

Prym Varieties of Tropical Plane Quintics

by

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Abstract

When considering an unramified double cover $\pi : C' \rightarrow C$ of nonsingular algebraic curves, the Prym variety (P, Θ) of the cover arises from the sheet exchange involution of C' via extension to the Jacobian $J(C')$. The Prym is defined to be the anti-invariant (odd) part of this induced map on $J(C')$, and it carries twice a principal polarization of $J(C')$. The pair (P, Θ) , where Θ is a representative of a theta divisor of $J(C')$ on P , makes the Prym a candidate for the Jacobian of another curve. In 1974, David Mumford proved that for an unramified double cover $\pi : C'_\eta \rightarrow C$ of a plane quintic curve, where η is a point of order two in $J(C)$, then the Prym (P, Θ) is not a Jacobian if the theta characteristic $L(\eta)$ is odd, L the hyperplane section.

We sought to find an analog of Mumford's result in the tropical geometry setting. We consider the Prym variety of certain unramified double covers of three types of tropical plane quintics. Applying the theory of lattice dicings, which give affine invariants of the lattice, we found that when the parity $\alpha(\mathcal{H}_3)$ is even, $\mathcal{H}_3$ the cycle associated to the hyperplane section and the analog to η in the classical setting, then the Prym is not a Jacobian, and is a Jacobian when the parity is odd.

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Chapter 1

Introduction

A primary goal in the field of algebraic geometry is to study algebraic varieties.

Definition 1.1: An **algebraic variety** over a field k is the common zero locus of a (finite) collection of polynomials in the ring $k[x_1, \dots, x_n]$; i.e. for $f_1, \dots, f_m \in k[x_1, \dots, x_n]$,

$$V(f_1, \dots, f_m) = \{(a_1, \dots, a_n) \in k^n \mid f_i(a_1, \dots, a_n) = 0, \forall 1 \leq i \leq m\}$$

If $f_1, \dots, f_m$ are homogeneous polynomials, then V can be thought to sit inside projective space $\mathbb{P}_k^{n-1}$.

Definition 1.2: An **algebraic curve** $C = V(f)$ is an algebraic variety of dimension 1.

Definition 1.3: A **hypersurface** is a subvariety of codimension 1 which is given by one polynomial. Plane curves are hypersurfaces of dimension 1.

We have a few choices to make when working with the above definitions, such as the number of variables in our ring, the collection of polynomials, and the field k . Common fields to work over include $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Q}$. It is also possible to work over a semifield (frequently characterized by the absence of the additive inverse axiom for fields), which is where we transition into the world of tropical geometry. Our story takes place in this world. This area

of mathematics is commonly described as a degeneration of algebraic geometry, where the objects which populate it - polyhedral complexes - are “combinatorial shadows” of those in the classical setting. (However, it is important to point out that not all tropical varieties come from an “upstairs” variety.) These objects carry a rich geometry, and several results on classical varieties - including Bezout’s, Abel-Jacobi, Index, and Riemann-Roch theorems - have been established for tropical varieties as well, and the establishment of a classical-to-tropical canon is a vibrant area of study in mathematics.

The goal of this paper is to make a contribution to that canon. We will consider three tropical plane quintic curves and their unramified double covers of a certain type. The route taken is one analogous to Mumford’s in the classical setting, where he classifies when an unramified double cover $\pi : C' \rightarrow C$ of an algebraic curve gives a Prym (P, Θ) that is a Jacobian. Specifically, the concluding theorem in section 7 of his paper *Prym Varieties I* states that if C is a nonhyperelliptic curve of genus 6, equipped with an odd theta characteristic L (requirement that $\dim \Gamma(L) \geq 3$ and odd), and $L(\eta)$ even, where η is the point of order 2 in $J(C)$ defining π (2.5), then (P, Θ) is a Jacobian. If C is a genus 6 curve but is not of the special type above, then (P, Θ) is not a Jacobian or a product of Jacobians. We will right away clarify that Θ is the canonical representation of twice the principal polarization of a theta divisor Θ' of $J(C')$ on P .

This classical result will be compared to the relationship between a special theta characteristic of our tropical plane quintics (of genus 6) and certain unramified double covers of them. Via this relationship, we are able to classify when a double cover will give Prym as a Jacobian of another curve. The layout is as follows. Chapter 2 introduces, in the classical sense, the theories of principally polarized Abelian varieties and Prym varieties. Chapter 3 then brings these theories into the tropical setting after a brief discussion of tropical curves. In Chapter 4 we introduce the main tool used in our classification - the lattice dicing. The importance of lattice dicings to our work is due to both the Jacobian and the Prym being defined by lattices. We then give a description of how we represented the unramified double

covers and thoroughly explain an example of the computations performed and present the results.

Chapter 2

Theory of Principally Polarized Abelian Varieties and Prym Varieties

The main reference in this section is David Mumford's *Prym Varieties 1*, where he goes through the theory of double covers of one algebraic curve over another; discusses two equivalent sets of data for principally polarized Abelian varieties; describes the Prym variety, theta divisors, and Jacobians; then finally focuses on these in the context of an unramified double cover. However, as his presentation is primarily algebraic, this section will also introduce some topological features of these double covers, the goal being to provide visual intuition. Additionally, although the goal is to end up where Mumford does and focus on unramified double covers, we will define some notions related to ramified double covers for the sake of thoroughness.

2.1 Double Covers of Curves

A double cover of an algebraic curve C can be defined (mostly) jargon-free as a “good” map from another algebraic curve C' to C , where C' has, if not exactly, almost twice as many points as C . To describe such maps more rigorously, what follows is a slew of definitions, some which will be used later, others which are included solely for clarity's sake.

Definitions

1. A **double cover** $\pi : C' \rightarrow C$ is a morphism of curves that is $2 : 1$, except possibly on a finite set. That is, the fiber over a point $p \in C$ consists of two unique points $q_\alpha, q_\beta \in C'$.
2. The points $\{p_1, \dots, p_k\} \in C$ with $\pi^{-1}(p_i) = q_i$ are the **branch points** of π .
3. The $\{q_i\}$ corresponding to the branch points are the **ramification points**.
4. The double cover is **unramified** if it has no branch points (equivalently, no ramification points).

A simple example of a double cover is illustrated in Figure 2.1.

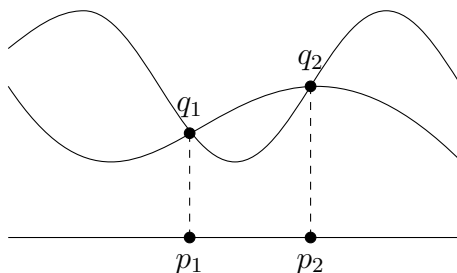


Figure 2.1: *Double Cover Example*

2.2 Divisors, Line Bundles & Theta Characteristics

Two important tools for studying algebraic varieties are divisors and line bundles, and there is a close connection between the two. Let C be an algebraic curve. There may be terminology and facts used with which some readers are unfamiliar and which are not elaborated on

here, but they will be returned to in later sections.

Definition 2.2 A **divisor** D on C is a formal linear combination of a finite number of points

$$D = \sum_{i=1}^n a_i p_i$$

where $a_i \in \mathbb{Z}$ and $p_i \in C$.

The set of divisors on C form an additive group.

1. A divisor is **effective** ($D \geq 0$) if $a_i \geq 0$ for all i .
2. The **degree** of a divisor is $\sum a_i$.
3. Divisors D and D' are **linearly equivalent** (written $D \sim D'$) if $D - D'$ is the divisor of a meromorphic function f on C . Then, if a divisor is the divisor of a meromorphic function, $D = (f)$, it is called **principal**.
4. (a) $|D|$ is the set of all effective divisors linearly equivalent to D . It is called the **complete linear series** of D .
 (b) Sheaf theoretically, we define the sheaf $\mathcal{O}(D)$ by $\Gamma(U, \mathcal{O}(D)) = \{f \in \mathcal{M}(U) | D + (f) \geq 0\}$, U open in C . The linear system $|D|$ is given by $\mathbb{P}H^0(C, \mathcal{O}(D))$.

Definition 2.3 A **line bundle** $\pi : L \rightarrow C$ is an open cover $\{U_\alpha\}$ of C together with trivializations $\phi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times k$. The set of line bundles on C can be given a multiplicative group structure under the tensor product. This group is called the *Picard group* of C , $Pic(C) = H^1(C, \mathcal{O}^*)$.

