

An Elementary  
Investigation of  
Rolling Disks.

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## Outline of the Text.

I The field and the part we are to consider

II Calculations with supposed conditions

1 Single weight at rim of disk

(a) Path

(b) Velocity

(c) Component velocities (hor. and vert.)

(d) Accelerations (hor. and vert.)

(e) Effect

2 Single weight at one fourth diameter from rim.

(a) Path

(b) Velocity

(c) Vertical component

(d) Vertical acceleration

(e) Effects

III A glance at other problems,

The real complexity of the facts and conditions present in every consideration of rolling bodies, has generally, if not universally, been underestimated by those who have had to deal with the problem; and while practically the problem is not so serious an obstacle to mechanical progress as theoretical investigation may tend to show, yet the inappreciation of the effects of unbalanced wheels, or the mistaken idea that unbalanced wheels were balanced, has led to serious and perplexing results in some lines of engineering.

I shall not attempt to solve all the combinations or problems that may arise in connection with this subject, or even to show whereby may be solved the entire question, but shall endeavor to do just two things:

First, To point out the conditions that do ~~exist~~ in a few instances of the simplest cases of unbalanced wheels.

Second, To show the effect of such conditions on the support or rail and upon the axle or shaft, if there be such effect, result or action.

The first may seem but a means to the second, but is enough of a problem in itself to warrant its being called a division of the subject.

The cases will be graphically investigated for want of apparatus, and because that method will show the theoretical results more clearly than words or pictures, besides being a more interesting problem. Little effort will be made to point out practical applications of the derived facts, but incidentally some instances may be mentioned where the question is of practical importance.

The field is too wide to attempt to develop tables or formulas to cover it, and therefore concrete instances only will be handled. This treatise will contain nothing new either in theory or practice; but will combine a number of interesting results, derived from theoretical investigation, and may possibly interest some one who has not solved the problem, or who has a liking for such a topic.

Let us suppose we have a perfectly round disk, that is, an ideal circular disk, of any practical diameter, and with a

reasonable thickness, just to aid the imagination in keeping it vertical on its edge.

Let us now assume to have an ideal level plane on which all our operations will take place. Our disk will now remain stationary on any part of the plane when acted on only by gravity, - our field assumed to be small enough to disregard the curvature of the earth. It will also remain in equilibrium on any point in its own circumference, or on any element which may be called the supporting element. If rolled along the plane, (or, since we assume the path to be straight, we may call it the rail if it helps the imagination) or support the pressure on that plane or support is constant. The question which arises in hydraulics as to velocity head and pressure head does not concern us here, because a velocity means a corresponding energy, and the pressure head is unaffected. This ~~would~~ seem to need no proof and it will be assumed that it does not, since it is the only way we can well imagine the condition to be.

The questions of vibrations of material may depend on velocity, but we shall suppose that also to be a foreign question.

Now let us bore a hole through our disk at the edge without altering the length or shape of the perimeter. This is, of course, practically impossible, but it may be approximated, and we may assume that we have the reality. Now will our arrangement balance, or to what extent will it balance? If left to itself, it will vibrate to and fro, or rock on an arc, the middle point of which is directly opposite the hole, unless it be placed so that the hole is in the perpendicular to the plane drawn through the centre of the disk; in which case, it will be in stable or unstable equilibrium as the hole is above or below the centre. If set rolling it will evidently roll over and over any number of times, and it is in one of these times that we will consider it.

We might admit, - or rather state, that the punctured disk weighs a constant quantity, regardless of its attitude; gravity being a constant effort, and if kept from assuming a position with the

hole directly above the center by some horizontal force, its vertical pressure remains constant. The question is this: -  
 "Do the static conditions of constant vertical pressure hold true during the time of rolling?"

First let us call our hole a positive quantity for convenience, or better perhaps, imagine a positive weight at the rim of an otherwise balanced (or even if we wish, weightless) disk, and concentrated at a point. If the disk rolls, the point will trace a path which is a cycloid, which is the black curve in figure I. This curve gives us direction, and direction only, of the travel of the point; that direction does not ~~show~~ the direction at successive intervals of time; that direction is with regard to the plane or some other fixed point. It does not show the direction with regard to any part of the disk, nor does it show the velocity in any way. How now does the velocity of the weight vary?

