AN INVESTIGATION OF THE STRAIT SERVO

by

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INTRODUCTION

Two goals of second order servo design are reduction of time delay and rise time without overshoot. In conventional design, rise time of the linear servo is shortened by using high gain. In this thesis, a certain non-linear characteristic will be used on a second order servo with further performance improvement. Strait (3) has shown that modulating the error signal with the absolute magnitude of the output velocity results in a system that has shorter rise time with less overshoot than is possible with a linear servo. Consider the second order linear servo with the control equation

$$y = (K/J)\,\frac{(x - y - dpy)}{p(p + a')} \tag{1}$$

$$a' = R/J$$

$$p = d/dt$$

where y is the output and x is the input. The system is completely characterized by constants R, J, K, and d. If a' is too small or if K is too large the system will overshoot. Conversely, if a' is too large or if K is too small the system will be overdamped and the rise time will be excessively long.

The error signal in the system represented by Eq. (1) is

$$E = (x - y - dpy)$$

If this is multiplied by $(1 + b|py|)$, Eq. (1) becomes

$$y = (K/J)\,\frac{(x - y - dpy)\,(1 + b|py|)}{p(p + a')} \tag{2}$$

If $b > 1$, then the gain of the system is increased during the transient period. Equation (2) is the control equation of the ideal Strait servo. A slight variation of Eq. (2) is

$$y = (K/J)\left[\frac{(x - y)\,(1 + b|py|) - d'py}{p(p + a')}\right] \tag{3}$$

Equation (3) is the control equation of the prototype Strait servo. The absolute magnitude of py is obtained by differentiating the output and feeding the result into a full-wave rectifier. Increase in gain due to the $(1 + b|py|)$ term results in a shorter rise time than the linear servo. Figure 1 compares the Strait servo response to the linear servo response.

Purposes of this thesis are to obtain a digital computer solution for Eq. (2) and Eq. (3) and to further investigate characteristics of systems represented by these control equations. Also investigated are certain variations of Eq. (2) and Eq. (3) using fractional derivatives. The response of each system to the unit ramp input and to step inputs of different amplitudes is evaluated using the analog computer and the digital computer.

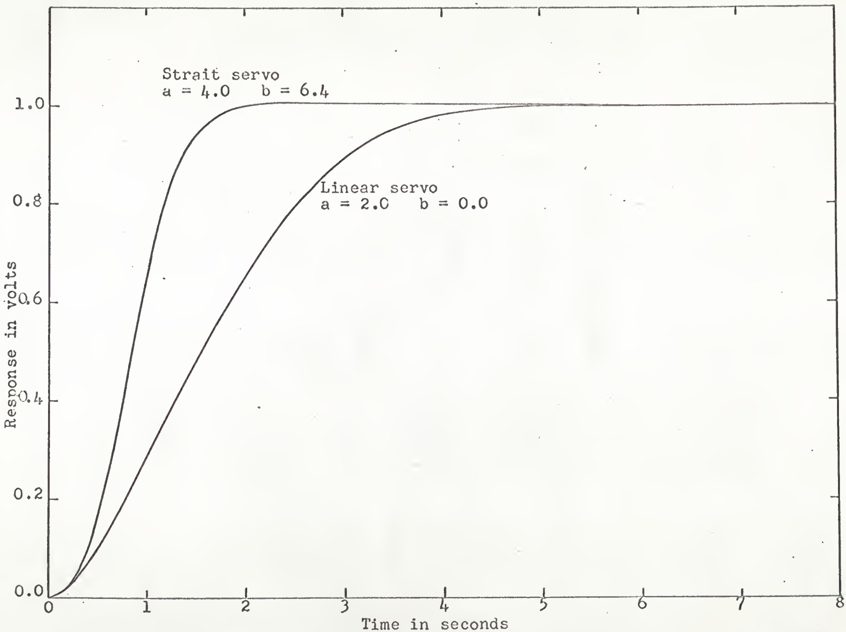

Fig. 1. Comparison of the Strait servo's and the linear servo's step responses.

NUMERICAL SOLUTION OF THE STRAIT SERVO

In order to compare the digital computer results with the analog computer results, we let K/J = 1. The ideal Strait servo can now be represented by the control equation

$$y = \frac{(x - y - dpy)\ (1 + b|py|)}{p(p + a')} \tag{4}$$

$$a' = R/J$$

where a', b, d, R, and J are constants. This is the control equation of the ideal Strait servo with K/J = 1. If d = 0 and a = a' + d', the control equation becomes

$$y = \frac{(x - y)\ (1 + b|py|)}{p(p + a)} \tag{5}$$

This is the control equation of the physical prototype Strait servo with K/J = 1. Therefore, only Eq. (4) need be programmed on the computer.

Equation (4) can be rewritten as

$$\ddot{y} + bd\dot{y}|\dot{y}| + (a' + d)\dot{y} + by|\dot{y}| + y - bx|\dot{y}| = x \tag{6}$$

This can be written as a vector differential equation of the form,

$$\begin{pmatrix} \dot{y} \\ \\ \ddot{y} \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ \\ 1 & c \end{pmatrix} \begin{pmatrix} y \\ \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 \\ \\ x \end{pmatrix} \tag{7}$$

$$c = a' + d + by\ \mathrm{sgn}(\dot{y}) - bx\ \mathrm{sgn}(\dot{y}) + bd\dot{y}\ \mathrm{sgn}(\dot{y})$$

Here, we need the definition

$$\operatorname{sgn}(\dot{y}) \equiv \begin{cases} 1 & \dot{y} > 0 \\ 0 & \dot{y} = 0 \\ -1 & \dot{y} < 0 \end{cases}$$

Equation (7) is of the form

$$\dot{\eta} + \mu\eta = \sigma$$

$$\eta = \begin{pmatrix} y \\ \dot{y} \end{pmatrix}$$

$$\mu = \begin{pmatrix} 0 & -1 \\ 1 & c \end{pmatrix}, \qquad \sigma = \begin{pmatrix} 0 \\ x \end{pmatrix}$$

Taking the Laplace transform of Eq. (8) and dividing by s yields,

$$\bar{\eta} + \frac{\overline{\mu\eta}}{s} = \frac{\bar{\sigma}}{s} \tag{9}$$

Since the equation cannot be formally solved for $\bar{\eta}$, Z transforms and trapezoidal convolution are used.

Using the trapezoidal convolution formula (Halijak (2))

*$Z(\overline{fg}) \doteq T(Z\bar{f})(Z\bar{g}) - 0.5T(Z\bar{f}g_0) - 0.5T(Z\bar{g}f_0)$, and the identity $Z(1/s) = 1/(1 - z)$, Eq. (9) can be evaluated. The Z transform of Eq. (9) yields

$$Z\bar{\eta} + 0.5T\,\frac{1+z}{1-z}\,Z\overline{\mu\eta} - 0.5T\,\frac{\mu_0\eta_0}{1-z} =$$

$$0.5T\,\frac{1+z}{1-z}\,Z\bar{\sigma} - 0.5T\,\frac{\sigma_0}{1-z} \;.$$

*T is the sampling interval in seconds.

Multiplying all terms by 1 - z yields,

$$(1 - z)Z\bar{\eta} + 0.5T(1 + z)Z\overline{\mu\eta} - 0.5T\mu_o\eta_o$$
$$= 0.5T(1 + z)Z\bar{\sigma} - 0.5T\sigma_o$$
$$= \xi_n \tag{10}$$

For a step function of magnitude x

$$\xi_n = x(0, \underline{T, T, T})$$

For a unit ramp function, one obtains

$$\xi_n = 0.5T(0, T, 3T, 5T, 7T, 9T, \ldots) \text{ and}$$

$$x_n = (0, T, 2T, 3T, \ldots)$$

Taking the inverse Z transform of Eq. (10) yields

$$\eta_n - \eta_{n-1} + 0.5T\mu_n\eta_n + 0.5T\mu_{n-1}\eta_{n-1} - 0.5T\mu_o\eta_o = \xi_n$$

Letting the initial conditions

$y(0) = \dot{y}(0) = 0$ and solving for η_n yields,

$$(I + 0.5T\mu_n)\eta_n = \xi_n + \eta_{n-1} - 0.5T\mu_{n-1}\eta_{n-1}$$
$$= \delta_n$$

Substituting the appropriate matrix for η and μ yields

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 0.5T\begin{pmatrix} 0 & -1 \\ 1 & c_n \end{pmatrix}\right]\begin{pmatrix} y_n \\ \dot{y}_n \end{pmatrix} = \begin{pmatrix} \delta_{1,n} \\ \delta_{2,n} \end{pmatrix} \tag{11}$$

$$c_n = a' + d + by_n \operatorname{sgn}(\dot{y}_n) + bd\dot{y}_n \operatorname{sgn}(\dot{y}_n) - bx_n \operatorname{sgn}(\dot{y}_n).$$

$$\delta_{1,n} = y_{n-1} + 0.5Ty_{n-1} \tag{12}$$

$$\delta_{2,n} = \xi_n + \dot{y}_{n-1} - 0.5Ty_{n-1} - 0.5Tc_{n-1}\dot{y}_{n-1} \tag{13}$$

$$c_{n-1} = a' + d + by_{n-1} \operatorname{sgn}(\dot{y}_{n-1}) + bd\dot{y}_{n-1} \operatorname{sgn}(\dot{y}_{n-1})$$

$$- bx_{n-1} \operatorname{sgn}(\dot{y}_{n-1}).$$

Equation (11) yields two simultaneous equations,

$$y_n - 0.5T\dot{y}_n = \delta_{1,n} \tag{14}$$

$$0.5Ty_n + \dot{y}_n + 0.5Tc_n\dot{y}_n = \delta_{2,n} \tag{15}$$

Substituting c_n into Eq. (15) yields

$$0.5Ty_n + \dot{y}_n + 0.5T\dot{y}_n \Big[a' + d + by_n\operatorname{sgn}(\dot{y}_n)$$

$$+ bd\dot{y}_n\operatorname{sgn}(\dot{y}_n) - bx_n\operatorname{sgn}(\dot{y}_n)\Big] = \delta_{2,n}$$

Solving for $\dot{y}_n$, where

$$y_n = \delta_{1,n} + 0.5T\dot{y}_n \text{ yields}$$

$$0.5T(\delta_{1,n} + 0.5T\dot{y}_n) + \dot{y}_n + 0.5T\dot{y}_n \Big[a' + d + b(\delta_{1,n} + 0.5T\dot{y}_n)$$

$$\operatorname{sgn}(\dot{y}_n) + bd\dot{y}_n\operatorname{sgn}(\dot{y}_n) - bx_n\operatorname{sgn}(\dot{y}_n)\Big] = \delta_{2,n}$$

$$0.5Tb(d + 0.5T)\, \dot{y}_n^2\operatorname{sgn}(\dot{y}_n) + \Big[1 + 0.25T^2 + 0.5T(a' + d) +$$

$$0.5Tb\,\delta_{1,n}\operatorname{sgn}(\dot{y}_n) - 0.5Tbx_n\operatorname{sgn}(\dot{y}_n)\Big]\dot{y}_n + 0.5T\delta_{1,n} - \delta_{2,n}$$

$$= 0 \tag{16}$$

Equation (16) is of the form

$$\alpha \dot{y}_n^2 + \beta \dot{y}_n + \gamma = 0$$

$$\alpha = 0.5Tb(d + 0.5T)\operatorname{sgn}(\dot{y}_n)$$

$$\beta = 1 + 0.25T^2 + 0.5T(a' + d) + 0.5Tb\delta_{1,n}\operatorname{sgn}(\dot{y}_n)$$

$$- 0.5Tbx_n\operatorname{sgn}(\dot{y}_n)$$

$$\gamma = 0.5T\delta_{1,n} - \delta_{2,n}$$

Solving for $\dot{y}_n$ yields

$$\dot{y}_n = \frac{-\beta \pm\sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} \tag{17}$$

where $\alpha \neq 0$

For a given set of initial conditions, $\delta_{1,n}$ and $\delta_{2,n}$, there are four possible values for $\dot{y}_n$. Two values can be obtained from Eq. (17) by assuming $\dot{y}_n > 0$, and two more values can be obtained by assuming $\dot{y}_n < 0$. Only one of these four possible values is correct. Since nothing can be predetermined about the value of $\dot{y}_n$, certain assumptions must be made. First, assume that the roots of Eq. (17) are well separated. Second, assume that the closest value of $\dot{y}_n$ to the initial condition $\dot{y}_{n-1}$ is the correct solution if $\dot{y}_n$ also satisfies the assumption of the sign $\dot{y}_n$. For example: Let $\dot{y}_{n-1}$ be the initial condition; let R_1 and R_2 be the roots of Eq. (17) for $\dot{y}_n > 0$; let R_3 and R_4 be the roots of Eq. (17) for $\dot{y}_n < 0$. If

$$|\dot{y}_{n-1} - R_1| - |\dot{y}_{n-1} - R_2| < 0 \tag{18}$$

then the closest root to y_{n-1} is R_1; if not, then the closest root is R_2. If the closest root is less than zero this root is not valid. The same procedure applies for R_3 and R_4 where the closest root to $\dot{y}_{n-1}$ is selected and if the closest root is greater than zero the root is invalid. Let the closest valid root (R_1 or R_2) to the initial condition $\dot{y}_{n-1}$ be AA. Let the closest valid root (R_3 or R_4) to the initial condition $\dot{y}_{n-1}$ be BB. Then if

$$|\dot{y}_{n-1} - AA| - |\dot{y}_{n-1} - BB| < 0 \tag{19}$$

the closest root is AA. If not, then the closest root is BB.