Briefly, the line bundle L is associated to a divisor $D \in C$ iff L has a nonzero global meromorphic section s where $D = (s)$, and for s a meromorphic section of a line bundle L , $L = [(s)]$. We write $L = [D]$. Via this correspondence, if $L_1 = [D_1], L_2 = [D_2]$ then $L_1 \otimes L_2 = [D_1 + D_2]$. Also, we can now say two divisors D and D' are linearly equivalent if

$$[D] = [D'].$$

Definition 2.4:

1. The **canonical bundle** K_C on a curve C is the cotangent bundle; the associated divisor class K is the **canonical class**; it has degree $2g - 2$.

An example is $K = -2p$ for any $p \in \mathbb{P}^1$. This is because any two points in $\mathbb{P}^1$ are linearly equivalent! That is, for any two points p_1, p_2 , there exists a meromorphic function on $\mathbb{P}^1$ with a single zero at p_1 and a single pole at p_2 . See [FK92] for proofs of the existence of meromorphic functions and differentials on a Riemann surface.

2. A **theta characteristic** Θ of C is a square root of the canonical bundle. That is, it is a line bundle L such that $L \otimes L = L^2 \cong K_C$.

The set of theta characteristics is a torsor over the points $J_2(C) = H_1(C, \mathbb{Z}/2) \cong (\mathbb{Z}/2\mathbb{Z})^{2g}$, line bundles L such that $L^2 = \mathcal{O}_C$ in $J(C) = \text{Pic}^0(C)$. Thus, a curve of genus g has 2^{2g} theta characteristics. Translating a chosen theta characteristic L by any $\eta \in J_2(C)$ will give another theta characteristic L' . We denote the translate as $L(\eta)$. A theta characteristic L is called **even** or **odd** when the dimension of the sheaf of global sections $\Gamma(C, L) \cong H^0(C, L)$ is even or odd.

Here is an example which includes the above concepts.

Example: If C has genus $g = 2$, then it is hyperelliptic and the degree two map $C \rightarrow \mathbb{P}^1$ has 6 ramification points $p_1, \dots, p_6$. Its affine part is given by the equation $y^2 = \prod_{i=1}^6 (x - p_i)$. The canonical divisor class is $K \sim 2(p_i), 1 \leq i \leq 6$. We know there must be a total of 16 theta characteristics because $J_2(C) \cong (\mathbb{Z}/2\mathbb{Z})^4$. Then there are 6 odd theta characteristics which are given by $\mathcal{O}(p_i), \Gamma(\mathcal{O}(p_i)) = 1$, (as in Definition 4b), and 10 even theta characteristics $\{(p_i + p_j - p_k), j < k, i \neq j \neq k\}$.

2.3 The Jacobian

The Jacobian $J(C)$ of a complex algebraic curve C of genus g is an object which is constructed by integrating abelian differentials $H^0(C, \Omega^1)$ over 1-cycles $H_1(C, \mathbb{Z})$. We have the coordinate free definition $J(C) = H^0(C, \Omega^1)^*/H_1(C, \mathbb{Z})$. One can also choose a canonical basis $\{\gamma_1, \dots, \gamma_{2g}\}$ of $H_1(C, \mathbb{Z})$ and with respect to this a normalized basis $\{\omega_1, \dots, \omega_g\}$ of $H^0(C, \mathbb{Z})$. Then vectors $(\int_{\gamma_i} \omega_1, \dots, \int_{\gamma_i} \omega_g)^T$ generate a lattice Λ in $\mathbb{C}^g$, and the Jacobian is then defined as $J(C) = \mathbb{C}^g/\Lambda$, a complex torus.

The curve C is mapped to $J(C)$ by fixing a base point p_0 and sending every $p \in C$ to the point $(\int_{p_0}^p \omega_1, \dots, \int_{p_0}^p \omega_g)$ modulo Λ in $J(C)$. Then, we can naturally extend the map to the group of divisors of degree 0 on C , yielding $D = \sum p_i - \sum q_i \mapsto (\sum \int_{p_i}^{q_i} \omega_1, \dots, \int_{p_i}^{q_i} \omega_g)$. Via the divisor-line bundle correspondence, we get the induced map $Pic^0(C) \rightarrow J(C)$. The Abel-Jacobi Theorem states that this map is an isomorphism. A great reference for these results is Griffiths-Harris' *Principles of Algebraic Geometry* [GH78].

2.4 Double Covers: Another Point of View

As we now have heavier machinery, we will return to unramified double covers and describe a construction which requires the language of divisors, line bundles, and the Jacobian.

The **norm map** $Nm_\pi : \mathcal{M}(C')^* \rightarrow \mathcal{M}(C)^*$ is a group homomorphism between meromorphic functions on C' to meromorphic functions on C . It is defined by

$$f(p) \mapsto \prod_{q \in \pi^{-1}(p)} f(q)^{v(q)}$$

where $v(q) = mult_\pi(q)$.

The norm map Nm_π induces maps on the groups of divisors and the Jacobians of the two curves. In fact, we get the commutative diagram in Figure 2.2.

$$\begin{array}{ccc}
J(C) & \xrightarrow{\pi^*} & J(C') \\
\cong \updownarrow & & \updownarrow \cong \\
\text{Pic}^0(C) & \xleftarrow{Nm} & \text{Pic}^0(C')
\end{array}$$

Figure 2.2: *Commutativity of Nm and π^**

It turns out that for any element $\eta \in J_2(C)$, we get a unique unramified double cover C'_η . The construction of such double covers is part of Appendix B in [ACGH85]. The connection between Nm and η is that the kernel of Nm intersects $J_2(C')$ in the points which are anti-invariant under an involution $\tau : C' \rightarrow C'$ exchanging the sheets, and η is in the kernel of π^* , so that η is represented by $L \in \text{Pic}^0(C)$ with $L^2 \cong \mathcal{O}_C$.

2.5 Principally Polarized Abelian Varieties

Principally polarized Abelian varieties (p.p.A.v's) are objects of great interest in algebraic geometry, and are closely related to the theory of the Jacobian of a curve, a double cover, and Prym varieties (introduced in Section 2.6), in both the classical and tropical settings (see 3.3).

Definition 2.5: An **Abelian variety** is a complex torus given by $\mathbb{C}^n/\Lambda$, where Λ is a lattice of rank $2n$, and the torus can be embedded in projective space so that it is a projective algebraic variety.

Now, if we have such a complex torus, then we can check the conditions of Kodaira's embedding theorem to determine if it is a projective variety. In the theory of the Jacobian of an algebraic curve (or i.e. a compact Riemann surface), the first and second Riemann bilinear relations prove that the period matrix of C makes $J(C)$ into an Abelian variety. ([GH78], Ch.2) We focus on $J(C)$ as an Abelian variety for the remainder of this text.

To round out the theory, we must define a principal polarization on $J(C)$.

Definition 2.6: A **polarization** of an Abelian variety of dimension n is a cohomology class $[\omega] \in H^2(J(C), \mathbb{Z})$ (Chern class) of an ample line bundle. In the case of $J(C)$, this is the skew-symmetric bilinear form

$$Q : H_1(J(C), \mathbb{Z}) \bigotimes_{\mathbb{Z}} H_1(J(C), \mathbb{Z}) \rightarrow \mathbb{Z}$$

which arises from the intersection pairing of cycles on C . On $J_2(C)$ it is called the Weil pairing, where the intersection of cycles is now on $H_1(C, \mathbb{Z}/2)$. The intersection matrix is

$$\begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$$

and it is a **principal polarization**.

The embedding of $J(C)$ into projective space involves theta functions and theta divisors.

Riemann's theta function:

$$\Theta(z; \Omega) = \sum_{n \in \mathbb{Z}^g} e^{\pi i^t n \Omega n} e^{2\pi i^t n z}$$

is holomorphic in the variables z, Ω , where $z \in \mathbb{C}^g$ and $\Omega \in M(\mathbb{C}, n)$ is symmetric and $\Im(\Omega) > 0$. The matrix Ω is the period matrix of $J(C)$, which is the matrix of the vectors which generate the lattice Λ defining $J(C)$. The theta divisor $\{\Theta(z) = 0\} \subset J(C)$ is an ample divisor on $J(C)$.

2.6 The Prym Variety

The Prym variety of a morphism of curves is another example of an Abelian variety, and the protagonist of our journey. We start with an unramified double cover $\pi : C' \rightarrow C$, where C' and C are nonsingular and complete curves. We can exchange the sheets of the cover C'

via an involution $\tau : C' \rightarrow C'$ (i.e. if the fiber over $p \in C$ consists of the two points q_1, q_2 , then $\tau(q_1) = q_2$). This involution can be extended to $J'(C')$, the Jacobian, and splits the Jacobian into two parts, which we will call J (even part; and, prophetically, the symbol used for the Jacobian of C) and P (odd part). Furthermore, P has a specific name: this is our **Prym variety** of C' . Let Θ be a representative of a theta divisor of $J(C')$ on P . When π is unramified, (P, Θ) turns out to be an Abelian variety with twice a principal polarization; in general this is not the case for ramified covers. ([Mum74], §1-3)

Mumford's classical result on Prym varieties and their relation to Jacobians is stated here:

Theorem

Let C be a curve of genus g , $\eta \in J_2$ define the double cover $\pi : C' \rightarrow C$, and (P, Θ) twice a principal polarization of C .