At every instant, each point in the disk is revolving (at a varying distance from it) about a point in the circumference, namely the point or element which for the

time being is in contact with the supporting plane, and therefore it follows that the total linear velocity of a point (as the one we are considering) varies as but one independent variable, which is its distance from the point of support. From the cycloid we find a number of these distances by taking points in the curve as centres and cutting the path of the centre of the centre of the disk by arcs of radii equal to the radius of the disk and dropping perpendiculars from these points of intersection to the ground line or locus of the points of support, and connecting the feet of these perpendiculars with their corresponding points in the cycloid. We now plot the curve which will show the total velocity for horizontal positions of the point by its ordinates, by taking the asymptotes of the point in the cycloid, for asymptotes and the real momentary radii of rotation already found, for ordinates. As we should expect it is greatest at point A figure I when it is twice the velocity of the centre of the disk, and for simplicity we may draw it to a scale which will show the velocity curve at its

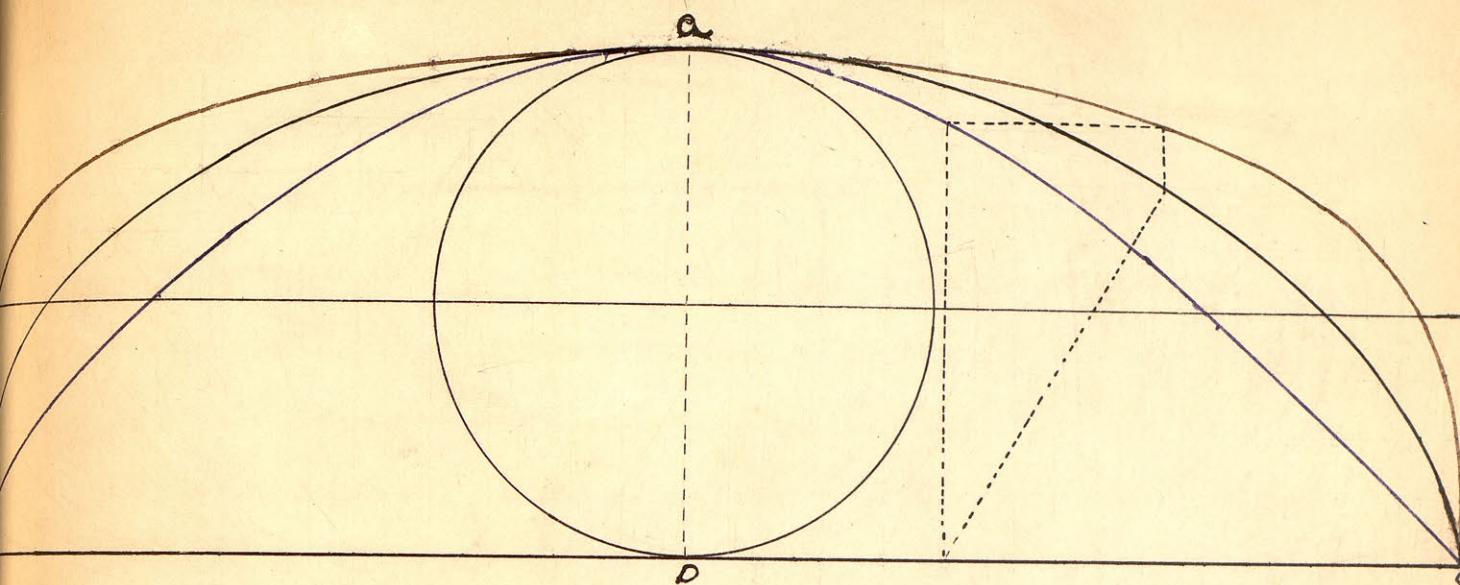


Fig. 1

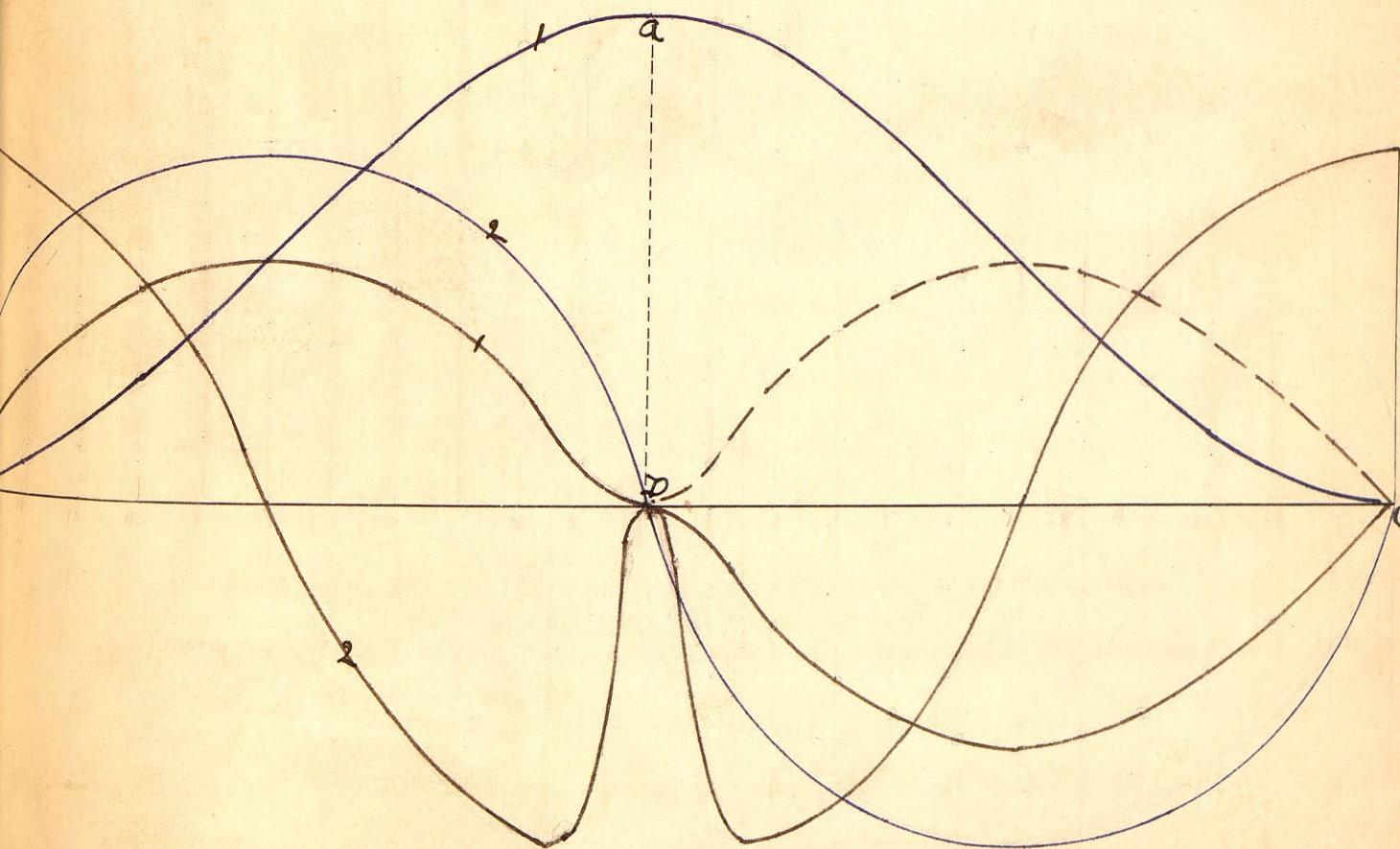


Fig 2

maximum equal to the maximum of the cycloid. At each end of the revolution the velocity is zero and it varies between zero and its maximum, reaching each value twice during each revolution. The curve on the whole resembles the cycloid but is not a cycloid and is outside of it. Now, lest we may want to use the false assumption later, we may state here, that when the point is as far from the plane as the centre of the disk is, it is not moving with the same velocity as the centre of the disk, though at these times the horizontal component of its velocity is equal to that of the centre of the disk, whose velocity curve is, of course, the straight line through the centre in any position and parallel to the X axis.

Another curve will soon become a necessity or at least a great convenience. It is a curve to represent by its ordinates the total linear velocity of our point or weight for any given supporting element. This curve is found by the two already determined curves by projecting the located points in the velocity curve horizontally

till they are vertically above their corresponding points of support, and drawing our curve through these new positions, as is shown in blue ink in figure I. We find this curve also similar to the cycloid but inside instead of outside as the first was. The exact meaning of this curve must be remembered when we use the curve.

We might now draw a tangent curve to our velocity curve; but it would reward us nothing, since the direction of action is not known, and it would be necessary for every point, to resolve the acceleration into horizontal and vertical components.

First let us resolve our velocity into vertical and horizontal components, and find a curve to represent each, taking first the vertical component.

