Multiparameter Hall-Littlewood vertex operators

by

Gabriel Necoechea

B.A., Roberts Wesleyan College, 2016

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas

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Abstract

We introduce a multiparameter generalization of Hall-Littlewood functions through the action of vertex operators. Properties of these operators, including commutation relations expressed through normal ordered products of quantum fields, are studied. We recover many important properties of the Schur functions, Schur Q-functions, and Hall-Littlewood functions at roots of unity, and we obtain analogues of some of these properties in the multiparameter case. We introduce a deformation of the KP hierarchy and prove that its linear part is integrable. In the process we show that the linear part of the deformed KP hierarchy coincides with the linear part of the usual KP hierarchy up to substitution of variables. When the vertex operators correspond to the Hall-Littlewood functions at a root of unity, we obtain a factorization of the vertex operators and compute commutation relations for the factors. The factorization suggests a possible generalization of the BKP hierarchy.

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Approved by:

Major Professor
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Chapter 1

Introduction

In this chapter, we introduce the symmetric functions that will be the building blocks and role models for our work in future chapters. We start by introducing the partitions that index symmetric functions. Then we define what a symmetric function is and discuss the algebraic structures on the symmetric functions. We pay particular attention to three classical families of symmetric functions, since these will be used later to build our multiparameter vertex operators. After introducing these classical families, we introduce three role-models: the Schur functions, the Schur Q-functions, and the Hall-Littlewood functions. Our discussion of Hall-Littlewood functions will include a special emphasis on the Schur Q-functions (the $t = -1$ case).

1.1 Preliminaries on Relevant Combinatorial Objects

Many of our objects of study are naturally indexed by partitions. We briefly recall concepts and terminology related to partitions and refer the reader to [17] for more details. A partition is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, ...)$ of non-negative integers, subject to the condition that only finitely many λ_i are nonzero. The λ_i which are nonzero are called *parts*, and the number of parts is called the *length* of the partition, denoted $\ell(\lambda)$. A partition may also be specified as $\lambda = (1^{m_1(\lambda)} 2^{m_2(\lambda)} \cdots)$, where $m_j(\lambda)$ is the number of parts of λ equal to j. The sum of the parts is denoted by $|\lambda|$. If $|\lambda| = d$, we write $\lambda \vdash d$ or $\lambda \in \mathcal{P}(d)$. We denote by $\mathcal{P}$ the set of all partitions: $\mathcal{P} = \bigcup_{d=0}^{\infty} \mathcal{P}(d)$. There exists a unique partition $\varnothing \in \mathcal{P}(0)$, which we shall call the empty partition.

A partition may be visualized through its *Young diagram*. Our convention is that a Young diagram is an arrangement of boxes in left-justified rows, such that the number of boxes in each row weakly decreases as we go from the top row to the bottom row. The Young diagram of the partition $\lambda = (5, 4, 3, 3, 1)$ is displayed in Figure 1.

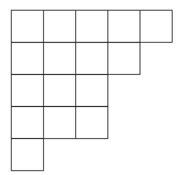


Figure 1.1: *The Young diagram associated to* $\lambda = (5, 4, 3, 3, 1)$

The Young diagram motivates a third symbolic description of a partition. The *code* of a partition is a bi-infinite string in letters 'U' and 'R' that is infinitely U to the left, infinitely R to the right, and of finite length in between. Any letter inclusively between the left-most R and the the first R in the infinite R-block is a *non-trivial* letter. A non-trivial letter can be thought of as labeling an Up Step ('U') or a Right Step ('R'), beginning from the bottom left corner of the Young diagram, so that reading the code from the left-most R to the right-most U traces out the eastern boundary of the Young diagram. For example, the code of $\lambda = (4, 3, 3, 1)$ is

$$\text{code}(\lambda) = \cdots URURRUURUR \cdots .$$

If λ is a partition, then its conjugate partition λ' is the partition whose Young diagram is obtained by reflecting λ's diagram about the main diagonal (in the usual matrix sense). More abstractly, the conjugate partition λ' may defined by the condition that the number of parts of λ' which are $\geq k$ is equal to λ_k for all $k \geq 1$.

A *Semistandard Young Tableaux* of shape $\lambda \vdash n$ is a filling of the boxes of the Young diagram of λ with the numbers $\{1, ..., n\}$ such that the numbers weakly increase along rows and strictly along columns. The collection of all semistandard Young tableaux of shape λ will be denoted by $\mathrm{SSYT}(\lambda)$. A *Standard Young Tableaux* is a semistandard Young tableaux which is strictly increasing along rows as well. The conditions of strictly increasing along both rows and columns have a nice interpretation: a standard Young tableaux of shape λ is a viable order in which one could add boxes one by one, beginning with the empty partition, such that the diagram that results at each step is the Young diagram of a partition. See Figure 2 for visual reference. The collection of all standard Young tableaux of shape λ will be denoted by $\mathrm{SYT}(\lambda)$.

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad\qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$$

Figure 1.2: *The standard Young tableaux for* $\lambda = (2, 1)$

There are a number of commonly used partial orders on partitions. One of these partial orders ($\leq_Y$) says that $\lambda \leq_Y \mu$ if and only if $\lambda_i \leq \mu_i$ for all i. If $\lambda \leq_Y \mu$, then the boxes which are in the Young diagram of μ but not in the Young diagram of λ form a shape, called a *skew diagram* or *skew shape* and denoted by $\mu \setminus \lambda$. A skew shape is a *vertical strip* if it does not contain more than one box in any single row. A skew shape is a *horizontal strip* if it does not contain more than one box in any single column. An example of the skew shape $(5, 4, 3) \setminus (3, 2, 2)$ is displayed in Figure 3.

$$\begin{array}{|c|c|c|c|c|}\hline & & & * & * \\ \hline & & * & * \\ \cline{1-4} & & * \\ \cline{1-3}\end{array}$$

Figure 1.3: *The diagram of* $(5, 4, 3)$*. The boxes in the skew shape* $(5, 4, 3) \setminus (3, 2, 2)$ *are labeled by* $*$*. Note this skew shape is neither a vertical nor a horizontal strip.*

In subsequent sections, we will freely identify a partition with its Young diagram.

A finite sequence α of non-negative integers that is not necessarily non-increasing will be called a *composition*. Whenever such usage is well-defined, we freely use partition terminology (e.g. "parts", "length", $|\alpha|$, etc.) when talking about compositions as well. For any composition α, there exists a (not necessarily unique) permutation $\tau \in S_{\ell(\alpha)}$ such that $\tau\alpha$ is a partition. This partition is unique, and we refer to it by $\mathrm{sort}(\alpha)$.

1.2 Families of Symmetric Functions

The symmetric group S_n acts on $\mathbb{C}[x_1, ..., x_n]$ by permuting the variables. The polynomials that are invariant under this action are called *symmetric polynomials.* The symmetric polynomials in n variables form an algebra Λ_n that is graded by the usual polynomial degree: $\Lambda_n = \bigoplus_{d=0}^{\infty} \Lambda_n^d$.

For $k \geq 1$, let

$$p_k = p_k(x_1, ..., x_n) = \sum_{i=1}^{n} x_i^k,$$

$$e_k = e_k(x_1, ..., x_n) = \sum_{1 \leq i_1 < i_2 < \ldots i_k \leq n} x_{i_1} \ldots x_{i_k},$$

and

$$h_k = h_k(x_1, ..., x_n) = \sum_{1 \leq i_1 \leq i_2 \leq \ldots i_k \leq n} x_{i_1} \ldots x_{i_k}.$$

Define also $e_0 = h_0 = 1$ and let $h_k = e_k = 0$ for all $k < 0$. Note that $e_k = 0$ if $k > n$. The p_k are called *power sum symmetric polynomials*; h_k are called *complete homogeneous symmetric polynomials*; and the e_k are called *elementary symmetric polynomials.*

For α a composition, let $e_\alpha = e_{\alpha_1} \ldots e_{\alpha_\ell}$. Define similarly p_α and h_α. Since multiplication in $\mathbb{C}[x_1, ..., x_n]$ is commutative, $e_{\mathrm{sort}(\alpha)} = e_\alpha$, $h_{\mathrm{sort}(\alpha)} = h_\alpha$, and $p_{\mathrm{sort}(\alpha)} = p_\alpha$. In fact, the following theorem holds (cf. [17], I.2).

Theorem 1.2.1. *The following are all bases of Λ_n^d:*

$$\{h_\lambda | \lambda \in \mathcal{P}(d), \lambda_1 \leq n\},$$

$$\{e_\lambda | \lambda \in \mathcal{P}(d), \lambda_1 \leq n\},$$

$$\{p_\lambda | \lambda \in \mathcal{P}(d), \lambda_1 \leq n\}.$$

Whenever a finite group G acts on some vector space of polynomials, a standard method of constructing G-invariant polynomials is to symmetrize, i.e. to sum over the G-orbit of a polynomial q, rescaling by the size of the stabilizer of q. Along these lines, define

$$m_\lambda = \sum_{\mathrm{sort}(\alpha)=\lambda} x^\alpha. \tag{1.1}$$

Then

$$\{m_\lambda | \lambda \in \mathcal{P}(d), \ell(\lambda) \leq n\}$$

is another basis of Λ_n^d. The m_λ are called *monomial symmetric polynomials.* Note that for each $r \geq 1$,

$$p_r = m_{(r)}, \qquad e_r = m_{(1^r)}. \tag{1.2}$$

The generating functions

$$H(u) = \sum_{k \geq 0} h_k u^k = \prod_{i=1}^n \frac{1}{1 - x_i u},$$

$$E(u) = \sum_{k \geq 0} e_k u^k = \prod_{i=1}^n (1 + x_i u),$$

and

$$P(u) = \sum_{k \geq 1} p_k u^{k-1}$$

satisfy the following relations (cf. [17], I.2, Eqns. 2.6, 2.10, 2.11):

$$H(u)E(-u) = 1, \tag{1.3}$$

$$P(u) = \frac{d}{du} \log H(u), \tag{1.4}$$

$$P(-u) = \frac{d}{du} \log E(u). \tag{1.5}$$

For our purposes, the number n of variables is at times irrelevant. Thus we work in the limit Λ of a particular inverse system (of graded rings) of the Λ_n. The elements in this inverse limit can be thought of as sequences of symmetric polynomials $\left\{f^{(j)} | f^{(j)} \in \Lambda_j\right\}_{j=1}^{\infty}$ satisfying the following stability condition:

$$f^{(j)}(x_1, ..., x_{j-1}, 0) = f^{(j-1)}(x_1, ..., x_{j-1}) \qquad \text{for all } j \geq 2.$$

The following theorems (cf. [17], I.2, pp. 18-19) allow us to work in this inverse limit.

Theorem 1.2.2. *Whenever* $\ell(\lambda) \leq n$*,*

$$e_\lambda(x_1, ..., x_{n-1}, 0) = e_\lambda(x_1, ..., x_{n-1})$$

$$h_\lambda(x_1, ..., x_{n-1}, 0) = h_\lambda(x_1, ..., x_{n-1})$$

$$p_\lambda(x_1, ..., x_{n-1}, 0) = p_\lambda(x_1, ..., x_{n-1})$$

Theorem 1.2.3. *The graded rings* $\{\Lambda_n\}_{n \in \mathbb{N}}$ *together with the natural projection maps*

$$\Lambda_m \to \Lambda_n$$

($m \geq n$) form an inverse system of graded rings. The inverse limit Λ *exists, and as a* $\mathbb{C}$*-algebra,*

$$\Lambda = \mathbb{C}[p_1, p_2, ...] = \mathbb{C}[e_1, e_2, ...] = \mathbb{C}[h_1, h_2, ...].$$

The elements of Λ are referred to as *symmetric functions.* The various generators ($e_k, p_k, etc.$) are referred to by their usual names, with “polynomials” replaced by the word “functions.” The generating functions $H(u)$, $E(u)$, and $P(u)$ still make sense with coefficients in Λ. The finite products are replaced with infinite products.

The vector space Λ is endowed with a bilinear symmetric form $\langle \ , \ \rangle : \Lambda \otimes \Lambda \to \mathbb{C}$ defined by the condition that

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda,\mu}.$$

This form is called the Hall inner product. Two bases $\{f_\lambda\}$ and $\{g_\lambda\}$ of Λ are dual with respect to this form if and only if they satisfy the *Cauchy identity*:

$$\prod_{i,j=1}^{\infty} \frac{1}{1 - x_i y_j} = \sum_{\lambda} f_\lambda(x) g_\lambda(y).$$

It is well-known that

$$\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda,\mu}, \tag{1.6}$$

where $z_\lambda = \prod_{j \geq 1} j^{m_j(\lambda)} m_j(\lambda)!$ and $m_j(\lambda)$ is the number of parts of λ equal to j. Consequently, for each $k \geq 1$, the multiplication operator p_k and the differential operator $k\frac{\partial}{\partial p_k}$ on $\Lambda = \mathbb{C}[p_1, p_2, ...]$ are adjoint operators with respect to the Hall inner product. Moreover, these operators realize Λ as a module of the Heisenberg algebra $\mathcal{H}$. $\mathcal{H}$ is an infinite-dimensional algebra with generators

$$\{B_k | k \in \mathbb{Z}_{\neq 0}\} \cup \{c\}$$

and relations

$$[B_k, B_\ell] = \ell \delta_{k,-\ell} \cdot c,$$

$$[B_k, c] = 0.$$

The $\mathcal{H}$-module structure on Λ is given by the assignment

$$c \mapsto 1,$$

$$B_k \mapsto k \frac{\partial}{\partial p_k},$$

$$B_{-k} \mapsto p_k,$$

for all $k \in \mathbb{N}$. In this way, Λ is an irreducible highest weight module of $\mathcal{H}$. The transport of various structures between Λ and other realizations of this canonical module are the historical inspiration for the present thesis. Because of this, we shall work primarily with the exponential forms of $H(u)$ and $E(u)$:

$$H(u) = \exp\left(\sum_{k\geq 1} \frac{p_k}{k} u^k\right),$$

and

$$E(u) = \exp\left(-\sum_{k\geq 1} \frac{(-1)^k p_k}{k} u^k\right).$$

For any $f \in \Lambda$, we get a multiplication operator on Λ. We denote by $f^\perp$ the operator adjoint to this multiplication operator. If we define $H^\perp(u) = \sum_{k\geq 0} h_k^\perp u^{-k}$ and $E^\perp(u) = \sum_{k\geq 0} e_k^\perp u^{-k}$, then

$$H^\perp(u) = \exp\left(\sum_{k\geq 1} \frac{\partial}{\partial p_k} u^{-k}\right),$$

and

$$E^\perp(u) = \exp\left(-\sum_{k\geq 1} (-1)^k \frac{\partial}{\partial p_k} u^{-k}\right).$$

1.3 Schur Functions

In addition to $h_\lambda, e_\lambda, p_\lambda$, and m_λ, there is another basis of great importance in Λ, namely the *Schur functions* s_λ. The purpose of this section is to recall several of the definitions, properties, and applications of Schur functions that will be relevant in what follows. For more details, we refer the reader to [17].