- a C hyperelliptic $\implies (P, \Theta)$ is a hyperelliptic Jacobian or a product of two.
- b $g = 3$, C not hyperelliptic $\implies (P, \Theta)$ is a two-dimensional Jacobian.
- c $g = 4$, C not hyperelliptic $\implies (P, \Theta)$ is a three-dimensional Jacobian; furthermore, Θ is singular $\iff P$ is a hyperelliptic Jacobian $\iff \exists$ an even theta characteristic L with $\Gamma(L) \neq (0)$ and $L(\eta)$ even.
- d If C is not hyperelliptic and $g \geq 5$, then $\dim(\text{Sing}\Theta) \leq g - 5$ and

$$\begin{array}{ll}
 & C \text{ trigonal} \\
 \dim(\text{Sing}(\Theta)) = g-5 \implies & C \text{ double cover of an elliptic curve} \\
 & g = 5 \text{ and } \exists \text{ even theta characteristic } L, \Gamma(L) \neq (0) \text{ and } L(\eta) \text{ even} \\
 & g = 6 \text{ and } \exists \text{ odd theta characteristic } L, \dim\Gamma(L) \geq 3 \text{ and } L(\eta) \text{ even}
 \end{array}$$

In relation to part *d* above, the following corollary is of importance to us:

Corollary

If $g \geq 5$ is none of the types in part (d) above, then the polarized Abelian variety (P, Θ) is neither a Jacobian nor a product of Jacobians. ***In particular, if C is a plane quintic and $L(\eta)$ is odd, where L is the hyperplane section, then (P, Θ) is not a Jacobian.***

A topological description of the Prym variety will be given, as seen in ([ACGH85], Appendix C). We want to choose for both C and C' normalized bases of the space of 1-cycles over $\mathbb{Z}$. We have the following relationships between these bases, depicted also in Figure 2.3.

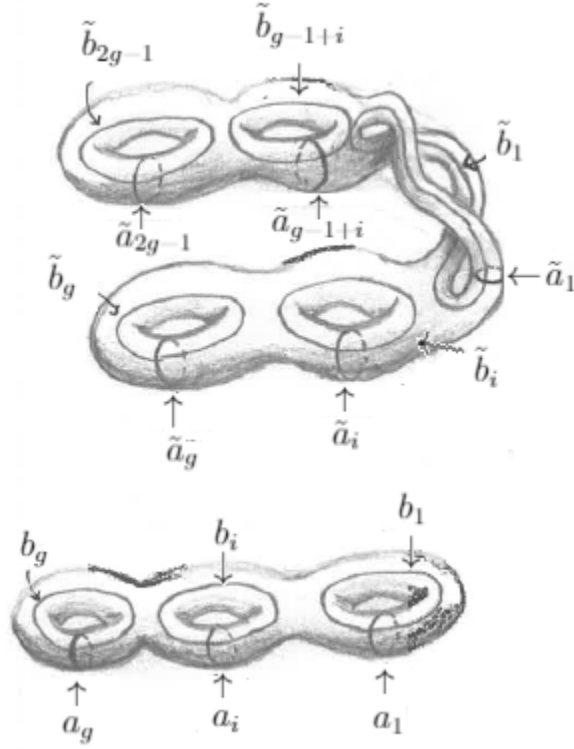


Figure 2.3: *Double Cover of a genus g algebraic curve (complex torus)*

$$\pi_* \tilde{a}_1 = a_1$$

$$\pi_* \tilde{b}_1 = 2b_1$$

$$\pi^*(a_i) = \tilde{a}_i + \tilde{a}_{g-1+i}, i \neq 1$$

$$\pi^*(b_i) = \tilde{b}_i + \tilde{b}_{g-1+i}, i \neq 1$$

Considering the involution $\tau : C' \rightarrow C'$ described above and the induced involutions on $H_1(C', \mathbb{Z})$ and $H^0(C', K)$, then the -1 eigenspaces (anti-invariant) are

$$H_1(C', \mathbb{Z})^- = \langle \tilde{a}_i - \tilde{a}_{i+g-1}, \tilde{b}_i - \tilde{b}_{i+g-1} | i = 2, \dots, g \rangle$$

$$H^{1,0}(C')^- = \langle \omega_i - \omega_{i+g-1} | i = 2, \dots, g \rangle$$

The the Prym variety is defined as

$$Prym(C', \tau) = \frac{(H^{1,0}(C')^-)^*}{H_1(C', \mathbb{Z})^-} \subset J(C')$$

The Prym has dimension $g - 1$ and the principal polarization given by intersection of the anti-invariant cycles is

$$\begin{pmatrix} 0 & 2I_{g-1} \\ -2I_{g-1} & 0 \end{pmatrix}$$

Recall from the previous section that the principal polarization of $J(C')$ is given by the intersection of cycles, and the matrix above is twice that via the normalized bases obtained. Thus, this makes the Prym variety of an unramified double cover of curves into an Abelian variety with twice the principal polarization.

Chapter 3

The Tropical Prym

3.1 Basic Notions

Instead of working over a field, we can work over the tropical semifield $\mathbb{T} = \mathbb{R} \cup \{\infty\}$ with the operations defined as follows: $\oplus$ is taking the maximum of two elements (e.g. $5 \oplus 8 = 5$); $\otimes$ is taking the usual sum of two elements (e.g. $-3 \otimes 14 = 11$). Thus, if we have a polynomial $p(x, y) = \bigoplus_{(i,j)} a_{ij} x^i y^j$, then in the tropical world it is interpreted as $p(x, y) = \max_{(i,j)} \{a_{ij} + ix + jy\}$.

What working over the tropical semifield does to an algebraic variety over an actual field is that it degenerates it in a certain sense (and we call this degenerate object a tropical variety), which one can see from the above description of a tropical polynomial. There are three main constructions to explain this, a useful one being that the tropical semifield operations are related to the operations on the regular extended reals by dequantization; that is, taking the $\lim_{t \rightarrow \infty} \log_t(t^a + t^b)$ where $t \geq e$ to give our tropical addition $a \oplus b$. It is clear that $\lim_{t \rightarrow \infty} \log_t(t^a t^b) = a \otimes b$. Recommended references for more explanation are [\[Mik06\]](#) and [\[MS15\]](#).

3.2 Plane Curves

We will be concerned with plane curves, which are tropical varieties given by tropical polynomials of two variables. The following proposition [MS15] is important in understanding the methods employed in this paper. (Refer to 1 for a definition of variety and hypersurface.)

Proposition 3.1: Given a tropical polynomial $p(x, y) = \bigoplus_{(i,j)} a_{ij} x^i y^j$, the hypersurface $V(p)$ is a plane tropical curve and satisfies the following conditions:

1. It is a finite graph embedded in $\mathbb{TP}^2$, the (extended) tropical projective plane.
2. It has edges both “unbounded” (ones which go to infinity) and bounded.
3. All of the edges have rational slope.
4. The balancing condition (sum of the primitive lattice vectors along each edge adjacent to a node is zero) is satisfied around each finite node.

The perspective we take is that a tropical curve can be represented by an equivalence class of metric graphs [MZ08]. There is a bijection between compact tropical curves and metric graphs. A metric graph is one which becomes a complete metric space after removing 1-valent vertices. Connected metric graphs Γ_1, Γ_2 are equivalent as tropical curves if after removing 1-valent vertices and the edges adjacent to them, and treating 2-valent vertices as interior points of edges, they are the same metric graph. When a tropical curve is viewed as a metric graph Γ , the genus is the first Betti number $g = b_1(\Gamma)$.

There are three types of plane quintics that we will be considering in this paper. Each is of genus $g = 6$. They will be referred to as Type A, Type B, and Type N throughout. Type N differs from A in B in that it is a nonplanar metric graph. We can contract all leaves of a tropical curve and not lose any information which involves the lengths of the cycles, which is needed to compute the Prym. Though there are other plane quintics, the Pryms are degenerations of the ones we are considering.

