THE MONOTONIC PROPERTY OF A SYSTEM OF LINEAR EQUATIONS AND ITS APPLICATIONS

by 45

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TABLE OF CONTENTS

NOMENCLATURE

A	$n \times n$ matrix of real elements
B	inverse of the matrix A
n	integer, number of elements in a row or column of a matrix
x	$n \times 1$ column matrix of real variables
c	$n \times 1$ column matrix of real constants
z	$n \times 1$ column matrix of real elements
q	$n \times 1$ column matrix of real elements
y	$n \times 1$ column matrix
w	$n \times 1$ column matrix
u	$n \times 1$ column matrix
d	$n \times 1$ column matrix
i,j,k	index integers with range $1,2,\ldots,n$
r,s	index integers
v	$n \times 1$ column matrix
g	$n \times 1$ column matrix
$b^{(k)},\bar{b}^{(k)}$	column vectors containing n elements
$y^{(k)},\bar{y}^{(k)}$	column vectors containing n elements
$h^{(k)}$	column vector containing n elements
$\bar{q}^{(k)},\bar{z}^{(k)}$	column vectors containing n elements
σ	parameter for determining vectors for monotonicity
Q^U,Q^L	$n \times n$ matrices verifying upper and lower bounds of the matrix B

INTRODUCTION

It is often impossible to determine exact solutions of systems of linear equations due to round off errors and approximate procedures which are necessary to attempt a solution.

An approximate solution of such a system has more meaning if the error of the approximate solution with respect to the exact solution is known. Therefore, if an exact solution of a system of linear equations of the form

$$Ax = c$$

cannot be determined, then it is desirable to be able to determine upper and lower bounds for x. When upper and lower bounds are determined, the maximum possible error in the approximate solution can be determined. However, it may not always be possible to determine bounds for the solution of such a system, but if the matrix A is a "monotonic" matrix, or more accurately, if the problem is "written as a problem of a monotonic type", then upper and lower bounds for x can be determined [1].

A necessary and sufficient condition for a matrix to be monotonic is that all elements of the inverse of the matrix be non-negative [1]. Since in general it is only possible to determine the inverse of a matrix approximately, the sign if the elements of the inverse is not definitely known. Therefore, theorems

[ ] Numbers in brackets designate references at end of report.

stating simpler conditions sufficient to assure the monotonicity of a matrix have been developed [1]. However, the existing theorems are not general enough to include all monotonic matrices.

This paper presents a theorem stating sufficient conditions to assure monotonicity which are less restrictive than the conditions of existing theorems. This theorem includes monotonic matrices not included under the existing theorems. It is also more readily applicable to practical problems than the existing theorems.

The theorem presented states conditions which, if satisfied, assure that all elements of the inverse of the matrix are positive. It is only necessary to determine an approximation to the inverse of the matrix to apply this theorem.

The monotonicity of a matrix is a very important property, for if bounds can be determined for x then it follows that bounds can be determined for the inverse of the monotonic matrix. With the bounds of the inverse of the monotonic matrix known, bounds for any case involving this matrix may be determined directly.

DEFINITION OF MONOTONIC SYSTEM

For the system of real linear equations $Ax = c$ the matrix A is called "monotonic" or "of monotonic type" [1] when for any real vector z such that

$$q = Az$$

and $$0 \leq q_i$$

implies that $$0 \leq z_i \quad \text{for all } i.$$

By reversing the sign of z it follows that

$$q_i \leq 0$$

implies that $$z_i \leq 0 \quad \text{for all } i$$

if A is a monotonic matrix. Therefore, $Az = 0$ implies that $z = 0$ and then the determinant of A cannot be zero.

BOUNDS FOR A MONOTONIC SYSTEM

Proof of Existence of Bounds for the Solution of a Monotonic System

The property of the matrix A being monotonic permits the determination of upper and lower bounds for the solution of the system $Ax = c$. If vectors y and w can be determined such that

$$q^{(y)} = Ay$$

$$\text{and} \quad q^{(w)} = Aw$$

$$\text{where} \quad q_i^{(y)} \leq c_i \leq q_i^{(w)}$$

then it follows that

$$y_i \leq x_i \leq w_i \quad \text{for all } i.$$

Proof:

Let $q^{(r)} = A(y-x) = Ay - Ax$

and $q^{(s)} = A(w-x) = Aw - Ax$

where $r = y-x$ and $s = w-x$. The above may be written as

$$q^{(r)} = Ay - Ax = q^{(y)} - c$$

$$q^{(s)} = Aw - Ax = q^{(w)} - c$$

so that $\quad q^{(y)} = q^{(r)} + c$

and $\quad q^{(w)} = q^{(s)} + c.$

The elements of the matrices are related as follows:

$$q_i^{(y)} = q_i^{(r)} + c_i$$

$$q_i^{(w)} = q_i^{(s)} + c_i$$

so that the relation

$$q_i^{(y)} \leq c_i \leq q_i^{(w)}$$

gives $$q_i^{(r)} + c_i \leq c_i \leq q_i^{(s)} + c_i \quad \text{for all } i.$$

Subtracting c_i from the above relation gives

$$q_i^{(r)} \leq 0 \leq q_i^{(s)}$$

and since the matrix A is monotonic, it follows that

$$y_i - x_i \leq 0 \leq w_i - x_i.$$

Then adding x_i to the above relation gives

$$y_i \leq x_i \leq w_i \quad \text{for all } i.$$

Therefore, w_i and y_i are upper and lower bounds respectively of x_i.

Therefore, if it can be shown that the matrix A of the system

$$Ax = c$$

is monotonic then upper and lower bounds of x can be determined.

Determination of Bounds for the Inverse of a Monotonic Matrix

Since upper and lower bounds can be determined for x for the system

$$Ax = c$$

if A is a monotonic matrix, then upper and lower bounds for the inverse of the matrix A can be determined also. This is shown as follows.

Form the vectors $q^{(k)}$ such that

$$q_i^{(k)} = \delta_{ik} = \begin{cases} 0 \text{ for } i \neq k \\ 1 \text{ for } i = k. \end{cases}$$

Form the vectors $y^{(k)}$ which correspond to the vectors $q^{(k)}$

$$Ay^{(k)} = q^{(k)}.$$

Since A is monotonic, upper and lower bounds can be determined for the vectors $y^{(k)}$ corresponding to the vectors $q^{(k)}$.

If an exact solution could be determined for the vectors $y^{(k)}$ corresponding to the vectors $q^{(k)}$ then the vectors $y^{(k)}$ would be identical to the columns of the inverse of the matrix A. Therefore, the matrix formed by using the upper bound vectors of $y^{(k)}$ as columns of the matrix would be the upper bound of the inverse matrix B^U and similarly the matrix formed using the lower bound vectors of $y^{(k)}$ would be the lower bound of the inverse matrix B^L. Then forming the product of the matrix A and the upper bound vectors of $y^{(k)}$, a matrix, denoted as Q^U, can be formed using the vectors resulting from these products as columns of the matrix. An identical procedure using the lower bound vectors of $y^{(k)}$ would form another matrix denoted as Q^L. Then all elements of the matrix Q^U will be greater than the corresponding elements of the unit matrix and all elements of the matrix Q^L will be less than the corresponding elements of the unit matrix which verifies that B^U and B^L are upper and lower bounds respectively of the inverse matrix.

Determination of Bounds on the Solution Using Bounds on the Inverse

Once the bounds are determined for the inverse matrix these bounds may be used to determine the bounds for the solution to this particular system. Therefore, the bounds for the solution may be determined easily for this same system with different conditions applied.

Form the product of the inverse matrix and the vector of constants of the system $Ax = c$ giving $x = Bc$. As is proven later, for the system to be monotonic all elements of the inverse matrix must be positive, which means that all elements of B are greater than zero. Consider also that the elements of the upper and lower bounds of B are all greater than zero. Then consider the possibility of all elements of c being negative which would produce negative elements for x since all elements of B are greater than zero. If the lower bound for x is to be calculated under these conditions the elements of the lower bound for x would be more negative than the elements of the exact solution of x. However, if the product of the lower bound for B and of c is formed under the above conditions the values calculated will be less negative than the exact solution for x since all elements of the lower bound for B are smaller than those for the exact inverse. Therefore, the vector calculated in this manner would be an upper bound for x. Similarly forming the above product with the upper bound for B would give the lower bound for x with all elements of c negative.

A similar result will occur if some elements of c are negative and the remainder of the elements are positive. Under these conditions all elements of x may be positive for both upper and lower bounds calculated using upper and lower bounds respectively of B and in fact the vectors calculated for upper and lower bounds of x in this manner may appear to be valid bounds. However, in forming the product of the original matrix A and these calculated bounds for x to calculate the corresponding upper and lower bounds for c, it will be discovered that the negative elements of c have become less negative in the lower bound vector of c and more negative in the upper bound vector of c. Thus the lower bound and upper bound have not been found for the negative elements of c even though the positive elements are satisfied.

In order to remedy this situation it is necessary to refer back to the condition where all elements of c were negative. It was concluded that the lower bound of x was determined by forming the product of the upper bound of B and the vector c. Therefore, when some of the elements of c are negative it is logical to be selective in the elements of either upper or lower bound of B used to determine the lower bound of x and consequently c. For the positive elements of c, products are formed with elements from the lower bound of B while for negative elements of c products are formed with elements from the upper bound of B. This will cause the negative elements of c to be more effective in making the negative elements of the lower bound of c more negative as shown by the following equations.

Elements of x are represented by

$$x_i = \sum_{j=1}^{n} b_{ij}\ c_j = b_{i1}\ c_1 + b_{i2}\ c_2 + \ .\ .\ .\ + b_{in}\ c_n$$

and from this the elements of c can be represented by

$$c_k = \sum_{i=1}^{n} a_{ki}\ (\sum_{j=1}^{n} b_{ij}\ c_j\ )$$

$$\begin{aligned} c_k = & \ a_{k1}\ (b_{11}\ c_1 + b_{12}\ c_2 + \ .\ .\ .\ + b_{1n}\ c_n\ ) \\ & + a_{k2}\ (b_{21}\ c_1 + b_{22}\ c_2 + \ .\ .\ .\ + b_{2n}\ c_n\ ) \\ & + \\ & \ . \\ & \ . \\ & \ . \\ & + a_{kn}\ (b_{n1}\ c_1 + b_{n2}\ c_2 + \ .\ .\ .\ + b_{nn}\ c_n\ ). \end{aligned}$$

Similarly, for calculating the upper bound of x when some elements of the vector c are negative, products of elements of the upper bound of B and the positive elements of c are formed and products of elements of the lower bound of B and the negative elements of c are formed in forming the matrix product of the matrix B and the vector c.

NECESSARY AND SUFFICIENT CONDITION FOR A SYSTEM TO BE MONOTONIC

A necessary and sufficient condition for the matrix A to be monotonic is that all elements of the inverse of A be positive [1]. A proof of this statement follows.

Proof:

Consider the system

$$A\,u = d$$

where the elements of A are a_{ij} and the elements of u and d are u_k and d_k respectively. Assuming this is a monotonic system, then

$$0 \leq d_k$$

implies $\quad 0 \leq u_k \quad$ for all k.

Let the inverse of A be denoted as B whose elements are b_{ij}. Premultiplying the system by B, it becomes

$$u = B\,d.$$

Then it must be proved that it is sufficient and also necessary that

$$0 \leq b_{ij} \quad \text{for all } i,j.$$

Certainly if all elements of B are non-negative and all elements of d are non-negative then the elements formed from the product of B and d

$$u_k = \sum_{j=1}^{n} b_{kj}\, d_j$$

are positive which satisfies the conditions of monotonicity. Therefore, it is sufficient that all elements of B be non-negative.

To prove that it is necessary that all elements of B be non-negative recall that for the system to be monotonic any vector d containing real elements such that

$$0 \leq d_k$$

implies that the vector u contains real elements such that

$$0 \leq u_k \quad \text{for all } k.$$

Assume that it is not necessary that all elements of B be non-negative and thus let one element be negative. Assume that b_{rs} ($1 < r < n$ and $1 < s < n$) is that negative element. Then let d be a vector such that

$$0 < d_k \text{ for } k = s$$

$$0 \equiv d_k \text{ for } k \neq s$$

and the system becomes

$$\begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_{r-1} \\ u_r \\ u_{r+1} \\ \cdot \\ \cdot \\ \cdot \\ u_{n-1} \\ u_n \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1s} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2s} & & b_{2n} \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ b_{r-1,1} & b_{r-1,2} & \dots & b_{r-1,s} & \dots & b_{r-1,n} \\ b_{r,1} & b_{r,2} & \dots & b_{r,s} & \dots & b_{r,n} \\ b_{r+1,1} & b_{r+1,2} & \dots & b_{r+1,s} & \dots & b_{r+1,n} \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ b_{n-1,1} & b_{n-1,2} & \dots & b_{n-1,s} & \dots & b_{n-1,n} \\ b_{n1} & b_{n2} & \dots & b_{ns} & \dots & b_{nn} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ d_s \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \end{bmatrix}$$

Consider the element u_r of the vector u. This is formed by the product of the r-th row of B and the vector d which gives

$$u_r = b_{rs}\, d_s.$$

Since d_s is positive and b_{rs} is negative u_r must be negative. However, the system is monotonic and the vector d satisfies the condition that $0 \leq d_k$ for all k which implies that the vector u must have $0 \leq u_k$ for all k. The above condition of u_r being negative is a contradiction of the monotonic system and therefore, B cannot have one negative element as was assumed.

It should be emphasized that the selection of the elements of the vector d is restricted only in that they must each be either equal to or greater than zero and then the elements of the vector u must be equal to or greater than zero for a monotonic system. From this it is concluded immediately that if B cannot have one negative element it certainly cannot have more than one. Therefore, it is necessary that all elements of the inverse matrix be non-negative.

EXISTING THEOREMS FOR DETERMINING IF A MATRIX IS MONOTONIC

As was stated previously, determination of the exact inverse of a matrix is difficult or even impossible for some matrices. Therefore, the criterion that all elements of the inverse matrix be positive is not a practical method of determining if a matrix is monotonic. It is desirable to arrive at more practical methods of determining if a system is monotonic.

Two existing theorems [1] which state sufficient conditions for determining if a matrix is monotonic are presented.

Theorem 1: If an $n \times n$ real matrix A is such that

1. $a_{jk} \leq 0$ for $j \neq k$,
2. A does not "decompose",
3. There exist non-zero vectors y and r such that $0 < y$, $0 \leq r$ and $Ay = r$,

then A is a monotonic matrix.

Theorem 2: If the coefficients a_{jk} of an $n \times n$ matrix A satisfy the conditions

1. sign distribution: $0 < a_{jj}$,

   $a_{jk} \leq 0$ for $j \neq k$,

2a. the "weak row sum criterion"

$$\sum_{k=1}^{n} a_{jk} \begin{cases} \geq 0 & \text{for } j = 1,2,\ldots,n \\ > 0 & \text{for at least one } j = j_o, \end{cases}$$

and

2b. the non-decomposition of A,

or instead of 2a. and 2b., the stronger condition

2c. the "ordinary row-sum criterion":

$$\sum_{k=1}^{n} a_{jk} > 0 \quad \text{for } j = 1,2,\ldots,n,$$

then A is monotonic and in particular det $A \neq 0$.

It should be emphasized that these two theorems state only sufficient conditions for the matrix to be monotonic and therefore are possibly more restrictive than is necessary. Both theorems require that all elements off the main diagonal of the matrix be negative or equal to zero. This is not a necessary criterion for a matrix to be monotonic as is demonstrated by the following example.