For a finite number n of variables, the Schur polynomial $s_\lambda(x_1, .., x_n)$ associated to a partition λ (with $\ell(\lambda) \leq n$) is defined by the formula

$$s_\lambda = \frac{\det\left(x_i^{\lambda_j+n-j}\right)_{1\leq i,j\leq n}}{\det\left(x_i^{n-j}\right)_{1\leq i,j\leq n}}.$$

This formula is equivalent to the sum

$$s_\lambda = \sum_{\sigma\in S_n} x_{\sigma(1)}^{\lambda_1} \cdots x_{\sigma(n)}^{\lambda_n} \prod_{1\leq i<j\leq n} \frac{x_{\sigma(i)}}{x_{\sigma(i)} - x_{\sigma(j)}}.$$

The one row and one column Schur functions coincide with the complete homogeneous and elementary symmetric functions, respectively:

$$s_{(r)} = h_r, \qquad s_{(1^r)} = e_r, \tag{1.7}$$

and in fact the expansion of any s_λ polynomial in terms of the h_k is known and given by the *Jacobi-Trudi* formula:

$$s_\lambda = \det\left(h_{\lambda_i+j-i}\right)_{1\le i,j\le \ell(\lambda)}. \tag{1.8}$$

It is not uncommon to take the Jacobi-Trudi formula as the definition of s_λ. There is an equivalent formulation in terms of elementary symmetric functions:

$$s_\lambda = \det(e_{\lambda'_i+j-i})_{1\le i,j\le \ell(\lambda')},$$

where λ' is the *conjugate partition* of λ.

The expansion of s_λ in power sums is also worth recalling. First recall that

$$H(u) = \exp\left(\sum_{k\ge 1}\frac{p_k}{k}u^k\right),$$

so that

$$h_m = \sum_{(1^{k_1}2^{k_2}\cdots m^{k_m})\vdash m}\frac{(x_1/1)^{k_1}}{k_1!}\frac{(x_2/2)^{k_2}}{k_2!}\cdots\frac{(x_m/m)^{k_m}}{k_m!} = \sum_{\mu\vdash m} z_\mu^{-1}p_\mu.$$

This formula generalizes to s_λ as

$$s_\lambda = \sum_{\mu\vdash|\lambda|}\frac{\chi_\lambda(\mu)p_\mu}{z_\mu},$$

where z_μ is as in Equation (1.6), χ_λ is the character of the irreducible representation of $S_{|\lambda|}$ indexed by λ, and $\chi_\lambda(\mu)$ is the evaluation of this character on the conjugacy class in S_ℓ corresponding to μ.

With respect to the Hall inner product, the Schur functions form an orthonormal basis

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}.$$

In fact, Λ is a positive self-dual Hopf algebra with coproduct Δ defined via

$$\Delta(m_\lambda) = \sum_{\lambda=\mathrm{sort}(\mu\cup\nu)} m_\mu \otimes m_\nu, \tag{1.9}$$

counit ϵ extended from the following rule on homogeneous components of Λ:

$$\epsilon(f) = \begin{cases} f & \deg(f) = 0, \\ 0 & \deg(f) > 0, \end{cases}$$

and antipode S defined by

$$S(f) = (-1)^d \omega(f) \qquad \text{for all } f \in \Lambda^d,$$

where ω is the involution of Λ that interchanges e_k and h_k for each $k \geq 1$. For an exposition of this Hopf algebra structure, we refer to the lecture notes [9].

1.4 Symmetric Functions with Parameter t

In this section we consider the algebra $\Lambda_t = \mathbb{C}(t) \otimes_{\mathbb{C}} \mathbb{C}[p_1, p_2, ...]$ of symmetric functions with a deformation parameter t. For more details, we again refer the reader to [17]. To begin, we recall that for any partition λ of length $\ell(\lambda) \leq n$, the Hall-Littlewood P polynomial in n variables is defined by

$$P_\lambda(x_1, ..., x_n; t) = \prod_{i\geq 0} \frac{1}{[m_i(\lambda)]_t!} \sum_{\sigma\in S_n} \left(x_{\sigma(1)}^{\lambda_1} \cdots x_{\sigma(n)}^{\lambda_n} \prod_{i<j} \frac{x_{\sigma(i)} - t x_{\sigma(j)}}{x_{\sigma(i)} - x_{\sigma(j)}} \right),$$

where

$$[m]_t! = \begin{cases} \prod_{j=1}^{m} \frac{1-t^j}{1-t} & t \neq 1 \\ m! & t = 1 \end{cases}$$

is the "t-analog" of factorial (extended to $t = 1$). When $t = 1$, the constant out front is the size of the stabilizer of x^λ under the S_n action on $\mathbb{C}[x_1, ..., x_n]$, so $P_\lambda(x; 1) = m_\lambda$. When $t = 0$, the constant is just a 1, and $P_\lambda(x; 0) = s_\lambda$. From the definition above, it follows that the one-row Hall-Littlewood polynomials are

$$P_{(r)}(x;t) = \frac{1}{[n-1]_t!} \sum_{i=1}^{n} x_i^r \prod_{j \neq i} \frac{x_i - tx_j}{x_i - x_j} \tag{1.10}$$

for any $r \geq 1$.

The Hall-Littlewood polynomials are stable:

Theorem 1.4.1. *(**[17], III.2, Thm. 2.5**) If $\ell(\lambda) \leq n-1$, then*

$$P_\lambda(x_1, ..., x_{n-1}, 0; t) = P_\lambda(x_1, ..., x_{n-1}; t).$$

There is a "t-deformed" version of the Hall inner product, with respect to which two bases $\{g_\lambda(x;t)\}$ and $\{f_\lambda(x;t)\}$ of symmetric functions with parameter are dual to each other if and only if they satisfy a t-deformed Cauchy identity:

$$\prod_{i,j} \frac{1 - tx_iy_j}{1 - x_iy_j} = \sum_\lambda f_\lambda(x;t) g_\lambda(y;t).$$

The P_λ are not generally self-dual. To this end, the Hall-Littlewood Q-polynomials in n variables are defined by

$$Q_\lambda(x;t) = b_\lambda(t) P_\lambda(x;t),$$

where

$$b_\lambda(t) = \prod_{i \geq 1} \prod_{j=0}^{m_i(\lambda)} (1 - t^j).$$

Note that $Q_\lambda(x;0) = s_\lambda$. The $t = -1$ case is the Schur Q-functions. Note also that $Q_\lambda(x;1) = 0$ unless λ is the empty partition. The $Q_\lambda(x;t)$ are also stable polynomials. The Hall-Littlewood P and Q functions satisfy the t-deformed "Cauchy identity":

$$\prod_{i,j} \frac{1 - tx_iy_j}{1 - x_iy_j} = \sum_\lambda P_\lambda(x;t)Q_\lambda(y;t).$$

At times we will refer to both $P_\lambda(x;t)$ and $Q_\lambda(x;t)$ as Hall-Littlewood functions.

In addition to the $t = 0$ and $t = 1$ cases, a number of other specializations of t are of interest.

1.4.1 Hall-Littlewood Functions at t = -1: Schur Q-functions

When $t = -1$, P_λ and Q_λ are the Schur P- and Q-functions, respectively. If λ has repeated parts, then $P_\lambda = Q_\lambda = 0$. The Schur P- and Q-functions were first utilized by Schur in his study of the projective representations of symmetric groups. A complex *projective representation* of a group G is a group homomorphism

$$G \to GL(V)/\mathbb{C}^\times,$$

where V is a vector space over $\mathbb{C}$, $GL(V)$ is the group of invertible linear transformations of V, and $\mathbb{C}^\times$ is the multiplicative group of nonzero complex numbers, thought of as a normal subgroup of $GL(V)$. Schur determined (cf. [22], [23], [24]) that when G is a finite group, the study of such homomorphisms "reduces" to the study of linear representations of certain central extensions of G. These central extensions are called *representation groups.* For general G, there is not a unique representation group. Indeed, for $G = S_n$, there are two non-isomorphic representation groups. However, the linear representations of one can be understood if the linear representations of the other are known, so that one need only study one such group–say $\widetilde{S_n}$, which has presentation given by the generators

$$z, \sigma_1, ..., \sigma_{n-1}$$

and the relations

$$\sigma_j^2 = z = (\sigma_j\sigma_{j+1})^3, \qquad 1 \le j \le n-1,$$

$$(\sigma_j\sigma_k)^2 = z, \qquad |j-k| > 2,$$

$$z^2 = 1 = z^{-1}\sigma_j^{-1}z\sigma_j, \qquad 1 \le j \le n-1.$$

An irreducible representation of $\widetilde{S_n}$ in which z acts trivially is essentially a linear representation of S_n. The remaining irreducible representations, in which z acts by -1, are called *spin representations*. As the characters of the symmetric group are related to the Schur functions, so the spin characters are related to the Schur Q-functions. That is, for λ a partition with no repeated parts,

$$Q_\lambda(x;t) = \sum_\mu \frac{2^{(\ell(\lambda)-\ell(\mu)+\epsilon(\lambda))/2}\widetilde{\chi}_\lambda(\mu)}{\widetilde{z}_\mu} p_\mu,$$

where the sum is over all $\mu \vdash |\lambda|$ with only odd nonzero parts, $\epsilon(\lambda) = 1$ if $n - \ell(\lambda)$ is odd and is zero otherwise, $\widetilde{\chi}_\lambda(\mu)$ is evaluation of irreducible spin character associated to λ on the conjugacy class associated to μ, and

$$\widetilde{z}_\mu = \prod_{j\ge1} \frac{j^{m_j(\lambda)}m_j(\lambda)!}{2}$$

Thanks to this perspective, several aspects of the theory of Schur P- and Q-functions have developed in analogy with the theory of Schur functions.

This is not the only analogy that exists between Schur functions and Schur Q-functions, however. Stembridge ([26]) mentions that there exist least four "natural" contexts in which Schur Q-functions appear. Each of these contexts has an analogue in the world of Schur functions. First, as stated in the previous paragraph, the Schur Q-functions encode characters for projective representations of S_n while the Schur functions encode characters for linear representations of S_n. The monograph [10] gives a detailed account of this projective representation theory. Second, it was shown in [25] that the Schur Q-functions on n variables are, up to scalar multiples, characters for the irreducible tensor representations of the queer Lie superalgebra $\mathfrak{q}(n)$. This is an analogue to the finding that Schur functions are characters for the finite-dimensional irreducible representations of the Lie algebra $\mathfrak{gl}(n)$. Moreover, it is the foundation of a "super-analogue" to the classical Schur-Weyl duality; see [2] for further discussion. Third ([20]), the cohomology of the isotropic Grassmannian Sp_{2n}/U_n is a quotient of the ring generated by Schur Q-functions, and the Schur Q-functions can be identified with the Schubert cycles. This is analogous to the classical identification of Schubert cycles of ordinary Grassmannians with Schur functions. Finally, the Schur functions are special solutions to a famous integrable hierarchy known as the KP hierarchy ([21]), while the Schur Q-functions are special solutions to another famous integrable hierarchy, the BKP hierarchy ([27]). It is this last analogy that motivates the latter portion of the thesis.

1.4.2 Hall-Littlewood Functions at t a primitive r-th root of unity

(cf. [17], III.7, Exercise 7 and [18]) Observe that if $t = \omega$ is a primitive r-th root of unity, then $b_\lambda(\omega) = \prod_{i \geq 1} \prod_{j=0}^{m_i(\lambda)} (1 - \omega^j) = 0$ whenever there exists some i such that $m_i(\lambda) \geq r$. This leads to the following lemma.

Lemma 1.4.2. *Let ω be a primitive r-th root of unity. Then*

$$Q_\lambda(x; \omega) = 0$$

if there exists a positive integer i such that $m_i(\lambda) \geq r$.

Moreover,

Theorem 1.4.3. *([18]) The $\mathbb{Q}(\omega)$-algebra $A_{(r)}$ generated by the $Q_\lambda(x; \omega)$ is isomorphic to the $\mathbb{Q}(\omega)$-algebra $B_{(r)}$ generated by $\{p_k | k \not\equiv 0 \bmod r\}$.*

Remark 1.4.4. Note that these results cover the case of Schur Q-functions (set $r = 2$). We can recover these results using the methods of Chapter 2.

Chapter 2

Multiparameter Vertex Operators

In this chapter, we introduce multiparameter vertex operators. To begin, we will recall the generating function of the Hall-Littlewood functions. Then we will reinterpret this in the language of the Hall-Littlewood vertex operators introduced in [12]. Our multiparameter vertex operators generalize the Hall-Littlewood vertex operators, and we study the symmetric functions they generate. We also study the normal order relations and Clifford-like relations of these formal distributions.

2.1 Generating Formal Distributions of Symmetric Functions

In this section, we generalize the Hall-Littlewood functions. The key to this generalization is to work on the level of generating functions. The following generating function can be found in [17]:

Theorem 2.1.1. *(**[17], III.2, Eqn (2.15)**) Let*

$$F(u) = \prod_i \frac{1 - tx_i u}{1 - x_i u} = H(u)E(-tu),$$

$$F(u_1, ..., u_\ell) = \prod_{1 \le i < j \le \ell} \frac{1 - (u_j/u_i)}{1 - t(u_j/u_i)} \prod_i F(u_i),$$

and denote by $[u^\lambda]F(u_1, ..., u_\ell)$ *the coefficient of* $u_1^{\lambda_1} \cdots u_\ell^{\lambda_\ell}$ *in the Laurent series*

$$F(u_1, ..., u_\ell) = \sum_{\alpha \in \mathbb{Z}^\ell} F_\alpha u^\alpha.$$

Then

$$[u^\lambda]F(u_1, ..., u_\ell) = Q_\lambda(x; t)$$

is the Hall-Littlewood Q-function associated to λ*, whenever* λ *is a partition of length* $\le \ell$*.*

In the $t = 0$ case, one recovers the Schur functions (indexed by partitions of bounded length). A quick proof of this case can be given by appealing to the Jacobi-Trudi formula. First we present a small Vandermonde-like calculation.

