

Predicting Buckling Load
Using Vibratory Data

by

CHEER GERM GO

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Department of Civil Engineering

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Approved by


Major Professor

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INTRODUCTION

Given a particular structure, simple experiments can be devised to determine its natural frequencies. These frequencies are known to be a function of the structure's support boundary conditions. Similarly, it is known that the critical buckling load is also a function of the structure's support boundary conditions. The feasibility for predicting buckling loads of elastic structures by using experimentally determined natural frequencies was investigated by Sweet, Genin and Mlakar (3) (4) (5). According to their theory, the characteristic equation which defines the vibration of the beam and the characteristic equation which governs its buckling need to be derived. The unknown boundary parameters are then predicted analytically by substituting the experimentally determined natural frequencies into the characteristic equation for the beam vibration. These restraint parameters are then substituted into the characteristic equation for buckling to compute the buckling load. In this paper, the matrix approach is used to determine these characteristic equations because the matrix algebra makes possible a formulation of the solution as a series of matrix operations suitable for a digital computer. But even more important is that, by using matrices, structures of all types can be analyzed through a general approach.

DESCRIPTION OF METHOD

Suppose that the fundamental frequencies are obtained experimentally. The displacement method of analysis is applied to determine the buckling load of a beam column whose support boundary conditions are not known in advance of service by using these frequencies. In the actual structure, the stiffness matrix relates displacements $\{V\}$ at a number of coordinates to the forces $\{F\}$ applied at the same coordinates by the equation

$$\{K\} \{V\} = \{F\}$$

The column vector $\{F\}$ depends on the loading on the structure. The elements of the matrix $\{K\}$ are forces corresponding to unit values of displacements. Therefore, structure stiffness $\{K\}$ depends on the type of structure. When the beam is vibrating freely, the elements of the matrix $\{K\}$ are functions of the angular frequency and its boundary conditions. The elements of column vector $\{F\}$ are zero. For a nontrivial solution to exist, the dynamic stiffness matrix $\{K\}$ must be singular. That is, the determinant of the stiffness matrix must be equal to zero. Therefore

$$|\{K\}| = 0 \tag{1}$$

The unknown boundary parameters can be predicted analytically by substituting experimentally determined frequencies into Eq. (1).

Similarly for the static (nonvibrating) case when the beam is subjected to an axial load, the forces $\{F_b\}$ and the displacements $\{V_b\}$ at coordinates are related by equation

$$\{K_b\} \{V_b\} = \{F_b\}$$

The elements of stiffness matrix $\{K_b\}$ are functions of the axial load and its boundary conditions. If the axial load corresponds to the critical buckling value, then it is possible to let the structure acquire some small displacements $\{\delta V_b\}$ without the application of transverse forces and the last

equation thus becomes

$$[K_b] [\delta V_b] = [0]$$

For a nontrivial solution to exist, the stiffness matrix $[K_b]$ must be singular. That is, the determinant

$$|[K_b]| = 0 \tag{2}$$

Suppose that the estimates of the unknown boundary parameters are identified by equation (1). Then, these estimates can be substituted into the characteristic equation (2) to compute the buckling load.

In this report, the deflection equation of the beam is used to evaluate its stiffness.

DEFLECTION OF VIBRATING BEAM SUBJECTED TO AXIAL LOAD

Consider the prismatic beam of Fig. 1 which is subjected to an axial load which is constant along the length of the member and does not vary with time. The end conditions for the beam are arbitrary, although they are pictured as simple supports for illustrative purposes.

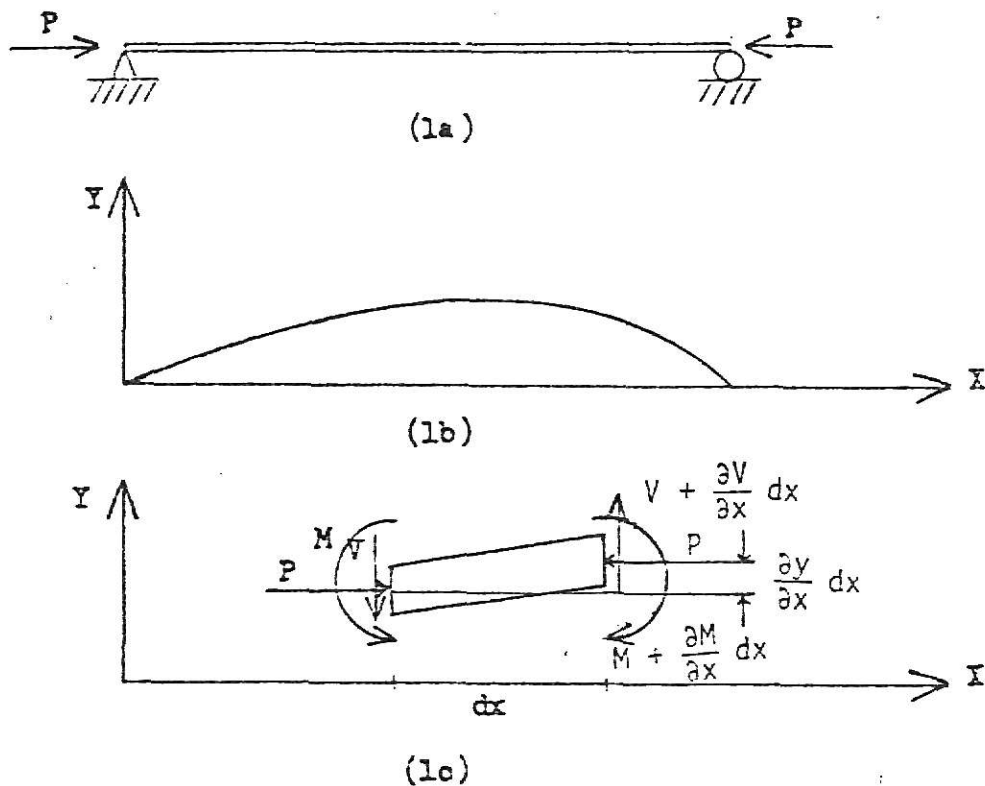


Fig. 1 Deflection of a beam subjected to an axial load

(1a) beam and loading

(1b) positive direction of deflection

(1c) positive directions of external and internal forces

Making the usual assumptions in the theory of bending, that plane transverse sections remain plane, and that the material obeys Hooke's law, we can easily show that

$$M = -EI \frac{\partial^2 y}{\partial x^2}$$

where y is the deflection and M is the bending moment at any section x (see fig. 1b). EI is the flexural rigidity, the positive direction of y is indicated in fig. 1b and the bending moment is considered positive if it causes tensile stress at the top face of the beam. When the free vibration of this beam is considered, an element of the beam of length dx is in equilibrium in a deflected position under the forces shown in fig. 1c. Where P is the axial load, V is the shear, M is the moment and m is the mass per unit length. The positive directions of P , y , M , V are as indicated in fig. 1c. Summing all forces acting vertically leads to the first dynamic equilibrium relationship

$$\left(V + \frac{\partial V}{\partial x} dx \right) - V = mdx \frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial V}{\partial x} = m \frac{\partial^2 y}{\partial t^2} \quad (4)$$

