A HISTORY OF THE FOUNDATION OF MATHEMATICS.

by

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A History of the Foundation of Mathematics.

A history of mathematics is instructive to the student. It shows him the paths in which mathematicians have gone to accomplish their results; it shows him the things which have made openings for great progress, such as an invention of a good notation. In the study he sees what problems have been solved or what cannot be solved and so does not waste his time on them. He sees the mistakes of others and avoids them.

The history inspires the teacher as well as the student. A teacher who can tell the history of various problems holds the interest of his class well. If he can make his pupil wonder at the deep thought placed on one problem he has done that pupil a great service.

Florian Cajori says:- "The history of mathematics is one of the large windows through which the philosophic eye looks into the past ages and traces the line of intellectual developement." It is plain to see that as civilization spreads and gets a firmer hold on the people the sciences deepen and so the developement of mathematics is an index to the history of civilization.

Accomplishments of the Babylonians

We look to the Babylonians for the first work of importance in mathematics. We will turn immediately to their system of notation and examine it. 𒁹 = 1. 𒌋 = 10. 𒈨 = 100. Two systems were used the additive and multiplicative. Numbers below 100 were expressed by adding values of the parts, 𒁹𒁹𒁹 = 3, 𒌋𒌋𒌋 = 30 etc. In writing the hundreds a symbol was placed to the left of the 100 and was to be multiplied by the 100. 𒁹𒁹𒁹𒁹 𒈨 = 400 𒌋𒈨 = 1000. This latter was then taken for a unit so that 𒌋𒌋𒈨 was not 20 x 100 but 10 x 1000. It is not known whether they had the 0 = zero or not. They used the sexagesimal system in fractions.

The Babylonians reckoned the year at 360 days. They divided the circle into 360 degrees. They divided the degree into 60 seconds. They divided the day into 24 hours; the hour into 60 minutes.

In arithmetic they made some advance but used the abacus for practical calculation. The arithmetical and geometrical progressions were in their acquaintance. In geometry they did little. The triangle and quadrangle were used in their augeries. They took π equal to 3. The lever and pulley received some attention. The progress in mechanics is further shown in their sun-dial and water-clock. They directed their greatest efforts toward astronomy. Their records reach back as far as 2234 B.C. They predicted eclipses and the return of comets with considerable accuracy. Here is an Assyrian astronomical report: —

"To the King, my lord, thy faithful servant, Mar-Istar."

"On the first day, as the new moon's-day of the month, Thammus declined, the moon was again visible over the planet Mercury, as I had already predicted to my master the King. I erred not."

Egyptians.

The old Greeks believed that to the Egyptians belonged the priority of invention in the mathematical sciences. The most ancient hand book of mathematics we have, the Ahmes Papyrus, was written by an Egyptian, **Ahmes**, before 1700 B.C. Its title was:– "Directions for obtaining Knowledge of all Dark Things." I was made up of results having but few rules of procedure. The Egyptians certainly had made some progress at an early date.

In geometry the Egyptians dealt with the practical and left the theoretical untouched. Their knowledge of geometry is seen in their constructions and surveying. Some of their problems involved proportion, but most of them were problems of construction. They failed to bring special cases under general rules.

Their system of notation was cumbrous. In practice they used pebbles and probably the abacus to count with. Here are some of their symbols:– 𓏺 = 1. 𓎆 = 10. 𓍢 = 100 𓆼 = 1000. 𓂭 = 10000. 𓆐 = 100 000. 𓁨 = 1 000 000. 𓍶 = 10 000 000. They used the additive system 𓎆𓎆𓎆𓏺𓏺𓏺 = 33. In fractions the Babylonians kept the denominator constant at 60, the Egyptians the numerator.

In the Ahmes Papyrus is seen the first record of problems involving an unknown quantity. It is called by them "hau" or heap. "Heap, its 1/7, its whole, it makes 19" i.e. $\frac{x}{7} + x = 19$. We see here the early beginning of algebra. The mechanical skill of the Egyptians was seen in the construction of pyramids and in the making of devices to build them. Their progress in astronomy is evident from their observations and from cycles for the adjustment of different reckonings of time.