Lemma 2.1.2.

$$\prod_{1 \le i < j \le \ell} \left(1 - \frac{u_j}{u_i}\right) = \det \left(u_i^{i-j}\right)_{1 \le i,j \le \ell}.$$

Proof. Recall the traditional Vandermonde determinant:

$$\prod_{1 \le i < j \le \ell} (u_j - u_i) = \det \left(u_i^{j-1}\right)_{1 \le i,j \le \ell},$$

which implies that

$$\prod_{1 \le i < j \le \ell} (u_i - u_j) = \det \left(u_i^{\ell-j}\right)_{1 \le i,j \le \ell},$$

since the latter matrix is obtained from the former matrix by turning the jth column into the $(\ell - j + 1)$-th column for each $1 \le j \le \ell$. It now follows that

$$\begin{aligned}
\prod_{1 \le i < j \le \ell} \left(1 - \frac{u_j}{u_i}\right) &= u_1^{-(\ell-1)} \cdots u_{\ell-1}^{-1} \prod_{1 \le i < j \le \ell} (u_i - u_j) \\
&= u_1^{-(\ell-1)} \cdots u_{\ell-1}^{-1} \det \left(u_i^{\ell-j}\right)_{1 \le i,j \le \ell} \\
&= u_1^{-(\ell-1)} \cdots u_{\ell-1}^{-1} \sum_{\sigma \in S_\ell} (-1)^\sigma u_1^{\ell-\sigma(1)} \cdots u_{\ell-1}^{\ell-\sigma(\ell-1)} u_\ell^{\ell-\sigma(\ell)} \\
&= \sum_{\sigma \in S_\ell} (-1)^\sigma u_1^{1-\sigma(1)} \cdots u_{\ell-1}^{\ell-1-\sigma(\ell-1)} u_\ell^{\ell-\sigma(\ell)} \\
&= \det \left(u_i^{i-j}\right)_{1 \le i,j \le \ell} .
\end{aligned}$$

□

Now we can prove the following theorem.

Theorem 2.1.3. *When $t = 0$, $F(u_1, ..., u_\ell)$ is a generating function for the Schur functions indexed by partitions of length $\le \ell$.*

Proof. (cf. [14]) When $t = 0$,

$$F(u) = \prod_i \frac{1}{1 - x_i u} = H(u),$$

hence

$$
\begin{aligned}
F(u_1, ..., u_\ell) &= \prod_{1 \le i < j \le \ell} \left(1 - \frac{u_j}{u_i}\right) \prod_i H(u_i) \\
&= \det\left(u_i^{i-j}\right)_{1 \le i,j \le \ell} \prod_i H(u_i) \\
&= \det\left(u_i^{i-j} H(u_i)\right)_{1 \le i,j \le \ell} \\
&= \sum_{\sigma \in S_\ell} (-1)^\sigma u_1^{1-\sigma(1)} \cdots u_\ell^{\ell-\sigma(\ell)} H(u_1) \cdots H(u_\ell) \\
&= \sum_{\sigma \in S_\ell} (-1)^\sigma \sum_\alpha u_1^{\alpha_1+1-\sigma(1)} \cdots u_\ell^{\alpha_\ell+\ell-\sigma(\ell)} h_{\alpha_1} \cdots h_{\alpha_\ell} \\
&= \sum_\lambda \sum_{\sigma \in S_\ell} (-1)^\sigma h_{\lambda_1 - 1 + \sigma(1)} \cdots h_{\lambda_\ell - \ell + \sigma(\ell)} u^\lambda \\
&= \sum_\lambda \det\left(h_{\lambda_i - i + j}\right)_{1 \le i,j \le \ell} u^\lambda.
\end{aligned}
$$

It follows by the Jacobi-Trudi formula (Equation (1.8)) that when λ is a partition,

$$[u^\lambda] F(u_1, ..., u_\ell) = s_\lambda.$$

□

2.2 Vertex Operator Realization of Symmetric Functions

Our focus on the finite-length generating function $F(u_1, ..., u_\ell)$ comes from the fact that they can be realized through the ℓ-fold application of certain formal distributions. The foundations for this realization are the following commutation relations, which may be found in Macdonald (I.5, Exercise 29):

Theorem 2.2.1. *([17], I.5, Exercise 29)*

$$\left(1-\frac{v}{u}\right)E^{\perp}(u)E(v)=E(v)E^{\perp}(u),$$
$$\left(1-\frac{v}{u}\right)H^{\perp}(u)H(v)=H(v)H^{\perp}(u),$$
$$H^{\perp}(u)E(v)=\left(1+\frac{v}{u}\right)E(v)H^{\perp}(u),$$
$$E^{\perp}(u)H(v)=\left(1+\frac{v}{u}\right)H(v)E^{\perp}(u).$$

Remark 2.2.2. Each equality is an equality of series expansions in powers of $u^k v^{-m}$ for $k, m \geq 0$.

Now define the following formal distribution on Λ

$$\psi^+(u) = H(u)E^{\perp}(-u). \tag{2.1}$$

A priori, the coefficient $[u^m]\psi^+(u)$ is an infinite sum for any $m \in \mathbb{Z}$. However, for each $f \in \Lambda$, there exists a positive integer $N = N(f)$ such that $e_k^{\perp}(f) = 0$ for $k \geq N$. Hence for any non-negative integer m and $f \in \Lambda$, $[u^m]\psi^+(u)(f) = \sum_{k=0}^{N(f)} h_{k+m}e_k^{\perp}(f)$ and $[u^{-m}]\psi^+(u)(f) = \sum_{k=m}^{N(f)} h_{k-m}e_k^{\perp}(f)$ are well-defined elements of Λ. Moreover, we see that $[u^{-m}]\psi^+(u)(f) = 0$ for $m > N(f)$. Using the commutation relations, one obtains the "vertex operator" realization of the Schur functions:

Theorem 2.2.3.

$$[u^{\lambda}]\psi^+(u_1)\cdots\psi^+(u_\ell)(1) = s_\lambda.$$

Proof. The first step is to write $\psi^+(u_1)\cdots\psi^+(u_\ell)$ in normal ordered form, i.e. so that all differentiation operators appear to the right of any multiplication operators. For each $1 \leq i \leq \ell$, we must move $E^{\perp}(-u_i)$ to the right of $H(u_j)$ for all $j > i$. By the commutation relations above, each move costs a factor of $1 - \frac{u_j}{u_i}$. To complete the proof, observe also that $E^{\perp}(u)(1) = 1$. We thus recover the generating function of Schurs functions. □

Remark 2.2.4. The formal distribution $\psi^+(u)$ is sometimes called the Bernstein creation operator. The realization of the Schur functions via $\psi^+(u)$ can found in [28].

If we define

$$\Gamma_t^+(u) = H(u)E(-tu)E^\perp(-u), \tag{2.2}$$

we obtain a similarly well-defined formal distribution of $\Lambda[t]$. Its action on any $f \in \Lambda[t]$ is well-defined for exactly the same reason that the coefficients of $\psi^+(u)$ act as well-defined operators on Λ. The same kind of commutation relations argument as in the proof of Theorem 2.2.3 shows that $\Gamma_t^+(u_1)\cdots\Gamma_t^+(u_1)(1)$ realizes the Hall-Littlewood functions (with partition length $\leq \ell$). This vertex operator realization of the Hall-Littlewood functions is found in ([12]).

2.3 Definition and Properties of Multiparameter HL Functions

We are now ready to generalize the Hall-Littlewood functions. Let a and b be sequences of complex numbers. If $a = (a_i)_{i=1}^\infty$ and $b = (b_i)_{i=1}^\infty$ have finitely many non-zero elements, then we define

$$\begin{aligned}\Gamma^+(u) &= \prod_i H(ua_i) \prod_i E(-ub_i)E^\perp(-u) \\ &= \exp\left(\sum_{k\geq 1}(p_k(a_1,a_2,...) - p_k(b_1,b_2,...))\frac{p_k}{k}u^k\right)\exp\left(-\sum_{k\geq 1}\frac{\partial}{\partial p_k}u^{-k}\right),\end{aligned} \tag{2.3}$$

where $p_k(a_1,a_2,...) = a_1^k + a_2^k + ...$ and $p_k(b_1,b_2,...) = b_1^k + b_2^k + ...$ are finite sums of complex numbers. The coefficients on $\Gamma^+(u)$ act as well-defined operators on Λ for exactly the same reason that the coefficients of $\psi^+(u)$ and $\Gamma_t^+(u)$ act as well-defined operators on their respective domains. Note that, since $p_k(a_1,a_2,...)$ and $p_k(b_1,b_2,...)$ are symmetric in their respective arguments, $\Gamma^+(u)$ has certain symmetries as well. Let

$$S_\infty = \{\sigma : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0} | \sigma(j) = j \text{ for all but finitely many } j\}.$$

One can specify a group S_∞ action on the space of all finite sequences by requiring that for any $\sigma \in S_\infty$ and any finite sequence a,

$$\sigma \cdot a = \sigma \cdot (a_1, a_2, ...) = (a_{\sigma(1)}, a_{\sigma(2)}, ...). \tag{2.4}$$

Thus $S_\infty \times S_\infty$ acts on pairs of sequences:

$$(\sigma, \tau) \cdot (a, b) = (\sigma \cdot a, \sigma \cdot b), \tag{2.5}$$

and the formal distribution $\Gamma^+(u)$ only depends upon the $S_\infty \times S_\infty$ orbit of a and b.

The analogue to $F(u_1, ..., u_\ell)$ is

$$T(u_1, ..., u_\ell) = \prod_{1 \le i < j \le \ell} \frac{a(u_j/u_i)}{b(u_j/u_i)} \prod_i T(u_i),$$

where $a(\zeta) = \prod_i (1 - a_i\zeta)$, $b(\zeta) = \prod_i (1 - b_i\zeta)$, and

$$T(u) = \prod_i H(ua_i) \prod E(-ub_i).$$

We denote by T_α a coefficient in the expansion

$$T(u_1, ..., u_\ell) = \sum_{\alpha \in \mathbb{Z}^\ell} T_\alpha u_1^{\alpha_1} \cdots u_\ell^{\alpha_\ell}.$$

If $a = (1, 0, ...)$ and $b = (t, 0, ...)$, we obtain Hall-Littlewood Q-functions. At $t = 0$, we recover the Schur functions. One of our goals is to show that many of the properties of the Hall-Littlewood Q-functions are shared by the T_λ for some more general choices of a and b.

In this spirit, recall that the s_λ and $Q_\lambda(t)$ arise as stable symmetric polynomials. The T_α are also stable. Precisely:

Lemma 2.3.1. *Restrict all E, H to finitely many variables $x_1, ..., x_n$. Then*

$$T_\alpha(x_1, ..., x_{n-1}, 0) = T_\alpha(x_1, ..., x_n).$$

Proof. Elementary symmetric polynomials and complete homogeneous symmetric polynomials are stable under setting $x_n = 0$, and $a(u_j/u_i)$ and $b(u_j/u_i)$ are independent of the number of variables. □

There is an explicit formula for one-row Hall-Littlewood polynomials, which in the $t = 0$ case reduces to $s_{(r)} = h_r$. (cf. [17], p.210). This formula can be generalized as follows (due to Rozhkovskaya and Naruse):

Lemma 2.3.2. *Assume that $a = (a_1, ..., a_M)$ and $(b_1, ..., b_M)$ have the same length (= number of non-zero elements) and that all the a_k are distinct. Then*

$$T_s(x_1, ..., x_n) = \sum_{k,i} x_i^s a_k^s (1 - b_k/a_k) \left(\prod_{(r,j) \neq (k,i)} \frac{a_k x_i - b_r x_j}{a_k x_i - a_r x_j} \right).$$

Proof. Under the hypotheses of the theorem,

$$H(ua_k) = \prod_{i=1}^{n} \frac{1}{1 - a_k x_i u},$$

$$E(-ub_k) = \prod_{i=1}^{n} (1 - b_k x_i u).$$

It follows that

$$\begin{aligned} T(u) &= \prod_{k=1}^{M} \prod_{i=1}^{n} \frac{1 - x_i b_k u}{1 - x_i a_k u} \\ &= \prod_{k=1}^{M} \prod_{i=1}^{n} \frac{u^{-1} - x_i b_k}{u^{-1} - x_i a_k}. \end{aligned}$$

Since all the a_k are assumed to be distinct, we may write

$$\prod_{k=1}^{M}\prod_{i=1}^{n}\frac{u^{-1}-x_ib_k}{u^{-1}-x_ia_k}=1+\sum_{k,i}\frac{\overline{A}_{ki}}{u^{-1}-x_ia_k}$$

and find via partial fractions that

$$\overline{A}_{ki}=(a_k-b_k)x_i\prod_{(r,j)\neq(k,i)}\frac{a_kx_i-b_rx_j}{a_kx_i-a_rx_j}.$$

Setting $A_{ki}=\overline{A}_{ki}/x_i$, we have

$$T(u)=1+\sum_{k,i}A_{ki}\frac{x_i}{u^{-1}-a_kx_i}.$$

Use the geometric series expansion in the region, $|a_kx_i|<1$

$$\frac{x_i}{u^{-1}-a_kx_i}=\sum_{s=0}^{\infty}x_i(a_kx_i)^su^{s+1},$$

to write

$$T(u)=1+\sum_{s=0}^{\infty}\sum_{k,i}(a_k-b_k)x_ia_k^sx_i^s\left(\prod_{(r,j)\neq(k,i)}\frac{a_kx_i-b_rx_j}{a_kx_i-a_rx_j}\right)u^{s+1}.$$

Thus in the expansion $T(u)=\sum_{s\geq 0}T_su^s$, the coefficient on u^s is

$$\sum_{k,i}x_i^sa_k^s(1-b_k/a_k)\left(\prod_{(r,j)\neq(k,i)}\frac{a_kx_i-b_rx_j}{a_kx_i-a_rx_j}\right),$$

as desired. □

Example 2.3.3. Suppose that $a=(1,0,...)$ and $b=(\omega,0,...)$, where ω is a primitive rth root of unity. Then

$$\begin{aligned}\Gamma^+(u) &= \exp\left(\sum_{k\geq 1}(1-\omega^k)\frac{p_k}{k}u^k\right)\exp\left(-\sum_{k\geq 1}\frac{\partial}{\partial p_k}u^{-k}\right)\\ &= \exp\left(\sum_{k\not\equiv 0 \bmod (r)}(1-\omega^k)\frac{p_k}{k}u^k\right)\exp\left(-\sum_{k\geq 1}\frac{\partial}{\partial p_k}u^{-k}\right),\end{aligned} \tag{2.6}$$

and

$$T_\lambda = Q_\lambda(x;\omega).$$

From the second line of Equation (2.6), it follows that $Q_\lambda(x;\omega) \in \mathbb{C}[p_k | k \not\equiv 0 \bmod r]$, as was claimed in Theorem 1.4.3. It is also readily apparent that $Q_\lambda(x;\omega) = 0$ if there exists a positive integer i such that $m_i(\lambda) \geq r$, as was claimed in Theorem 1.4.2.

