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APPLICATION OF THE METHOD OF PARTICULAR
SOLUTIONS TO INDUSTRIAL MANAGEMENT SYSTEMS

by

SHIH-TSU SUN

B.S.(Ind.Mgt.), Cheng Kung University
Tainan, Taiwan, China, 1966

9589

A MASTER'S THESIS

submitted in partial fulfillment of the
requirements for degree

MASTER OF SCIENCE

Department of Industrial Engineering

KANSAS STATE UNIVERSITY

Manhattan, Kansas

1972

Approved by:


Major Professor

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1972
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CHAPTER 1

INTRODUCTION

1.1 INITIAL AND BOUNDARY VALUE PROBLEMS

Calculus of variations and maximum principle are two approaches frequently used for solving dynamic optimization problems. When these methods are used, the problem generally reduced to a system of nonlinear differential equations of the two-point boundary value type. In general, problems of this type cannot be solved analytically. Various numerical methods have been derived to solve a nonlinear differential equation of the two-point boundary value type, such as methods based on successive approximations and methods based on iterative procedures.

For a system of differential equations, if the given conditions are specified at both ends of an interval, the problem is known as two-point boundary value problems. If all conditions are specified at the same point, it is called initial value problems. If conditions are imposed on more than two points, the problem is known as multi-point boundary value problem.

By introducing additional variables, an ordinary differential equation of higher order can be replaced by a set of first order differential equations. Therefore, in this thesis only systems of first order differential equations will be discussed.

There are various methods for solving initial value problems. These can be separated into single-step methods such as the Runge-Kutta scheme and multiple-step methods such as the predictor-corrector schemes [4,5].

Because some of the initial conditions are not given, the numerical solutions of the two-point boundary value problem cannot be obtained easily.

This chapter gives a brief review of some frequently used techniques available for solving boundary value problems. These techniques are the method of complementary functions and the method of adjoint functions developed by Goodman and Lance [6], and the quasilinearization method developed by Bellman and Kalaba [8].

1.2 THE METHOD OF COMPLEMENTARY FUNCTIONS AND THE METHOD OF ADJOINT FUNCTIONS

Goodman and Lance [6], discussed two commonly used methods for solving linear two-point boundary value problems. These two methods are the method of adjoint functions and the method of complementary functions. Each method leads to a system of linear algebraic equations for the missing initial (final) conditions of the two-point boundary value problems.

Consider a general system of linear ordinary differential equations in matrix form

$$\dot{Y} = A(t)Y + F(t) \quad (1-1)$$

where the dot denotes the differentiation with respect to the independent variable t . Y is an $n \times 1$ matrix with components $Y_i(t)$, $A(t)$ is an $n \times n$ matrix and $F(t)$ is an $n \times 1$ matrix with components $f_i(t)$.

We shall assume that the γ initial conditions given at $t = t_0$ are

$$Y_i(t_0) = Y_i^0 \quad i = 1, 2, \dots, \gamma \quad (1-2)$$

and the $n-\gamma$ final conditions given at $t = t_f$ are

$$y_{i_m}(t_f) = y_{i_m}^1 \quad m = 1, 2, \dots, n-\gamma \quad (1-3)$$

(A) The method of Adjoint Functions

The adjoint system of Eq. (1-1) is

$$\dot{X} = -A^T(t)X \quad (1-4)$$

where $X(t)$ is $n \times 1$ vector with components $x_i(t)$ and T indicates the transpose. The x_i 's and y_i 's are related by

$$\frac{d}{dt} \left(\sum_{i=1}^n x_i y_i \right) = \sum_{i=1}^n x_i f_i \quad (1-5)$$

Eq. (5) can be obtained as the following

$$\frac{d}{dt} \left(\sum_{i=1}^n x_i y_i \right) = \frac{d}{dt} (X^T Y) = X^T \dot{Y} + \dot{X}^T Y \quad (1-6)$$

Substitute Eq. (1) and (4) into Eq. (6)

$$\begin{aligned} \frac{d}{dt} \left(\sum_{i=1}^n x_i y_i \right) &= X^T (AY + F) - (A^T X)^T Y \\ &= X^T AY + X^T F - X^T AY \\ &= X^T F = \sum_{i=1}^n x_i f_i \end{aligned} \quad (1-7)$$

Integrating Eq. (7) over the interval (t_0, t_f) , we have

$$\sum_{i=1}^n x_i(t_f)y_i(t_f) - \sum_{i=1}^n x_i(t_0)y_i(t_0) = \int_{t_0}^{t_f} \sum_{i=1}^n x_i(t)f_i(t)dt \quad (1-8)$$

Now we integrate the adjoint system Eq. (1-4) backward $n-\gamma$ times with the following final conditions

$$\begin{aligned} x_{m,i}(t_f) &= 0 & i &\neq m + \gamma \\ &= 1 & i &= m + \gamma \\ & & i &= 1, 2, \dots, n \\ & & m &= 1, 2, \dots, n - \gamma \end{aligned} \quad (1-9)$$

Let ${}_m x_i(t)$ denote the value of x_i obtained from the m^{th} integration of the adjoint system and substituting this ${}_m x_i(t)$ into Eq. (1-8), we have

$$\sum_{i=1}^n {}_m x_i(t_0)y_i(t_0) - y_{i_m}(t_f) = -\int_{t_0}^{t_f} \sum_{i=1}^n {}_m x_i(t)f_i(t)dt \quad (1-10)$$

$$m = 1, 2, \dots, n - \gamma$$

There are $n - \gamma$ simultaneous linear algebraic equations for the $n - \gamma$ unknown initial conditions $y_{i_m}(t_0)$. Once the missing initial conditions are obtained, the problem becomes an initial value problem.

Total number of integrations by the method of adjoint functions are $n - \gamma$ for the adjoint system plus one for the original given system.

(B) The method of complementary functions

The homogeneous equation derived from Eq. (1) is

$$\dot{U} = A(t)U \quad (1-11)$$

Integrate this equation forward $n - \gamma$ times with the initial conditions

$$\begin{aligned} U_{m,i}(t_0) &= 0 & i &\neq \gamma + m \\ &= 1 & i &= \gamma + m \\ & & m &= 1, 2, \dots, n - \gamma \end{aligned}$$

Let this integration results be represented by ${}_m U_i(t)$, where the subscript m represents the m^{th} integration of $U_i(t)$. The nonhomogeneous equations

$$\dot{V} = A(t)V + F(t) \quad (1-12)$$

is then integrated forward once with the initial conditions

$$\begin{aligned} v_i(t_0) &= y_i(t_0) & i &= 1, 2, \dots, \gamma \\ &= 0 & i &= \gamma + 1, \dots, n \end{aligned}$$

Let $v_i(t)$ denote the result of this integration.

Now the general solution of Eq. (1), with the given conditions, can be written as

$$y_i(t) = v_i(t) + \sum_{m=1}^{n-\gamma} A_{mm} u_i(t) \quad (1-13)$$

$$i = 1, 2, \dots, n$$

Substituting Eq. (1-3) into Eq. (1-13), $n - \gamma$ integration constants a_m can be solved by the $n - \gamma$ linear algebraic equations:

$$y_{i_m}(t_f) = v_{i_m}(t_f) + \sum_{m=1}^{n-\gamma} a_m u_{i_m}(t_f) \quad (1-14)$$

$$m = 1, 2, \dots, n - \gamma$$

By letting $t = t_0$ in Eq. (13), it is seen that the missing initial conditions $y_{\gamma+m}(t_0) = a_m$, $m = 1, 2, \dots, n-\gamma$. By this method, also we require $n - \gamma + 1$ integrations.

S. M. Roberts and J. S. Shipman [7], have compared these two methods according to the ease of programming, computation time, accuracy and stability. There seems to be no real advantage of one method over the other. But, if a problem is stable on integrating forward, and is unstable on integrating backward, the method of complementary functions is preferred. Otherwise, the method of adjoint functions is preferred.

The method of adjoint functions can also be applied to nonlinear system [6], [7]. It is called shooting method in common use. More detail

discussions can be found in references [6] and [7].

1.3 Quasilinearization

In this section we discuss a powerful method for solving nonlinear two-point boundary value problems. The quasilinearization method is essentially a generalized Newton-Raphson method for functional equations [4]. By this method, the nonlinear differential equations are solved recursively by a series of linear differential equations. The linear equations is obtained by applying the Taylor's series expansion with second and higher order terms omitted.

Consider a general nonlinear, ordinary, differential system

$$\begin{aligned}\dot{x}_i &= f_i(x, t) & (1-15) \\ i &= 1, 2, \dots, m \\ t_0 &\leq t \leq t_f\end{aligned}$$

where the dot denotes differentiation with respect to independent variable t . Boundary conditions are given as

$$\begin{aligned}x_i(t_0) &= x^0 & i = 1, 2, \dots, \gamma \\ x_i(t_f) &= x^1 & i = \gamma + 1, \dots, m\end{aligned} \quad (1-16)$$

Choosing a set of initial approximation x_{i0} , by the Taylor's series expansion, Eq. (15) can be linearized as

$$\dot{x}_{i, n+1} = f_i(x_{i,n}, t) + J_{(x_{i,n})}(x_{i,n+1} - x_{i,n}) \quad (1-17)$$

$$i = 1, 2, \dots, m$$

$$n = 0, 1, 2, \dots$$

where

$$J_{(x_{i,n})} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_m} \\ \dots & \dots & \dots & \dots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \dots & \frac{\partial f_m}{\partial x_m} \end{bmatrix} \quad (1-18)$$

For the initial approximations $x_{i,0}$, the values of $f_i(x_{i,0}, t)$ and jacobins $J_{(x_{i,0})}$ can be obtained. The only unknown in Eq. (17) then is $x_{i,1}$. This linearized equation with boundary conditions given by Eq. (16) can be solved by either method discussed in the previous section.

After the unknown $x_{i,n+1}$ was obtained, set it as initial approximations for the next iteration. This procedure can be continued until the desired accuracy is satisfied.

The main advantage of this method is, if the procedure converges, it converges quadratically to the solution of the original system. The main difficulty is in obtaining the integration constants, ill conditioned equations can make the method such as complementary functions useless.

More detailed discussions of quasilinearization can be found in Ref. [4].

CHAPTER 2

THE METHOD OF PARTICULAR SOLUTIONS

2.1 INTRODUCTION

Many techniques have been derived for solving the two-point boundary value problems. In the previous chapter, the method of adjoint functions and the method of complementary functions [6] have been discussed. Both methods have one common characteristic: each requires to solve two differential systems, that is the original system and the derived system. This derived system is the homogeneous system in the method of complementary functions and the adjoint equations in the method of adjoint functions.

By the method of particular solutions, several sets of particular solutions are obtained from the original system, then linear combination is used. It has the following advantages as compared with the previous techniques: (1) it requires only the original differential system; (2) it employs stepsize control technique to speed the rate of convergence; (3) in a physical problem, each particular solution represents a possible trajectory satisfies the initial conditions.

The method of particular solutions can be applied not only to linear systems, but also to nonlinear systems. First, the nonlinear system is linearized into a system which is linear in the perturbation about a nominal function. Then, applying the method of particular solutions to find the perturbation leading to a new nominal function. The procedure is employed iteratively, until the desired accuracy is obtained.

Since when solving a nonlinear system, the system must be linearized first, we will discuss how to handle a linear system first. Then in

sections 3, 4 and 5, nonlinear system was discussed. Two different algorithms for solving nonlinear systems are summarized in section 6, discussion and conclusion are given in section 7. More detailed discussion can be found in Ref. [1,2]. The following discussion follows the work of Miele and Iyer [1, 2, 3].

2.2 LINEAR SYSTEM

Consider the linear system

$$\dot{X} = M(t)X + N(t) \quad (2-1)$$

$$0 \leq t \leq 1$$

where t is the independent variable, the matrices $\dot{X}$ and X are $n \times 1$; the matrices M and N are $n \times n$ and $n \times 1$ respectively. At the initial time $t = 0$, p scalar components of x are given, that is

$$BX(0) = A \quad (2-2)$$

matrices B and A are $p \times n$ and $p \times 1$ respectively, and components of A are a_i , $i = 1, 2, \dots, p$. At the final time $t = 1$, q scalar relations must be satisfied in matrix form as

$$CX(1) = D \quad (2-3)$$

where C and D are given matrices with $q \times n$ and $q \times 1$. Finally, assume that

$$p + q = n \quad \text{and} \quad p \geq q$$

Now we can formulate the following problem: Find the functions

$$x_i = x_i(t) \quad i = 1, 2, \dots, n$$

satisfying the differential system (2-1), the given initial conditions (2-2) and final conditions (2-3).

To solve this problem by the method of particular solutions, Eq. (2-1) is integrated forward $q + 1$ times with $q + 1$ sets of initial conditions

$$x_{ij}(0) = a_i \quad j = 1, 2, \dots, q + 1, \quad i = 1, 2, \dots, p \quad (2-4)$$

$$x_{p+k,j}(0) = k_{ij} \quad j = 1, 2, \dots, q + 1, \quad k = 1, 2, \dots, q$$

where δ_{ij} denotes the kronecker delta. From these integrations, we obtain the function

$$x_{pj} = x_{pj}(t) \quad j = 1, 2, \dots, q + 1 \quad (2-5)$$

where x_{pj} is $n \times 1$ matrix and subscript pj denotes j^{th} set of particular solutions of Eq. (2-1), therefore,

$$\dot{x}_{pj} = M(t)x_{pj} + N(t) \quad j = 1, 2, \dots, q + 1 \quad (2-6)$$

Now we introduce $q + 1$ undetermined scalar constants k_j and form the linear combination

$$X = \sum_{j=1}^{q+1} k_j X_{pj} \quad (2-7)$$

We will prove whether, by an appropriate choice of the constants k_j , this linear combination can satisfy the differential system (2-1), the initial condition (2-2) and the final condition (2-3).

Substitute Eq. (2-7) into Eq. (2-1), we obtain the relation

$$\sum_{j=1}^{q+1} k_j [\dot{X}_{pj} - M(t)X_{pj}] = N(t) \quad (2-8)$$

From Eq. (2-6) and Eq. (2-8), it is obvious that if

$$\sum_{j=1}^{q+1} k_j = 1 \quad (2-9)$$

Eq. (2-7) will satisfy the differential system (2-1). By substituting Eq. (2-7) into Eq. (2-2), we obtain

$$\sum_{j=1}^{q+1} k_j [BX_{pj}(0)] = A \quad (2-10)$$

which, in the light of Eq. (2-6), can be written as

$$\sum_{j=1}^{q+1} k_j A = A \quad (2-11)$$

this means the initial conditions are satisfied providing the constants k_j are consistent with Eq. (2-9).

Finally, by substituting Eq. (2-7) into final condition (2-3) we obtain the relation

$$\sum_{j=1}^{q+1} k_j [CX_{pj}(1)] = D \quad (2-12)$$

which is equivalent to q scalar equations.

Hence, we have $q + 1$ simultaneous linear algebraic equations (2-9) and (2-12), from which $q + 1$ unknown constants k_j can be solved. After that, the general solutions $X(t)$ of the linear, two-point boundary value problem can be obtained from Eq. (2-7).

Notice if $p < q$, it is convenient to integrate Eq. (2-1) backward in order to reduce the computational efforts. More detail discussions can be found in Ref. (2).

2.3 NONLINEAR SYSTEM

Consider a system described by the differential Eqs.

$$\dot{X} - F(x, t) = 0 \quad 0 \leq t \leq 1 \quad (2-13)$$

subject to the boundary conditions

$$f[x(0)] = 0 \quad (2-14)$$

$$g[x(1)] = 0 \quad (2-15)$$

$$h[x(0), x(1)] = 0 \quad (2-16)$$

Where $\dot{X}$ and F are $n \times 1$ matrix, f is $p \times 1$ and g is $q \times 1$ matrix, h is $\gamma \times 1$ matrix, with $p + q + \gamma = n$. t is an independent variable. The prescribed initial time is $t = 0$ and final time is $t = 1$. The dot denotes differentiation with respect to time t .

It is assumed that F is a continuous function of the arguments X and t , also the first derivatives of the function f, g, h with respect to vector X exists and are continuous. The problem is to find the continuous vector function $X(t)$ which solves Eqs. (2-13) through (2-15).

Since the systems are nonlinear, the approximate method will be used. In this connection, consider the continuous function $X(t)$, which satisfies the initial conditions but not necessarily the differential system and the other boundary conditions.

Define P function as

$$P = \int_0^1 (\dot{X} - F)^T (\dot{X} - F) dt + (f^T f + g^T g + h^T h) \quad (2-17)$$

which measures the cumulative error in the differential equation (2-14) and boundary conditions (2-15) and (2-16). For any $X(t)$ satisfying Eqs. (2-14) through (2-16), $P = 0$; otherwise $P > 0$. When approximate methods are being used, they must ultimately lead to a state such that

$$P \leq \epsilon \quad (2-18)$$

where ϵ is a small, prescribed value.

In order to solve the proposed problem, consider a nominal function $X(t)$ and a varied function $\tilde{X}(t)$ such that

$$\tilde{X}(t) = X(t) + \Delta X(t) \quad (2-19)$$

Where $\Delta X(t)$ denotes the perturbation of X at a certain time t . The passage from the nominal function to the varied function causes the cumulative error to change, later we shall prove that the function P has a descent property.

In order to find the varied function from the nominal function, first, linearize Eq. (2-14) through Eq. (2-16) and obtain the following system:

$$\Delta \dot{X} - F_x^T(x, t)\Delta X + [\dot{X} - F(x, t)] = 0 \quad (2-20)$$

$$f_{x(0)}^T \Delta X + f[x(0)] = 0 \quad (2-21)$$

$$g_{x(1)}^T \Delta X + g[x(1)] = 0 \quad (2-22)$$

$$h_{x(0)}^T \Delta X(0) + h_{x(1)}^T \Delta X(1) + h[x(0), x(1)] = 0 \quad (2-23)$$

to prevent the variations $\Delta X(t)$ becoming too large, the above system is imbed into a more general system

$$\Delta \dot{X} - F_x^T(x, t)\Delta X + \alpha[\dot{X} - F(x, t)] = 0$$

$$f_{x(0)}^T \Delta X + \alpha f[x(0)] = 0 \quad (2-24)$$

$$g_{x(1)}^T \Delta X + \alpha g[x(1)] = 0$$

$$h_{x(0)}^T \Delta X(0) + h_{x(1)}^T \Delta X(1) + \alpha h[x(0), x(1)] = 0$$

where α denotes a scaling factor (stepsize) in the range

$$0 \leq \alpha \leq 1 \quad (2-25)$$

Next, introducing the auxiliary variable

$$A = \frac{\Delta X}{\alpha} \quad (2-26)$$

rewrite Eq. (2-24) as

$$\dot{A} - F_x^T(x, t)A + [\dot{X} - F(x, t)] = 0 \quad (2-27)$$

$$f_{x(0)}^T A(0) + f[x(0)] = 0 \quad (2-28)$$

$$g_{x(1)}^T A(1) + g[x(1)] = 0 \quad (2-29)$$

$$h_{x(0)}^T A(0) + h_{x(1)}^T A(1) + h[x(0), x(1)] = 0 \quad (2-30)$$

For a given nominal function $X(t)$, the vector $\dot{X} - F$ and matrices F_x^T , $f_{x(0)}^T$, $g_{x(1)}^T$, $h_{x(0)}^T$, $h_{x(1)}^T$, $f[x(0)]$, $g[x(1)]$, $h[x(0), x(1)]$ are known quantities. Therefore, Eq. (2-27) through (2-30) form a linear, two-point boundary value problem with dependent variable $A(t)$.

Apply the method of particular solutions, integrate Eq. (2-27) forward $q + \gamma + 1$ times with the following initial conditions:

$$A_{ij}(0) = a_{ij} \quad j = 1, 2, \dots, q + \gamma + 1, \quad i = 1, 2, \dots, p \quad (2-31)$$

$$A_{p+k,j}(0) = \delta_{kj} \quad j = 1, 2, \dots, q + \gamma + 1, \quad k = 1, 2, \dots, q + \gamma \quad (2-32)$$

Where the first subscript denotes variable number and subscript j denotes the j^{th} set of particular solutions. Equation (2-31) is obtained by solving Eq. (2-28). δ_{kj} denotes Kronecker delta.

After performing the integration, following functions are obtained:

$$A_{pj} = A_{pj}(t) \quad j = 1, 2, \dots, q + \gamma + 1 \quad (2-33)$$

Where p_j denotes the j^{th} set of particular solutions.

Now introducing $q + \gamma + 1$ undetermined scalar constants k_j and form the linear combination

$$A(t) = \sum_{j=1}^{q+\gamma+1} k_j A_{pj}(t) \quad (2-34)$$

This linear combination satisfies the Eq. (2-27) through (2-30) providing

$$\sum_{j=1}^{q+\gamma+1} k_j = 1$$

$$\sum_{j=1}^{q+\gamma+1} k_j [g_{x(1)}^T A_{pj}(1)] + g = 0 \quad (2-35)$$

$$\sum_{j=1}^{q+\gamma+1} k_j [h_{x(0)}^T A_{pj}(0) + h_{x(1)}^T A_{pj}(1)] + h = 0$$

Equation (2-35) gives $q + \gamma + 1$ simultaneous linear algebraic equations and can be solved for $q + \gamma + 1$ unknown constants k_j . After that, $A(t)$ can be obtained from Eq. (2-34). If stepsize α is specified, the perturbation $\Delta X(t)$ can be obtained from Eq. (2-26) and the varied function $\tilde{X}(t)$ can be computed from Eq. (2-19).

2.4 DETERMINATION OF THE STEPSIZE

After we obtain $A(t)$, Eq. (2-19) is a function of stepsize α . The performance index p becomes a function of the form

$$p = p(\alpha) \quad (2-36)$$

Assume the minimum of $p(\alpha)$ exists, the corresponding stepsize which gives the minimum p value can be determined by

$$P\alpha(\alpha) = 0 \quad (2-37)$$

Since the rigorous determination of α takes time on a computer, we will solve optimum α as follows: assign the value 1 to the stepsize α , this is acceptable only if

$$P(\alpha) < P(0) \quad (2-38)$$

Otherwise, the previous value of α must be replaced by some smaller value in the range zero and one (for example, by a bisection process). The process is repeated until the inequality (2-38) is satisfied.

Since the above procedure is guaranteed by the decent property of P function, we will discuss this property in the next section.

2.5 DESCENT PROPERTY OF PERFORMANCE INDEX

The performance index P has been defined by Eq. (2-17). The first variation of P function is given by

$$\begin{aligned}
\delta P = & 2 \int_0^1 [\dot{X} - F]^T [\Delta \dot{X} - F_x^T \Delta X] dt + \\
& 2[f^T f_{x(0)}^T \Delta X(0) + g^T g_{x(1)}^T \Delta X(1) + h^T h_{x(0)}^T \Delta X(0) \\
& + h^T h_{x(1)}^T \Delta X(1)] \quad (2-39)
\end{aligned}$$

Substitute Eq. (2-24) into (2-39), we obtain

$$\begin{aligned}
\delta P = & -2\alpha \int_0^1 [\dot{X} - F]^T [\dot{X} - F] dt - 2\alpha[f^T f + g^T g + h^T h] \\
= & -2\alpha \left\{ \int_0^1 [\dot{X} - F]^T [\dot{X} - F] dt + f^T f + g^T g + h^T h \right\} \\
= & -2\alpha P \quad (2-40)
\end{aligned}$$

Since $P > 0$ and α has the range as given by Eq. (2-25), if α is sufficiently small, the decrease of the P function is guaranteed, that is

$$\tilde{P} < P \quad (2-41)$$

2.6 ALGORITHMS FOR NONLINEAR SYSTEMS

(1) Scheme A

- (a) Assume a nominal function $X(t)$ consistent with the initial conditions and possibly, with the final conditions.
- (b) Along the interval of integration, compute the vector $\dot{X} - F$ and the matrix F_x . On the boundary, compute the vector

f, g, h and the matrices $f_{x(0)}, g_{x(1)}, h_{x(0)}$ and $h_{x(1)}$.

- (c) Determine the $q + \gamma + 1$ particular solutions $A_{pj}(t)$ by forward integration of the differential equation (3-27), subject to the initial conditions given by Eqs. (2-31) and (2-32).
- (d) Compute the $q + \gamma + 1$ scalar constants k_j solving simultaneous linear equations (2-35).
- (e) Determine the function $A(t)$ by Eq. (2-34).
- (f) Determine the stepsize in such a way that Ineq. (2-38) is satisfied. To this effect, perform a bisection process on α with starting value $\alpha = 1$.
- (g) Compute the varied function $\tilde{X}(t)$ from Eq. (2-19).
- (h) With $\tilde{X}(t)$ known, the iteration is completed. The varied function $\tilde{X}(t)$ becomes the nominal function $X(t)$ for next iteration and return to step (a).
- (i) Terminate the algorithm if the stopping condition Eq. (2-18) is satisfied.

(2) Scheme B

- (a) Assume a nominal initial value $X(0)$ consistent with the given initial conditions. Integrate Eq. (2-14) forward to obtain the nominal function $X(t)$, for $0 \leq t \leq 1$.
- (b) For the nominal function $X(t)$, compute the matrix F_x as a function of time t , also compute matrices $f_{x(0)}, g_{x(1)}, h_{x(0)}, h_{x(1)}$ and vector f, g, h .
- (c) Determine the $q + 1$ particular solutions $A_{pj}(t)$ by forward integration of the differential equation (2-27).

- (d) Compute the $q + 1$ constants k_j by solving the linear algebraic system (2-35).
- (e) Determine the initial value $A(o)$ from Eq. (2-34) applied at $t = 0$.
- (f) Determine the stepsize α in such a way that Ineq. (2-38) is satisfied. To this effect, perform a bisection process on α , start from $\alpha = 1$.
- (g) Find the initial points of the varied function by

$$\tilde{X}(o) = X(o) + \alpha A(o)$$

Then, integrate Eq. (2-15) forward with this initial condition to obtain $\tilde{X}(t)$, $0 \leq t \leq 1$.

- (h) $\tilde{X}(t)$ becomes the nominal function for the next iteration. Go to step (b) and start algorithm again.
- (i) The algorithm is terminated when the stopping condition Eq. (2-18) is satisfied.

There are two main differences between scheme (A) and scheme (B). First, in scheme (A), the initial approximation $X(t)$ for $0 \leq t \leq 1$, has the value consistent with the given initial value $X(o)$. In scheme (B), $X(t)$ is obtained by forward integration of the original system with initial condition $X(o)$. Second, the linearized differential system is a non-homogeneous system in scheme (A) and is a homogeneous system in scheme (B).

2.7 DISCUSSION AND CONCLUSION

In preceding sections, a general method for solving nonlinear two-

point boundary value problems is presented. It is assumed that the differential system has order n and is subject to p initial conditions, q final conditions and γ mixed boundary conditions, where $p + q + \gamma = n$.

The method is based on the consideration of the performance index P , which is the cumulative error in the differential equations and the boundary conditions.

The algorithm has the form $\tilde{X}(t) = X(t) + \Delta X(t)$, where $X(t)$ is a nominal function satisfying the initial conditions. The differential system is linearized about $X(t)$, then, the stepsize α , $0 \leq \alpha \leq 1$, is introduced to each forcing term. The method of particular solutions is then employed to obtain the perturbation $\Delta X(t) = \alpha A(t)$, leading from the nominal function $X(t)$ to the varied function $\tilde{X}(t)$.

Since the cumulative error P has a decent property, if α is sufficiently small, the passage of nominal function to the varied function will decrease the value of P . The stepsize is determined by a one-dimensional search until the inequality $\tilde{P} < P$ is satisfied. This can be achieved through a bisection process starting from $\alpha = 1$. The algorithm is terminated when performance index P function is smaller than a prescribed value.

There are three main advantages of this method. First, it employs only the original differential system. Second, stepsize control is used to speed the convergence rate. Third, in physical problems, each particular solution represents a physically possible trajectory even though it satisfies only the initial conditions.

The method of particular solutions requires more undetermined constants as compared with the methods mentioned in chapter one. Another drawback is that the performance index P function does not guarantee

convergence. In other words, as the number of iterations increases, we cannot guarantee that $P \rightarrow 0$. This is due to the fact that the convergence depends on the analytical nature of the functions F , f , g and h . Furthermore, the convergence also depends on the nominal function $X(t)$ used to start the algorithm.

CHAPTER THREE

APPLICATION TO AN ADVERTISEMENT PROBLEM

3.1 DEVELOPMENT OF THE MODEL

The well known diffusion model [10] is used for advertising. Consider a group of people in which only certain members possesses a particular piece of information about a firm's product. Suppose that the total number of persons in this group remains constant and the diffusion of information occurs only through personal contact. The number of contacts made by an average informed person in an arbitrary time is given by a contact coefficient. In a contact, the contactee receives information if he does not already have it. If he already has it, the contact is wasted in the sense that it does not increase the number of informed people.

Let

N = The total number of persons in the group.

C = Contact coefficient = The number of contacts made by one informed person per unit time.

$K(t)$ = The number of informed persons at time t .

$K(t)/N$ = Proportion of informed persons at time t .

$CK(t)dt$ = Contacts made during a time interval dt .

The number of informed persons at time t , $K(t)$, can be obtained by solving the following differential equations

$$\frac{dK(t)}{dt} = CK(t) \left(1 - \frac{K(t)}{N}\right) \quad (3-1)$$

Suppose now that the firm can increase the number of its contacts by

spending money on advertising. Specifically, it can increase the number of contacts made by the informed people by an additional number A per unit time. Thus

$$\frac{dK(t)}{dt} = K(t)(C + A(t))\left(1 - \frac{K(t)}{N}\right) \quad (3-2)$$

If each informed person buys C_q units of the firm's product, and $S(t)$ represents the sale at time t , then

$$S(t) = C_q K(t) \quad (3-3)$$

If $C_q = 1$, and substitute for $K(t)$ in Eq. (3-2)

$$\frac{dS(t)}{dt} = S(t)(C + A(t))\left(1 - \frac{S(t)}{N}\right) \quad (3-4)$$

The rate of change of the inventory level $I(t)$ can be presented by

$$\frac{dI(t)}{dt} = P(t) - S(t) \quad (3-5)$$

where $P(t)$ is the production rate at time t . The production rate is assumed as

$$P(t) = a + bt \quad (3-6)$$

Where a and b are given constants.