Example 1.

$$A = \begin{bmatrix} 1/4 & 0 & -1/2 \\ -1/8 & 1/2 & -1/4 \\ 1/8 & -1/2 & 5/4 \end{bmatrix}$$

Letting the inverse of A be denoted as B

$$B = \begin{bmatrix} 4 & 2 & 2 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

All elements of the inverse are non-negative which satisfies the requirements for A to be monotonic.

AN ADDITIONAL THEOREM FOR DETERMINING MONOTONICITY

Now another theorem for determining if a system is monotonic will be presented. The theorem presented is very adaptable to practical applications and less restrictive than the two previously stated theorems. This theorem states only sufficient conditions for monotonicity.

Matrix multiplication is a straight forward procedure so it is possible to assume a vector v and to determine accurately the corresponding vector q for the system

$$q = Av.$$

However, the reverse problem of assuming a vector q and determining the corresponding vector v for the above system is not so straight forward and often times the accuracy of v is not satisfactory.

If it were possible to choose a set of vectors

$$v^{(k)}$$

such that $$0 \le v_i^{(k)} \text{ for all } i,k$$

and if the corresponding vectors $q^{(k)}$ resulting from the above system would be

$$q^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \quad q^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \text{ etc. to } q^{(n)} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

or in general

$$q_i^{(k)} = \delta_{ik} = \begin{cases} 1 \text{ for } i = k, \\ 0 \text{ for } i \neq k \end{cases}$$

this would be sufficient to prove that A is monotonic. However, this is generally not possible since there is a unique vector $v^{(k)}$ corresponding to each vector $q^{(k)}$.

The system

$$Av = q$$

can be considered as representing a physical system such as a beam with loads applied at discrete points. The vector q would represent the loads applied to the beam and the vector v would represent the deflections of points on the beam.

Considering the system as representing a loaded beam, it is intuitive that positive and negative loading can be applied to the beam with the resulting deflections all positive. This concept will be used to replace zero elements in the vectors $q^{(k)}$ above. Using this approach a theorem stating conditions for determining if A is monotonic will be proven.

Statement of the Theorem

Theorem: If a set of real vectors $w^{(k)}$ can be constructed with

$$0 \leq w_i^{(k)} \quad \text{for all } i,k$$

and the product

$$Aw^{(k)} = g^{(k)} \quad \text{for A an } n \times n \text{ real matrix}$$

exists such that

$$g_i^{(k)} \leq 0 \quad \text{for } i \neq k$$

and $$0 < g_k^{(k)}$$

THE FOLLOWING PAGES CONTAIN CROOKED TYPING AND IS THE BEST POSSIBLE IMAGE AVAILABLE and $$0 < \sum_{i=1}^{n} g_k^{(i)} \quad \text{for all } k$$

then A is a monotonic matrix.

Proof of the Theorem

Consider that vectors $w^{(k)}$ have been constructed such that

$$0 \leq w_i^{(k)} \quad \text{for all } i,k$$

and the corresponding vectors $g^{(k)}$ are such that

$$g_i^{(k)} \leq 0 \quad \text{for } i \neq k$$

and $$0 < g_k^{(k)}$$

and $$0 < \sum_{i=1}^{n} g_k^{(i)} \quad \text{for all } k$$

where $$g^{(k)} = Aw^{(k)}.$$

To prove that A is monotonic it is sufficient to show that all elements of the inverse of A are non-negative.

Again let the inverse of A be denoted as B and then pre-multiplying the above system by B gives

$$w^{(k)} = Bg^{(k)}.$$

Then with the elements of B denoted as b_{ij} the elements of $w^{(k)}$ are

$$w_i^{(k)} = \sum_{j=1}^{n} b_{ij}\, g_j^{(k)}.$$

Assume that only one element of B is negative. Let that element be b_{rs}. Consider the vector $g^{(s)}$ so the system becomes

$$\begin{bmatrix} w_1^{(s)} \\ w_2^{(s)} \\ \cdot \\ \cdot \\ \cdot \\ w_r^{(s)} \\ \cdot \\ \cdot \\ \cdot \\ w_n^{(s)} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & \cdot & \cdot & \cdot & b_{1s} & \cdot & \cdot & \cdot & b_{1n} \\ b_{21} & b_{22} & \cdot & \cdot & \cdot & b_{2s} & \cdot & \cdot & \cdot & b_{2n} \\ \cdot & \cdot & & & & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot & & & & \cdot \\ b_{r1} & b_{r2} & \cdot & \cdot & \cdot & b_{rs} & \cdot & \cdot & \cdot & b_{rn} \\ \cdot & \cdot & & & & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot & & & & \cdot \\ b_{n1} & b_{n2} & \cdot & \cdot & \cdot & b_{ns} & \cdot & \cdot & \cdot & b_{nn} \end{bmatrix} \begin{bmatrix} g_1^{(s)} \\ g_2^{(s)} \\ \cdot \\ \cdot \\ \cdot \\ g_r^{(s)} \\ \cdot \\ \cdot \\ \cdot \\ g_n^{(s)} \end{bmatrix}$$

Then determine the element $w_r^{(s)}$ as

$$w_r^{(s)} = b_{r1}\, g_1^{(s)} + b_{r2}\, g_2^{(s)} + \ldots + b_{rs}\, g_s^{(s)} + \ldots + b_{rn}\, g_n^{(s)}.$$

From previous statements concerning the elements of $g^{(k)}$ and of B it is concluded that all terms in $w_r^{(s)}$ are non-positive so

$$w_r^{(s)} < 0$$

which is a contradiction since the vectors $w^{(k)}$ were constructed with all elements non-negative. It is concluded that B cannot have only one negative element.

Next it is assumed that two elements of B are negative. Considering the above proof concerning one negative element of B it is intuitive that assuming two elements of B to be negative and the elements being located in separate rows just results in in two proofs identical to the above proof. Therefore, it is not possible for two rows or all rows of B to have only one negative element. Then it is assumed that the two negative elements of B are in the same row. Let these elements be b_{rs} and b_{rt}. All other elements of B are positive.

Then consider the two elements $w_r^{(s)}$ and $w_r^{(t)}$

$$w_r^{(s)} = b_{r1}\, g_1^{(s)} + b_{r2}\, g_2^{(s)} + \dots + b_{rs}\, g_s^{(s)} + \dots + b_{rt}\, g_t^{(s)} + \dots + b_{rn}\, g_n^{(s)}$$

$$w_r^{(t)} = b_{r1}\, g_1^{(t)} + b_{r2}\, g_2^{(t)} + \dots + b_{rs}\, g_s^{(t)} + \dots + b_{rt}\, g_t^{(t)} + \dots + b_{rn}\, g_n^{(t)}.$$

Add these two equations to get $w_r^{(s)} + w_r^{(t)}$

$$w_r^{(s)} + w_r^{(t)} = b_{r1}\, g_1^{(s)} + b_{r1}\, g_1^{(t)} + b_{r2}\, g_2^{(s)} + b_{r2}\, g_2^{(t)} + \dots + b_{rs}\, g_s^{(s)} + b_{rs}\, g_s^{(t)} + \dots + b_{rt}\, g_t^{(s)} + b_{rt}\, g_t^{(t)} + \dots + b_{rn}\, g_n^{(s)} + b_{rn}\, g_n^{(t)}$$

$$w_r^{(s)} + w_r^{(t)} = b_{r1}\, (g_1^{(s)} + g_1^{(t)}) + b_{r2}\, (g_2^{(s)} + g_2^{(t)}) + \dots + b_{rs}\, (g_s^{(s)} + g_s^{(t)}) + \dots + b_{rt}\, (g_t^{(s)} + g_t^{(t)}) + \dots + b_{rn}\, (g_n^{(s)} + g_n^{(t)}).$$

From the preceding statements concerning the elements of the vectors $g^{(k)}$ and the elements of B it is concluded immediately that all terms on the right side of the equation are non-positive except

$$b_{rs}\, (g_s^{(s)} + g_s^{(t)}) \text{ and } b_{rt}\, (g_t^{(s)} + g_t^{(t)}).$$

From the conditions required of the sum of the elements $g_k^{(i)}$ it follows that

$$0 < (g_s^{(t)} + g_s^{(s)}) \quad \text{and} \quad 0 < (g_t^{(s)} + g_t^{(t)})$$

which means that

$$b_{rs}\ (g_s^{(s)} + g_s^{(t)}) < 0$$

and $$b_{rt}\ (g_t^{(s)} + g_t^{(t)}) < 0\ .$$

It follows that

$$(w_r^{(s)} + w_r^{(t)}) < 0$$

and then at least one of $w_r^{(s)}$ or $w_r^{(t)}$ must be negative. This is a contradiction since the vectors $w^{(k)}$ were constructed with all elements non-negative.

Therefore, it is concluded that one row of B cannot have only two negative elements. Considering that two rows of B may each have two negative elements it is concluded that this would result in two proofs the same as the above proof for one row with two negative elements. It is concluded that two rows and in fact all n-rows cannot have only two negative elements.

The above proof for two negative elements of one row of B may be generalized for all elements of one row of B. Assume that all elements of the r-th row of B are negative. Then consider the r-th element of each of the vectors $w^{(k)}$.

$$w_r^{(1)} = b_{r1}\ g_1^{(1)} + b_{r2}\ g_2^{(1)} + \ldots + b_{rn}\ g_n^{(1)}$$
$$w_r^{(2)} = b_{r1}\ g_1^{(2)} + b_{r2}\ g_2^{(2)} + \ldots + b_{rn}\ g_n^{(2)}$$

$$\vdots$$

$$w_r^{(n)} = b_{r1}\ g_1^{(n)} + b_{r2}\ g_2^{(n)} + \ldots + b_{rn}\ g_n^{(n)}$$

Summing the equations and combining terms on the right side of the equation gives

$$w_r^{(1)} + w_r^{(2)} + \cdots + w_r^{(n)} = b_{r1} \, (g_1^{(1)} + g_1^{(2)} + \cdots + g_1^{(n)}) + b_{r2} \, (g_2^{(1)} + g_2^{(2)} + \cdots + g_2^{(n)}) + \cdots + b_{rn} \, (g_n^{(1)} + g_n^{(2)} + \cdots + g_n^{(n)}).$$

From the conditions stated for the elements of the vectors $g^{(k)}$ it is concluded that the sums in parenthesis are all positive. Since it was assumed that all elements of the r-th row of B were negative, the right side of the equation is negative. This requires that at least one of the terms $w_r^{(k)}$ is negative which is a contradiction, as all elements of the vectors $w^{(k)}$ were constructed non-negative. Therefore, it is concluded that all elements of any one row of B cannot be negative. This conclusion is valid for two or more rows of B also since this would only require two or more proofs identical to the one above.

This completes the proof of the theorem.

Conditions for Failure of the Theorem

The preceding theorem states sufficient conditions for a matrix to be monotonic but a matrix may be monotonic and the conditions of the theorem still not be satisfied. One case of a monotonic matrix which does not satisfy the conditions is the monotonic matrix whose inverse has one or more elements equal to zero.

Consider the inverse matrix B again and let the element b_{rs} be zero and determine $w_r^{(s)}$

$$w_r^{(s)} = b_{r1}\, g_1^{(s)} + b_{r2}\, g_2^{(s)} + \ldots + b_{rs}\, g_s^{(s)} + \ldots + b_{rn}\, g_n^{(s)}.$$

With $b_{rs} = 0$ and all other terms satisfying the conditions of the theorem the right side of the equation is negative so

$$w_r^{(s)} < 0.$$

Therefore, it is impossible to form vectors $w^{(k)}$ with all non-negative elements and yet the matrix which has B as its inverse can satisfy the conditions of monotonicity that all the elements of the inverse be non-negative. It is not necessary that $b_{rs} \equiv 0$ for the theorem to fail. If b_{rs} is a small positive value such that the product $b_{rs}\, g_s^{(s)}$ is less than the absolute value of the sum of all the other terms, then $w_r^{(s)} < 0$ so the conditions of the theorem cannot be satisfied.

PRACTICAL APPLICATION OF THE THEOREM

The construction of a set of vectors $w^{(k)}$ which satisfy the conditions of the theorem and whose product with the matrix A produces the vectors $g^{(k)}$ which satisfy the theorem could result in a long process of trial and error. A procedure of finding vectors $w^{(k)}$ and $g^{(k)}$, which satisfy the theorem, using an approximation to the inverse of the matrix is presented at this point.

An approximation to the inverse of the $n \times n$ matrix A denoted as B^o is determined. Then form the vectors $h^{(k)}$ such that

$$h^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \quad h^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \text{ etc. to } h^{(n)} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

or $\qquad h_j^{(k)} = 0$ for $j \neq k$

and $\qquad h_k^{(k)} = 1$ for all k.

Premultiplying the vectors $h^{(k)}$ by the matrix B^o forms the vectors $\bar{y}^{(k)}$ whose elements are identical to those of B^o

$$\bar{y}^{(k)} = B^o h^{(k)} \text{ for all k.}$$

Premultiplying the vectors $\bar{y}^{(k)}$ by A forms the vectors $\bar{g}^{(k)}$

$$A\bar{y}^{(k)} = \bar{g}^{(k)}.$$

Normalizing the vectors $\bar{b}^{(k)}$ forms the vectors $b^{(k)}$

$$b^{(k)} = \frac{\bar{b}^{(k)}}{\bar{b}_k^{(k)}}$$

and also form
$$y^{(k)} = \frac{\bar{y}^{(k)}}{\bar{b}_k^{(k)}} .$$

If B^o is a reasonably accurate approximation of the inverse of A then

$$b_j^{(k)} \approx h_j^{(k)} \quad j = 1,2,\ldots,n, \quad k = 1,2,\ldots,n.$$

Form the vectors $\bar{z}^{(k)}$ such that

$$\bar{z}^{(k)} = y^{(k)} - (1+\sigma) \sum_{\substack{j=1 \\ j\neq k \\ \text{for } 0 \leq b_j^{(k)}}}^{n} b_j^{(k)} y^{(j)} - (1-\sigma) \sum_{\substack{j=1 \\ j\neq k \\ \text{for } b_j^{(k)} < 0}}^{n} b_j^{(k)} y^{(j)}$$

and premultiplying the vectors $\bar{z}^{(k)}$ by A forming the vectors $\bar{q}^{(k)}$

$$A\bar{z}^{(k)} = \bar{q}^{(k)}.$$

The preceding equation for the vectors $\bar{z}^{(k)}$ when multiplied by A becomes

$$\bar{q}^{(k)} = b^{(k)} - (1+\sigma) \sum_{\substack{j=1 \\ j\neq k \\ \text{for } 0 \leq b_j^{(k)}}}^{n} b_j^{(k)} b^{(j)} - (1-\sigma) \sum_{\substack{j=1 \\ j\neq k \\ \text{for } b_j^{(k)} < 0}}^{n} b_j^{(k)} b^{(j)} .$$

The elements of the vectors $\tilde{q}^{(k)}$ become

$$\tilde{q}_i^{(k)} = b_i^{(k)} - (1+\sigma) \sum_{\substack{j=1 \\ j\neq k \\ \text{for } 0 \leq b_j^{(k)}}}^{n} b_j^{(k)}\, b_i^{(j)} - (1-\sigma) \sum_{\substack{j=1 \\ j\neq k \\ \text{for } b_j^{(k)} < 0}}^{n} b_j^{(k)}\, b_i^{(j)} .$$

