)(\xi) &= \int_{-\infty}^{+\infty} c_\delta \int_{-\delta}^{\delta} \frac{u(x+z)-u(x)}{|z|^\beta}\, dz\, e^{-ix\xi}\, dx, \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} c_\delta\, \frac{1}{|z|^\beta}\, \chi_{[-\delta,\delta]}(z) \Big(u(x+z)-u(x)\Big) e^{-ix\xi}\, dz\, dx.\end{aligned}$$

Applying Fubini's theorem, we obtain

$$\mathcal{F}(L_\delta u) = \int_{-\infty}^{+\infty} c_\delta\, \frac{1}{|z|^\beta}\, \chi_{[-\delta,\delta]}(z) \left[\int_{-\infty}^{+\infty} u(x+z) e^{-ix\xi}\, dx - \int_{-\infty}^{+\infty} u(x) e^{-ix\xi}\, dx\right] dz$$

$$= \int_{-\infty}^{+\infty} c_\delta\, \frac{1}{|z|^\beta}\, \chi_{[-\delta,\delta]}(z) \left(e^{i\xi z}\, \hat{u}(\xi) - \hat{u}(\xi)\right)\, dz, \tag{3.2}$$

where in the last equation we used the shift property of the Fourier transform. Thus (3.2) becomes

$$\mathcal{F}(L_\delta u)(\xi) = c_\delta \int_{-\delta}^{\delta} \frac{e^{i\xi z}-1}{|z|^\beta}\, dz\, \hat{u}(\xi).$$

Therefore, the Fourier multiplier of L_δ is given by

$$\begin{aligned}m_\delta(\xi) &= c_\delta \int_{-\delta}^{\delta} \frac{e^{i\xi z}-1}{|z|^\beta}\, dz \\ &= c_\delta \int_{-\delta}^{\delta} \frac{\cos(\xi z)-1}{|z|^\beta}\, dz + \; i\, c_\delta \int_{-\delta}^{\delta} \frac{\sin(\xi z)}{|z|^\beta}\, dz.\end{aligned}$$

Using (1.4), the second integral above vanishes, which implies that

$$m_\delta(\xi) = c_\delta \int_{-\delta}^{\delta} \frac{\cos(\xi z)-1}{|z|^\beta}\, dz.$$

□

The multipliers of the nonlocal Laplacian converge to the multipliers of the Laplacian. This is given in the following result.

Theorem 3.1. *Let $\xi \in \mathbb{R}$. Then*

$$\lim_{\delta\to 0^+} m_\delta(\xi) = -|\xi|^2.$$

Proof. Using Taylor's theorem, there exist $\alpha_{z,\xi}$ with $\alpha_{z,\xi} \le |z\xi|$ such that

$$\cos(\xi z) - 1 = -\frac{(\xi z)^2}{2!} + \sin(\alpha_{z,\xi})\frac{(\xi z)^3}{3!}. \tag{3.3}$$

Substituting (3.3) in (3.1), we obtain

$$\begin{aligned} m_\delta(\xi) &= c_\delta \int_{-\delta}^{\delta} \frac{\cos(\xi z) - 1}{|z|^\beta}\, dz, \\ &= -c_\delta \int_{-\delta}^{\delta} \frac{z^2}{2!\,|z|^\beta}\, dz\; |\xi|^2 + c_\delta \int_{-\delta}^{\delta} \frac{\sin(\alpha_{z,\xi}) z^3}{3!\,|z|^\beta}\, dz\; \xi^3 \\ &= -|\xi|^2 + c_\delta \int_{-\delta}^{\delta} \frac{\sin(\alpha_{z,\xi}) z^3}{3!\,|z|^\beta}\, dz\; \xi^3, \end{aligned}$$

where in the last equation we applied (1.10). Therefore,

$$\left| m_\delta(\xi) + |\xi|^2 \right| \le c_\delta \int_{-\delta}^{\delta} \frac{|\sin(\alpha_{z,\xi})|\; |z|^3}{3!\,|z|^\beta}\, dz\; |\xi|^3 \le c_\delta \int_{-\delta}^{\delta} \frac{|z|^{3-\beta}}{3!}\, dz\; |\xi|^3.$$

Since

$$c_\delta \int_{-\delta}^{\delta} \frac{|z|^{3-\beta}}{3!}\, dz = \frac{2}{3!}\,\frac{3-\beta}{4-\beta}\,\delta,$$

it follows that

$$\left| m_\delta(\xi) + |\xi|^2 \right| \le \frac{1}{3}\,\frac{3-\beta}{4-\beta}\,|\xi|^3\,\delta. \tag{3.4}$$