The second equilibrium relationship is obtained by summing moments about the elastic axis at the right hand face of the segment.

$$M - \left(M + \frac{\partial M}{\partial x} dx \right) + V dx + P \frac{\partial y}{\partial x} dx = 0$$

$$\frac{\partial M}{\partial x} - V - P \frac{\partial y}{\partial x} = 0$$

$$\frac{\partial^2 M}{\partial x^2} - \frac{\partial V}{\partial x} - P \frac{\partial^2 y}{\partial x^2} = 0 \quad (5)$$

Substituting equations (3), (4), into equation (5), we obtain the differential equation of the vibration beam

$$EI \frac{\partial^4 y}{\partial x^4} + P \frac{\partial^2 y}{\partial x^2} + m \frac{\partial^2 y}{\partial t^2} = 0 \quad (6)$$

One form of solution of this equation can be obtained by separation of variables, assuming that the solution has the form

$$y(x, t) = \phi(x) \cdot k(t) \quad (7)$$

Substituting Eq. (7) into Eq. (6)

$$EI \frac{d^4 \phi}{dx^4} \cdot k(t) + P \frac{d^2 \phi(x)}{dx^2} \cdot k(t) + m \phi(x) \frac{d^2 k(t)}{dt^2} = 0$$

$$EI \frac{d^4 \phi(x)/dx^4}{\phi(x)} + P \frac{d^2 \phi(x)/dx^2}{\phi(x)} = -m \frac{d^2 k(t)/dt^2}{k(t)} \quad (8)$$

Hence the Eq. (8) must be equal to a constant, because if the expression on the left is not constant, then changing x will presumably change the value of this expression but certainly not that on the right, since the expression on the right does not depend on x . Similarly, if the expression on the right is not constant, changing t will presumably change the value of this expression but certainly not that on the left. Thus

$$EI \frac{d^4 \phi(x)/dx^4}{\phi(x)} + P \frac{d^2 \phi(x)/dx^2}{\phi(x)} = -m \frac{d^2 k(t)/dt^2}{k(t)} = \text{Constant} \quad (9)$$

from which are obtained the two independent equations

$$\frac{d^2 k(t)}{dt^2} + \omega^2 k(t) = 0 \quad (10)$$

$$EI \frac{d^4 \phi(x)}{dx^4} + P \frac{d^2 \phi}{dx^2} - m \omega^2 \phi(x) = 0 \quad (11)$$

Where the constant has been replaced by $m\omega^2$, ω is the angular frequency. This shows that a constant axial force does not affect the simple harmonic character of the free vibrations. Equation (11) leads to frequency and mode-shape expressions for the freely vibrating beam, in which the axial force P is a parameter. When equation (11) is divided by EI , it can be written

$$\frac{d^4 \phi(x)}{dx^4} + g^2 \frac{d^2 \phi(x)}{dx^2} - a^4 \phi(x) = 0 \quad (12)$$

where

$$a^4 = \frac{m\omega^2}{EI}$$

$$g^2 = \frac{P}{EI}$$

Assuming $\phi(x) = e^{sx}$ and (13)

substituting equation (13) into equation (12)

$$s^4 e^{sx} + g^2 s^2 e^{sx} - a^4 e^{sx} = 0$$

$$(s^4 + g^2 s^2 - a^4) e^{sx} = 0$$

$$s = \pm i\delta, \pm \epsilon$$

where $\delta = \sqrt{(a^4 + \frac{g^4}{4})^{\frac{1}{2}} + \frac{g^2}{2}}$

$$\epsilon = \sqrt{(a^4 + \frac{g^4}{4})^{\frac{1}{2}} - \frac{g^2}{2}}$$

$$\phi_1(x) = e^{i\delta x} = \cos(\delta x) + i \sin(\delta x)$$

$$\phi_2(x) = e^{-i\delta x} = \cos(-\delta x) + i \sin(-\delta x) = \cos \delta x - i \sin(\delta x)$$

$$\phi_3(x) = e^{\epsilon x}$$

$$\phi_4(x) = e^{-\epsilon x}$$

let $\phi_5(x) = \frac{1}{2}[\phi_1(x) + \phi_2(x)] = \cos(\delta x)$

$$\phi_6(x) = \frac{1}{2i}[\phi_1(x) - \phi_2(x)] = \sin(\delta x)$$

$$\phi_7(x) = \frac{1}{2}[\phi_3(x) + \phi_4(x)] = \frac{1}{2}(e^{\epsilon x} + e^{-\epsilon x}) = \cosh(\epsilon x)$$

$$\phi_8(x) = \frac{1}{2}[\phi_3(x) - \phi_4(x)] = \frac{1}{2}(e^{\epsilon x} - e^{-\epsilon x}) = \sinh(\epsilon x)$$

the solution of equation (12) can be written as

$$\phi(x) = D_1 \phi_5(x) + D_2 \phi_6(x) + D_3 \phi_7(x) + D_4 \phi_8(x)$$

$$= D_1 \cos(\delta x) + D_2 \sin(\delta x) + D_3 \cosh(\epsilon x) + D_4 \sinh(\epsilon x) \quad (14)$$

Equation (14) defines the shape of the vibrating beam for any value of axial force which might be specified.