It is zero at B and C, (Figure I or II) obviously and varies with the length of the length momentary radius of rotation, and also with the cosine of the angle which that radius makes with the rail or plane. Proof of this is given in figure III with explanations on same sheet. For given points, the ordinates of this curve may

The vertical component of the velocity of a point in a rolling disk varies as the cosine of the angle between the momentary radius of rotation and the path on which the disk rolls.

Proof:

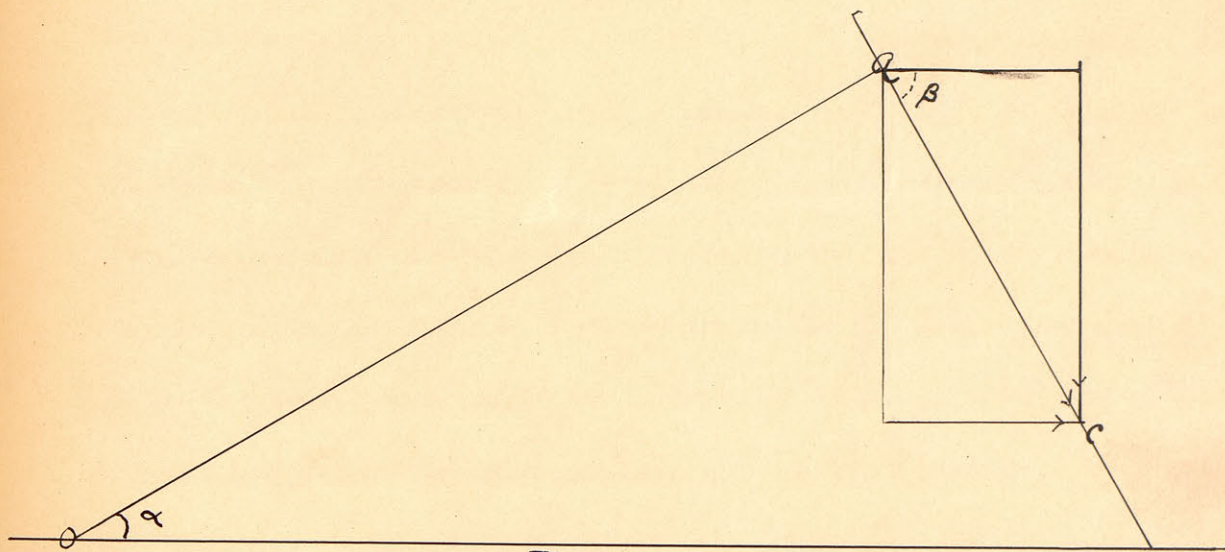


FIG. 3

Let  $O$  be the momentary centre,  $A$  the point and  $OA$  the momentary radius.

The direction of the movement at this moment is that of  $A.C.$  the vertical component of which varies as the sine of  $\beta$  and the horizontal component as the cosine of  $\beta$ .

The cosine and sine of  $\alpha$  equal respectively the sine and cosine of  $\beta$ .

$\therefore$  The vertical component of the motion or velocity in a vertical direction varies as the cosine of the angle between the horizontal and the momentary radius and the horizontal component varies as the sine of the same angle.

be found approximately from previous curves, by measuring by measuring and finding the cosine the angles of these radii, and multiplying by the length of the radii to any scale. We draw this curve and find it to be zero at A, B and C, figure I and II. It must be remembered that the ordinates of this curve show the vertical components of the velocity of the weight for each point of support on the rail. There is a maximum at about seven twenty-fourths of a revolution each way from the central position, or if we consider direction or sign as well as quantity it will throw the right half of our curve below the X axis if the disk rolls from left to right. Now, since we have a minimum at D, figure II, when we consider only quantity, it becomes a point of inflection. This has a peculiar effect on a succeeding curve, but by carefully working over the ground several times, I am convinced that at the point A (figure I) the point does make the motion necessary to give this curve a point of inflection at D (figure II).

A new condition confronts us. The velocity up or down or horizontally has no effect upon the rail or upon the axle. It is the change of velocity that has an influence, — the creating

and destroying of momentum or kinetic energy. This Change of Velocity may obviously be represented by the tangent to the curve representing vertical velocity. This curve is already quite complex, and whether it has an equation or not, it will hardly pay us to attempt to discover. The curve of the first derivative may be found approximately by drawing tangents at several points, and drawing to an enlarged scale the curve whose ordinates are the tangents of the angles these lines subtend upon the  $X$  axis. This is the brown curve number two (figure II). This curve will also represent the pressure on the rail, more or less than the action of gravity, - the higher the curve, the higher the pressure etc. Note that this curve is slightly peculiar. Beginning with the point where the weight is at rest, namely at B, as the wheel is turning, it is in the midst of a flow upon the rail. This flow or high pressure decreases slowly, and then more rapidly till about one fifth of a revolution is completed, when it reaches 0. It continues to decrease rapidly becoming more slowly till nearly half the revolution is completed, when a minimum is reached and the curve comes suddenly up to zero.