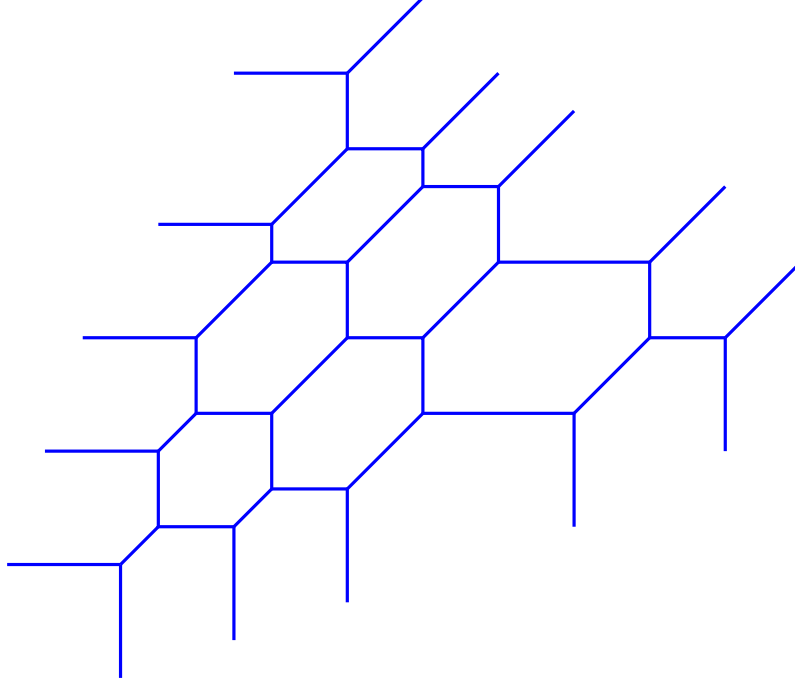


Figure 3.1: *Plane Quintic, Type A*

3.3 The Tropical Jacobian and Tropical Prym

The tropical Jacobian is defined analogously to the classical Jacobian. For C a tropical curve of genus g locally embedded in $\mathbb{R}^2$ as a balanced tropical curve, its Jacobian is the real torus $\mathbb{R}^g/\Lambda$, where $\Omega(C)^* \cong \mathbb{R}^g$ is the space of real-valued linear functionals on the space of global 1-forms $\Omega(C)$, and $\Lambda \cong \mathbb{Z}^g$ is the lattice obtained by embedding the vector space of integral 1-cycles $H_1(C, \mathbb{Z})$ into $\mathbb{R}^g$ by integrating over them. One way to view the elements of $\Omega(C)$ are as global objects which are locally restrictions of 1-forms df to C , where $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is linear. This translates for metric graphs to ω being a current, which must satisfy Kirchoff's law, a result of the balancing condition around a vertex. Thus, the 1-forms are constant on the interior of edges of C . Corresponding to an oriented edge $e \in C$ is the linear functional $l_e \in \Omega(C)^*$, which evaluates an element $\omega \in \Omega(C)$ on the primitive tangent vector parallel to e . One result we get is the tropical Abel-Jacobi Theorem: a divisor $D = \sum a_i p_i$ is mapped to $J(C)$ by fixing a point $q \in C$, then integrating forms in $\Omega(C)$ along each path q to p_i .

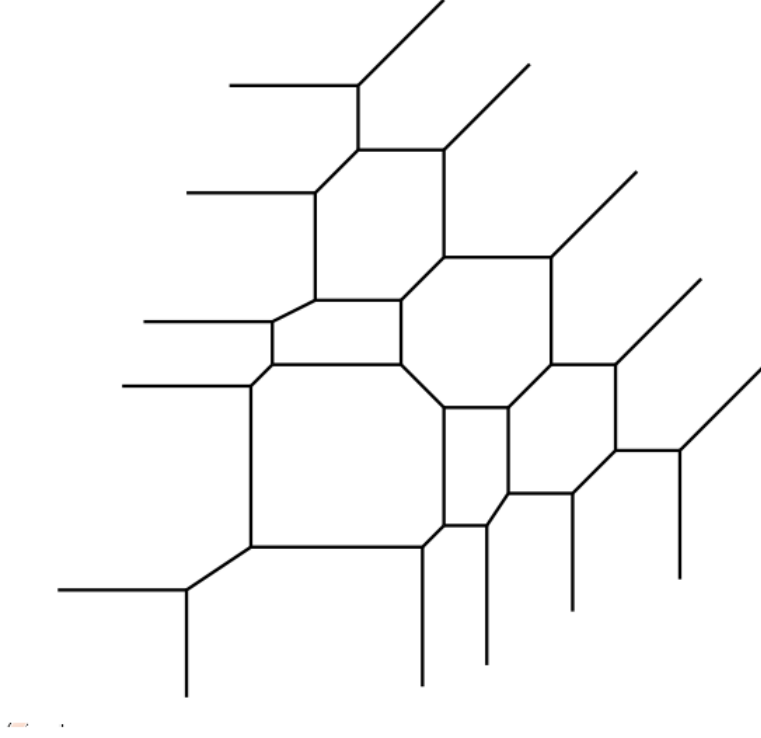


Figure 3.2: *Plane Quintic, Type B*

This gives the Abel-Jacobi map $\mu^d : \text{Div}^d(C) \rightarrow J(C)$, as any choice of paths will differ by some cycle $\gamma_\alpha \in \Lambda$.

Another perspective on $J(C)$ for a tropical curve C is a symmetric bilinear form Q on Λ , descending from the metric on C . The form Q is defined by declaring $Q(\gamma, \gamma) = \text{length}(\gamma)$ for γ a simple path on C , and is then extended bilinearly to any pair of paths γ_1, γ_2 . This form can be written as $Q = \sum_e \beta_e (l_e)^2$, with $\beta_e > 0$ the length of edge e . An example of Q for a tropical curve is in Figure 3.5. The collection $\{l_e\}$ is a totally unimodular system (see 4.1) in $\Omega(C)^*$. Also, we see that Q is positive definite because it gives isomorphism between Λ and the integral 1-forms $\Omega_{\mathbb{Z}}(C)$. ([MZ08], p.218). This shows that $J(C)$ is a principally polarized (tropical) Abelian variety.

We take the positive, symmetric bilinear form Q on cycles defined above, and let

$$\theta(x) = \max_{\lambda \in \Lambda} \{Q(\lambda, x) - \frac{1}{2}Q(\lambda, \lambda), x \in \Omega(C)^*\}$$

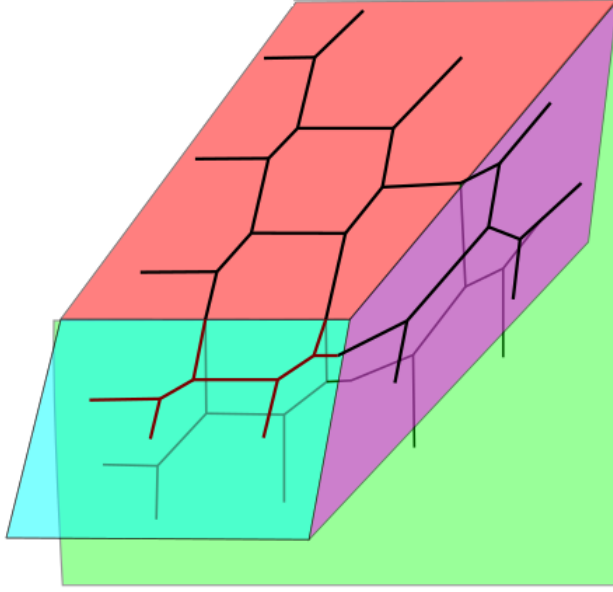


Figure 3.3: *Nonplanar tropical curve*

which is a section of a polarization line bundle on $J(C)$. It is a tropical analog of the theta function. Then the theta divisor $[\Theta]$ on $J(C)$ is the corner locus of $\Theta(x)$. The principal polarization comes from the tropical theta divisor $[\Theta]$.

An important fact for this work is that there is a *special* even theta characteristic on a tropical curve [Zha10]. The special theta characteristic $\mathcal{K}_0$ is the negative of the universal element $\kappa \in \text{Pic}^{g-1}(C)$ such that by the Abel-Jacobi map, $\mu^g(D_\lambda) + \kappa = \lambda$ for all $\lambda \in J(C)$ ($\mathcal{K}_0 = -\kappa$), where D_λ is the pullback divisor $\mu^*(\lambda)$ of λ . This is the only non-effective theta characteristic. The theta characteristics form a torsor over $H_1(C, \mathbb{Z}/2)$; thus, the other tropical theta characteristics are in bijection with simple cycles on C .