For the case of $j=i$ and $i\neq k$ in the sum the equation becomes

$$\tilde{q}_i^{(k)} = b_i^{(k)} - (1+\sigma) \sum_{\substack{j=1 \\ j\neq k \\ j\neq i \\ \text{for } 0 \leq b_j^{(k)}}}^{n} b_j^{(k)}\, b_i^{(j)} - (1-\sigma) \sum_{\substack{j=1 \\ j\neq k \\ j\neq i \\ \text{for } b_j^{(k)} < 0}}^{n} b_j^{(k)}\, b_i^{(j)} - \underset{\text{for } 0 \leq b_i^{(k)}}{(1+\sigma)\, b_i^{(k)}\, b_i^{(i)}} - \underset{\text{for } b_i^{(k)} < 0}{(1-\sigma)\, b_i^{(k)}\, b_i^{(i)}}$$

simplifying further

$$\tilde{q}_i^{(k)} = -\sigma \left| b_i^{(k)} \right| - (1+\sigma) \sum_{\substack{j=1 \\ j\neq k \\ j\neq i \\ \text{for } 0 \leq b_j^{(k)}}}^{n} b_j^{(k)}\, b_i^{(j)} - (1-\sigma) \sum_{\substack{j=1 \\ j\neq k \\ j\neq i \\ \text{for } b_j^{(k)} < 0}}^{n} b_j^{(k)}\, b_i^{(j)} .$$

This equation may be simplified still further to give

$$\bar{q}_i^{(k)} = -\sigma \left(\left| b_i^{(k)} \right| + \sum_{\substack{j=1 \\ j\neq k \\ j\neq i}}^{n} \left| b_j^{(k)} \right| b_i^{(j)} \right) - \sum_{\substack{j=1 \\ j\neq k \\ j\neq i}}^{n} b_j^{(k)} b_i^{(j)} \qquad \text{for } i\neq k.$$

The k-th elements of the vectors $\bar{q}^{(k)}$ become

$$\bar{q}_k^{(k)} = b_k^{(k)} - (1+\sigma) \sum_{\substack{j=1 \\ j\neq k \\ \text{for } 0 \leq b_j^{(k)}}}^{n} b_j^{(k)} b_k^{(j)} - (1-\sigma) \sum_{\substack{j=1 \\ j\neq k \\ \text{for } b_j^{(k)} < 0}}^{n} b_j^{(k)} b_k^{(j)} ,$$

or

$$\bar{q}_k^{(k)} = b_k^{(k)} - \sigma \sum_{\substack{j=1 \\ j\neq k}}^{n} \left| b_j^{(k)} \right| b_k^{(j)} - \sum_{\substack{j=1 \\ j\neq k}}^{n} b_j^{(k)} b_k^{(j)} ,$$

and since $b_k^{(k)} = 1$

$$\bar{q}_k^{(k)} = 1 - \sigma \sum_{\substack{j=1 \\ j\neq k}}^{n} \left| b_j^{(k)} \right| b_k^{(j)} - \sum_{\substack{j=1 \\ j\neq k}}^{n} b_j^{(k)} b_k^{(j)} .$$

The value of σ has not been specified. If the products in the sums are small second order terms the value of σ may be selected arbitrarily such that the vectors $\bar{q}^{(k)}$ will satisfy the theorem.

After forming the new vectors the following checks must be made.

1. Do the elements of the vectors $z^{(k)}$ satisfy

$$0 \leq z_i^{(k)} \quad \text{for all } i,\ k?$$

2. Do the elements of the vectors $\tilde{q}^{(k)}$ satisfy

$$\tilde{q}_i^{(k)} \leq 0 \quad \text{for } i \neq k,$$

$$0 < \tilde{q}_k^{(k)}$$

$$0 < \sum_{i=1}^{n} \tilde{q}_k^{(i)} \quad \text{for all } k?$$

If these conditions are satisfied then it is concluded that the matrix A is monotonic.

EXAMPLE PROBLEMS

The theorem for monotonicity was applied to three example problems. The examples are systems of equations for finite difference solutions to differential equations. Either the exact solution to the differential equation or a finite difference approximate solution is known for all examples [2,3,4].

Example No. 1 is the system of finite difference equations for a two dimensional elasticity problem [2]. Example No. 2 is the system of finite difference equations for the temperature distribution in a corrugated wall under steady state conditions [4]. Example No. 3 is the system of finite difference equations for the deflection of a beam on an elastic foundation with a uniformly distributed load applied [3].

The systems of equations resulting for each example are presented on the following pages. The resulting vectors proving monotonicity of each example are tabulated following each example.

In addition the following are tabulated for Example No. 1;

1. upper and lower bounds for the inverse of the coefficient matrix and corresponding verification matrices Q^U and Q^L,
2. upper and lower bounds for the solution vector and corresponding verification vectors c^U and c^L,

3. maximum possible error of the solution resulting from the average of the upper and lower bounds of the solution vector.

$$
\begin{bmatrix}
21 & -16 & 2 & -8 & 4 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-8 & 22 & -8 & 2 & -8 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -8 & 22 & 0 & 2 & -8 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-8 & 4 & 0 & 20 & -16 & 2 & -8 & 4 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & -8 & 2 & -8 & 21 & -8 & 2 & -8 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 2 & -8 & 1 & -8 & 21 & 0 & 2 & -8 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & -8 & 4 & 0 & 20 & -16 & 2 & -8 & 4 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 2 & -8 & 2 & -8 & 21 & -8 & 2 & -8 & 2 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 2 & -8 & 1 & -8 & 21 & 0 & 2 & -8 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & -8 & 4 & 0 & 20 & -16 & 2 & -8 & 4 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 & -8 & 2 & -8 & 21 & -8 & 2 & -8 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & -8 & 1 & -8 & 21 & 0 & 2 & -8 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -8 & 4 & 0 & 21 & -16 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & -8 & 2 & -8 & 22 & -8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & -8 & 1 & -8 & 22
\end{bmatrix}
\begin{bmatrix}
\phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \\ \phi_9 \\ \phi_{10} \\ \phi_{11} \\ \phi_{12} \\ \phi_{13} \\ \phi_{14} \\ \phi_{15}
\end{bmatrix}
=
\begin{bmatrix}
16.4 \\ 15.12 \\ 8.52 \\ -3.6 \\ -2.38 \\ 0.32 \\ 0.0 \\ 0.72 \\ 1.92 \\ 0.0 \\ 0.72 \\ 1.92 \\ 0.0 \\ 0.72 \\ 1.92
\end{bmatrix}
$$

EXAMPLE NO. 1

TABLE I

Vectors $\tilde{z}^{(k)}$ for monotonicity of example no. 1

k = 1	k = 2	k = 3	k = 4	k = 5	k = 6	k = 7	k = 8
0.1092355	0.1111475	0.0335884	0.0924466	0.1211750	0.0427997	0.0588537	0.0851871
0.0555742	0.1240675	0.0452408	0.0603128	0.1102460	0.0455051	0.0422434	0.0712898
0.0167942	0.0452406	0.0706763	0.0206871	0.0445110	0.0418393	0.0150434	0.0284670
0.0924438	0.1206198	0.0413732	0.2134391	0.2638310	0.0928571	0.1620892	0.2274975
0.0605864	0.1102428	0.0445104	0.1319171	0.2534118	0.1009987	0.1136170	0.1976526
0.0214033	0.0455101	0.0418417	0.0464374	0.1010111	0.1079692	0.0422435	0.0823231
0.0588539	0.0844853	0.0300872	0.1620947	0.2272396	0.0844699	0.2521255	0.3241340
0.0426023	0.0713014	0.0284722	0.1137722	0.1976840	0.0823208	0.1620898	0.3024138
0.0159390	0.0296935	0.0191685	0.0426037	0.0828164	0.0625252	0.0588538	0.1221318
0.0297020	0.0440444	0.0155711	0.0890162	0.1315366	0.0497191	0.1620892	0.2274944
0.0222281	0.0357519	0.0136624	0.0657689	0.1090449	0.0449769	0.1136170	0.1976489
0.0082908	0.0143445	0.0073663	0.0248676	0.0449889	0.0270437	0.0422434	0.0823211
0.0093530	0.0139137	0.0047487	0.0297032	0.0444577	0.0165757	0.0588537	0.0851855
0.0069570	0.0109537	0.0039672	0.0220237	0.0357539	0.0143403	0.0422435	0.0712385
0.0023743	0.0039670	0.0017921	0.0077857	0.0136627	0.0073644	0.0150433	0.0284662

σ required to get vector

2.0	2.0	2.0	2.0	0.5	10.5	2.0	11.0

TABLE I (Cont'd)

Vectors $\tilde{z}^{(k)}$ for monotonicity of example no. 1

k = 9	k = 10	k = 11	k = 12	k = 13	k = 14	k = 15
0.0318742	0.0296975	0.0444581	0.0165785	0.0093514	0.0139115	0.0047439
0.0296913	0.0220194	0.0357543	0.0143425	0.0069557	0.0109520	0.0039673
0.0191674	0.0077840	0.0136629	0.0073654	0.0023738	0.0039664	0.0017922
0.0851956	0.0890023	0.1315379	0.0497255	0.0296981	0.0440392	0.0155715
0.0828082	0.0657582	0.1090460	0.0449821	0.0222251	0.0357479	0.0136628
0.0625289	0.0248632	0.0449894	0.0270462	0.0082896	0.0143429	0.0073665
0.1176976	0.1620788	0.2272415	0.0844769	0.0588487	0.0844785	0.0300878
0.1221385	0.1137594	0.1976857	0.0823265	0.0425985	0.0712960	0.0284727
0.1185856	0.0425981	0.0828172	0.0625283	0.0159374	0.0296912	0.0191688
0.0851957	0.2134278	0.2638331	0.0928626	0.0924385	0.1206135	0.0413738
0.0828082	0.1319078	0.2534137	0.1010027	0.0605828	0.1102377	0.0445110
0.0625284	0.0464330	0.1010119	0.1079713	0.0214018	0.0455078	0.0418420
0.0318744	0.0924416	0.1211760	0.0428024	0.1092331	0.1111442	0.0335888
0.0296915	0.0603088	0.1102469	0.0455075	0.0555725	0.1240648	0.0452412
0.0191673	0.0206853	0.0445114	0.0418405	0.0167934	0.0452392	0.0706765

σ required to get vector

7.0	6.0	0.5	6.0	7.5	6.0	0.5

TABLE II

Vectors $\tilde{q}^{(k)}$ for monotonicity of example no. 1

k = 1	k = 2	k = 3	k = 4	k = 5	k = 6	k = 7	k = 8
0.99999638	-0.294×10^{-5}	-0.600×10^{-6}	-0.454×10^{-6}	-0.194×10^{-6}	-0.179×10^{-5}	-0.387×10^{-6}	-0.826×10^{-5}
-0.521×10^{-7}	0.99999825	-0.109×10^{-5}	-0.715×10^{-6}	-0.131×10^{-5}	-0.152×10^{-4}	-0.153×10^{-5}	-0.233×10^{-4}
-0.671×10^{-6}	-0.118×10^{-5}	0.99999619	-0.782×10^{-7}	-0.522×10^{-6}	-0.832×10^{-5}	-0.447×10^{-7}	-0.457×10^{-5}
-0.781×10^{-5}	-0.121×10^{-4}	-0.167×10^{-5}	1.0000086	-0.659×10^{-5}	-0.323×10^{-4}	-0.126×10^{-4}	-0.107×10^{-3}
-0.410×10^{-6}	-0.430×10^{-5}	-0.354×10^{-6}	-0.149×10^{-6}	0.99999927	-0.366×10^{-8}	-0.738×10^{-6}	-0.222×10^{-7}
-0.522×10^{-7}	-0.200×10^{-5}	-0.185×10^{-5}	-0.335×10^{-6}	-0.281×10^{-6}	0.99999965	-0.116×10^{-5}	-0.447×10^{-6}
-0.551×10^{-6}	-0.329×10^{-5}	-0.119×10^{-6}	-0.163×10^{-5}	-0.834×10^{-6}	-0.162×10^{-4}	0.99999760	-0.332×10^{-4}
-0.108×10^{-6}	-0.104×10^{-6}	-0.115×10^{-6}	-0.166×10^{-5}	-0.213×10^{-5}	-0.568×10^{-5}	-0.367×10^{-5}	1.0000025
-0.292×10^{-6}	-0.156×10^{-5}	-0.539×10^{-6}	-0.913×10^{-6}	-0.119×10^{-5}	-0.149×10^{-4}	-0.356×10^{-5}	-0.263×10^{-4}
-0.200×10^{-5}	-0.367×10^{-5}	-0.522×10^{-6}	-0.371×10^{-5}	-0.314×10^{-5}	-0.250×10^{-4}	-0.127×10^{-4}	-0.102×10^{-3}
-0.787×10^{-6}	-0.108×10^{-5}	-0.305×10^{-6}	-0.298×10^{-7}	-0.749×10^{-6}	-0.714×10^{-5}	-0.171×10^{-6}	-0.284×10^{-4}
-0.125×10^{-6}	-0.138×10^{-6}	-0.447×10^{-7}	-0.842×10^{-6}	-0.764×10^{-6}	-0.293×10^{-5}	-0.190×10^{-5}	-0.966×10^{-5}
-0.871×10^{-7}	-0.372×10^{-7}	-0.927×10^{-9}	-0.670×10^{-7}	-0.659×10^{-6}	-0.114×10^{-5}	-0.115×10^{-5}	-0.125×10^{-4}
-0.345×10^{-6}	-0.440×10^{-6}	-0.160×10^{-6}	-0.149×10^{-6}	-0.127×10^{-6}	-0.221×10^{-5}	-0.440×10^{-6}	-0.136×10^{-4}
-0.638×10^{-7}	-0.335×10^{-6}	-0.214×10^{-7}	-0.525×10^{-6}	-0.335×10^{-6}	-0.210×10^{-5}	-0.901×10^{-6}	-0.522×10^{-5}

TABLE II (Cont'd)

Vectors $\tilde{q}^{(k)}$ for monotonicity of example no. 1

k = 9	k = 10	k = 11	k = 12	k = 13	k = 14	k = 15	$\sum_{k=1}^{n} \tilde{q}_i^{(k)}$
-0.167×10^{-5}	-0.371×10^{-8}	-0.149×10^{-7}	-0.372×10^{-8}	-0.233×10^{-6}	-0.458×10^{-6}	-0.424×10^{-7}	0.99997933
-0.333×10^{-5}	-0.137×10^{-5}	-0.320×10^{-6}	-0.273×10^{-5}	-0.215×10^{-5}	-0.224×10^{-5}	-0.114×10^{-6}	0.99994279
-0.142×10^{-5}	-0.119×10^{-6}	-0.168×10^{-7}	-0.149×10^{-6}	-0.559×10^{-6}	-0.583×10^{-11}	-0.494×10^{-7}	0.99997849
-0.203×10^{-4}	-0.270×10^{-4}	-0.328×10^{-5}	-0.160×10^{-4}	-0.707×10^{-5}	-0.121×10^{-4}	-0.261×10^{-6}	0.99974262
-0.440×10^{-5}	-0.775×10^{-5}	-0.931×10^{-7}	-0.618×10^{-6}	-0.199×10^{-5}	-0.130×10^{-5}	-0.461×10^{-7}	0.99997709
-0.224×10^{-7}	-0.198×10^{-5}	-0.149×10^{-6}	-0.559×10^{-6}	-0.959×10^{-6}	-0.570×10^{-6}	-0.820×10^{-7}	0.99998921
-0.101×10^{-4}	-0.155×10^{-4}	-0.127×10^{-5}	-0.139×10^{-4}	-0.245×10^{-5}	-0.694×10^{-5}	-0.298×10^{-7}	0.99989162
-0.792×10^{-5}	-0.150×10^{-4}	-0.147×10^{-5}	-0.224×10^{-5}	-0.840×10^{-6}	-0.301×10^{-5}	-0.932×10^{-8}	0.99995855
0.99999683	-0.437×10^{-5}	-0.628×10^{-6}	-0.570×10^{-5}	-0.922×10^{-6}	-0.278×10^{-5}	-0.164×10^{-6}	0.99993301
-0.205×10^{-4}	1.0000093	-0.519×10^{-5}	-0.212×10^{-4}	-0.249×10^{-4}	-0.170×10^{-4}	-0.736×10^{-6}	0.99976704
-0.220×10^{-5}	-0.258×10^{-5}	1.0000047	-0.810×10^{-5}	-0.392×10^{-7}	-0.113×10^{-4}	-0.192×10^{-6}	0.99994160
-0.843×10^{-5}	-0.880×10^{-5}	-0.559×10^{-7}	1.0000012	-0.109×10^{-5}	-0.244×10^{-5}	-0.790×10^{-6}	0.99996316
-0.146×10^{-5}	-0.881×10^{-5}	-0.909×10^{-6}	-0.209×10^{-5}	0.99999562	-0.812×10^{-5}	-0.371×10^{-6}	0.99995824
-0.417×10^{-6}	-0.332×10^{-5}	-0.566×10^{-6}	-0.592×10^{-5}	-0.184×10^{-5}	0.99999526	-0.224×10^{-6}	0.99996553
-0.213×10^{-5}	-0.297×10^{-5}	-0.745×10^{-7}	-0.200×10^{-5}	-0.327×10^{-5}	-0.812×10^{-5}	0.99999788	0.99996981