2.4 Multiparameter Analogue of Clifford Algebra

In addition to $\Gamma^+(u)$, there is a second formal distribution we will make use of:

$$\Gamma^-(u) = \prod_i E(-ua_i)H(ub_i)H^\perp(u). \tag{2.7}$$

This formal distribution will play a key role in defining bilinear identities generalizing integrable hierarchies appearing later in this thesis. In particular, for the special case when $a = (1, 0, ...)$ and $b = (0, 0, ...)$ we use a distinguished name,

$$\psi^-(u) = E(-u)H^\perp(u). \tag{2.8}$$

In the Hall-Littlewood case, $\Gamma^+(u)$ and $\Gamma^-(u)$ were shown by Jing ([13]) to satisfy a deformed version of Clifford anticommutator relations. To derive the relations, we make use of normal order relations. A formal distribution will be said to be in *normal order*, if all differentiation operators occur to the right of all multiplication operators. Explicitly,

$$:\Gamma^+(z)\Gamma^+(w):=\prod_i H(za_i)\prod_i E(-zb_i)\prod_i H(wa_i)E(-wb_i)E^\perp(-z)E^\perp(-w).$$

$$:\Gamma^-(z)\Gamma^-(w):=\prod_i E(-za_i)\prod_i H(zb_i)\prod_i E(-wa_i)H(wb_i)H^\perp(-z)H^\perp(-w).$$

$$:\Gamma^+(z)\Gamma^-(w):=\prod_i H(za_i)\prod_i E(-zb_i)\prod_i E(-wa_i)H(wb_i)E^\perp(-z)H^\perp(-w).$$

$$:\Gamma^-(z)\Gamma^+(w):=\prod_i E(-za_i)\prod_i H(zb_i)\prod_i H(wa_i)E(-wb_i)H^\perp(z)E^\perp(-w).$$

The following normal order relations hold:

Lemma 2.4.1.

$$\Gamma^+(z)\Gamma^+(w)=:\Gamma^+(z)\Gamma^+(w):\frac{a(w/z)}{b(w/z)} \tag{2.9}$$

$$\Gamma^-(z)\Gamma^-(w)=:\Gamma^-(z)\Gamma^-(w):\frac{a(w/z)}{b(w/z)} \tag{2.10}$$

$$\Gamma^+(z)\Gamma^-(w)=:\Gamma^+(z)\Gamma^-(w):\frac{b(w/z)}{a(w/z)} \tag{2.11}$$

$$\Gamma^-(z)\Gamma^+(w)=:\Gamma^-(z)\Gamma^+(w):\frac{b(w/z)}{a(w/z)} \tag{2.12}$$

Proof. Use Theorem 2.2.1. □

Using these normal order relations, we can generalize Proposition 2.12 in [12] to the case where $a=(1,0,...)$ and $b=(b_1,...,b_M)$. The proof by Jing can be adapted to this more general case.

Theorem 2.4.2. *Let $a=(1,0,0,...)$ and $b=(b_1,...,b_M)$. Then*

$$zb(w/z)\Gamma^+(z)\Gamma^+(w)+wb(z/w)\Gamma^+(w)\Gamma^+(z)=0, \tag{2.13}$$

$$zb(w/z)\Gamma^-(z)\Gamma^-(w)+wb(z/w)\Gamma^-(w)\Gamma^-(z)=0, \tag{2.14}$$

$$z^{-1}b(z/w)\Gamma^+(z)\Gamma^-(w)+w^{-1}b(w/z)\Gamma^-(w)\Gamma^+(z)=\prod_{i=1}^M(1-b_i)^2\delta(z,w), \tag{2.15}$$

where $\delta(z,w) = \sum_{k\in\mathbb{Z}} z^k w^{-k-1}$ *is the formal delta distribution.*

Equivalently, if we write $\Gamma^+(u) = \sum_{m\in\mathbb{Z}} \Gamma^+_m u^m$ *and* $\Gamma^-(u) = \sum_{m\in\mathbb{Z}} \Gamma^-_m u^{-m}$*, then for all* $n, m \in \mathbb{Z}$*,*

$$\sum_{k=0}^{M} (-1)^k e_k(b_1, ..., b_M) \left(\Gamma^+_{m+k-1}\Gamma^+_{n-k} + \Gamma^+_{n+k-1}\Gamma^+_{m-k}\right) = 0, \tag{2.16}$$

$$\sum_{k=0}^{M} (-1)^k e_k(b_1, ..., b_M) \left(\Gamma^-_{n-k}\Gamma^-_{m+k-1} + \Gamma^-_{m-k}\Gamma^-_{n+k-1}\right) = 0, \tag{2.17}$$

$$\sum_{k=0}^{M} (-1)^k e_k(b_1, ..., b_M) \left(\Gamma^-_{k+m}\Gamma^+_{k-n} + \Gamma^+_{n-k}\Gamma^-_{k-m}\right) = \prod_{i=1}^{M}(1-b_i)^2 \delta_{m,-n-1}, \tag{2.18}$$

where $\delta_{i,j}$ *is the Kronecker delta, which is* 1 *if* $i = j$ *and is* 0 *otherwise.*

Proof. First, we prove equations (2.13) and (2.16). Equations (2.14) and (2.17) are similarly verified. Observe that when $a = (1, 0, ...)$,

$$wa(z/w) = -za(w/z),$$

so that Lemma 2.4.1 gives

$$zb(w/z)\Gamma^+(z)\Gamma^+(w) + wb(z/w)\Gamma^+(z)\Gamma^+(w) = (za(w/z) + wa(z/w)) : \Gamma^+(z)\Gamma^+(w) := 0,$$

as desired for Equation (2.13).

To obtain Equation (2.16), observe that

$$zb(w/z) = z\sum_{k=0}^{M}(-1)^k \left(\frac{w}{z}\right)^k e_k(b_1, ..., b_M),$$

$$wb(z/w) = w\sum_{k=0}^{M}(-1)^k \left(\frac{z}{w}\right)^k e_k(b_1, ..., b_M),$$

and equate coefficients of $z^m w^n$ in the following equation:

$$z\sum_{k=0}^{M}(-1)^k\left(\frac{w}{z}\right)^k e_k(b_1,...,b_M)\sum_{r\in\mathbb{Z}}\Gamma_r^+z^r\sum_{s\in\mathbb{Z}}\Gamma_s^+w^s$$
$$+\,w\sum_{k=0}^{M}(-1)^k\left(\frac{z}{w}\right)^k e_k(b_1,...,b_M)\sum_{p\in\mathbb{Z}}\Gamma_p^+w^p\sum_{q\in\mathbb{Z}}\Gamma_q^+z^q=0.$$

To obtain Equation (2.15), use normal order relations to write

$$\begin{aligned}
&z^{-1}b(z/w)\Gamma^+(z)\Gamma^-(w)+w^{-1}b(w/z)\Gamma^-(w)\Gamma^+(z)\\
&\qquad=b(z/w)b(w/z):\Gamma^+(z)\Gamma^-(w):\left(z^{-1}\frac{1}{1-w/z}+w^{-1}\frac{1}{1-z/w}\right)\\
&\qquad=b(z/w)b(w/z):\Gamma^+(z)\Gamma^-(w):\left(z^{-1}\sum_{k\geq 0}(w/z)^k+w^{-1}\sum_{k\geq 0}(z/w)^k\right)\\
&\qquad=b(z/w)b(w/z):\Gamma^+(z)\Gamma^-(w):\delta(z,w)\\
&\qquad=b(z/w)b(w/z):\Gamma^+(z)\Gamma^-(z):\delta(z,w)\\
&\qquad=\prod_{i=1}^{M}(1-b_i)^2\delta(z,w),
\end{aligned}$$

where the last equality comes from $H(u)E(-u)=1$ and the property of $\delta(z,w)$ that

$$f(z)\delta(z,w)=f(w)\delta(z,w)$$

for any formal distribution $f(z)$.

□

Remark 2.4.3. Recall that $\Gamma^{\pm}(u)$ only depends on the $S_\infty\times S_\infty$ orbit of the sequences a and b with respect to the action given by Equation (2.5). Thus the statement of Theorem 2.4.2 is true if one replaces $a=(1,0,...)$ with any sequence of the form $(\delta_{i,j})_{i=1}^{\infty}$ for some fixed $j\geq 1$. (Here $\delta_{i,j}$ is again Kronecker delta, which we defined in the statement of Theorem 2.4.2.) In the case when $a=(\delta_{i,j})_{i=1}^{\infty}$ for some fixed j, the proof of Theorem 2.4.2 is still valid without modification. In Appendix A, we show that the statement of Theorem 2.4.2

does not hold if we replace the sequence $a = (1, 0, ...)$ with a sequence that is not in its S_∞ orbit (with respect to the action given by (2.4)).

Remark 2.4.4. At least in certain cases, the relations (2.16)-(2.18) in Theorem 2.4.2 are similar to the quadratic relations of infinite-dimensional Clifford algebras. The case $b = (0, 0, ...)$ corresponds to the *type A* infinite-dimensional Clifford algebra. The type A infinite-dimensional Clifford algebra has generators

$$\left\{\Upsilon_n^{\pm} | n \in \mathbb{Z}\right\}$$

and relations

$$\Upsilon_m^{\pm}\Upsilon_n^{\pm} + \Upsilon_n^{\pm}\Upsilon_m^{\pm} = 0, \qquad \Upsilon_m^{\pm}\Upsilon_n^{\mp} + \Upsilon_n^{\mp}\Upsilon_m^{\pm} = \delta_{m,-n}.$$

On the other hand, specializing the relations (2.16) - (2.18) to the case $b = (0, 0, ...)$ yields

$$\Gamma_{m-1}^{+}\Gamma_n^{+} + \Gamma_{n-1}^{+}\Gamma_m^{+} = 0,$$

$$\Gamma_n^{-}\Gamma_{m-1}^{-} + \Gamma_m^{-}\Gamma_{n-1}^{-} = 0,$$

$$\Gamma_m^{-}\Gamma_{-n}^{+} + \Gamma_n^{+}\Gamma_{-m}^{-1} = \delta_{m,-n-1}.$$

The case $b = (-1, 0, ...)$ corresponds to the *type B* infinite-dimensional Clifford algebra. The type B infinite-dimensional Clifford algebra has generators

$$\left\{\widetilde{\Upsilon}_n | n \in \mathbb{Z}\right\}$$

and relations

$$\widetilde{\Upsilon}_m\widetilde{\Upsilon}_n + \widetilde{\Upsilon}_n\widetilde{\Upsilon}_m = 2(-1)^m\delta_{m,-n}.$$

Meanwhile, specializing (2.16) - (2.18) to the case $b = (-1, 0, 0, ...)$ yields

$$\Gamma_m^{+}\Gamma_n^{+} + \Gamma_{n-1}^{+}\Gamma_m^{+} + \Gamma_m^{+}\Gamma_{n-1}^{+} + \Gamma_n^{+}\Gamma_{m-1}^{+} = 0,$$

$$\Gamma^-_n\Gamma^-_{m-1}+\Gamma^-_m\Gamma^-_{n-1}+\Gamma^-_{n-1}\Gamma^-_m+\Gamma^-_{m-1}\Gamma^-_n=0,$$

$$\Gamma^-_m\Gamma^-_{-n}+\Gamma^+_n\Gamma^-_{-m}+\Gamma^-_{m+1}\Gamma^+_{-n+1}+\Gamma^+_{n-1}\Gamma^-_{-m+1}=4\delta_{m,-n-1}.$$

Chapter 3

Integrable Hierarchies and Multiparameter Vertex Operators

In this chapter, we recall the KP hierarchy, suggest its deformation, and prove that the linear part of this deformation is equivalent to the linear part of the KP hierarchy (up to substitution of variables). We also show a way to construct KP tau functions.

3.1 The KP Hierarchy

Let P be a polynomial in an arbitrary number of variables, and f, g sufficiently regular functions in the same number of (other) variables $x_1, x_2, \ldots$. Then the *Hirota derivative* $Pf \cdot g$ is defined by the formula

$$Pf \cdot g = P\left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \ldots\right) f(x+u)g(x-u)|_{u=0}.$$

This formalism was introduced by Hirota in order to study soliton equations. Among such equations is the KdV (Korteweg-de Vries) equation

$$u_t + u_{xxx} - 6uu_x = 0,$$

and the KP (Kadomtsev-Petviashvili) equation, a higher-dimensional analogue of KdV equation. The KP equation is just one equation in an infinite system (or "hierarchy") of differential equations which can all be written in terms of Hirota derivatives. This system, called the *KP hierarchy*, was extensively studied in the 1980s by members of the Kyoto School ([16],[5],[3],[6],[7],[4],[8],[11]), who discovered connections between the hierarchy and representations of infinite-dimensional Lie algebras. The KP hierarchy has been studied further in recent years and is implicated in representation theory, enumerative geometry, and other areas of mathematics.