(This is because of the point of inflection in the velocity curve.) Here there is a maximum, and for an instant, the pressure remains constant at 0, when a lift begins rapidly till it reaches a second minimum, - also very near the middle of the revolution, and then it rises to a maximum in the reverse of the way it came down on the other side. These minima may be sufficient to lift the entire body from the rail as has been the case in locomotives where the whole locomotive was lifted by the action of a poorly arranged counter balance.

Even if it be incorrect that the tangent curve should rise to zero, the result is plainly a lift at the middle of the stroke whether it is interrupted by an instantaneous cessation of lift or not. Ordinarily all the effects of vertical acceleration will be exerted on the rail, - the case where the whole body is lifted being an exception.

The horizontal acceleration is the part that influences the shaft or axle. A curve to represent this, may be produced by the same process by which we found the vertical acceleration curve, except that

the horizontal velocity varies with the sine of the angles between the radii and the base line, instead of with the cosine, and the acceleration curve must be adjusted to the velocity curve. At the beginning of the stroke, the horizontal component of velocity is zero, but it, slowly at first, and then more rapidly, increases to a maximum at *a*. (Figure II) and then slowly, and then rapidly, decreases to a minimum at *c*. In fact it is a simple sine curve or a cosine curve as we choose to place our Y axis, but is in no place minus, that is, it is a cosine curve with a constant added to the ordinates to bring the minima into the X axis. Its tangent curve will now be a sine curve also; but is half above and half below zero, showing that it both pushes and pulls, or accelerates and retards the shaft, and according to our curve, when the acceleration is positive, it must be a retardation to the axle. This curve is the blue curve (number 2) Figure II. In other words, the result will be a jerking to and fro of the point or axle.

If the rail should bend at each stroke, or the weight were regularly lifted, it might be interesting to find the resultant action of the horizontal and

vertical action on the shaft. The total quantity could be found by drawing a curve tangent to the total velocity curve and the direction for any point is fairly well shown (reversed however) already in that curve.