To avoid square root operations on the computer, Eq. (18) can be rewritten as follows:

Since $|\dot{y}_{n-1} - R_1|$ and $|\dot{y}_{n-1} - R_2|$ are strictly positive, then if

$$|\dot{y}_{n-1} - R_1|^2 - |\dot{y}_{n-1} - R_2|^2 < 0$$

or if

$$\dot{y}_{n-1}^2 - 2\dot{y}_{n-1}R_1 + R_1^2 - \dot{y}_{n-1}^2 + 2\dot{y}_{n-1}R_2 - R_2^2 < 0$$

and so if

$$R_1^2 - R_2^2 + 2\dot{y}_{n-1}(R_2 - R_1) < 0,$$

then R_1 is the closest root to the initial condition. Likewise for R_3, R_4, and AA, BB.

$$R_3^2 - R_4^2 + 2\dot{y}_{n-1}(R_4 - R_3) < 0$$

$$AA^2 - BB^2 + 2\dot{y}_{n-1}(BB - AA) < 0$$

It is possible that the root closest to the initial condition may not be the correct root. It is for this reason that analog computer solutions should be used to verify numerical solutions of the non-linear differential equation.

By using Equations (12), (13), (14), and (17) and the root selection process described above, an iteration process can be set up on a digital computer to solve the Strait servo for a ramp or step input. Figures 2 and 3 show the program in block diagram form.

The program was written in IBM FORTRAN language and tested using IBM 1620 FORGO. The final execution of the program was made using IBM 1410 FORTRAN. An IBM 1410 computer requires about two minutes to cover ten seconds of solution time for one set of parameters.

The computer program for ramp response differs only slightly from the step response program. The input variable x of Eq. (5) becomes the iteration

$$x_o = 0.0$$

$$x_n = x_{n-1} + T$$

$$n = 0(1)100$$

By using the iteration for T_n in the step response program, it is obvious that

$$x_n = T_n$$

Also the sequence

$$\xi_n = 0.5T(\ 0,\ T,\ 3T,\ 5T,\ \ldots)$$

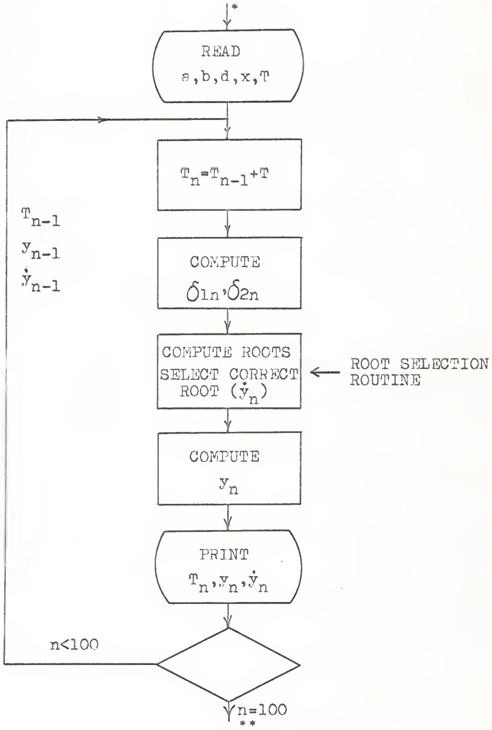

Fig. 2. Block diagram for the digital computer program.

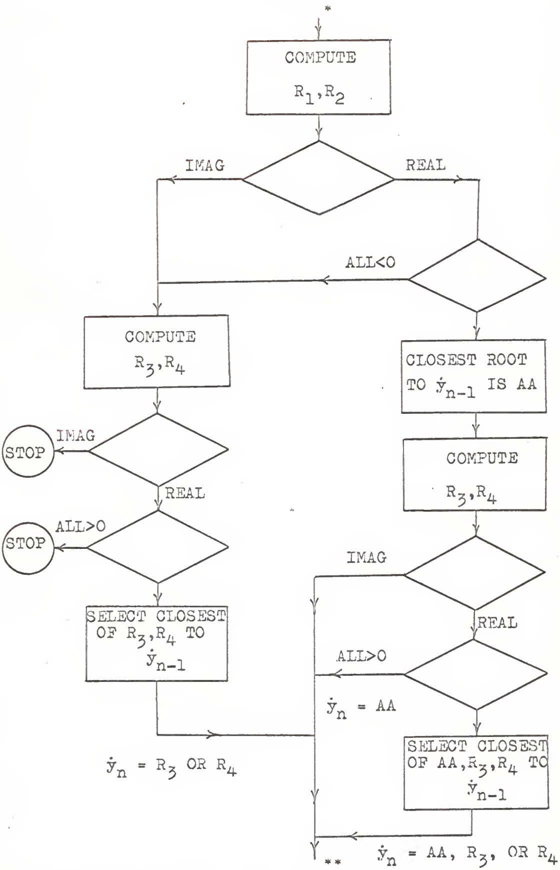

Fig. 3. Block diagram of the root selection routine.

must be generated. This is done in the following manner:

$$z_1 = T$$

$$z_n = z_{n-1} + 2T$$

$$n = 0(1)100$$

$$\xi_n = 0.5TZ_n$$

Notice that the value of ξ_n at $n = 0$ cannot be obtained from this iteration. Therefore, the first iteration in the solution of the Strait servo ramp response must be done separately. Figure 4 shows additional steps that must be added to the step response program in order to obtain the ramp response.

Computer time required for a complete set of iterations covering ten seconds of solution time is about the same as the step response.

The completed FORTRAN program for the Strait servo step response and ramp response is shown in the Appendix.

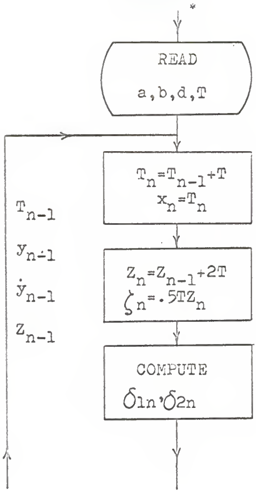

Fig. 4. Program modifications for the ramp response.

ANALOG COMPUTER SOLUTION OF THE STRAIT SERVO

The control equation

$$y = (K/J)\,\frac{(x - y - dpy)\,(1 + b|py|)}{p(p + a')}$$

$$a' = R/J$$

is set up on the analog computer using the diagram shown in Fig. 5. The plant transfer function

$$M(p) = \frac{K/J}{p(p + a)}$$

is simulated using two amplifiers in order to avoid differentiating to obtain p^2y. Access to this term is necessary for the next investigation which uses fractional integrators. The gain to inertia ratio K/J is set equal to one in order to keep the rise time within the limits of the x-y plotter.

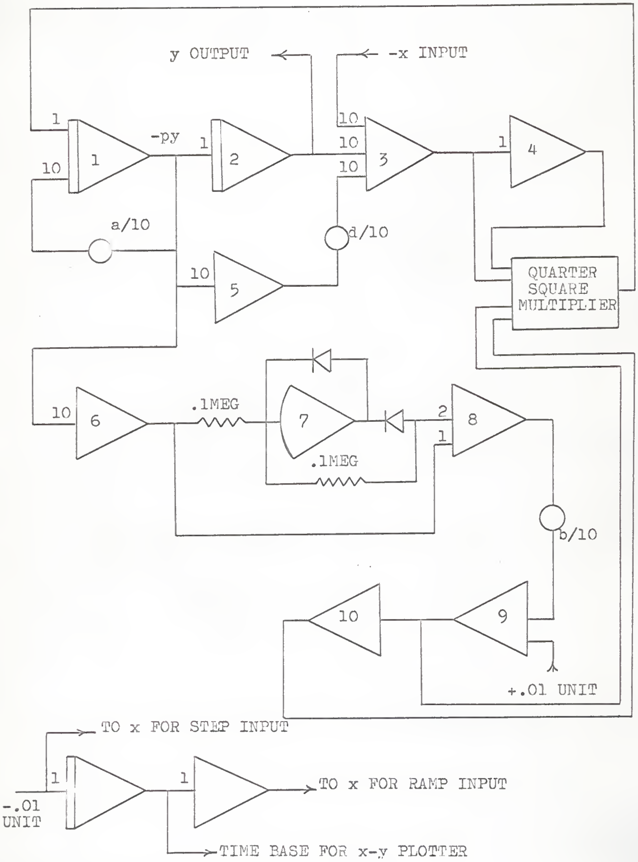

Fig. 5. Analog computer circuit diagram.

THE PHYSICAL PROTOTYPE STRAIT SERVO

The practical servo, proposed by Strait, does not include dpy in the error signal prior to multiplication. This servo subtracts the py term after multiplication. The control equation of the physical prototype servo is

$$\ddot{y} = (K/J)\,\frac{(x - y)(1 + b|py|) - d'py}{p(p + a')} \tag{3}$$

The d'py term, when multiplied by K/J, will add to a'py and increase the system damping. In the analog and digital solution of this equation K/J was set equal to one and the damping term was

$$a = d' + a'.$$

The step response of this servo was very good for optimum values of a and b. The rise time was at least half that of the linear second order servo. However, the servo was very sensitive to changes in parameters. Both a and b had to be quite large to achieve good results. If either a was too small or b was too large, then overshoot would result. Conversely, if either a was too large or b was too small, then considerable time was required to reach zero error. These responses are shown in Figs. 6, 7, 8.

Another shortcoming of this system is its non-linear response to step inputs of different amplitudes. If the system is optimized for a unit step input and a two-unit step input is applied, the system will overshoot. If a half-unit step input is applied, the rise time becomes very long. This means that the system response is different for different step input amplitudes.

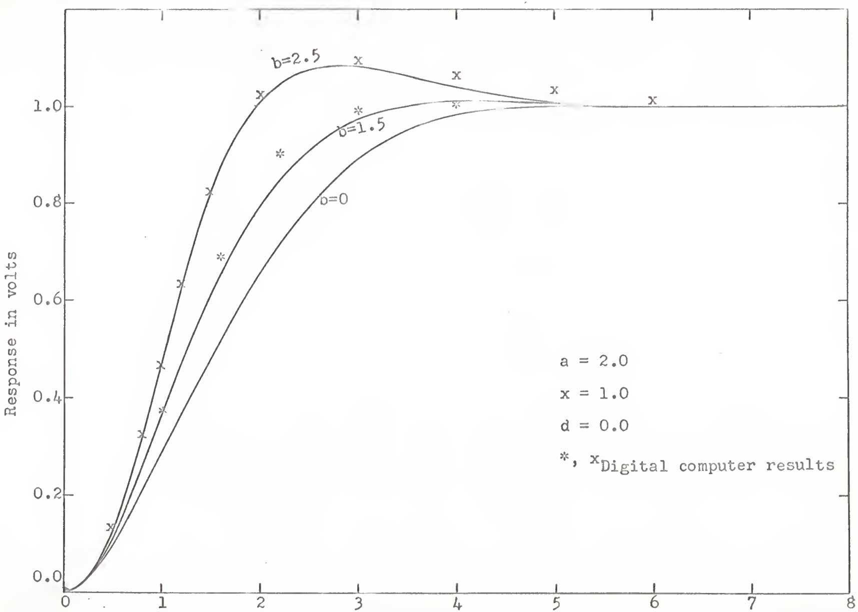

Fig. 6. Prototype Strait servo step response.