Why did the progress cease after going so high in such remote times? Probably because the early discoveries were entered in their sacred books and in after ages it was heretical to change these books. So stagnation came and the Greeks took up the work.

Greeks.

From the Egyptians the Greeks received their impetus toward the study of mathematics. But it was as Plato said, "Whatever we Greeks receive, we improve and perfect". The Greeks carried geometry far beyond the Egyptians, caring not alone for practical results but loving the science as a science and finding pleasure in ideal relations.

Ionic School.

Thales (640-546 B.C.) was the founder of Greek mathematics. He learned the lore he introduced in Greece in Egypt. Among his pupils at the Ionic school, which he founded are these:- **Anaximander**, **Anaximenes**, and **Anaxagoras**. Another geometer of this time was **Oenopides of Chios**. Two theorems are ascribed to him. "From a point without to draw a perpendicular to a given line"; and "To draw an angle on a line equal to a given angle". To Thales are attributed these:- "The circle is bisected by its diameter. The angles at the base of an isosceles triangle are equal. If two straight lines cut one another the opposite angles are equal. The angle in a semi-circle is a right angle. A triangle is determined if it base and base-angles be given".

Pythagoreans.

Pythagoras of Samos (580-500 B.C.) next took up the work. After wide study and travel he went to Croton and founded his school. All of his pupils were bound in a brotherhood for study. The brotherhood took a stand in politics and became an object of suspicion. It was broken up by the plebeian party. The brotherhood existed for 200 years more however. **Philolaus**, a late pythagorean wrote a book on their doctrines. He gave first to the world the teachings of the school.

The geometry of the school had to do with areas. Among important demonstrations are these:- "The sum of the squares on the two sides of a right triangle is equal to the square on the hypothenuse. The sum of the angles of a triangle is equal to two right angles. The plane about a point is filled with six equilateral triangles, four squares, or three regular hexagons". They could construct a polygon equal in area to another and similar to a third. They were acquainted with the tetraedron, octaedron, icosaedron, cube. They had no demonstration on the circle.

Archytas (428-347 B.C.) was the first to apply geometry to mechanics and to treat the latter methodically. He forwarded the doctrine of proportion

Sophist School.

When the Greeks defeated the Persians they formed their cities into a league with Athens as a center. Athens founded schools. Teachers, called **Sophists**, came from Sicily where Pythagorean doctrines had spread. They took up the geometry of the circle. Many discoveries were made in attempts to solve these three famous problems:- "To trisect an arc or an angle;" "To double the cube;" and "To square the circle". **Hippias of Elis** invented the quadratrix with which he solved the first of these three. **Hippocrates of Chios** worked with the last. He transferred the theory of proportion from numbers to magnitudes. He wrote a text book, the elements. **Antiphon** and **Bryson of Heraclea** introduced the method of exhaustion in attempts to square the circle. This method was afterwards used in finding the ratio of the areas of two curvilinear plane figures. **Democritus of Abdera** (460-370 B.C.) worked at this time though not at the sophist school. He wrote on incommensurable lines, on geometry, on numbers, and on perspective. None of these are extant.

Platonic School.

After the Peloponnesian War (431-404 B.C.) Athens sank politically but rose in learning. **Plato** (429-348 B.C.) after travelling in Egypt, Italy, and Sicily founded a school in Athens, (389 B.C.) He did little original work but improved methods, made definitions, and invented analysis as a method of proof. It had be used instinctively before. Plato duplicated the cube and took up the study of the prism, pyramid, cylinder, and cone. As a result **Menaechmus** invented the conic sections parabola, ellipse, and hyperbola. **Dinostratus** squared the circle by means of the quadratrix of Hippias.

The fame of Plato's academy is partly due to the school founded at Cyzicus by **Eudoxus**, probably the greatest mathematician of this time. He is called the father of scientific astronomy. He increased the number of theorems. He proved that the pyramid is exactly one third of a prism, and a cone one third of a cylinder, having equal base and altitude. He used the method of exhaustion. **Menaechmus, Dinostratus, Athenaeus**, and **Helicon** were his pupils.