The problem is to find the advertising rate $A(t)$ such that the

following net profit is maximized

$$J = \int_{t_0}^{t_f} [C_s S(t) - C_I (I_m - I(t))^2 - C_A S(t) A^2(t)] dt \quad (3-7)$$

where C_s is the revenue from the sale of one unit of the product, C_I is the inventory carrying cost, and I_m can be considered as the capacity for the storage of inventory and C_A is the cost of advertisement.

3.2 DEFINITION OF THE PROBLEM

Maximize

$$J = \int_{t_0}^{t_f} [C_s S(t) - C_I (I_m - I(t))^2 - C_A S(t) A^2(t)] dt \quad (3-7)$$

subject to

$$P(t) = a + b(t) \quad (3-8)$$

$$\frac{dI(t)}{dt} = P(t) - S(t) \quad (3-9)$$

$$\frac{dS(t)}{dt} = S(t) (C + A(t) (1 - \frac{S(t)}{N})) \quad (3-10)$$

with boundary conditions

$$I(t_0) = I^0 \quad (3-11)$$

$$S(t_0) = S^0$$

3.3 FORMULATION OF THE PROBLEM

The above problem can be solved by calculus of variations with the help of the method of particular solutions. Detail treatment of the calculus of variations can be found in Ref. [4, 13].

Equations (3-8) and (3-9) can be combined as

$$\frac{dI(t)}{dt} = (a + bt) - S(t) \quad (3-12)$$

The problem now has two state variables I and S and one control variable A . To solve the problem, the Lagrange Multipliers L_1, L_2 are introduced. Define the following functions

$$F = [L_1(\dot{I} - a - bt + S) + L_2(\dot{S} - S(C + A + \frac{SC}{N} - \frac{SA}{N})) + C_S S - C_I(I_m - I)^2 - C_A SA^2] \quad (3-13)$$

$$J = \int_{t_0}^{t_f} F dt \quad (3-14)$$

Where $\dot{I}$ and $\dot{S}$ represent the differentiation of I and S with respect to t .

Equation (3-14) will be maximized, this is a Bolza problem in the calculus of variations [4]. The Euler - Lagrange equations can now be written as

$$\frac{d}{dt}(\frac{\partial F}{\partial \dot{I}}) - \frac{\partial F}{\partial I} = 0 \quad (3-15)$$

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{S}}\right) - \frac{\partial F}{\partial S} = 0 \quad (3-16)$$

$$\frac{\partial F}{\partial A} = 0 \quad (3-17)$$

that is

$$\frac{dL_1}{dt} = 2X_I(I_m - I) \quad (3-18)$$

$$\frac{dL_2}{dt} = L_1 + C_S - cL_2 + \frac{3L_2^2 S^2}{4C_A N^2} - \frac{L_2^2 S}{C_A N} + \frac{L_2^2}{4C_A} + \frac{2CL_2 S}{N} \quad (3-19)$$

$$A = \frac{L_2}{2C_A} \left(\frac{S}{N} - 1 \right) \quad (3-20)$$

After substituting Eq. (3-20) into Eq. (3-10), the control variable A can be eliminated and obtain the following equations

$$\frac{dI}{dt} = a + bt - S \quad (3-21)$$

$$\frac{dS}{dt} = CS - \frac{CS^2}{N} + \frac{S^2 L_2^2}{C_A N} - \frac{SL_2}{2C_A} - \frac{S^3 \lambda_2}{2C_A N^2} \quad (3-22)$$

$$\frac{dL_1}{dt} = 2C_I(I_m - I) \quad (3-23)$$

$$\frac{dL_2}{dt} = L_1 + C_S - cL_2 + \frac{3L_2^2 S^2}{4C_A N^2} - \frac{L_2^2 S}{C_A N} + \frac{L_2^2}{4C_A} + \frac{2CSL_2}{N} \quad (3-24)$$

There are four equations, but only two boundary conditions are given by Eq. (3-11). We need boundary conditions for the Lagrange Multipliers L_1 and L_2 , they can be obtained by using the following nature boundary conditions

$$\frac{\partial F}{\partial \dot{I}} = 0 \qquad \frac{\partial F}{\partial \dot{S}} = 0 \qquad (3-25)$$

for $t = t_0$ and $t = t_f$

Because the boundary conditions at $t = t_0$ are given, only the conditions at $t = t_f$ are needed. By Eq. (3-25), we obtain

$$L_1(t_f) = 0 \qquad (3-26)$$

$$L_2(t_f) = 0$$

Now we have nonlinear differential equations (3-21) through (3-24) and boundary values Eq. (3-11) and Eq. (3-26). In order to maximize the objective function Eq. (3-7), this nonlinear two-point boundary value problem must be solved first. We shall use the method of particular solutions to solve this two-point boundary value problem.

3.4 COMPUTATIONAL PROCEDURE BY ALGORITHM SCHEME (A)

Equations (3-21) through (3-24) can be written as

$$\begin{bmatrix} \dot{I} \\ \dot{S} \\ \dot{L}_1 \\ \dot{L}_2 \end{bmatrix} = \begin{bmatrix} a + bt + s \\ CS - \frac{CS^2}{N} + \frac{S^2 L_2}{C_A N} - \frac{SL_2}{2C_A} - \frac{S^3 L_2}{2C_A N^2} \\ 2C_I(I_m - I) \\ L_1 + C_S - cL_2 + \frac{3L_2^2 S^2}{4C_A N^2} - \frac{L_2^2 S}{C_A N} + \frac{L_2^2}{4C_A} + \frac{2CSL_2}{N} \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} \quad (3-27)$$

Also rewrite boundary conditions as

$$f_1[x_1(t_0)] = I(t_0) - I^0$$

$$f_2[x_2(t_0)] = S(t_0) - S^0$$

(3-28)

$$g_1[x_3(t_f)] = L_1(t_f)$$

$$g_2[x_4(t_f)] = L_2(t_f)$$

To solve this problem by the method of particular solutions with scheme (A), we first assume the following nominal functions

$$I(t) = I^0, \quad S(t) = S^0, \quad L_1(t) = 0, \quad L_2(t) = 0 \quad (3-29)$$

$$\text{for } t_0 \leq t \leq t_f$$

Then, compute the following matrix

$$F_x^T(x,t) = \begin{bmatrix} \frac{\partial F_1}{\partial I} & \frac{\partial F_1}{\partial S} & \frac{\partial F_1}{\partial L_1} & \frac{\partial F_1}{\partial L_2} \\ \frac{\partial F_2}{\partial I} & \frac{\partial F_2}{\partial S} & \frac{\partial F_2}{\partial L_1} & \frac{\partial F_2}{\partial L_2} \\ \frac{\partial F_3}{\partial I} & \frac{\partial F_3}{\partial S} & \frac{\partial F_3}{\partial L_1} & \frac{\partial F_3}{\partial L_2} \\ \frac{\partial F_4}{\partial I} & \frac{\partial F_4}{\partial S} & \frac{\partial F_4}{\partial L_1} & \frac{\partial F_4}{\partial L_2} \end{bmatrix} =$$

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & C - \frac{2CS}{N} + \frac{2SL_2}{C_A N} - \frac{L_2}{2C_A} - \frac{3S^2 L_2}{2C_A N^2} & 0 & \frac{S^2}{C_A N} - \frac{S}{2C_A} - \frac{S^3}{2C_A N^2} \\ -2C_I & 0 & 0 & 0 \\ 0 & \frac{3SL_2^2}{2C_A N^2} - \frac{L_2^2}{C_A N} + \frac{2CL_2}{N} & 1 & -C + \frac{3S^2 L_2}{2C_A N^2} - \frac{2L_2 S}{C_A N} + \frac{L_2}{2C_A} + \frac{2CS}{N} \end{bmatrix} \quad (3-30)$$

According to Eq. (2-20), the linearized system can be written as

$$\Delta \dot{I} + \Delta S + [\dot{I} - F_1] = 0$$

$$\begin{aligned} \dot{\Delta S} - \left[C - \frac{2CS}{N} + \frac{2SL_2}{C_A N} - \frac{L_2}{2C_A} - \frac{3S^2 L_2}{2C_A N^2} \right] \Delta S - \\ \left[\frac{S^2}{C_A N} - \frac{S}{2C_A} - \frac{S^3}{2C_A N^2} \right] \Delta L_2 + [\dot{S} - F_2] = 0 \end{aligned} \quad (3-31)$$

$$\dot{\Delta L_1} + 2C_I \Delta I + [\dot{L_1} - F_3] = 0$$

$$\begin{aligned} \dot{\Delta L_2} - \left[\frac{3SL_2^2}{2C_A N^2} - \frac{L_2^2}{C_A N} + \frac{2CL_2}{N} \right] \Delta S - \Delta L_1 - \\ \left[-C + \frac{3S^2 L_2}{2C_A N^2} - \frac{2L_2 S}{C_A N} + \frac{L_2}{2C_A} + \frac{2CS}{N} \right] \Delta L_2 + [\dot{L_2} - F_4] = 0 \end{aligned}$$

After that, we linearize the boundary conditions, since

$$f_I^T(t_0) = f_S^T(t_0) = 1$$

$$g_{L_1}^T(t_f) = g_{L_2}^T(t_f) = 1 \quad (3-32)$$

According to Eq. (2-21) and (2-22), we obtain the following equations

$$\Delta I(t_0) + I(t_0) - I^0 = 0$$

$$\Delta S(t_0) + S(t_0) - S^0 = 0 \quad (3-33)$$

$$\Delta L_1(t_f) + L_1(t_f) = 0$$

$$\Delta L_2(t_f) + L_2(t_f) = 0$$

Introducing stepsize α to each forcing term of Eq. (3-32) and (3-33), also introducing the auxiliary variables

$$A_1 = \Delta I / \alpha \qquad A_3 = \Delta L / \alpha \qquad (3-34)$$

$$A_2 = \Delta S / \alpha \qquad A_4 = \Delta L_2 / \alpha$$

Rewrite Equation (3-32) and (3-33) as follows

$$\begin{aligned} \dot{A}_1 + A_2 + [\dot{I} - F] &= 0 \\ \dot{A}_2 - [C - \frac{2CS}{N} + \frac{2SL_2}{C_A N} - \frac{L_2}{2C_A} - \frac{3S^2 L_2}{2C_A N^2}] A_2 \\ - [\frac{S^2}{C_A N} - \frac{S}{2C_A} - \frac{S^3}{2C_A N^2}] A_4 + [\dot{S} - F_2] &= 0 \end{aligned} \qquad (3-35)$$

$$\begin{aligned} \dot{A}_3 + 2C_I A_1 + [\dot{L}_1 - F_3] &= 0 \\ \dot{A}_4 - [\frac{3SL_2^2}{2C_A N^2} - \frac{L_2^2}{C_A N} + \frac{2CL_2}{N}] A_2 - A_3 - \\ [-C + \frac{3S^2 L_2}{2C_A N^2} - \frac{2L_2 S}{C_A N} + \frac{L_2}{2C_A} + \frac{2CS}{N}] A_4 + [\dot{L}_2 - F_4] &= 0 \end{aligned}$$

$$A_1(t_0) = 0$$

$$A_2(t_0) = 0 \qquad (3-36)$$

$$A_3(t_f) + L_1(t_f) = 0$$

$$A_4(t_f) + L_2(t_f) = 0$$

Equations (3-35) and (3-36) are corresponding to Eqs. (2-27) through (2-29). For the assumed nominal functions, the matrix $F_x^T(x, t)$ and $\dot{I} - F_1$, $\dot{S} - F_2$, $\dot{L}_1 - F_3$, $\dot{L}_2 - F_4$, $L_1(t_f)$, $L_2(t_f)$ are known quantities. Therefore, equations (3-35) and (3-36) form an ordinary, linear, two-point boundary value problem.

From Eq. (3-36) we notice that there are two initial and two final conditions, that is, $p = 2$ and $q = 2$. By the method of particular solutions, $q + 1 = 3$ sets of particular solutions are integrated with the initial conditions

$$A_{ij}(0) = 0 \quad j = 1, 2, 3. \quad i = 1, 2. \quad (3-37)$$

$$A_{k+2}(0) = \delta_{kj} \quad j = 1, 2, 3. \quad k = 1, 2.$$

Let $A_{pj}^i(t)$ denote the results we obtained where i denotes variables, pj denotes the j^{th} set of particular solutions.

The undetermined constants k_1 , k_2 , k_3 can be solved by the linear algebraic equations

$$k_1 + k_2 + k_3 = 1$$

$$k_1 A_{p_1}^3(t_f) + k_2 A_{p_2}^3(t_f) + k_3 A_{p_3}^3(t_f) = -L_1(t_f) \quad (3-38)$$

$$k_1 A_{p_1}^4(t_f) + k_2 A_{p_2}^4(t_f) + k_3 A_{p_3}^4(t_f) = -L_2(t_f)$$

with k_1, k_2, k_3 known, we form the linear combination

$$A^i(t) = \sum_{j=1}^3 k_j A_{p_j}^i(t) \quad i = 1, 2, 3, 4 \quad (3-39)$$

$$t_0 \leq t \leq t_f$$

After that, varied functions can be formed as

$$\begin{aligned} \tilde{I}(t) &= I(t) + \alpha A^1(t) \\ \tilde{S}(t) &= S(t) + \alpha A^2(t) \\ \tilde{L}_1(t) &= L_1(t) + \alpha A^3(t) \\ \tilde{L}_2(t) &= L_2(t) + \alpha A^4(t) \end{aligned} \quad (3-40)$$

The performance index P of this problem is

$$P = \int_{t_0}^{t_f} [(\dot{I} - F_1)^2 + (\dot{S} - F_2)^2 + (\dot{L}_1 - F_3)^2 + (\dot{L}_2 - F_4)^2] dt +$$

$$[L_1(t_f)]^2 + [L_2(t_f)]^2 \quad (3-41)$$

The stepsize α can be calculated with Eqs. (3-40) and (3-41) in such a way that Ineq. (2-43) is satisfied. Once stepsize is known, the varied functions can be obtained by Eq. (3-40). Then the profit and advertisement can be obtained. This complete one iteration, further iterations are allowed until desired accuracy is satisfied.

3.5 COMPUTATIONAL PROCEDURE BY ALGORITHM SCHEME (B)

By scheme (B), assume the nominal initial values consistent with the given initial conditions.

$$\begin{aligned} I(t_0) &= I^0 & S(t_0) &= S^0 \\ L_1(t_0) &= 0 & L_2(t_0) &= 0 \end{aligned} \quad (3-42)$$

With this initial values, integrate Eq. (3-27) forward to obtain the nominal functions $I(t)$, $S(t)$, $L_1(t)$, $L_2(t)$ for $t_0 \leq t \leq t_f$. For these nominal functions, we know that $\dot{I} - F_1 = 0$, $\dot{S} - F_2 = 0$, $\dot{L}_1 - F_3 = 0$ and $\dot{L}_2 - F_4 = 0$. Therefore, Eq. (3-35) becomes a system of homogeneous equations:

$$\begin{aligned} \dot{A}_1 + A_2 &= 0 \\ \dot{A}_2 - \left[C - \frac{2CS}{N} + \frac{2SL_2}{2C_A N} - \frac{L_2}{2C_A} - \frac{3S^2 L_2}{2C_A N^2} \right] A_2 &= 0 \end{aligned} \quad (3-43)$$

$$\left[\frac{S^2}{C_A N} - \frac{S}{2C_A} - \frac{S^3}{2C_A N^2} \right] A_4 = 0$$

$$\dot{A}_3 + 2C_I A_1 = 0$$

(3-43)

$$\dot{A}_4 - \left[\frac{3SL_2^2}{2C_A N^2} - \frac{L_2^2}{C_A N} + \frac{2CL_2}{N} \right] A_2 - A_3 -$$

$$\left[-C + \frac{3S^2 L_2}{2C_A N^2} - \frac{2L_2 S}{C_A N} + \frac{L_2}{2C_A} + \frac{2CS}{N} \right] A_4 = 0$$

Boundary conditions were given by Eq. (3-36).

This linearized system will be integrated forward $q + 1$ times with the initial conditions Eq. (3-37). Actually, only q sets of particular solutions are required by scheme (B). This is because Eq. (3-44) is a homogeneous system, and the third set of initial values are equal to zero. Therefore, the third set of particular solution

$$A_{p3}^i(t) = 0$$

$$\text{for } i = 1, 2, 3, 4$$

(3-44)

$$t_0 \leq t \leq t_f$$

Since we need only two sets of particular solutions, the undetermined constants reduced to two and can be obtained by following equations

$$k_1 A_{p_1}^3(t_f) + k_2 A_{p_2}^3(t_f) + L_1(t_f) = 0 \quad (3-45)$$

$$k_1 A_{p_1}^4(t_f) + k_2 A_{p_2}^4(t_f) + L_2(t_f) = 0$$

When k_1 and k_2 are known, form the linear combination

$$A^i(t) = \sum_{j=1}^2 k_j A_{p_j}^i(t) \quad i = 1, 2, 3, 4 \quad (3-46)$$

The initial values of the varied functions were formed as

$$\begin{aligned} \tilde{I}(t_0) &= I(t_0) + \alpha A^1(t_0) \\ \tilde{S}(t_0) &= S(t_0) + \alpha A^2(t_0) \\ \tilde{L}_1(t_0) &= L_1(t_0) + \alpha A^3(t_0) \\ \tilde{L}_2(t_0) &= L_2(t_0) + \alpha A^4(t_0) \end{aligned} \quad (3-47)$$

Once stepsize α was calculated by P function and satisfies Ineq. (2-43), the initial values of the varied functions can be computed from Eq. (3-47). With these initial values, integrate Eq. (3-27) forward to obtain the nominal functions for next iteration. Advertisement and profit were calculated by Eq. (3-21) and (3-7) respectively. Iterations were continued until desired accuracy was obtained.

3.6 NUMERICAL ASPECTS

In order to solve this problem, the constants were assumed to have the following values

$$a = 70 \quad C_A = 1.5 \quad I(t_0) = I^0 = 20$$

$$b = 100 \quad C_I = 0.15 \quad S(t_0) = S^0 = 20$$

$$c = 2 \quad N = 150 \quad t_0 = 0, \quad t_f = 1$$

$$C_S = 10 \quad I_m = 50 \quad \Delta t = 0.01$$

Nominal functions of scheme (A)

$$I(t) = 20 \quad L_1(t) = 0$$

$$S(t) = 20 \quad L_2(t) = 0$$

$$\text{for } t_0 \leq t \leq t_f$$

Initial nominal values of scheme (B)

$$I(t_0) = 20 \quad L_1(t_0) = 0$$

$$S(t_0) = 20 \quad L_2(t_0) = 0$$

The initial values for three sets of particular solutions are

	$I(t_0)$	$S(t_0)$	$L_1(t_0)$	$L_2(t_0)$	
set 1	0	0	1	0	
set 2	0	0	0	1	(3-48)
set 3	0	0	0	0	

3.7 COMPUTATIONAL ASPECTS

(I) Scheme A

With the initial conditions given by (3-48), Eq. (3-35) was integrated forward by fourth order Runge-Kutta integration method with 100 grid points and double precision accuracy. Three sets of particular solutions were obtained.

Constants k_j were obtained by solving simultaneous equations (3-38) by direct substitution and elimination. In order to test the effectiveness of stepsize control, the problem was solved through two processes. (1) With stepsize control process, (2) Without stepsize control process.

Stepsize was set equal to one for the process without stepsize control. For the process with stepsize control, stepsize was calculated in such a way that Eq. (3-41) was minimized and Ineq. (2-43) being satisfied. In search for α , we proceeded by two ways, one is by bisection method, the other is by Fabonacci search technique.

By bisection method, α was first set to zero and calculate $P(0)$ by Simpson's integration method. Then set $\alpha = 1$ and calculate

$P(1)$, if $P(1) < P(0)$, stepsize has the value one. Otherwise, bisect the α value until the inequality $P(\alpha) < P(0)$ is satisfied.

By Fibonacci search technique, define ϵ as the least interval between two points of the last experiment. Let n be the number of experiments, the first experiment was placed at

$$x_1 = \frac{F_{n-1} + (-1)^n \epsilon}{F_n}$$

$$x_2 = a + b - x_1$$

where a and b are the extreme points of the range of α . F_i is the i^{th} Fibonacci number in the Fibonacci series:

$$F_0 = 1$$

$$F_1 = 1$$

$$F_K = F_{K-1} + F_{K-2} \quad K = 2, 3, \dots$$

In this thesis, $\epsilon = 0.01$ and $n = 5$, therefore, for the first experiment, $x_1 = 0.624$ and $x_2 = 0.376$.

The IBM - 360/50 Computer system was used for all computations. Computer program was given in Appendix.

(II) Scheme B

Computational aspects for scheme (B) is essentially the same as

scheme (A). Also the problem was solved through process with stepsize control and process without stepsize control.

3.8 RESULTS

The Advertisement problem was solved by the following methods:

- (1) The method of particular solutions, scheme (A) with and without stepsize control.
- (2) The method of particular solutions, scheme (B) with and without stepsize control.
- (3) Quasilinearization with the method of complementary functions.
- (4) Quasilinearization with the method of adjoint functions.

For the purpose of comparison, the results obtained by the above methods were presented in this section.

- (1) Results obtained by the method of particular solutions scheme (A) without stepsize control.

Convergence was produced by 13 iterations, optimal profit was $J = 587.58$ and optimal initial and final values are

$$I(0) = 20.$$

$$I(1) = 66.596$$

$$S(0) = 20.$$

$$S(1) = 115.750$$

$$A(0) = 4.$$

$$A(1) = 0$$

The convergence rates of sales, inventory and advertisement are shown in Fig. (1) through Fig. (3). Figure (4) shows the convergence rates

of the profit function. Also the convergence rates of these variables are tabulized in Table (1). The IBM 360/50 computer took about five minutes to complete 17 iterations for this problem with FORTRAN IV G level complier.

(2) By the method of particular solutions scheme (A) With stepsize control.

(a) Stepsize determined by bisection method.

Convergence was produced by 12 iterations, the optimal profit was $J = 587.59$. Optimal initial and final values are

$$I(0) = 20.$$

$$I(1) = 66.595$$

$$S(0) = 20.$$

$$S(1) = 115.750$$

$$A(0) = 4.$$

$$A(1) = 0$$

The convergence rates of sales, inventory and advertisement are tabulized in Table (5). IBM 360/50 computer took 7.02 minutes to complete 15 iterations with FORTRAN IV G level complier.

(b) Stepsize determined by Fabonacci search

Convergence was produced by 11 iterations, optimal values are the same as obtained by bisection method. The computer took 10.98 minutes to complete 17 iterations. The convergence rates of sales, inventory and advertisement are tabulized in Table (6).

(3) By the method of particular solution scheme (B) without stepsize control.

Convergence was produced by six iterations, optimal profit was $J = 587.71$, optimal initial and final conditions are

$$I(0) = 20$$

$$I(1) = 66.157$$

$$S(0) = 20$$

$$S(1) = 115.690$$

$$A(0) = 3.983$$

$$A(1) = 0$$

Convergence rates of sales, inventory and advertisement are shown in Fig. (5) through Fig. (7). Figure (8) shows the convergence of profit function. In table (2-1) the convergence rates of these variables are tabulized.

(4) By the method of particular solutions scheme (B) with step-size control.

Stepsize was determined by bisection method. Convergence was produced by five iterations, optimal profit was $J = 587.71$. Optimal initial and final values are

$$I(0) = 20.$$

$$I(1) = 66.156$$

$$S(0) = 20.$$

$$S(1) = 115.69$$

$$A(0) = 3.983$$

$$A(1) = 0.$$

Convergence rates of sales, inventory, advertisement and profit are tabulized in Table (2-2).

(5) By quasilinearization with the method of complementary functions.

Convergence was produced by five iterations. The optimal value was $J = 587.80$ and optimal initial and final values are

$$I(0) = 20.$$

$$I(1) = 66.159$$

$$S(0) = 20.$$

$$S(1) = 115.69$$

$$A(0) = 3.981$$

$$A(1) = 0.$$

Convergence rates of sales, inventory and advertisement, profit are shown in Fig. (9) through Fig. (12). Also they are tabulized in Table 3.

(6) By quasilinearization with the method of adjoint functions.

Convergence was produced by five iterations. The optimal profit was $J = 587.74$ and the optimal initial and final values are

$$I(0) = 20.$$

$$I(1) = 65.844$$

$$S(0) = 20.$$

$$S(1) = 116.100$$

$$A(0) = 4.023$$

$$A(1) = 0.$$

Convergence rates of sales, inventory, advertisement and profit are shown in Fig. (13) through Fig. (16). Also they are tabulized in Table 4.

For the purpose of comparison, we listed in Table 7, the optimal

of I, S, A and J obtained by each method. Also the iterations required for convergence and the stepsize α , the cumulative error P were listed.

3.9 DISCUSSION

The Advertisement problem has been solved by four computational procedures:

- (1) The method of particular solutions, scheme (A) with and without stepsize control.
- (2) The method of particular solutions, scheme (B) with and without stepsize control.
- (3) Quasilinearization with the method of complementary functions.
- (4) Quasilinearization with the method of adjoint functions.

It was observed that by the method of particular solutions, scheme (A) without stepsize control, the number of iterations required for convergence are three times more than that required by quasilinearization. From Table 3, it was observed that the values of state variables decrease monotonically to the optimal value. But, in Table 1, those values oscillates as they converge to the optimal. Therefore, the method of particular solutions appears to require more iterations for this particular problem.

As the problem was solved by scheme (A) with stepsize control, if the stepsize was determined by bisection method, 12 iterations were required for five digit accuracy. When Fabonacci search technique was used, 11 iterations were required. The cumulative error P decreased rapidly during the beginning iterations and then gradually slow down. By

bisection method, stepsize was 0.5 at the first iteration and stay at 1 for later iterations, this is because $P(1) < P(0)$ at later iterations. Because of these properties, there is no significant improvement when stepsize control was applied to this particular problem.

By the method of particular solution scheme (B), 6 iterations were required for convergence. Scheme (B) has a faster convergence rate than scheme (A), this is because scheme (B) has a better initial approximation than scheme (A). If a better estimate for the initial values of Lagrange multipliers was made, a better result can be expected. When stepsize control was applied, the iterations required for convergence were reduced from 6 to 5.

If quasilinearization with the method of complementary functions was used, it requires only four iterations. No significant difference was observed when quasilinearization with the method of adjoint functions was used.

For this advertisement problem, it seems that the method of particular solutions is not as powerful as the quasilinearization method. But it still is an applicable technique for solving boundary value problems, especially if scheme (B) with stepsize control was used.

In this problem, the control variables can be eliminated from the performance equations. Next chapter we will apply the method of particular solutions to a problem with six state variables and three control variables, which cannot be eliminated from the performance equations.

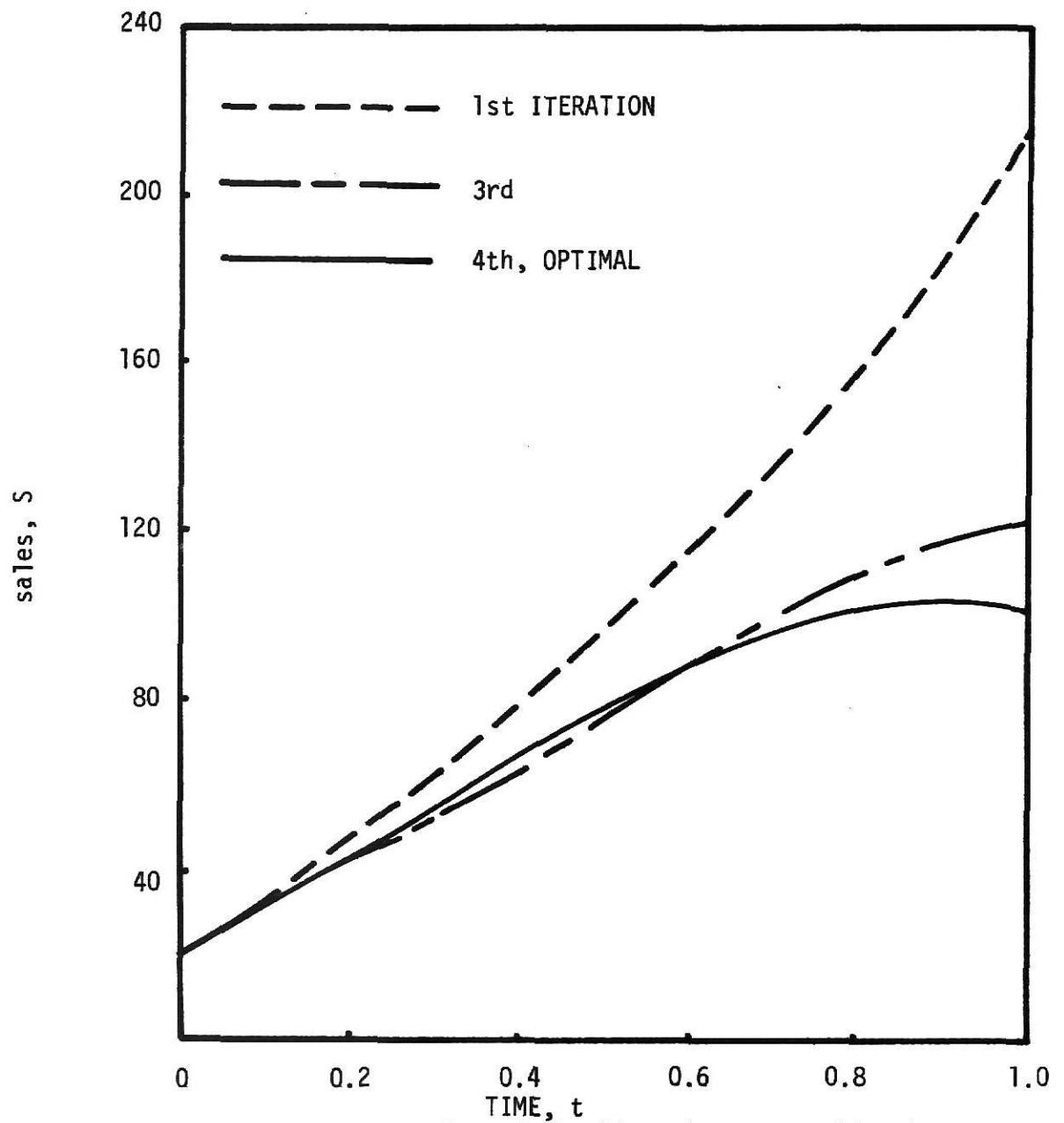


Fig. 1 Convergence rate of sales of the Advertisement problem by Particular solutions method, scheme (A).