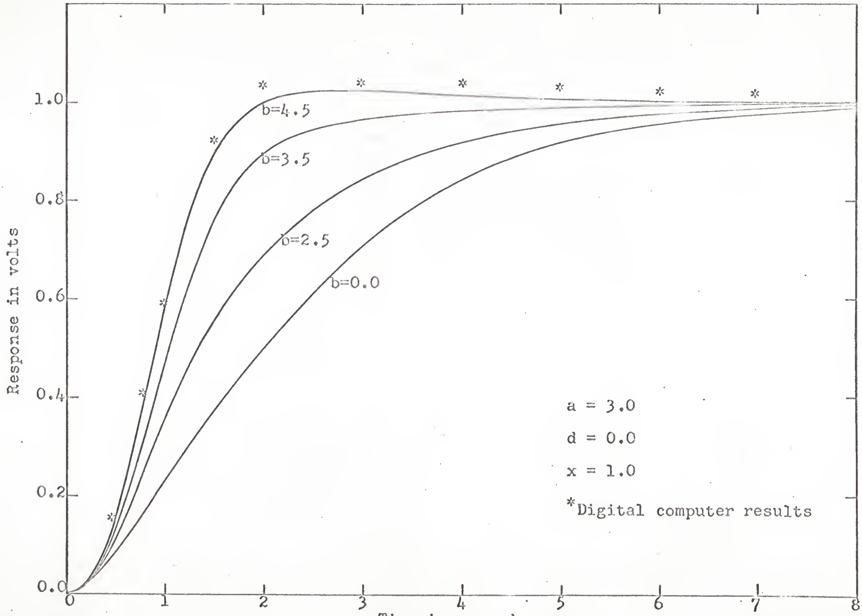

Fig. 7. Prototype Strait servo step response.

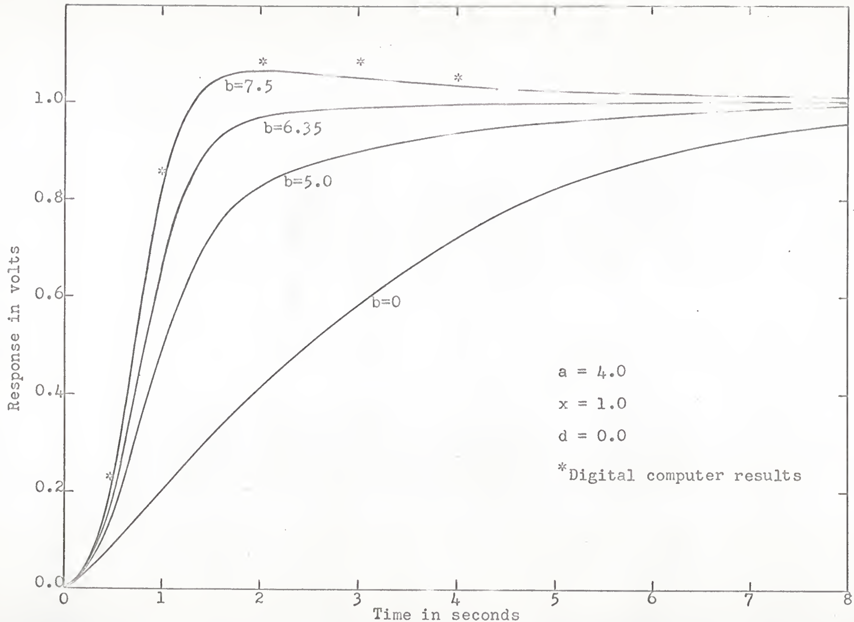

Fig. 8. Prototype Strait servo step response.

Figure 9 shows the response of the prototype Strait servo to three different input levels for a given set of parameters. The reason for a different response for each input level is the system is represented by a non-linear differential equation. Further investigation showed that if the system is optimized for a unit step input, the value of b must be reduced by one-half for a two-unit step input and increased by a factor of two for a one-half unit step input. This requires that

$$b' = b/\,x \qquad (20)$$

Figure 10 shows the effect of the application of Eq. (20) to the Strait servo. This is further supported by the digital computer results shown in Table 1. Unfortunately, Eq. (20) cannot be easily realized because for very small values of x, b' would have to be extremely large.

Table 1. Relation between x and b. d = 0.0, T = 0.10, a = 4.0.

t(sec)	y/x for x = 1 and b = 6.4	y/x for x = 2 and b = 3.2	y/x for x = 0.5 and b = 12.8
0.5	0.17902798	0.17902798	0.17027982
1.0	0.67695884	0.6769588	0.67695884
1.5	0.94844969	0.9484496	0.94944968
2.0	1.0031946	1.0031946	1.00319466
2.5	1.0102353	1.0102353	1.01023538

The ramp response of this system was found to be oscillatory. For optimum values of a and b with the step input, the

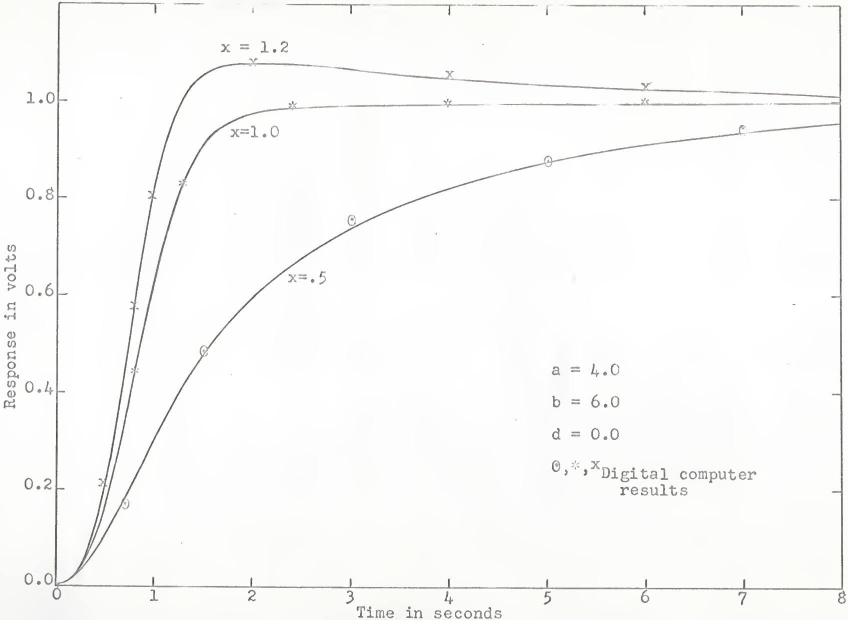

Fig. 9. Prototype Strait servo step response.

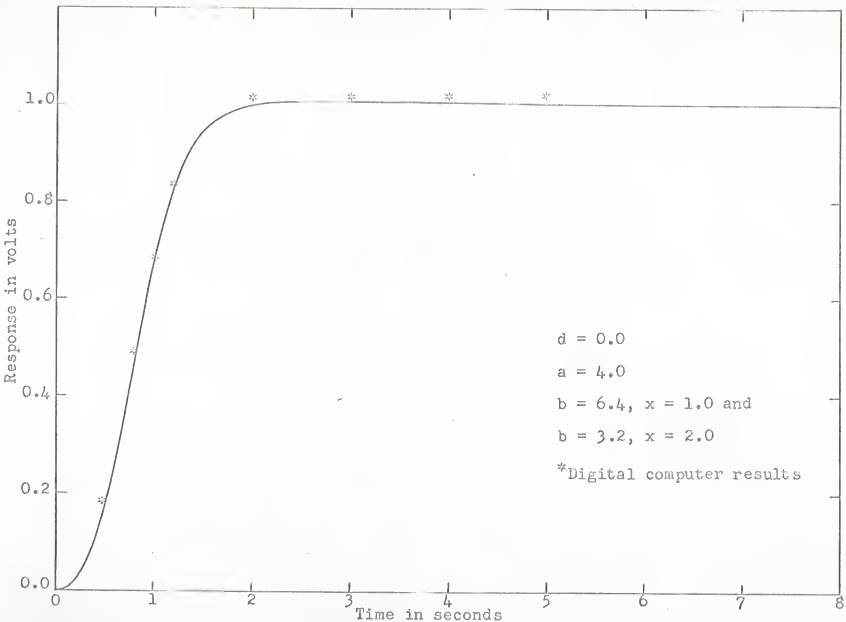

Fig. 10. Relation between b and x for the prototype Strait servo.

ramp response was more like a smoothed staircase. Figure 11 shows the unit ramp response of the prototype Strait servo. These oscillations indicated that the error signal was oscillating and that a constant velocity error could never be reached. This shows clearly in the digital computer solution where the velocity is always changing in magnitude. The unit ramp response on the analog computer was difficult to obtain because large velocities overloaded operational amplifiers.

If the gain to inertia ratio K/J is greater than one, then the rise time becomes shorter and effects of changing parameters and input levels are minimized. Also, the large time constant of the x-y recorder will enter in the solution to distort observations.

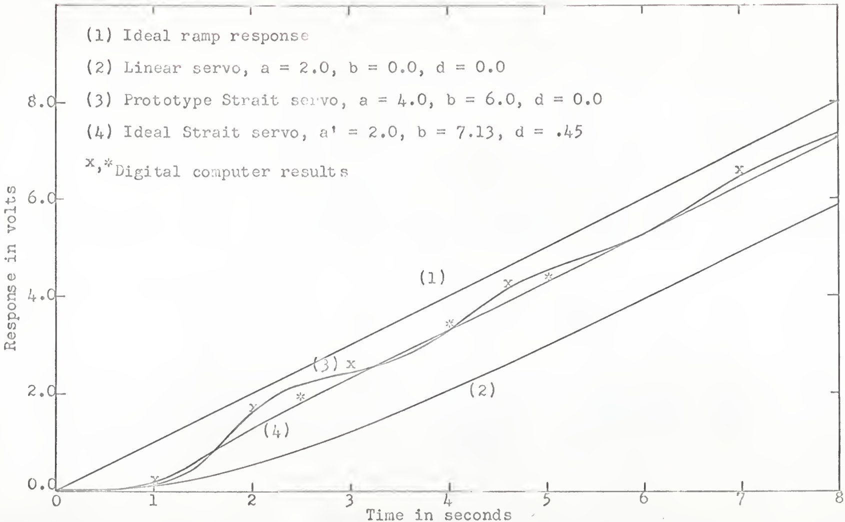

Fig. 11. Ramp response of linear and nonlinear servos.

THE IDEAL STRAIT SERVO

The original servo proposed by Strait included in the error signal a dpy term. This entire error term was multiplied by $(1 + b|py|)$. The resulting control equation is

$$y = (K/J)\,\frac{(x - y - dpy)\,(1 + b|py|)}{p(p + a')} \tag{2}$$

For the purpose of analysis using slowly responding measuring equipment such as the x-y plotter, let $K/J = 1$. In this system the gain and the damping are both increased during the transient time. This results in a slightly longer rise time than if the dpy term were removed from the multiplication operation.

With the dpy term inserted, the system is very stable. This system is sensitive to changes in the parameter d only. Variations in parameters a and b do not effect the step response of this system as much as in the prototype servo. Figure 12 shows the effect of different values of b on the step response. This system is also less sensitive to changes in step input voltages. That is, the rise time for a unit step input and a half unit step input are nearly the same. Figure 13 shows the response of the servo to various input amplitudes. The relationship between the input x and the parameter b is the same as that of the prototype servo. This relationship is

$$b' = b/\,x \tag{20}$$

This is proven in Fig. 14 and Table 2.