The hyperplane section, represented by the cycle $\mathcal{H}_3$, for tropical plane quintics is another theta characteristic. It corresponds to the simple cycle indicated by the bold cycles in Figure 3.4 (3 disjoint cycles of the quintics). It is $\mathcal{H}_3$ by which we will determine if the Prym of the double cover is a Jacobian or not.

An unramified double cover of a metric graph C can be constructed by taking two disjoint

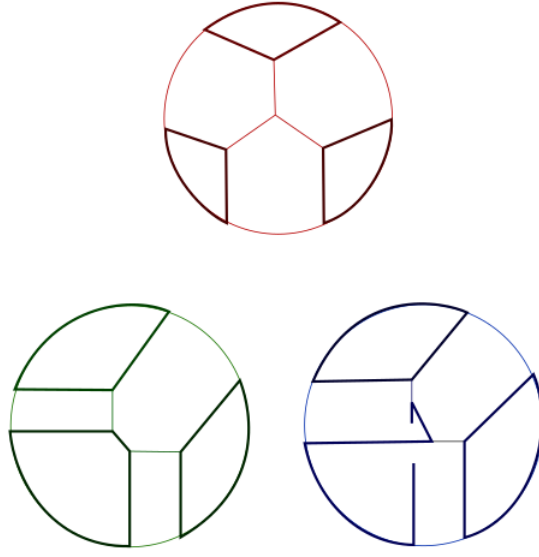
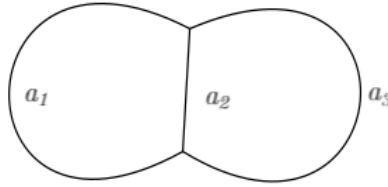


Figure 3.4: *The three plane quintics of concern after contracting unbounded edges*



$$Q = a_1(l_1)^2 + a_2(l_2)^2 + a_3(l_3)^2$$

Figure 3.5: *A tropical curve with edge lengths a_1, a_2, a_3 . The bilinear form Q is shown underneath, where l_i is the linear functional associated to the corresponding edge.*

copies of C , cutting corresponding pairs of edges, and gluing them criss-cross. This will be explained in more detail in Chapter 4. Then under an involution, the cycles which are anti-invariant in C' correspond to cycles in C for which the pairing $\alpha : H_1(C, \mathbb{Z}) \rightarrow \mathbb{Z}/2$ is even. There are other double covers of metric graphs of tropical curves which are not simple which will not be considered.

The tropical Prym is defined analogously to the classical Prym. The Prym of a tropical curve is also a tropical Abelian variety given by a lattice in $\mathbb{R}^{g-1}$. The quadratic form is similar to the form for the Jacobian, but it is restricted to the anti-invariant cycles. That is,

every edge in C contributes twice its length to the evaluation on an anti-invariant form. A preview of results is that if the parity $\alpha(\mathcal{H}_3)$ is odd, then the Prym lattice is the Jacobian of another graph; if $\alpha(\mathcal{H}_3)$ is even, it is not a Jacobian. This is analogous to Mumford's result in the corollary in section [2.6](#).

Chapter 4

Computation of the Prym Variety for Tropical Plane Quintics

The main tool used in the computation of the Prym variety of the tropical plane quintics is that of lattice dicings. This chapter will further explain the basic notions of lattice dicings and present the classification of the maximal dicings of $\mathbb{R}^5$, largely in reference to the paper by Erdahl and Ryshkov [ER94]. Then, an example of computing the Prym of a double cover is given, followed by the results of the computations.

4.1 Unimodular Systems

Underlying the theory of lattice dicings is that of unimodular systems. Here, a brief discussion of the definitions.

Definition 5.1: A set of vectors $R \subseteq V$ is a **unimodular system** (U-system) if every $r \in R$ has an integral representation in every basis $B \subseteq R$. We say that a U-system is **maximal** if one cannot add a non-zero vector to R such that the new system is unimodular.

A trivial example of a U-system, then, would be the standard basis $\{e_i\}$ of $V = \mathbb{R}^n$. However,

this is not a maximal U-system, as we could add the vector $v = e_i - e_j$ for any i, j and get a unimodular system $R \cup \{v\}$.

An equivalent definition is

Definition 1.2: R is a U-system if for any subsystem B which generates R (i.e. $\mathbb{R}B = \mathbb{R}R$), the lattice $\mathbb{Z}B$ does not depend on B .

If we choose a basis of R , then we can write the vectors of R as column vectors in this basis; we will always get a **totally unimodular matrix** $U(R)$; that is, all maximal minors are 0 or ± 1 . Then the study of U-systems corresponds to the study of totally unimodular matrices. We will now transition to lattice dicings.

4.2 Lattice Dicings and Classification

The idea of a lattice dicing of $\mathbb{R}^n$ was presented by Gauss in an 1831 paper, where he described a dicing of the plane as a tiling by identical parallelepipeds with the vertices forming a 2-dimensional lattice. However, Gauss' definition for dicings of n -dimensional real-space only includes dicings which consist of exactly n hyperplane D -families with n linearly independent normal vectors. One can generalize to include dicings which fulfill the conditions of a Gaussian dicing of $\mathbb{R}^n$, but may have more than n D -families.

Definition 5.4: A **lattice dicing** $\mathcal{D}$ is a finite set of parallel hyperplanes in $\mathbb{R}^n$, where the set satisfies the following two conditions:

1. *Nondegeneracy:* There must be n linearly independent normal vectors among the D -families.
2. *Vertex Regularity:* Each vertex of the dicing must have exactly one hyperplane from each of the r D -families of $\mathcal{D}$ pass through it.

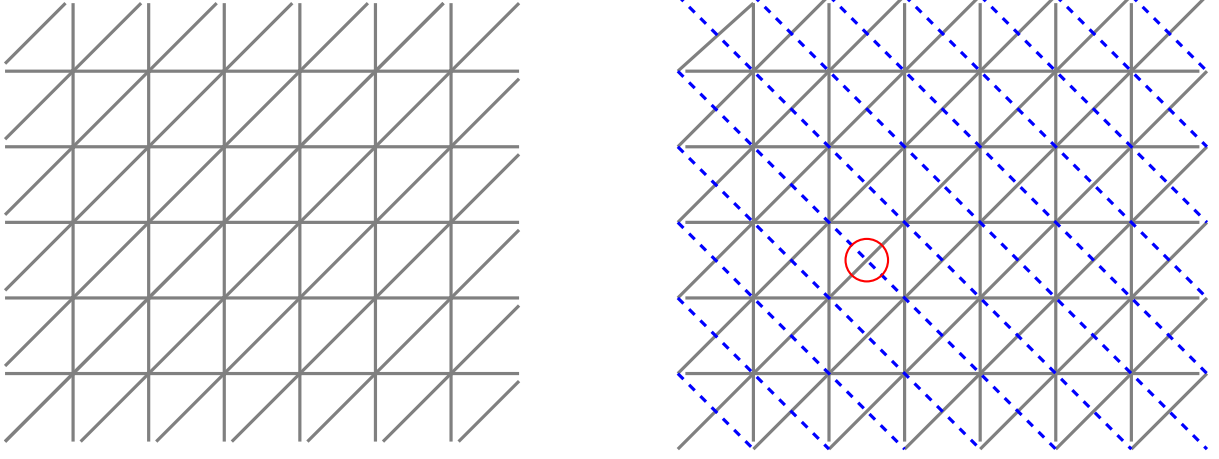


Figure 4.1: *The picture on the left shows a lattice dicing of $\mathbb{R}^2$. The figure on the right is a non-example; it is not vertex regular.*

One property that is immediate from these definitions is that given an inclusion of dicings $\mathcal{D}_1 \subseteq \mathcal{D}_2 \subseteq \mathcal{D}$, the dicings have the same vertex set. A property which is not as obvious but is important to our work is that the lattice dicings of n -dimensional real-space can be ordered by inclusion, so that any lattice dicing fits in a chain which terminates with a maximal lattice dicing (so the dicings in a chain have a common vertex set). Thus, Erdahl and Ryshkov went to task to find the maximal dicings of $\mathbb{R}^n$ for $n \leq 5$, subsequently giving all subdicings of these as well.

They use a partition of the cone $\mathcal{P}(n)$ of positive quadratic forms on n variables by what are called perfect domains. A **perfect domain** is a set of forms $\{\sum_{p \in \mathcal{P}} \lambda_p (\mathbf{p}\mathbf{x})^2 | \lambda_p > 0\}$. The vectors $\mathbf{p}$ are perfect vectors, which are integral and primitive. For each n , there are a finite number of perfect domains (classified by arithmetic class), each one being an open polyhedral cone of full dimension $\binom{n+1}{2}$. The perfect domains fit together facet-to-facet. The way that the affine classes of maximal dicings of $\mathbb{R}^n$ are determined, then, is by first producing a list of the maximal dicings for each arithmetic class of perfect domains in $\mathcal{P}(n)$, then removing any dicings of one domain which are affinely equivalent to subdicings of another.