Let $\psi^+(u)$ be given as in Equation (2.1) and $\psi^-(u)$ be given as in Equation (2.8). These distributions are written in terms of the generators $(p_1, p_2, p_3, ...)$ of Λ. For $t = (t_1, t_2, ...)$ some sequence of variables, let $\psi^{\pm}(u)[\boldsymbol{t}]$ denote the distributions where each p_i has been substituted for t_i. Define

$$\mathcal{B}[\boldsymbol{t}] = \bigoplus_{m\in\mathbb{Z}} z^m \mathbb{C}[t_1, t_2, ...].$$

The KP hierarchy may be recovered from the *bosonic bilinear identity*

$$[u^0](R(u)\psi^+(u)[\widetilde{\boldsymbol{p}}] \otimes R^{-1}(u)\psi^-(u)[\widetilde{\boldsymbol{q}}])(\tau(\widetilde{p}) \otimes \tau(\widetilde{q})) = 0. \tag{3.1}$$

Here $\widetilde{p} = (p_1, \frac{1}{2}p_2, \frac{1}{3}p_3, ...)$ and $\widetilde{q} = (q_1, \frac{1}{2}q_2, \frac{1}{3}q_3, ...)$ may be thought of as two copies of ("normalized") power sum symmetric functions, $\tau(\widetilde{p}) \in \mathcal{B}[\widetilde{\boldsymbol{p}}]$, $\tau(\widetilde{q}) \in \mathcal{B}[\widetilde{\boldsymbol{q}}]$, and $R(u)$ and $R^{-1}(u)$ operate on $\mathcal{B}[\widetilde{\boldsymbol{p}}]$ and $\mathcal{B}[\widetilde{\boldsymbol{q}}]$ respectively by

$$R(u)(z^m f) = z^{m+1}u^{m+1} f,$$

and

$$R^{-1}(u)(z^m f) = z^{m-1}u^{-m} f,$$

for any $z^m f \in \mathcal{B}[\widetilde{\boldsymbol{p}}] \cup \mathcal{B}[\widetilde{\boldsymbol{q}}]$. Let us briefly recall the procedure by which one recovers the KP hierarchy from the bosonic bilinear identity. The full details can be found in [15]. By the definitions of $\psi^+(u)$ and $\psi^-(u)$ (Equations (2.1) and (2.8)),

$$\begin{aligned}(R(u)\psi^+(u)[\widetilde{\boldsymbol{p}}] \otimes R^{-1}(u)\psi^-(u)[\widetilde{\boldsymbol{q}}])(\tau(\widetilde{p}) \otimes \tau(\widetilde{q}))\\ = u\exp\left(\sum_{j\geq 1} u^j \left(\frac{p_j}{j} - \frac{q_j}{j}\right)\right) \exp\left(-\sum_{j\geq 1} u^{-j} \left(\frac{\partial}{\partial p_j} - \frac{\partial}{\partial q_j}\right)\right) \tau(\widetilde{p}) \otimes \tau(\widetilde{q}).\end{aligned} \tag{3.2}$$

One now introduces substitutions to turn the above expression into a collection of Hirota derivatives. To this end, introduce x and y such that

$$x - y = \widetilde{p} \quad \text{and} \quad x + y = \widetilde{q}.$$

Then $\widetilde{p} - \widetilde{q} = -2y$ and $\frac{\partial}{\partial p_j} - \frac{\partial}{\partial q_j} = \frac{-1}{j}\frac{\partial}{\partial y_j}$, so that the right hand side of (3.2) becomes

$$u\exp\left(\sum_{j\geq 1} u^j(-2y)\right) \exp\left(-\sum_{j\geq 1} \frac{u^{-j}}{j}\frac{\partial}{\partial y_j}\right) \tau(x-y)\tau(x+y). \tag{3.3}$$

With the help of a Taylor expansion and the introduction of

$$S_k(\zeta) = \sum_{k\geq 0} S_k(\zeta)u^k = \exp\left(\sum_{k\geq 1} \zeta_k u^k\right),$$

we write

$$\begin{aligned}u\exp\left(\sum_{j\geq 1} u^j(-2y)\right)\exp\left(-\sum_{j\geq 1} \frac{u^{-j}}{j}\frac{\partial}{\partial y_j}\right) \tau(x-y)\tau(x+y)\\ = u\left(\sum_{j=0} u^j S_j(-2y)\right)\left(\sum_{j=0}^{\infty} u^{-j} S_j(\widetilde{x})\exp\left(\sum_{s\geq 1} y_s x_s\right) \tau(x)\cdot\tau(x)\right).\end{aligned} \tag{3.4}$$

It follows that Equation (3.1) is equivalent to

$$\sum_{j=0}^{\infty} S_j(-2y)S_{j+1}(\widetilde{x})\prod_{s=1}^{\infty}\left(\sum_{k=0}^{\infty}\frac{(x_s y_s)^k}{k!}\right)\tau(x)\cdot\tau(x) = 0, \tag{3.5}$$

and equating coefficients of a monomial $y_{i_1}^{k_1}\cdots y_{i_j}^{k_j}$ with 0 gives a partial differential equation. The totality of all such partial differential equations is the KP hierarchy. A non-zero solution $\tau \in z^m\Lambda$ to Equation (3.5) is called a *polynomial tau function* of the KP hierarchy. More generally, solutions in the formal completion $\widehat{z^m\Lambda}$ are just called *tau functions*. It is known ([21]) that Schur polynomials are polynomial tau functions of the KP hierarchy.

3.2 Deformed KP Hierarchy

Now we are going to make a deformed KP hierarchy. If we repeat the calculations for Equations (3.1)-(3.2) with $\psi^+(u)$ replaced by $\Gamma^+(u)$ and $\psi^-(u)$ replaced by $\Gamma^-(u)$, then our expression of interest is

$$u\exp\left(\sum_{j\geq 1} u^j(p_j(A)-p_j(B))\left(\frac{p_j}{j}-\frac{q_j}{j}\right)\right)\exp\left(-\sum_{j\geq 1} u^{-j}\left(\frac{\partial}{\partial q_j}-\frac{\partial}{\partial p_j}\right)\right), \tag{3.6}$$

where $p_j(A)$ is evaluation at $x_i = a_i$ and $p_j(B)$ is evaluation at $x_i = b_i$.

If we repeat the calculations for Equations (3.3)-(3.5) with $\psi^\pm(u)$ replaced by $\Gamma^\pm(u)$, we arrive at a new Hirota system:

$$[y_{i_1}^{k_1}\cdots y_{i_j}^{k_j}\cdots]\sum_{j=0}^{\infty} S_j(-2y\theta')S_{j+1}\prod_{s=1}^{\infty}\left(\sum_{k=0}^{\infty}\frac{(x_s y_s)^k}{k!}\right)\tau(x)\cdot\tau(x) = 0, \tag{3.7}$$

where $\theta'_m = p_m(A) - p_m(B)$.

3.3 The Linear Part of the Deformed KP Hierarchy

The *linear part* of this Hirota system is the totality of equations arising from looking at the equations corresponding to linear monomials $[y_m]$ for some m:

$$(x_m x_1 - 2\theta'_m S_{m+1}(\widetilde{x}))\tau\cdot\tau = 0, \qquad m \geq 1. \tag{3.8}$$

In fact, we prefer to make the substitution

$$\theta_m = \frac{1-\theta_m'}{\theta_m'}, \tag{3.9}$$

and study the system

$$(\theta_m x_m x_1 + x_m x_1 - 2S_{m+1}(\widetilde{x}))\tau \cdot \tau = 0, \qquad m \geq 1. \tag{3.10}$$

Remark 3.3.1. Of course, the Substitution 3.9 is valid only if $\theta_m' \neq 0$. If $\theta_m' = 0$, then the mth-equation in System (3.8) becomes

$$x_m x_1 \tau \cdot \tau = 0. \tag{3.11}$$

Solutions to Hirota systems that include equations of the form of Equation (3.11) necessarily have asymmetries in the variables $x_1, x_2, \ldots$, as the following lemma makes precise.

Lemma 3.3.2. *Let* $\left(\theta_m'\right)_{m=1}^{\infty}$ *be a sequence of real numbers. Let* $I = I(\theta') = \{i_1 < i_2 < \ldots\} = \left\{m \in \mathbb{Z}_{>0} | \theta_m' = 0\right\}$. *Let*

$$\boldsymbol{x}_I = (x_{i_1}, x_{i_2}, \ldots)$$

denote the variables (among $x_1, x_2, \ldots$*) indexed by elements of* I *and let*

$$\boldsymbol{x}_{I^c} = (x_1, \ldots, x_{i_1-1}, x_{i_1+1}, \ldots, x_{i_2-1}, \ldots)$$

denote the variables (among $x_1, x_2, \ldots$*) indexed by elements of* $\mathbb{Z}_{>0} \setminus I$. *Then a solution* τ *to System (3.8) can be written as*

$$\tau(x_1, x_2, \ldots) = \begin{cases} f(\boldsymbol{x}_I) g(\boldsymbol{x}_{I^c}), & \theta_1' \neq 0 \\ f(\boldsymbol{x}_{I \setminus \{1\}}) g(\boldsymbol{x}_{I^c \cup \{1\}}), & \theta_1' = 0 \end{cases}$$

for some functions f *and* g.

Proof. If $\theta_m' = 0$, then the mth-equation in System (3.8) becomes

$$x_m x_1 \tau \cdot \tau = 0.$$

Unpacking the Hirota notation, this says that

$$\tau_{x_1,x_m}\tau - \tau_{x_1}\tau_{x_m} = 0,$$

i.e., on supp τ, $\frac{\partial}{\partial x_m}\left(\frac{\tau_{x_1}}{\tau}\right) = 0$. Hence

$$\tau_{x_1}/\tau = u(\boldsymbol{x}_{I^c}).$$

Using the logarithmic derivative, we see that

$$\frac{\partial}{\partial x_1}\ln(\tau) = u(\boldsymbol{x}_{I^c}),$$

so that

$$\tau = \exp\left(\int u(\boldsymbol{x}_{I^c})\ dx_1\right).$$

If $\theta_1' \neq 0$, then $1 \notin I$, and there exist functions $U(\boldsymbol{x}_{I^c})$ and $V(\boldsymbol{x}_I)$ for which

$$\int u(\boldsymbol{x}_{I^c})\ dx_1 = U\left(\boldsymbol{x}_{I^c}\right) + V\left(\boldsymbol{x}_I\right). \tag{3.12}$$

Therefore,

$$\tau = \exp\left(U\left(\boldsymbol{x}_{I^c}\right)\right)\exp\left(V\left(\boldsymbol{x}_I\right)\right),$$

which yields the desired result by setting

$$f(\boldsymbol{x}_I) = \exp\left(V\left(\boldsymbol{x}_I\right)\right), \qquad g(\boldsymbol{x}_{I^c}) = \exp\left(U\left(\boldsymbol{x}_{I^c}\right)\right).$$

On the other hand if $\theta_1' = 0$, then $1 \in I$, and

$$\int u(\boldsymbol{x}_{I^c})\, dx_1 = U\left(\boldsymbol{x}_{I^c \cup \{1\}}\right) + V\left(\boldsymbol{x}_{I \setminus \{1\}}\right).$$

Therefore,

$$\tau = \exp\left(U\left(\boldsymbol{x}_{I^c \cup \{1\}}\right)\right) \exp\left(V\left(\boldsymbol{x}_{I \setminus \{1\}}\right)\right),$$

yielding the desired result by setting

$$f(\boldsymbol{x}_{I \setminus \{1\}}) = \exp\left(V\left(\boldsymbol{x}_{I \setminus \{1\}}\right)\right), \qquad g(\boldsymbol{x}_{I^c \cup \{1\}}) = U\left(\boldsymbol{x}_{I^c \cup \{1\}}\right).$$

□

3.4 Reduction of the Linear Part of the Deformation

In this section, we shall show that the deformed hierarchy given by (3.10) is integrable. In fact, we shall show that a hierarchy of the form (3.10) reduces to the linear part of the KP hierarchy under substitution of variables. To this end, denote by

$$\begin{aligned} KP_m(\widetilde{x}) = KP_m(x_1, ..., x_{m+1}) &= x_m x_1 - 2S_{m+1}(\widetilde{x}) \\ &= x_m x_1 - 2 \sum_{k_1 + 2k_2 + ... = m+1} \frac{x_1^{k_1}}{1^{k_1} k_1!} \frac{x_2^{k_2}}{2^{k_2} k_2!} \cdots \frac{x_m^{k_m}}{m^{k_m} k_m!} \end{aligned}$$

the polynomial showing up in the Hirota form of the mth equation of the linear part of KP hierarchy. It is a polynomial in $x_1, x_2, ..., x_{m+1}$. We can now state the theorem.

Theorem 3.4.1. *Let $\theta_1, \theta_2, \theta_3, ...$ be a sequence of complex numbers. The bilinear hierarchy*

$$(\theta_m x_m x_1 + KP_m(\widetilde{x}))\tau \cdot \tau = 0 \qquad\qquad m \geq 1$$

is integrable.