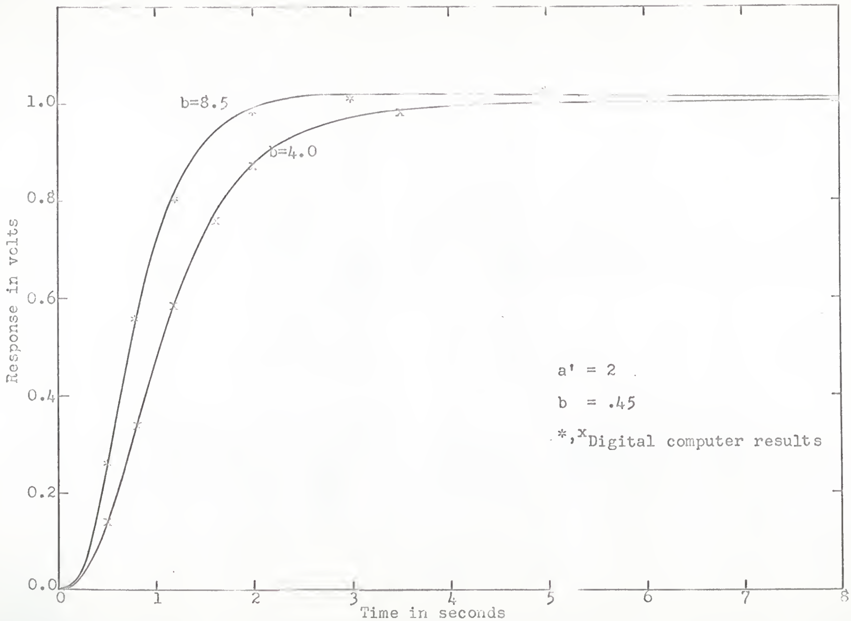

Fig. 12. Ideal Strait servo step response.

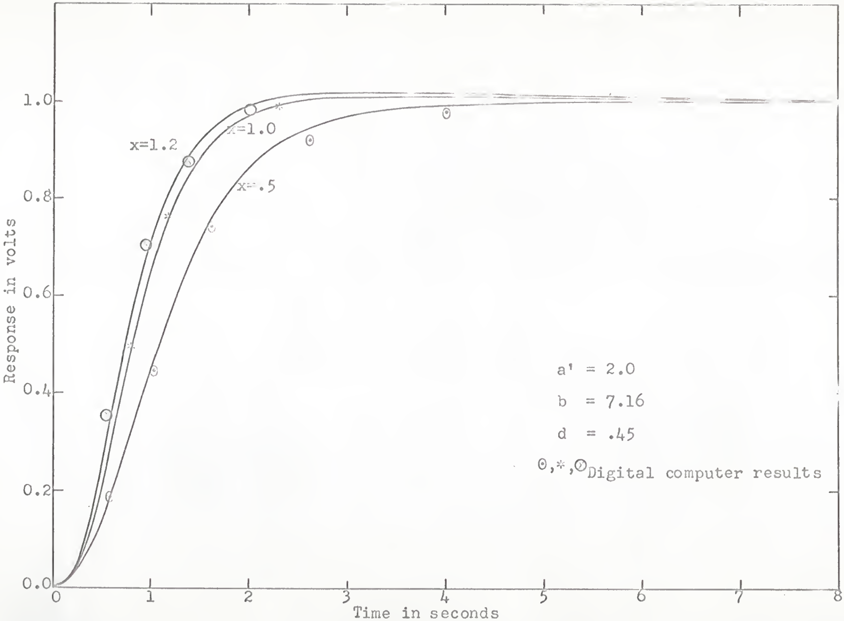

Fig. 13. Ideal Strait servo step response.

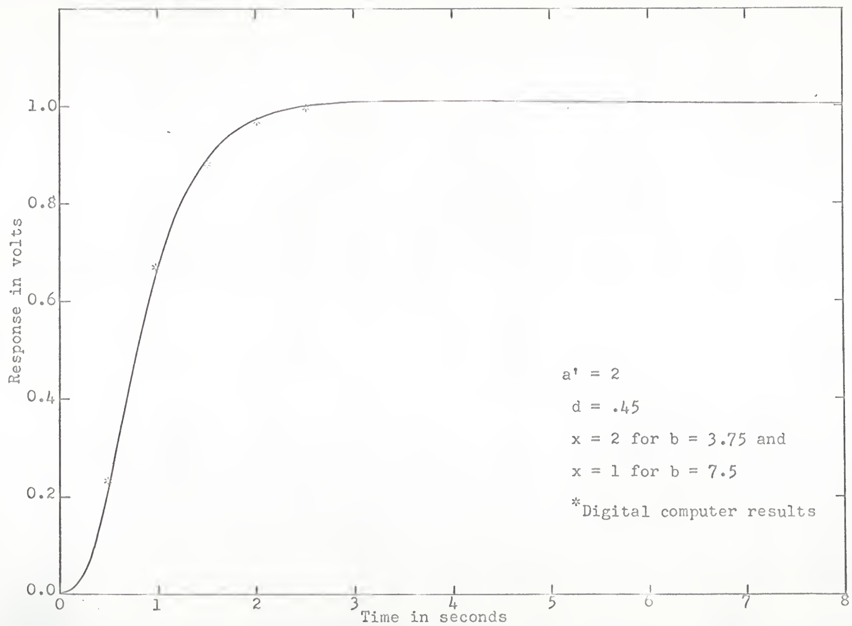

Fig. 14. Relation between b and x for the ideal Strait servo.

Table 2. Relation between x and b. $d = 0.45$, $T = 0.10$, $a = 2.0$.

t(sec)	y/x for x = 1 and b = 7.5	y/x for x = 2 and b = 3.75
0.5	0.22075316	0.22075316
1.0	0.65590762	0.6559067
1.5	0.87734833	0.8773483
2.0	0.96149644	0.9614964
2.5	0.99119428	0.9911942

The ramp response of the ideal Strait servo is nearly the response of a zero velocity error system. The oscillatory behavior is noticeable only in the digital computer solution where the magnitude of velocity oscillations is less than 10^{-4} units. The ramp response of this servo compared with the prototype servo and the linear servo is shown in Fig. 11. If good ramp response is required, then the ideal Strait servo will give better results than the prototype servo. However, the theoretical servo is more difficult to construct because the dpy term cannot be easily and economically multiplied by $(1 + b|py|)$.

USE OF FRACTIONAL INTEGRATORS AND DIFFERENTIATORS TO IMPROVE THE STRAIT SERVO RESPONSE

The $\sqrt{s}$ operator appears in the solution of RC transmission lines, heat flow, neutron diffusion in a nuclear reactor core, and in other physical problems. The $\sqrt{s}$ and $\sqrt{1/s}$ operators can be approximated on the analog computer by using the network designed by Carlson (1). This network is a lattice of resistors and capacitors whose values are such that RC is one computer unit of time. Figure 15 shows the computer amplifier connected as a fractional integrator. Carlson found that cascading five lattice networks, shown in Fig. 15, gave a good approximation to $\sqrt{1/s}$. Since differentiation is to be avoided on the analog computer, only the $\sqrt{1/s}$ operator will be used. Digital computer solutions of differential equations involving $\sqrt{s}$ were avoided because of difficulty in obtaining the Z transform of $\sqrt{1/s}$. Therefore, the following investigation relies entirely on analog computer simulation.

Use of fractional integrations in the Strait servo yields two new control equations:

$$y = (K/J)\,\frac{(x - y - dpy)\,(1 + b|p^{1/2}y|)}{p(p + a')} \tag{21}$$

$$K/J = 1 \qquad\qquad a' = R/J$$

$$y = (K/J)\,\frac{(x - y - dpy)\,(1 + b|p^{3/2}y|)}{p(p + a')} \tag{22}$$

$$K/J = 1 \qquad\qquad a' = R/J$$

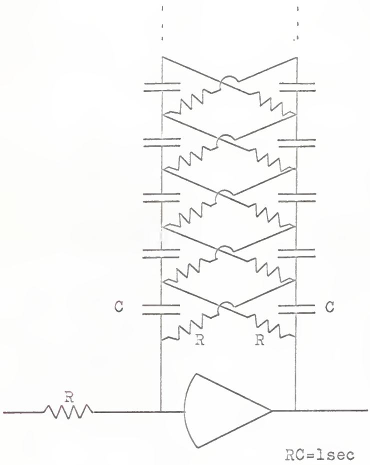

Fig. 15. Fractional integrator $\sqrt{1/s}$

If $d = 0$ the form of Eq. (21) and Eq. (22) is similar to the physical prototype servo. To obtain $p^{1/2}y$, amplifier 6 in Fig. 5 is replaced with a fractional integrator. To obtain $p^{3/2}y$, it is necessary to first obtain p^2y and then this p^2y is fed into a fractional integrator to obtain $p^{3/2}y$. Necessary modifications to the analog computer circuit to obtain $p^{3/2}y$ are shown in Fig. 16.

The step response of Eq. (21) with $d = 0$ is shown in Figs. 17, 18, 19, and 20. The rise time of this servo is slightly longer than the prototype servo as indicated by Fig. 21. The servo constructed from control equation (21) had the same problems as the prototype servo. Changes in parameters greatly effected the response of the servo using Eq. (21), and high damping and large values of b were required to obtain good results. Also the effect of different input amplitudes on the rise time was the same as for the prototype servo. Figure 22 shows the effect of the step input amplitude on the rise time.

If a is reduced and $d > 0$, then the response of this servo is improved. Also the effects of changing parameters are reduced as shown in Fig. 23. The rise time of the servo where $d > 0$ is much less effected by the step input voltage than if $d = 0$. This characteristic is shown in Fig. 24.

The ramp response of Eq. (21) is shown in Fig. 25. The response is nearly the same for all combinations of parameters. Low values of b are used to prevent overloading the operational amplifiers. Figure 25 also shows that a very small velocity error can be obtained using the fractional derivative.

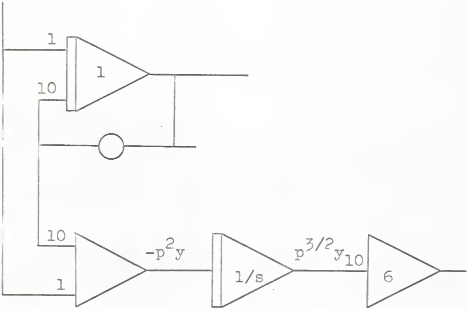

Fig. 16. Modifications to the analog computer circuit to obtain $p^{3/2}y$.

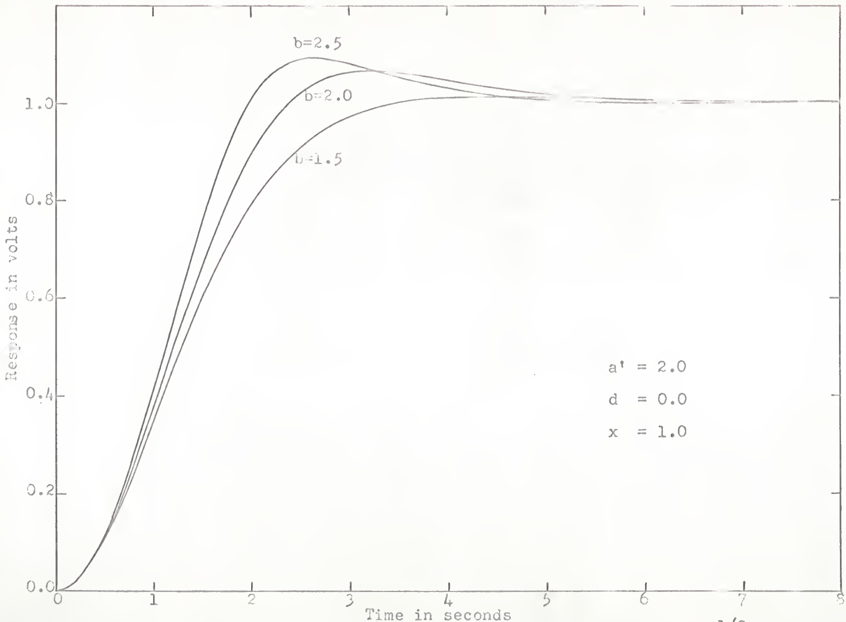

Fig. 17. Step response of the prototype Strait servo using $p^{1/2}y$.