Plato was called a maker of mathematicians. Here are some of them. **Thaetetus, Leodamus, Neocleides, Leon, Thaedeus of Magnesia, Hermotimus of Colophon, Amyclas of Heraclea, Cyzicenus, and Philippus of Mende.** **Aristaeus** wrote on conic sections and regular solids. His works contain a summary of the work of the Platonic School. **Aristotle** though not a professed mathematician improved many difficult definitions. He probably wrote a work, Mechanica

The First Alexandrian School.

Egypt was Ptolemy's share of the Alexandrian Empire when it fell. He built the University of Alexandria and encouraged all learning. **Euclid**, known by his book on geometry, the Elements, came and opened the school of mathematics. This book is still used as a text book. Comparitively few of the theorems are his own. There are thirteen books in the Elements. The first four are on plane geometry. The fifth is on proportion; the sixth, geometry of similar figures; the next three are on arithmetic; tenth, theory of incommensurables; the next three are on stereometry. There are two more books on solid geometry but probably they are not his. **Conon, Dositheus,** and **Zeuxippus** followed him.

Archimedes of Syracuse (287?-212 B.C.), the greatest mathematician of Greece, studied in the Alexandrian school then returned and became King Hieron's friend by making war engines. Here are his extant works:- 1st. Two books on Equiponderance of Planes; 2nd. Two books on the Sphere and Cylinder; 3rd, Mensuration of the circle; 4th, On Spirals; 5th, Conoids and Spheroids; 6th, Sand counter; 7th Two books on Floating Bodies; 8th. Fifteen Lemmas. He is the author of the first good, sound work on mechanics.

Apollonius of Perga comes about forty years after Archimedes. He ranks next to him among Greek mathematicians. We know him by his work on Conic Sections, a work in eight books, of which we have seven. The last three books were entirely new thought. He wrote other books:- On Contacts, Plane Loci, Sections of an Area, Inclinations, Determinate Sections.

After Apollonius there was a decline. Among the mathematicians are these:- **Eratosthenes; Nicomedes,** inventor of the conchoid; **Diocles,** cissoid; **Perseus,** wrote on spires; **Zenodorus; Hypsicles,** author probably of the 14th and 15th books of Euclid; **Hipparchus,** the greatest astronomer of antiquity, originated the science of trigonometry.

calculated a table of chords; **Heron the Elder**, invented Heron's fountain; **Geminus**; **Theodosius**; **Dionysodorus**.

The Second Alexandrian School.

The Christian era marks the beginning of this school. There is an increased interest in the theory of numbers. **Serenus of Antissa** was connected with the school. He solved problems on the cone and cylinder and wrote on their sections. **Menelaus of Alexandria** worked and wrote on spherical triangles. **Claudius Ptolemaeus**, a great astronomer, wrote the Almagest and Geographica. The Almagest for a foundation for all astronomy down to Copernicus. He developed trigonometry, especially spherical trigonometry, for astronomical use. He calculated a table of chords. For 150 years no geometer except **Sextus Julius Africanus** appeared. But in this interval **Nicomachus** and **Theon of Smyrna** worked on the theory of numbers.

Pappus (b 340? A.D.) was the last great mathematician of this school. He wrote many commentaries on works of the masters. His Mathematical Collections in eight books is all we have however. Information on the treatises of some of the great Greek mathematicians which are now lost, is given and so is invaluable. Some of his original work was fine. Here is a theorem of his:- "The volume generated by the revolution of a plane curve which lies wholly on one side of the axis, equals the area of the curve multiplied by the circumference of its center of gravity."

Theon of Alexandria brought out an edition of Euclid's Elements with notes. He wrote a commentary on the Almagest. His daughter **Hypatia** (see Kingsley's Hypatia) was the last teacher of prominence of this school. She was said to be an abler philosopher than her father. Other geometers were **Proclus**, **Damascius**, **Eutocius**, and **Simplicius**. Two others worked here **Iamblichus** and **Porphyrius**.

In the year 529 A.D. Justinian closed the schools.

We have noted the growth of geometry. Astronomy with the aid of algebra and trigonometry was making rapid strides. All of the learning was passed over to the Arabs in the 9th century and they brought it into the west of Europe. Till the 17th Century Greek geometry ruled; when Descartes, Newton, Leibnitz the best of its scholars superceded it.