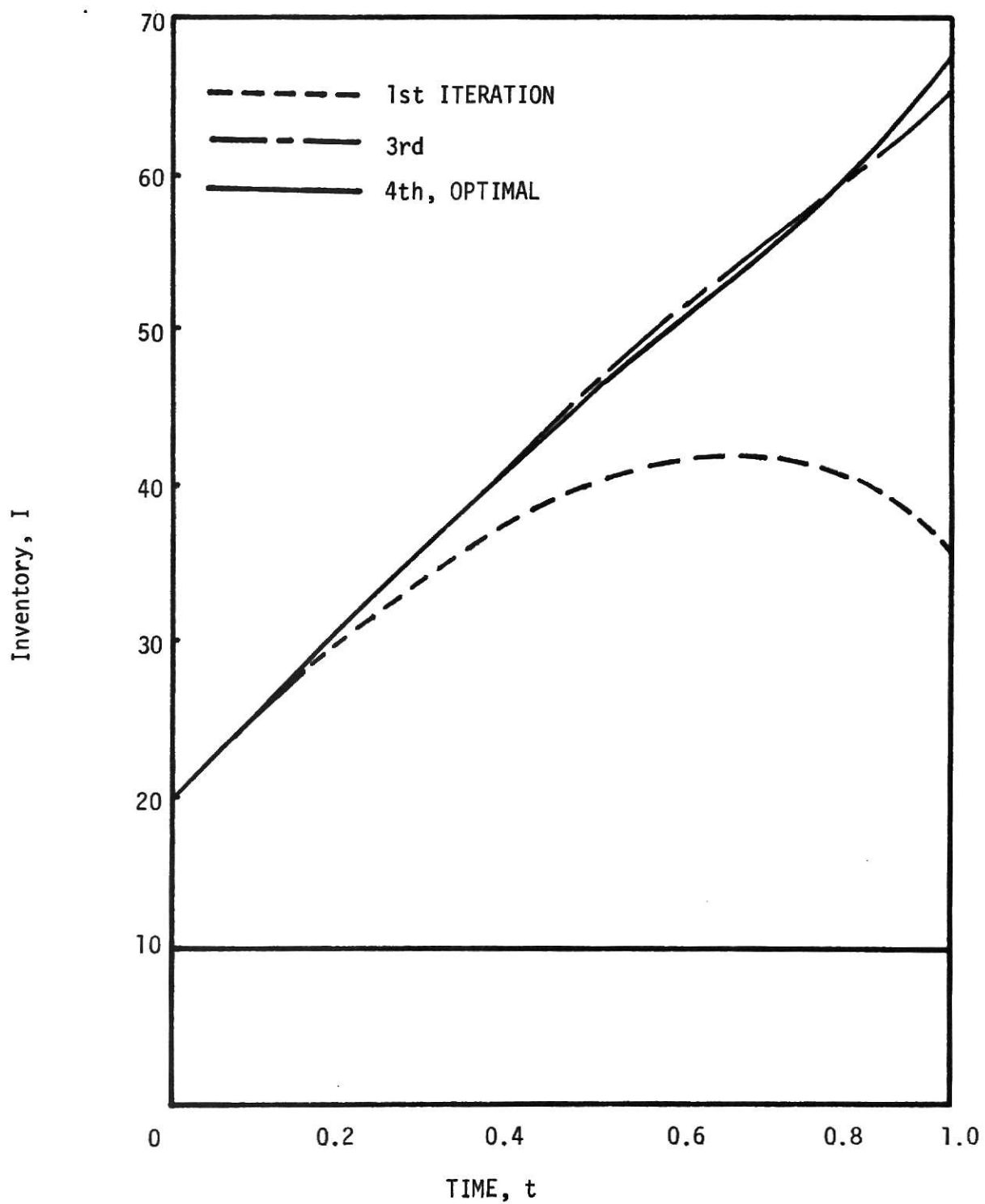


Fig. 2 Convergence rate of Inventory of the Advertisement problem by particular solutions method, scheme (A).

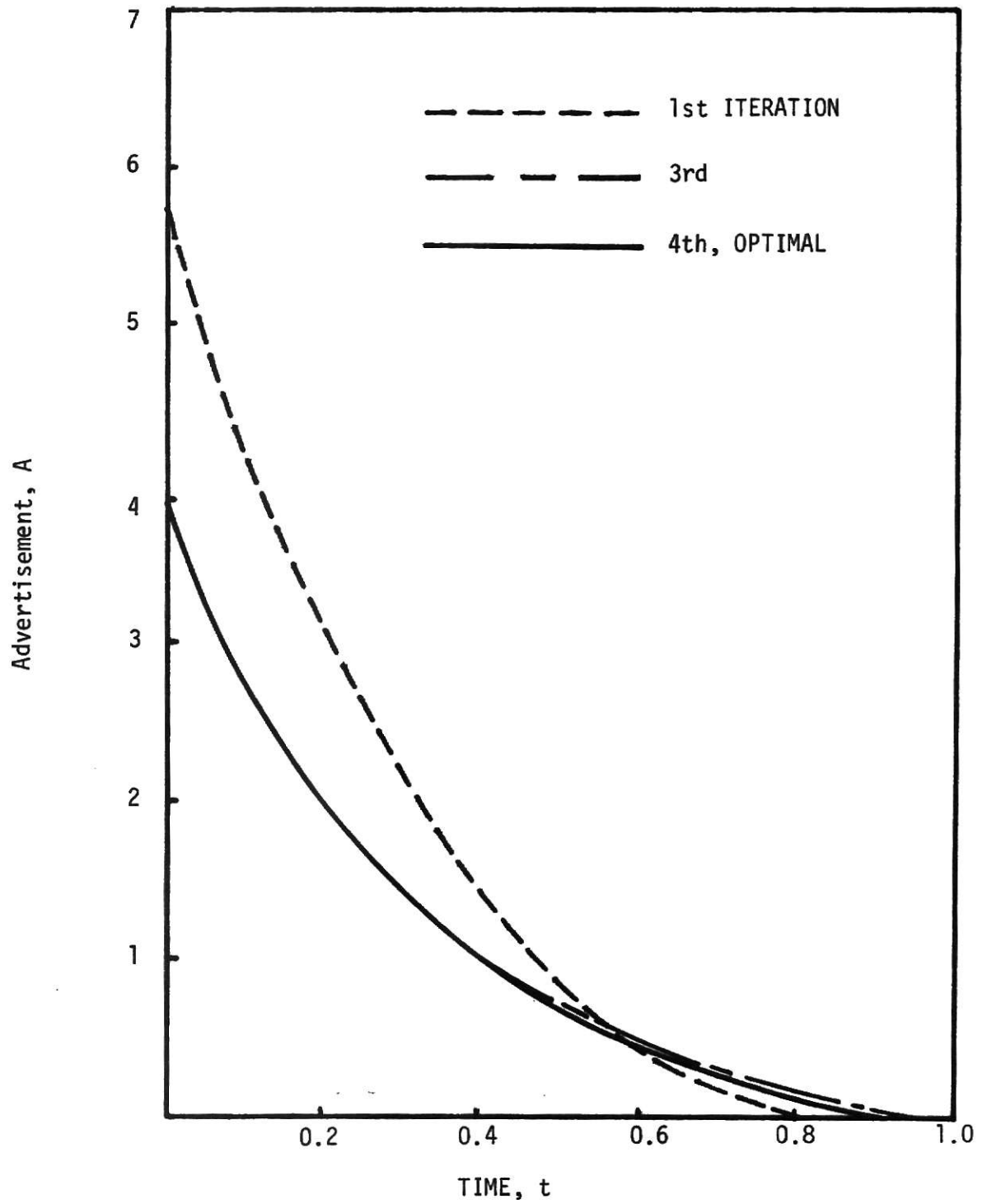


Fig. 3 Convergence rate of advertisement of the Advertisement problem by particular solution method, scheme (A).

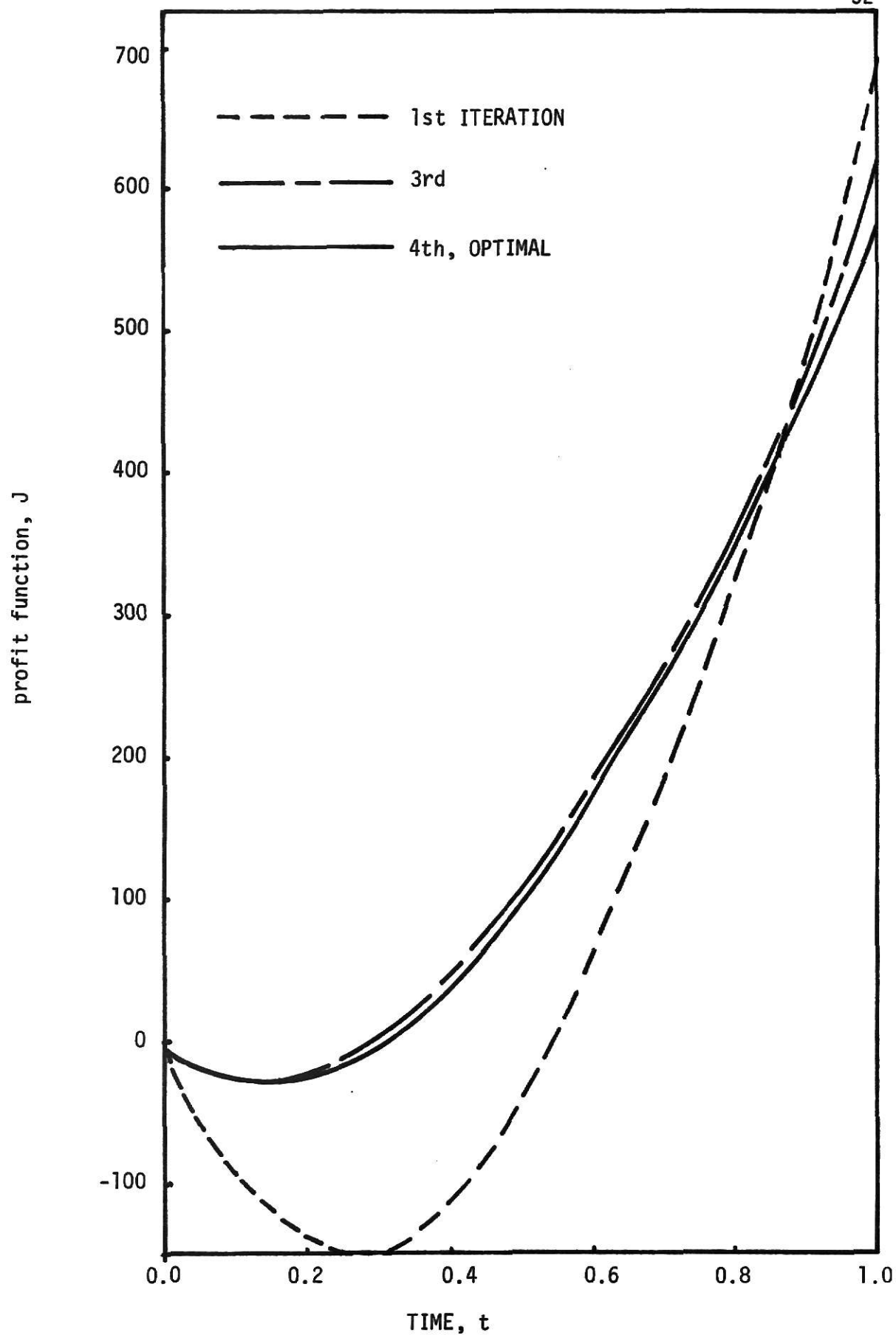


Fig. 4 Convergence rate of profit of the Advertisement problem by particular solution method, scheme (A).

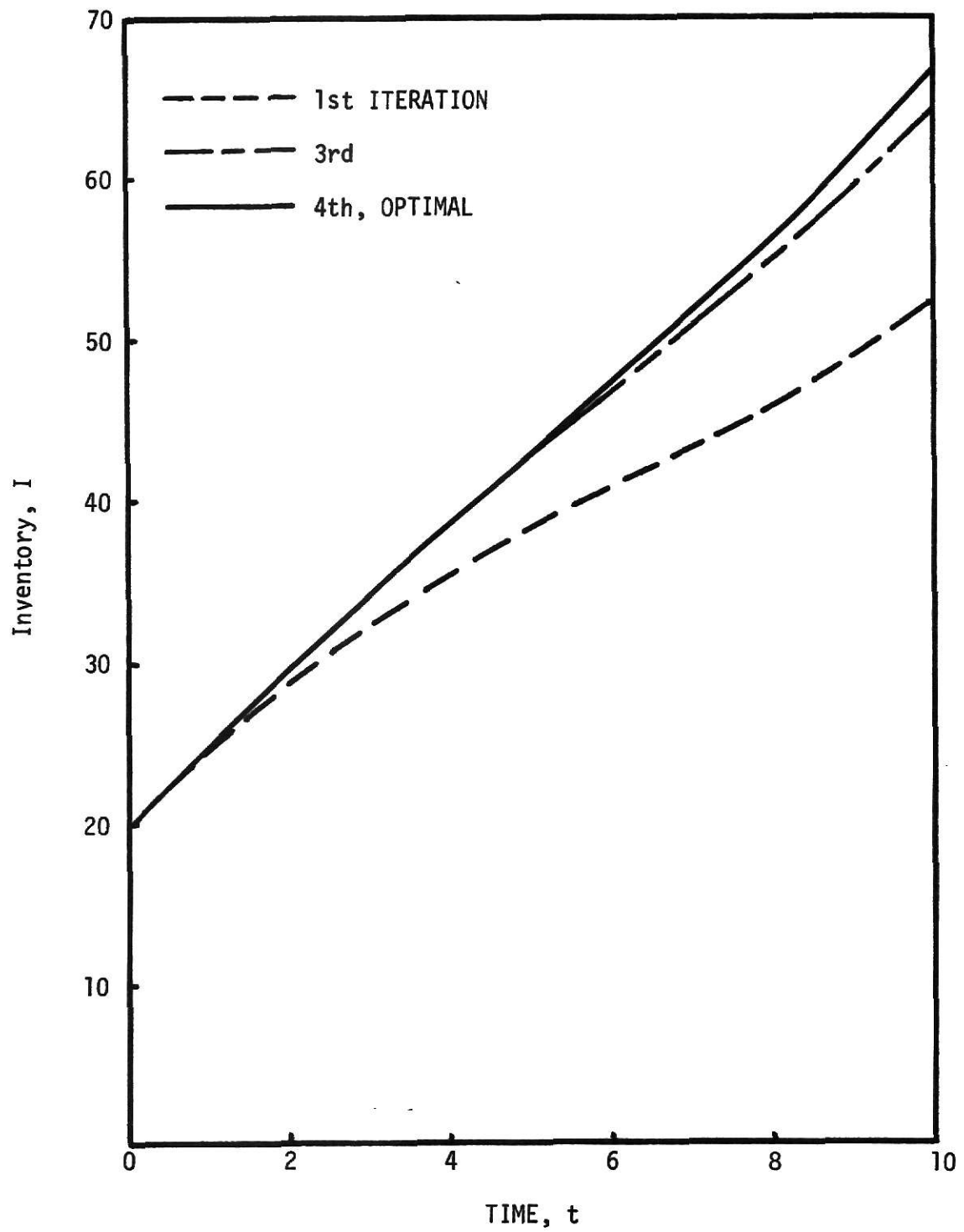


Fig. 5 Convergence rate of Inventory of the Advertisement problem by particular solutions method, scheme (B).

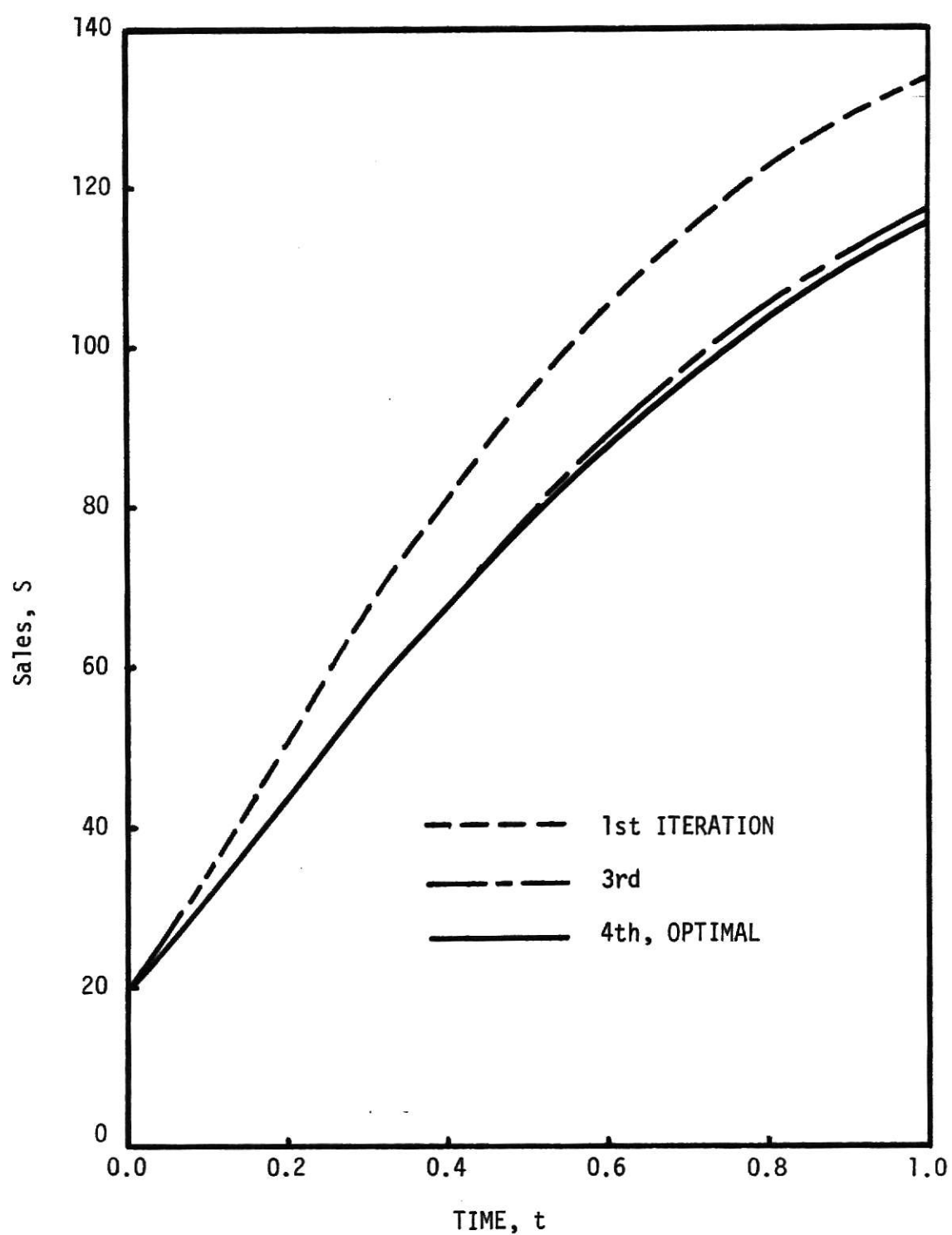


Fig. 6 Convergence rate of sales of the Advertisement problem by particular solutions method, scheme (B).

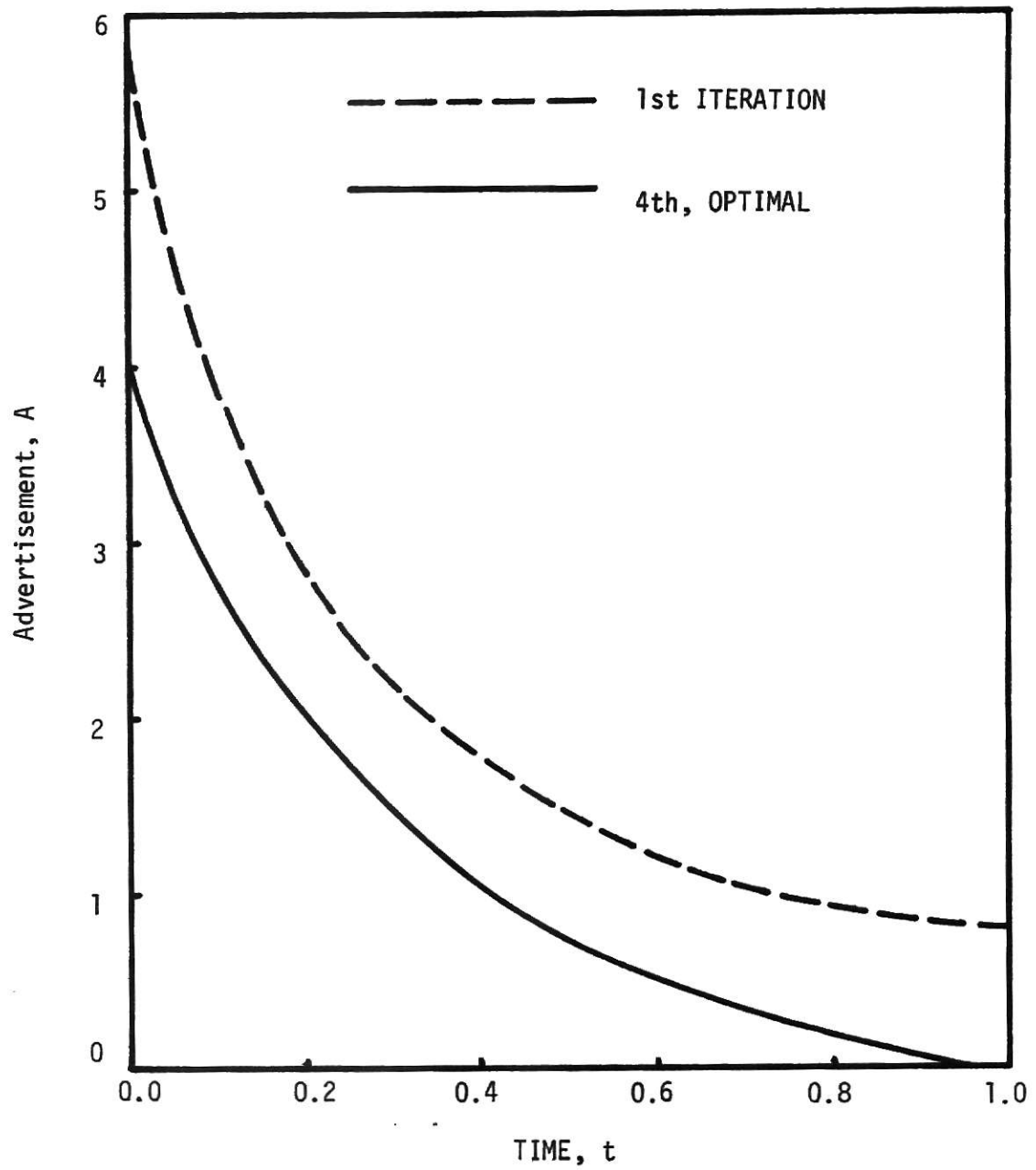


Fig. 7 Convergence of advertisement of the Advertisement problem by particular solutions method, scheme (B).

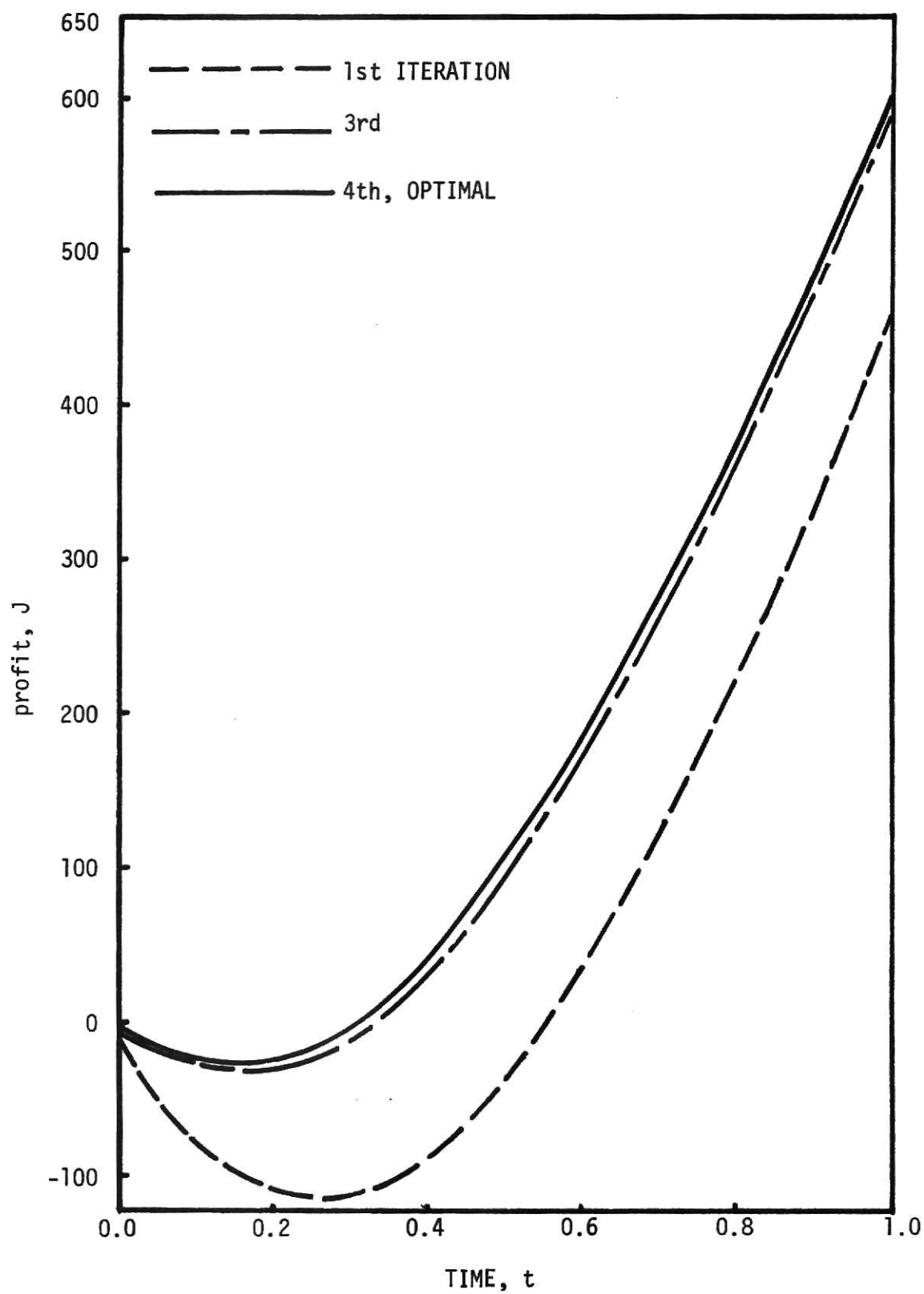


Fig. 8 Convergence of profit of the Advertisement problem by particular solutions method, scheme (B).

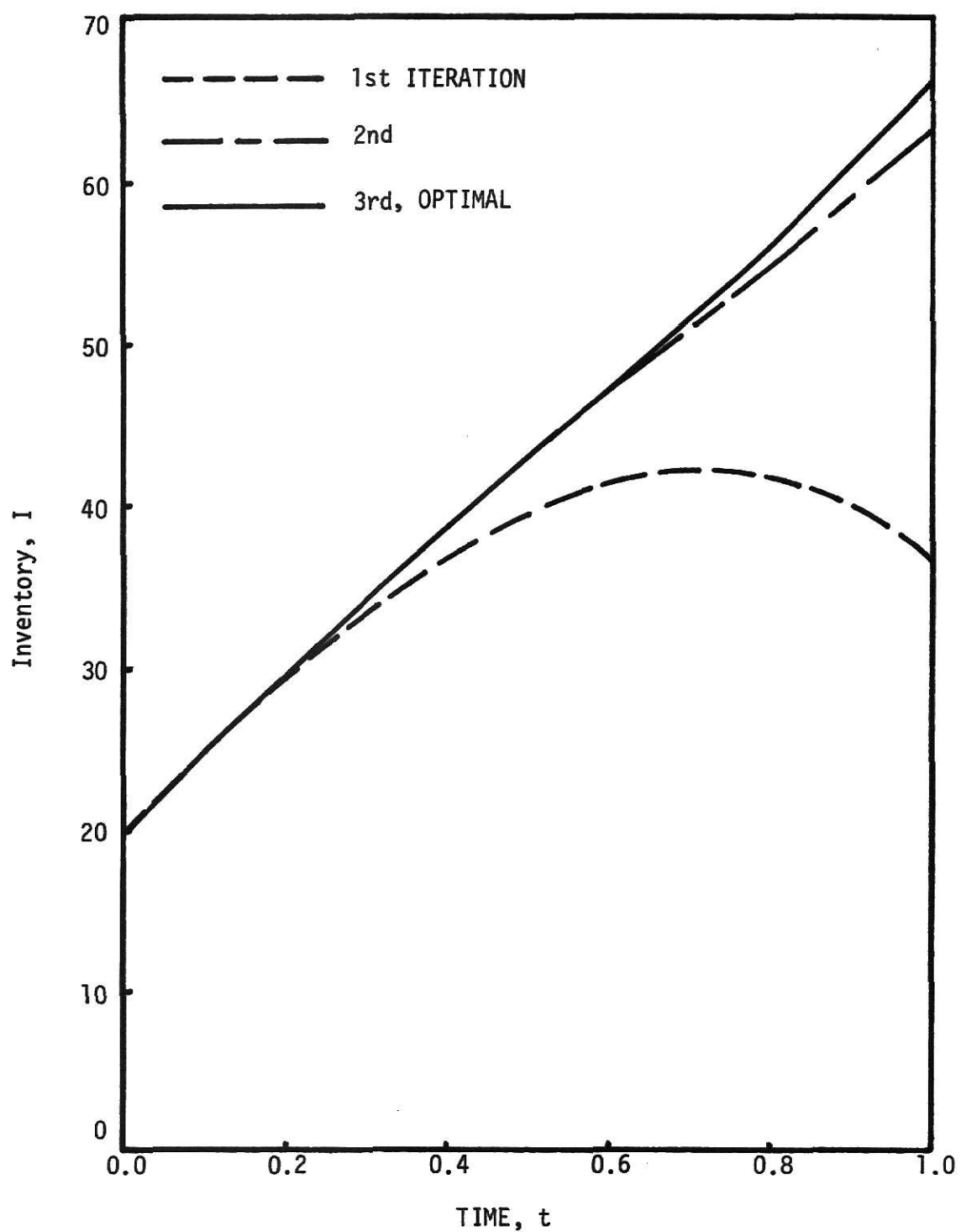


Fig. 9 Convergence of Inventory of the Advertisement problem by quasilinearization with the method of complementary functions.