TABLE III

Lower bound for the inverse matrix for example no. 1

0.1090175	0.1109256	0.0335213	0.0922621	0.1209331	0.0427142	0.0587362	0.0850171
0.0554632	0.1238198	0.0451505	0.0601923	0.1100260	0.0454142	0.0421591	0.0711475
0.0167606	0.0451502	0.0705352	0.0206458	0.0444221	0.0417557	0.0150133	0.0284102
0.0922592	0.1203790	0.0412905	0.2130130	0.2633044	0.0926717	0.1617656	0.2270434
0.0604655	0.1100228	0.0444215	0.1316537	0.2529060	0.1007970	0.1133902	0.1972581
0.0213605	0.0454192	0.0417581	0.0463447	0.1008094	0.1077536	0.0421591	0.0821588
0.0587364	0.0843166	0.0300271	0.1617712	0.2267860	0.0843012	0.2516223	0.3234871
0.0425172	0.0711590	0.0284153	0.1135451	0.1972894	0.0821564	0.1617662	0.3018102
0.0159072	0.0296342	0.0191302	0.0425186	0.0826511	0.0624004	0.0587363	0.1218880
0.0296426	0.0439565	0.0155400	0.0888385	0.1312740	0.0496198	0.1617657	0.2270403
0.0221836	0.0356804	0.0136351	0.0656375	0.1088272	0.0448871	0.1133902	0.1972543
0.0082742	0.0143158	0.0073516	0.0248179	0.0448991	0.0269897	0.0421591	0.0821568
0.0093343	0.0138859	0.0047392	0.0296439	0.0443689	0.0165426	0.0587362	0.0850155
0.0069431	0.0109318	0.0039592	0.0219797	0.0356825	0.0143117	0.0421591	0.0711461
0.0023695	0.0039590	0.0017885	0.0077701	0.0136353	0.0073496	0.0150132	0.0284093

TABLE III (Cont'd)

Lower bound for the inverse matrix for example no. 1

0.0318105	0.0296382	0.0443694	0.0165453	0.0093327	0.0138837	0.0047393
0.0296319	0.0219754	0.0356829	0.0143138	0.0069418	0.0109301	0.0039594
0.0191291	0.0077684	0.0136355	0.0073507	0.0023690	0.0039584	0.0017885
0.0850255	0.0888246	0.1312753	0.0496262	0.0296387	0.0439512	0.0155404
0.0826428	0.0656269	0.1088283	0.0448923	0.0221807	0.0356765	0.0136354
0.0624040	0.0248136	0.0448996	0.0269922	0.0082730	0.0143142	0.0073517
0.1174626	0.1617553	0.2267879	0.0843083	0.0587312	0.0843098	0.0300277
0.1218947	0.1135323	0.1972911	0.0821621	0.0425134	0.0711536	0.0284158
0.1183489	0.0425131	0.0826518	0.0624034	0.0159056	0.0296319	0.0191305
0.0850256	0.2130018	0.2633064	0.0926772	0.0922539	0.1203727	0.0412912
0.0826429	0.1316445	0.2529078	0.1008010	0.0604619	0.1100176	0.0444221
0.0624035	0.0463403	0.1008102	0.1077557	0.0213591	0.0454169	0.0417584
0.0318108	0.0922570	0.1209341	0.0427169	0.1090151	0.1109223	0.0335217
0.0296322	0.0601884	0.1100268	0.0454166	0.0554615	0.1238171	0.0451508
0.0191290	0.0206439	0.0444225	0.0417570	0.0167598	0.0451488	0.0705354

TABLE IV

Upper bound for the inverse matrix for example no. 1

0.1097864	0.1117095	0.0337580	0.0929122	0.1217902	0.0430237	0.0591545	0.0856377
0.0558549	0.1246935	0.0454688	0.0606167	0.1108047	0.0457411	0.0424595	0.0716650
0.0168791	0.0454690	0.0710318	0.0207914	0.0447365	0.0420528	0.0151205	0.0286166
0.0929128	0.1212334	0.0415826	0.2145140	0.2651703	0.0933432	0.1629157	0.2286922
0.0608939	0.1108025	0.0447353	0.1325819	0.2546949	0.1015207	0.1141968	0.1986825
0.0215120	0.0457412	0.0420526	0.0466717	0.1015227	0.1085180	0.0424595	0.0827516
0.0591538	0.0849172	0.0302398	0.1629127	0.2283959	0.0849168	0.2534062	0.3258236
0.0428195	0.0716653	0.0286164	0.1143466	0.1986884	0.0827518	0.1629151	0.3039788
0.0160204	0.0298452	0.0192655	0.0428190	0.0832373	0.0628474	0.0591545	0.1227653
0.0298543	0.0442709	0.0156504	0.0894665	0.1322085	0.0499870	0.1629158	0.2286923
0.0223421	0.0359354	0.0137319	0.0661017	0.1096010	0.0452167	0.1141969	0.1986826
0.0083334	0.0144182	0.0074037	0.0249936	0.0452186	0.0271857	0.0424596	0.0827517
0.0094013	0.0139856	0.0047730	0.0298538	0.0446853	0.0166664	0.0591546	0.0856377
0.0069929	0.0110102	0.0039875	0.0221354	0.0359369	0.0144181	0.0424596	0.0716650
0.0023866	0.0039875	0.0018012	0.0078253	0.0137327	0.0074038	0.0151205	0.0286166

TABLE IV (Cont'd)

Upper bound for the inverse matrix for example no. 1

0.0320402	0.0298543	0.0446833	0.0166665	0.0094009	0.0139854	0.0047730
0.0298449	0.0221358	0.0359353	0.0144181	0.0069927	0.0110101	0.0039875
0.0192655	0.0078254	0.0137320	0.0074038	0.0023865	0.0039875	0.0018012
0.0856375	0.0894679	0.1322032	0.0499872	0.0298534	0.0442705	0.0156504
0.0832347	0.0661027	0.1095971	0.0452168	0.0223414	0.0359351	0.0137319
0.0628472	0.0249940	0.0452168	0.0271857	0.0083331	0.0144180	0.0074037
0.1183055	0.1629142	0.2283889	0.0849171	0.0591527	0.0849165	0.0302398
0.1227650	0.1143478	0.1986832	0.0827520	0.0428186	0.0716648	0.0286165
0.1191868	0.0428195	0.0832351	0.0628475	0.0160200	0.0298449	0.0192655
0.0856374	0.2145152	0.2651639	0.0933434	0.0929115	0.1212326	0.0415826
0.0832346	0.1325828	0.2546907	0.1015210	0.0608931	0.1108018	0.0447354
0.0628472	0.0466721	0.1015210	0.1085182	0.0215117	0.0457409	0.0420527
0.0320401	0.0929127	0.1217873	0.0430238	0.1097858	0.1117089	0.0337580
0.0298449	0.0606170	0.1108026	0.0457412	0.0558545	0.1246930	0.0454689
0.0192655	0.0207916	0.0447357	0.0420528	0.0168790	0.0454688	0.0710319

TABLE V

Lower bound verification matrix Q^L for example no. 1

$$
\begin{bmatrix}
0.99800053 & -0.294\times10^{-5} & -0.599\times10^{-6} & -0.454\times10^{-6} & -0.193\times10^{-6} & -0.178\times10^{-5} & -0.387\times10^{-6} & -0.825\times10^{-5} \\
-0.520\times10^{-7} & 0.99800239 & -0.109\times10^{-5} & -0.714\times10^{-6} & -0.131\times10^{-5} & -0.152\times10^{-4} & -0.152\times10^{-5} & -0.232\times10^{-4} \\
-0.669\times10^{-6} & -0.118\times10^{-5} & 0.99800033 & -0.781\times10^{-7} & -0.520\times10^{-6} & -0.830\times10^{-5} & -0.446\times10^{-7} & -0.456\times10^{-5} \\
-0.779\times10^{-5} & -0.121\times10^{-4} & -0.167\times10^{-5} & 0.99801273 & -0.657\times10^{-5} & -0.322\times10^{-4} & -0.125\times10^{-4} & -0.107\times10^{-3} \\
-0.409\times10^{-6} & -0.429\times10^{-5} & -0.353\times10^{-6} & -0.149\times10^{-6} & 0.99800341 & -0.365\times10^{-8} & -0.736\times10^{-6} & -0.221\times10^{-7} \\
-0.521\times10^{-7} & -0.200\times10^{-5} & -0.185\times10^{-5} & -0.335\times10^{-6} & -0.281\times10^{-6} & 0.99800379 & -0.116\times10^{-5} & -0.446\times10^{-6} \\
-0.550\times10^{-6} & -0.329\times10^{-5} & -0.119\times10^{-6} & -0.163\times10^{-5} & -0.833\times10^{-6} & -0.162\times10^{-4} & 0.99800174 & -0.331\times10^{-4} \\
-0.108\times10^{-6} & -0.104\times10^{-6} & -0.115\times10^{-6} & -0.166\times10^{-5} & -0.213\times10^{-5} & -0.567\times10^{-5} & -0.366\times10^{-5} & 0.99800663 \\
-0.291\times10^{-6} & -0.155\times10^{-5} & -0.538\times10^{-6} & -0.911\times10^{-6} & -0.119\times10^{-5} & -0.149\times10^{-4} & -0.355\times10^{-5} & -0.262\times10^{-4} \\
-0.199\times10^{-5} & -0.366\times10^{-5} & -0.521\times10^{-6} & -0.370\times10^{-5} & -0.313\times10^{-5} & -0.249\times10^{-4} & -0.127\times10^{-4} & -0.102\times10^{-3} \\
-0.786\times10^{-6} & -0.108\times10^{-5} & -0.304\times10^{-6} & -0.298\times10^{-7} & -0.747\times10^{-6} & -0.713\times10^{-5} & -0.171\times10^{-6} & -0.283\times10^{-4} \\
-0.125\times10^{-6} & -0.138\times10^{-6} & -0.446\times10^{-7} & -0.840\times10^{-6} & -0.762\times10^{-6} & -0.293\times10^{-5} & -0.190\times10^{-5} & -0.964\times10^{-5} \\
-0.869\times10^{-7} & -0.372\times10^{-7} & -0.926\times10^{-9} & -0.669\times10^{-7} & -0.658\times10^{-6} & -0.113\times10^{-5} & -0.115\times10^{-5} & -0.125\times10^{-4} \\
-0.344\times10^{-6} & -0.439\times10^{-6} & -0.160\times10^{-6} & -0.149\times10^{-6} & -0.126\times10^{-6} & -0.220\times10^{-5} & -0.439\times10^{-6} & -0.136\times10^{-4} \\
-0.637\times10^{-7} & -0.335\times10^{-6} & -0.214\times10^{-7} & -0.524\times10^{-6} & -0.335\times10^{-6} & -0.210\times10^{-5} & -0.900\times10^{-6} & -0.520\times10^{-5}
\end{bmatrix}
$$

TABLE V (Cont'd)

Lower bound verification matrix Q^L for example no. 1

-0.167×10^{-5}	-0.370×10^{-8}	-0.149×10^{-7}	-0.372×10^{-8}	-0.232×10^{-6}	-0.457×10^{-6}	-0.423×10^{-7}
-0.332×10^{-5}	-0.137×10^{-5}	-0.320×10^{-6}	-0.272×10^{-5}	-0.214×10^{-5}	-0.223×10^{-5}	-0.113×10^{-6}
-0.141×10^{-5}	-0.119×10^{-6}	-0.167×10^{-7}	-0.149×10^{-6}	-0.558×10^{-6}	-0.582×10^{-11}	-0.493×10^{-7}
-0.203×10^{-4}	-0.270×10^{-4}	-0.327×10^{-5}	-0.160×10^{-4}	-0.705×10^{-5}	-0.121×10^{-4}	-0.260×10^{-6}
-0.439×10^{-5}	-0.773×10^{-5}	-0.929×10^{-7}	-0.617×10^{-6}	-0.199×10^{-5}	-0.130×10^{-5}	-0.460×10^{-7}
-0.223×10^{-7}	-0.197×10^{-5}	-0.149×10^{-6}	-0.558×10^{-6}	-0.957×10^{-6}	-0.569×10^{-6}	-0.818×10^{-7}
-0.100×10^{-4}	-0.155×10^{-4}	-0.127×10^{-5}	-0.138×10^{-4}	-0.245×10^{-5}	-0.692×10^{-5}	-0.297×10^{-7}
-0.790×10^{-5}	-0.150×10^{-4}	-0.147×10^{-5}	-0.223×10^{-5}	-0.838×10^{-6}	-0.300×10^{-5}	-0.930×10^{-8}
0.99800097	-0.436×10^{-5}	-0.626×10^{-6}	-0.569×10^{-5}	-0.921×10^{-6}	-0.277×10^{-5}	-0.164×10^{-6}
-0.204×10^{-4}	0.99801345	-0.518×10^{-5}	-0.212×10^{-4}	-0.248×10^{-4}	-0.170×10^{-4}	-0.734×10^{-6}
-0.219×10^{-5}	-0.257×10^{-5}	0.99800885	-0.808×10^{-5}	-0.391×10^{-7}	-0.113×10^{-4}	-0.191×10^{-6}
-0.842×10^{-5}	-0.878×10^{-5}	-0.558×10^{-7}	0.99800530	-0.109×10^{-5}	-0.243×10^{-5}	-0.788×10^{-6}
-0.145×10^{-5}	-0.879×10^{-5}	-0.907×10^{-6}	-0.209×10^{-5}	0.99799976	-0.810×10^{-5}	-0.370×10^{-6}
-0.416×10^{-6}	-0.331×10^{-5}	-0.565×10^{-6}	-0.590×10^{-5}	-0.183×10^{-5}	0.99799941	-0.223×10^{-6}
-0.213×10^{-5}	-0.297×10^{-5}	-0.743×10^{-7}	-0.200×10^{-5}	-0.326×10^{-5}	-0.810×10^{-5}	0.99800202