STIFFNESS OF VIBRATING BEAM DUE TO DYNAMIC EFFECTS

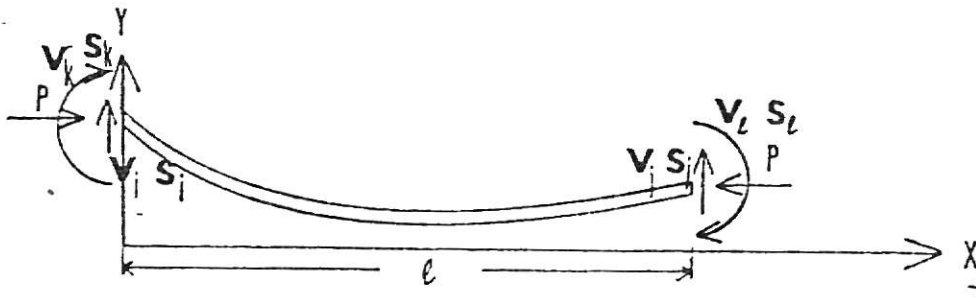


Fig. 2 Boundary forces and displacements of uniform beam segment

In Fig. 2, it is evident that when the axial force vanishes, that is $P = 0$, so that $g = 0$, then $\delta = \epsilon = a$ and equation (14) reverts to

$$Y(x) = D_1 \sin(ax) + D_2 \cos(ax) + D_3 \sinh(ax) + D_4 \cosh(ax) \quad (15)$$

Equation (15) will now be used to derive the stiffness matrix of a vibration beam corresponding to the coordinates i, j, k and l in fig. 2. The displacements $[V]$ at the four coordinates

$$V_i = Y(0)$$

$$V_j = Y(l)$$

$$V_k = -Y'(0)$$

$$V_l = -Y'(l)$$

are related to the constants $[D]$ by the equation

$$\begin{bmatrix} V_i/l \\ V_j/l \\ V_k \\ V_l \end{bmatrix} = \begin{bmatrix} Y(0)/l \\ Y(l)/l \\ -Y'(0) \\ -Y'(l) \end{bmatrix} = \begin{bmatrix} 0 & 1/l & 0 & 1/l \\ s/l & c/l & \bar{s}/l & \bar{c}/l \\ -a & 0 & -a & 0 \\ -ac & as & -a\bar{c} & -a\bar{s} \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{bmatrix} \quad (16)$$

where

$$S = \sin(a\ell)$$

$$c = \cos(a\ell)$$

$$\bar{s} = \sinh(a\ell)$$

$$\bar{c} = \cosh(a\ell)$$

The forces $[S]$ at the four coordinates are the shear and bending moment at $x = 0$ and $x = \ell$, thus

$$\begin{bmatrix} S_i \\ S_j \\ S_k \\ S_\ell \end{bmatrix} = EI \begin{bmatrix} Y''''(0) \cdot \ell \\ -Y''''(\ell) \cdot \ell \\ Y'''(0) \\ -Y'''(\ell) \end{bmatrix}$$

Differentiating equation (15) and substituting into the last equation gives

$$\begin{bmatrix} S_i \ell \\ S_j \ell \\ S_k \\ S_\ell \end{bmatrix} = EIa^2 \begin{bmatrix} -a\ell & 0 & a\ell & 0 \\ ac\ell & -as\ell & -a\bar{c}\ell & -a\bar{s}\ell \\ 0 & -1 & 0 & 1 \\ s & c & -\bar{s} & -\bar{c} \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{bmatrix} \quad (17)$$

Solving for $[D]$ from Eq. (16) and substituting into Eq. (17) results in

$$\begin{bmatrix} S_i \ell \\ S_j \ell \\ S_k \\ S_\ell \end{bmatrix} = EIa^2 \begin{bmatrix} -a\ell & 0 & a\ell & 0 \\ ac\ell & -as\ell & -a\bar{c}\ell & -a\bar{s}\ell \\ 0 & -1 & 0 & 1 \\ s & c & -\bar{s} & -\bar{c} \end{bmatrix} \begin{bmatrix} 0 & 1/\ell & 0 & 1/\ell \\ \bar{s}/\ell & c/\ell & \bar{s}/\ell & \bar{c}/\ell \\ -a & 0 & -a & 0 \\ ac & as & -a\bar{c} & -a\bar{s} \end{bmatrix}^{-1} \begin{bmatrix} V_i/\ell \\ V_j/\ell \\ V_k \\ V_\ell \end{bmatrix}$$

When the inversion and multiplication are carried the final form becomes

$$\begin{bmatrix} S_i \\ S_j \\ S_k \\ S_l \end{bmatrix} = \frac{EI}{l} \begin{bmatrix} \gamma & -\bar{\gamma} & -\beta & -\bar{\beta} \\ -\bar{\gamma} & \gamma & \bar{\beta} & \beta \\ -\beta & \bar{\beta} & \alpha & \bar{\alpha} \\ -\bar{\beta} & \beta & \bar{\alpha} & \alpha \end{bmatrix} \begin{bmatrix} V_i/l \\ V_j/l \\ V_k \\ V_l \end{bmatrix} \quad (18)$$

where

$$\alpha = \frac{s\bar{c} - c\bar{s}}{d} (al) \quad , \quad \bar{\alpha} = \frac{\bar{s} - s}{d} (al)$$

$$\beta = \frac{s\bar{s}}{d} (al)^2 \quad , \quad \bar{\beta} = \frac{\bar{c} - c}{d} (al)^2$$

$$\gamma = \frac{s\bar{c} + c\bar{s}}{d} (al)^3 \quad , \quad \bar{\gamma} = \frac{\bar{s} + s}{d} (al)^3$$

$$d = 1 - c\bar{c}$$

Summarizing the above discussion, it is seen that the natural frequencies are considered as parameters affecting the stiffness of the member. According to the stiffness matrix in Eq. (18), the end-forces corresponding to unit end-displacement of a prismatic member due to dynamic effects are shown in Fig. 3.