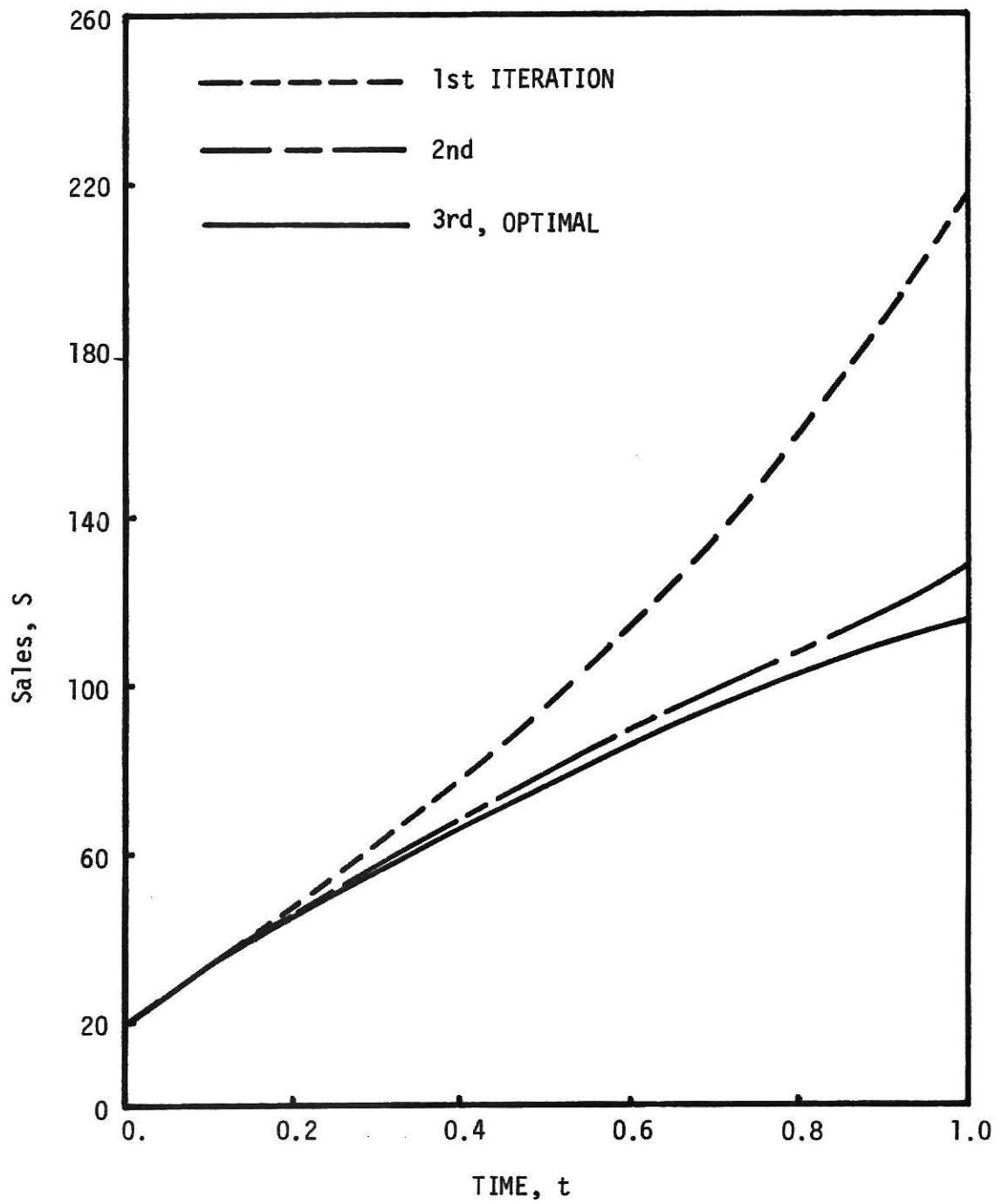


Fig. 10 Convergence rate of sales of the Advertisement problem by quasilinearization with the method of complementary functions.

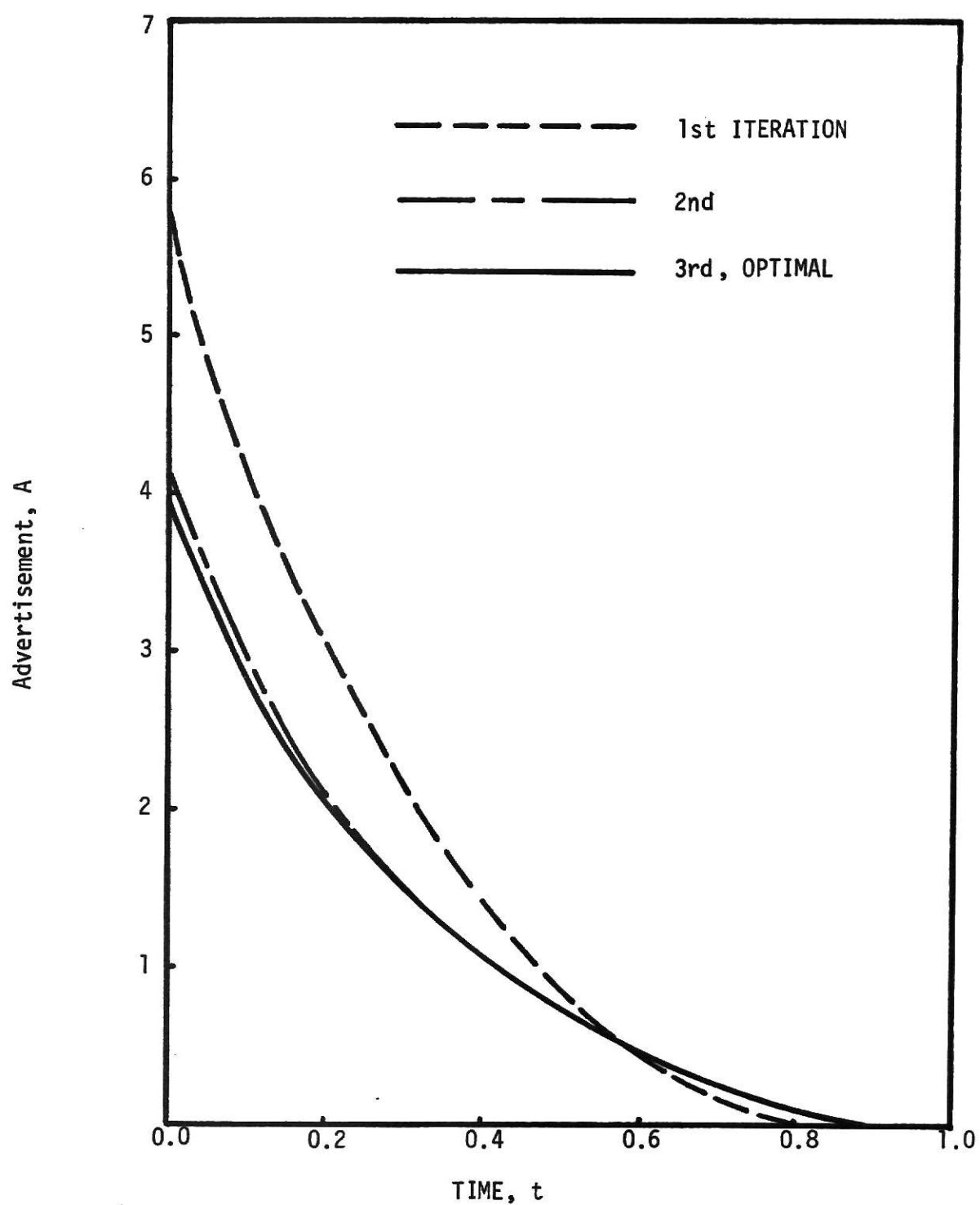


Fig. 11 Convergence rate of advertisement of the Advertisement problem by quasilinearization with the method of complementary functions.

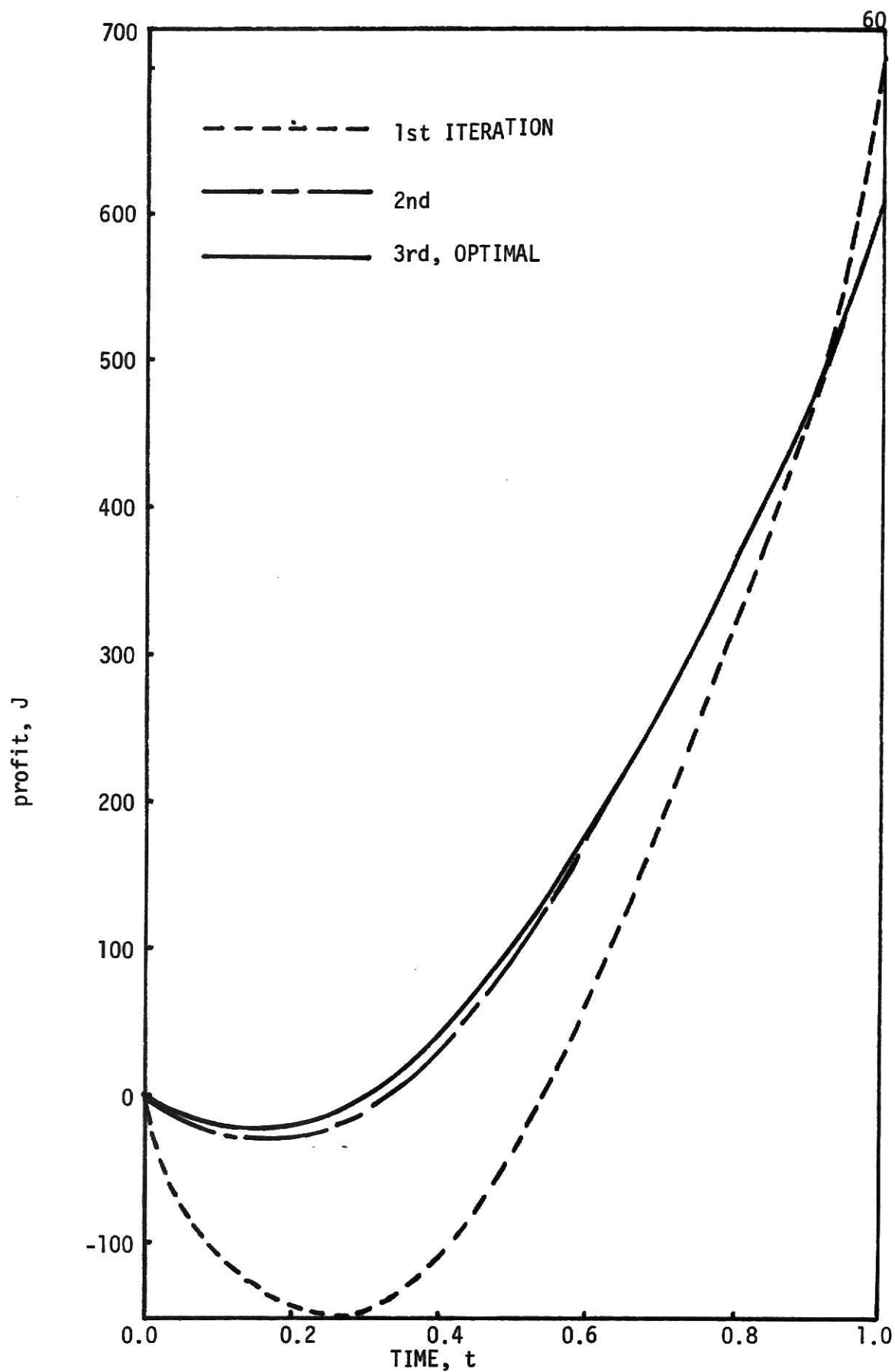


Fig. 12 Convergence rate of profit of the Advertisement problem by quasilinearization with the method of complementary functions.

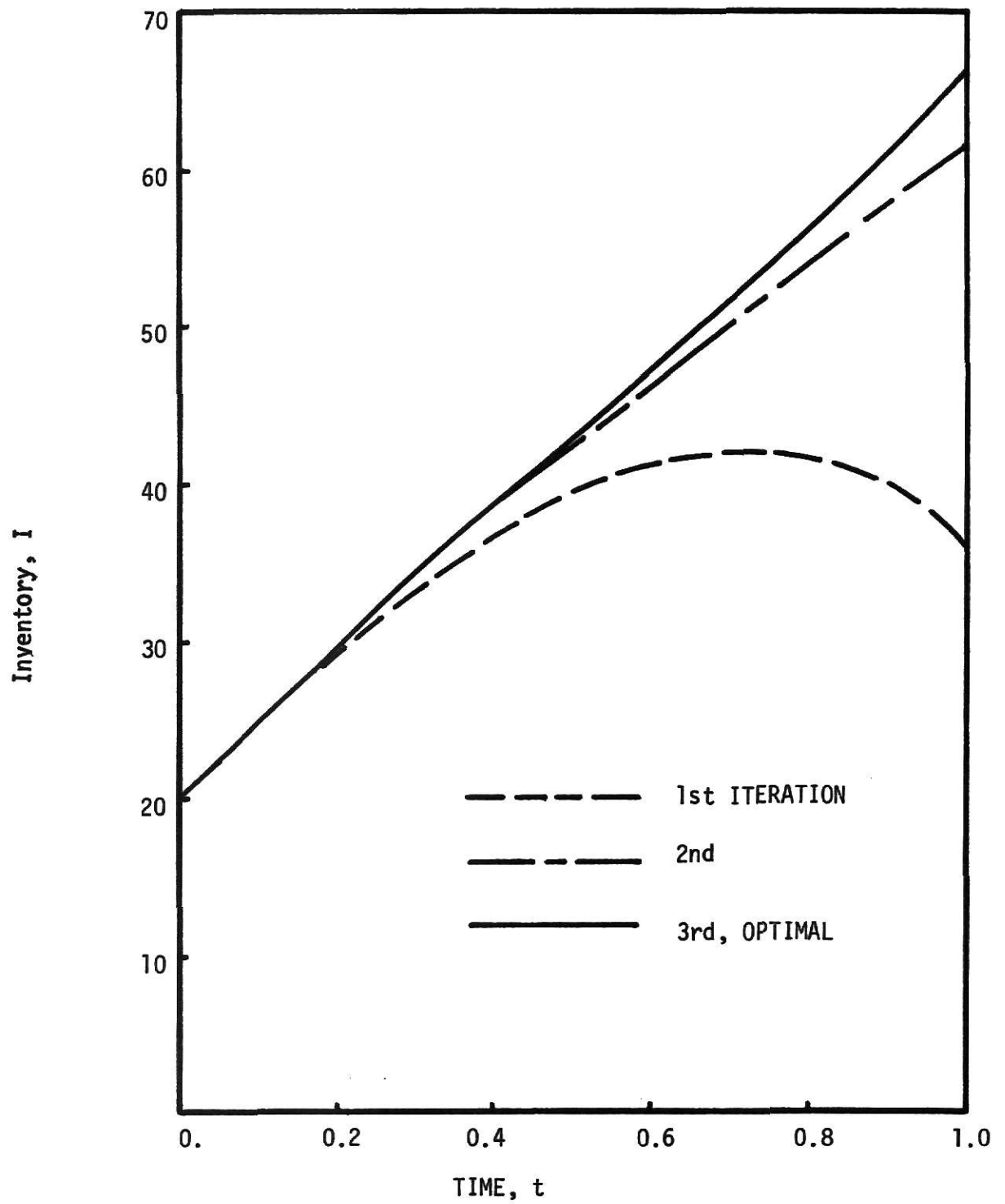


Fig. 13 Convergence rate of Inventory of the Advertisement problem by quasilinearization with the method of adjoint functions.

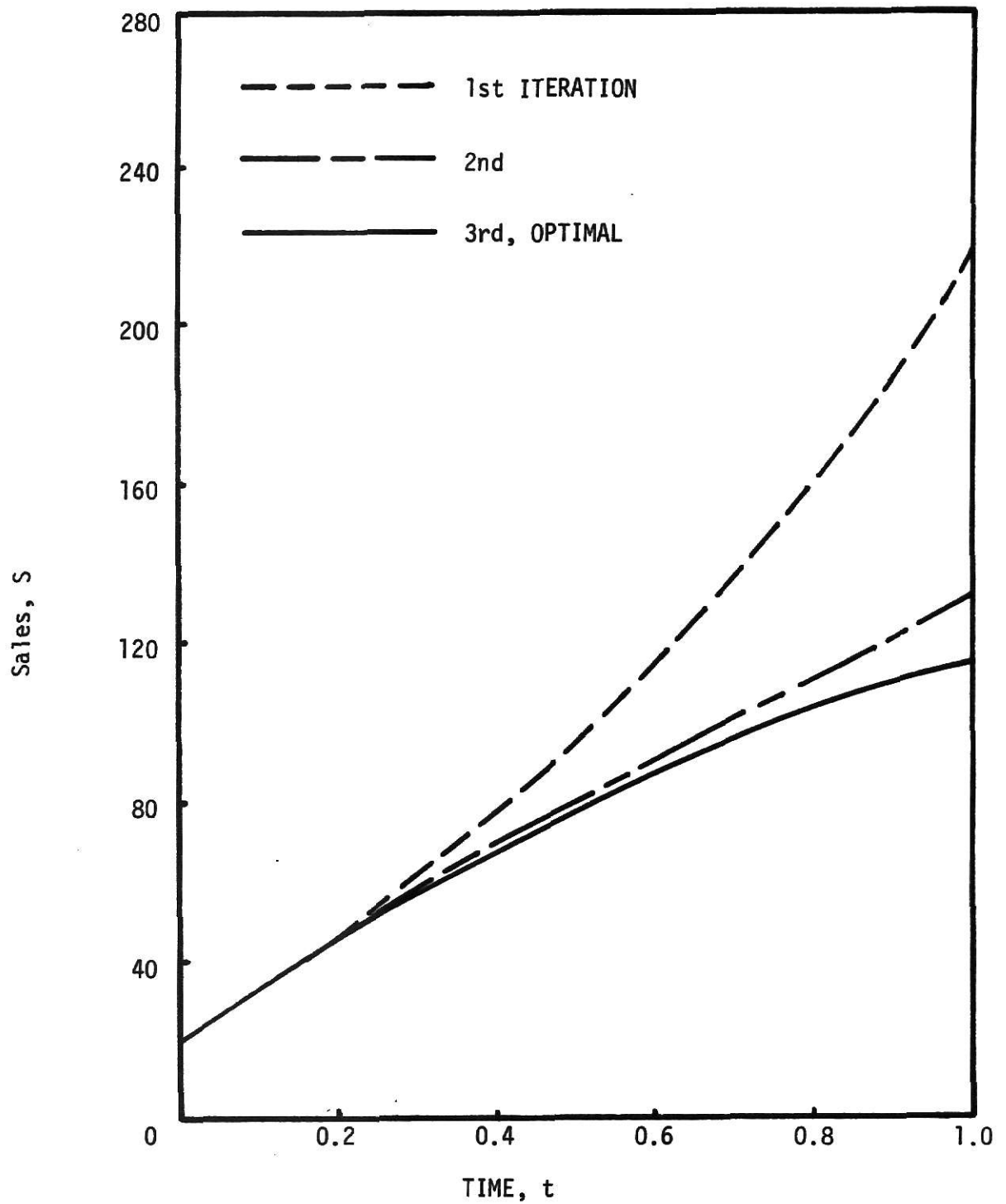


Fig. 14 Convergence rate of sales of the Advertisement problem by quasilinearization with the method of adjoint functions.

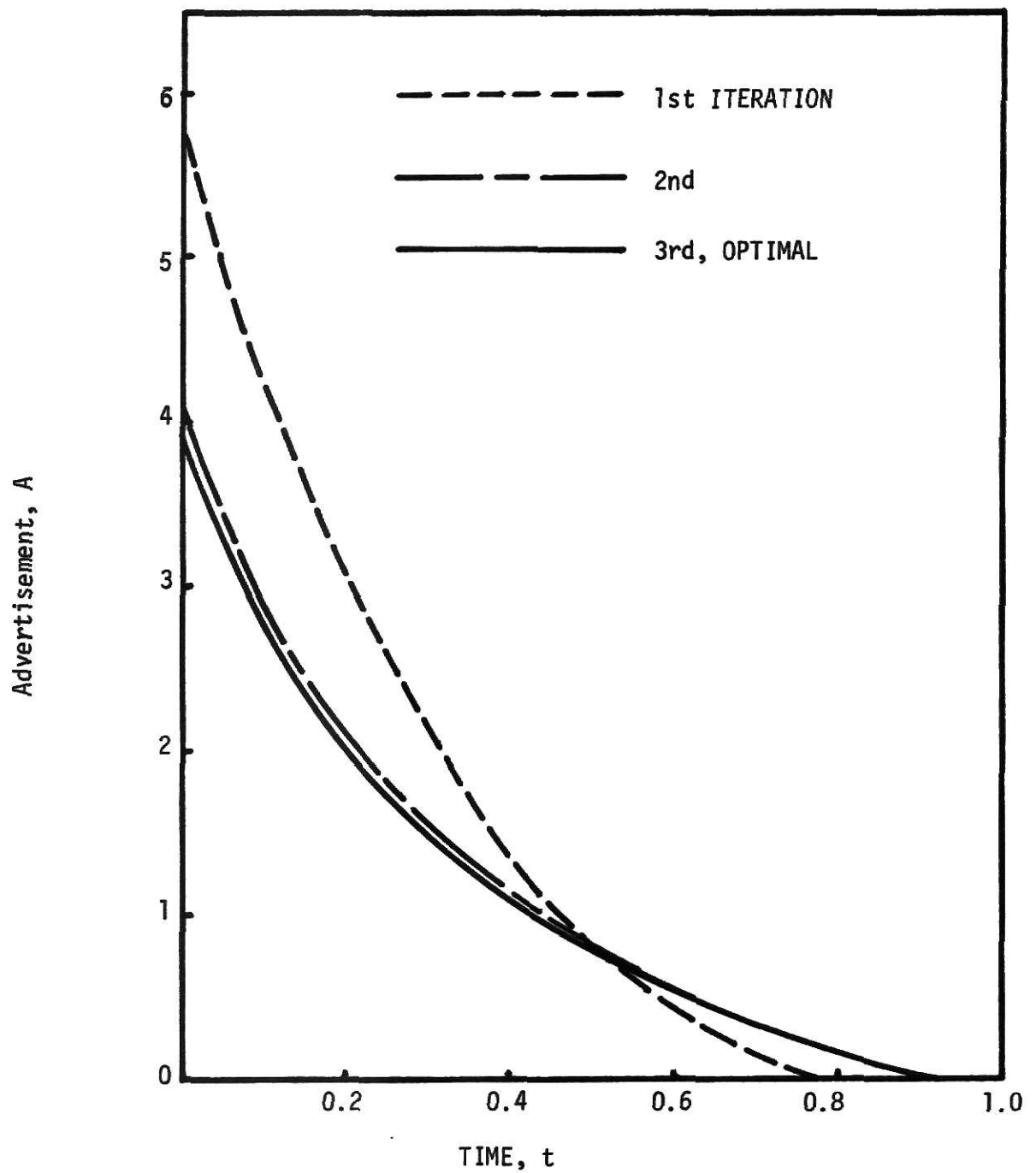


Fig. 15 Convergence rate of advertisement of the Advertisement problem by quasilinearization with the method of adjoint functions.

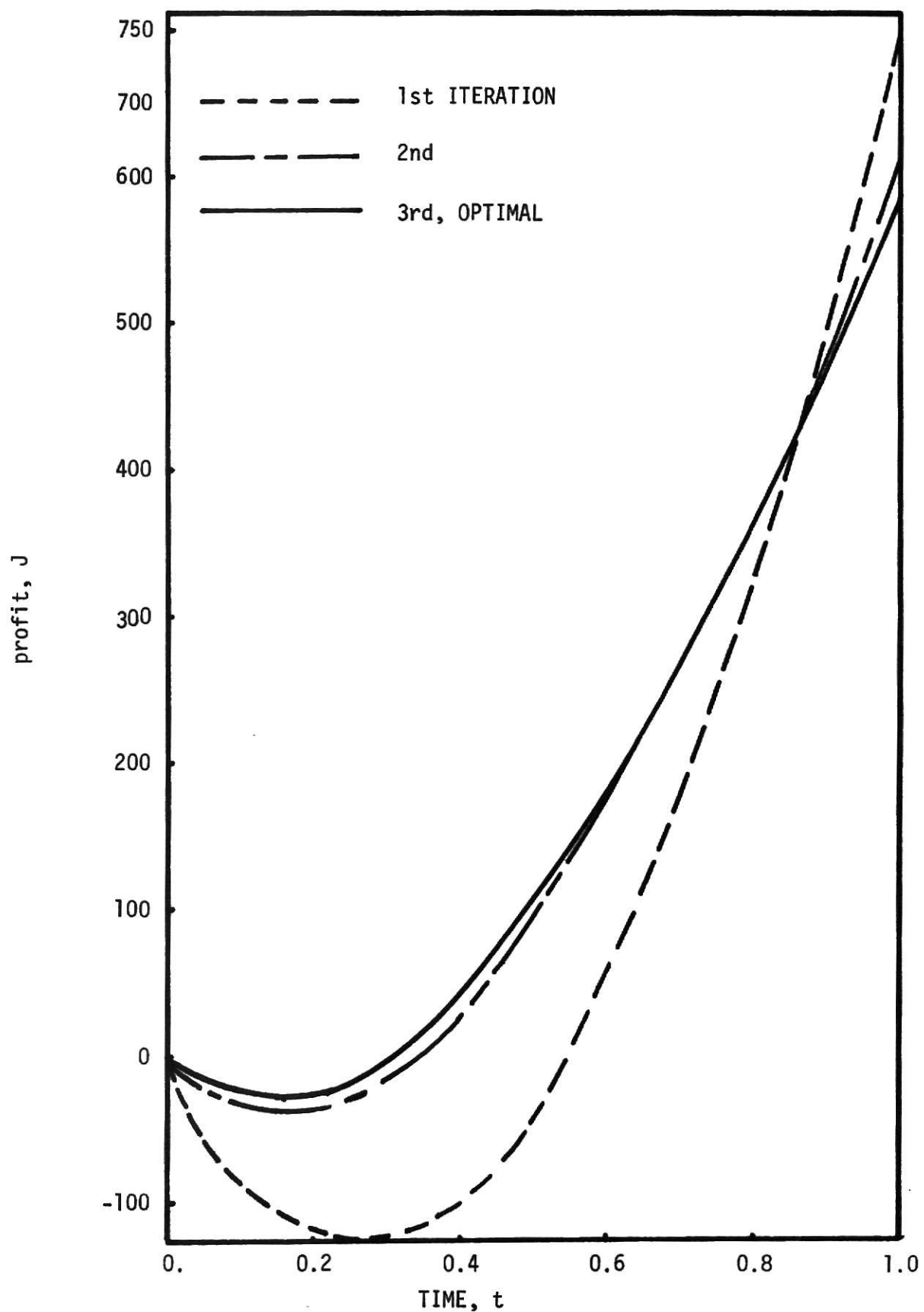


Fig. 16 Convergence rate of profit of the Advertisement problem by quasilinearization with the method of adjoint functions.

Table 1. Convergence rate of the Advertisement problem by the method of particular solutions, scheme (A) without stepsize control.

Iteration	$I(t_f)$	$S(t_f)$	$A(o)$	J
0	20.000	20.00		
1	36.878	216.39	5.7902	703.51
2	75.600	109.24	4.8914	427.89
3	65.023	125.51	4.0194	606.59
4	65.307	116.91	3.8991	608.22
5	66.936	114.64	4.0008	583.39
6	66.726	115.76	4.0139	585.29
7	66.539	115.90	3.9992	588.34
8	66.597	115.74	3.9981	587.90
9	66.606	115.73	4.0004	587.45
10	66.598	115.75	4.0004	587.54
11	66.594	115.75	4.0004	587.61
12	66.596	115.75	4.0004	587.58
13	66.596	115.75	4.0004	587.58

Table 2-1. Convergence rate of the Advertisement problem by the method of particular solutions, scheme (B) without stepsize control.

Iteration	$I(t_f)$	$S(t_f)$	$A(o)$	J
0	106.330	41.85		
1	52.095	133.19	5.8033	455.60
2	59.988	124.14	4.7199	566.28
3	64.853	117.58	4.1315	586.80
4	66.107	115.76	3.9889	587.71
5	66.158	115.69	3.9834	587.71
6	66.157	115.69	3.9834	587.71

Table 2-2. Convergence rate of the Advertisement problem by the method of particular solutions, scheme (B) with stepsize control.

Iteration	$I(t_f)$	$S(t_f)$	$A(o)$	J	a	P
0	106.330	41.85				
1	75.472	101.61	2.9016	554.85	0.5	12.8440
2	62.820	120.43	4.3684	581.68	1	7.1907
3	65.773	116.25	4.0265	587.63	1	0.0802
4	66.156	115.69	3.9835	587.71	1	0.0090

Table 3. Convergence rate of the Advertisement problem by quasilinearization with the method of complementary functions.