TABLE VI

Upper bound verification matrix Q^U for example no. 1

1.0050215	0.936×10^{-6}	0.324×10^{-6}	0.487×10^{-7}	0.430×10^{-5}	0.180×10^{-6}	0.659×10^{-6}	0.125×10^{-5}
0.642×10^{-6}	1.0050234	0.547×10^{-6}	0.809×10^{-6}	0.467×10^{-5}	0.115×10^{-5}	0.158×10^{-5}	0.141×10^{-5}
0.571×10^{-6}	0.326×10^{-6}	1.0050213	0.505×10^{-7}	0.217×10^{-5}	0.453×10^{-6}	0.490×10^{-11}	0.232×10^{-6}
0.633×10^{-5}	0.395×10^{-5}	0.895×10^{-6}	1.0050338	0.265×10^{-4}	0.233×10^{-5}	0.132×10^{-4}	0.488×10^{-5}
0.103×10^{-5}	0.191×10^{-5}	0.936×10^{-7}	0.625×10^{-6}	1.0050244	0.902×10^{-6}	0.109×10^{-5}	0.753×10^{-6}
0.393×10^{-7}	0.489×10^{-6}	0.958×10^{-6}	0.101×10^{-6}	0.113×10^{-5}	1.0050248	0.114×10^{-5}	0.240×10^{-6}
0.966×10^{-6}	0.143×10^{-5}	0.150×10^{-6}	0.119×10^{-5}	0.575×10^{-5}	0.809×10^{-6}	1.0050228	0.177×10^{-5}
0.937×10^{-8}	0.869×10^{-6}	0.243×10^{-7}	0.655×10^{-6}	0.613×10^{-5}	0.554×10^{-6}	0.373×10^{-5}	1.0050277
0.148×10^{-6}	0.382×10^{-6}	0.361×10^{-6}	0.532×10^{-6}	0.345×10^{-5}	0.120×10^{-5}	0.408×10^{-5}	0.164×10^{-5}
0.153×10^{-5}	0.940×10^{-6}	0.262×10^{-6}	0.191×10^{-5}	0.117×10^{-4}	0.119×10^{-5}	0.128×10^{-4}	0.491×10^{-5}
0.650×10^{-6}	0.243×10^{-6}	0.187×10^{-6}	0.119×10^{-5}	0.165×10^{-6}	0.318×10^{-6}	0.192×10^{-5}	0.142×10^{-5}
0.137×10^{-6}	0.674×10^{-7}	0.101×10^{-6}	0.412×10^{-6}	0.313×10^{-5}	0.183×10^{-6}	0.188×10^{-5}	0.629×10^{-6}
0.135×10^{-6}	0.562×10^{-7}	0.679×10^{-7}	0.786×10^{-7}	0.255×10^{-6}	0.936×10^{-7}	0.201×10^{-5}	0.884×10^{-6}
0.322×10^{-6}	0.176×10^{-6}	0.103×10^{-6}	0.142×10^{-6}	0.771×10^{-6}	0.168×10^{-6}	0.517×10^{-6}	0.150×10^{-5}
0.140×10^{-8}	0.749×10^{-7}	0.122×10^{-7}	0.275×10^{-6}	0.131×10^{-5}	0.109×10^{-6}	0.876×10^{-6}	0.270×10^{-6}

TABLE VI (Cont'd)

Upper bound verification matrix Q^U for example no. 1

0.176×10^{-6}	0.487×10^{-7}	0.824×10^{-7}	0.449×10^{-7}	0.655×10^{-7}	0.124×10^{-6}	0.276×10^{-7}
0.247×10^{-6}	0.247×10^{-6}	0.397×10^{-6}	0.277×10^{-6}	0.120×10^{-6}	0.139×10^{-6}	0.842×10^{-7}
0.936×10^{-7}	0.112×10^{-7}	0.618×10^{-7}	0.693×10^{-7}	0.749×10^{-8}	0.243×10^{-7}	0.515×10^{-7}
0.202×10^{-5}	0.453×10^{-5}	0.335×10^{-5}	0.134×10^{-5}	0.505×10^{-6}	0.101×10^{-5}	0.292×10^{-6}
0.910×10^{-6}	0.186×10^{-5}	0.947×10^{-6}	0.599×10^{-7}	0.155×10^{-6}	0.105×10^{-6}	0.987×10^{-7}
0.483×10^{-6}	0.371×10^{-6}	0.210×10^{-6}	0.711×10^{-7}	0.842×10^{-7}	0.599×10^{-7}	0.824×10^{-7}
0.149×10^{-5}	0.291×10^{-5}	0.272×10^{-5}	0.197×10^{-5}	0.532×10^{-6}	0.889×10^{-6}	0.899×10^{-7}
0.134×10^{-5}	0.145×10^{-5}	0.608×10^{-6}	0.431×10^{-6}	0.243×10^{-7}	0.917×10^{-6}	0.393×10^{-7}
1.0050220	0.749×10^{-6}	0.990×10^{-6}	0.623×10^{-6}	0.538×10^{-7}	0.273×10^{-6}	0.300×10^{-6}
0.144×10^{-5}	1.0050346	0.569×10^{-5}	0.178×10^{-5}	0.218×10^{-5}	0.224×10^{-5}	0.829×10^{-6}
0.711×10^{-6}	0.921×10^{-6}	1.0050299	0.145×10^{-5}	0.889×10^{-6}	0.191×10^{-5}	0.245×10^{-6}
0.734×10^{-6}	0.932×10^{-6}	0.775×10^{-6}	1.0050264	0.133×10^{-6}	0.253×10^{-6}	0.869×10^{-6}
0.109×10^{-6}	0.148×10^{-5}	0.226×10^{-5}	0.236×10^{-6}	1.0050208	0.794×10^{-6}	0.507×10^{-6}
0.300×10^{-7}	0.225×10^{-7}	0.177×10^{-5}	0.106×10^{-5}	0.730×10^{-6}	1.0050204	0.359×10^{-6}
0.185×10^{-6}	0.472×10^{-6}	0.165×10^{-6}	0.200×10^{-6}	0.251×10^{-6}	0.696×10^{-6}	1.0050231

TABLE VII

Upper and lower bounds for the solution vector x^U and x^L and corresponding verification vectors c^U and c^L, mean value of the solution vector x^M, maximum possible error of x^M, and the constant vector c for example no. 1

x^U	x^L	c^U	c^L	x^M	maximum % error of x^M	c
3.3775455	3.3451059	16.482368	16.367132	3.3613257	0.4848	16.4
2.9029025	2.8757993	15.195978	15.089737	2.8893509	0.4712	15.12
1.4909321	1.4779451	8.5627980	8.5029191	1.4844386	0.4393	8.52
2.9296277	2.8892813	-3.5926440	-3.6186771	2.9094545	0.6982	-3.6
2.5325820	2.4995546	-2.3751965	-2.3920449	2.5160683	0.6606	-2.38
1.3202463	1.3052158	0.32162760	0.31930926	1.3127311	0.5757	0.32
2.3251811	2.2928392	0.00005748	-0.00015855	2.3090102	0.7052	0.0
2.0408518	2.0140021	0.72364918	0.71851896	2.0274270	0.6665	0.72
1.1047732	1.0920320	1.9296630	1.9160807	1.5040564	0.5833	1.92
1.5424184	1.5224932	0.00007973	-0.0003057	1.5324558	0.6543	0.0
1.3927432	1.3758529	0.72364645	0.71847893	1.3842981	0.6138	0.72
0.80561724	0.79714755	1.9296630	1.9161285	0.80138240	0.5312	1.92
0.63803987	0.63048865	0.00000995	-0.00002621	0.63426426	0.5988	0.0
0.60927538	0.60262864	0.72362952	0.71852012	0.60595201	0.5514	0.72
0.39638593	0.39273261	1.9296489	1.9161344	0.39455927	0.4651	1.92

$$
\begin{bmatrix}
15 & -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-5 & 25 & -5 & 0 & 0 & -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -5 & 25 & -5 & 0 & 0 & -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -5 & 25 & -5 & 0 & 0 & -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -10 & 35 & 0 & 0 & 0 & -20 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -20 & 0 & 0 & 0 & 40 & -20 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -10 & 0 & 0 & -10 & 40 & -10 & 0 & 0 & -10 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -10 & 0 & 0 & -10 & 40 & -10 & 0 & 0 & -10 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -10 & 0 & 0 & -10 & 40 & -10 & 0 & 0 & -10 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -20 & 40 & 0 & 0 & 0 & -20 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -20 & 0 & 0 & 0 & 40 & -10 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -10 & 0 & 0 & -5 & 30 & -5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -10 & 0 & 0 & -5 & 30 & -5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -10 & 0 & 0 & -5 & 30 & -5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -10 & 20
\end{bmatrix}
\begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_6 \\ t_7 \\ t_8 \\ t_9 \\ t_{10} \\ t_{11} \\ t_{12} \\ t_{13} \\ t_{14} \\ t_{15} \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1000 \\ 1000 \\ 1000 \\ 1000 \\ 1000 \end{bmatrix}
$$

EXAMPLE NO. 2

TABLE VIII

Vectors $\bar{z}^{(k)}$ for monotonicity of example no. 2

k = 1	k = 2	k = 3	k = 4	k = 5	k = 6	k = 7	k = 8
0.8385070	0.5155386	0.1965244	0.0898776	0.0288555	0.1760349	0.1886036	0.0975510
0.2577595	0.7733086	0.2947867	0.1348165	0.0432833	0.2640527	0.2829056	0.1463270
0.0982512	0.2947874	0.6424490	0.2371101	0.0687767	0.1587400	0.3401748	0.2027341
0.0449304	0.1348162	0.2371092	0.6261750	0.1495985	0.0867735	0.2122790	0.2972442
0.0288482	0.0865667	0.1375535	0.2991981	0.4386476	0.0593921	0.1510027	0.2405217
0.1760216	0.5281072	0.3174803	0.1735480	0.0593923	0.4927448	0.4428750	0.2156763
0.0942885	0.2829061	0.3401744	0.2122797	0.0755012	0.2214371	0.6028450	0.2850501
0.0487771	0.1463642	0.2027722	0.2972812	0.1202843	0.1078680	0.2851090	0.5214868
0.0280206	0.0840837	0.1221643	0.2105095	0.1928341	0.0605495	0.1581154	0.2723131
0.0188566	0.0565864	0.0820234	0.1403361	0.1272988	0.0407960	0.1065985	0.1824888
0.0541085	0.1623658	0.1979974	0.1411799	0.0535517	0.1263966	0.3432215	0.2003478
0.0278771	0.0836520	0.1116417	0.1401609	0.0632044	0.0627124	0.1671985	0.2313402
0.0156011	0.0468181	0.0663086	0.1052236	0.0851061	0.0341423	0.0897517	0.1447595
0.0096932	0.0290892	0.0418828	0.0701628	0.0617642	0.0210426	0.0550816	0.0926793
0.0048466	0.0145446	0.0209414	0.0350813	0.0308821	0.0105213	0.0275408	0.0463390

σ required to get vector

4.0	1.5	2.0	1.5	2.0	1.5	2.0	4.0

TABLE VIII (Cont'd)

Vectors $\tilde{z}^{(k)}$ for monotonicity of example no. 2

k = 9	k = 10	k = 11	k = 12	k = 13	k = 14	k = 15
0.0560412	0.0188552	0.0541217	0.0557682	0.0312123	0.0193929	0.0048416
0.0840620	0.0282829	0.0811827	0.0836524	0.0468184	0.0290893	0.0072624
0.1221358	0.0409987	0.0989986	0.1116424	0.0663093	0.0418830	0.0104570
0.2104747	0.0701484	0.0705896	0.1401610	0.1052238	0.0701628	0.0175229
0.3856216	0.1272626	0.0535517	0.1264095	0.1702129	0.1235288	0.0308556
0.1210666	0.0407810	0.1263965	0.1254258	0.0682853	0.0420854	0.0105069
0.1580715	0.0532812	0.1716105	0.1671993	0.0897522	0.0550816	0.0137514
0.2723110	0.0912412	0.1001989	0.2313767	0.1447986	0.0927010	0.0231518
0.5696263	0.1876588	0.0584206	0.1511361	0.2452607	0.1810945	0.0452381
0.3753485	0.4314224	0.0401245	0.1078007	0.2014965	0.3504167	0.0875426
0.1168022	0.0401090	0.3608490	0.2003523	0.0796156	0.0436570	0.0108986
0.1511052	0.0538839	0.1001754	0.4670108	0.1389579	0.0644650	0.0161012
0.2452216	0.1007130	0.0398075	0.1389579	0.4645351	0.1577319	0.0394055
0.1810707	0.1751860	0.0218285	0.0644653	0.1577324	0.5197392	0.1298937
0.0905354	0.0875929	0.0109142	0.0322326	0.0788662	0.2598696	0.5649470

σ required to get vector

4.0	4.0	2.0	0.5	0.5	2.0	4.5

TABLE IX

Vectors $\tilde{q}^{(k)}$ for monotonicity of example no. 2

k = 1	k = 2	k = 3	k = 4	k = 5	k = 6	k = 7	k = 8
1.0000010	-0.805×10^{-6}	-0.104×10^{-6}	-0.894×10^{-7}	-0.158×10^{-7}	-0.313×10^{-6}	-0.119×10^{-6}	-0.596×10^{-6}
-0.179×10^{-5}	1.0000014	-0.283×10^{-6}	-0.536×10^{-6}	-0.419×10^{-7}	-0.492×10^{-6}	-0.253×10^{-6}	-0.119×10^{-5}
-0.555×10^{-5}	-0.241×10^{-13}	1.0000002	-0.134×10^{-6}	-0.335×10^{-6}	-0.224×10^{-6}	-0.149×10^{-6}	-0.477×10^{-6}
-0.707×10^{-6}	-0.849×10^{-6}	-0.373×10^{-6}	1.0000021	-0.313×10^{-6}	-0.268×10^{-6}	-0.417×10^{-6}	-0.429×10^{-5}
-0.293×10^{-5}	-0.134×10^{-6}	-0.671×10^{-6}	-0.566×10^{-6}	0.99999976	-0.335×10^{-7}	-0.298×10^{-6}	-0.444×10^{-4}
-0.986×10^{-5}	-0.536×10^{-6}	-0.894×10^{-6}	-0.536×10^{-6}	-0.149×10^{-7}	0.99999964	-0.119×10^{-5}	-0.493×10^{-4}
-0.432×10^{-5}	-0.238×10^{-6}	-0.143×10^{-5}	-0.268×10^{-6}	-0.133×10^{-6}	-0.116×10^{-5}	0.99999952	-0.444×10^{-4}
-0.832×10^{-5}	-0.107×10^{-5}	-0.894×10^{-6}	-0.417×10^{-6}	-0.745×10^{-6}	-0.386×10^{-6}	-0.179×10^{-5}	0.99999946
-0.581×10^{-6}	-0.743×10^{-6}	-0.328×10^{-6}	-0.107×10^{-5}	-0.328×10^{-6}	-0.553×10^{-6}	-0.209×10^{-6}	-0.438×10^{-5}
-0.123×10^{-5}	-0.298×10^{-7}	-0.708×10^{-6}	-0.179×10^{-6}	-0.168×10^{-5}	-0.447×10^{-7}	-0.298×10^{-7}	-0.296×10^{-4}
-0.200×10^{-4}	-0.894×10^{-6}	-0.894×10^{-6}	-0.805×10^{-6}	-0.522×10^{-7}	-0.179×10^{-6}	-0.253×10^{-5}	-0.493×10^{-4}
-0.580×10^{-6}	-0.254×10^{-6}	-0.439×10^{-12}	-0.224×10^{-6}	-0.885×10^{-7}	-0.235×10^{-6}	-0.447×10^{-7}	-0.197×10^{-4}
-0.231×10^{-5}	-0.894×10^{-7}	-0.847×10^{-6}	-0.447×10^{-6}	-0.196×10^{-6}	-0.754×10^{-7}	-0.335×10^{-6}	-0.444×10^{-4}
-0.925×10^{-6}	-0.428×10^{-7}	-0.494×10^{-7}	-0.201×10^{-6}	-0.114×10^{-6}	-0.466×10^{-7}	-0.149×10^{-7}	-0.142×10^{-6}
-0.298×10^{-7}	-0.224×10^{-7}	-0.149×10^{-7}	-0.179×10^{-6}	-0.186×10^{-7}	-0.168×10^{-7}	-0.149×10^{-7}	-0.129×10^{-5}