Proof. We want to show that for $m \geq 1$, $(\theta_m x_m x_1 + KP_m(\widetilde{x}))\tau \cdot \tau$ is just $KP_m(\widetilde{y})\tau \cdot \tau$ for some transformed variables $\widetilde{y} = (y_1, y_2/2, y_3/3, ...)$. We argue by induction that for $m \geq 1$ this change of variables takes the form

$$y_{m+1} = x_{m+1} + f_{m+1}(y_1, ..., y_m, \theta_1, ..., \theta_m),$$

where f_{m+1} is a polynomial. To establish the base case ($m = 1$), set $y_1 = x_1$ and solve for y_2 in the equation

$$(1+\theta_1)x_1^2 - x_1^2 - x_2 = y_1^2 - y_1^2 - y_2.$$

Now assume the inductive hypothesis, namely, that for $1 \leq j \leq m$,

$$y_j = x_j + f_j(x_1, ..., x_{j-1}, \theta_1, ..., \theta_{j-1})$$

for some polynomial f_j, and that under this substitution of variables,

$$(\theta_m x_m x_1 + KP_m(\widetilde{x}))\tau(y) \cdot \tau(y) = KP_m(\widetilde{y})\tau(y) \cdot \tau(y). \tag{3.13}$$

Then

$$KP_m(\widetilde{y}) = (x_m + f_m)x_1 - 2 \sum_{\substack{k_1+2k_2+...=m+1 \\ k_{m+1}=0}} \left(\frac{(x_1+f_1)^{k_1}}{1^{k_1}k_1!} \frac{(x_2+f_2)^{k_2}}{2^{k_2}k_2!} \cdots \frac{(x_m+f_m)^{k_m}}{m^{k_m}k_m!} \right) - \frac{2}{m+1} y_{m+1},$$

while

$$\theta_m x_m x_1 + KP_m(\widetilde{x}) = \theta_m x_m x_1 + x_m x_1 - 2 \sum_{\substack{k_1+2k_2+...=m+1 \\ k_{m+1}=0}} \left(\frac{x_1^{k_1}}{1^{k_1}k_1!} \frac{x_2^{k_2}}{2^{k_2}k_2!} \cdots \frac{x_m^{k_m}}{m^{k_m}k_m!} \right) - \frac{2}{m+1} x_{m+1}.$$

Solving for y_{m+1} in the equation $KP_m(\widetilde{y}) = \theta_m x_m x_1 + KP_m(\widetilde{x})$, we find that

$$y_{m+1} = x_{m+1} + \frac{m+1}{2} f_m x_1 - \frac{m+1}{2}\theta_m x_m x_1 + (m+1) \sum_{(1^{m+1})<\lambda<(m+1)} A_\lambda, \tag{3.14}$$

where

$$A_\lambda = \left(\frac{x_1^{k_1}}{1^{k_1}k_1!}\cdots\frac{x_m^{k_m}}{m^{k_m}k_m!}\right) - \left(\frac{(x_1+f_1)^{k_1}}{1^{k_1}k_1!}\cdots\frac{(x_m+f_m)^{k_m}}{m^{k_m}k_m!}\right),$$

$\lambda = (1^{k_1}2^{k_2}\cdots m^{k_m}) \vdash m+1$, and the inequality is with respect to the dominance order. (Note that, a priori, the right-most sum should be over all $\lambda \neq (m+1)$, but $A_{(1^{m+1})} = 0$ since $f_1 = 0$.) The second, third, and fourth terms on the right hand side of Equation (3.14) do not depend on x_{m+1} or θ_{m+1} at all, and in fact these terms are polynomials in $x_1, ..., x_m$ and $f_1, ..., f_m$. Thus $y_{m+1} = x_{m+1} + f_{m+1}(x_1, ..., x_m, \theta_1, ...\theta_m)$, as desired.

From here, we can conclude that our system is integrable: for all $m \geq 1$, if $\tau(y)$ is a KP tau function for $KP(\widetilde{y})$, then

$$(\theta_m x_m x_1 + KP_m(\widetilde{x}))\tau(y)\cdot\tau(y) = KP_m(\widetilde{y})\tau(y)\cdot\tau(y) = 0, \tag{3.15}$$

hence $\tau(y)$ is a solution to this deformed hierarchy. $\square$

Now we compute some of the f_j.

$$f_2 = -\theta_1 x_1^2. \tag{3.16}$$

$$f_3 = \frac{3}{2}f_2x_1 - \frac{3}{2}\theta_2x_2x_1 + 3A_{(2,1)} = -\frac{3}{2}\theta_2x_2x_1. \tag{3.17}$$

$$f_4 = 2f_3x_1 - 2\theta_3x_3x_1 + 4\left(A_{(3,1)} + A_{(2^2)}\right) = \theta_2x_2x_1^2 - 2\theta_3x_3x_1 + \theta_1x_1^2x_2 - \frac{1}{2}\theta_1^2x_1^4. \tag{3.18}$$

If the sequence $(\theta_m)_{m=1}^\infty$ is identically zero, then the deformed hierarchy reduces trivially to the KP hierarchy. The converse is also true.

Lemma 3.4.2. *Let $\theta_2, \theta_3, ...$ be a sequence of complex numbers. Then $f_j = 0$ for all $j \geq 1$ if and only if $\theta_j = 0$ for all $j \geq 1$.*

Proof. If $\theta_m = 0$ for all $m \geq 2$, then System (3.10) is clearly the linear part of System (3.5).

Conversely, suppose that there exists m for which $\theta_m \neq 0$, and moreover let m be minimal among such indices. Then $f_{m+1} = -\frac{m+1}{2}\theta_m x_m x_1$ is non-zero. $\square$

Observe the following:

Lemma 3.4.3. *With respect to the grading* $\deg(x_j) = j$*, the polynomial* A_λ *is homogeneous of degree* $|\lambda| = m+1$ *in the variables* $x_1, x_2, ...$ *for* $m \geq 1$*. It follows that, for* $m \geq 1$*,* f_{m+1} *is also homogeneous of degree* $m+1$ *in the variables* $x_1, x_2, ...$

Proof. Write $\lambda = (1^{k_1}...m^{k_m}) \vdash m+1$. To prove the result for A_λ, suppose for the sake of induction that, with respect to the grading $\deg(x_j) = j$, f_k is homogeneous of degree k in the variables $x_1, x_2, ...$, for $2 \leq k \leq m$. A term of A_λ is, up to constant, of the form

$$x_1^{k_1} \cdots x_m^{k_m} - (x_1 - f_1)^{k_1} \cdots (x_m - f_m)^{k_m}.$$

By the inductive hypothesis, each term of the form $(x_j - f_j)^{k_j}$ is of degree jk_j, So that each term of the product is of degree

$$k_1 + 2k_2 + ... + mk_m = m+1.$$

The base case is established by Equation (3.16), which completes the proof. $\square$

3.5 A Scheme for Getting some KP Tau Functions

In the last section, we replaced $\psi^\pm(u)$ with $\Gamma^\pm(u)$ to deform the KP hierarchy. We showed that the linear part of this deformed hierarchy coincided with the linear part of the KP hierarchy under a substitution of variables. In this section, we show a way of constructing KP tau functions.

Let a and b be sequences of complex numbers for which $p_k(a) - p_k(b) \neq 0$ for all $k \geq 1$. Then we can construct an injective map

$$\epsilon_{a,b}[\boldsymbol{t}] : \mathbb{C}[t_1, t_2, ...] \to \mathbb{C}[t_1, t_2, ...]$$

via

$$t_k \mapsto (p_k(a) - p_k(b))t_k \tag{3.19}$$

The differential operator $\widehat{D}_k[\boldsymbol{t}] = \frac{1}{p_k(a)-p_k(b)}\frac{\partial}{\partial t_k}$ on $\mathcal{B}$ is related to $D_k[\boldsymbol{t}] = \frac{\partial}{\partial t_k}$ by

$$\widehat{D}_k = \epsilon_{a,b} \circ D_k \circ \epsilon_{a,b}^{-1}, \tag{3.20}$$

which is well-defined since $\epsilon_{a,b}$ is injective. Writing

$$\Omega = [u^0]\left(R(u)\psi^+(u)[\widetilde{\boldsymbol{p}}] \otimes R^{-1}(u)\psi^-(u)[\widetilde{\boldsymbol{q}}]\right),$$

with $R^{\pm 1}(u)$ and $\psi^{\pm}(u)$ as in Section 3.1, we can rewrite the KP hierarchy (Equation (3.1)) as

$$\Omega(\tau(\widetilde{p}) \otimes \tau(\widetilde{q})) = 0.$$

We can construct an *a priori* new hierarchy by introducing

$$\Omega_{a,b}[\boldsymbol{p},\boldsymbol{q}] := \epsilon_{a,b}[\boldsymbol{p},\boldsymbol{q}] \circ \Omega \circ (\epsilon_{a,b}^{-1}[\widetilde{\boldsymbol{p}}] \otimes \epsilon_{a,b}^{-1}[\widetilde{\boldsymbol{q}}]). \tag{3.21}$$

By the relationship between $\widehat{D_k}$ and D_k, it follows that

$$\Omega_{a,b}[\boldsymbol{p},\boldsymbol{q}] = [u^0]\left(\Phi_{a,b}^+(u)[\widetilde{\boldsymbol{p}}] \otimes \Phi_{a,b}^-(u)[\widetilde{\boldsymbol{q}}]\right), \tag{3.22}$$

where

$$\Phi_{a,b}^{\pm}(u)[\boldsymbol{t}] = R^{\pm}(u)\exp\left(\pm\sum_{k\geq 0}\frac{\epsilon_{a,b}(t_k)}{k}u^k\right)\exp\left(\mp\sum_{k\geq 0}\widehat{D_k}[\boldsymbol{t}]u^{-k}\right). \tag{3.23}$$

Remark 3.5.1. Note that $\Phi_{a,b}^{\pm}(u)$ looks very similar to $\Gamma^{\pm}(u)$ (cf. Equation (2.3)), but the differential operators are different: in $\Phi_{a,b}^{\pm}(u)$, the differential portion is $\exp\left(\mp\sum_{k\geq 0}\widehat{D}_k u^{-k}\right)$, while in $\Gamma^{\pm}(u)$, the differential portion is $\exp\left(\mp\sum_{k\geq 0} D_k u^{-k}\right)$.

By construction, the following diagram commutes.

$$
\begin{tikzcd}
\mathcal{B}[\widetilde{\boldsymbol{p}}] \otimes \mathcal{B}[\widetilde{\boldsymbol{q}}] \arrow[r, "\epsilon_{a,b} \otimes \epsilon_{a,b}"] \arrow[d, "\Omega"'] & \mathcal{B}[\widetilde{\boldsymbol{p}}] \otimes \mathcal{B}[\widetilde{\boldsymbol{q}}] \arrow[d, "\Omega_{a,b}"] \\
\mathcal{B}[\widetilde{\boldsymbol{p}}, \widetilde{\boldsymbol{q}}] \arrow[r, "\epsilon_{a,b}"] & \mathcal{B}[\widetilde{\boldsymbol{p}}, \widetilde{\boldsymbol{q}}]
\end{tikzcd}
$$

It follows that $\epsilon_{a,b}(s_\lambda) = S_\lambda^{a,b}$ is a tau function of the hierarchy arising from the $\Omega_{a,b}$ bilinear identity,

$$\Omega_{a,b}\left(\tau \otimes \tau\right) = 0. \tag{3.24}$$

This hierarchy coincides with the KP hierarchy, so that $S_\lambda^{a,b}$ is a KP tau function. More precisely, we have the following theorem.

Theorem 3.5.2. *Let a, b be sequences of complex numbers for which $p_k(a) - p_k(b) \neq 0$. Let $\epsilon_{a,b}$ be defined as in Equation (3.19), $\widehat{D}_k$ as in Equation (3.20), and $\Omega_{a,b}$ be as in Equation (3.21). Then System (3.24) is the KP hierarchy.*

Proof. By Equations 3.22 and 3.23, System (3.24) is equivalent to the condition

$$
\begin{aligned}
&u\exp\left(\sum_{j\geq 1} u^j \frac{1}{j}\left(\epsilon_{a,b}(p_j) - \epsilon_{a,b}(q_j)\right)\right) \\
&\quad\cdot \exp\left(-\sum_{j\geq 1} u^{-j}\left(\frac{1}{p_j(A) - p_j(B)}\left(\frac{\partial}{\partial p_j} - \frac{\partial}{\partial q_j}\right)\right)\right)\tau(\widetilde{p})\tau(\widetilde{q}) = 0.
\end{aligned} \tag{3.25}
$$

Introduce new variables $x = (x_1, x_2, ...)$ and $y = (y_1, y_2, ...)$ so that

$$\left(p_k(a) - p_k(b)\right)\widetilde{p}_k = x_k - y_k, \qquad \left(p_k(a) - p_k(b)\right)\widetilde{q}_k = x_k + y_k,$$

for all $k \geq 1$. Then

$$\left(p_k(a) - p_k(b)\right)\widetilde{p}_k - \left(p_k(a) - p_k(b)\right)\widetilde{q}_k = -2y_k,$$

and

$$-\frac{\partial}{\partial y_k} = \frac{1}{p_k(a) - p_k(b)} \left(\frac{\partial}{\partial \widetilde{p}_k} - \frac{\partial}{\partial \widetilde{q}_k} \right),$$

for all $k \geq 1$. Hence Equation (3.25) becomes

$$u\exp\left(\sum_{j\geq 1} u^j(-2y)\right) \exp\left(-\sum_{j\geq 1} \frac{u^{-j}}{j}\frac{\partial}{\partial y_j}\right) \tau(x-y)\tau(x+y) = 0.$$

This is exactly Equation (3.3), which is the KP hierarchy. □

Example 3.5.3. (cf. [19]) As an example, let $a = (1, 0, ...)$ and $b = (t, 0, ...)$ for $|t| < 1$. Then $S_\lambda^{a,b}$ are the symmetric functions in $\mathcal{B}[t]$ which are dual to s_λ with respect to the t-deformed Hall inner product (cf. [17], III, (4.5)).

Chapter 4

Hall-Littlewood Vertex Operators at Roots of Unity

In this chapter, we study the Hall-Littlewood vertex operator (cf. Equation (2.2)) in the case that the parameter is a primitive r-th root of unity. When $r = 2$, the Hall-Littlewood Q-functions are the Schur Q-functions of Section 1.4.1, which are known to be polynomial tau functions for the BKP hierarchy. The BKP hierarchy can be defined in terms of the vertex operators, and the fact that the Schur Q-functions are solutions can be deduced from the type B Clifford relations. In this chapter, we study relations for the Hall-Littlewood vertex operators when the parameter is a primitive root of unity. In this case, there is a natural way of factorizing the formal distributions, and we calculate normal order relations for these factors.

4.1 Refined Commutation Relations at Roots of Unity

Let ω be a primitive r-th root of unity. The Hall-Littlewood vertex operator

$$\Gamma^{+}(u) = \Gamma^{+}_{\omega}(u) = H(u)E(-u\omega)E^{\perp}(-u)$$

factorizes as

$$\Gamma_\omega^+(u) = \prod_{j=0}^{r-1} T_j(u) E_j^\perp(-u), \tag{4.1}$$

where for $j = 1, ..., r-1$,

$$T_j(u) = \exp\left(\sum_{k\geq 0} \frac{1-\omega^j}{rk+j} p_{rk+j} u^{rk+j}\right),$$

$$E_j^\perp(-u) = \exp\left(-\sum_{k\geq 0} \frac{\partial}{\partial p_{rk+j}} u^{-(rk+j)}\right),$$

$$T_0(u) = \exp\left(\sum_{k\geq 1} \frac{(1-\omega^0)}{rk} p_{rk} u^{rk}\right) = 1,$$

and

$$E_0^\perp(-u) = \exp\left(-\sum_{k\geq 1} \frac{\partial}{\partial p_{rk}} u^{-rk}\right).$$

It is immediate that if $i \neq j$ then $[T_i(u), E_j^\perp(-v)] = 0$. Furthermore, the action of $E_j^\perp(-v)$ on the subalgebra of $\Lambda[\omega]$ generated by $\{p_k | k \not\equiv j \mod r\}$ is trivial. We wish to determine the commutation relations

$$[T_j(u), E_j^\perp(-v)], \qquad 1 \leq j \leq r-1.$$

We rely on the repeated application of two key facts:

(I) $\exp\left(a\frac{\partial}{\partial x} f(x)\right) = f(x+a)$,

(II) $x \mapsto x + a$ is a ring homomorphism of $A[x]$ if A is a commutative algebra.