Iteration	$I(t_f)$	$S(t_f)$	$A(o)$	J
0	20.000	20.00		
1	36.869	216.39	5.7902	703.49
2	63.050	129.04	4.0867	612.26
3	66.129	115.88	3.9803	588.06
4	66.155	115.69	3.9816	587.80
5	66.155	115.69	3.9816	587.80

Table 4. Convergence rate of the Advertisement problem by quasilinearization with the method of adjoint functions.

Iteration	$I(t_f)$	$S(t_f)$	$A(o)$	J
0	20.000	20.00		
1	36.868	216.39	5.7902	703.50
2	62.252	130.39	4.2176	607.49
3	65.618	116.67	4.0430	588.27
4	65.836	116.11	4.0248	587.74
5	65.836	116.10	4.0248	587.74

Table 5. Convergence rates of the Advertisement problem, scheme (A)
with stepsize control and bisection.

Iteration	$I(t_f)$	$S(t_f)$	$A(o)$	J	α	P
0	20.000	20.00				
1	28.434	118.20	2.8951	469.60	0.5	4404.800
2	52.825	140.30	3.3037	769.95	1	2494.600
3	69.476	109.34	4.0249	552.76	1	234.830
4	66.609	117.47	4.0395	585.30	1	27.670
5	66.315	116.19	3.9871	591.77	1	29.040
6	66.580	115.62	3.9941	585.10	1	0.604
7	66.641	115.68	4.0027	586.90	1	0.311
8	66.596	115.77	4.0011	587.54	1	0.263
9	66.588	115.76	3.9998	587.70	1	0.255
10	66.596	115.75	4.0001	587.59	1	0.254
11	66.595	115.75	4.0002	587.57	1	0.253
12	66.595	115.75	4.0002	587.59	1	0.253

Table 6. Convergence rate of the Advertisement problem, scheme (A)
with stepsize control and Fibonacci.

Iteration	$I(t_f)$	$S(t_f)$	$A(o)$	J	α	P
0	20.000	20.00				
1	28.501	118.98	2.9183	472.08	0.504	4424.400
2	44.013	132.13	3.1614	663.55	0.632	1441.800
3	62.056	118.72	3.7424	626.51	0.880	69.676
4	65.504	115.48	3.9237	594.31	0.632	8.944
5	66.604	115.23	3.9990	584.88	0.752	2.236
6	66.657	115.72	0.4008	586.10	0.880	0.450
7	66.584	115.82	0.4002	587.57	0.752	0.292
8	66.578	115.78	3.9995	587.84	0.880	0.259
9	66.591	115.75	3.9998	587.67	0.752	0.253
10	66.596	115.75	0.4001	587.60	0.632	0.253
11	66.596	115.75	0.4001	587.58	0.632	0.253

Table 7. Comparison of optimal results and convergence rates, for the advertisement problem.

METHODS	INVENTORY $I(t_f)$	SALES $S(t_f)$	ADVERTISEMENT $A(t_0)$	PROFIT $J(t_f)$	STEP SIZE α	CUMULATIVE ERROR, P	ITERATIONS REQUIRED
Particular Solution Method Scheme A Without Stepsize Control	66.596	115.750	4.000	587.58			13
Particular Solution Method Scheme A With Stepsize Control. Bisection Method.	66.596	115.750	4.000	587.59	1	0.253	12
Particular Solution Method Scheme A With Stepsize Control. Fibonacci search.	66.596	115.750	4.000	587.58	0.632	0.253	11
Particular Solution Method Scheme B With- out Stepsize Control	66.157	115.690	3.983	587.71			6
Particular Solution Method Scheme B with Stepsize Control. Bisection Method.	66.156	115.690	3.983	587.71	1	0.009	5

Table 7. Comparison of optimal results and convergence rates, for the advertisement problem. (cont.)

METHODS	INVENTORY $I(t_f)$	SALES $S(t_f)$	ADVERTISEMENT $A(t_0)$	PROFIT $J(t_f)$	STEP SIZE α	CUMULATIVE ERROR, P	ITERATIONS REQUIRED
Quasilinearization with superposition principle.	66.155	115.69	3.982	587.80			5
Quasilinearization with the method of adjoint functions.	66.844	116.10	4.024	587.74			5

CHAPTER 4

APPLICATION TO AN ADVERTISEMENT AND PRODUCTION PROBLEM

To show the effectiveness of the method of particular solutions, we now apply the technique to a large-dimensional boundary value problem, an advertising and production scheduling problem with twelve dimensions will be solved.

4.1 DEVELOPMENT OF THE MODEL

Consider the manufacturing process shown in Fig. 17.

Let us assume that raw materials containing A, B and C is available in unlimited quantity from another part of the manufacturing process. B is the most valuable product and C is the least valuable. Furthermore, B can be produced from A by chemical reaction. During this reaction, a certain amount of B also decomposes into C, since C is less valuable, this decomposition is undesirable. A, C have an unlimited market at a fixed price and that they can be sold as soon as manufactured, therefore no inventory for A, C will be considered. The amount of B produced and decomposed can be controlled by the reactor temperature T . In here B is a new product which needs advertisement to boost the sales, and to protect against fluctuations in demand, an inventory will be assumed for B.

The problem is: considering the costs of adjusting the temperature in two reactors, costs of advertising, the storage of B, and the relative sale values of A, B and C, what are the optimal advertising and production rates of B?

Let x_i and y_i , $i = 1, 2$ represents the concentration of A and B

respectively. Under steady state conditions, from material balance, we have

$$\begin{array}{rcl} \text{amount of} & = & \text{amount of} + \text{amount of A} \\ \text{A in} & & \text{A out} \quad \text{transformed to B} \end{array} \quad (4-1)$$

$$\begin{array}{rcl} \text{amount of} & = & \text{amount of} + \text{amount of B} - \text{amount of B} \\ \text{B in} & & \text{B out} \quad \text{transformed to C} \quad \text{produced from A} \end{array} \quad (4-2)$$

let

V_i = Volume of chemical reactor i , $i = 1, 2$.

q = flow rate

K_{ai} = reaction rate constant of the first reaction in reactor i .

K_{bi} = reaction rate constant of the second reaction in reactor i .

G_a, G_b = frequency constants of the first and second reactions, respectively.

E_a, E_b = Activation energies of the first and second reactions, respectively.

R = Gas constant

T_i = Temperature in reactor i

The kinetics of the reactions can now be written as

$$qx_0 = qx_1 + V_1 K_{a_1} x_1$$

$$\text{or } q(x_0 - x_1) - V_1 K_{a_1} x_1 = 0$$

at steady state. Under unsteady state we have

$$V_1 \frac{dx_1}{dt} = q(x_0 - x_1) - V_1 K_{a_1} x_1 \quad (4-3)$$

similary, equation (4-2) can be expressed as

$$qy_0 = qy_1 + V_1 k_{b_1} y_1 - V_1 k_{a_1} x_1$$

$$q(y_0 - y_1) - v_1 k_{b_1} y_1 + v_1 K_{a_1} x_1 = 0$$

at a steady state. Under unsteady state we have

$$V_1 \frac{dy_1}{dt} = q(y_0 - y_1) - v_1 K_{b_1} y_1 + v_1 K_{a_1} x_1 \quad (4-4)$$

with similar arguments, the kinetics of the reactions in the second reactor can be written as

$$V_2 \frac{dx_2}{dt} = q(x_1 - x_2) - v_2 K_{a_2} x_2 \quad (4-5)$$

$$\text{and } V_2 \frac{dy_2}{dt} = q(y_1 - y_2) - v_2 k_{b_2} y_2 + v_2 k_{a_2} x_2 \quad (4-6)$$

The reaction rate constants are defined as

$$\begin{aligned} k_{a_1} &= G_a \exp\left(-\frac{E_a}{RT_1}\right) & k_{b_1} &= G_b \exp\left(-\frac{E_b}{RT_1}\right) \\ k_{a_2} &= G_a \exp\left(-\frac{E_a}{RT_2}\right) & k_{b_2} &= G_b \exp\left(-\frac{E_b}{RT_2}\right) \end{aligned} \quad (4-7)$$

We know that B is the desired product and it needs inventory and advertisement. The performance equation for the inventory is

$$\begin{aligned} \text{Rate of change of inventory} &= \text{production rate} - \text{sales rate} \\ \frac{dI}{dt} &= qy_2 - S \end{aligned} \quad (4-8)$$

Where I is inventory and S is sales.

The performance equation for advertisement is the same as equation (3-2) in previous chapter. Let $C_q = 1$. According to equation (3-3) we have

$$S(t) = K(t)$$

$$\frac{dS}{dt} = S(C + A) \left(1 - \frac{S}{N}\right) \quad (4-9)$$

Where C is the contact coefficient, A is the advertisement and N represents the total number of people in the group.

Equations (4-3), (4-4), (4-5), (4-6), (4-8), (4-9) describe the system completely. We have six state variables, x_1 , x_2 , y_1 , y_2 , I and S and three control variables T_1 , T_2 and A .

The management wishes to obtain the values of three control variables such that the following profit function J is maximized.

$$\text{profit} = \int_{t_0}^{t_f} [\text{revenue of B} + \text{revenue of A} + \\ \text{revenue of C} - \text{inventory cost} - \\ \text{advertisement cost} - \text{mgt. cost}] dt$$

that is

$$J = \int_{t_0}^{t_f} \{C_1 S + C_2 q x_2 + C_3 q (1 - x_2 - y_2) - C_I (I_m - I)^2 \\ - C_A A^2 S^2 - C_T [(T_m - T_1)^2 + (T_1 - T_2)^2]\} dt$$

4.2 DEFINITION OF THE PROBLEM

Maximize the functional

$$J = \int_{t_0}^{t_f} \{C_1 S + C_2 q x_2 + C_3 q (1 - x_2 - y_2) - C_I (I_m - I)^2 - C_A A^2 S^2 - C_T [(T_m - T_1)^2 + (T_1 - T_2)^2]\} dt \quad (4-10)$$

subject to

$$v_1 \frac{dx_1}{dt} = q(x_0 - x_1) - v_1 k_{a_1} x_1 \quad (4-11)$$

$$v_1 \frac{dy_1}{dt} = q(y_0 - y_1) - v_1 k_{b_1} y_1 + v_1 k_{a_1} x_1 \quad (4-12)$$

$$v_2 \frac{dx_2}{dt} = q(x_1 - x_2) - v_2 k_{a_2} x_2 \quad (4-13)$$

$$v_2 \frac{dy_2}{dt} = q(y_1 - y_2) - v_2 k_{b_2} y_2 + v_2 k_{a_2} x_2 \quad (4-14)$$

$$\frac{dI}{dt} = qy_2 - S \quad (4-15)$$

$$\frac{ds}{dt} = S(C + A)(1 - \frac{S}{N}) \quad (4-16)$$

with boundary conditions

$$\begin{aligned} x_1(t_0) &= x_1^0 & y_1(t_0) &= y_1^0 \\ x_2(t_0) &= x_2^0 & y_2(t_0) &= y_2^0 \\ I(t_0) &= I^0 & S(t_0) &= S^0 \end{aligned} \quad (4-17)$$

$$I(t_f) = I^f$$

4.3 FORMULATION OF THE PROBLEM

It was required to find the optimal value of the state variables and control variables so that the objective function is maximized. The procedure for obtaining the solution remains essentially the same.

Equations (4-11) through (4-16) can be rewritten as

$$\dot{X}_1 = \frac{q}{V_1} (X_0 - X_1) - G_a e^{-\frac{E_a}{RT_1}} X_1 = F_1 \quad (4-18)$$

$$\dot{Y}_1 = \frac{q}{V_1} (Y_0 - Y_1) - G_b e^{-\frac{E_b}{RT_1}} Y_1 + G_a e^{-\frac{E_a}{RT_1}} X_1 = F_2 \quad (4-19)$$

$$\dot{X}_2 = \frac{q}{V_2} (X_1 - X_2) - G_a e^{-\frac{E_a}{RT_2}} X_2 = F_3 \quad (4-20)$$

$$\dot{Y}_2 = \frac{q}{V_2} (Y_1 - Y_2) - G_b e^{-\frac{E_b}{RT_2}} Y_2 + G_a e^{-\frac{E_a}{RT_2}} X_2 = F_4 \quad (4-21)$$

$$\dot{I} = qY_2 - S = F_5 \quad (4-22)$$

$$\dot{S} = (CS + AS) \left(1 - \frac{S}{N}\right) = F_6 \quad (4-23)$$

Introducing Lagrange multipliers λ_i , $i = 1, 2, \dots, 6$, and constant multipliers Q_j , $j = 1, 2, \dots, 7$ and define the following functions.

$$F = \{\lambda_1 \left[\dot{X}_1 - \frac{q}{V_1} (X_0 - X_1) + G_a e^{-\frac{E_a}{RT_1}} X_1 \right] +$$

$$\lambda_2 \left[\dot{Y}_1 - \frac{q}{V_1} (Y_0 - Y_1) + G_b e^{-\frac{E_b}{RT_1}} Y_1 - G_a e^{-\frac{E_a}{RT_1}} X_1 \right] +$$

$$\lambda_3 \left[\dot{X}_2 - \frac{q}{V_2} (X_1 - X_2) + G_a e^{-\frac{E_a}{RT_2}} X_2 \right] +$$

$$\lambda_4 \left[\dot{Y}_2 - \frac{q}{V_2} (Y_1 - Y_2) + G_b e^{-\frac{E_b}{RT_2}} Y_2 - G_a e^{-\frac{E_a}{RT_2}} X_2 \right] +$$

$$\lambda_5 \left[\dot{I} - qY_2 + S \right] +$$

$$\lambda_6 \left[\dot{S} - CS - AS + \frac{CS^2}{N} + \frac{AS^2}{N} \right] +$$

$$C_1 S + C_2 q X_2 + C_3 q (1 - X_2 - Y_2) - C_I (I_m - I)^2 - C_A A^2 S^2$$

$$- C_T [(T_m - T_1)^2 + (T_1 - T_2)^2] \} \quad (4-24a)$$

and

$$\begin{aligned} G = & \{ Q_1 [X_1(t_0) - X_1^0] + Q_2 [Y_1(t_0) - Y_1^0] + Q_3 [X_2(t_0) - X_2^0] \\ & + Q_4 [Y_2(t_0) - Y_2^0] + Q_5 [I(t_0) - I^0] \\ & + Q_6 [I(t_f) - I^f] + Q_7 [S(t_0) - S^0] \} \end{aligned} \quad (4-24b)$$

Then, apply the Euler-Lagrange equation

$$\frac{d}{dt} \left(\frac{\partial F}{\partial \dot{Y}_i} \right) - \frac{\partial F}{\partial Y_i} = 0$$

$$\frac{d}{dt} \left(\frac{\partial F}{\partial \dot{Y}_i} \right) - \frac{\partial F}{\partial X_i} = 0$$

$$\frac{\partial F}{\partial Z} = 0$$

these equations can be applied to equations (4-24) and (4-25) to obtain the relationships for the six Lagrange multipliers

$$\dot{\lambda}_1 = q \left(\frac{\lambda_1}{v_1} - \frac{\lambda_3}{v_2} \right) + (\lambda_1 - \lambda_2) G_a e^{-\frac{E_a}{RT_1}} = F_7 \quad (4-24)$$

$$\dot{\lambda}_2 = q \left(\frac{\lambda_2}{v_1} - \frac{\lambda_4}{v_2} \right) + \lambda_2 G_b e^{-\frac{E_b}{RT_1}} = F_8 \quad (4-25)$$

$$\dot{\lambda}_3 = \frac{\lambda_3 q}{v_2} + (\lambda_3 - \lambda_4) G_a e^{-\frac{E_a}{RT_2}} + q(C_2 - C_3) = F_9 \quad (4-26)$$

$$\dot{\lambda}_4 = \frac{\lambda_4 q}{v_2} + \lambda_4 G_b e^{-\frac{E_b}{RT_2}} - q(C_3 + \lambda_5) = F_{10} \quad (4-27)$$

$$\dot{\lambda}_5 = 2C_I I_m - 2C_I I = F_{11} \quad (4-28)$$

$$\dot{\lambda}_6 = C_1 + \lambda_5 - C\lambda_6 - A\lambda_6 + \frac{2CS\lambda_6}{N} + \frac{2AS\lambda_6}{N}$$

$$- 2C_A A^2 S = F_{12} \quad (4-29)$$

and

$$A = \frac{\lambda_6}{2C_A} \left(\frac{1}{N} - \frac{1}{S} \right) \quad (4-30)$$

$$\frac{(\lambda_1 - \lambda_2) G_a E_a X_1}{RT_1^2} e^{-\frac{E_a}{RT_1}} + \frac{\lambda_2 G_b E_b y_1}{RT_1^2} e^{-\frac{E_b}{RT_1}}$$

$$-2C_T (2T_1 - T_2 - T_m) = 0 \quad (4-31)$$

$$\frac{(\lambda_3 - \lambda_4) G_a E_a X_2}{RT_2^2} e^{-\frac{E_a}{RT_2}} + \frac{\lambda_4 G_b E_b Y_2}{RT_2^2} e^{-\frac{E_b}{RT_2}} +$$

$$2C_T(T_1 - T_2) = 0 \quad (4-32)$$

A can be eliminated from all performance equations, if we substitute equation (4-30) into the performance equations. But, the control variables T_1 and T_2 , appear implicitly in the above equations and cannot be eliminated.

Substituting Eq. (4-30) into equations (4-23) and (4-29), we obtain

$$\frac{dS}{dt} = CS - \frac{CS^2}{N} + \frac{S\lambda_6}{C_A N} - \frac{\lambda_6}{2C_A} - \frac{S^2 \lambda_6}{2C_A N^2} \quad (4-33)$$

$$\frac{d\lambda_6}{dt} = C_1 + \lambda_5 - C\lambda_6 - \frac{\lambda_6^2}{2C_A N} + \frac{2CS\lambda_6}{N} + \frac{S\lambda_6^2}{2C_A N^2} \quad (4-34)$$

Equation (4-18) through (4-21), Eq. (4-24) through (4-28) and Eqs. (4-33), (4-34) represents the system. For these 12 differential equations we have only 7 boundary conditions, given by Eq. (4-17). The additional 5 boundary conditions can be obtained by applying the transversality conditions

$$\left. \frac{\partial G}{\partial X_i} \right|_{t_0} - \left. \frac{\partial F}{\partial X_i} \right|_{t_0} = 0$$

or

$$\left. \frac{\partial G}{\partial X_i} \right|_{t_f} - \left. \frac{\partial F}{\partial X_i} \right|_{t_f} = 0$$

to equations (4-24) and (4-25), we obtain

$$\lambda_1(t_f) = 0$$

$$\lambda_2(t_f) = 0$$

$$\lambda_3(t_f) = 0 \tag{4-35}$$

$$\lambda_4(t_f) = 0$$

$$\lambda_6(t_f) = 0$$

Now there are 12 boundary conditions, a two-point boundary value problem was formed.

4.4 COMPUTATIONAL PROCEDURE BY THE METHOD OF PARTICULAR SOLUTIONS, SCHEME (A).

To solve the above system by the method of particular solutions scheme (A), the procedure is the same as solving the two dimensional advertisement problem. First, assume the following nominal functions

$$x_1(t) = x_1^0 \quad \lambda_1(t) = 0$$

$$x_2(t) = x_2^0 \quad \lambda_2(t) = 0$$

$$I(t) = I^0 \quad \lambda_3(t) = 0$$

(4-36)

$$y_1(t) = y_1^0 \quad \lambda_4(t) = 0$$

$$y_2(t) = y_2^0 \quad \lambda_5(t) = 0$$

$$S(t) = S^0 \quad \lambda_6(t) = 0$$

$$\text{for } t_0 \leq t \leq t_f$$

Then we linearize Eqs. (4-18) through (4-22), Eqs. (4-24) through (4-28) and Eqs. (4-33), (4-34), and obtain the following equations

$$\Delta \dot{I} - q \Delta y_2 + \Delta S + (\dot{I} - q y_2 + S) = 0$$

(4-37)

$$\Delta \dot{S} + \Delta S \left(-C + \frac{2CS}{N} - \frac{\lambda_6}{C_A N} + \frac{S \lambda_6}{C_A N^2} \right)$$

$$+ \Delta \lambda_6 \left(\frac{-S}{C_A N} + \frac{1}{2C_A} + \frac{S^2}{2C_A N^2} \right) + (\dot{S} - CS + \frac{CS^2}{N}$$

(4-38)

$$- \frac{S \lambda_6}{C_A N} + \frac{\lambda_6}{2C_A} + \frac{S^2 \lambda_6}{2C_A N^2}) = 0$$

$$\begin{aligned} \Delta \dot{x}_1 + \Delta x_1 \left(\frac{q}{v_1} + G_a e^{-\frac{E_a}{RT_1}} \right) + \left(\dot{x}_1 - \frac{q}{v} (x_0 - x_1) \right. \\ \left. + G_a e^{-\frac{E_a}{RT_1}} x_1 \right) = 0 \end{aligned} \quad (4-39)$$

$$\begin{aligned} \Delta \dot{x}_2 + \Delta x_1 \left(-\frac{q}{v_2} \right) + \Delta x_2 \left(\frac{q}{v_2} + G_a e^{-\frac{E_a}{RT_2}} \right) + \\ \left(\dot{x}_2 - \frac{q}{v_2} (x_1 - x_2) + G_a e^{-\frac{E_a}{RT_2}} x_2 \right) = 0 \end{aligned} \quad (4-40)$$

$$\begin{aligned} \Delta \dot{y}_1 + \Delta y_1 \left(\frac{q}{v_1} + G_b e^{-\frac{E_b}{RT_1}} \right) + \Delta x_1 \left(-G_a e^{-\frac{E_a}{RT_1}} \right) \\ + \left(\dot{y}_1 - \frac{q}{v_1} (y_0 - y_1) + G_b e^{-\frac{E_b}{RT_1}} y_1 - G_a e^{-\frac{E_a}{RT_1}} x_1 \right) = 0 \end{aligned} \quad (4-41)$$

$$\begin{aligned} \Delta \dot{y}_2 + \Delta y_1 \left(-\frac{q}{v_2} \right) + \Delta y_2 \left(\frac{q}{v_2} + G_b e^{-\frac{E_b}{RT_2}} \right) - \Delta x_2 \left(G_a e^{-\frac{E_a}{RT_2}} \right) \\ + \left(\dot{y}_2 - \frac{q}{v_2} (y_1 - y_2) + G_b e^{-\frac{E_b}{RT_2}} y_2 - G_a e^{-\frac{E_a}{RT_2}} x_2 \right) = 0 \end{aligned} \quad (4-42)$$

$$\begin{aligned} \Delta \dot{\lambda}_1 + \Delta \lambda_1 \left(-\frac{q}{v_1} - G_a e^{-\frac{E_a}{RT_1}} \right) + \Delta \lambda_2 \left(G_a e^{-\frac{E_a}{RT_1}} \right) + \Delta \lambda_3 \frac{q}{v} \\ + \left[\dot{\lambda}_1 - q \left(\frac{\lambda_1}{v_1} - \frac{\lambda_3}{v_2} \right) - (\lambda_1 - \lambda_2) G_a e^{-\frac{E_a}{RT_1}} \right] = 0 \end{aligned} \quad (4-43)$$

$$\begin{aligned} \Delta \dot{\lambda}_2 + \Delta \lambda_2 \left(-\frac{q}{v_1} - G_b e^{-\frac{E_b}{RT_1}} \right) + \Delta \lambda_4 \frac{q}{v_2} + \left[\dot{\lambda}_2 - \right. \\ \left. q \left(\frac{\lambda_2}{v_1} - \frac{\lambda_4}{v_2} \right) - \lambda_2 G_b e^{-\frac{E_b}{RT_1}} \right] = 0 \end{aligned} \quad (4-44)$$

$$\Delta \dot{\lambda}_3 + \Delta \lambda_3 \left(-\frac{q}{v_2} - G_a e^{\frac{-E_a}{RT_2}} \right) + \Delta \lambda_4 \left(G_a e^{\frac{-E_a}{RT_2}} \right) \quad (4-45)$$

$$+ \left[\lambda_3 - \frac{\lambda_3 q}{v_2} - (\lambda_3 - \lambda_4) G_a e^{\frac{-E_a}{RT_2}} + q(C_2 - C_3) \right] = 0$$

$$\Delta \dot{\lambda}_4 + \Delta \lambda_4 \left(-\frac{q}{v_2} - G_b e^{\frac{-E_b}{RT_2}} \right) + \Delta \lambda_5 q + \quad (4-46)$$

$$\left[\lambda_4 - \frac{\lambda_4 q}{v_2} - \lambda_4 G_b e^{\frac{-E_b}{RT_2}} + q(C_3 + \lambda_5) \right] = 0$$

$$\Delta \dot{\lambda}_5 + 2C_I \Delta I + (\dot{\lambda}_5 - 2C_I I_m - 2C_I I) = 0 \quad (4-47)$$

$$\Delta \dot{\lambda}_6 - \Delta \lambda_5 + \Delta \lambda_6 \left(C + \frac{\lambda_6}{C_A N} - \frac{2CS}{N} - \frac{S\lambda_6}{C_A N^2} \right) + \quad (4-48)$$

$$\left[\dot{\lambda}_6 - C_1 - \lambda_5 + C\lambda_6 + \frac{\lambda_6^2}{2C_A N} - \frac{2CS\lambda_6}{N} - \frac{S\lambda_6^2}{2C_A N^2} \right] = 0$$

After that, linearize the boundary conditions and we obtain:

$$\Delta I(t_0) = 0 \quad \Delta \lambda_1(t_f) + \lambda_1(t_f) = 0$$

$$\Delta S(t_0) = 0 \quad \Delta \lambda_2(t_f) + \lambda_2(t_f) = 0$$

$$\Delta x_1(t_0) = 0 \quad \Delta \lambda_3(t_f) + \lambda_3(t_f) = 0$$

$$\Delta x_2(t_0) = 0 \quad \Delta \lambda_4(t_f) + \lambda_4(t_f) = 0$$

$$\Delta y_1(t_0) = 0 \quad \Delta \lambda_6(t_f) + \lambda_6(t_f) = 0$$

$$\Delta y_2(t_0) = 0 \quad \Delta I(t_f) + [I(t_f) - I^f] = 0$$

(4-49)

Now introducing stepsize α to each forcing term of equation (4-37) through (4-48) and equation (4-49). Also introducing the auxiliary variables

$$\begin{aligned}
 A_1 &= \Delta I / \alpha & A_7 &= \lambda_1 / \alpha \\
 A_2 &= \Delta S / \alpha & A_8 &= \lambda_2 / \alpha \\
 A_3 &= X_1 / \alpha & A_9 &= \lambda_3 / \alpha \\
 A_4 &= y_1 / \alpha & A_{10} &= \lambda_4 / \alpha \\
 A_5 &= X_2 / \alpha & A_{11} &= \lambda_5 / \alpha \\
 A_6 &= y_2 / \alpha & A_{12} &= \lambda_6 / \alpha
 \end{aligned} \tag{4-50}$$

Then the equations (4-37) through (4-48) can be rewritten as

$$\dot{A}_1 - qA_6 + A_2 + (\dot{I} - qy_2 + S) = 0 \tag{4-51}$$

$$\begin{aligned}
 \dot{A}_2 + \left(-C + \frac{2CS}{N} - \frac{\lambda_6}{C_A N} + \frac{S\lambda_6}{C_A N^2}\right) A_2 + \left(-\frac{S}{C_A N} + \frac{1}{2C_A} + \frac{S^2}{2C_A N^2}\right) A_{12} \\
 + \left(\dot{S} - CS + \frac{CS^2}{N} - \frac{S\lambda_6}{C_A N} + \frac{\lambda_6}{2C_A} + \frac{S^2 \lambda_6}{2C_A N^2}\right) = 0
 \end{aligned} \tag{4-52}$$

$$\dot{A}_3 + \left(\frac{q}{v_1} + G_a e^{\frac{-E_a}{RT_1}} \right) A_3 + \left(\dot{x}_1 - \frac{q}{v} (x_0 - x_1) + G_a e^{\frac{-E_a}{RT_1}} x_1 \right) = 0 \quad (4-53)$$

$$\dot{A}_4 + \left(\frac{q}{v_1} + G_b e^{\frac{-E_b}{RT_1}} \right) A_4 - G_a e^{\frac{-E_a}{RT_1}} A_3 +$$

$$\left(\dot{y}_1 - \frac{q}{v_1} (y_0 - y_1) + G_b e^{\frac{-E_b}{RT_1}} y_1 - G_a e^{\frac{-E_a}{RT_1}} x_1 \right) = 0 \quad (4-54)$$

$$\dot{A}_5 - \frac{q}{v_2} A_3 + \left(\frac{q}{v_2} + G_a e^{\frac{-E_a}{RT_2}} \right) A_5 + \left(\dot{x}_2 - \frac{q}{v} (x_1 - x_2) \right.$$

$$\left. + G_a e^{\frac{-E_a}{RT_2}} x_2 \right) = 0 \quad (4-55)$$

$$\dot{A}_6 - \frac{q}{v_2} A_4 + \left(\frac{q}{v_2} + G_b e^{\frac{-E_b}{RT_2}} \right) A_6 - G_a e^{\frac{-E_a}{RT_2}} A_5$$

$$+ \left[\dot{y}_2 - \frac{q}{v_2} (y_1 - y_2) + G_b e^{\frac{-E_b}{RT_2}} y_2 - G_a e^{\frac{-E_a}{RT_2}} x_2 \right] = 0 \quad (4-56)$$

$$\dot{A}_7 - \left(\frac{q}{v_1} + G_a e^{\frac{-E_a}{RT_1}} \right) A_7 + G_a e^{\frac{-E_a}{RT_1}} A_8 + \frac{q}{v_2} A_9$$

$$+ \left[\dot{\lambda}_1 - q \left(\frac{\lambda_1}{v_1} - \frac{\lambda_3}{v_2} \right) - (\lambda_1 - \lambda_2) G_a e^{\frac{-E_a}{RT_1}} \right] = 0 \quad (4-57)$$

$$\begin{aligned} \dot{A}_8 - \left(\frac{q}{v_1} + G_b e^{\frac{-E_b}{RT_1}} \right) A_8 + \frac{q}{v_2} A_{10} + \left[\dot{\lambda}_2 - q \left(\frac{\lambda_2}{v_1} - \frac{\lambda_4}{v_2} \right) \right. \\ \left. - \lambda_2 G_b e^{\frac{-E_b}{RT_1}} \right] = 0 \end{aligned} \quad (4-58)$$

$$\begin{aligned} \dot{A}_9 - \left(\frac{q}{v_2} + G_a e^{\frac{-E_a}{RT_2}} \right) A_9 + G_a e^{\frac{-E_a}{RT_2}} A_{10} \\ + \left[\dot{\lambda}_3 - \frac{\lambda_3 q}{v_2} - (\lambda_3 - \lambda_4) G_a e^{\frac{-E_a}{RT_2}} - q(C_2 - C_3) \right] = 0 \end{aligned} \quad (4-59)$$

$$\begin{aligned} \dot{A}_{10} - \left(\frac{q}{v_2} + G_b e^{\frac{-E_b}{RT_2}} \right) A_{10} + q A_{11} + \left[\dot{\lambda}_4 - \frac{\lambda_4 q}{v_2} \right. \\ \left. - \lambda_4 G_b e^{\frac{-E_b}{RT_2}} + q(C_3 + \lambda_5) \right] = 0 \end{aligned} \quad (4-60)$$

$$\dot{A}_{11} + 2C_I A_1 + (\dot{\lambda}_5 - 2C_I I_m - 2C_I I) = 0 \quad (4-61)$$

$$\begin{aligned} \dot{A}_{12} - A_{11} + \left(C + \frac{\lambda_6}{C_A N} - \frac{2CS}{N} - \frac{S\lambda_6}{C_A N^2} \right) A_{12} + \left[\dot{\lambda}_6 \right. \\ \left. - C_1 - \lambda_5 + C\lambda_6 + A\lambda_6 - \frac{2CS\lambda_6}{N} - \frac{2AS\lambda_6}{N} + 2C_A A^2 S \right] = 0 \end{aligned} \quad (4-62)$$

Also equation (5-49) can be written as:

$$\begin{aligned}
 A_1(t_0) &= 0 & A_7(t_f) + \lambda_1(t_f) &= 0 \\
 A_2(t_0) &= 0 & A_8(t_f) + \lambda_2(t_f) &= 0 \\
 A_3(t_0) &= 0 & A_9(t_f) + \lambda_3(t_f) &= 0 \\
 A_4(t_0) &= 0 & A_{10}(t_f) + \lambda_4(t_f) &= 0 \\
 A_5(t_0) &= 0 & A_{12}(t_f) + \lambda_5(t_f) &= 0 \\
 A_6(t_0) &= 0 & A_1(t_f) + I(t_f) &= I^f
 \end{aligned} \tag{4-63}$$

For the assumed nominal functions (4-36), we can calculate the coefficient and constant terms in equation (4-51) through (4-62). Hence, the system forms a linear nonhomogeneous boundary value problem, here we have six initial conditions and six final conditions. By the method of particular solutions, we have to solve $q + 1 = 7$ sets of particular solutions with the initial conditions:

$$\begin{aligned}
 A_{ij}(t_0) &= 0 \quad j = 1, 2, \dots, 7, \quad i = 1, 2, \dots, 6 \\
 A_{k+6,j}(t_0) &= \delta_{kj} \quad j = 1, 2, \dots, 7, \quad k = 1, 2, \dots, 6
 \end{aligned} \tag{4-64}$$

The results obtained are $A_{pj}^i(t)$, where i denotes variables, pj denotes

the j^{th} set of particular solutions.