TABLE IX (Cont'd)

Vectors $\tilde{q}^{(k)}$ for monotonicity of example no. 2

k = 9	k = 10	k = 11	k = 12	k = 13	k = 14	k = 15	$\sum_{k=1}^{n} \tilde{q}_i^{(k)}$
$-0.247\text{x}10^{-6}$	$-0.818\text{x}10^{-7}$	$-0.326\text{x}10^{-7}$	$-0.279\text{x}10^{-7}$	$-0.168\text{x}10^{-7}$	$-0.102\text{x}10^{-7}$	$-0.483\text{x}10^{-7}$	0.99999845
$-0.692\text{x}10^{-7}$	$-0.685\text{x}10^{-6}$	$-0.885\text{x}10^{-7}$	$-0.142\text{x}10^{-6}$	$-0.931\text{x}10^{-8}$	$-0.950\text{x}10^{-7}$	$-0.145\text{x}10^{-6}$	0.99999555
$-0.477\text{x}10^{-6}$	$-0.559\text{x}10^{-7}$	$-0.745\text{x}10^{-7}$	$-0.298\text{x}10^{-7}$	$-0.633\text{x}10^{-7}$	$-0.652\text{x}10^{-8}$	$-0.162\text{x}10^{-5}$	0.99999104
$-0.286\text{x}10^{-5}$	$-0.804\text{x}10^{-6}$	$-0.801\text{x}10^{-7}$	$-0.179\text{x}10^{-6}$	$-0.194\text{x}10^{-6}$	$-0.177\text{x}10^{-7}$	$-0.798\text{x}10^{-6}$	0.99998997
$-0.518\text{x}10^{-4}$	$-0.469\text{x}10^{-4}$	$-0.466\text{x}10^{-7}$	$-0.596\text{x}10^{-7}$	$-0.596\text{x}10^{-7}$	$-0.894\text{x}10^{-6}$	$-0.448\text{x}10^{-5}$	0.99984656
$-0.477\text{x}10^{-6}$	$-0.432\text{x}10^{-5}$	$-0.298\text{x}10^{-6}$	$-0.179\text{x}10^{-6}$	$-0.820\text{x}10^{-7}$	$-0.931\text{x}10^{-7}$	$-0.362\text{x}10^{-11}$	0.99993184
$-0.296\text{x}10^{-4}$	$-0.493\text{x}10^{-5}$	$-0.894\text{x}10^{-6}$	$-0.596\text{x}10^{-7}$	$-0.298\text{x}10^{-7}$	$-0.130\text{x}10^{-6}$	$-0.866\text{x}10^{-5}$	0.99990318
$-0.336\text{x}10^{-4}$	$-0.771\text{x}10^{-5}$	$-0.410\text{x}10^{-6}$	$-0.298\text{x}10^{-6}$	$-0.209\text{x}10^{-6}$	$-0.931\text{x}10^{-8}$	$-0.644\text{x}10^{-5}$	0.99993715
1.0000023	$-0.381\text{x}10^{-5}$	$-0.156\text{x}10^{-6}$	$-0.298\text{x}10^{-6}$	$-0.387\text{x}10^{-6}$	$-0.328\text{x}10^{-6}$	$-0.315\text{x}10^{-5}$	0.99998600
$-0.273\text{x}10^{-12}$	0.99999988	$-0.931\text{x}10^{-7}$	$-0.104\text{x}10^{-13}$	$-0.119\text{x}10^{-12}$	$-0.715\text{x}10^{-6}$	$-0.931\text{x}10^{-4}$	0.99987247
$-0.395\text{x}10^{-4}$	$-0.105\text{x}10^{-4}$	0.99999940	$-0.238\text{x}10^{-6}$	$-0.298\text{x}10^{-7}$	$-0.633\text{x}10^{-7}$	$-0.974\text{x}10^{-5}$	0.99986467
$-0.740\text{x}10^{-5}$	$-0.544\text{x}10^{-6}$	$-0.978\text{x}10^{-6}$	1.0000004	$-0.209\text{x}10^{-6}$	$-0.312\text{x}10^{-6}$	$-0.145\text{x}10^{-6}$	0.99996971
$-0.493\text{x}10^{-4}$	$-0.549\text{x}10^{-4}$	$-0.102\text{x}10^{-7}$	$-0.358\text{x}10^{-6}$	0.99999958	$-0.969\text{x}10^{-6}$	$-0.189\text{x}10^{-4}$	0.99982641
$-0.148\text{x}10^{-4}$	$-0.173\text{x}10^{-4}$	$-0.233\text{x}10^{-7}$	$-0.801\text{x}10^{-7}$	$-0.894\text{x}10^{-7}$	1.0000002	$-0.379\text{x}10^{-4}$	0.99992850
$-0.340\text{x}10^{-10}$	$-0.238\text{x}10^{-6}$	$-0.186\text{x}10^{-8}$	$-0.149\text{x}10^{-7}$	$-0.298\text{x}10^{-7}$	$-0.596\text{x}10^{-7}$	1.000000	0.99999849

$$
\begin{bmatrix}
405 & -324 & 81 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-324 & 486 & -324 & 81 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
81 & -324 & 486 & -324 & 81 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 81 & -324 & 486 & -324 & 81 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 81 & -324 & 486 & -324 & 81 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 81 & -324 & 486 & -324 & 81 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 81 & -324 & 486 & -324 & 81 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 81 & -324 & 486 & -324 & 81 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 81 & -324 & 486 & -324 & 81 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 81 & -324 & 486 & -324 & 81 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 81 & -324 & 486 & -324 & 81 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 81 & -324 & 486 & -324 & 81 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 81 & -324 & 486 & -324 & 81 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 81 & -324 & 567 & -324 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 162 & -648 & 486
\end{bmatrix}
\begin{bmatrix}
\theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \\ \theta_6 \\ \theta_7 \\ \theta_8 \\ \theta_9 \\ \theta_{10} \\ \theta_{11} \\ \theta_{12} \\ \theta_{13} \\ \theta_{14} \\ \theta_{15}
\end{bmatrix}
=
\begin{bmatrix}
1\text{x}10^{-4} \\ 1\text{x}10^{-4} \\ 1\text{x}10^{-4} \\ 1\text{x}10^{-4} \\ 1\text{x}10^{-4} \\ 1\text{x}10^{-4} \\ 1\text{x}10^{-4} \\ 1\text{x}10^{-4} \\ 1\text{x}10^{-4} \\ 1\text{x}10^{-4} \\ 1\text{x}10^{-4} \\ 1\text{x}10^{-4} \\ 1\text{x}10^{-4} \\ 1\text{x}10^{-4} \\ 1\text{x}10^{-4}
\end{bmatrix}
$$

EXAMPLE NO. 3

TABLE X

Vectors $\tilde{z}^{(k)}$ for monotonicity of example no. 3

k = 1	k = 2	k = 3	k = 4	k = 5	k = 6	k = 7	k = 8
17.74848	34.26409	49.54850	63.60355	76.43020	88.03054	98.40577	107.55917
34.26409	67.29699	97.86744	125.97870	151.63389	174.83579	195.58772	213.89172
49.54850	97.86744	143.72718	185.89798	224.38447	259.19107	290.32191	317.78081
63.60335	125.97870	185.89798	242.13295	293.45516	339.87042	381.38416	418.00142
76.43020	151.63389	224.38447	293.45516	357.61890	415.64826	467.54993	513.33011
88.03054	174.83597	259.19107	339.87042	415.64826	485.29841	547.59420	602.54290
98.40577	195.58772	290.32191	381.38416	467.54993	547.59420	620.29138	684.41522
107.55719	213.89172	317.78081	418.00142	513.33011	602.54290	684.41522	757.72173
115.48595	229.75028	341.57123	449.72676	552.99440	650.15113	739.97324	821.23624
122.19309	243.16546	361.69622	476.56420	586.54778	690.42474	786.97215	874.96622
127.67951	254.13903	378.15843	498.51724	613.99454	723.36880	825.41771	918.91809
131.94594	262.67248	390.96006	515.58877	635.33826	748.98752	855.31474	953.09722
134.99297	268.76696	400.10283	527.78108	650.58175	767.28421	876.66703	977.50783
136.82102	272.42332	405.58799	535.09580	659.72702	778.26126	889.47730	992.15297
137.43034	273.64204	407.41629	537.53393	662.77531	781.92011	893.74720	997.03448

σ required to get vector

1.5	2.0	3.5	4.5	2.0	1.5	1.5	1.5

TABLE X (Cont'd)

Vectors $z^{(k)}$ for monotonicity of example no. 3

k = 9	k = 10	k = 11	k = 12	k = 13	k = 14	k = 15
115.4860	122.1931	127.6795	131.9459	134.9930	136.8210	68.71517
229.7503	243.1655	254.1390	262.6725	268.7670	272.4233	136.82102
341.5712	361.6962	378.1584	390.9601	400.1028	405.5880	203.70814
449.7268	476.5642	498.5172	515.5888	527.7811	535.0958	268.76696
552.9944	586.5478	613.9945	635.3383	650.5817	659.7270	331.38765
650.1511	690.4247	723.3688	748.9875	767.2842	778.2613	390.96006
739.9732	786.9722	825.4177	855.3147	876.6670	889.4773	446.87360
821.2362	874.9662	918.9181	953.0972	977.5078	992.1530	498.51724
892.7147	953.1822	1002.6457	1041.1112	1068.5832	1085.0650	545.27937
953.1822	1020.3942	1075.3753	1118.1317	1148.6684	1166.9889	586.54778
1002.6457	1075.3753	1135.8802	1182.9325	1216.5374	1236.6989	621.70957
1041.1112	1118.1317	1182.9325	1234.2859	1270.9630	1292.9676	650.15113
1068.5832	1148.6684	1216.5374	1270.9630	1310.7161	1334.5663	671.25807
1085.0650	1166.9889	1236.6989	1292.9676	1334.5663	1360.2646	684.41522
1090.5587	1173.0956	1243.4191	1300.3023	1342.5161	1368.8304	689.00655

σ required to get vector

4.0	3.0	1.5	3.0	5.0	3.5	4.0

TABLE XI

Vectors $\bar{q}^{(k)}$ for monotonicity of example no. 3

k = 1	k = 2	k = 3	k = 4	k = 5	k = 6	k = 7	k = 8
1.0000000	-0.12×10^{-12}	-0.14×10^{-11}	-0.26×10^{-10}	-0.26×10^{-12}	-0.24×10^{-12}	-0.26×10^{-12}	-0.28×10^{-12}
-0.64×10^{-13}	1.0000000	-0.43×10^{-13}	-0.51×10^{-11}	-0.12×10^{-12}	-0.57×10^{-13}	-0.45×10^{-12}	-0.57×10^{-13}
-0.96×10^{-13}	-0.56×10^{-12}	1.0000000	-0.74×10^{-10}	-0.28×10^{-12}	-0.91×10^{-12}	-0.74×10^{-12}	-0.80×10^{-12}
-0.36×10^{-13}	-0.43×10^{-13}	-0.76×10^{-12}	1.0000000	-0.34×10^{-12}	-0.11×10^{-12}	-0.40×10^{-12}	-0.17×10^{-12}
-0.11×10^{-13}	-0.50×10^{-13}	-0.59×10^{-12}	-0.71×10^{-11}	1.0000000	-0.11×10^{-12}	-0.34×10^{-12}	-0.11×10^{-12}
-0.17×10^{-12}	-0.26×10^{-12}	-0.14×10^{-10}	-0.21×10^{-9}	-0.80×10^{-12}	1.0000000	-0.80×10^{-12}	-0.80×10^{-12}
-0.82×10^{-13}	-0.15×10^{-12}	-0.57×10^{-13}	-0.39×10^{-10}	-0.34×10^{-12}	-0.68×10^{-12}	1.0000000	-0.74×10^{-12}
-0.29×10^{-12}	-0.98×10^{-12}	-0.20×10^{-10}	-0.33×10^{-9}	-0.31×10^{-11}	-0.19×10^{-11}	-0.26×10^{-11}	1.0000000
-0.39×10^{-12}	-0.23×10^{-12}	-0.91×10^{-12}	-0.11×10^{-11}	-0.45×10^{-12}	-0.57×10^{-12}	-0.14×10^{-11}	-0.97×10^{-12}
-0.28×10^{-13}	-0.42×10^{-12}	-0.38×10^{-11}	-0.19×10^{-11}	-0.74×10^{-12}	-0.28×10^{-12}	-0.85×10^{-12}	-0.15×10^{-11}
-0.18×10^{-12}	-0.16×10^{-12}	-0.73×10^{-11}	-0.11×10^{-9}	-0.45×10^{-12}	-0.45×10^{-12}	-0.23×10^{-12}	-0.11×10^{-11}
-0.64×10^{-13}	-0.19×10^{-11}	-0.18×10^{-11}	-0.11×10^{-9}	-0.10×10^{-11}	-0.68×10^{-12}	-0.16×10^{-11}	-0.15×10^{-11}
-0.71×10^{-14}	-0.24×10^{-12}	-0.16×10^{-10}	-0.18×10^{-10}	-0.12×10^{-11}	-0.45×10^{-12}	-0.14×10^{-11}	-0.15×10^{-11}
-0.11×10^{-12}	-0.80×10^{-12}	-0.56×10^{-11}	-0.17×10^{-9}	-0.34×10^{-12}	-0.11×10^{-11}	-0.11×10^{-11}	-0.91×10^{-12}
-0.23×10^{-12}	-0.11×10^{-12}	-0.15×10^{-11}	-0.10×10^{-10}	-0.15×10^{-11}	-0.40×10^{-12}	-0.91×10^{-12}	-0.36×10^{-11}

TABLE XI (Cont'd)

Vectors $\tilde{q}^{(k)}$ for monotonicity of example no. 3

k = 9	k = 10	k = 11	k = 12	k = 13	k = 14	k = 15	$\sum_{k=1}^{n} \tilde{q}_i^{(k)}$
-0.21×10^{-10}	-0.24×10^{-11}	-0.28×10^{-12}	-0.40×10^{-12}	-0.20×10^{-11}	-0.49×10^{-11}	-0.84×10^{-11}	1.000000
-0.44×10^{-11}	-0.45×10^{-12}	-0.17×10^{-12}	-0.11×10^{-11}	-0.99×10^{-10}	-0.16×10^{-11}	-0.15×10^{-12}	1.000000
-0.70×10^{-10}	-0.54×10^{-11}	-0.97×10^{-12}	-0.38×10^{-11}	-0.56×10^{-9}	-0.23×10^{-10}	-0.81×10^{-11}	1.000000
-0.32×10^{-11}	-0.20×10^{-11}	-0.28×10^{-12}	-0.85×10^{-12}	-0.82×10^{-10}	-0.34×10^{-12}	-0.10×10^{-10}	1.000000
-0.48×10^{-10}	-0.11×10^{-11}	-0.63×10^{-12}	-0.11×10^{-12}	-0.16×10^{-9}	-0.69×10^{-11}	-0.39×10^{-11}	1.000000
-0.32×10^{-10}	-0.12×10^{-10}	-0.68×10^{-12}	-0.11×10^{-10}	-0.11×10^{-8}	-0.25×10^{-10}	-0.30×10^{-10}	1.000000
-0.91×10^{-12}	-0.21×10^{-11}	-0.57×10^{-12}	-0.63×10^{-12}	-0.15×10^{-10}	-0.74×10^{-11}	-0.71×10^{-11}	1.000000
-0.13×10^{-9}	-0.29×10^{-10}	-0.32×10^{-11}	-0.27×10^{-10}	-0.32×10^{-8}	-0.73×10^{-10}	-0.74×10^{-10}	1.000000
1.0000000	-0.16×10^{-11}	-0.23×10^{-12}	-0.51×10^{-12}	-0.24×10^{-10}	-0.80×10^{-11}	-0.24×10^{-11}	1.000000
-0.84×10^{-10}	1.0000000	-0.16×10^{-11}	-0.14×10^{-10}	-0.14×10^{-8}	-0.39×10^{-10}	-0.19×10^{-10}	1.000000
-0.63×10^{-10}	-0.67×10^{-11}	1.0000000	-0.67×10^{-11}	-0.34×10^{-9}	-0.80×10^{-11}	-0.26×10^{-10}	1.000000
-0.85×10^{-10}	-0.14×10^{-10}	-0.20×10^{-11}	1.0000000	-0.13×10^{-8}	-0.26×10^{-10}	-0.21×10^{-10}	1.000000
-0.88×10^{-10}	-0.97×10^{-11}	-0.17×10^{-11}	-0.60×10^{-11}	1.0000000	-0.13×10^{-10}	-0.38×10^{-10}	1.000000
-0.54×10^{-10}	-0.83×10^{-11}	-0.11×10^{-11}	-0.73×10^{-11}	-0.91×10^{-12}	1.0000000	-0.20×10^{-10}	1.000000
-0.73×10^{-11}	-0.18×10^{-11}	-0.91×10^{-12}	-0.27×10^{-11}	-0.97×10^{-10}	-0.27×10^{-11}	1.0000000	1.000000

RESULTS

Only one of the examples presented satisfied the existing two theorems for monotonicity. However, it was possible to show that all three examples were monotonic using the new theorem presented in this report.