Greek Arithmetic.

The Greeks distinguished between the science of numbers, called arithmetica, and the art of calculation, called logistica. At first they counted on their fingers or with pebbles. Pythagoras first introduced the abacus. The first Gracian numerical symbols were the Herodianic signs. These were replaced by the alphabetic numerals:-

α	β	γ	δ	ε	ϛ	ζ	η	θ	ι
1	2	3	4	5	6	7	8	9	10

κ	λ	μ	ν	ξ	ο	π	ϙ	ρ	σ	τ	υ	φ	χ	ψ	ω	ϡ	͵α	͵β	͵γ	$\overset{\alpha}{M}$	$\overset{\beta}{M}$	$\overset{\gamma}{M}$
20	30	40	50	60	70	80	90	100	200	300	400	500	600	700	800	900	1000	2000	3000	10 000	20 000	30 000

etc. 10 000, 20 000, 30 000 etc. In writing a horizontal line indicated the number $\overline{\delta M\, ,\gamma \chi o \eta} = 43,678$. Fractions were denoted by first writing the numerator marked with an accent and then the denominator marked with two accents and written twice $\iota\gamma'\kappa\theta''\kappa\theta'' = 13/29$. Addition, subtraction, and multiplication were performed, except by expert mathematicians, on the abacus. They could not give to the world a good symbolism. We will pass to arithmetica.

Pythagoras first draws our attention. He called numbers odd or even. He further divided them into heteromecia, triangular, perfect, defective, excessive, and amicable. The Pythagoreans devoted themselves to proportion. a, b, c, d are in arithmetical proportion when $a-b=c-d$; in geometrical proportion when $a:b::c:d$; in harmonic when $a-b:b-c=a:c$; in musical when $a:\frac{a+b}{2}=\frac{2ab}{a+b}:b$. In connection with arithmetic Pythagoras made some geometrical investigations. In the study of the isosceles right triangle it was found that a number exactly equal to the hypothenuse could not be obtained. The first first incommensurable ratio was that of the side of a square to its diagonal, as $1:\sqrt{2}$. The theory of irrational quantities thus arose. **~~Theodorus~~ of Cyrene, Theaetetus,** and **Euclid** generalized this theory.

The 7th, 8th, and 9th books of **Euclid's** Elements were arithmetica. How much was original is not known. It treats of numbers in continued proportion. Many definitions and a method of finding the G.C.D. were given.

Eratosthenes (275-194 B.C.) invented a seive for finding prime numbers. **Hypsicles** worked on polygonal numbers and arithmetical numbers. **Nicomachus** about 100 A.D. comes next. He wrote the first exhaustive work on arithmetic treated independently of geometry.

It was entitled Introductio Arithmetica. Here is a proposition from it:- "Cubical numbers are always equal to the sum of successive odd numbers". $8 = 2^3 = 3+5$. $27 = 3^3 = 7+9+11$. **Theon of Smyrna** worked and wrote. Here is a proposition of his:- "Every square number or that number less one is divisible by three or four or both" **Iamblichus** has this theorem:- "If we add any three consecutive numbers of which the highest is divisible by three, then add the digits of that sum, and so on, the final sum will be six". $61+62+63=186$. $1+8+6=15$, $1+5=6$. **Theon of Smyrna**, **Nicomachus**, and **Thymarides** have problems, given as puzzles, which are algebraic. "A mule and a donkey were walking along laden with corn. The mule said to the donkey:- 'If you gave me one measure, I should carry twice as much as you. If I gave you one, we should both carry equal burdens'. Tell me their burdens, O most learned master of geometry.