After that, compute the undetermined constants k_j by solving the following linear algebraic equations

$$\sum_{j=1}^7 k_j = 1$$

$$\sum_{j=1}^7 k_j A_{pj}^7(t_f) = -\lambda_1(t_f)$$

$$\sum_{j=1}^7 k_j A_{pj}^8(t_f) = -\lambda_2(t_f)$$

$$\sum_{j=1}^7 k_j A_{pj}^9(t_f) = -\lambda_3(t_f) \quad (4-65)$$

$$\sum_{j=1}^7 k_j A_{pj}^{10}(t_f) = -\lambda_4(t_f)$$

$$\sum_{j=1}^7 k_j A_{pj}^{12}(t_f) = -\lambda_6(t_f)$$

$$\sum_{i=1}^7 k_i A_{pi}^1(t_f) = -[I(t_f) - I^f]$$

With k_i , $i = 1, 2, \dots, 7$ known, we form the linear combination

$$A^i(t) = \sum_{j=1}^7 k_j A_j^i(t) \quad (4-66)$$

$$t_i \leq t \leq t_f$$

$$i = 1, 2, \dots, 12$$

then varied functions can be formed

$$\tilde{I}(t) = I(t) + \alpha A^1(t)$$

$$\tilde{S}(t) = S(t) + \alpha A^2(t)$$

$$\tilde{x}(t) = x_1(t) + \alpha A^3(t)$$

$$\tilde{y}_1(t) = y_1(t) + \alpha A^4(t)$$

$$\tilde{x}_2(t) = x_2(t) + \alpha A^5(t) \quad (4-67)$$

$$\tilde{y}_2(t) = y_2(t) + \alpha A^6(t)$$

$$\tilde{\lambda}_1(t) = \lambda_1(t) + \alpha A^7(t)$$

$$\tilde{\lambda}_2(t) = \lambda_2(t) + \alpha A^8(t)$$

$$\tilde{\lambda}_3(t) = \lambda_3(t) + \alpha A^9(t)$$

$$\tilde{\lambda}_4(t) = \lambda_4(t) + \alpha A^{10}(t)$$

$$\tilde{\lambda}_5(t) = \lambda_5(t) + \alpha A^{11}(t)$$

$$\tilde{\lambda}_6(t) = \lambda_6(t) + \alpha A^{12}(t)$$

After obtaining the solution for the six state variables and six Lagrange multipliers, the profit J and control variables A , T_1 and T_2 can be obtained by solving equations (4-10), (4-30), (4-31) and (4-32).

The varied function then becomes the nominal function for the next iteration. Further iterations are allowed until desired accuracy is obtained.

4.5 NUMERICAL ASPECTS

The above problem was solved with following numerical constants

$$G_a = 0.535 \times 10^{11} \text{ per minute} \quad GT = 0.001$$

$$G_b = 0.461 \times 10^{18} \text{ per minute} \quad CA = 0.01$$

$$E_a = 18000 \text{ cal/mole} \quad C = 1.$$

$$E_b = 30000 \text{ cal/mole} \quad C_1 = 5$$

$$R = 2 \text{ cal/mole } ^0K \quad C_2 = C_3 = 0$$

$$q = 60 \text{ gal./min.}$$

$$C_I = 1.$$

$$v_1 = v_2 = 12 \text{ gallons}$$

$$x_0(t) = 0.53$$

$$0 \leq t \leq t_f$$

$$I_m = 10 \text{ gallons}$$

$$y_0(t) = 0.43$$

$$T_m = 340^{\circ}\text{K}$$

$$t_i = 0.0$$

$$N = 100$$

$$t_f = 1.0$$

$$\Delta t = 0.02$$

The boundary conditions are

$$x_1(0) = 0.53$$

$$I(0) = 1.0$$

$$x_2(0) = 0.53$$

$$S(0) = 0.1$$

$$y_1(0) = 0.43$$

$$I(1) = 10.$$

$$y_2(0) = 0.43$$

Since the control variables T_1 and T_2 appear implicitly in Eq. (4-31) and (4-32), initial approximations of T_1 and T_2 were given as following in order to solve the equations.

$$T_{1,n=0}(t) = T_{2,n=0}(t) = 345.$$

$$\text{For } 0 \leq t \leq 1$$

The nominal functions for the 1st iteration were assumed as

$$x_1(t) = 0.53 \qquad \lambda_1(t) = 0$$

$$y_1(t) = 0.43 \qquad \lambda_2(t) = 0$$

$$x_2(t) = 0.53 \qquad \lambda_3(t) = 0$$

$$y_2(t) = 0.43 \qquad \lambda_4(t) = 0$$

$$I(t) = 1 + 9t \qquad \lambda_5(t) = 0$$

$$S(t) = 0.1 \qquad \lambda_6(t) = 0$$

The initial values for the seven sets of particular solutions are

	A ₁	A ₂	A ₃	A ₄	A ₅	A ₆	A ₇	A ₈	A ₉	A ₁₀	A ₁₁	A ₁₂
Set 1	0	0	0	0	0	0	1	0	0	0	0	0
2	0	0	0	0	0	0	0	1	0	0	0	0
3	0	0	0	0	0	0	0	0	1	0	0	0
4	0	0	0	0	0	0	0	0	0	1	0	0
5	0	0	0	0	0	0	0	0	0	0	1	0
6	0	0	0	0	0	0	0	0	0	0	0	1
7	0	0	0	0	0	0	0	0	0	0	0	0

4.6 COMPUTATIONAL ASPECTS

Basically, the computational aspect was the same as that mentioned in chapter 3. With the initial conditions given in section 4.5, equations (4-51) through (4-62) were integrated by using fourth order Runge-Kutta integration technique with 50 grid points. Seven sets of particular solutions were obtained.

In order to solve seven undetermined constants, a matrix inversion subroutine SIMQ, supplied by IBM was used.

If we solve this problem with stepsize control, the P function is

$$\begin{aligned}
 P = \int_{t_0}^{t_f} [& (\dot{x}_1 - F_1)^2 + (\dot{y}_1 - F_2)^2 + (\dot{x}_2 - F_3)^2 + (\dot{y}_2 - F_4)^2 + \\
 & (\dot{I} - F_5)^2 + (\dot{S} - F_6)^2 + (\dot{\lambda}_1 - F_7)^2 + (\dot{\lambda}_2 - F_8)^2 + \\
 & (\dot{\lambda}_3 - F_9)^2 + (\dot{\lambda}_4 - F_{10})^2 + (\dot{\lambda}_5 - F_{11})^2 + (\dot{\lambda}_6 - F_{12})^2] dt
 \end{aligned}$$

$$\begin{aligned}
& + [\lambda_1(t_f)]^2 + [\lambda_2(t_f)]^2 + [\lambda_3(t_f)]^2 + [\lambda_4(t_f)]^2 \\
& + [\lambda_5(t_f)]^2 + [\lambda_6(t_f)]^2
\end{aligned} \tag{4-68}$$

Where F_i , $i = 1, 2, \dots, 6$ are functions listed in Eqs. (4-18) through (4-29). This cumulative error P is a function of stepsize α and can be computed by Simpson's integration method. Optimal α which minimizes the value of P can be obtained by search technique. Since the problem has 12 nonlinear differential equations, the calculation of P function and the searching for optimal stepsize require a lot of computer time. Therefore, for this particular problem, only the process without stepsize control was applied.

As we set the stepsize equal to one, the varied function was obtained by Eq. (4-67). After that, the advertisement at each grid point was computed from Eq. (4-30). For simplicity, the following approximation was used to calculate the total profit.

$$\begin{aligned}
J = \int_{t_0}^{t_f} \{ & C_1 S + C_2 q x_2 + C_3 q (1 - x_2 - y_2) - C_I (I_m - I)^2 \\
& - C_A A^2 S^2 - C_T [(T_m - T_1)^2 + (T_1 - T_2)^2] \} \Delta t
\end{aligned}$$

Since T_1 and T_2 in equations (4-31) and (4-32) are implicit, Newton-Raphson method was used for root finding. To start Newton-Raphson method, initial approximations for T_1 and T_2 were assumed to be 350.0. With these values, T_1 and T_2 at the first grid point of the first iteration

were solved. At other grid points, solutions of T_1 and T_2 at previous grid points were used as the initial approximations. Same procedure was repeated for the next iteration. The accuracy to check Newton-Raphson's convergence was 0.1.

All these computations complete one iteration. Further iterations were allowed until convergence was obtained.

4.7 RESULTS

Results obtained by the method of particular solutions scheme A without stepsize control are presented in this section.

The convergence rates of three control variables and six state variables are shown in Fig. 18 through Fig. 24. Also they are tabulized in Table 8 and Table 9.

The optimal profit is $J = 80.195$, and optimal initial and final values are

$$x_1(0) = 0.53$$

$$x_1(1) = 0.477$$

$$y_1(0) = 0.43$$

$$y_1(1) = 0.460$$

$$x_2(0) = 0.53$$

$$x_2(1) = 0.415$$

$$y_2(0) = 0.43$$

$$y_2(1) = 0.487$$

$$I(0) = 1.0$$

$$I(1) = 10.00$$

$$S(0) = 0.1$$

$$S(1) = 34.78$$

$$A(0) = 370.36$$

$$A(1) = 0.$$

$$T_1(0) = 362.43$$

$$T_1(1) = 340.$$

$$T_2(0) = 365.70$$

$$T_2(1) = 340.$$

4.8 DISCUSSION

The advertisement and production scheduling problem was solved by the method of particular solutions, scheme (A) without stepsize control.

This problem has been solved with quasilinearization method by SHAH [13]. For comparison, the iterations required by quasilinearization and by the method of particular solutions were listed in Table 10. (5 digit accuracy is used for all variables). It was observed that, the quasilinearization has a better convergence rate than the method of particular solutions for this particular problem.

The problem has been tried by scheme (B) of the method of particular solutions. By assuming the missing initial conditions for the Lagrange multipliers and combined with the given initial values; the original nonlinear system was integrated forward to obtain the nominal functions. Because these nominal functions for Lagrange multipliers were poor estimations, difficulty appears when solving T_1 and T_2 by Newton Raphson method. This causes scheme (B) to be useless when solving this particular problem.

It was observed that the method of particular solutions, although

is not as powerful as quasilinearization; still is an applicable tool even applied to a complicated problem.

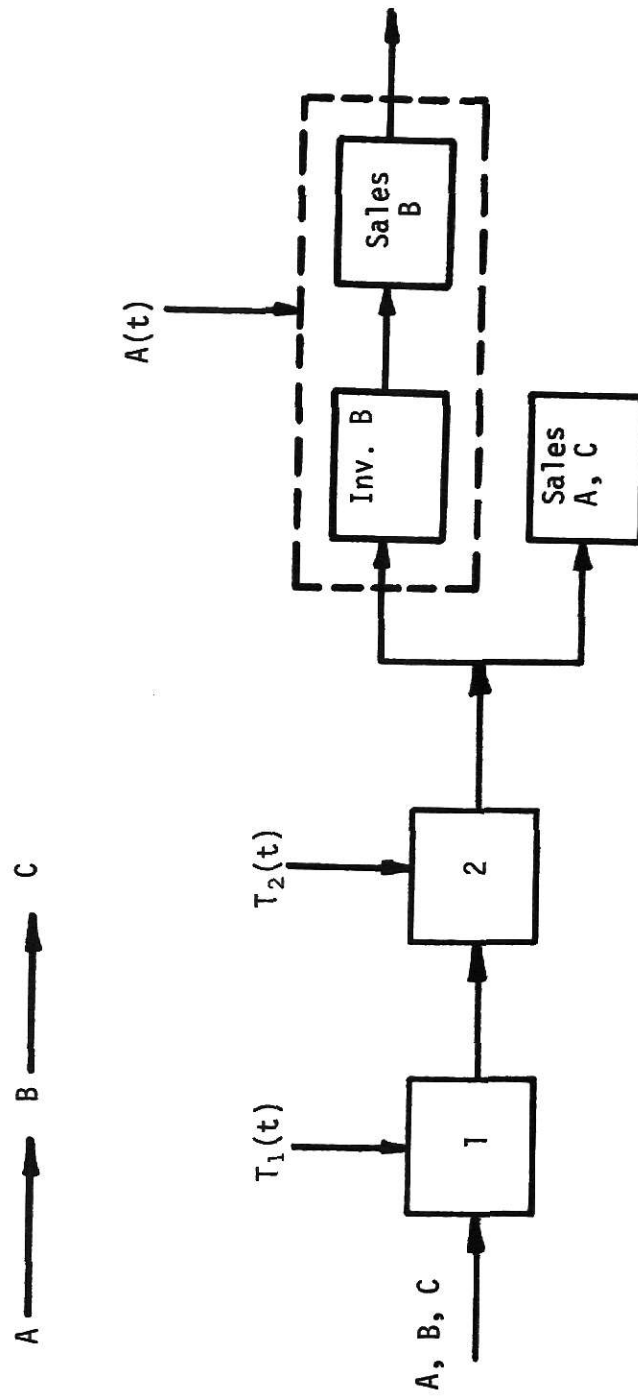


Fig. 17 Schematic presentation of production scheduling problem.

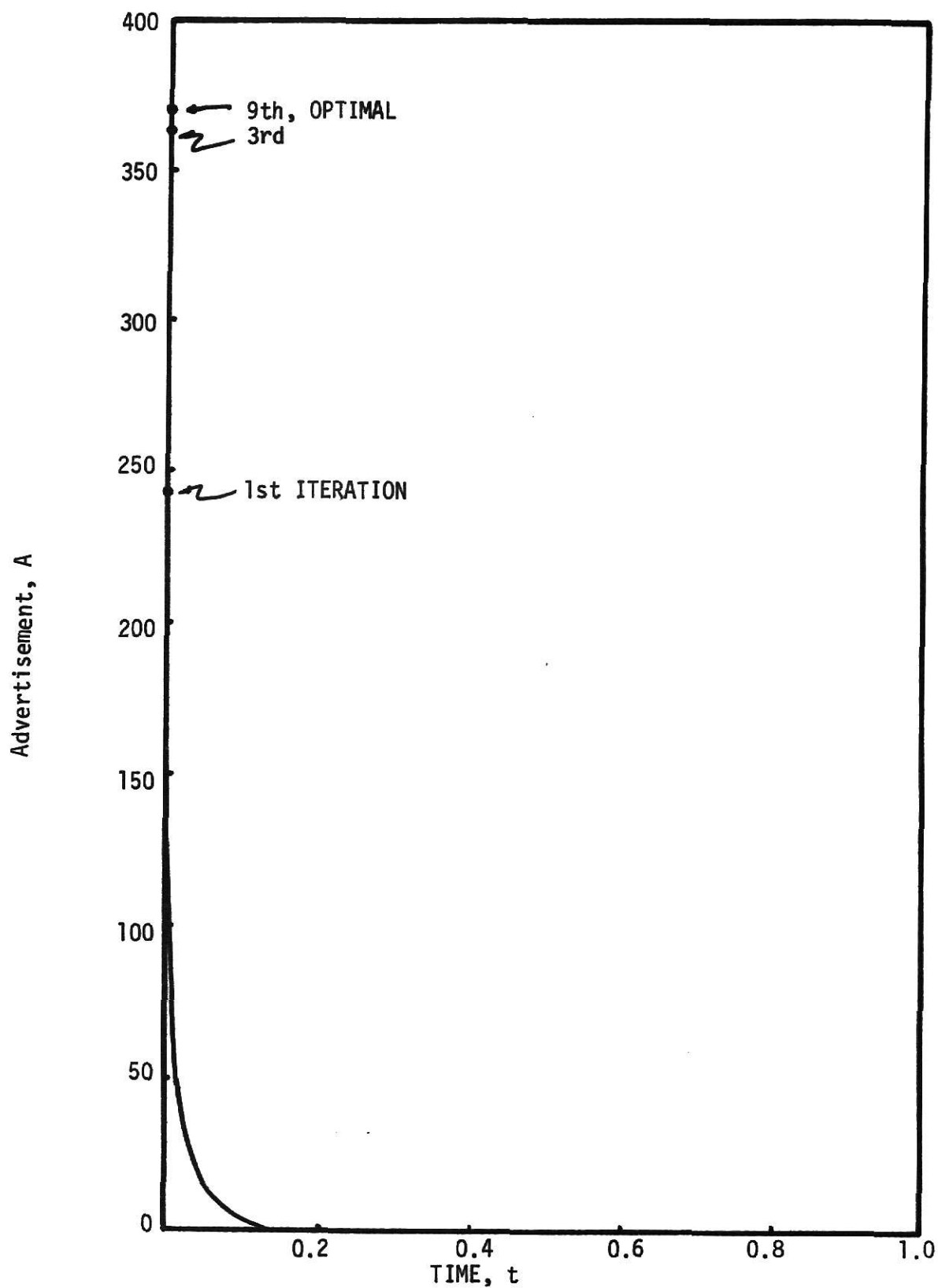


Fig. 18 Convergence rate of Advertisement of the production scheduling problem by the method of particular solutions, scheme (A) without step-size control.

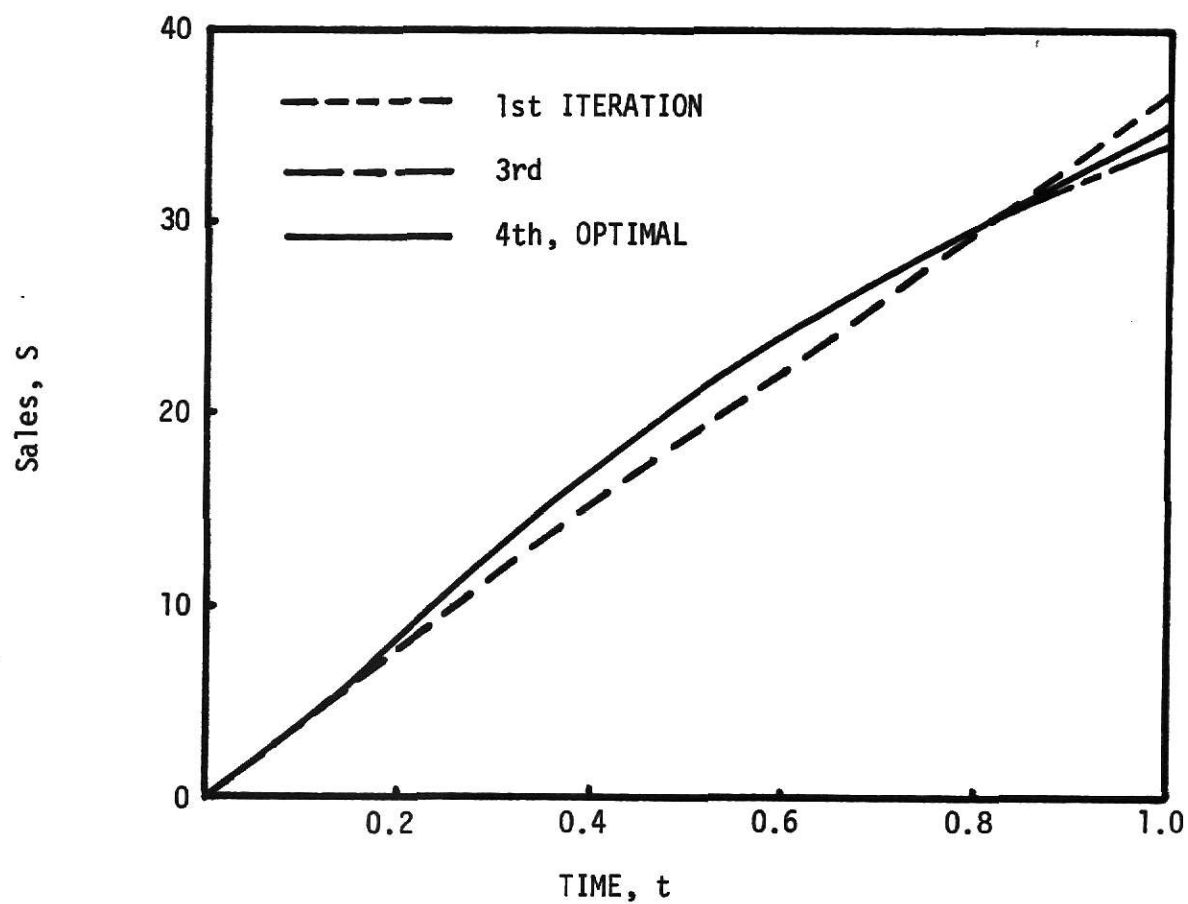


Fig. 19 Convergence rate of sales of the production scheduling problem by the method of particular solutions, scheme (A) without stepsize control.

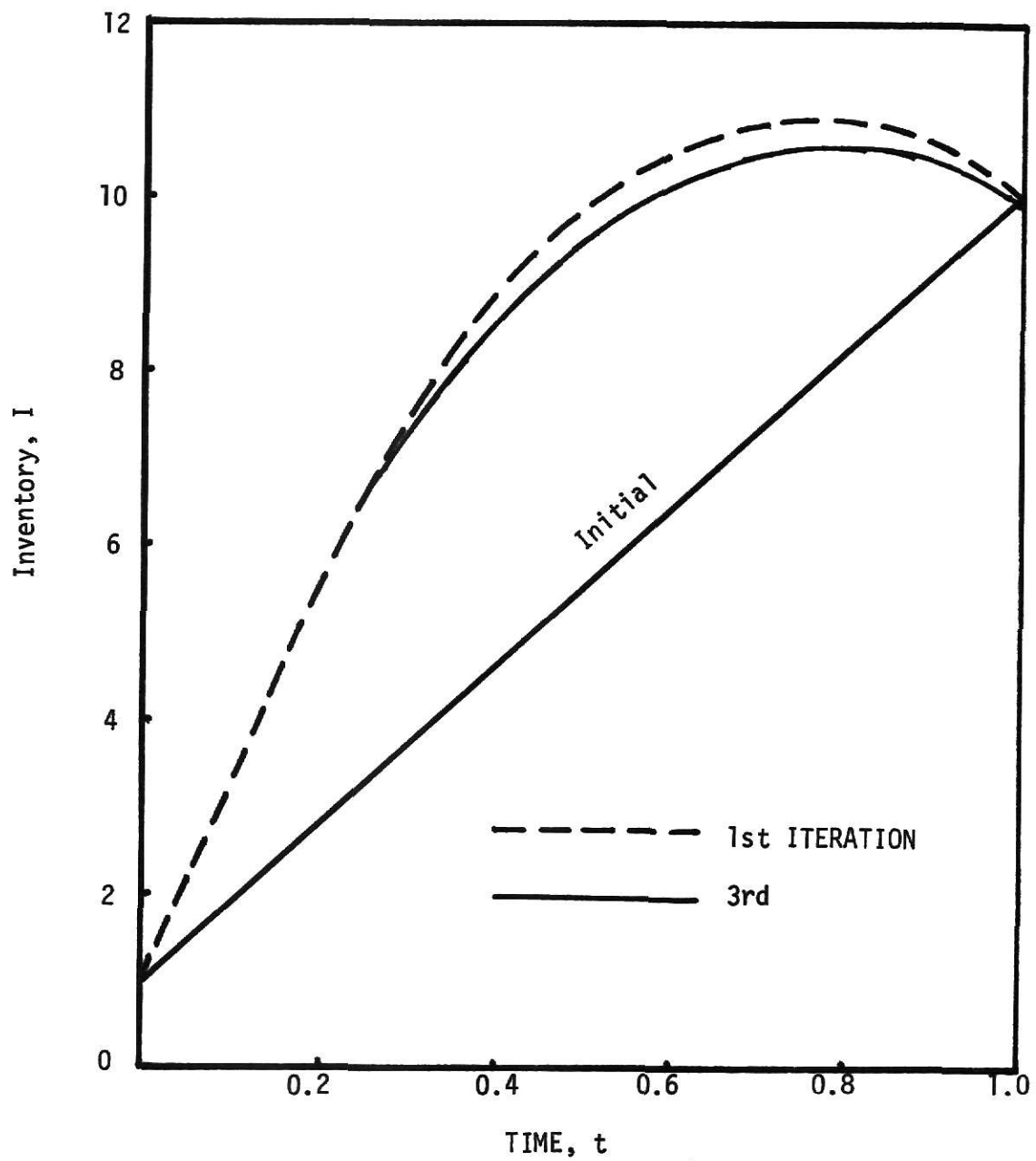


Fig. 20 Convergence rate of Inventory of the production scheduling problem by the method of particular solutions, scheme (A) without stepsize control.

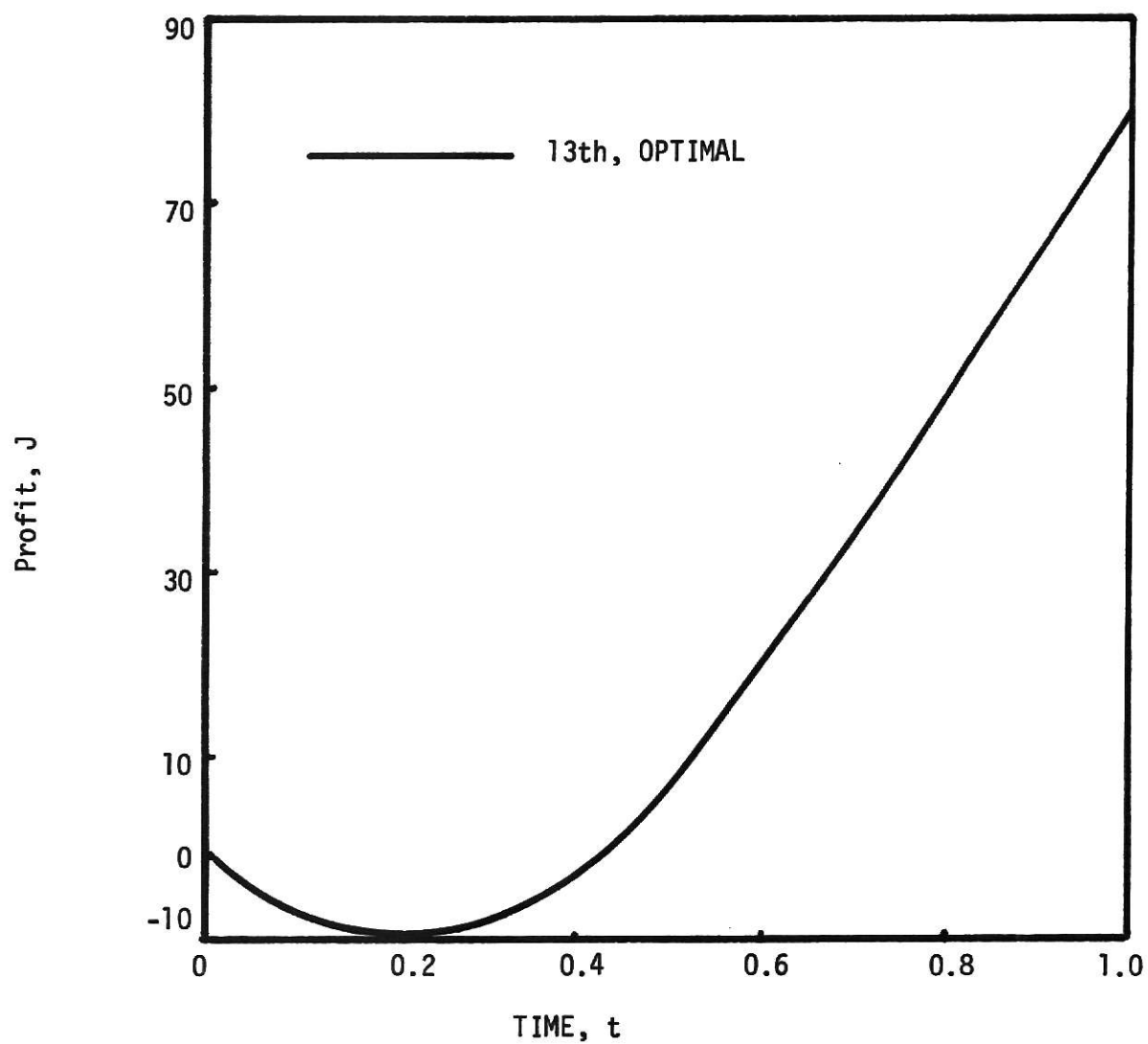


Fig. 21 Convergence rate of profit function of the production scheduling problem, by the method of particular solutions, scheme (A) without step-size control.