A quadratic form defines a Delaunay decomposition on $\mathbb{R}^n$ by the empty sphere condition, i.e. by finding the largest sphere which will be held rigidly by the lattice points [ER87]. If this

decomposition comes from a lattice dicing, a rare occurrence, then the dicing vectors form a totally unimodular system. *This is an affine invariant of the quadratic form. Therefore, to show that two quadratic forms which are both dicings are not equivalent, it is enough to show that the corresponding totally unimodular systems are not affinely equivalent.* The former is important because we will find that the linear functionals associated to the edges of our graph form a totally unimodular system. With them, we write a quadratic form and yield a lattice dicing.

Let $\{e_i\}_{i=1}^5$ be the standard basis of $\mathbb{R}^5$. The three perfect domains in $\mathbb{R}^5$ and the perfect vectors in those domains are:

Table 4.1: *Perfect Domains in $\mathbb{R}^5$*

$\Pi_1 = \{\sum_{0 \leq i < j \leq 5} a_{ij}(x_i - x_j)^2\}$	$\mathbf{v}_{0i} = e_i, \mathbf{v}_{ij} = e_i - e_j$
$\Pi_2 = \{\sum_{1 \leq i < j \leq 5} b_{ij}(x_i + x_j)^2 + c_{ij}(x_i - x_j)^2\}$	$\mathbf{v}_{ij} = e_i - e_j, \mathbf{w}_{ij} = e_i + e_j$
$\Pi_3 = \{\sum_{0 \leq i < j \leq 5} d_{ij}(y_i + y_j)^2\}$	$\mathbf{w}_{ij} = e_i + e_j, \mathbf{u}_{0i} = e_i - e_1 - e_2 - e_3 - e_4 - e_5$

A subset of perfect vectors is realized as a graph on 5 vertices in Π_2 and a graph on 6 vertices in Π_1, Π_3 . There is an edge between vertex i and vertex j if the subset contains a perfect vector with subscript ij . In Π_2 , there are two possibilities for an edge between vertices: edges representing $\mathbf{v}_{ij}$ are called red (dotted in Figure 4.2); edges representing $\mathbf{w}_{ij}$ are called black. We can map a facet of $\Pi_3(5)$ to a facet of $\Pi_2(5)$ via a change of variables, thus sending a graph on 6 vertices to one on 5 vertices. The resulting graph is equivalent as a representation of the dicing. The affine classes of maximal dicings of $\mathbb{R}^5$ are produced in the following theorem:

Theorem 4.2: [ER94] The affine classes of maximal lattice dicings of $\mathbb{R}^5$ are the dicings denoted $A_5, \mathcal{D}_2^1, \mathcal{D}_2^2, E_5$, where the subscript refers to the perfect domain in which the dicing resides, and the superscript is the numbering used by Erdahl and Ryshkov to distinguish between maximal dicings in the same domain. These four classes are depicted in Figure 4.2.

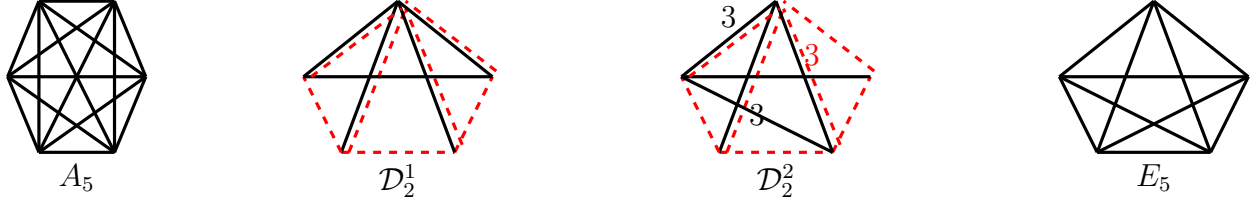


Figure 4.2: *Maximal dicings of $\mathbb{R}^5$*

Removing a dicing vector from E_5 gives the dicing G_5 , which can be seen to be equivalent to a subdicing of F_5 (See figure 4.3). Returning to the context of tropical plane quintics and the discussion on page 24, we consider the above theorem. When viewed as metric graphs, each curve has 15 edges (see Figure 3.4). The set of linear functionals associated to the edges form a totally unimodular system and give a lattice dicing. Due to three relations between the lengths of the edges, we will be reduced to 12 parameters for our lattice. Our computations show that these lattices are subdicings of one of the four types given. *Figures 4.3-4.6* show the lattice dicings which appear in our computations of the Pryms. If the dicing is the Jacobian of a graph, we display that graph. Figure 4.3 shows the dicing F_5 , which is obtained from A_5 by removing a triangle, and its subdicings. Figures 4.4 and 4.5 show the dicings B_5 and C_5 , subdicings which appear in the computations, along with the planar genus 5 graphs whose Jacobians are those dicings. We also show their subdicings. The two non-planar graphs of genus 5 are pictured in Figure 4.6. These four graphs of genus 5 are the only ≥ 3 connected trivalent graphs.

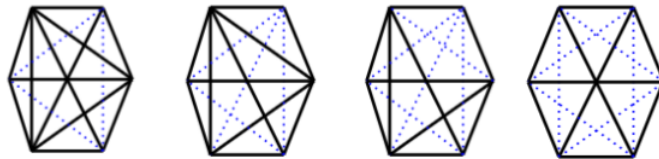


Figure 4.3: *The lattice F_5 and subdicings F_5^1 , F_5^2 , G_5*

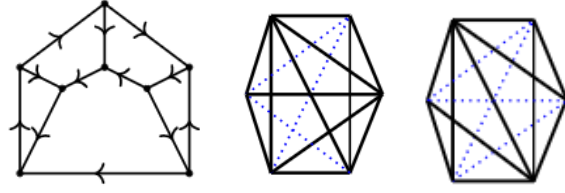


Figure 4.4: Above: Planar graph I , its Jacobian dicing B_5 , subdicing B_5^1

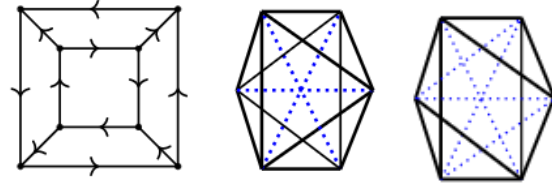


Figure 4.5: Planar Graph II , its Jacobian dicing C_5 , subdicing C_5^1

4.3 Computation Example

We will now present an example of how we computed the Prym lattice of a double cover. First, let us introduce how we choose to represent a double cover of a plane quintic of Type A, Type B, or Type N.

An unramified double cover C' of a metric graph C can be constructed by gluing two copies of C together at a specified set of edges. We cut these edges and then glue the pieces in a criss-cross fashion. We can pick orientations on the sheets so that for a simple cycle

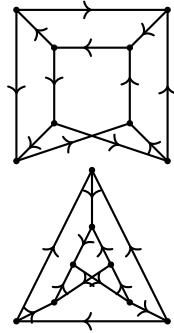


Figure 4.6: Planar Graph I , Genus 5, and Dicing

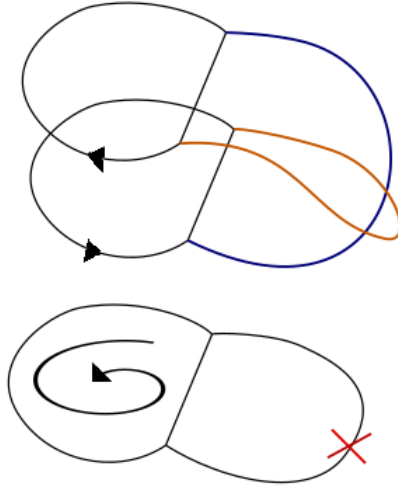


Figure 4.7: Edge where a cut is made is marked with an edge on the downstairs curve. Gluing of the two sheets is shown above. An anti-invariant cycle is demonstrated with the orientation of arrows.