Save for **Diophantus** (246-330 A.D) algebra would have been unknown to the Greeks. Of his works we have some parts of his Polygonal numbers, and seven of thirteen books on Arithmetica. If we except the Ahmes papyrus this is the oldest book on algebra extant. He introduced the idea of expressing an algebraic equation in algebraic symbols. He was the first to state that the product of two negative numbers was positive. His only idea of a negative number was as a difference i.e. $2x-10$. ↑ was the subtraction sign; ι was equals; They was no sign for addition except juxtaposition. He used (S) for unknown quantities. In the solution of simultaneous equations only one symbol was used for the unknown quantities. He used the method of tentative assumption. He solved determinate equations of the second degree, but all but one book of his his work was devoted to quadratic equations of this form $Ax^2+Bx+C=y^2$ or of two simultaneous equations of this form. He reduced all equations to this form. He never accepted as a result a negative quantity or an irrational one. He was always satisfied with one solution though his problem admitted of an indefinite number. General methods were to him unknown. Euler, La Grange and Gauss received no aid in the formulation of general methods from him. They began at the bottom in the work of independent analysis and built up.

The Romans

No where is the contrast between Greek and Roman shown more clearly than in mathematics. The Romans did not rise to the rank of an imitator. Nothing but the practical was valuable to the Roman. They reckoned on their fingers, with the abacus, or with tables prepared for the purpose. All of their mathematics came from the Etruscans. In fractions they used the duodecimal system. They solved problems relative to the payment of interest. Their geometry was a mass of empirical rules which they used in surveying. It was practical. Treatises on it have come down to this time.

Not till the 5th century, when the Roman empire was falling did an interest arise in Greek science. School books were compiled from the elements of Greek authors. They were very poor. Foremost among the writers here was **Boethius** (d. 524 A.D.). He wrote Institutus Arithmetica, On the Consolation of Philosophy, and a Geometry in several books. The last may not be his. At any rate it, (though copied) and the first are deficient.

The Hindoos.

Manuscripts show lofty heights attained but the path of developement is no longer traceable. Mathematics was always a slave to astronomy. It was in the hands of the priests who put their problems and results in mystic verse and did not preserve their proofs. The bent of the Hindoo was arithmetical. Numerical symbolism, and the science of numbers, and algebra attained far greater heights here than in Greece; but there is hardly any geometry that cannot by traced to the Greeks. Their astronomy was influenced by the Greek.

Of the great Indian astronomers and incidentally mathematicians are these:- **Aryabhatta** (476 A.D.) author of Aryabhattiyam. The third chapter is devoted to mathematics. **Brahmagupta** (b. 598 A.D.) wrote Brahma-sphuta-siddhanta (Revised system of Brahma). It is considered the greatest work of the Hindoos

An anonymous work ranks next, Surya-Siddhanta (Knowledge from the sun). **Cridhara** wrote Ganita-sara (Quintessence of Calculation). **Padmanabka** wrote an algebra. **Bhaskara Acarya** in 1150 wrote Siddhantaciromani. From this time no advance is made.

The greatest achievement of the Hindoos was the invention of the principal of position in writing numbers. Of all mathematical inventions it has contributed most to the general progress of intelligence. We generally speak of our system as of Arabic origin, but it is Hindoo. The nine digits came first, the (0) zero later. The whole was complete before the time of Aryabhatta.

They early exhibited great skill in calculating even with large numbers. They gave problems as puzzles. "Beautiful maiden with beaming eyes, tell me, as thou understandest the right method of inversion which is the number which multiplied by three, then increased by 3/4 of the product, divided by 7, diminished by 1/3 of the quotient, multiplied by itself, diminished by 52, the square root extracted, addition of 8, and division by 10 gives the number 2?" [4] They solved problems in interest, discount, partnership, alligation, summation of arithmetical and geometrical series, square and cube root. They invented chess the profoundest of all games. In algebra they went far beyond anything done up to their time, indeed their work stood till recent times. They were the first to recognize the existence of absolutely negative quantities. They didn't confine their operations to rational numbers. They didn't distinguish between numbers and magnitudes. They saw that quadratics had two roots. They recognized as infinity a fraction with (0) as a denominator. The saw that there was no square root to a negative quantity. They advanced in the solution of indeterminate forms. General methods of solution were introduced. They sought all possible answers. Remarkable is their solution of these two forms $xy = ax + by + c$ and $cy^2 = ax^2 + b$. The cyclic method was used. When the Arabs lost this method they lost something which caused modern mathematicians much thought and hard work to replace.