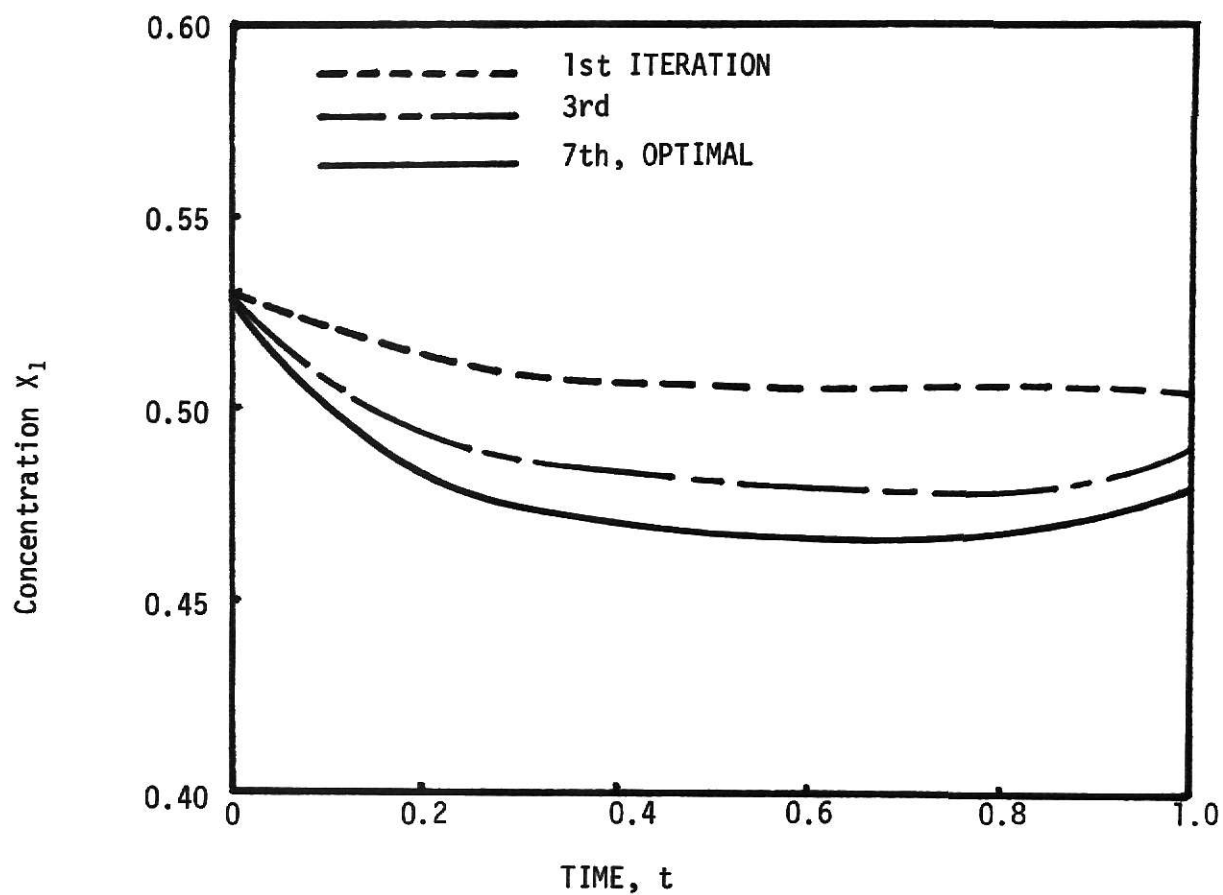


Fig. 22 Convergence rate of X_1 of the production scheduling problem, by the method of particular solutions, scheme (A) without stepsize control.

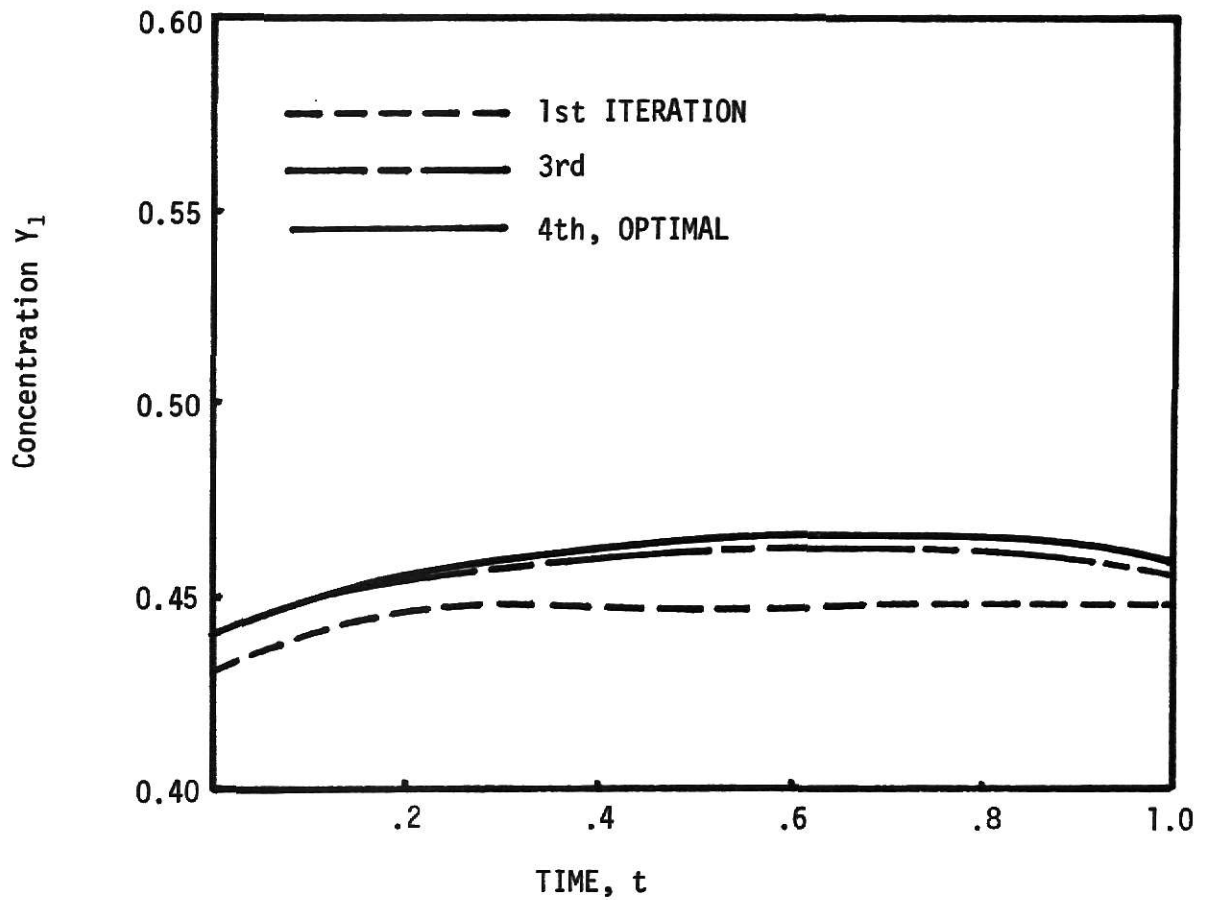


Fig. 23 Convergence rate of Y_1 of the production scheduling problem, by the method of particular solutions, scheme (A) without stepsize control.

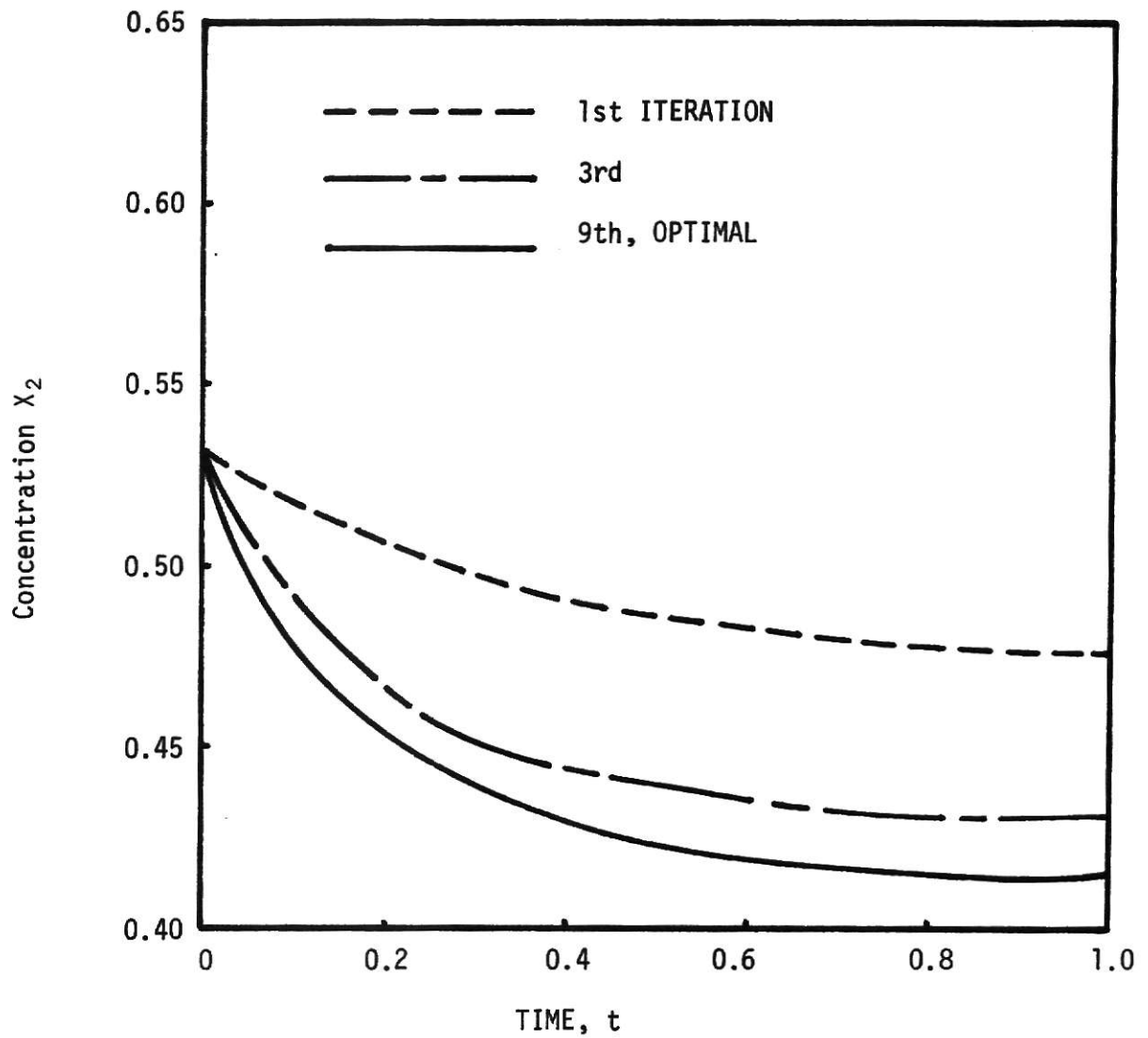


Fig. 24 Convergence rate of X_2 of the production scheduling problem, by the method of particular solutions, scheme (A) without stepsize control.

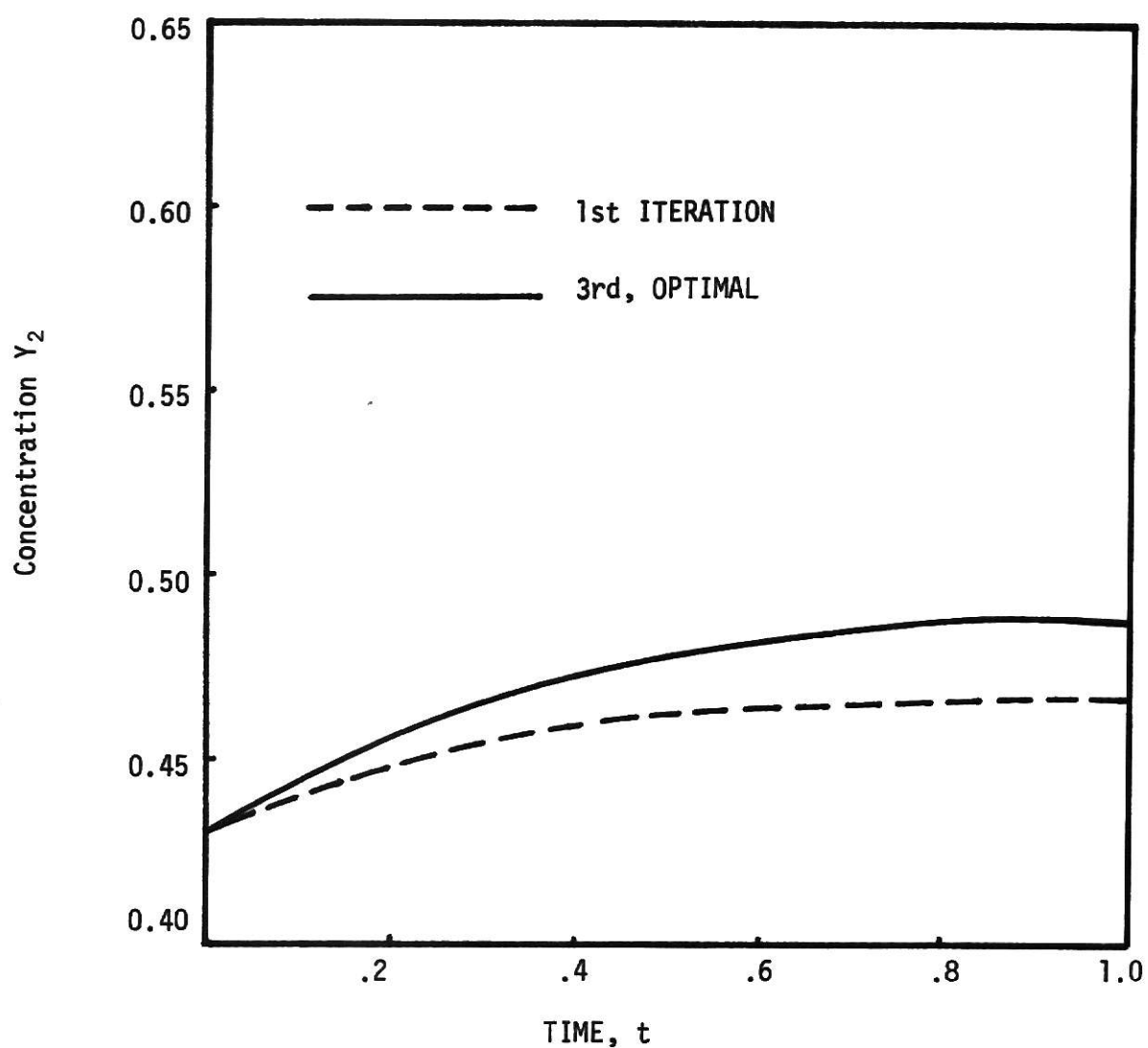


Fig. 25 Convergence rate of Y_2 of the production scheduling problem, by the method of particular solutions, scheme (A) without stepsize control.

Table 8. Convergence rates of the production scheduling problem, by the method of particular solutions without stepsize control.

Iteration	$x_1(1)$	$y_1(1)$	$x_2(1)$	$y_2(1)$	$I(1)$	$S(1)$	$A(o)$	$J(1)$	$T_1(1)$	$T_2(1)$
1	0.5048	0.4498	0.4815	0.4676	10.012	36.370	243.59	78.507	340.01	340.05
2	0.4608	0.4604	0.3759	0.4791	10.000	34.292	356.82	79.003	340.10	339.05
3	0.4864	0.4584	0.4326	0.4869	10.000	34.652	365.77	79.902	339.99	339.97
4	0.4732	0.4606	0.4056	0.4875	10.000	34.770	370.60	80.120	340.02	340.02
5	0.4793	0.4600	0.4194	0.4875	10.000	34.777	369.80	80.183	340.00	340.00
6	0.4758	0.4606	0.4123	0.4875	10.000	34.794	370.55	80.200	340.00	340.00
7	0.4774	0.4604	0.4157	0.4875	10.000	34.790	370.27	80.204	340.00	340.00
8	0.4768	0.4604	0.4141	0.4875	10.000	34.790	370.42	80.203	340.00	340.00
9	0.4776	0.4604	0.4151	0.4875	10.000	34.790	370.35	80.200	340.00	340.00
10	0.4776	0.4604	0.4145	0.4875	10.000	34.790	370.38	80.197	340.00	340.00
11	0.4776	0.4604	0.4145	0.4875	10.000	34.790	370.35	80.196	340.00	340.00
12	0.4776	0.4604	0.4151	0.4875	10.000	34.790	370.36	80.195	340.00	340.00
13	0.4776	0.4604	0.4151	0.4875	10.000	34.790	370.36	80.195	340.00	340.00

Table 9. Comparison of the convergence rates by quasilinearization and the method of particular solutions.

Variable	Number of Iterations*	
	Quasilinearization	Particular Solutions
$x_1(t_f)$	6	10
$y_1(t_f)$	3	8
$x_2(t_f)$	6	13
$y_2(t_f)$	3	5
$I(t_f)$	2	3
$S(t_f)$	3	8
$J(t_f)$	8	13
$T_1(t_f)$	6	6
$T_2(t_f)$	6	6
$A(t_0)$	8	13

* 4 digit accuracy for x_1 , y_1 , x_2 , y_2 .

5 digit accuracy for I , S , J , A , T_1 , T_2 .

CHAPTER 5

CONCLUSION

Two numerical examples have been solved by the method of particular solutions, which were presented in previous chapters. It suggests that the method of particular solutions may be an applicable technique for solving nonlinear two-point boundary value problems. Because the intimate relations between the boundary value problem and optimization control, this technique can also be a useful tool for solving dynamic optimization problems.

There are three main advantages of this method. First, it employs only the original differential systems. Second, it employs stepsize control to speed the rate of convergence. Third, in physical problem, each particular solution represents a possible trajectory even if it satisfies only the initial conditions.

When solving the mathematical models in previous two chapters, the method of particular solutions has a slower convergence rate than the quasilinearization method. Other disadvantage is that the particular solutions method requires more undetermined constants than the quasilinearization method.

Computationally, the method of particular solutions can be proceeded by two algorithms. Scheme (A): the nominal function $X(t)$ is updated according to $\tilde{X}(t) = X(t) + \alpha A(t)$. Scheme (B): the initial value $X(0)$ is updated according to $\bar{X}(0) = X(0) + \alpha A(0)$, then integrate the original system forward to obtain $X(t)$. If the original system is a stable system, scheme (B) has a better convergence rate than scheme (A). This is because the nominal function of scheme (B) was obtained by the integration

of the original system.

The performance index P function measures the cumulative error in the differential system. P function has a descent property and is a function of stepsize α . The optimal value of α which minimizes the P function, can be determined by one dimensional search. Theoretically, when stepsize control was employed, the convergence rate will be improved. In this thesis, bisection method and Fabonacci search technique have been employed to search optimum stepsize. It is believed that if a more sophisticated search technique had been used, a better result can be expected. But, this will be traded by more computer times.

In chapter three, the effectiveness of stepsize control has been tested. For that particular problem, the importance of stepsize control was not significant. This is because the inequality $P(1) < P(0)$ has been satisfied at each iteration (only in the first iteration $\alpha = 0.5$). However, the stepsize control is a guiding tool during the progression of the algorithm.

For illustrative purposes, only two problems have been considered. It is believed that this method, although not as powerful as quasilinearization, still provides an effective approach in solving two-point boundary value problems.

APPENDIX

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**PARM
C
C      THIS PROGRAM IS FOR THE ADVERTISEMENT PROBLEM BY THE PSM SCHEME (A)
C      WITHOUT STEP SIZE CONTROL
C
      DOUBLE PRECISION X(20,101),Y(20,101),Z(20,101),M(20,101),XA(20,101)
      1,YA(20,101),ZA(20,101),MA(20,101),A1(101),A2(101),A3(101),A4(101)
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      66,A517(101),A518(101),A519(101),A520(101),A521(101),A522(101),A523(101),A524(101)
      67,A525(101),A526(101),A527(101),A528(101),A529(101),A530(101),A531(101),A532(101)
      68,A533(101),A534(101),A535(101),A536(101),A537(101),A538(101),A539(101),A540(101)
      69,A541(101),A542(101),A543(101),A544(101),A545(101),A546(101),A547(101),A548(101)
      70,A549(101),A550(101),A551(101),A552(101),A553(101),A554(101),A555(101),A556(101)
      71,A557(101),A558(101),A559(101),A560(101),A561(101),A562(101),A563(101),A564(101)
      72,A565(101),A566(101),A567(101),A568(101),A569(101),A570(101),A571(101),A572(101)
      73,A573(101),A574(101),A575(101),A576(101),A577(101),A578(101),A579(101),A580(101)
      74,A581(101),A582(101),A583(101),A584(101),A585(101),A586(101),A587(101),A588(101)
      75,A589(101),A590(101),A591(101),A592(101),A593(101),A594(101),A595(101),A596(101)
      76,A597(101),A598(101),A599(101),A600(101),A601(101),A602(101),A603(101),A604(101)
      77,A605(101),A606(101),A607(101),A608(101),A609(101),A610(101),A611(101),A612(101)
      78,A613(101),A614(101),A615(101),A616(101),A617(101),A618(101),A619(101),A620(101)
      79,A621(101),A622(101),A623(101),A624(101),A625(101),A626(101),A627(101),A628(101)
      80,A629(101),A630(101),A631(101),A632(101),A633(101),A634(101),A635(101),A636(101)
      81,A637(101),A638(101),A639(101),A640(101),A641(101),A642(101),A643(101),A644(101)
      82,A645(101),A646(101),A647(101),A648(101),A649(101),A650(101),A651(101),A652(101)
      83,A653(101),A654(101),A655(101),A656(101),A657(101),A658(101),A659(101),A660(101)
      84,A661(101),A662(101),A663(101),A664(101),A665(101),A666(101),A667(101),A668(101)
      85,A669(101),A670(101),A671(101),A672(101),A673(101),A674(101),A675(101),A676(101)
      86,A677(101),A678(101),A679(101),A680(101),A681(101),A682(101),A683(101),A684(101)
      87,A685(101),A686(101),A687(101),A688(101),A689(101),A690(101),A691(101),A692(101)
      88,A693(101),A694(101),A695(101),A696(101),A697(101),A698(101),A699(101),A700(101)
      89,A701(101),A702(101),A703(101),A704(101),A705(101),A706(101),A707(101),A708(101)
      90,A709(101),A710(101),A711(101),A712(101),A713(101),A714(101),A715(101),A716(101)
      91,A717(101),A718(101),A719(101),A720(101),A721(101),A722(101),A723(101),A724(101)
      92,A725(101),A726(101),A727(101),A728(101),A729(101),A730(101),A731(101),A732(101)
      93,A733(101),A734(101),A735(101),A736(101),A737(101),A738(101),A739(101),A740(101)
      94,A741(101),A742(101),A743(101),A744(101),A745(101),A746(101),A747(101),A748(101)
      95,A749(101),A750(101),A751(101),A752(101),A753(101),A754(101),A755(101),A756(101)
      96,A757(101),A758(101),A759(101),A760(101),A761(101),A762(101),A763(101),A764(101)
      97,A765(101),A766(101),A767(101),A768(101),A769(101),A770(101),A771(101),A772(101)
      98,A773(101),A774(101),A775(101),A776(101),A777(101),A778(101),A779(101),A780(101)
      99,A781(101),A782(101),A783(101),A784(101),A785(101),A786(101),A787(101),A788(101)
      100,A789(101),A790(101),A791(101),A792(101),A793(101),A794(101),A795(101),A796(101)
      101,A797(101),A798(101),A799(101),A800(101),A801(101),A802(101),A803(101),A804(101)
      102,A805(101),A806(101),A807(101),A808(101),A809(101),A810(101),A811(101),A812(101)
      103,A813(101),A814(101),A815(101),A816(101),A817(101),A818(101),A819(101),A820(101)
      104,A821(101),A822(101),A823(101),A824(101),A825(101),A826(101),A827(101),A828(101)
      105,A829(101),A830(101),A831(101),A832(101),A833(101),A834(101),A835(101),A836(101)
      106,A837(101),A838(101),A839(101),A840(101),A841(101),A842(101),A843(101),A844(101)
      107,A845(101),A846(101),A847(101),A848(101),A849(101),A850(101),A851(101),A852(101)
      108,A853(101),A854(101),A855(101),A856(101),A857(101),A858(101),A859(101),A860(101)
      109,A861(101),A862(101),A863(101),A864(101),A865(101),A866(101),A867(101),A868(101)
      110,A869(101),A870(101),A871(101),A872(101),A873(101),A874(101),A875(101),A876(101)
      111,A877(101),A878(101),A879(101),A880(101),A881(101),A882(101),A883(101),A884(101)
      112,A885(101),A886(101),A887(101),A888(101),A889(101),A890(101),A891(101),A892(101)
      113,A893(101),A894(101),A895(101),A896(101),A897(101),A898(101),A899(101),A900(101)
      114,A901(101),A902(101),A903(101),A904(101),A905(101),A906(101),A907(101),A908(101)
      115,A909(101),A910(101),A911(101),A912(101),A913(101),A914(101),A915(101),A916(101)
      116,A917(101),A918(101),A919(101),A920(101),A921(101),A922(101),A923(101),A924(101)
      117,A925(101),A926(101),A927(101),A928(101),A929(101),A930(101),A931(101),A932(101)
      118,A933(101),A934(101),A935(101),A936(101),A937(101),A938(101),A939(101),A940(101)
      119,A941(101),A942(101),A943(101),A944(101),A945(101),A946(101),A947(101),A948(101)
      120,A949(101),A950(101),A951(101),A952(101),A953(101),A954(101),A955(101),A956(101)
      121,A957(101),A958(101),A959(101),A960(101),A961(101),A962(101),A963(101),A964(101)
      122,A965(101),A966(101),A967(101),A968(101),A969(101),A970(101),A971(101),A972(101)
      123,A973(101),A974(101),A975(101),A976(101),A977(101),A978(101),A979(101),A980(101)
      124,A981(101),A982(101),A983(101),A984(101),A985(101),A986(101),A987(101),A988(101)
      125,A989(101),A990(101),A991(101),A992(101),A993(101),A994(101),A995(101),A996(101)
      126,A997(101),A998(101),A999(101),A1000(101),A1001(101),A1002(101),A1003(101),A1004(101)
      127,A1005(101),A1006(101),A1007(101),A1008(101),A1009(101),A1010(101),A1011(101),A1012(101)
      128,A1013(101),A1014(101),A1015(101),A1016(101),A1017(101),A1018(101),A1019(101),A1020(101)
      129,A1021(101),A1022(101),A1023(101),A1024(101),A1025(101),A1026(101),A1027(101),A1028(101)
      130,A1029(101),A1030(101),A1031(101),A1032(101),A1033(101),A1034(101),A1035(101),A1036(101)
      131,A1037(101),A1038(101),A1039(101),A1040(101),A1041(101),A1042(101),A1043(101),A1044(101)
      132,A1045(101),A1046(101),A1047(101),A1048(101),A1049(101),A1050(101),A1051(101),A1052(101)
      133,A1053(101),A1054(101),A1055(101),A1056(101),A1057(101),A1058(101),A1059(101),A1060(101)
      134,A1061(101),A1062(101),A1063(101),A1064(101),A1065(101),A1066(101),A1067(101),A1068(101)
      135,A1069(101),A1070(101),A1071(101),A1072(101),A1073(101),A1074(101),A1075(101),A1076(101)
      136,A1077(101),A1078(101),A1079(101),A1080(101),A1081(101),A1082(101),A1083(101),A1084(101)
      137,A1085(101),A1086(101),A1087(101),A1088(101),A1089(101),A1090(101),A1091(101),A1092(101)
      138,A1093(101),A1094(101),A1095(101),A1096(101),A1097(101),A1098(101),A1099(101),A1100(101)
      139,A1101(101),A1102(101),A1103(101),A1104(101),A1105(101),A1106(101),A1107(101),A1108(101)
      140,A1109(101),A1110(101),A1111(101),A1112(101),A1113(101),A1114(101),A1115(101),A1116(101)
      141,A1117(101),A1118(101),A1119(101),A1120(101),A1121(101),A1122(101),A1123(101),A1124(101)
      142,A1125(101),A1126(101),A1127(101),A1128(101),A1129(101),A1130(101),A1131(101),A1132(101)
      143,A1133(101),A1134(101),A1135(101),A1136(101),A1137(101),A1138(101),A1139(101),A1140(101)
      144,A1141(101),A1142(101),A1143(101),A1144(101),A1145(101),A1146(101),A1147(101),A1148(101)
      145,A1149(101),A1150(101),A1151(101),A1152(101),A1153(101),A1154(101),A1155(101),A1156(101)
      146,A1157(101),A1158(101),A1159(101),A1160(101),A1161(101),A1162(101),A1163(101),A1164(101)
      147,A1165(101),A1166(101),A1167(101),A1168(101),A1169(101),A1170(101),A1171(101),A1172(101)
      148,A1173(101),A1174(101),A1175(101),A1176(101),A1177(101),A1178(101),A1179(101),A1180(101)
      149,A1181(101),A1182(101),A1183(101),A1184(101),A1185(101),A1186(101),A1187(101),A1188(101)
      150,A1189(101),A1190(101),A1191(101),A1192(101),A1193(101),A1194(101),A1195(101),A1196(101)
      151,A1197(101),A1198(101),A1199(101),A1200(101),A1201(101),A1202(101),A1203(101),A1204(101)
      152,A1205(101),A1206(101),A1207(101),A1208(101),A1209(101),A1210(101),A1211(101),A1212(101)
      153,A1213(101),A1214(101),A1215(101),A1216(101),A1217(101),A1218(101),A1219(101),A1220(101)
      154,A1221(101),A1222(101),A1223(101),A1224(101),A1225(101),A1226(101),A1227(101),A1228(101)
      155,A1229(101),A1230(101),A1231(101),A1232(101),A1233(101),A1234(101),A1235(101),A1236(101)
      156,A1237(101),A1238(101),A1239(101),A1240(101),A1241(101),A1242(101),A1243(101),A1244(101)
      157,A1245(101),A1246(101),A1247(101),A1248(101),A
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```