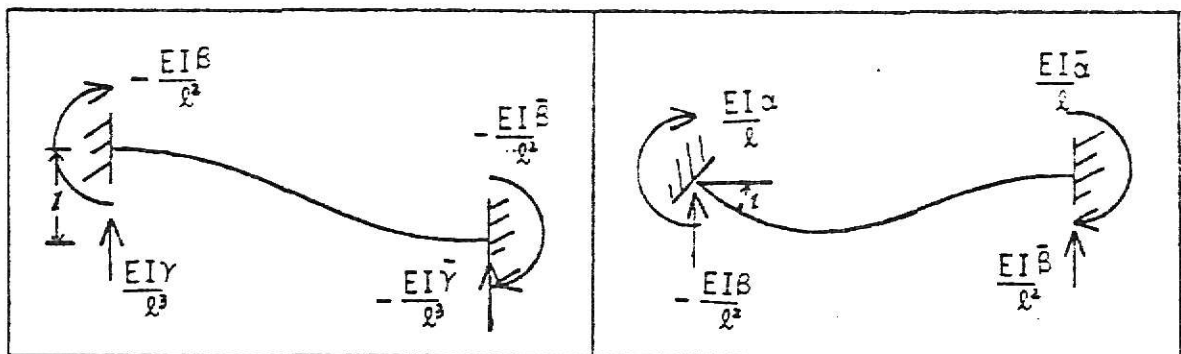


Fig. 3 End-forces caused by end displacements of a prismatic member due to dynamic effects

STIFFNESS OF STATIC BEAM SUBJECTED TO AN AXIAL LOAD

In Fig. 2, for the static (nonvibrating) case (where $\omega = 0$ so that $a = 0$) $\delta = g$ while $\epsilon = 0$. Thus, the solution of equation (12) corresponding to these conditions ($\delta = g$ and $\epsilon = 0$) can be written as

$$y(x) = D_1 \sin(gx) + D_2 \cos(gx) + D_4 x + D_3 \quad (19)$$

Similarly, expressing the displacements and rotations at two ends (nodal points) of the segment in matrix form is

$$\begin{bmatrix} V_{i/l} \\ V_{j/l} \\ V_k \\ V_l \end{bmatrix} = \begin{bmatrix} Y(0)/l \\ Y(l)/l \\ -Y'(0) \\ -Y'(l) \end{bmatrix} = \begin{bmatrix} 0 & 1/l & 0 & 1/l \\ s'/l & c'/l & 1 & 1/l \\ -u/l & 0 & -1/l & 0 \\ -uc'/l & us'/l & -1 & 0 \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ D_4 \\ D_3 \end{bmatrix} \quad (20)$$

where

$$u = gl = l\sqrt{P/EI}$$

$$s' = \sin u$$

$$c' = \cos u$$

The nodal forces at the two ends of the beam segment can be written in matrix form as

$$\begin{bmatrix} S_{i/l} \\ S_{j/l} \\ S_k \\ S_l \end{bmatrix} = EI \begin{bmatrix} l \left[Y''''(0) + \frac{u^2}{l^2} Y'(0) \right] \\ -l \left[Y''''(l) + \frac{u^2}{l^2} Y'(l) \right] \\ Y''(0) \\ -Y''(l) \end{bmatrix}$$

where

$$\frac{u^2}{l^2} = P/EI$$

Differentiating equation (19) and substituting in the last equation gives

$$\begin{bmatrix} S_{i/l} \\ S_{j/l} \\ S_k \\ S_l \end{bmatrix} = EI \begin{bmatrix} 0 & 0 & u^2/l & 0 \\ 0 & 0 & -u^2/l & 0 \\ 0 & -u^2/l^2 & 0 & 0 \\ s'u^2/l^2 & c'u^2/l^2 & 0 & 0 \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ D_4 \\ D_3 \end{bmatrix} \quad (21)$$

Solving for $[D]$ from equation (20) and substituting into Eq. (21) gives

$$\begin{bmatrix} S_{i/l} \\ S_{j/l} \\ S_k \\ S_l \end{bmatrix} = EI \begin{bmatrix} 0 & 0 & u^2/l & 0 \\ 0 & 0 & -u^2/l & 0 \\ 0 & -u^2/l^2 & 0 & 0 \\ s'u^2/l^2 & c'u^2/l^2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1/l & 0 & 1/l \\ s'/l & c'/l & 1 & 1/l \\ -u/l & 0 & -1/l & 0 \\ -uc'/l & us'/l & -1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} V_{i/l} \\ V_{j/l} \\ V_k \\ V_l \end{bmatrix}$$

When the inversion and multiplication are carried the final form becomes

$$\begin{bmatrix} S_{i/l} \\ S_{j/l} \\ S_k \\ S_l \end{bmatrix} = \frac{EI}{l} \begin{bmatrix} \bar{a} & -\bar{a} & -\bar{b} & -\bar{b} \\ -\bar{a} & \bar{a} & \bar{b} & \bar{b} \\ -\bar{b} & \bar{b} & \bar{d} & \bar{d} \\ -\bar{b} & \bar{b} & \bar{d} & \bar{f} \end{bmatrix} \begin{bmatrix} V_{i/l} \\ V_{j/l} \\ V_k \\ V_l \end{bmatrix} \quad (22)$$

where

$$\begin{aligned} \bar{a} &= \frac{u^3 s'}{2-2c'-us'} & \bar{b} &= \frac{u^2(1-c')}{2-2c'-us'} \\ \bar{d} &= \frac{u(u-s')}{2-2c'-us'} & \bar{f} &= \frac{u(s'-uc')}{2-2c'-us'} \end{aligned}$$

Summarizing the above discussion, it is seen that the axial load is considered as a parameter affecting the stiffness of the member. According to the stiffness matrix in Eq. (22), the end forces according to unit end-displacement of a prismatic member subjected to an axial compressive force

are shown in Fig. 4.

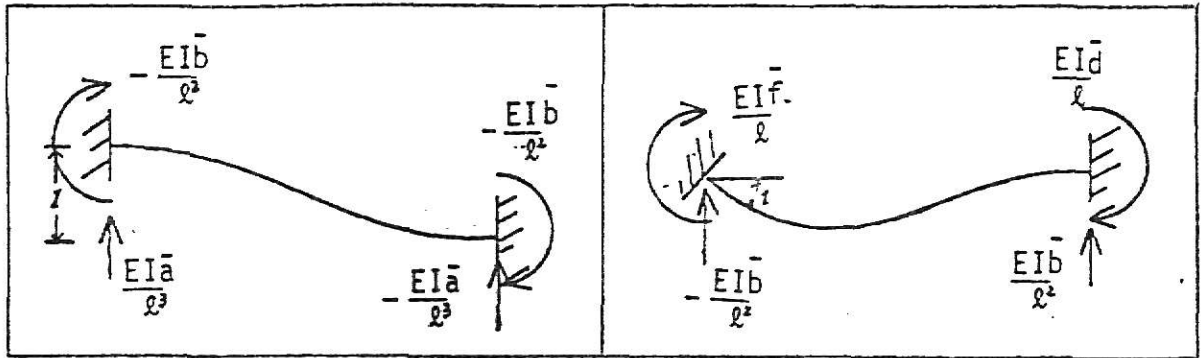


Fig. 4 End forces caused by end displacements of prismatic member subjected to axial compressive force

ILLUSTRATIVE EXAMPLE

Consider a uniform beam which is restrained against translation and rotation at its two ends. Denote torsional spring constants by K_k , K_ℓ and translational spring constants by K_j , K_j as shown in Fig. 5.