”downstairs” on C , the corresponding cycles in the double cover have opposite orientations on the two sheets. These will be the cycles which are anti-invariant under the sheet exchange involution (deck transformation). (See Figure 4.7)

We encode an unramified double cover of one of our tropical curves by marking the edges where we glue the two sheets together as above with a red “X”. Then, we count the cycles for which we exchange sheets an even number of times or stayed on one sheet for the entire cycle. This is indicated by marking the edges with crosses with opposite orientations of either side of the cross. Under the parity $\alpha : H_1(C, \mathbb{Z}) \rightarrow \mathbb{Z}/2$, the parity of $\alpha(\gamma)$ for a simple cycle γ in C' determines if the cycle is even or odd. An odd cycle in C' corresponds to an even cycle in C . There are other double covers of metric graphs of tropical curves which are not simple and will not be considered. Additionally, we can push crosses to the outside edges as in Figure 4.3 and have an equivalent cover, as this will not change the map α .

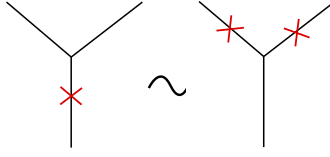


Figure 4.8: *Equivalence of crossings*

Notation

- E_{ij} : Edge connecting vertex i to vertex j (see figure 4.3 for vertex labels; they are omitted in other figures).
- $\mathcal{H}_3$: The cycle associated to the hyperplane section.
- $\mathcal{S}$: the set which contains the edges in $\mathcal{H}_3$ that have crossings. The parity of $\mathcal{S}$ is the parity of $\alpha(\mathcal{H}_3)$.

The anti-invariant cycles on the double cover C' correspond to even cycles on C . We orient the edges of the two sheets of C' in opposite directions. This is represented on C by labeling the orientation on the top branch with arrows, and then reverse the arrow on the other side of a cross.

The graph we will look at is of Type A.

We have crossings on edges E_{12} , E_{34} and E_{45} , so the parity of the cover is even. We will choose a vertex where all cycles will begin and end, indicated in the figure. Figure 4.9 shows the graph with the crossings; 4.10 shows the edge length relations.

Excluding vertex 0, each outer vertex of the graph will correspond to a unique cycle. Thus, we will have five cycles in total. We start with cycle 1: we choose to first travel along edge y and label the orientation of the edge and write 1 next to it so that we remember our cycle passes through it. We continue to do this until we reach the outer vertex noted in the figure. Then, we choose to travel either clockwise or anti-clockwise around the perimeter based on how many crossings we are allowed to pass through. In this example, we must

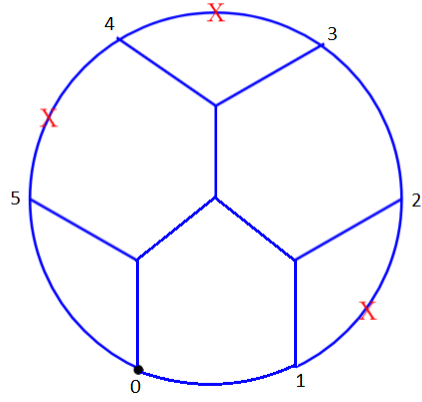


Figure 4.9: *Example encoded graph*

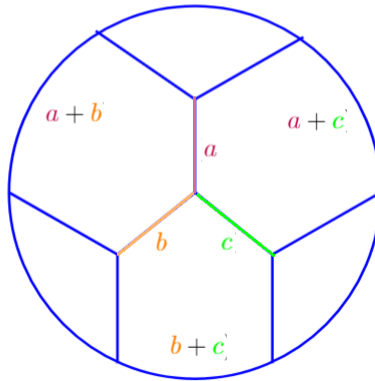
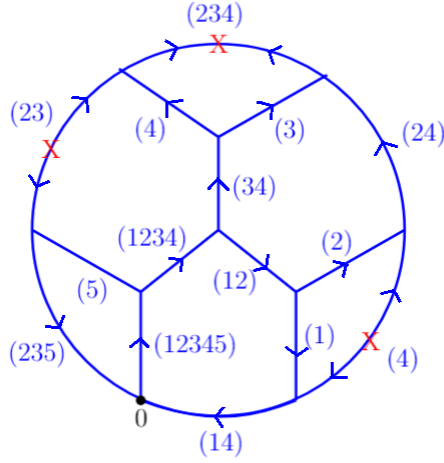


Figure 4.10: *Edge length relations*



The notation for the linear functional (234) means the vector $(0, 1, 1, 1, 0)$

Figure 4.11: *Complete orientation of cycles and linear functional labels*

immediately return to our starting point by traveling along E_{01} - remembering to label it with a 1 - as the other option would have us make three crossings, which would not give an even cycle. We are done with cycle 1.

We begin cycle 2 in the same way, but instead travel to the second outer vertex as indicated. Here, we must travel anti-clockwise so that we make two crossings, one each on E_{34} and E_{45} (so that we move to the bottom sheet, then back to the top). The orientation switches both times and we re-enter vertex 0 with the correct orientation. Again, we must label each edge along which we travel with a 2.

Once all of the cycles have been drawn and recorded, we then turn our attention to the edge labelings. These labelings are linear functionals which we evaluate on each cycle, and which give us our lattice for the Prym. We have 15 linear functionals on our graphs (one for each edge); they form a totally unimodular system. We can write the symmetric, bilinear, quadratic form Q for the system:

$$\begin{aligned}
Q &= \sum_{\text{edges } e \in C} \beta_{l_e}(l_e)^2 = \\
&= a(\mathbf{34})^2 + b(\mathbf{1234})^2 + c(\mathbf{12})^2 + \beta_{12}(\mathbf{12})^2 + \beta_2(\mathbf{2})^2 + \beta_{12345}(\mathbf{12345})^2 + \beta_1(\mathbf{1})^2 + \beta_{235}(\mathbf{235})^2 + \\
&\quad + \beta_4^1(\mathbf{4})^2 + \beta_4^2(\mathbf{4})^2 + \beta_3(\mathbf{3})^2 + \beta_{234}(\mathbf{234})^2 + (a+b)(\mathbf{23})^2 + (a+c)(\mathbf{24})^2 + (b+c)(\mathbf{14})^2
\end{aligned}$$

where β_{l_e} is the length of edge e , with a superscript if two or more edges have the same associated functional. Collecting terms with the same coefficient, we get:

$$\begin{aligned}
Q &= a((\mathbf{34})^2 + (\mathbf{23})^2 + (\mathbf{24})^2) + b((\mathbf{1234})^2 + (\mathbf{23})^2 + (\mathbf{14})^2) + c((\mathbf{12})^2 + (\mathbf{24})^2 + (\mathbf{14})^2) + \\
&\quad + \beta_{12}(\mathbf{12})^2 + \beta_2(\mathbf{2})^2 + \beta_{12345}(\mathbf{12345})^2 + \beta_1(\mathbf{1})^2 + \beta_{235}(\mathbf{235})^2 + \beta_4^1(\mathbf{4})^2 + \beta_4^2(\mathbf{4})^2 + \beta_3(\mathbf{3})^2 + \beta_{234}(\mathbf{234})^2
\end{aligned}$$

We use the following identity:

$$(l_1 + l_2)^2 + (l_2 + l_3)^2 + (l_1 + l_3)^2 = l_1^2 + l_2^2 + l_3^2 + (l_1 + l_2 + l_3)^2$$

This identity is used to check for linear dependence among the functionals corresponding to the indicated edges. We can rewrite our form Q :

$$\begin{aligned}
Q &= a((\mathbf{3})^2 + (\mathbf{4})^2 + (\mathbf{2})^2 + (\mathbf{234})^2) + b((\mathbf{1234})^2 + (\mathbf{23})^2 + (\mathbf{14})^2) + c((\mathbf{1})^2 + (\mathbf{2})^2 + (\mathbf{4})^2 + (\mathbf{124})^2) + \\
&\quad + \beta_{12}(\mathbf{12})^2 + \beta_2(\mathbf{2})^2 + \beta_{12345}(\mathbf{12345})^2 + \beta_1(\mathbf{1})^2 + \beta_{235}(\mathbf{235})^2 + \beta_4^1(\mathbf{4})^2 + \beta_4^2(\mathbf{4})^2 + \beta_3(\mathbf{3})^2 + \beta_{234}(\mathbf{234})^2
\end{aligned}$$

We have 12 linear functionals after considering these relations. We associate these 12 functionals with perfect vectors in the first perfect domain on 5 variables, $\Pi_1(5)$, and realize them as a graph. A cycle of vectors corresponds to $v_1 + \dots + v_k = 0$. The perfect vectors are as follows: $a_{0i} = e_i, 1 \leq i \leq 5$ and $a_{ij} = e_j - e_i, 1 \leq i < j \leq 5$, where $\{e_i\}$ is the standard basis for $\mathbb{R}^n$. We then associate each number $0, \dots, 5$ to a vertex and draw the edges corresponding to each perfect vector by connecting vertices k and l if we have the vector a_{kl} . The missing

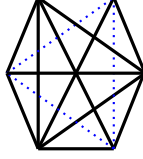


Figure 4.12: *Lattice dicing F_5*

edges are indicated by dotted lines.