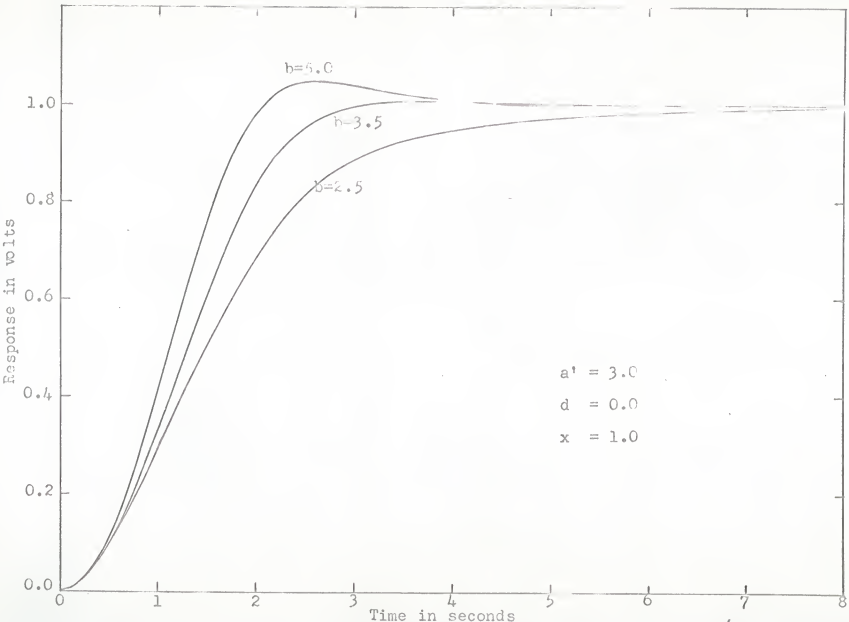

Fig. 18. Step response of the prototype Strait servo using $p^{1/2}y$.

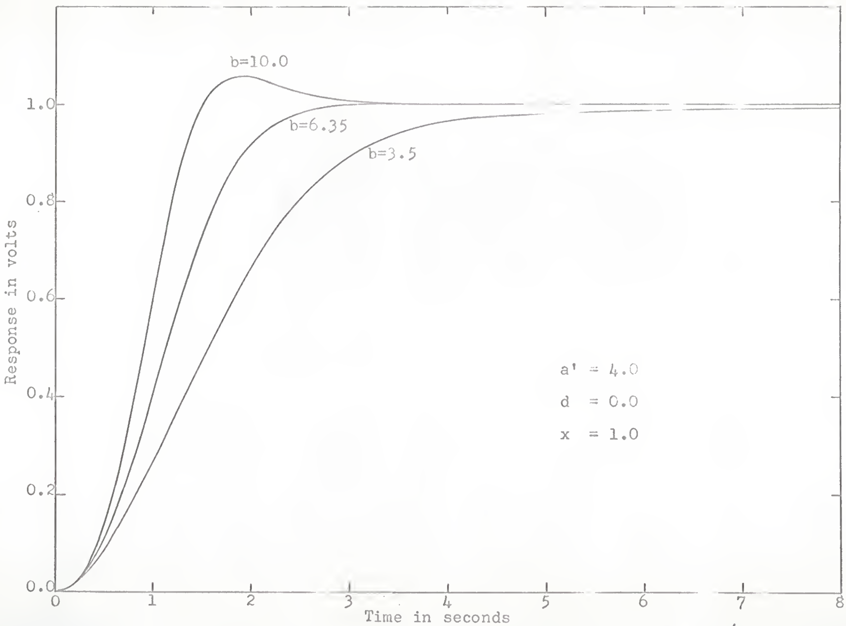

Fig. 19. Step response of the prototype Strait servo using $p^{1/2}y$.

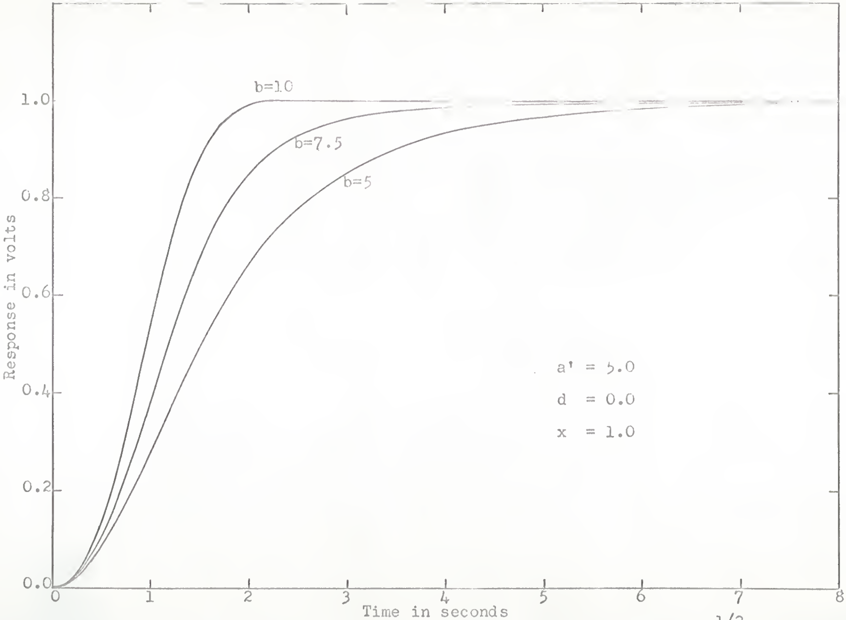

Fig. 20. Step response of the prototype Strait servo using $p^{1/2}y$.

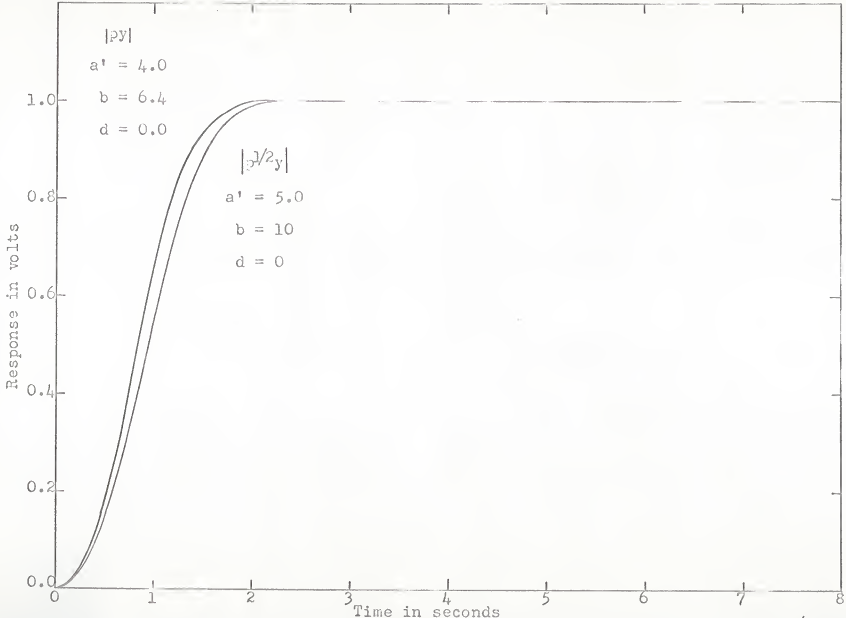

Fig. 21. Comparison of the step response of Strait servos using py and $p^{1/2}y$.

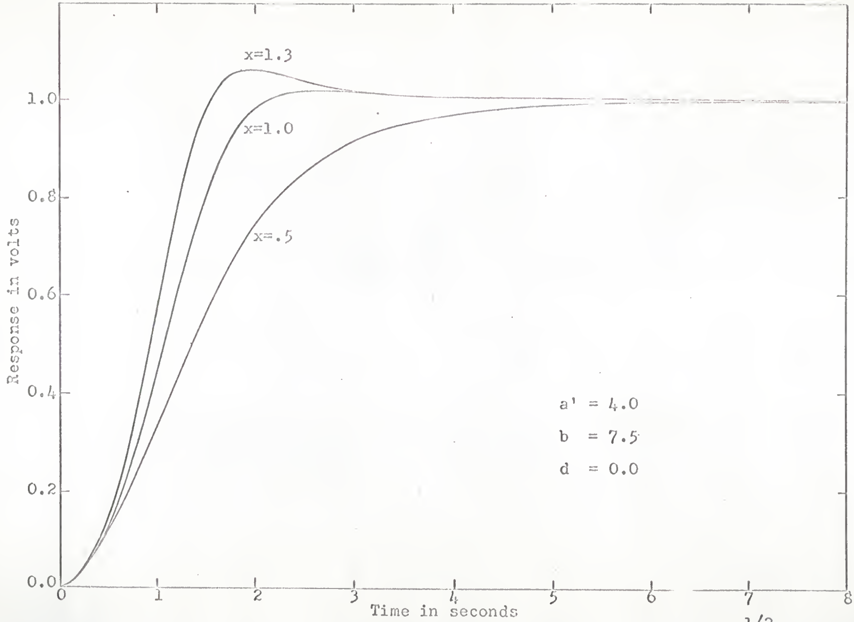

Fig. 22. Step response of the prototype Strait servo using $p^{1/2}y$.

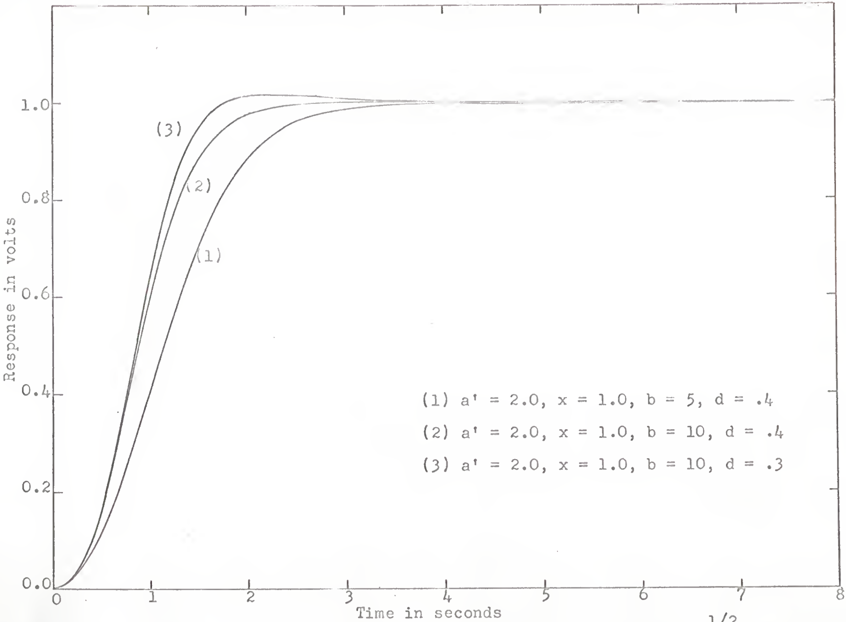

Fig. 23. Step response of the ideal Strait servo using $p^{1/2}y$.

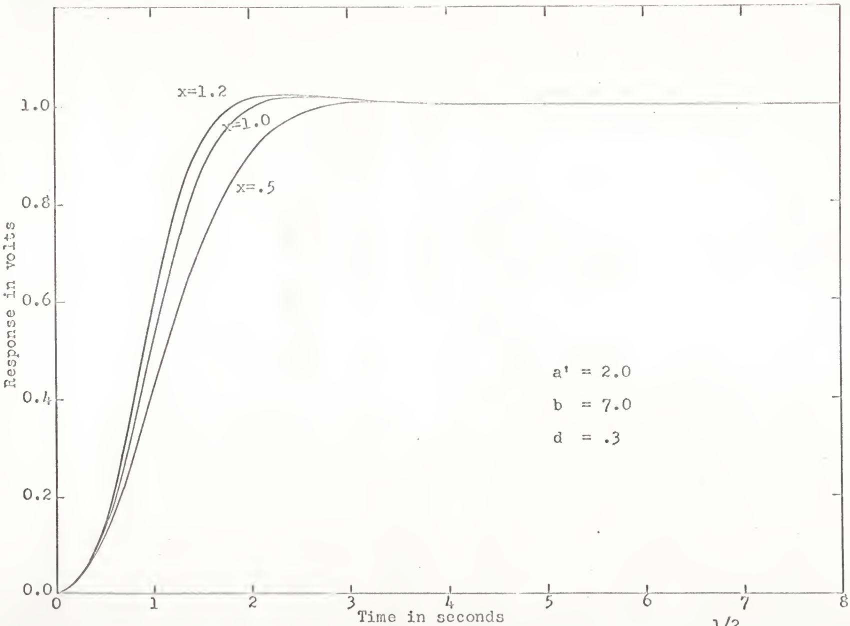

Fig. 24. Step response of the ideal Strait servo using $p^{1/2}y$.