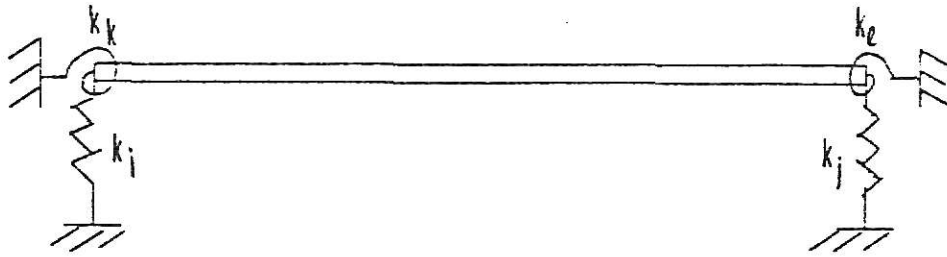


Fig. 5 Beam column model

By referring to Fig. 3, the dynamic stiffness matrix $[K]$ relates displacements by the equation

$$[K] [V] = [0] \quad (23)$$

where

$$[K] = \frac{EI}{l} \begin{bmatrix} \gamma + \frac{K_j l^3}{EI} & -\bar{\gamma} & -\beta & -\bar{\beta} \\ -\bar{\gamma} & \gamma + \frac{K_j l^3}{EI} & \bar{\beta} & \beta \\ -\beta & \bar{\beta} & \alpha + \frac{K_k l}{EI} & \bar{\alpha} \\ -\bar{\beta} & \beta & \bar{\alpha} & \alpha + \frac{K_\ell l}{EI} \end{bmatrix}$$

By referring to Eq. (1), the characteristic equation for the beam vibration can be obtained by letting the determinant of the dynamic stiffness matrix equal to zero. That is

$$| [K] | = 0 \quad (24)$$

Similarly, the characteristic equation for buckling is

$$[K_b] = 0$$

where

$$[K_b] = \frac{EI}{\ell} \begin{bmatrix} \bar{a} + \frac{K_j \ell^3}{EI} & -\bar{a} & -\bar{b} & -\bar{b} \\ -\bar{a} & \bar{a} + \frac{K_j \ell^3}{EI} & \bar{b} & \bar{b} \\ -\bar{b} & \bar{b} & \bar{f} + \frac{K_k \ell^2}{EI} & \bar{d} \\ -\bar{b} & \bar{b} & \bar{d} & \bar{f} + \frac{K_\ell \ell^2}{EI} \end{bmatrix}$$

In cases where end conditions are specified differently than in Fig. 5, Eqs. (23) and (24) may be reduced to a simpler form. For instance, in reference (5) by Sweet, Genin and Mlakar, experiments were performed to evaluate the buckling loads of 10 beam columns. One end of the beam column was bolted with steel blocks to the head of a universal testing machine. The other end of each specimen was connected to a load cell with similar blocks and a platform. Each specimen of 4150 steel, having a measured Young's modulus of $21.27 \times 10^4 \text{ Mpa}$ was cut from 12.7mm by 25.4mm bar stock so as to give a nominal unsupported length of 902mm. The dimensionless mass parameter of the load cell was found to be $M = 3.290$.

By performing vibration tests, the averages of natural frequencies for each specimen were detected. Following these tests, the specimen was buckled, the load being applied quasistatically and the buckling loads were determined. The mathematical model was considered as a uniform beam which is restrained against translation and rotation at its two ends.

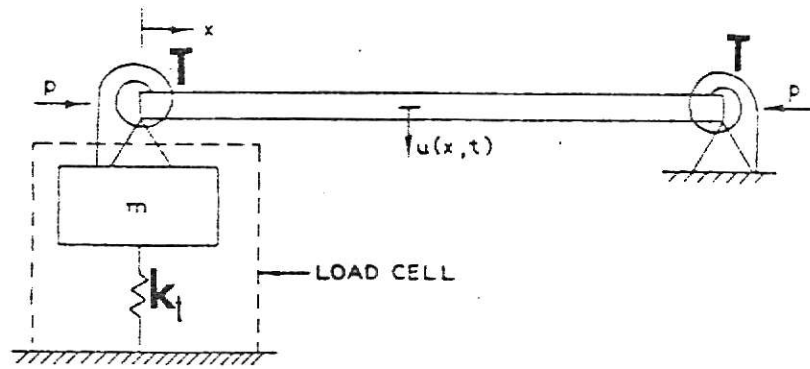


Fig. 6 Beam column model

The torsional restraint spring constants are denoted as T and the translational spring constant by K_t as shown in Fig. 4. Notice in the figure that the left end of the beam column is connected to a load cell which is treated as a concentrated mass m . By substituting T for K_k and K_ℓ , $K_t - m\omega^2$ for K_i (See appendix I) and using the displacements $\{V\}$ of the four coordinates as shown in Fig. 6, the Eq. (23) becomes

$$\frac{EI}{l} \begin{bmatrix} \gamma + \frac{(K_t - m\omega^2)l^3}{EI} & -\bar{\gamma} & -\beta & -\bar{\beta} \\ -\bar{\gamma} & \gamma + K_j & \bar{\beta} & \beta \\ -\beta & \bar{\beta} & \alpha + \frac{Tl}{EI} & \bar{\alpha} \\ -\bar{\beta} & \beta & \bar{\alpha} & \alpha + \frac{Tl}{EI} \end{bmatrix} \begin{bmatrix} V_i/l \\ V_j/l \\ V_k \\ V_\ell \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Since the displacement V_j is prevented by the support, the most compact form of stiffness equation is determined by deleting the column and row corresponding to this coordinate. That is

$$\frac{EI}{l} \begin{bmatrix} \gamma + \frac{(K_t - m\omega^2)l^3}{EI} & -\beta & -\bar{\beta} \\ -\beta & \alpha + \frac{Tl}{EI} & \bar{\alpha} \\ -\bar{\beta} & \bar{\alpha} & \alpha + \frac{Tl}{EI} \end{bmatrix} \begin{bmatrix} V_i/l \\ V_k \\ V_l \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (25)$$