This dicing is identified as the subdicing F_5 of the maximal dicing A_5 of $\mathbb{R}^5$.

Proposition 4.1 The dicing G_5 is not a subdicing of B_5 , C_5 , $\mathcal{D}_2^1$, or $\mathcal{D}_2^2$.

Proof The dicing G_5 has 9 lattice vectors, and is missing two disjoint triangles from A_5 . If it were a subdicing of either B_5 or C_5 , then we could obtain G_5 by removing 3 edges from either one. However, it is clear that this is impossible, see Figures 4.3 and 4.4.

It will be shown that deleting an edge from E_5 does not produce a subdicing of $\mathcal{D}_2^1, \mathcal{D}_2^2$. We consider affine triangles in those two dicings. An **affine triangle** is a set of 3 zone vectors $\{v_1, v_2, v_3\}$ such that $v_1 + v_2 + v_3 = 0$. Erdahl and Ryshkov define the invariants I_3 for the maximal dicings of concern to record the number of triangles each edge is a member of in the graph realization of the zone vectors. We can count to see that

1. $\mathcal{D}_2^1$ has all 12 edges in 2 triangles
2. $\mathcal{D}_2^2$ has 9 edges in 2 triangles and 3 in 3 triangles
3. E_5 , hence G_5 has no triangles

We want to verify that if we remove 1 edge from E_5 , then will have at least one affine triangle after removing 3 edges from either $\mathcal{D}_2^1$ or $\mathcal{D}_2^2$. Affine triangles, when considered in the second perfect domain, correspond to what are called red triangles:

concerning the graphs in Figure 4.2, these are triangles with either 3 red dotted edges (perfect vector $e_i - e_j$ between vertex i and vertex j) or exactly 2 black edges (perfect vectors $e_i + e_j$ between vertex i and j).

Consider first the dicing $\mathcal{D}_2^1$. All 12 edges are part of exactly 2 red triangles. Then there are a total of $(12 \times 2)/3 = 8$ triangles. Removing a single edge kills at most 2 triangles, so we will have at least 2 triangles remaining. So $G_5 \notin \mathcal{D}_2^1$.

Now, consider $\mathcal{D}_2^2$. It contains a total of $(9 \times 2 + 3 \times 3)/3 = 9$ triangles. Removing 3 edges can kill at most 8 of them, as the three edges contained in 3 triangles (see Figure 4.2) form an affine triangle. Then we are left with at least one red triangle. Hence, $G_5 \notin \mathcal{D}_2^2$.

This allows us to state our main result.

Theorem: The dicings F_5 and E_5 are not subdicings of B_5, C_5, D_2^1 , or D_2^2 . Thus, if the Prym of the unramified double cover of a tropical plane quintic of Type A, Type B, or Type N has the lattice F_5, E_5, G_5 (see tables in Section 4.4), then it is not a Jacobian.

4.4 Results

The following tables list the results of the computations. We display the lattice which corresponds to the cardinality of $\mathcal{S}$ (hence, the parity $\alpha(\mathcal{H}_3)$). If applicable, the graph and its Jacobian that are given by that dicing are also listed. For Types B and N, we distinguish between two cases when $\alpha(\mathcal{H}_3)$ is 1 or 2: we say $\mathcal{S}$ is **asymmetric** if it contains the edge which lies on the axis of symmetry for the graph. Otherwise, $\mathcal{S}$ is asymmetric.

The main result is that if $\alpha(\mathcal{H}_3)$ is odd, then the Prym is a Jacobian.

Table 4.2: *Type A*

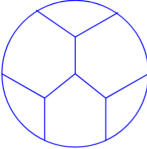
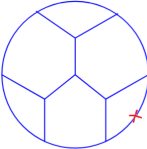
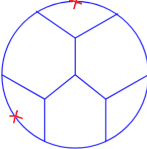
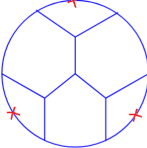
$\mathcal{S}$	Dicing
	F_5
	B_5
	F_5
	C_5

Table 4.3: *Type B*

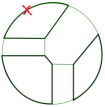
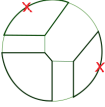
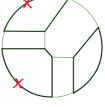
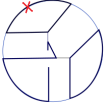
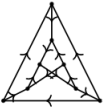
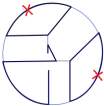
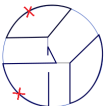
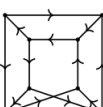
S	Dicing
	F_5^1
 	B_5, B_5^1 B_5^1, B_5^2
 	F_5^2 F_5^1
	F_5^3

Table 4.4: *Type N*

S	Dicing
	F_5, F_5^1
 	 B_5
 	E_5 F_5
	

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Appendix A

Further Computation Examples

Two more computation examples will be briefly presented here so as to illustrate a few differences which occur among the types of graphs and when the parity of the number of crossings is even.

Example 1:

The graph shown in Figure [A.2](#) represents a double cover of Type B and has 4 crossings, with one crossing on the symmetric edge in the set $\mathcal{H}_3$. One feature to note on this graph is that the even number of crossings makes it impossible for a cycle starting at an arbitrary vertex to travel first to any other vertex of choosing; we must choose a vertex which allows us to then travel along a path of edges on the perimeter so that we meet zero or an even number of crossings. Thus, there is a so-called "grouping" of vertices. Two of the cycles begin at the vertex labeled v , three at the vertex labeled w . We will still choose our vertex "0" as usual to be consistent with notation, but in this case none of our cycles begin there.

The edges which are related by length give relations between the linear functionals associated to those edges:

$$(345)^2 + (134)^2 + (15)^2 = (1)^2 + (5)^2 + (34)^2 + (1345)^2$$

The dicing given by the 12 perfect vectors associated to the functionals is the subdicing B_5 . We saw on page 16 that this is the Jacobian of a graph of genus 5.

Example 2:

The next computation example is done for a double cover of a graph of Type N with a crossing on each edge in $\mathcal{H}_3$, so that $|\mathcal{S}| = 3$. The parity $\alpha(\mathcal{H}_3)$ is odd, so we expect that the Prym is a Jacobian. We again choose vertex "0" as usual, which will be the starting point for every cycle, as there is an odd number of crossings. The relations among the edges are

$$(135)^2 + (45)^2 + (134)^2 = (4)^2 + (5)^2 + (13)^2 + (1345)^2$$

$$(124)^2 + (23)^2 + (134)^2 = (2)^2 + (3)^2 + (14)^2 + (1234)^2$$

The functionals give us a dicing which is the Jacobian of the graph in Figure A.1.

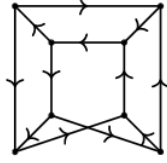


Figure A.1: *Graph for Example A.2*

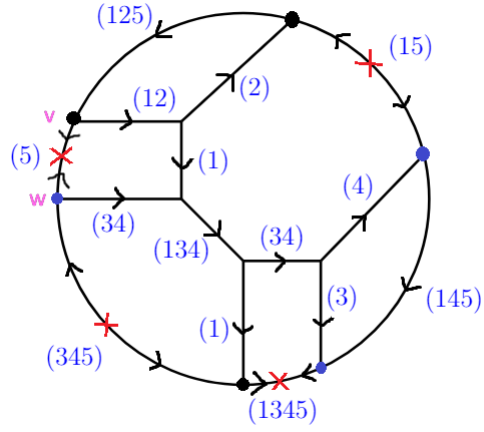


Figure A.2: *Four crossings, $|S| = 1$ with crossing on antisymmetric edge*

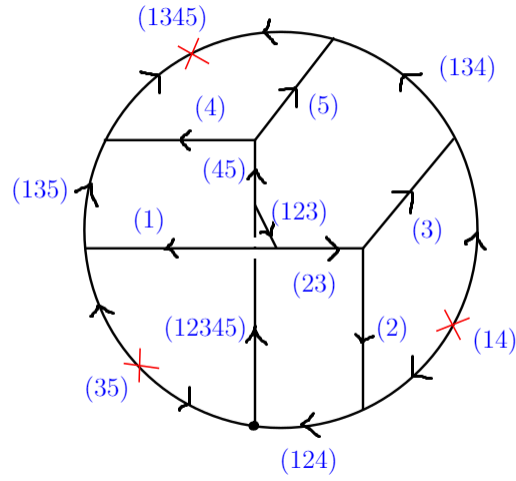


Figure A.3: *Three crossings, $|S| = 3$*