In addition to determining the monotonicity of all three examples, the bounds for the inverse and also the bounds for the solution were determined for example no. 1. Determination of these bounds was a simple procedure after it was shown that the system was monotonic.

It should be mentioned that it is possible to skip the step of normalizing the vectors under certain conditions of application of the theorem. This step was skipped in all three of the examples presented in this report.

It was possible to skip this step because the first approximation to the inverse in each example was near enough to the exact inverse that the product of this approximation to the inverse and the original matrix resulted in a matrix almost equal to the unit matrix. Thus the vectors in the proof of monotonicity corresponding to the columns of this approximate unit matrix remained in this form which is the main reason for normalizing the vectors.

The first approximation to the inverse for each of the examples presented contained all positive elements. Under these conditions the application of the theorem presented was a straight forward procedure except for one condition which created a problem. If the first approximation to the unit matrix, formed by the product of the first approximation to the inverse and the original matrix, had zero elements it was often difficult to satisfy the conditions of the theorem.

Examining the equation for determining the elements of the vectors, which satisfy the theorem, at one stage of its simplification it is found that

$$\tilde{q}_i^{(k)} = -\sigma\left|b_i^{(k)}\right| - (1+\sigma)\sum_{\substack{j=1\\ j\neq k\\ j\neq i\\ \text{for } 0 \leq b_j^{(k)}}}^{n} b_j^{(k)}\, b_i^{(j)} - (1-\sigma)\sum_{\substack{j=1\\ j\neq k\\ j\neq i\\ \text{for } b_j^{(k)} < 0}}^{n} b_j^{(k)}\, b_i^{(j)} .$$

Then if

$$\left|b_j^{(k)}\right| << 1 \quad \text{for all } j\neq k \text{ and } j\neq i$$

and $\qquad b_i^{(k)} = 0$

it is impossible to state definitely if $\tilde{q}_i^{(k)}$ will be negative or positive regardless of the value of σ. However, even though $b_i^{(k)} = 0$, in general $\tilde{q}_i^{(k)}$ will not be equal to zero so when all elements of the vector $\tilde{q}^{(k)}$ have been calculated this vector can replace the vector $b^{(k)}$ in the above equation. This will force the new vector $\tilde{q}^{(k)}$ to have all negative elements for large enough σ.

Another condition which may cause some problems in application of the theorem is the condition of some elements of the first approximation to the inverse being negative. For the examples presented in this report this condition did not exist. An attempt was made to produce this condition for example no. 1 by changing some of the smallest elements of the first approximation to the inverse to negative. However, this produced such a large error in the resulting approximate unit matrix that some of the off diagonal elements were very large compared to others and in fact some off diagonal elements were near unity while the diagonal elements remained at unity. Since not all of the off diagonal elements were small second order terms the procedure diverged making it impossible to get any satisfactory results. Therefore, it cannot be stated what problems if any would occur if this condition did exist.

The resulting upper and lower bounds to the solution of example no. 1 are presented in the examples section of the report. The corresponding upper and lower bounds on the constant vector are also presented as a verification that the solution upper and lower bounds presented are valid bounds.

The computer program used to determine the monotonicity of examples no. 2 and no. 3 is presented in Appendix A. The computer program used to determine the monotonicity and also find the bounds for example no. 1 is presented in Appendix B. The part of the program which determines the bounds of example no. 1 is in general just an addition to the program for determining the monotonicity of examples no. 2 and no. 3.

CONCLUSIONS

The theorem presented in this report makes it possible to determine the monotonicity of systems which do not satisfy the conditions of the two existing theorems. However, since the conditions of the theorem presented are only sufficient conditions for monotonicity, there may be monotonic systems which do not satisfy this theorem.

It is a simple step from the method presented for determining the monotonicity of a system to the determination of bounds for the solution to the system. Therefore, the method presented has practical applications in finding bounds on the solution as well as determining the monotonicity of the system.

It may be possible to determine bounds for the solution of systems which are not monotonic. It is possible that the method presented for determining the vectors to prove that a system is monotonic could be used to determine the bounds of the inverse matrix for a system which is not monotonic. This is an area for additional study.

The method presented works very satisfactorily when the first approximation to the inverse has all positive elements. However, if the first approximation to the inverse has negative elements some problems may be encountered in application of the method presented and alteration of the method may be required to produce satisfactory results. This is an area for further study.

The bounds calculated for the solution of example no. 1 were very accurate. Selecting the mean value of the bounds as the solution to example no. 1, it is shown in Table VII that the maximum possible error is less than one percent for all elements of the solution vector. From this it is concluded that this method of calculating the bounds of a monotonic system is very accurate as well as simple to apply.

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APPENDIX A

Computer program for determining the monotonicity of examples no. 2 and no. 3.

```
      DOUBLE PRECISION X,E,BE,BEE,BU,BUC,BBC,BUCS,BBN,BBP
      DIMENSION B(26,27), BSTOR(26,27), BB(26,27)
      DIMENSION  NROW(26), NCOL(26), ABA(26), CONST(26)
      DIMENSION BU(26,27),BBC(26,27),BUC(26,27)
      DIMENSION BBN(26,27),BBP(26,27),BUCS(26)
  481 FORMAT(5X,7HROW NO.,I3,5HSUM =,E16.8)
  482 FORMAT(5X,9HEPSILON =,E16.8)
  483 FORMAT(10X,7HCOL NO.,I3,3X,10H**********  )
  484 FORMAT(1H1,20X,25HAPPROX UNIT MATRIX CHANGE)
  485 FORMAT(1H1,20X,14HCHANGE INVERSE)
  486 FORMAT(1H1,20X,18HAPPROX UNIT MATRIX)
  487 FORMAT(21X,22HTHE ORIGINAL EQUATIONS  )
  488 FORMAT(10X,7HROW NO.,I3,3X,10H**********     )
  489 FORMAT(21X,6HANSWER,21X,8HRESIDUAL)
  490 FORMAT(1HL,19X,8HRESIDUAL)
  491 FORMAT(21X,6HANSWER,22X,6HCHANGE)
  492 FORMAT(5X,11HEQUATIONS =,I3,10X,6HRANK =,I3  )
  493 FORMAT(5X,13HDETERMINANT =,E25.18)
  494 FORMAT(1H1,20X,11HTHE INVERSE   )
  495 FORMAT(1H1)
  496 FORMAT(E26.18,5I6)
  497 FORMAT(     10X,2E28.18,I5)
  498 FORMAT(4I5)
  499 FORMAT(4E16.8)
  500 FORMAT(8E16.8)
      READ(1,498)LN,NSOLVE
      LNN=LN+1
      DET=1.0
      DO 510 I=1,LN
      DO 510 J=1,LNN
  510 READ(1,500)  B(I,J)
      WRITE(3,487)
      DO 2471 I=1,LN
      WRITE(3,488)  I
 2471 WRITE(3,500)  (B(I,J),J=1,LNN)
      DO 2001 I=1,LN
      NROW(I)=I
 2001 NCOL(I)=I
      DO 2004 I = 1, LN
      DO 2004 J = 1, LNN
 2004 BSTOR(I,J) = B(I,J)
      DO 2000 ILEFT=1,LN
C     SEARCH FOR LARGEST ELEMENT ********************************
      BIG=0.0
      DO 2200 J=ILEFT,LN
      DO 2200 I=ILEFT,LN
      BURP=B(I,J)
      BURPE=BURP
      IF(BURP)  2031,2032,2032
 2031 BURPE=-BURP
 2032 IF(BIG-BURPE)  2039,2200,2200
 2039 BIG=BURPE
```

```
      IROW=I
      JCOL=J
 2200 CONTINUE
      IF(IROW-ILEFT)2092,2091,2092
 2092 DET=-DET
 2091 IF(JCOL-ILEFT)2094,2093,2094
 2094 DET=-DET
 2093 CONTINUE
C     INTERCHANGE TWO ROWS ***********************************
      DO 2050  J=1,LN
      D=B(IROW,J)
      B(IROW,J)=B(ILEFT,J)
 2050 B(ILEFT,J)=D
      KEEP=NROW(ILEFT)
      NROW(ILEFT)=NROW(IROW)
      NROW(IROW)=KEEP
C     INTERCHANGE TWO COLUMNS ********************************
      DO 2051  I=1,LN
      D=B(I,JCOL)
      B(I,JCOL)=B(I,ILEFT)
 2051 B(I,ILEFT)=D
      KEEP=NCOL(ILEFT)
      NCOL(ILEFT)=NCOL(JCOL)
      NCOL(JCOL)=KEEP
C     NORMALIZING AND ZEROING COLUMN *************************
      DIV=B(ILEFT,ILEFT)
      DET=DET*DIV
      IRANK=ILEFT-1
      IF(DIV)  2351,2350,2351
 2351 DIV=1.0/DIV
      DO 2098 I=1,LN
 2098 B(I,LNN)=0.0
      B(ILEFT,LNN)=1.0
      DO 2060 K=1,LNN
 2060 B(ILEFT,K)=B(ILEFT,K)*DIV
      DO 2080 I=1,LN
      IF(I-ILEFT)  2065,2080,2065
 2065 AIJ=-B(I,ILEFT)
      DO 2070 K=1,LNN
 2070 B(I,K)=B(I,K)+AIJ*B(ILEFT,K)
 2080 CONTINUE
      DO 2029 I=1,LN
 2029 B(I,ILEFT)=B(I,LNN)
 2000 CONTINUE
      IRANK=LN
 2350 WRITE(3,493)  DET
      WRITE(3,492)  LN,IRANK
      IF(LN.GT.IRANK)  GO TO 2011
      WRITE(3,494)
C     INVERSE HAS BEEN GENERATED  ****************************
      DO 2095 J=1,LN
      DO 2095 I = 1, LN
```

```
 2095 BB(I,J) = B(I,J)
      DO 2085 J=1,LN
      K=NROW(J)
      DO 2085 I = 1, LN
 2085 B(I,K) = BB(I,J)
      DO 2018 I=1,LN
      DO 2018 J = 1, LN
 2018 BB(I,J) = B(I,J)
      DO 2002 I=1,LN
      K=NCOL(I)
      DO 2002 J = 1, LN
 2002 B(K,J) = BB(I,J)
      DO 2020 I = 1, LN
      DO 2020 J = 1, LN
 2020 BB(I,J) = B(I,J)
C     INVERSE BB(I,J)    ****************************************
      DO 2468 I=1,LN
      WRITE(3,488)  I
 2468 WRITE(3,500)  (B(I,J),J=1,LN)
      DO 2022 I = 1, LN
      DO 2022 J = 1, LNN
 2022 B(I,J) = BSTOR(I,J)
      WRITE(3,486)
      DO 2600 K=1,LN
      WRITE(3,483) K
      DO 2602 I=1,LN
      BU(I,K)=0.0
      DO 2604 J=1,LN
 2604 BU(I,K)=BU(I,K)+B(I,J)*BB(J,K)
 2602 CONTINUE
C     APPROXIMATE UNIT MATRIX BU(I,K) ***************************
      WRITE(3,500) (BU(I,K),I=1,LN)
 2600 CONTINUE
      M=LN-1
      X=0.5
      WRITE(3,485)
      DO 2620 K=1,LN
      E=0.0
 2611 E=E+X
      BE=-(1.0-E)
      BEE=-(1.0+E)
      DO 2610 I=1,LN
      BBN(I,K)=0.0
      BBP(I,K)=0.0
      DO 2612 J=1,LN
      IF(J.EQ.K.OR.BU(J,K).EQ.0.0) GO TO 2612
      IF(BU(J,K).LT.0.0) BBN(I,K)=BBN(I,K)+BU(J,K)*BB(I,J)
      IF(BU(J,K).GT.0.0) BBP(I,K)=BBP(I,K)+BU(J,K)*BB(I,J)
 2612 CONTINUE
      BBC(I,K)=BE*BBN(I,K)+BEE*BBP(I,K)+BB(I,K)
C     INVERSE CHANGE BBC(I,K)***********************************
 2610 CONTINUE
```

```
      NONEG=0
      DO 2622 I=1,LN
      BUC(I,K)=0.0
      DO 2624 J=1,LN
 2624 BUC(I,K)=BUC(I,K)+B(I,J)*BBC(J,K)
C     UNIT MATRIX CHANGE BUC(I,K) ********************************
      IF(BUC(I,K).LT.0.0) NONEG=NONEG+1
      BU(I,K)=BUC(I,K)
 2622 CONTINUE
      IF(NONEG.LT.M.AND.E.LT.20.0) GO TO 2611
      WRITE(3,483) K
      WRITE(3,500) (BBC(I,K),I=1,LN)
      WRITE(3,482) E
 2620 CONTINUE
      WRITE(3,484)
      DO 2618 K=1,LN
      WRITE(3,483) K
 2618 WRITE(3,500) (BUC(I,K),I=1,LN)
      WRITE(3,484)
      DO 2614 K=1,LN
      BUCS(K)=0.0
      DO 2616 I=1,LN
 2616 BUCS(K)=BUCS(K)+BUC(K,I)
 2614 WRITE(3,481) K,BUCS(K)
C     UNIT MATRIX CHANGE ROW SUM BUCS(K) ***********************
 2011 CONTINUE
      STOP
      END
```