For a nontrivial solution to exist, the determinant of the stiffness matrix should be zero. It yields the equation which determines the natural frequencies of the beam in Fig. 6. That is

$$\begin{vmatrix} \gamma + \frac{(K_t - m\omega^2)l^3}{EI} & -\beta & -\bar{\beta} \\ -\beta & \alpha + \frac{Tl}{EI} & \bar{\alpha} \\ -\bar{\beta} & \bar{\alpha} & \alpha + \frac{Tl}{EI} \end{vmatrix} = 0 \quad (26)$$

Similarly, by referring to Fig. 4, it is seen that the characteristic equation which governs the buckling is

$$\begin{vmatrix} \bar{a} + \frac{K_t l^3}{EI} & -\bar{b} & -\bar{b} \\ -\bar{b} & \bar{f} + \frac{Tl}{EI} & \bar{d} \\ -\bar{b} & \bar{d} & \bar{f} + \frac{Tl}{EI} \end{vmatrix} = 0 \quad (27)$$

Two independent equations for determining the restraint parameters

$(\frac{K_t l^3}{EI}, \frac{Tl}{EI})$ are obtained by substituting the two frequencies $\hat{\Omega}$ from Reference (5) and listed in TABLE I, into the characteristic Eq. (26). These restraint parameters were substituted into the characteristic Eq. (27) to compute the estimate of the buckling load P_m . The value P_m obtained by matrix approach is the same as the buckling load P_r predicted by classical

methods in Reference (5), which are listed in Table I. However, the matrices furnish the most convenient mathematical language for expressing the structure's character and provide a convenient means for carrying out numerical calculations on the computer.

specimen	$\hat{\Omega}_1 = \left(\frac{m\omega_1}{EI}\right)^{\frac{1}{4}}\ell$	$\hat{\Omega}_2 = \left(\frac{m\omega_2}{EI}\right)^{\frac{1}{4}}\ell$	$\frac{T\ell}{EI}$	$\frac{K_t\ell^3}{EI}$	$\hat{P}_m = \left(\frac{P_m}{EI}\right)^{\frac{1}{2}}\ell$	$\hat{P}_r = \left(\frac{P_r}{EI}\right)^{\frac{1}{2}}\ell$
1	4.326	7.076	9.275	883.82	5.253	5.253
2	4.362	7.110	10.106	883.52	5.351	5.351

Table I. Comparison of buckling loads predicted by matrix approach and classical method.

DISCUSSION OF OTHER END CONDITIONS

For a mathematical model as shown in Fig. 5, the natural frequencies of the beam are defined by the characteristic equation

$$\begin{vmatrix} \gamma + \frac{K_1 l^2}{EI} & -\bar{\gamma} & -\beta & -\bar{\beta} \\ -\bar{\gamma} & \gamma + \frac{K_j l^3}{EI} & \bar{\beta} & \beta \\ -\beta & \bar{\beta} & \alpha + \frac{K_k l}{EI} & \bar{\alpha} \\ -\bar{\beta} & \beta & \bar{\alpha} & \alpha + \frac{K_l l}{EI} \end{vmatrix} = 0$$

The value of the determinant may be written in the following form:

$$\begin{vmatrix} \gamma & -\bar{\gamma} & -\beta & -\bar{\beta} \\ -\bar{\gamma} & \gamma & \bar{\beta} & \beta \\ -\beta & \bar{\beta} & \alpha & \bar{\alpha} \\ -\bar{\beta} & \beta & \bar{\alpha} & \alpha \end{vmatrix} + \left(\frac{K_1 l^3}{EI} + \frac{K_j l^3}{EI} \right) \begin{vmatrix} \gamma & \bar{\beta} & \beta \\ \bar{\beta} & \alpha & \bar{\alpha} \\ \beta & \bar{\alpha} & \alpha \end{vmatrix} + \left(\frac{K_k l}{EI} + \frac{K_l l}{EI} \right) \begin{vmatrix} \gamma & -\bar{\gamma} & -\bar{\beta} \\ -\bar{\gamma} & \gamma & \beta \\ -\bar{\beta} & \beta & \alpha \end{vmatrix} \\ + \frac{K_1 l^3}{EI} \cdot \frac{K_j l^3}{EI} \begin{vmatrix} \alpha & \bar{\alpha} \\ \bar{\alpha} & \alpha \end{vmatrix} + \left(\frac{K_1 l^3}{EI} \cdot \frac{K_k l}{EI} + \frac{K_j l^3}{EI} \cdot \frac{K_l l}{EI} \right) \begin{vmatrix} \gamma & \beta \\ \beta & \alpha \end{vmatrix} \\ + \left(\frac{K_1 l^3}{EI} \cdot \frac{K_l l}{EI} + \frac{K_j l^3}{EI} \cdot \frac{K_k l}{EI} \right) \begin{vmatrix} \gamma & -\bar{\beta} \\ -\bar{\beta} & \alpha \end{vmatrix} + \frac{K_k l}{EI} \cdot \frac{K_l l}{EI} \begin{vmatrix} \gamma & -\bar{\gamma} \\ -\bar{\gamma} & \gamma \end{vmatrix} \\ + \left(\frac{K_1 l^3}{EI} \cdot \frac{K_j l^3}{EI} \cdot \frac{K_k l}{EI} + \frac{K_1 l^3}{EI} \cdot \frac{K_j l^3}{EI} \cdot \frac{K_l l}{EI} \right) |\alpha| + \left(\frac{K_1 l^3}{EI} \cdot \frac{K_k l}{EI} \cdot \frac{K_l l}{EI} \right. \\ \left. + \frac{K_j l^3}{EI} \cdot \frac{K_k l}{EI} \cdot \frac{K_l l}{EI} \right) |\gamma| + \frac{K_1 l^3}{EI} \cdot \frac{K_j l^3}{EI} \cdot \frac{K_k l}{EI} \cdot \frac{K_l l}{EI} = 0 \quad (28)$$