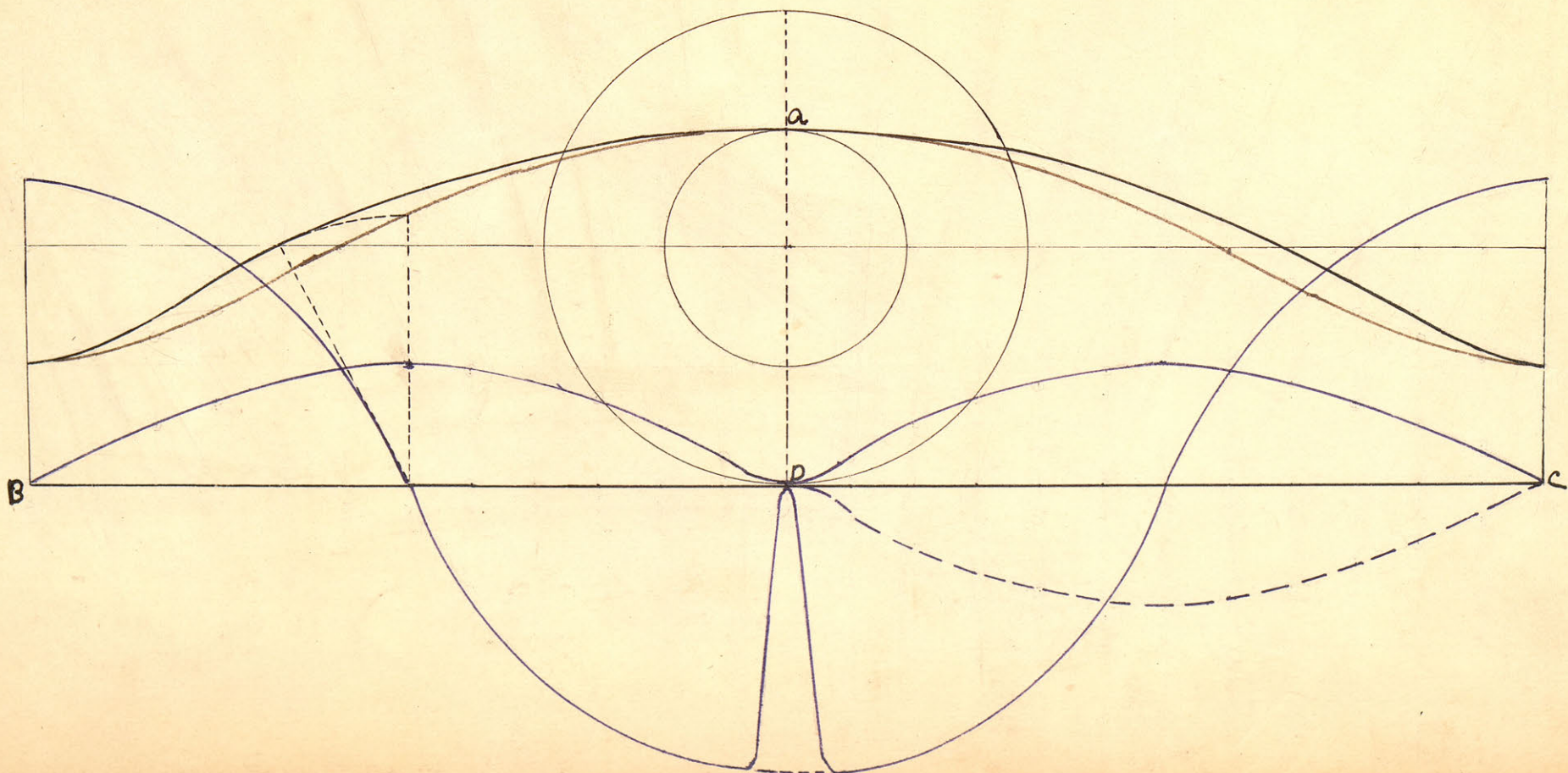
We have now spoken only of the simplest possible condition, that of a single weight at a point in the circumference of a perfect disk.

Suppose we had more than one weight. Suppose they were not symmetrically arranged. Suppose they were not on the rim. Suppose they were of unequal ~~magnitudes~~ magnitude. The case at once becomes more complex; but we will draw one

nearly as simple, that of a single weight at half the radius from the centre of the disk. Its path is shown in black in figure IV. Limiting the curve to show velocity for horizontal

positions of the point, we may draw first the ~~vertical component and also the tangent curve~~ by curve for velocity for points of support, and find the vertical component and also the tangent curve by the same plan we followed before when the point was at the rim, and we find that while the actual path and total velocity curve differ from the first example, the resultant action

Fig. 4



on the rail is the same except that it is but half as great. Now if we double the mass of our point or weight the result would be identical, which we should probably have guessed at first, but which the first investigation might have led us to doubt.

A case more practical than all this far considered is that in which there are reciprocating parts vertically supported, as in the locomotive, and which act horizontally on the crank. Part of these parts rest on the crank pin, and are easily balanced by a simple counter balance, and ~~therefore~~ no difficulty has arisen. If the wheel rotates uniformly at any speed, the velocity of the reciprocating parts which are not resting on the crank pin, may be represented by a sine curve, and its action on that pin and thence on the shaft, by the cosine curve. If we put a counter balance on the wheel to counteract this action, we can destroy it, but on the rail, the counter balance is left to act unmodified as any unbalanced weight. This may be remedied by lightening the rotating parts and adjusting the steam valves to cushion the piston and rod;

but so long as the cylinder is horizontal, there must be an action on either the rail or the shaft or both. If the cylinder is made vertical, the action on the rail is easily counterbalanced away; but the forward and backward action on the crank remains.

Another interesting example is to imagine the tendency of the velocity of the weight to become a constant. There must be such a tendency. If it exerts its fullest influence what will happen? If carried to extreme it is an impossibility for the reason that at B and C, figure I, the velocity is zero, and if it were always zero, the disk could not roll. Since the vertical action concerns gravity alone, does the horizontal component of the velocity tend to be a constant, and if so, to what extent and what would be this action ~~of~~ on the shaft or the disk? Would the disk be compelled to slide instead of rolling at B and C? These are just hints at the directions this study might take; but which time will not permit. If nothing more has been done ~~it is~~ hoped that we have seen that there is an interesting problem here and that it may be much more complex.