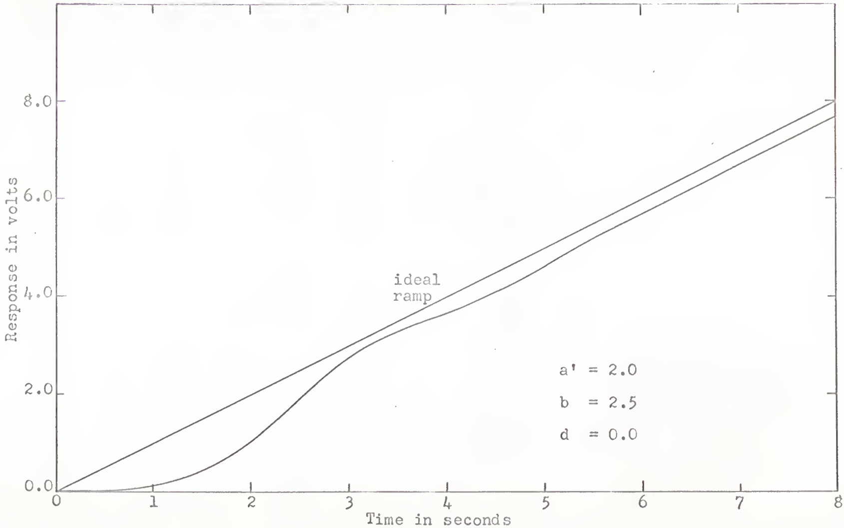

Fig. 25. Ramp response of the prototype Strait servo using $p^{1/2}y$.

Comparisons of the ramp responses of all servos investigated can be made by referring to Fig. 26.

The step response of Eq. (22) shows a very short rise time compared with the other servos. For a given value of a' and with $d = 0$ the value of b needed to give good response is very critical. If b is too large, the system will overshoot. If b is too small, a long time will be required to reach zero error. Figures 27, 28, 29, and 30 show the unit step response for $d = 0$ and for various combinations of a and b. If the system is optimized for a unit step input and if a step input of less than one unit amplitude is applied, then the response of the system will be different and a longer rise time will be encountered. Figure 31 shows the effect of different input levels on the system response.

If $d \neq 0$ the system response shows greater stability but the rise time is degraded considerably. For this case no improvement can be seen over the ideal servo. Figure 32 shows the response of Eq. (22) for $d > 0$. Because of the difficulty in obtaining $p^{3/2}y$, including dpy in the error signal prior to multiplying by $1 + b\,|p^{3/2}y|$, and the poor results obtained, Eq. (22) is not practical. In order to obtain good rise time with this system a sacrifice is made in stability and range of linear response to different input levels.

If good ramp response is required, Eq. (22) with $d = 0$ will not yield good results because of the instability caused by using $p^{3/2}y$. Figure 33 shows that the response is very unstable even for low values of b. If larger values of b are used, the analog computer solution becomes difficult because of operational amplifier overload.

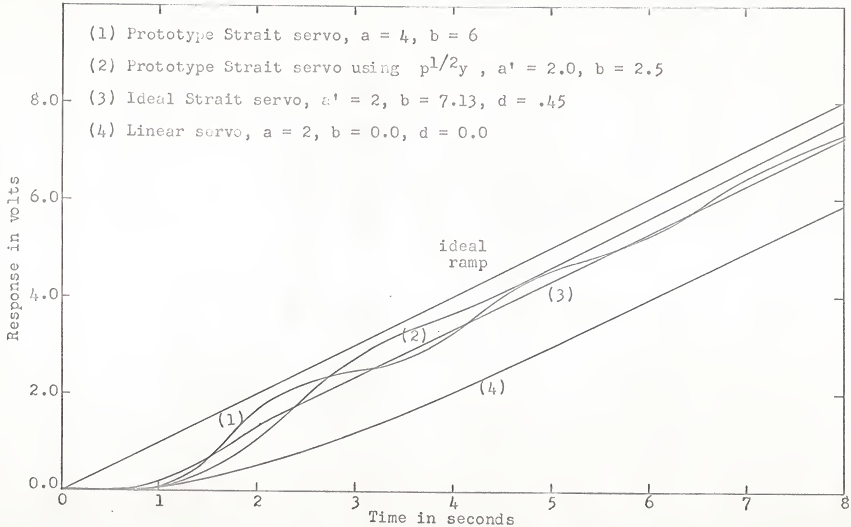

Fig. 26. Comparison of the ramp response of various Strait servos.

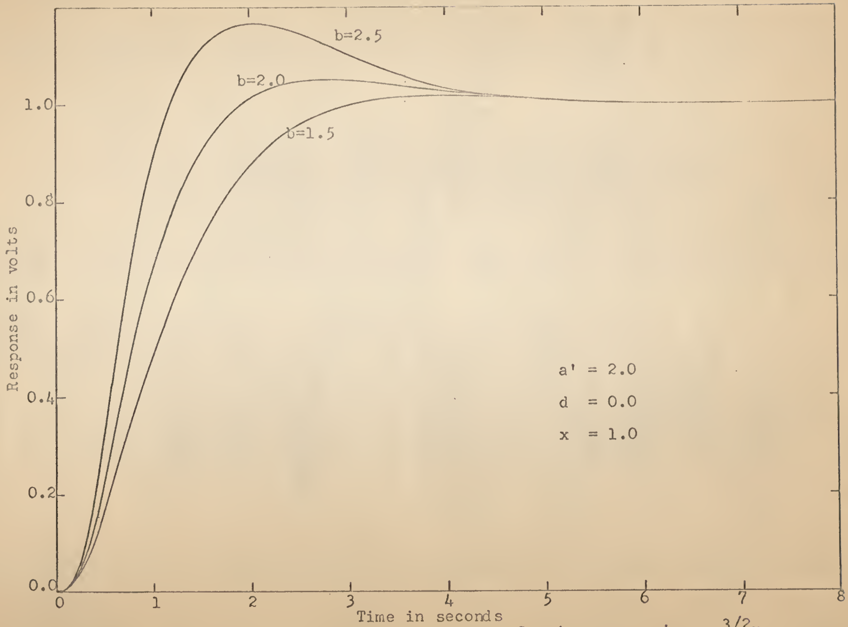

Fig. 27. Step response of the prototype Strait servo using $p^{3/2}y$.

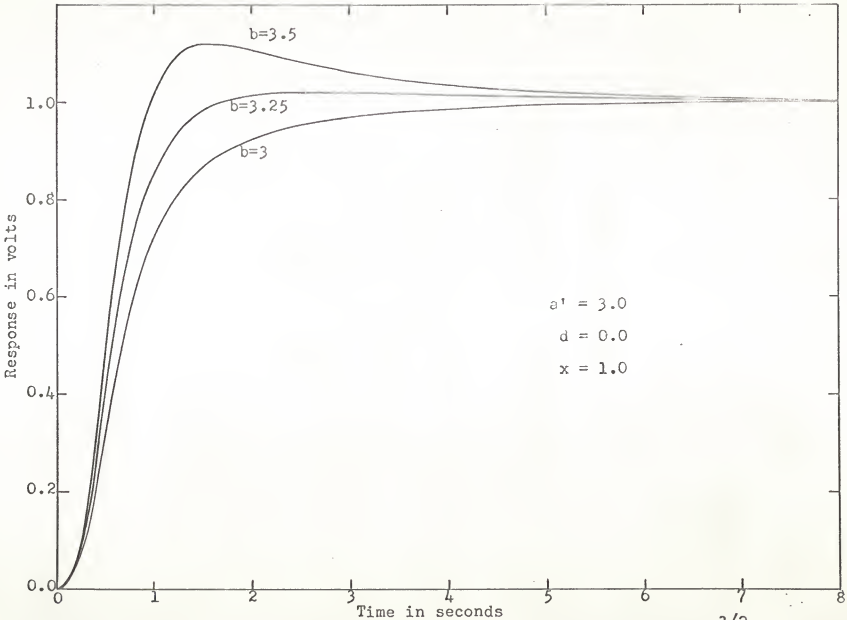

Fig. 28. Step response of the prototype Strait servo using $p^{3/2}y$.

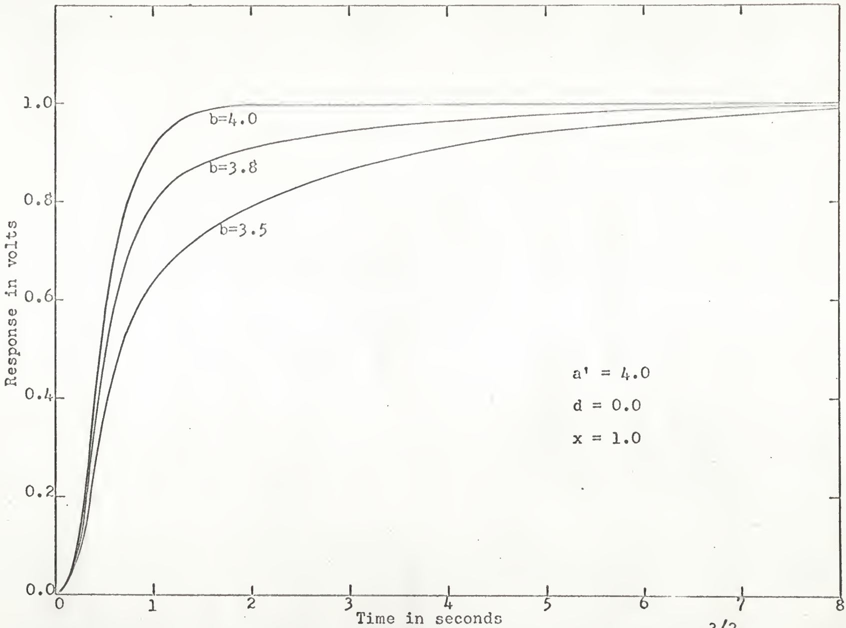

Fig. 29. Step response of the prototype Strait servo using $p^{3/2}y$.

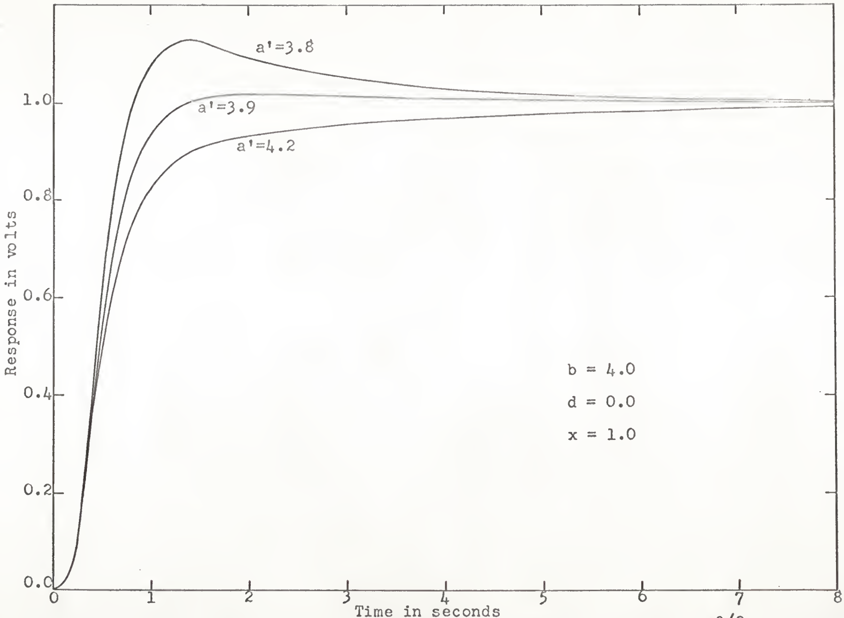

Fig. 30. Step response of the prototype Strait servo using $p^{3/2}y$.

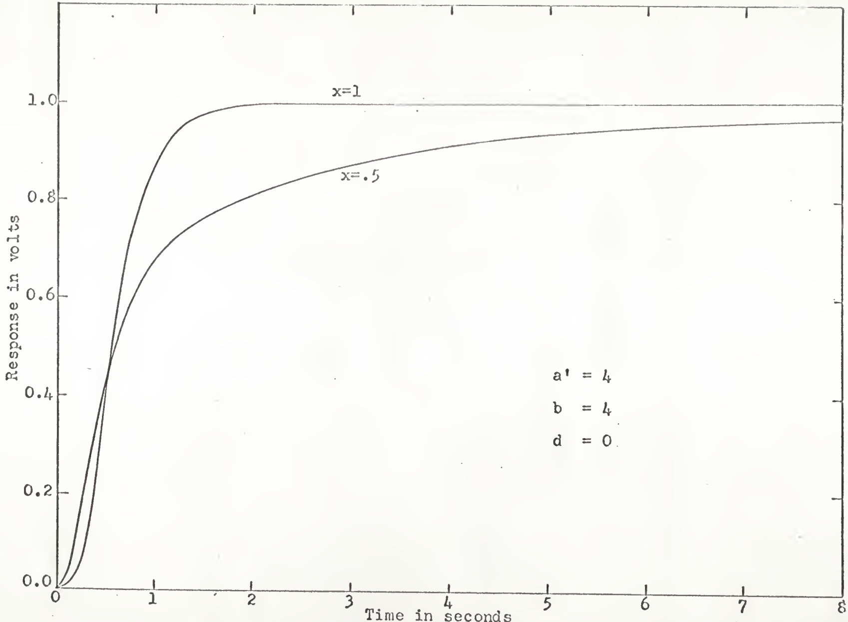

Fig. 31. Step response of the prototype Strait servo using $p^{3/2}y$.