APPENDIX B

Computer program for determining the monotonicity and for finding the bounds on the inverse and the bounds on the solution of example no. 1.

```
      DOUBLE PRECISION X,E,BE,BEE,BU,BUC,BBC,BUCS,BDL,BBL,BUL
      DOUBLE PRECISION BUCC,BDU,BBU,BUU,BS,YL,CONL,YU,CONU,BBCC
      DIMENSION B(26,27), BSTOR(26,27), BB(26,27)
      DIMENSION  NROW(26), NCOL(26), ABA(26), CONST(26)
      DIMENSION BBN(26,27),BBP(26,27),BUCS(26),BBL(26,27)
      DIMENSION BU(26,27),BBC(26,27),BUC(26,27),BUL(26,27)
      DIMENSION BBCC(26,27),BUCC(26,27),BUU(26,27),BBU(26,27)
      DIMENSION BS(26)
      DIMENSION YL(26),YU(26),CONL(26),CONU(26)
  472 FORMAT(20X,20HSOLUTION UPPER BOUND)
  473 FORMAT(20X,20HCONSTANT UPPER BOUND)
  474 FORMAT(1H1,20X,20HSOLUTION LOWER BOUND)
  475 FORMAT(20X,20HCONSTANT LOWER BOUND)
  476 FORMAT(5X,43HTERMS IN SUM FORMING ONE ELEMENT OF BU(I,K))
  477 FORMAT(1H1,20X,23HUNIT MATRIX UPPER BOUND)
  478 FORMAT(1H1,20X,19HINVERSE UPPER BOUND)
  479 FORMAT(1H1,20X,19HINVERSE LOWER BOUND)
  480 FORMAT(1H1,20X,23HUNIT MATRIX LOWER BOUND)
  481 FORMAT(5X,7HROW NO.,I3,5HSUM =,E16.8)
  482 FORMAT(5X,9HEPSILON =,E16.8)
  483 FORMAT(10X,7HCOL NO.,I3,3X,10H**********   )
  484 FORMAT(1H1,20X,25HAPPROX UNIT MATRIX CHANGE)
  485 FORMAT(1H1,20X,14HCHANGE INVERSE)
  486 FORMAT(1H1,20X,18HAPPROX UNIT MATRIX)
  487 FORMAT(21X,22HTHE ORIGINAL EQUATIONS   )
  488 FORMAT(10X,7HROW NO.,I3,3X,10H**********      )
  489 FORMAT(21X,6HANSWER,21X,8HRESIDUAL)
  490 FORMAT(1HL,19X,8HRESIDUAL)
  491 FORMAT(21X,6HANSWER,22X,6HCHANGE)
  492 FORMAT(5X,11HEQUATIONS =,I3,10X,6HRANK =,I3   )
  493 FORMAT(5X,13HDETERMINANT =,E25.18)
  494 FORMAT(1H1,20X,11HTHE INVERSE    )
  495 FORMAT(1H1)
  496 FORMAT(E26.18,5I6)
  497 FORMAT(     10X,2E28.18,I5)
  498 FORMAT(4I5)
  499 FORMAT(4E16.8)
  500 FORMAT(8E16.8)
  501 FORMAT(5E25.16)
      READ(1,498)LN,NSOLVE
      LNN=LN+1
      DET=1.0
      DO 510 I=1,LN
      DO 510 J=1,LNN
  510 READ(1,500)  B(I,J)
      WRITE(3,487)
      DO 2471 I=1,LN
      WRITE(3,488)  I
 2471 WRITE(3,500)  (B(I,J),J=1,LNN)
      DO 2001 I=1,LN
      NROW(I)=I
 2001 NCOL(I)=I
```

```
      DO 2004 I = 1, LN
      DO 2004 J = 1, LNN
 2004 BSTOR(I,J) = B(I,J)
      DO 2000 ILEFT=1,LN
C     SEARCH FOR LARGEST ELEMENT ********************************
      BIG=0.0
      DO 2200 J=ILEFT,LN
      DO 2200 I=ILEFT,LN
      BURP=B(I,J)
      BURPE=BURP
      IF(BURP)  2031,2032,2032
 2031 BURPE=-BURP
 2032 IF(BIG-BURPE)  2039,2200,2200
 2039 BIG=BURPE
      IROW=I
      JCOL=J
 2200 CONTINUE
      IF(IROW-ILEFT)2092,2091,2092
 2092 DET=-DET
 2091 IF(JCOL-ILEFT)2094,2093,2094
 2094 DET=-DET
 2093 CONTINUE
C     INTERCHANGE TWO ROWS ***********************************
      DO 2050  J=1,LN
      D=B(IROW,J)
      B(IROW,J)=B(ILEFT,J)
 2050 B(ILEFT,J)=D
      KEEP=NROW(ILEFT)
      NROW(ILEFT)=NROW(IROW)
      NROW(IROW)=KEEP
C     INTERCHANGE TWO COLUMNS ********************************
      DO 2051 I=1,LN
      D=B(I,JCOL)
      B(I,JCOL)=B(I,ILEFT)
 2051 B(I,ILEFT)=D
      KEEP=NCOL(ILEFT)
      NCOL(ILEFT)=NCOL(JCOL)
      NCOL(JCOL)=KEEP
C     NORMALIZING AND ZEROING COLUMN ***************************
      DIV=B(ILEFT,ILEFT)
      DET=DET*DIV
      IRANK=ILEFT-1
      IF(DIV)  2351,2350,2351
 2351 DIV=1.0/DIV
      DO 2098 I=1,LN
 2098 B(I,LNN)=0.0
      B(ILEFT,LNN)=1.0
      DO 2060 K=1,LNN
 2060 B(ILEFT,K)=B(ILEFT,K)*DIV
      DO 2080 I=1,LN
      IF(I-ILEFT)  2065,2080,2065
 2065 AIJ=-B(I,ILEFT)
```

```
      DO 2070 K=1,LNN
 2070 B(I,K)=B(I,K)+AIJ*B(ILEFT,K)
 2080 CONTINUE
      DO 2029 I=1,LN
 2029 B(I,ILEFT)=B(I,LNN)
 2000 CONTINUE
      IRANK=LN
 2350 WRITE(3,493)  DET
      WRITE(3,492)  LN,IRANK
      IF(LN.GT.IRANK)  GO TO 2011
      WRITE(3,494)
C     INVERSE HAS BEEN GENERATED  ********************************
      DO 2095 J=1,LN
      DO 2095 I = 1, LN
 2095 BB(I,J) = B(I,J)
      DO 2085 J=1,LN
      K=NROW(J)
      DO 2085 I = 1, LN
 2085 B(I,K) = BB(I,J)
      DO 2018 I=1,LN
      DO 2018 J = 1, LN
 2018 BB(I,J) = B(I,J)
      DO 2002 I=1,LN
      K=NCOL(I)
      DO 2002 J = 1, LN
 2002 B(K,J) = BB(I,J)
      DO 2020 I = 1, LN
      DO 2020 J = 1, LN
 2020 BB(I,J) = B(I,J)
C     INVERSE BB(I,J)     ****************************************
      DO 2468 I=1,LN
      WRITE(3,488)  I
 2468 WRITE(3,500)  (B(I,J),J=1,LN)
      DO 2022 I = 1, LN
      DO 2022 J = 1, LNN
 2022 B(I,J) = BSTOR(I,J)
      WRITE(3,486)
      DO 2600 K=1,LN
      WRITE(3,483) K
      DO 2602 I=1,LN
      BU(I,K)=0.0
      DO 2604 J=1,LN
 2604 BU(I,K)=BU(I,K)+B(I,J)*BB(J,K)
 2602 CONTINUE
C     APPROXIMATE UNIT MATRIX BU(I,K) ***************************
      WRITE(3,500) (BU(I,K),I=1,LN)
 2600 CONTINUE
      WRITE(3,476)
      DO 2601 J=1,LN
 2601 BS(J)=B(3,J)*BB(J,1)
      WRITE(3,501) (BS(J),J=1,LN)
      M=LN-1
```

```
      X=0.5
      WRITE(3,485)
      DO 2620 K=1,LN
      E=0.0
 2611 E=E+X
      BE=-(1-E)
      BEE=-(1+E)
      DO 2610 I=1,LN
      BBC(I,K)=0.0
      DO 2612 J=1,LN
      IF(J.EQ.K) GO TO 2612
      IF(BU(J,K).LT.0.0) BBC(I,K)=BBC(I,K)+BE*BU(J,K)*BB(I,J)
      IF(BU(J,K).GE.0.0) BBC(I,K)=BBC(I,K)+BEE*BU(J,K)*BB(I,J)
 2612 CONTINUE
      BBC(I,K)=BBC(I,K)+BB(I,K)
C     INVERSE CHANGE BBC(I,K)************************************
 2610 CONTINUE
      NONEG=0
      DO 2622 I=1,LN
      BUC(I,K)=0.0
      DO 2624 J=1,LN
 2624 BUC(I,K)=BUC(I,K)+B(I,J)*BBC(J,K)
C     UNIT MATRIX CHANGE BUC(I,K) ******************************
      IF(BUC(I,K).LT.0.0) NONEG=NONEG+1
 2622 CONTINUE
      IF(NONEG.LT.M) GO TO 2611
      WRITE(3,483) K
      WRITE(3,500) (BBC(I,K),I=1,LN)
      WRITE(3,482) E
 2620 CONTINUE
      WRITE(3,484)
      DO 2618 K=1,LN
      WRITE(3,483) K
 2618 WRITE(3,500) (BUC(I,K),I=1,LN)
      WRITE(3,484)
      DO 2614 K=1,LN
      BUCS(K)=0.0
      DO 2616 I=1,LN
 2616 BUCS(K)=BUCS(K)+BUC(K,I)
 2614 WRITE(3,481) K,BUCS(K)
C     UNIT MATRIX CHANGE ROW SUM BUCS(K) **********************
      WRITE(3,479)
      BDL=1.002
      DO 2626 K=1,LN
      DO 2619 I=1,LN
 2619 BBL(I,K)=BBC(I,K)/BDL
C     INVERSE LOWER BOUND BBL(I,K) ***************************
      WRITE(3,483) K
 2626 WRITE(3,500) (BBL(I,K),I=1,LN)
      WRITE(3,480)
      DO 2630 K=1,LN
      DO 2632 I=1,LN
```

```
      BUL(I,K)=0.0
      DO 2634 J=1,LN
 2634 BUL(I,K)=BUL(I,K)+B(I,J)*BBL(J,K)
C     UNIT MATRIX LOWER BOUND BUL(I,K)**************************
 2632 CONTINUE
      WRITE(3,483) K
 2630 WRITE(3,500) (BUL(I,K),I=1,LN)
      WRITE(3,485)
      DO 2720 K=1,LN
      E=0.0
 2711 E=E-X
      BE=-(1-E)
      BEE=-(1+E)
      DO 2710 I=1,LN
      BBCC(I,K)=0.0
      DO 2712 J=1,LN
      IF(J.EQ.K) GO TO 2712
      IF(BU(J,K).LT.0.0) BBCC(I,K)=BBCC(I,K)+BE*BU(J,K)*BB(I,J)
      IF(BU(J,K).GE.0.0) BBCC(I,K)=BBCC(I,K)+BEE*BU(J,K)*BB(I,J)
 2712 CONTINUE
      BBCC(I,K)=BBCC(I,K)+BB(I,K)
C     INVERSE CHANGE TO FIND UPPER BOUND BBCC(I,K) *************
 2710 CONTINUE
      NOPOS=0
      DO 2722 I=1,LN
      BUCC(I,K)=0.0
      DO 2724 J=1,LN
 2724 BUCC(I,K)=BUCC(I,K)+B(I,J)*BBCC(J,K)
C     UNIT MATRIX CHANGE TO FIND UPPER BOUND BUCC(I,K) *********
      IF(BUCC(I,K).GT.0.0) NOPOS=NOPOS+1
 2722 CONTINUE
      IF(NOPOS.LT.LN.AND.E.GT.-10.0) GO TO 2711
      WRITE(3,483) K
      WRITE(3,500) (BBCC(I,K),I=1,LN)
      WRITE(3,482) E
 2720 CONTINUE
      WRITE(3,484)
      DO 2714 K=1,LN
      WRITE(3,483) K
 2714 WRITE(3,500) (BUCC(I,K),I=1,LN)
      WRITE(3,478)
      BDU=0.995
      DO 2726 K=1,LN
      DO 2718 I=1,LN
 2718 BBU(I,K)=BBCC(I,K)/BDU
C     INVERSE UPPER BOUND BBU(I,K)******************************
      WRITE(3,483) K
 2726 WRITE(3,500) (BBU(I,K),I=1,LN)
      WRITE(3,477)
      DO 2730 K=1,LN
      DO 2732 I=1,LN
      BUU(I,K)=0.0
```

```
      DO 2734 J=1,LN
 2734 BUU(I,K)=BUU(I,K)+B(I,J)*BBU(J,K)
C     UNIT MATRIX UPPER BOUND BUU(I,K) *************************
 2732 CONTINUE
      WRITE(3,483) K
 2730 WRITE(3,500) (BUU(I,K),I=1,LN)
      WRITE(3,474)
      DO 2800 I=1,LN
      YL(I)=0.0
      DO 2802 J=1,LN
      IF(B(J,LNN).GT.0.0) YL(I)=YL(I)-BBU(I,J)*B(J,LNN)
      IF(B(J,LNN).LE.0.0) YL(I)=YL(I)-BBL(I,J)*B(J,LNN)
 2802 CONTINUE
 2800 CONTINUE
C     SOLUTION LOWER BOUND YL(I) *******************************
      WRITE(3,500) (YL(I),I=1,LN)
      WRITE(3,475)
      DO 2804 I=1,LN
      CONL(I)=0.0
      DO 2806 J=1,LN
 2806 CONL(I)=CONL(I)+B(I,J)*YL(J)
 2804 CONTINUE
C     CONSTANT LOWER BOUND CONL(I) *****************************
      WRITE(3,500) (CONL(I),I=1,LN)
      WRITE(3,472)
      DO 2808 I=1,LN
      YU(I)=0.0
      DO 2810 J=1,LN
      IF(B(J,LNN).GT.0.0) YU(I)=YU(I)-BBL(I,J)*B(J,LNN)
      IF(B(J,LNN).LE.0.0) YU(I)=YU(I)-BBU(I,J)*B(J,LNN)
 2810 CONTINUE
 2808 CONTINUE
C     SOLUTION UPPER BOUND YU(I) *******************************
      WRITE(3,500) (YU(I),I=1,LN)
      WRITE(3,473)
      DO 2812 I=1,LN
      CONU(I)=0.0
      DO 2814 J=1,LN
 2814 CONU(I)=CONU(I)+B(I,J)*YU(J)
 2812 CONTINUE
C     CONSTANT UPPER BOUND CONU(I)*****************************
      WRITE(3,500) (CONU(I),I=1,LN)
 2011 CONTINUE
      STOP
      END
```

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THE MONOTONIC PROPERTY OF A SYSTEM
OF LINEAR EQUATIONS AND ITS APPLICATIONS

by

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This report presents a theorem which states sufficient conditions to assure that a system is monotonic. The theorem presented is less restrictive than the two known existing theorems.

The theorem was applied to three example problems. Only one of the example problems satisfied the conditions of the two existing theorems. However, all three examples satisfied the conditions of the theorem presented.

In addition, the bounds were determined for the solution of one of the examples since it is assured that bounds for the solution can be determined if the system is monotonic. The bounds were determined by first determining the bounds for the inverse matrix and then using these bounds to determine the bounds for the solution. The bounds determined in this manner were very accurate.

The method presented for determining if a system is monotonic was used to determine the bounds on the inverse. This made it a simple procedure to determine the bounds for the solution once the monotonicity of the system was determined. Therefore, the method presented has practical applications for determining bounds as well as proving that a system is monotonic.