Similarly, the characteristic equation for buckling is

$$\begin{vmatrix} \bar{a} & -\bar{a} & -\bar{b} & -\bar{b} \\ -\bar{a} & \bar{a} & \bar{b} & \bar{b} \\ -\bar{b} & \bar{b} & \bar{f} & \bar{d} \\ -\bar{b} & \bar{b} & \bar{d} & \bar{f} \end{vmatrix} + \left(\frac{K_i l^3}{EI} + \frac{K_j l^3}{EI} \right) \begin{vmatrix} \bar{a} & \bar{b} & \bar{b} \\ \bar{b} & \bar{f} & \bar{d} \\ \bar{b} & \bar{d} & \bar{f} \end{vmatrix} + \left(\frac{K_k l}{EI} + \frac{K_l l}{EI} \right)$$

$$\begin{vmatrix} \bar{a} & -\bar{a} & -\bar{b} \\ -\bar{a} & \bar{a} & \bar{b} \\ -\bar{b} & \bar{b} & \bar{f} \end{vmatrix} + \frac{K_i l^3}{EI} \cdot \frac{K_j l^3}{EI} \begin{vmatrix} \bar{f} & \bar{d} \\ \bar{d} & \bar{f} \end{vmatrix} + \left(\frac{K_i l^3}{EI} \cdot \frac{K_k l}{EI} + \frac{K_j l^3}{EI} \cdot \frac{K_l l}{EI} \right) \begin{vmatrix} \bar{a} & \bar{b} \\ \bar{b} & \bar{f} \end{vmatrix}$$

$$+ \left(\frac{K_i l^3}{EI} \cdot \frac{K_l l}{EI} + \frac{K_j l^3}{EI} \cdot \frac{K_k l}{EI} \right) \begin{vmatrix} \bar{a} & \bar{b} \\ \bar{b} & \bar{f} \end{vmatrix} + \frac{K_k l}{EI} \cdot \frac{K_l l}{EI} \begin{vmatrix} \bar{a} & -\bar{a} \\ -\bar{a} & \bar{a} \end{vmatrix}$$

$$+ \left(\frac{K_i l^3}{EI} \cdot \frac{K_j l^3}{EI} \cdot \frac{K_k l}{EI} + \frac{K_i l^3}{EI} \cdot \frac{K_j l^3}{EI} \cdot \frac{K_l l}{EI} \right) \begin{vmatrix} \bar{a} \\ \bar{a} \end{vmatrix} + \left(\frac{K_i l^3}{EI} \cdot \frac{K_k l}{EI} \cdot \frac{K_l l}{EI} \right.$$

$$\left. + \frac{K_j l^3}{EI} \cdot \frac{K_k l}{EI} \cdot \frac{K_l l}{EI} \right) \begin{vmatrix} \bar{f} \\ \bar{f} \end{vmatrix} + \frac{K_i l^3}{EI} \cdot \frac{K_j l^3}{EI} \cdot \frac{K_k l}{EI} \cdot \frac{K_l l}{EI} = 0 \quad (29)$$

Apparently, there is no easy way to evaluate the four restraint constants

$\left(\frac{K_i l^3}{EI}, \frac{K_j l^3}{EI}, \frac{K_k l}{EI}, \frac{K_l l}{EI} \right)$ by substituting four natural frequencies of different modes into the nonlinear Eq. (28). However, it is seen obviously

that the nine parameters $\left(\frac{K_i l^3}{EI} + \frac{K_j l^3}{EI}, \frac{K_k l}{EI} + \frac{K_l l}{EI}, \frac{K_i l^3}{EI} \cdot \frac{K_j l^3}{EI}, \frac{K_i l^3}{EI} \cdot \frac{K_k l}{EI} + \frac{K_j l^3}{EI} \cdot \frac{K_l l}{EI}, \frac{K_k l}{EI} \cdot \frac{K_l l}{EI}, \frac{K_i l^3}{EI} \cdot \frac{K_l l}{EI} + \frac{K_j l^3}{EI} \cdot \frac{K_k l}{EI}, \frac{K_i l^3}{EI} \cdot \frac{K_k l}{EI} \cdot \frac{K_l l}{EI}, \frac{K_j l^3}{EI} \cdot \frac{K_k l}{EI} \cdot \frac{K_l l}{EI}, \frac{K_i l^3}{EI} \cdot \frac{K_j l^3}{EI} \cdot \frac{K_k l}{EI} \cdot \frac{K_l l}{EI} \right)$

$$+ \frac{K_i \ell^3}{EI} \cdot \frac{K_j \ell^3}{EI} \cdot \frac{K_\ell \ell}{EI}, \frac{K_i \ell^3}{EI} \cdot \frac{K_k \ell}{EI} \cdot \frac{K_\ell \ell}{EI} + \frac{K_j \ell^3}{EI} \cdot \frac{K_k \ell}{EI} \cdot \frac{K_\ell \ell}{EI}, \frac{K_i \ell^3}{EI} \cdot \frac{K_j \ell^3}{EI} \cdot \frac{K_k \ell}{EI} \cdot \frac{K_\ell \ell}{EI})$$

can be identified easily by substituting nine frequencies of different modes into Eq. (28) and the buckling load can be predicted by substituting these nine parameters into the Eq. (29). It is evident that the number of natural frequencies needed for predicting the buckling load depends on the type of mathematical model. In TABLE II, some types of mathematical model and the number of its natural frequencies needed for predicting buckling load are listed.

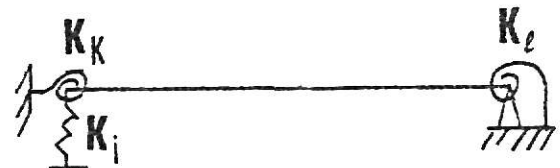
mathematical model	Natural frequencies for predicting buckling load
	9
	6
	6

TABLE II. Natural frequencies for predicting buckling load.

CONCLUSIONS

The procedures for determining the buckling load by using vibratory data in this paper and in papers (3), (4), (5) are the same except the derivation of the characteristic equation which governs the frequencies and the characteristic equation which governs the buckling. However, the matrix approach presented in this paper avoids a large amount of computations. Furthermore, it offers a general approach and provides a convenient means for carrying out numerical calculations on the computer.