Theorem 4.1.1. $E_j^\perp(-v)T_j(u) = \prod_k \exp\left(-\frac{1-\omega^j}{rk+j}\left(\frac{u}{v}\right)^{rk+j}\right) T_j(u) E_j^\perp(-v)$, *where k ranges over all non-negative integers for $j \neq 0$ and ranges over all positive integers for $j = 0$.*

Proof. In the $j = 0$ case, the claim is that $[T_0(u), E_0^\perp(-v)] = 0$, which follows immediately from the fact that $T_0(u) = 1$.

If $j \neq 0$, use Fact (I), to see that

$$\begin{aligned}
E_j^{\perp}(-v)T_j(u) &= E_j^{\perp}(-v)\prod_{k\geq 0}\exp\left(\frac{1-\omega^j}{rk+j}p_{rk+j}u^{rk+j}\right)\\
&= \prod_{k\geq 0}\exp\left(\frac{1-\omega^j}{rk+j}\left(p_{rk+j}-v^{-(rk+j)}\right)u^{rk+j}\right)\\
&= \frac{\prod_{k\geq 0}\exp\left(\frac{1-\omega^j}{rk+j}p_{rk+j}u^{rk+j}\right)}{\prod_{k\geq 0}\exp\left(\frac{1-\omega^j}{rk+j}\left(\frac{u}{v}\right)^{rk+j}\right)}\\
&= T_j(u)\cdot\prod_{k\geq 0}\exp\left(-\frac{1-\omega^j}{rk+j}\left(\frac{u}{v}\right)^{rk+j}\right).
\end{aligned}$$

At the same time, Fact (II) implies that, for arbitrary f,

$$\begin{aligned}
E_j^{\perp}(-v)\left(T_j(u)\left(f\right)\right) &= \left(E_j^{\perp}(-v)T_j(u)\right)\cdot\left(E_j^{\perp}(-v)(f)\right)\\
&= \left(T_j(u)\cdot\prod_{k\geq 0}\exp\left(-\frac{1-\omega^j}{rk+j}\left(\frac{u}{v}\right)^{rk+j}\right)\right)\cdot\left(E_j^{\perp}(-v)\left(f\right)\right).
\end{aligned} \tag{4.2}$$

On the other hand,

$$T_j(u)E_j^{\perp}(-v)(f) = T_j(u)\cdot\left(E_j^{\perp}(-v)(f)\right), \tag{4.3}$$

and comparing Equations (4.2) and (4.3) shows that

$$E_j^{\perp}(-v)T_j(u)(f) = \prod_{k\geq 0}\exp\left(-\frac{1-\omega^j}{rk+j}\left(\frac{u}{v}\right)^{rk+j}\right)T_j(u)E_j^{\perp}(-v)(f).$$

Since f is arbitrary, this concludes the proof. □

Note that Theorem 4.1.1 involves hypergeometric series of the form

$$\sum_k \frac{1-\omega^j}{rk+j}\left(\frac{u}{v}\right)^{rk+j}, \tag{4.4}$$

where $k \geq 0$ if $j \neq 0$ or $k \geq 1$ if $j = 0$ (in which case Series (4.4) is zero, since $\omega^0 = 1$).

Example 4.1.2. In the particular case $r = 2$, we consider Series (4.4) with $j = 0$ (for which the series is zero) and $j = 1$, for which we have the series

$$\sum_{k\geq 0} \frac{2}{2k+1} \left(\frac{u}{v}\right)^{2k+1} = \tanh^{-1}(u/v).$$

Note also that by adding the $j = 0$ and $j = 1$ series, we recover

$$\sum_{k\geq 1} \frac{1-(-1)^{2k}}{2k} \left(\frac{u}{v}\right)^{2k} + \sum_{k\geq 0} \frac{2}{2k+1} \left(\frac{u}{v}\right)^{2k+1} = \ln \frac{1+u/v}{1-u/v}, \tag{4.5}$$

provided $|u/v| \leq 1$ and $u/v \neq 1$. In particular, exponentiating Equation (4.5) and combining with Theorem 4.1.1 will recover Equation (2.9) from Lemma 2.4.1, in the special case $a = (1, 0, ...)$ and $b = (-1, 0, ...)$.

In Theorem 4.1.1, we exponentiate hypergeometric series, so it is natural to express these hypergeometric series in terms of various logarithms. To this end, we recall the following series expansion, valid for $|z| < 1, z \neq 1$:

$$-\ln(1-z) = \sum_{k\geq 1} \frac{z^k}{k},$$

and we let

$$Q_s(z) = \sum_{k\geq 0} \frac{z^{rk+s}}{rk+s}, \qquad s = 1, ..., r-1,$$

and

$$Q_0(z) = \sum_{k\geq 1} \frac{z^{rk}}{rk}.$$

See that

$$Q_s(\omega^m z) = \omega^{ms} Q_s(z),$$

so that by writing

$$L_s = L_s(z) = \sum_{k\geq 1} \frac{(\omega^s z)^k}{k},$$

we obtain a linear system

$$\boldsymbol{M}\boldsymbol{Q} = \boldsymbol{L}, \tag{4.6}$$

where $\boldsymbol{M} = \left(\omega^{kj}\right)_{0\leq k,j\leq r-1}$, $Q = [Q_0...Q_{r-1}]^T$, and $\boldsymbol{L} = [L_0...L_{r-1}]^T$. In fact, this is just the *Discrete Fourier Transform (DFT)*, and the matrix M has as inverse

$$\boldsymbol{M}^{-1} = \left(\frac{1}{r}\omega^{-jm}\right)_{0\leq j,m\leq r-1}. \tag{4.7}$$

Thus

$$Q_j = \sum_{m=0}^{r-1}\frac{1}{r}\omega^{-jm}L_m, \qquad 0\leq j\leq r-1. \tag{4.8}$$

We now present two new proofs of Equation (2.9) in Lemma 2.4.1 for the special case $a = (1,0,...)$ and $b = (\omega,0,...)$.

Lemma 4.1.3. *If* $a = (1,0,...)$ *and* $b = (\omega,0,...)$*, then* $\Gamma^+(z)\Gamma^+(w) =: \Gamma^+(z)\Gamma^+(w) : \frac{1-w/z}{1-\omega w/z}$.

Remark 4.1.4. To put $\Gamma^+(z)\Gamma^+(w)$ into normal order, move $E_{j_1}^{\perp}(-z)$ to the right of $T_{j_2}(w)$ for all $0\leq j_1, j_2\leq r-1$. There is only commutator cost to this move when $j_1 = j_2$, so it suffices to show that

$$\prod_{j=0}^{r-1} E_j^{\perp}(-z)T_j(w) = \frac{\exp\left(L_1(w/z)\right)}{\exp\left(L_0(w/z)\right)}\prod_{j=0}^{r-1} T_j(w)E_j^{\perp}(-z).$$

Verifying this equation is the goal of each proof.

Proof. In the first proof, we apply Theorem 4.1.1 and the first two equations of System (4.6).

$$\begin{aligned}
\prod_{j=0}^{r-1} E_j^{\perp}(-z)T_j(w) &= \prod_{j=0}^{r-1} \frac{\exp\left(\omega^j Q_j(w/z)\right)}{\exp\left(Q_j(w/z)\right)} T_j(w)E_j^{\perp}(-z) \\
&= \frac{\exp\left(\sum_{j=0}^{r-1} \omega^j Q_j(w/z)\right)}{\exp\left(\sum_{j=0}^{r-1} Q_j(w/z)\right)} \prod_{j=0}^{r-1} T_j(w)E_j^{\perp}(-z) \\
&= \frac{\exp\left(L_1(w/z)\right)}{\exp\left(L_0(w/z)\right)} \prod_{j=0}^{r-1} T_j(w)E_j^{\perp}(-z)
\end{aligned}$$

□

Proof. In the second proof, we apply Theorem 4.1.1, Equation (4.8), and the following two properties of the transition coefficients $\left(\frac{1}{r}\omega^{-jm}\right)_{0\leq j,m\leq r-1}$:

(P1) $\sum_{j=0}^{r-1} \frac{1}{r}\omega^{-jm} = \delta_{m,0}$,

(P2) $\frac{\omega^j}{r}\omega^{-jm} = \frac{1}{r}\omega^{-j(m-1)}$, to make the following calculations:

$$
\begin{aligned}
\prod_{j=0}^{r-1} E_j^{\perp}(-z)T_j(w) &= \prod_{j=0}^{r-1} \frac{\exp\left(\omega^j Q_j(w/z)\right)}{\exp\left(Q_j(w/z)\right)} T_j(w)E_j^{\perp}(-z) \\
&= \prod_{j=0}^{r-1} \frac{\exp\left(\omega^j \sum_{m=0}^{r-1} r^{-1}\omega^{-jm} L_m(w/z)\right)}{\exp\left(\sum_{m=0}^{r-1} r^{-1}\omega^{-jm} L_m(w/z)\right)} T_j(w)E_j^{\perp}(-z) \\
&= \prod_{j=0}^{r-1} \frac{\exp\left(\sum_{m=0}^{r-1} \omega^j r^{-1}\omega^{-jm} L_m(w/z)\right)}{\exp\left(\sum_{m=0}^{r-1} r^{-1}\omega^{-jm} L_m(w/z)\right)} T_j(w)E_j^{\perp}(-z) \\
&= \prod_{j=0}^{r-1} \frac{\exp\left(\sum_{m=0}^{r-1} r^{-1}\omega^{-j(m-1)} L_m(w/z)\right)}{\exp\left(\sum_{m=0}^{r-1} r^{-1}\omega^{-jm} L_m(w/z)\right)} T_j(w)E_j^{\perp}(-z) \\
&= \prod_{j=0}^{r-1} \frac{\exp\left(\sum_{m=0}^{r-1} r^{-1}\omega^{-j(m-1)} L_m(w/z)\right)}{\exp\left(\sum_{m=0}^{r-1} r^{-1}\omega^{-j(m-1)} L_{m-1}(w/z)\right)} T_j(w)E_j^{\perp}(-z) \\
&= \prod_{m=0}^{r-1} \exp\left((L_m - L_{m-1}) \sum_{j=0}^{r-1} r^{-1}\omega^{-j(m-1)}\right) \prod_{j=0}^{r-1} T_j(w)E_j^{\perp}(-z) \\
&= \prod_{m=0}^{r-1} \exp\left((L_m - L_{m-1})\, \delta_{m-1,0}\right) \prod_{j=0}^{r-1} T_j(w)E_j^{\perp}(-z) \\
&= \exp\left(L_1(w/z) - L_0(w/z)\right) \prod_{j=0}^{r-1} T_j(w)E_j^{\perp}(-z) \\
&= \frac{\exp\left(L_1(w/z)\right)}{\exp\left(L_0(w/z)\right)} \prod_{j=0}^{r-1} T_j(w)E_j^{\perp}(-z).
\end{aligned}
$$

$\square$

For a fixed primitive r-th root ω, the factorization of $\Gamma_{\omega}^{+}(u)$ in Equation (4.1) gives rise to r formal distributions $\Gamma_{[0]}^{+}(u), ..., \Gamma_{[r-1]}^{+}(u)$ that share properties in common with each other and with $\Gamma^+_\omega(u)$. Precisely, define

$$\Gamma^+_{[j]}(u) = T_j(u)E^\perp_j(-u), \qquad 0 \le j \le r-1, \tag{4.9}$$

where $T_j(u)$ and $E^\perp_j(-u)$ are as in the factorization in Equation (4.1). For fixed r, j, denote by $\mathbb{C}[p_{rk+j}]$ the subalgebra of $\mathbb{C}[p_1, p_2, ...]$ generated by $\{p_k | k \equiv j \mod r\}$. The following theorem is immediate.

Theorem 4.1.5. *Let ω be a fixed primitive r-th root of unity, and let $\Gamma_{[0]}(u), ...\Gamma^+_{[r-1]}(u)$ be defined as in Equation (4.9). Then the following hold.*

(i) For $0 \le j \le r-1$, for any $\lambda = (\lambda_1, ...\lambda_\ell) \in \mathbb{Z}^\ell$, $[u^\lambda]\Gamma^+_{[j]}(u_1)\cdots\Gamma^+_{[j]}(u_\ell)(1) \in \mathbb{C}[p_{rk+j}]$.

(ii) Let $\langle\ ,\ \rangle$ be the Hall inner product on Λ (cf. Chapter 1). For any $f \in \Lambda$ and $g \in \mathbb{C}[p_{rk}] \setminus \mathbb{C}$, if $\langle f, g\rangle = 0$, then $\langle [u^m]\,\Gamma^+_{[0]}(f), g\rangle = 0$ for all $m \in \mathbb{Z}$.