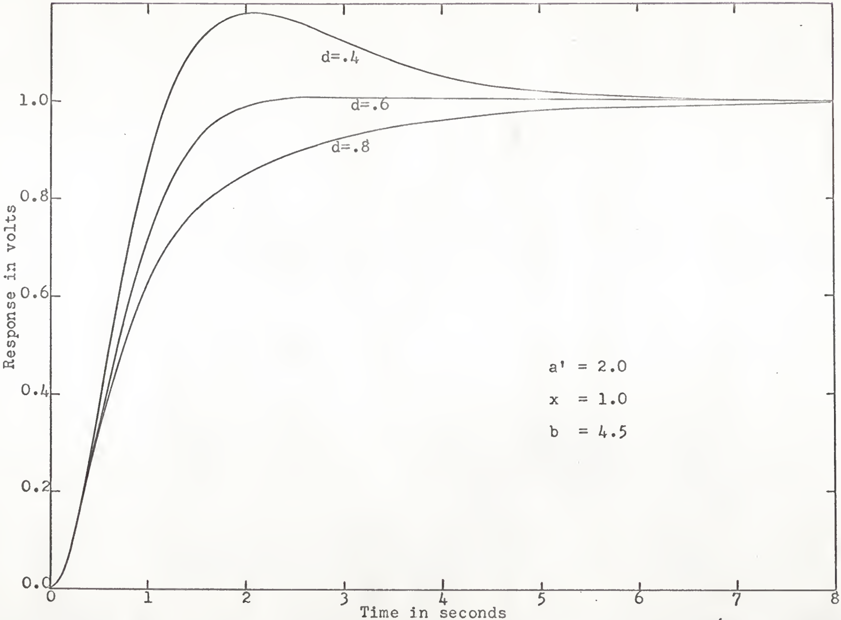

Fig. 32. Step response of the ideal Strait servo using $p^{3/2}y$.

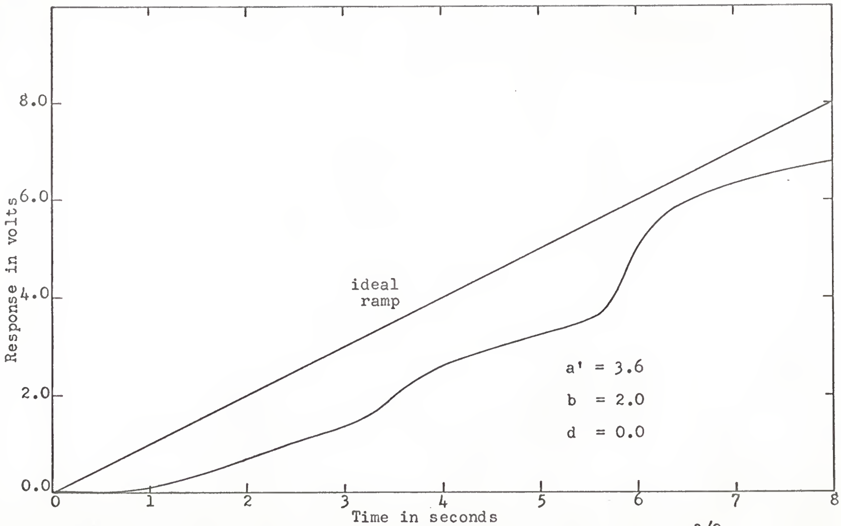

Fig. 33. Ramp response of the prototype Strait servo using $p^{3/2}y$.

THE EFFECT OF GAIN ON SERVO RESPONSE

Until now the system gain K and inertia J has been such that $K/J = 1$. A slight variation on Eq. (21) demonstrates the effect of increasing the gain constant K and the effect of decreasing the gain during the transient period instead of increasing the gain as is done in the Strait servo. The modified Eq. (21) becomes

$$y = (K/J)\,\frac{(x - y - dpy)\,(0.5 + b|p^{1/2}y \pm 0.5|)}{p(p + a')} \tag{23}$$

$$a' = R/J$$

If the input is positive and $p^{1/2}y > 0$ (i.e., no overshoot) and if $b|p^{1/2}y + 0.5|$ is used, the gain is increased during the transient period. The steady state gain is

$$K_s = 0.5(K/J)(1 + b)$$

Since $b > 1$ and $K/J = 1$, then K_s is greater than one. This means that Eq. (23) is the same as Eq. (21) with $K/J = 0.5(1 + b)$. Because $K/J > 1$, a short rise time will be obtained and errors due to the x-y recorder may enter into the solution. If $|p^{1/2}y - 0.5|$ is used or if $p^{1/2}y < 0$, the gain is decreased during the transient period. This results in poor response for the system for both the step and ramp response. Figures 34 and 37 compare responses of the two variations of Eq. (23). Figure 35 shows the effect of increasing the gain K on the system behavior. Gain increase results in a desirable decrease of rise time. Also the

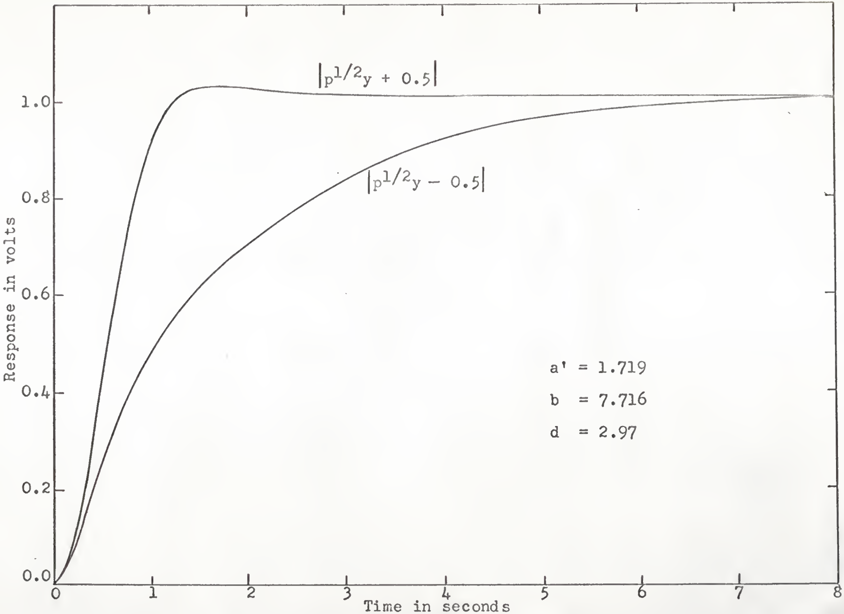

Fig. 34. Step response of the Strait servo with gain increased and with gain decreased during the transient.

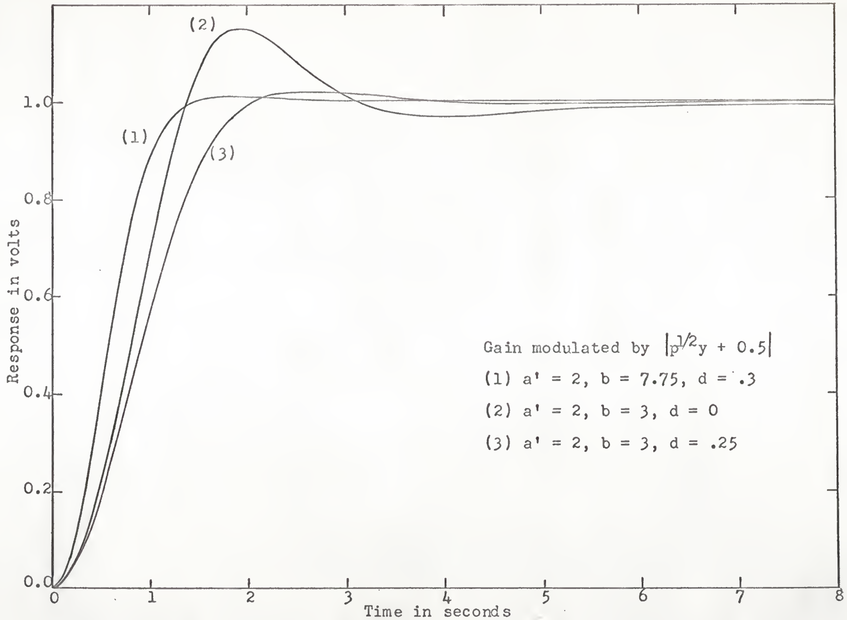

Fig. 35. The effect of gain on the Strait servo step response.

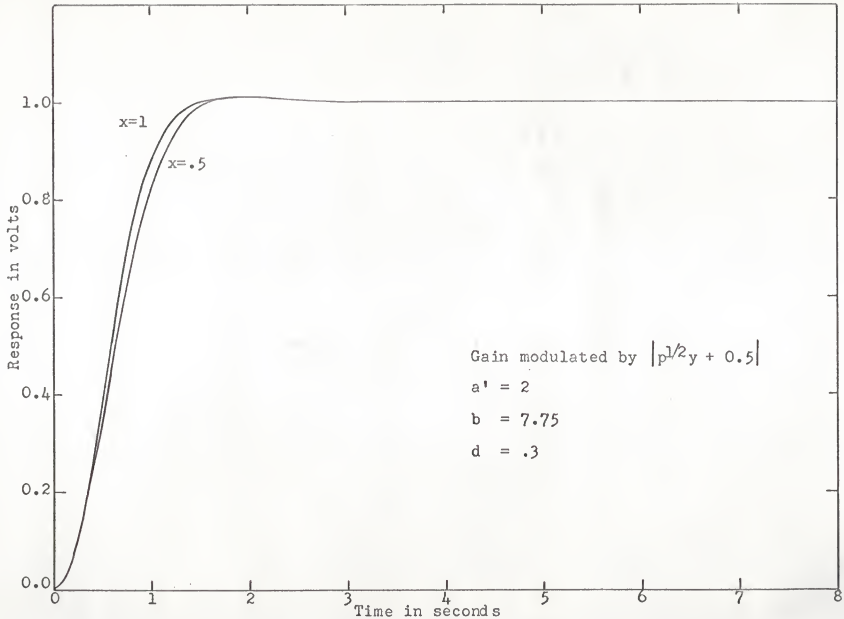

Fig. 36. The effect of gain and input voltage on rise time linearity.

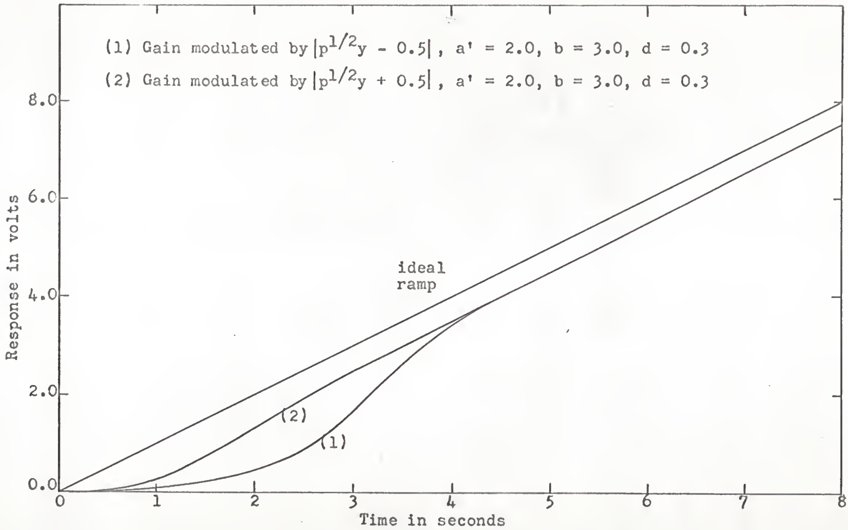

Fig. 37. The effect of gain on the Strait servo ramp response.

effects of different input levels on rise time variation is minimized. Figure 36 shows the step response of Eq. (23) for different input levels.