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APPENDIX I

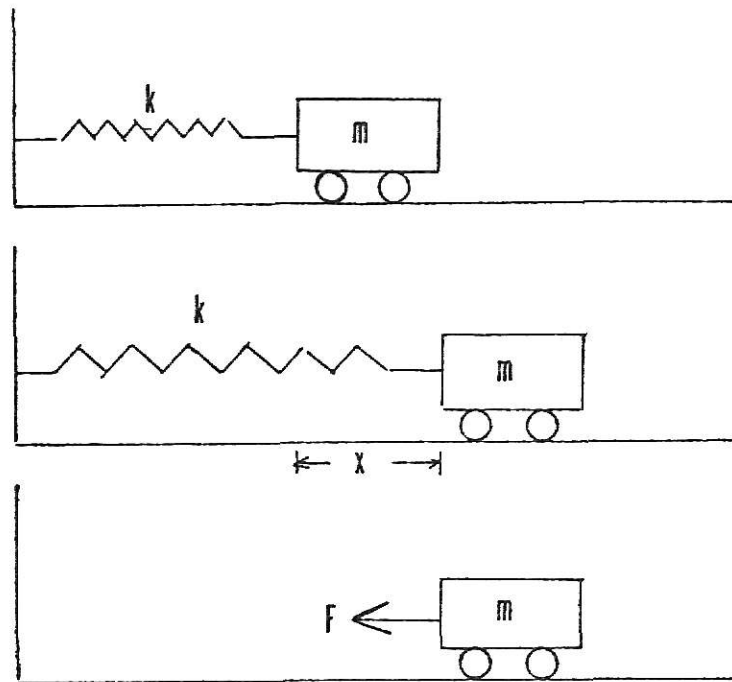


Fig. A (a) balance condition
 (b) rigid body moves a distance x
 (c) free body diagram of the rigid body

Consider a rigid body which is connected to a weightless spring. Denote the mass of this body by m and the spring constant by K as shown in Fig. A. The free vibration of the body can be expressed by equation

$$Kx + m\ddot{x} = 0 \quad (1)$$

The solution of Eq. (1) can be written as

$$\begin{aligned} x(t) &= A \sin \omega t + B \cos \omega t \\ &= Q \cos(\omega t - \theta) \end{aligned} \quad (2)$$

where $\omega = \sqrt{\frac{K}{m}}$ and θ is the phase of angle.

The force in the spring is

$$F = -Kx \quad (3)$$

Substituting Eq. (3) and (2) into Eq. (1) gives

$$-F = -m\ddot{x}$$

$$F = m\ddot{x} = -m\omega^2 Q \cos(\omega t - \theta) = -m\omega^2 x$$

$$\text{or } \frac{F}{x} = -m\omega^2 \quad (4)$$

From Eq. (4), it is seen that the dynamic stiffness of the mass m due to free vibration is $-m\omega^2$.

APPENDIX II NOTATION

E = Young's modulus	V_b = displacement
F = force	V_i = vertical displacement
I = moment of inertia	V_j = vertical displacement
K = stiffness	V_k = rotation
M = moment	V_ℓ = rotation
P = axial force	S_i = shear force
V = shear force	S_j = shear force
$a = (m\omega^2/EI)^{1/2}$	S_k = bending moment
$c = \cos(a\ell)$	S_ℓ = bending moment
$d = 1 - c\bar{c}$	P_m = buckling load predicted by matrix method
$g = (P/EI)^{1/2}$	P_r = buckling load predicted by classical method in reference (5)
ℓ = length of the beam	$\bar{a} = u^3 s' / (2 - 2c' - us')$
m = mass per unit length; mass	$\bar{b} = u^2 (1 - c') / (2 - 2c' - us')$
$s = \sin(a\ell)$	$\bar{c} = \cos h(a\ell)$
t = time	$\bar{d} = u(u - s') / (2 - 2c' - us')$
$u = g\ell$	$\bar{f} = u(s' - uc') / (2 - 2c' - us')$
$\alpha = (s\bar{c} - c\bar{s})a\ell/d$	$\bar{s} = \sin h(a\ell)$
$\beta = s\bar{s}(a\ell)^2/d$	$\bar{\alpha} = (\bar{s} - s)a\ell/d$
$\gamma = (s\bar{c} - c\bar{s})(a\ell)^3/d$	$\bar{\beta} = (\bar{c} - c)(a\ell)^2/d$
ω = angular frequency	$\bar{\gamma} = (\bar{s} + s)(a\ell)^3/d$
$\delta = (a^4 + g^4/4)^{1/2} + g^2/2$	
$\epsilon = (a^4 + g^4/4)^{1/2} - g^2/2$	
K_b = stiffness	
F_b = force	

$$c' = \cos u$$

$$s' = \sin u$$

$$T = \text{rotational spring constant}$$

$$K_i, K_j, K_k, K_\ell = \text{spring constants}$$

$$\delta V_b = \text{small displacement}$$

$$\omega_1 = \text{the first angular frequency}$$

$$\omega_2 = \text{the second angular frequency}$$

$$\hat{\Omega}_1 = (m\omega_1^2/EI)^{1/4} \ell$$

$$\hat{\Omega}_2 = (m\omega_2^2/EI)^{1/4} \ell$$

$$\hat{P}_m = (P_m/EI)^{1/2} \ell$$

$$\hat{P}_r = (P_r/EI)^{1/2} \ell$$

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Predicting Buckling Load
Using Vibratory Data

by

CHEER GERM GO

B.C.E., Tamkang College, Taiwan, 1974

AN ABSTRACT OF A MASTER'S REPORT

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Department of Civil Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

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ABSTRACT

Given a particular structure, simple experiments can be devised to determine its natural frequencies. These frequencies are known to be a function of the structure's support boundary conditions. Similarly, it is known that the critical buckling load is also a function of the structure's support boundary conditions. The procedures for determining the buckling load by using vibratory data in this paper and in papers (3), (4), (5) are the same except for the derivation of the characteristic equation which governs the frequencies and the characteristic equation which governs the buckling. However, the matrix approach presented in this report avoids a large amount of computations. Furthermore, it offers a general approach and provides a convenient means for carrying out numerical calculations on the computer.