Their geometry was deficient. They had some empirical rules. They did no original work. They found the correct value of $\pi = \frac{3927}{1250}$. They did much work but is is unimportant.

In trigonometry they divided the circle into quadrants of 90° each. They reckoned with sine and versed sine. The obtained sines for angles at intervals of 3° 45'. No treatise on the trigonometry of the triangle is extant. In astronomy they solved plane and spherical triangles

Although the Arabs lost many Indian problems and proofs, many methods and discoveries, yet the tone of our algebra and arithmetic is Indian.

The Arabs.

Wonderful was the ease with which the nomadic Arabs took up the love of science after conquering everything in their march of conquest. Bagdad and Cordova became centers of learning.

In 772 an Indian astronomer translated some astronomical tables, containing a table of sines, into the Arabic. In the same year the Indian Siddhanta was translated. About the same time the Indian numerals came to be used by the Arabs. Here is the influence of the Indian upon the Arab.

From Syria came Greek scholars to Bagdad. At the beginning of the 10th century all important Greek philosophic, medical, mathematical, and astronomical works could be read in Arabic. These were revised again and again before satisfactory, however. Not till the 10th century did they begin their scanty original work.

Mohammed ben Musa Al Hoverezmie did work in algebra and arithmetic. His is the first algebra called by that name. It is a combination of Hindoo and Greek. The three sons of **Musa ben Sakir** wrote several works. **Tabit ben Korra** (836-901) translated Apollonius, Archimedes, Euclid, Ptolemy, Theodosius. The first original work of the Arabs appears in his work on amicable numbers. **Al Battani** foremost astronomer of the 9th century expressed Greek geometrical operations algebraically. He added to the known formulas on oblique-angled triangles. **Abul Wefa** (940-998) discovered the variation of the moon, introduced the tangent into trigonometry, calculated a table of tangents and wrote on Geometric Constructions. **Al Kuhi**, **Al Sagani**, **Al Biruni**, and **Abul Gud** worked on the trisection of angles. **Abu Mohammed Al Hogendi** discovered that the sum of two cubes can never be a cube. **Al Karhi's** (11th century) treatise on algebra is the greatest Arab work. He was the first to solve equations of this form $x^{2n}+ax^n=b$. He made both arithmetical and geometrical proofs for quadratic equations. He was the first Arab to prove theorems on the summation of series. We see no indeterminate analysis. **Al Mahani, Abu Gafar Al Hazin, Al Hasan ben Al Haitam** worked on various problems.

The greatest achievement of the Arabs in algebra was the solution of problems by intersecting conics. There was accomplished by **Omar al Hayyami** in 1079 A.D, the solution of cubic equations. Descartes and Baker knew nothing of this so they had to solve anew.

Mathematics now declines. The crusades (1100-1300) give the Europeans ideas of the achievements of the Arabs, but the Arabs gained not. The Mongolian hordes (1256); the Tartars (14th century) come and conquer. Science declined in this turmoil. **Nazir Eddin** (1201-1274), **Ulug Beg** (1393-1449), **Al Kaschi, Beha Eddin** (1547-1622) were among the mathematicians of this time of turmoil.

In Egypt **Ben Junus, Ibn Al Haitam, Abul Hasan Ali**, rekindle the scientific fires. But they soon burn out.

In Spain **Al Madshriti** (d. 1007) was the first mathematician. He wrote on amicable numbers. Schools were founded by his pupils at Cordova, Dainia, and Granada. **Gabir ben Aflah** of Seville was the only great astronomer. He wrote an astronomy in nine books. In it he has great independence of thought in his spherical trigonometry. He derived some new formulas. In 1492 the Moors lost their foothold in Spain and they were gone. But their work lingered.

Their chief service was that they adopted the learning of Greece and India and kept it with scrupulous care. They transmitted to the Europeans the valuable treasures of antiquity.

Upon this work of centuries the modern men Vista, Descartes, Newton, Euler, LaGrange, Laplace, Napier and Briggs have built up our great system of mathematics. More general rules were needed. Logarithms, Analytical Geometry, and Calculus were needed. We have them and with them we have advanced.

Philip Fox.