4.2 Further Questions and Future Directions

The factorization in Equation (4.1) leads naturally to consideration of the $\mathbb{C}$-subalgebras $\mathbb{C}[p_{rk}], \mathbb{C}[p_{rk+1}], ..., \mathbb{C}[p_{rk+r-1}]$ of Λ. When $r = 2$, there is a certain sense in which this "decomposition" is canonical. We sketch an outline and refer the reader to [1] for full details (often in greater generality). Recall that Λ is a Hopf algebra. In fact, it is a connected graded Hopf algebra, which is to say that $\Lambda^0 \cong \mathbb{C}$ and the coproduct Δ (cf. Equation (1.9)) respects the grading: $\Delta(\Lambda^d) \subset \bigoplus_{m+n=d} \Lambda^m \otimes \Lambda^n$. A *character* of Λ is a homomorphism of unital algebras $\Lambda \to \mathbb{C}$. The set $\mathbb{X}(\Lambda)$ of all characters of Λ forms a group under convolution

$$(\phi, \psi) \mapsto m_{\mathbb{C}} \circ (\phi \otimes \psi) \circ \Delta,$$

where Δ is the coproduct of Λ and $m_{\mathbb{C}}$ is just the usual product on $\mathbb{C}$. With respect to this convolution product, the inverse of $\phi : \Lambda \to \mathbb{C}$ is given by precomposition with the antipode

S of Λ:

$$\phi^{-1} = \phi \circ S.$$

A graded Hopf algebra, $\Lambda = \bigoplus \Lambda^d$ has an automorphism $\alpha : \Lambda \to \Lambda$ defined by

$$h \mapsto (-1)^d h, \qquad h \in \Lambda^d.$$

The automorphism α induces induces an automorphism (an involution) on $\mathbb{X}(\Lambda)$ by

$$\phi \mapsto \overline{\phi} = \phi \circ \alpha.$$

By the group structure on $\mathbb{X}(\Lambda)$, one may define $\phi \in \mathbb{X}(\Lambda)$ to be *even* if

$$\overline{\phi} = \phi,$$

and *odd* if

$$\overline{\phi} = \phi^{-1}.$$

Given two characters ϕ, ψ, $\mathcal{S}(\phi, \psi)$ is defined to be the largest graded cosubalgebra of Λ such that for all $h \in \mathcal{S}(\phi, \psi)$, $\phi(h) = \psi(h)$. In fact, this is a graded Hopf subalgebra of Λ. There is a character $\zeta : \Lambda \to \mathbb{C}$ (referred to in [1] as the *zeta character* of Λ) defined by its values on the monomial symmetric functions (cf. (1.1)) as

$$m_\lambda \mapsto \begin{cases} 1 & \lambda = (n) \text{ for some } n \geq 1 \text{ or } \lambda = \varnothing, \\ 0 & \text{otherwise}, \end{cases} \tag{4.10}$$

and the *even and odd subalgebras* of Λ are, by definition,

$$\Lambda_+ = \mathcal{S}(\overline{\zeta}, \zeta)$$

and

$$\Lambda_- = \mathcal{S}(\overline{\zeta}, \zeta^{-1}),$$

respectively. In [1] it is shown that

$$\Lambda_+ = \mathbb{C}[p_{2k}], \qquad \Lambda_- = \mathbb{C}[p_{2k+1}], \tag{4.11}$$

and as cocommutative Hopf algebras

$$\Lambda \cong \Lambda_+ \otimes \Lambda_-.$$

To verify the claim that $\Lambda_- = \mathbb{C}[p_{2k-1}]$, consider that combining the definition of S (cf. (1.9)) with Equations (1.4) and (1.5) yields

$$S(p_\lambda) = (-1)^{\ell(\lambda)} p_\lambda,$$

while (4.10) and (1.2) imply that $\zeta(p_\lambda) = 1$, hence

$$\begin{aligned} \overline{\zeta(p_\lambda)} = \zeta^{-1}(p_\lambda) &\iff (-1)^{|\lambda|} = (-1)^{\ell(\lambda)} \\ &\iff |\lambda| \equiv \ell(\lambda) \mod 2. \end{aligned}$$

Now for ω a primitive r-th root of unity, define an automorphism α_ω of Λ by action on homogeneous elements as follows:

$$h \mapsto \omega^d h, \qquad h \in \Lambda^d.$$

This induces an automorphism on $\mathbb{X}(\Lambda)$ by precomposition:

$$\phi \mapsto \overline{\phi}_\omega = \phi \circ \alpha_\omega.$$

For fixed $j \leq 0 \leq r-1$, denote by S_j the $\mathbb{C}$-algebra morphism of Λ extended from

$$1 \mapsto 1, \qquad p_n \mapsto \omega^j p_n.$$

The following theorem generalizes (4.11).

Theorem 4.2.1. *Let ω be a fixed primitive root of unity. Then $\mathcal{S}\left(\overline{\zeta}_\omega, \zeta \circ S_j\right) = \mathbb{C}[p_{rk+j}]$.*

Proof. By definition, $\overline{\zeta}_\omega(p_n) = \omega^n$, and $\zeta \circ S_j = \omega^j$, so that

$$\begin{aligned} \overline{\zeta}_\omega(p_\lambda) = \zeta \circ S_j &\iff \omega^{|\lambda|} \equiv \omega^{j\ell(\lambda)} \mod r \\ &\iff |\lambda| \equiv j\ell(\lambda) \mod r. \end{aligned}$$

If each part of λ is congruent to $j \mod r$, then $\lambda \equiv j\ell(\lambda) \mod r$, so that $\mathbb{C}[p_{rk+j}]$ is contained in the equalizer of $\overline{\zeta}_\omega$ and $\zeta \circ S_j$. On the other hand, for any integer $q \not\equiv j \mod r$, the partition $\widetilde{\lambda} = \mathrm{sort}\,(\lambda \cup (q))$ has $|\widetilde{\lambda}| - j(\ell(\widetilde{\lambda}) \equiv |\lambda| + q - j(\ell(\lambda)) - j \equiv q - j \not\equiv 0 \mod r$. Thus $\mathbb{C}[p_{rk+j}]$ is maximal subcoalgebra for which $\overline{\zeta}_\omega = \zeta \circ S_j$. □

Another direction for future explorations comes from considering the BKP hierarchy, which is connected to the $r = 2$ case. Let $\Gamma^+_B(u) = \Gamma^+(u)$ for the choice $a = (1, 0, ...)$ and $b = (-1, 0, ...)$. If we write

$$\Gamma^+_B(u) = \sum_{m \in \mathbb{Z}} \Gamma^+_{m,B} u^m,$$

then the *BKP hierarchy* is defined by the following bilinear identity:

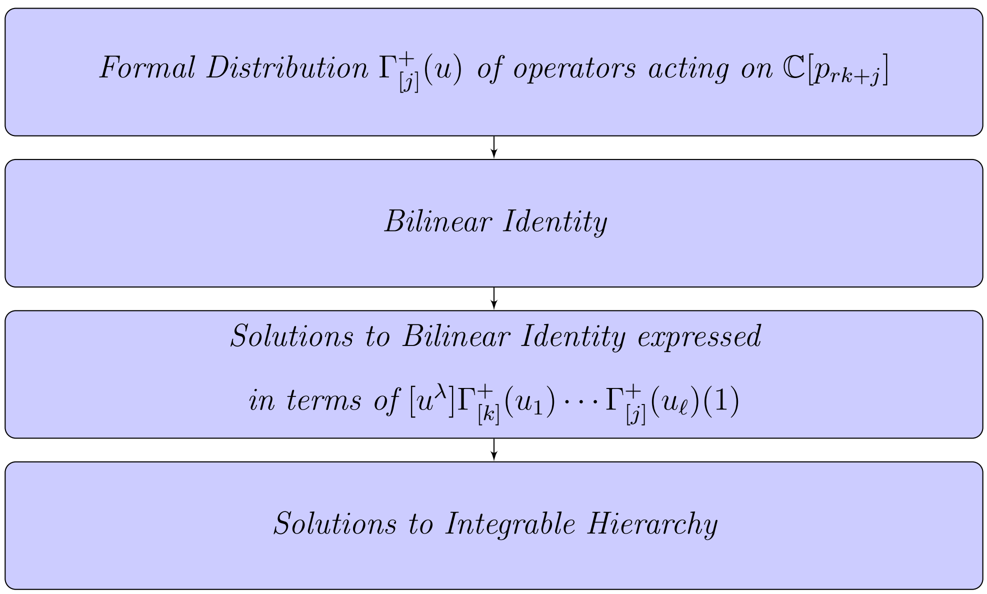
$$\Omega_B\left(\tau \otimes \tau\right) = 0, \tag{4.12}$$

where $\tau = \tau(p_1, p_3, ...)$, $\Omega_B = [u^0]\left(\Gamma^+_B(u) \otimes \Gamma^+_B(-u)\right)$, and we restrict the action of $\Gamma^+_B(\pm u)$ to $\mathbb{C}[p_{2k+1}]$. A *polynomial tau function* for the BKP hierarchy is a non-zero solution $\tau \in \mathbb{C}[p_1, p_3, ...]$. You ([27]) showed that the Schur Q-functions are polynomial tau functions of the BKP hierarchy. Using the language of this chapter, the bilinear identity for the BKP hierarchy can be written in terms of the formal distribution $\Gamma^+_{[1]}$ for $\omega = -1$. The polynomial tau functions for this hierarchy can be obtained by the vertex operator realization (cf. [13])

$$Q_\lambda = [u^\lambda]\Gamma^+_{[1]}(u_1)\cdots\Gamma^+_{[1]}(u_\ell)(1).$$

We ask if this can be generalized to $r > 2$.

Question 4.2.2. *Can we realize the following schematic?*



Question 4.2.3. *Can we realize a schematic similar to that of Question 4.2.2 with any of the following changes:*

1. *Replace Formal Distribution $\Gamma^+_{[j]}(u)$ with Formal Distribution $\Gamma^+_\omega(u)$ in the first and fourth blocks;*

2. *Replace 'Bilinear' with 'Multilinear,' to reflect the move from $r = 2$ to more general r. For example, consider like $\Gamma^+_\omega(u) \otimes \Gamma^+_\omega(\omega u) \otimes \Gamma^+_\omega(\omega^2 u)$ for ω a third root of unity.*

Addressing Questions 4.2.2 and 4.2.3 is beyond the scope of this text.

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Appendix A

A Fine-Tuning Lemma

Lemma A.0.1. $wa(z/w) = -za(w/z) \iff$ *there exists* i *such that* $a_j = \delta_{i,j}$ *for all* j. *(WLOG take* $i = 1$*.)*

Proof. We will make use of the following expansion:

$$z^\ell w^\ell (wa(z/w) + za(w/z)) = \sum_{J,K \geq 0} c_{JK} z^J w^K.$$

Since

$$\begin{aligned} z^\ell w^\ell (wa(z/w) + za(w/z)) = &\sum_k (-1)^k z^{\ell+k} w^{\ell+1-k} e_k(a) \\ &+ \sum_k (-1)^k z^{\ell+1-k} w^{\ell+k} e_k(a), \end{aligned}$$

it follows that

- $c_{JK} \neq 0$ only if $J + K = 2\ell + 1$ and $J, K \neq 0$;
- $c_{JK} \neq 0$ only if $k = J - \ell \geq 0$ and $K = \ell + 1 - k$ or $J = \ell + 1 - k \geq 0$ and $K = \ell + k$ for some $0 \leq k \leq \ell$.

Therefore, we get that

$$c_{JK} = \begin{cases} (-1)^{J-\ell} e_{J-\ell} + (-1)^{\ell+1-J} e_{\ell+1-J} & \text{if } J+K = 2\ell+1, J, K \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Now if we assume $wa(z/w) = -za(w/z)$ and equate coefficients of

$$\sum_{J,K} c_{JK} z^J w^K = 0,$$

then by letting J range from 2ℓ to $\ell+1$, we get the following system of equations:

$$\begin{gathered} (-1)^{\ell} e_{\ell}(a) + (-1)^{-\ell+1} e_{-\ell+1}(a) = 0, \\ \vdots \\ (-1)^{\ell-i} e_{\ell-i}(a) + (-1)^{-\ell+1+i} e_{-\ell+1+i}(a) = 0, \\ \vdots \\ (-1)^{1} e_{1}(a) + (-1)^{0} e_{0}(a) = 0. \end{gathered}$$

In each equation except for the last, the second term vanishes. With this observation in hand, the top equation implies that some $a_j = 0$. Without loss of generality, assume it is a_ℓ. Then we reduce the second equation to the equation

$$e_{\ell-1}(a_1, ..., a_{\ell-1}, 0) = 0,$$

and by stability we may argue just as we did with the top equation, yielding $a_{\ell-1} = 0$. Continuing in this line of argument all the way to the bottom equation, we find that $a_i = 0$ for $i > 1$. The bottom equation in the system then reduces to

$$a_1 = 1.$$

$\square$

Appendix B

Example Calculations: Discrete Fourier Decomposition of Hypergeometric Series

$r = 2$

$Q_0 = \frac{1}{2}\left(L_0 + L_1\right)$

$Q_1 = \frac{1}{2}\left(L_0 - L_1\right)$

$r = 3$

$Q_0 = \frac{1}{3}\left(L_0 + L_1 + L_2\right)$

$Q_1 = \frac{1}{3}\left(L_0 + \omega^2 L_1 + \omega^1 L_2\right)$

$Q_2 = \frac{1}{3}\left(L_0 + \omega^1 L_1 + \omega^2 L_2\right)$

$r = 5$

$Q_0 = \frac{1}{5}\left(L_0 + ... + L_4\right)$

$Q_1 = \frac{1}{5}\left(L_0 + \omega^4 L_1 + \omega^3 L_2 + \omega^2 L_3 + \omega L_4\right)$

$Q_2 = \frac{1}{5}\left(L_0 + \omega^3 L_1 + \omega L_2 + \omega^4 L_3 + \omega^2 L_4\right)$

$Q_3 = \frac{1}{5}\left(L_0 + \omega^2 L_1 + \omega^4 L_2 + \omega L_3 + \omega^3 L_4\right)$

$Q_4 = \frac{1}{5}\left(L_0 + \omega L_1 + \omega^2 L_2 + \omega^3 L_3 + \omega^4 L_4\right)$

$r = 7$

$Q_0 = \frac{1}{7}(L_0 + ... + L_6)$

$Q_1 = \frac{1}{7}\left(L_0 + \omega^6 L_1 + \omega^5 L_2 + ... + \omega L_6\right)$

$Q_2 = \frac{1}{7}\left(L_0 + \omega^5 L_1 + \omega^3 L_2 + \omega L_3 + \omega^6 L_4 + \omega^4 L_5 + \omega^2 L_6\right)$

$Q_3 = \frac{1}{7}\left(L_0 + \omega^4 L_1 + \omega L_2 + \omega^5 L_3 + \omega^2 L_4 + \omega^6 L_5 + \omega^3 L_6\right)$

$Q_4 = \frac{1}{7}\left(L_0 + \omega^3 L_1 + \omega^6 L_2 + \omega^2 L_3 + \omega^5 L_4 + \omega L_5 + \omega^4 L_6\right)$

$Q_5 = \frac{1}{7}\left(L_0 + \omega^2 L_1 + \omega^4 L_2 + \omega^6 L_3 + \omega L_4 + \omega^3 L_5 + \omega^5 L_6\right)$

$Q_6 = \frac{1}{7}\left(L_0 + \omega L_1 + \omega^2 L_2 + ... + \omega^6 L_6\right)$