The results of this investigation apply to all servos investigated so far. That is, it does not matter whether py, $p^{3/2}y$, or $p^{1/2}y$ is used to modulate the error signal.

CONCLUSION

Strait (3) has shown that multiplying the error signal of a second order servo by $1 + b|py|$ improves the performance of the system. There are two problems associated with this non-linear servo. These are: an oscillatory ramp response where a constant velocity error can never be reached, and a dependence of the rise time on the input voltage. Further investigation showed that the optimum value for b is inversely proportional to the magnitude of the input x, and if this relation could be incorporated into the Strait servo the rise time would be the same for all values of x.

Investigations using fractional integrators showed that multiplying the error signal by $1 + b|p^{1/2}y|$ results in a servo with a smaller constant velocity error than the Strait servo. If the error signal is multiplied by $1 + b|p^{3/2}y|$, a shorter step response rise time is possible than with the Strait servo; however, the ramp response is more oscillatory than the Strait servo.

In all non-linear servos investigated in this thesis, inclusion of velocity feedback in the error signal prior to multiplication resulted in a more stable system than if velocity feedback were not included. Also by including velocity feedback in the error signal the non-linear effect of the input magnitude on the rise time was reduced.

Included in this thesis is a numerical solution of the Strait servo which was executed on a digital computer. The method of discretization used was trapezoidal convolution.

ACKNOWLEDGMENT

The author wishes to express his sincere appreciation to Dr. Charles A. Halijak of Kansas State University for supplying the topic, directing the investigation, and guidance in writing this thesis.

REFERENCES

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Simulation of the fractional derivative operator $\sqrt{s}$ and the fractional integral operator $\sqrt{1/s}$. Proceedings of the Central States Simulation Council Meeting on Extrapolation of Analog Computation Methods, Special Report Number 10, Kansas State University Bulletin, July, 1961, 45:1-22.

(2) Halijak, C. A.
Digital approximation of differential equations by trapezoidal convolution. Proceedings of the Central States Simulation Council Meeting on Extrapolation of Analog Computation Methods, Special Report Number 10, Kansas State University Bulletin, July, 1961, 45:83-94.

(3) Strait, B. G.
A position servomechanism with the gain modulated by the output velocity. Unpublished master's thesis, Kansas State University, 1960.

APPENDIX

```
C       STRAIT SERVO STEP RESPONSE
C       U=STEP INPUT AMPLITUDE
        DIMENSION Y(100),DY(100),X(100)
     8  FORMAT(4F4.2,F2.2)
    79  FORMAT(1HL,15X,F5.2,5X,F5.2,5X,F5.2,5X,F5.2,5X,F3.2)
    92  FORMAT(1HK,15X,4HTIME,9X,8HPOSITION,12X,8HVELOCITY)
    89  FORMAT(1H/,15X,F5.2,5X,E15.8,5X,E15.8)
    55  FORMAT(1H/,15X,6HTILT 5)
    66  FORMAT(1H/,15X,7HTILT 81)
    59  READ(1,8)A,B,D,U,T
        IF(B)58,58,57
    57  WRITE(3,79)A,B,D,U,T
        Q=.5*T*B*(D+.5*T)
    51  R=1.+.5*T*(A+D)+.25*T**2
        Y(1)=0.0
        DY(1)=0.0
        WRITE(3,92)
        X(1)=0.0
    52  DO 40 I=2,100
        X(I)=X(I-1)+T
        D1=Y(I-1)+.5*T*DY(I-1)
        IF(DY(I-1))54,53,53
    53  DPOS=1.-.5*T*(A+D+B*Y(I-1)-B*U+B*D*DY(I-1))
        D2=T*U-.5*T*Y(I-1)+DPOS*DY(I-1)
        GO TO 56
    54  DNEG=1.-.5*T*(A+D-B*Y(I-1)+B*U-B*D*DY(I-1))
        D2=T*U-.5*T*Y(I-1)+DNEG*DY(I-1)
    56  A1=Q
        B1=R+.5*T*B*D1-.5*T*B*U
        C1=.5*T*D1-D2
        A2=-Q
        B2=R-.5*T*B*D1+.5*T*B*U
        C2=C1
        IF(B1**2-4.*A1*C1)3,2,2
     2  SQ=SQRT(B1**2-4.*A1*C1)
        R1=(-B1+SQ)/(2.*A1)
        R2=(-B1-SQ)/(2.*A1)
        IF(R1)91,1,1
    91  IF(R2)3,1,1
     1  IF(R1**2-R2**2+2.*Y(I-1)*(R2-R1))22,33,33
    22  AA=R1
        GO TO 7
    33  AA=R2
        GO TO 7
     3  IF(B2**2-4.*A2*C2)5,4,4
     4  SR=SQRT(B2**2-4.*A2*C2)
        R3=(-B2+SR)/(2.*A2)
        R4=(-B2-SR)/(2.*A2)
        IF(R3)34,34,35
    35  IF(R4)34,34,81
```

```
34 IF(R3**2-R4**2+2.*Y(I-1)*(R4-R3))6,9,9
 6 DY(I)=R3
   GO TO 36
 9 DY(I)=R4
36 Y(I)=D1+.5*T*DY(I)
   WRITE(3,89)X(I),Y(I),DY(I)
   GO TO 40
 7 IF(B2**2-4.*A2*C2)16,10,10
10 SR=SQRT(B2**2-4.*A2*C2)
   R3=(-B2+SR)/(2.*A2)
   R4=(-B2-SR)/(2.*A2)
   IF(R3)41,41,69
69 IF(R4)41,41,16
41 IF(R3**2-R4**2+2.*Y(I-1)*(R4-R3))12,13,13
12 BB=R3
   GO TO 75
13 BB=R4
75 IF(AA**2-BB**2+2.*Y(I-1)*(BB-AA))14,15,15
14 DY(I)=AA
   GO TO 77
15 DY(I)=BB
77 Y(I)=D1+.5*T*DY(I)
   WRITE(3,89)X(I),Y(I),DY(I)
   GO TO 40
16 DY(I)=AA
   Y(I)=D1+.5*T*DY(I)
   WRITE(3,89)X(I),Y(I),DY(I)
   GO TO 40
 5 WRITE(3,55)
   GO TO 40
81 WRITE(3,66)
40 CONTINUE
   GO TO 59
58 STOP
   END
```

```
C     STRAIT SERVO RAMP RESPONSE
      DIMENSION Y(100),DY(100),ZTA(100),X(100)
    8 FORMAT(3F4.2,F2.2)
   79 FORMAT(1HL,15X,F5.2,5X,F5.2,5X,F5.2,5X,F3.2)
   92 FORMAT(1HK,15X,4HTIME,9X,8HPOSITION,12X,8HVELOCITY)
   89 FORMAT(1H/,15X,F5.2,5X,E15.8,5X,E15.8)
   55 FORMAT(1H/,15X,6HTILT 5)
   66 FORMAT(1H/,15X,7HTILT 81)
   59 READ(1,8)A,B,D,T
      IF(B)58,58,57
   57 WRITE(3,79)A,B,D,T
      Q=.5*T*B*(D+.5*T)
   51 R=1.+.5*T*(A+D)+.25*T**2
      Y(1)=0.0
      DY(1)=0.0
      WRITE(3,92)
      A0=Q
      B0=R-.5*B*T**2
      C0=-.5*T**2
      DY(2)=(-B0+SQRT(B0**2-4.*A0*C0))/(2.*A0)
      Y(2)=.5*T*DY(2)
      WRITE(3,89)T,Y(2),DY(2)
      X(1)=0.0
      X(2)=T
      ZTA(1)=0.0
      ZTA(2)=T
      DO 40 I=3,100
      X(I)=X(I-1)+T
      ZTA(I)=ZTA(I-1)+2.*T
      Z=.5*T*ZTA(I)
      D1=Y(I-1)+.5*T*DY(I-1)
      IF(DY(I-1))54,53,53
   53 DPOS=1.-.5*T*(A+D+B*Y(I-1)-B*X(I)+B*D*DY(I-1))
      D2=Z-.5*T*Y(I-1)+DPOS*DY(I-1)
      GO TO 56
   54 DNEG=1.-.5*T*(A+D-B*Y(I-1)+B*X(I)-B*D*DY(I-1))
      D2=Z-.5*T*Y(I-1)+DNEG*DY(I-1)
   56 A1=Q
      B1=R+.5*T*B*D1-.5*T*B*X(I)
      C1=.5*T*D1-D2
      A2=-Q
      B2=R-.5*T*B*D1+.5*T*B*X(I)
      C2=C1
      IF(B1**2-4.*A1*C1)3,2,2
    2 SQ=SQRT(B1**2-4.*A1*C1)
      R1=(-B1+SQ)/(2.*A1)
      R2=(-B1-SQ)/(2.*A1)
      IF(R1)91,1,1
   91 IF(R2)3,1,1
    1 IF(R1**2-R2**2+2.*Y(I-1)*(R2-R1))22,33,33
   22 AA=R1
```

```
      GO TO 7
   33 AA=R2
      GO TO 7
    3 IF(B2**2-4.*A2*C2)5,4,4
    4 SR=SQRT(B2**2-4.*A2*C2)
      R3=(-B2+SR)/(2.*A2)
      R4=(-B2-SR)/(2.*A2)
      IF(R3)34,34,35
   35 IF(R4)34,34,81
   34 IF(R3**2-R4**2+2.*Y(I-1)*(R4-R3))6,9,9
    6 DY(I)=R3
      GO TO 36
    9 DY(I)=R4
   36 Y(I)=D1+.5*T*DY(I)
      WRITE(3,89)X(I),Y(I),DY(I)
      GO TO 40
    7 IF(B2**2-4.*A2*C2)16,10,10
   10 SR=SQRT(B2**2-4.*A2*C2)
      R3=(-B2+SR)/(2.*A2)
      R4=(-B2-SR)/(2.*A2)
      IF(R3)41,41,69
   69 IF(R4)41,41,16
   41 IF(R3**2-R4**2+2.*Y(I-1)*(R4-R3))12,13,13
   12 BB=R3
      GO TO 75
   13 BB=R4
   75 IF(AA**2-BB**2+2.*Y(I-1)*(BB-AA))14,15,15
   14 DY(I)=AA
      GO TO 77
   15 DY(I)=BB
   77 Y(I)=D1+.5*T*DY(I)
      WRITE(3,89)X(I),Y(I),DY(I)
      GO TO 40
   16 DY(I)=AA
      Y(I)=D1+.5*T*DY(I)
      WRITE(3,89)X(I),Y(I),DY(I)
      GO TO 40
    5 WRITE(3,55)
      GO TO 40
   81 WRITE(3,66)
   40 CONTINUE
      GO TO 59
   58 STOP
      END
```

AN INVESTIGATION OF THE STRAIT SERVO

by

DON RUSSELL OPDYCKE

B. S., Kansas State University, 1966

AN ABSTRACT OF A MASTER'S THESIS

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Electrical Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1967

The ideal Strait servo has a control equation

$$y = M(p)(x - y - dpy)(1 + b|py|)$$

Because of difficulties in physically realizing this servo, the control equation is modified so that

$$y = M(p)\ (x - y)(1 + b|py|) - dpy$$

Equation (2) is the prototype Strait servo. In either case a servo with performance superior to that of the linear servo results.

Investigation of the ramp and step responses of the Strait servo shows that the ramp response is oscillatory and that the rise time is non-linear with respect to input magnitude. It has been shown that the value of b must be inversely proportional to the input x in order to obtain the same rise time for all input magnitudes. Further improvement on Strait servo performance is obtained by using $|p^{1/2}y|$ instead of $|py|$. This results in an almost zero velocity error servo with only a slightly longer rise time than the Strait servo.

This investigation shows that if the Strait servo is to operate only within a small range of step input voltages, then the prototype Strait servo will yield good results. If a wide range of step inputs is expected or if a ramp input is expected, then it may be necessary to use the fractional derivative $p^{1/2}y$ or to use the ideal Strait servo or both.

The ideal and prototype Strait servos were simulated on digital and analog computers. Servo inputs are step inputs of various magnitudes and the unit ramp input. The effects of varying parameters on the servo response for each input were observed.