The result follows from taking the limit as $\delta \to 0^+$ in (3.4). □

Theorem 3.2. *The multipliers m_δ are non-positive and bounded; that is*

(a) for any $\xi \in \mathbb{R}, \delta > 0$,

$$m_\delta(\xi) \le 0,$$

(b) for fixed $\delta > 0$ and any $\xi \in \mathbb{R}$,

$$|m_\delta(\xi)| \leq 4\,\frac{3-\beta}{1-\beta}\,\frac{1}{\delta^2}.$$

Proof. (a) Since $c_\delta = \frac{3-\beta}{\delta^{3-\beta}} > 0$ and $\cos(z\xi) - 1 \leq 0$, for any $z, \xi \in \mathbb{R}$, it follows that

$$m_\delta(\xi) = c_\delta \int_{-\delta}^{\delta} \frac{\cos(z\xi) - 1}{|z|^\beta}\,dz \leq 0.$$

(b) We observe

$$|m_\delta(\xi)| \leq c_\delta \int_{-\delta}^{\delta} \frac{|\cos(z\xi) - 1|}{|z|^\beta}\,dz \leq 2c_\delta \int_{-\delta}^{\delta} \frac{1}{|z|^\beta}\,dz = 4\,\frac{3-\beta}{1-\beta}\,\frac{1}{\delta^2}.$$

□

3.2 Nonlocal diffusion

In this section, we consider a nonlocal version of the heat equation.

3.2.1 The heat equation

First, we review the Fourier transform method for solving the heat equation in $\mathbb{R}$. Consider the heat equation

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t), \\ u(x,0) = u_0(x), \end{cases} \tag{3.5}$$

for $x \in \mathbb{R}$ and $t > 0$. By applying the Fourier transform, we obtain

$$\frac{\partial \hat{u}}{\partial t}(\xi,t) = \mathcal{F}\left(\frac{\partial^2 u}{\partial x^2}\right)(\xi,t) = -|\xi|^2\,\hat{u}(\xi,t).$$

Thus, $\hat{u}$ solves

$$\begin{cases} \frac{\partial \hat{u}}{\partial t}(\xi,t) = -|\xi|^2\,\hat{u}(\xi,t), \\ \hat{u}(\xi,0) = \hat{u}_0(\xi), \end{cases} \tag{3.6}$$

for $\xi \in \mathbb{R}$ and $t > 0$. For fixed $\xi \in \mathbb{R}$, the solution to (3.6) is given by

$$\hat{u}(\xi, t) = e^{-|\xi|^2 t}\, \hat{u}_0(\xi). \tag{3.7}$$

Therefore, the solution to (3.5) is given by applying the inverse Fourier transform

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-|\xi|^2 t} \hat{u}_0(\xi) e^{ix\xi}\, d\xi.$$

3.2.2 A nonlocal heat equation

Consider a nonlocal analogue to (3.5)

$$\begin{cases} \frac{\partial u_\delta}{\partial t}(x,t) = L_\delta u_\delta(x,t), \\ u_\delta(x,0) = u_0(x), \end{cases} \tag{3.8}$$

for $x \in \mathbb{R}$ and $t > 0$. Here the solution u_δ is parameterized by the nonlocality parameter $\delta > 0$. By applying the Fourier transform, we obtain

$$\frac{\partial \hat{u}_\delta}{\partial t}(\xi, t) = \mathcal{F}(L_\delta u_\delta)(\xi, t) = m_\delta(\xi) \hat{u}_\delta(\xi, t),$$

where m_δ is multiplier given by (3.1). Thus, $\hat{u}_\delta$ solves

$$\begin{cases} \frac{\partial \hat{u}_\delta}{\partial t}(\xi,t) = m_\delta(\xi) \hat{u}_\delta(\xi,t), \\ \hat{u}_\delta(\xi,0) = \hat{u}_0(\xi), \end{cases} \tag{3.9}$$

for $\xi \in \mathbb{R}$ and $t > 0$. For fixed ξ, the solution to (3.9) is given by

$$\hat{u}_\delta(\xi, t) = e^{m_\delta(\xi) t}\, \hat{u}_0(\xi). \tag{3.10}$$

We observe that using Theorem (3.1)

$$\begin{aligned}\lim_{\delta\to 0^+} \hat{u}_\delta(\xi,t) &= \lim_{\delta\to 0^+} e^{m_\delta(\xi)t}\ \hat{u}_0(\xi),\\ &= e^{-|\xi|^2 t}\hat{u}_0(\xi),\\ &= \hat{u}(\xi,t),\end{aligned}$$

where $\hat{u}$ is the solution to (3.6). By using (3.10) and applying the inverse Fourier transform, we obtain an integral representation for the solution of the nonlocal heat equation

$$u_\delta(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{m_\delta(\xi)t}\hat{u}_0(\xi)e^{ix\xi}\ d\xi.$$

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