A1P2(I)=A1(I)
A2P2(I)=A2(I)
A3P3(I)=A3(I)
A4P2(I)=A4(I)
30 CONTINUE
C
C      SOLVE THE THIRD SET PARTICULAR SOLUTIONS
C
      A1(I)=0.
      A2(I)=0.
      A3(I)=0.
      A4(I)=0.
      CALL RAUCE
      CC=0
      I=1,101
      A1P3(I)=A1(I)
      A2P3(I)=A2(I)
      A3P3(I)=A3(I)
      A4P3(I)=A4(I)
40 CONTINUE
C
C      PRINT OUT THREE SETS OF PARTICULAR SOLUTIONS
C
      N=J-1
      WRITE(6,25) N
      WRITE(6,26)
      WRITE(6,27)
      WRITE(6,28)
      WRITE(6,29)
      WRITE(6,30)
      WRITE(6,31)
      WRITE(6,32)
      WRITE(6,33)
      WRITE(6,34)
      WRITE(6,35)
      WRITE(6,36)
      WRITE(6,37)
      WRITE(6,38)
      WRITE(6,39)
      WRITE(6,40)
      WRITE(6,41)
      WRITE(6,42)
      DO 16 I=1,101,5
16 WRITE(6,18) I,A1P1(I),A2P1(I),A3P1(I),A4P1(I),A1P2(I),A2P2(I),A3P2
      1(I),A4P2(I)
      WRITE(6,13)
      WRITE(6,14)
      DO 17 I=1,101,5
17 WRITE(6,9) I,A1P3(I),A2P3(I),A3P3(I),A4P3(I)
35 FORMAT(/,' ITERATION NUMBER ',I4)
36 FORMAT(130(' '))
34 FORMAT(/,19X,' INITIAL VALUE FOR SET 1 ',35X,' INITIAL VALUE FOR S
      ET 2 ')
33 FORMAT(74X,' A1P1(0)=0. ',63X,' A1P2(0)=0. ')
32 FORMAT(74X,' A2P1(0)=0. ',63X,' A2P2(0)=0. ')
37 FORMAT(74X,' A3P1(0)=1. ',63X,' A3P2(0)=0. ')
38 FORMAT(74X,' A4P1(0)=0. ',63X,' A4P2(0)=1. ')
11 FORMAT(/,21X,' PARTICULAR SOLUTION SET 1 ',32X,' PARTICULAR SOLUTI
      ON SET 2 ')
12 FORMAT(/,3X,'T=',5X,'A1P1(T) ',7X,' A2P1(T) ',6X,' A3P1(T) ',6X,' A
      4P1(T) ',6X,' A1P2(T) ',6X,' A2P2(T) ',6X,' A3P2(T) ',6X,' A4P2(T)
      ',)
13 FORMAT(/,21X,' PARTICULAR SOLUTION SET 3 ')
14 FORMAT(/,3X,'T=',5X,' A1P3(T) ',7X,' A2P3(T) ',5X,' A3P3(T) ',6X,'
      A4P3(T) ',)
18 FORMAT(1X,14,F15.5)
19 FORMAT(1X,14,F15.5)
C
C      SOLVE THE CONSTANTS K1,K2,K3
C

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      A1P2(I)=A1(I)
      A2P2(I)=A2(I)
      A3P2(I)=A3(I)
      A4P2(I)=A4(I)
70 CONTINUE
C
C      SOLVE THE THIRD SET PARTICULAR SOLUTIONS
C
      A1(I)=0.
      A2(I)=0.
      A3(I)=0.
      A4(I)=0.
      CALL RANGE
      DO 40 I=1,101
      A1P3(I)=A1(I)
      A2P3(I)=A2(I)
      A3P3(I)=A3(I)
      A4P3(I)=A4(I)
40 CONTINUE
C
C      PRINT OUT THREE SETS OF PARTICULAR SOLUTIONS
C
      N=J-1
      WRITE(16,25) N
      WRITE(16,26)
      WRITE(16,24)
      WRITE(16,23)
      WRITE(16,22)
      WRITE(16,21)
      WRITE(16,20)
      WRITE(16,19)
      WRITE(16,18)
      DO 16 I=1,101,5
16 WRITE(16,18) I,A1P1(I),A2P1(I),A3P1(I),A4P1(I),A1P2(I),A2P2(I),A3P2
      1(I),A4P2(I)
      WRITE(16,13)
      WRITE(16,14)
      DO 17 I=1,101,5
17 WRITE(16,13) I,A1P3(I),A2P3(I),A3P3(I),A4P3(I)
35 FORMAT(1,' ITERATION NUMBER ',I4)
36 FORMAT(130(' '))
37 FORMAT(19X,' INITIAL VALUE FOR SET 1 ',35X,' INITIAL VALUE FOR S
      1 ',2,' )
38 FORMAT( 74X,' A1P1(I)=0. ',63X,' A1P2(I)=0. ',
39 FORMAT( 74X,' A2P1(I)=0. ',63X,' A2P2(I)=0. ',
40 FORMAT( 74X,' A3P1(I)=0. ',63X,' A3P2(I)=0. ',
41 FORMAT( 74X,' A4P1(I)=0. ',63X,' A4P2(I)=0. ',
42 FORMAT(19X,' PARTICULAR SOLUTION SET 1 ',32X,' PARTICULAR SOLUTI
      ON SET 1 ')
43 FORMAT(19X,' A1P1(I)=0. ',63X,' A1P2(I)=0. ',63X,' A3P1(I)=0. ',63X,' A
      3P2(I)=0. ',63X,' A2P1(I)=0. ',63X,' A2P2(I)=0. ',63X,' A4P2(I)
      1 ')
44 FORMAT(19X,' PARTICULAR SOLUTION SET 3 ')
45 FORMAT(19X,' A1P3(I)=0. ',63X,' A1P2(I)=0. ',63X,' A3P3(I)=0. ',63X,' A
      3P3(I)=0. ',63X,' A2P3(I)=0. ',63X,' A2P3(I)=0. ',63X,' A4P3(I)=0. ',63X,'
      14,4(15.5)
46 FORMAT(14,4(15.5)
      SOLVE THE CONSTANTS K1,K2,K3

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      -A3P2(I,I)-A3P1(I,I)
      F=A3P3(I,I)-A3P1(I,I)
      G=A3P2(I,I)-A4P1(I,I)
      H=A3P3(I,I)-A4P1(I,I)
      K2=(F+A2P1(I,I)-G+A3P1(I,I))/(F+G-E+H)
      K2=(-F+K3-A3P1(I,I))/E
      K1=1.-K2-DK3
      WRITE(6,*) DK1,DK2,DK3
49  FORMAT(1H,*, 'K1= ',E15.5,2X,*, 'K2= ',E15.5,2X,*, 'K3= ',E15.5,/)
C
      SOLVE FOR AA1,AA2,AA3,AA4
      DO 50 I=1,101
        AA1(I)=CK1*ALP1(I)+CK2*ALP2(I)+CK3*ALP3(I)
        AA2(I)=CK1*ALP1(I)+CK2*ALP2(I)+CK3*ALP3(I)
        AA3(I)=DK1*A3P1(I)+DK2*A3P2(I)+DK3*A3P3(I)
        AA4(I)=DK1*A4P1(I)+DK2*A4P2(I)+DK3*A4P3(I)
50  PRINT OUT AA1(I),AA2(I),AA3(I),AA4(I)
C
      WRITE(6,*)
52  FORMAT(1H,*, 'T=5X,*, AA1(T), '9X,*, AA2(T), '8X,*, AA3(T), '8X,*, AA4
      1(T), *)
      DO 60 I=1,101,5
60  WRITE(6,*) I,AA1(I),AA2(I),AA3(I),AA4(I)
55  FORMAT(1H,14,2X,6E15.5)
      ALPH=1.
180  DO 70 I=1,101
      X(I,I)=X(J-1,I)+ALPH*AA1(I)
      Y(I,I)=Y(J-1,I)+ALPH*AA2(I)
      Z(I,I)=Z(J-1,I)+ALPH*AA3(I)
70  W(I,I)=W(J-1,I)+ALPH*AA4(I)
C
      CALCULATION OF CONTROL VARIABLE AND PROFIT
177  PIR=0.
      PIR=0.
      PIR=0.
      DC 200 I=1,101
      ACVT(I)=(W(I,I)/(2.*CU))*(Y(I,I)/P-1.)
      P2(I)=PIR+DT*(CS*Y(I,I))
      PIR=P2(I)
      PR2(I)=PIR+DT*(CI*((IIP-X(I,I))+2))
      P2R=P2(I)
      PR3(I)=PIR+DT*(CL*Y(I,I)*(ACVT(I)+2))
      PR3=P3(I)
      PRFT(I)=PR2(I)-PR3(I)
200  CONTINUE
C
      PRINT OUT X(I),Y(I),Z(I),W(I),ACVT(I),PRFT(I)
      WRITE(6,*)
53  FORMAT(1H,*, 'T=7X,*, INVENTORY', 6X,*, PRODUCTION', 4X,*, ADVERTISEMENT',
      15X,*, PROFIT', 7X,*, MULTIPLIER 1', 5X,*, MULTIPLIER 2',)
      DO 75 I=1,101
75  WRITE(6,*) I, X(I,I), Y(I,I), Z(I,I), W(I,I), ACVT(I), PRFT(I), Z(I,I), W(I,I)
C
      ASSIGN THE NEW CAPITAL FUNCTION
      DO 80 I=1,101
      X(I,I)=X(I,I)
      Y(I,I)=Y(I,I)
      Z(I,I)=Z(I,I)
      W(I,I)=W(I,I)
80  W(I,I)=W(I,I)
C
      START NEXT ITERATION
500  CONTINUE
      STOP
      END

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$IPARV
SUBMULTIPLY RANGE
DOUBLE PRECISION X(20,101),Y(20,101),Z(20,101),W(20,101),A1(101),A
12(101),A(101),A4(101),A5(101),E2(101),E3(101),E4(101),F1(101),F2
1(101),F4(101),F4(101),G1(101),G2(101),G3(101),G4(101),H1(101),H2(1
101),H3(101),H4(101)
COMMON A,B,C,CS,CL,C1,A1M,P,DT,J,X,Y,Z,W,A1,A2,A3,A4
DC 100 I=1,100
YA=1-1
T=DT*TA
U=X(J-1,I)
V=Y(J-1,I)
Q=Z(J-1,I)
S=W(J-1,I)
CALL C1F(X,C1CT,I,J-1)
CALL D1F(Y,D1CT,I,J-1)
CALL D1F(Z,D11DT,I,J-1)
CALL C1F(W,D12DT,I,J-1)
C1=(C*V)-C*(V**2)/P*(V**2)*S/(CU*P)-(V*S)/(2*CU)-(V**3)*S/(2*CU*
1(P**2))
C2=C-(2*CU*V)/P*(2*CU*V)/(CU*P)-S/2*CU-(3*CU*(V**2))/(2*CU*(P**2
1))
C3=(V**2)/(CU*P)-V/(2*CU)-(V**3)/(2*CU*(P**2))
C4=-2*CT
C5=12*CT*(A1M-U)
C6=A*V*(S**2)/(2*CU*(P**2))-(S**2)/(CU*P)*(2*CU*S)/P
C7=-C*(3*CU*(V**2)*S)/(2*CU*(P**2))-(2*CU*V)/(CU*P)*S/(2*CU*(2*CU
1*V)/P
C8=J*CS-(C*S)+3*(S**2)*(V**2)/(4*CU*(P**2))-(S**2)*V/(CU*P)+(S**
12)/(4*CU)+(2*CU*S*V)/P
E1(I)=(-2(I)-C1CT+A*(B*V))-V)*DT
F1(I)=(C*-A2(I)+C3*A4(I)-CSCT+C1)*DT
G1(I)=(C4*A1(I)-CL1CT+C5)*DT
H2(I)=(C7*A2(I)+A3(I)+C7*A4(I)-DL2DT+C8)*DT
Z2(I)=(-A2(I)+F1(I)/2)-C1CT+A*(B*V*DT/2))-V)*DT
F2(I)=(C7*(A2(I)+F1(I)/2)+C3*(A4(I)+F1(I)/2))-CSCT+C1)*DT
G2(I)=(C4*(A1(I)+E1(I)/2))-DL1CT+C5)*DT
H2(I)=(C4*(A2(I)+F1(I)/2)+(A3(I)+G1(I)/2)+C7*(A4(I)+H1(I)/2))-DL
1CT+C8)*DT
F3(I)=(-A2(I)+F2(I)/2)-C1CT+A*(B*V*DT/2))-V)*DT
F4(I)=(-2*(A2(I)+F2(I)/2)+C3*(A4(I)+H2(I)/2))-CSCT+C1)*DT
S3(I)=(C4*(A1(I)+E2(I)/2))-DL1CT+C5)*DT
H3(I)=(C4*(A2(I)+F2(I)/2)+(A3(I)+G2(I)/2)+C7*(A4(I)+H2(I)/2))-CL
2DT+C8)*DT
Z4(I)=(-A2(I)+F3(I))-C1CT+A*(B*V*DT))-V)*DT
F4(I)=(C7*(A2(I)+F3(I))+C3*(A4(I)+H3(I))-CSCT+C1)*DT
G4(I)=(C4*(A1(I)+E3(I))-DL1CT+C5)*DT
H4(I)=(C4*(A2(I)+F3(I)+(A3(I)+G3(I))+C7*(A4(I)+H3(I))-DL2DT+C8)*D
1Y
A1(I)=A1(I)+(E1(I)+2*E2(I)+2*E3(I)+E4(I))/6*
A2(I)=A2(I)+(F1(I)+2*F2(I)+2*F3(I)+F4(I))/6*
A3(I)=A3(I)+(G1(I)+2*G2(I)+2*G3(I)+G4(I))/6*
A4(I)=A4(I)+(H1(I)+2*H2(I)+2*H3(I)+H4(I))/6*
200 CONTINUE
RETURN
END

```

```

11PARM
C
C
C
      THIS SUBROUTINE USING STIRLING'S FORMULA FOR NUMERICAL DIFFERENTIATION

      SUBROUTINE DIFFIV,DVDTA,I,J)
      DIMENSION V(20,10)
      DT=0.01
      IF(I.EQ.1) GO TO 2
      IF(I.EQ.2) GO TO 1
      IF(I.EQ.100) GO TO 1
      IF(I.EQ.101) GO TO 1
      DV1=V(J,I-1)-V(J,I-2)
      1 DV2=V(J,I)-V(J,I-1)
      IF(I.EQ.101) GO TO 5
      2 DV3=V(J,I+1)-V(J,I)
      IF(I.EQ.1) GO TO 3
      IF(I.EQ.2) GO TO 4
      IF(I.EQ.100) GO TO 4
      DV4=V(J,I+2)-V(J,I+1)
      DV12=CV2-CV1
      DV32=CV3-CV2
      DV43=CV4-CV3
      DV321=CV32-CV12
      DV433=CV43-CV32
      IF(I.EQ.1) GO TO 7
      IF(I.EQ.59) GO TO 7
      5 DVCTA=DV7/DT
      GO TO 6
      3 DVCTA=DV7/DT
      GO TO 6
      4 DVCTA=((1./DT)*((DV2+DV3)/2.))
      GO TO 6
      7 DVCTA=((1./DT)*((DV2+DV3)/2.--(DV321+DV433)/12.))
      6 RETURN
      END

```



```

C C C
SOLVE THE SECOND SET PARTICULAR SOLUTIONS

A1(1)=0.
A2(1)=0.
A3(1)=0.
A4(1)=1.
CALL RANGE
DC 3C I=1,101
A1P2(1)=A1(1)
A2P2(1)=A2(1)
A3P2(1)=A3(1)
A4P2(1)=A4(1)
30 CONTINUE

C C C
SOLVE THE THIRD SET PARTICULAR SOLUTIONS

A1(1)=0.
A2(1)=0.
A3(1)=0.
A4(1)=0.
CALL RANGE
DC 4C I=1,101
A1P3(1)=A1(1)
A2P3(1)=A2(1)
A3P3(1)=A3(1)
A4P3(1)=A4(1)
40 CONTINUE
WRITE(6,74)
WRITE(6,75)
WRITE(6,76)
WRITE(6,77)
WRITE(6,78)
WRITE(6,79)
WRITE(6,80)
C 16 I=1,101,5
16 WRITE(6,81) A1P1(1),A2P1(1),A3P1(1),A4P1(1),A1P2(1),A2P2(1),A3P2(1),A4P2(1)
1(1),A4P3(1)
WRITE(6,83)
WRITE(6,84)
C 17 I=1,101,20
17 WRITE(6,85) A1P3(1),A2P3(1),A3P3(1),A4P3(1)
15 FORMAT(' ', ITERATION NUMBER ',14)
36 FORMAT('3C(***))
34 FORMAT(/,19X,' INITIAL VALUE FOR SET 1 ',35X,' INITIAL VALUE FOR SET 1FT 2 ')
33 FORMAT('74X,' A1(1)=0.,',63X,' A1P2(1)=0.,')
32 FORMAT('74X,' A2(1)=0.,',63X,' A2P2(1)=0.,')
37 FORMAT('74X,' A3P1(1)=1.,',63X,' A3P2(1)=0.,')
36 FORMAT('74X,' A4P1(1)=0.,',63X,' A4P2(1)=1.,')
11 FORMAT('121X,' PARTICULAR SOLUTION SET 1 ',32X,' PARTICULAR SOLUTION 10X SET 2 ')
12 FORMAT('121X,' A1P1(1)',7X,' A2P1(1)',6X,' A3P1(1)',6X,' A4P1(1)',6X,' A1P2(1)',6X,' A2P2(1)',6X,' A3P2(1)',6X,' A4P2(1)',6X,' A1P3(1)',6X,' A2P3(1)',6X,' A3P3(1)',6X,' A4P3(1)',6X,' A1P4(1)',6X,' A2P4(1)',6X,' A3P4(1)',6X,' A4P4(1)',6X,' A1P5(1)',6X,' A2P5(1)',6X,' A3P5(1)',6X,' A4P5(1)',6X,' A1P6(1)',6X,' A2P6(1)',6X,' A3P6(1)',6X,' A4P6(1)',6X,' A1P7(1)',6X,' A2P7(1)',6X,' A3P7(1)',6X,' A4P7(1)',6X,' A1P8(1)',6X,' A2P8(1)',6X,' A3P8(1)',6X,' A4P8(1)',6X,' A1P9(1)',6X,' A2P9(1)',6X,' A3P9(1)',6X,' A4P9(1)',6X,' A1P10(1)',6X,' A2P10(1)',6X,' A3P10(1)',6X,' A4P10(1)',6X,' A1P11(1)',6X,' A2P11(1)',6X,' A3P11(1)',6X,' A4P11(1)',6X,' A1P12(1)',6X,' A2P12(1)',6X,' A3P12(1)',6X,' A4P12(1)',6X,' A1P13(1)',6X,' A2P13(1)',6X,' A3P13(1)',6X,' A4P13(1)',6X,' A1P14(1)',6X,' A2P14(1)',6X,' A3P14(1)',6X,' A4P14(1)',6X,' A1P15(1)',6X,' A2P15(1)',6X,' A3P15(1)',6X,' A4P15(1)',6X,' A1P16(1)',6X,' A2P16(1)',6X,' A3P16(1)',6X,' A4P16(1)',6X,' A1P17(1)',6X,' A2P17(1)',6X,' A3P17(1)',6X,' A4P17(1)',6X,' A1P18(1)',6X,' A2P18(1)',6X,' A3P18(1)',6X,' A4P18(1)',6X,' A1P19(1)',6X,' A2P19(1)',6X,' A3P19(1)',6X,' A4P19(1)',6X,' A1P20(1)',6X,' A2P20(1)',6X,' A3P20(1)',6X,' A4P20(1)',6X,' A1P21(1)',6X,' A2P21(1)',6X,' A3P21(1)',6X,' A4P21(1)',6X,' A1P22(1)',6X,' A2P22(1)',6X,' A3P22(1)',6X,' A4P22(1)',6X,' A1P23(1)',6X,' A2P23(1)',6X,' A3P23(1)',6X,' A4P23(1)',6X,' A1P24(1)',6X,' A2P24(1)',6X,' A3P24(1)',6X,' A4P24(1)',6X,' A1P25(1)',6X,' A2P25(1)',6X,' A3P25(1)',6X,' A4P25(1)',6X,' A1P26(1)',6X,' A2P26(1)',6X,' A3P26(1)',6X,' A4P26(1)',6X,' A1P27(1)',6X,' A2P27(1)',6X,' A3P27(1)',6X,' A4P27(1)',6X,' A1P28(1)',6X,' A2P28(1)',6X,' A3P28(1)',6X,' A4P28(1)',6X,' A1P29(1)',6X,' A2P29(1)',6X,' A3P29(1)',6X,' A4P29(1)',6X,' A1P30(1)',6X,' A2P30(1)',6X,' A3P30(1)',6X,' A4P30(1)',6X,' A1P31(1)',6X,' A2P31(1)',6X,' A3P31(1)',6X,' A4P31(1)',6X,' A1P32(1)',6X,' A2P32(1)',6X,' A3P32(1)',6X,' A4P32(1)',6X,' A1P33(1)',6X,' A2P33(1)',6X,' A3P33(1)',6X,' A4P33(1)',6X,' A1P34(1)',6X,' A2P34(1)',6X,' A3P34(1)',6X,' A4P34(1)',6X,' A1P35(1)',6X,' A2P35(1)',6X,' A3P35(1)',6X,' A4P35(1)',6X,' A1P36(1)',6X,' A2P36(1)',6X,' A3P36(1)',6X,' A4P36(1)',6X,' A1P37(1)',6X,' A2P37(1)',6X,' A3P37(1)',6X,' A4P37(1)',6X,' A1P38(1)',6X,' A2P38(1)',6X,' A3P38(1)',6X,' A4P38(1)',6X,' A1P39(1)',6X,' A2P39(1)',6X,' A3P39(1)',6X,' A4P39(1)',6X,' A1P40(1)',6X,' A2P40(1)',6X,' A3P40(1)',6X,' A4P40(1)',6X,' A1P41(1)',6X,' A2P41(1)',6X,' A3P41(1)',6X,' A4P41(1)',6X,' A1P42(1)',6X,' A2P42(1)',6X,' A3P42(1)',6X,' A4P42(1)',6X,' A1P43(1)',6X,' A2P43(1)',6X,' A3P43(1)',6X,' A4P43(1)',6X,' A1P44(1)',6X,' A2P44(1)',6X,' A3P44(1)',6X,' A4P44(1)',6X,' A1P45(1)',6X,' A2P45(1)',6X,' A3P45(1)',6X,' A4P45(1)',6X,' A1P46(1)',6X,' A2P46(1)',6X,' A3P46(1)',6X,' A4P46(1)',6X,' A1P47(1)',6X,' A2P47(1)',6X,' A3P47(1)',6X,' A4P47(1)',6X,' A1P48(1)',6X,' A2P48(1)',6X,' A3P48(1)',6X,' A4P48(1)',6X,' A1P49(1)',6X,' A2P49(1)',6X,' A3P49(1)',6X,' A4P49(1)',6X,' A1P50(1)',6X,' A2P50(1)',6X,' A3P50(1)',6X,' A4P50(1)',6X,' A1P51(1)',6X,' A2P51(1)',6X,' A3P51(1)',6X,' A4P51(1)',6X,' A1P52(1)',6X,' A2P52(1)',6X,' A3P52(1)',6X,' A4P52(1)',6X,' A1P53(1)',6X,' A2P53(1)',6X,' A3P53(1)',6X,' A4P53(1)',6X,' A1P54(1)',6X,' A2P54(1)',6X,' A3P54(1)',6X,' A4P54(1)',6X,' A1P55(1)',6X,' A2P55(1)',6X,' A3P55(1)',6X,' A4P55(1)',6X,' A1P56(1)',6X,' A2P56(1)',6X,' A3P56(1)',6X,' A4P56(1)',6X,' A1P57(1)',6X,' A2P57(1)',6X,' A3P57(1)',6X,' A4P57(1)',6X,' A1P58(1)',6X,' A2P58(1)',6X,' A3P58(1)',6X,' A4P58(1)',6X,' A1P59(1)',6X,' A2P59(1)',6X,' A3P59(1)',6X,' A4P59(1)',6X,' A1P60(1)',6X,' A2P60(1)',6X,' A3P60(1)',6X,' A4P60(1)',6X,' A1P61(1)',6X,' A2P61(1)',6X,' A3P61(1)',6X,' A4P61(1)',6X,' A1P62(1)',6X,' A2P62(1)',6X,' A3P62(1)',6X,' A4P62(1)',6X,' A1P63(1)',6X,' A2P63(1)',6X,' A3P63(1)',6X,' A4P63(1)',6X,' A1P64(1)',6X,' A2P64(1)',6X,' A3P64(1)',6X,' A4P64(1)',6X,' A1P65(1)',6X,' A2P65(1)',6X,' A3P65(1)',6X,' A4P65(1)',6X,' A1P66(1)',6X,' A2P66(1)',6X,' A3P66(1)',6X,' A4P66(1)',6X,' A1P67(1)',6X,' A2P67(1)',6X,' A3P67(1)',6X,' A4P67(1)',6X,' A1P68(1)',6X,' A2P68(1)',6X,' A3P68(1)',6X,' A4P68(1)',6X,' A1P69(1)',6X,' A2P69(1)',6X,' A3P69(1)',6X,' A4P69(1)',6X,' A1P70(1)',6X,' A2P70(1)',6X,' A3P70(1)',6X,' A4P70(1)',6X,' A1P71(1)',6X,' A2P71(1)',6X,' A3P71(1)',6X,' A4P71(1)',6X,' A1P72(1)',6X,' A2P72(1)',6X,' A3P72(1)',6X,' A4P72(1)',6X,' A1P73(1)',6X,' A2P73(1)',6X,' A3P73(1)',6X,' A4P73(1)',6X,' A1P74(1)',6X,' A2P74(1)',6X,' A3P74(1)',6X,' A4P74(1)',6X,' A1P75(1)',6X,' A2P75(1)',6X,' A3P75(1)',6X,' A4P75(1)',6X,' A1P76(1)',6X,' A2P76(1)',6X,' A3P76(1)',6X,' A4P76(1)',6X,' A1P77(1)',6X,' A2P77(1)',6X,' A3P77(1)',6X,' A4P77(1)',6X,' A1P78(1)',6X,' A2P78(1)',6X,' A3P78(1)',6X,' A4P78(1)',6X,' A1P79(1)',6X,' A2P79(1)',6X,' A3P
```

```

C
C
C      SOLVE THE CONSTANTS K1,K2,K3
C
      SPINE=W(J-1,101)*A3P1(101)-Z(J-1,101)*A4P1(101)
      VJFAV=A4P1(101)*A3P2(101)-A4P2(101)*A3P1(101)
      CK2=5*INF/YJCAN
      PK1=(-Z(J-1,101)-CK2*A3P2(101))/A3P1(101)
      WRTT(6,9) CK1,CK2
49  FORMAT(1H, ' K1=',E15.5,2X, ' K2=',E15.5)
C
C      SOLVE FOR AA1(C),AA2(0),AA3(0),AA4(0)
C
      AA1=DK1*A1P1(1)+CK2*A1P2(1)
      AA2=CK1*A2P1(1)+CK2*A2P2(1)
      AA3=DK1*A3P1(1)+CK2*A3P2(1)
      AA4=DK1*A4P1(1)+CK2*A4P2(1)
      WRTT(6,1) AA1,AA2,AA3,AA4
51  FORMAT(1, 'AA1=',E15.5,2X, 'AA2=',E15.5,2X, 'AA3=',E15.5,2X, 'AA4=',E1
15.5)
      JJ=0
      NN=1
      ALPH=C.
180 CONTINUE
      R1(1)=X(J-1,1)+ALPH*AA1
      R2(1)=Y(J-1,1)+ALPH*AA2
      R3(1)=Z(J-1,1)+ALPH*AA3
      R4(1)=W(J-1,1)+ALPH*AA4
      CALL RGKT
      DC 502 1=1,101
      X(J,1)=R1(1)
      Y(J,1)=R2(1)
      Z(J,1)=R3(1)
      W(J,1)=R4(1)
502 CONTINUE
      IF(JJ.EC.1) GOTO 177
      DO 71 I=1,101
      IT=I-1
      TT=CT
      CALL DIFF(X,DICTA,I,J)
      CALL DIFF(Y,DSDTA,I,J)
      CALL DIFF(Z,DL1CTA,I,J)
      CALL DIFF(W,DL2CTA,I,J)
      CCL=C*Y (J,I)-C*Y (J,I**2)/P*(Y (J,I**2)*W (J,I)/(CU*P)-Y (J,I)
18W (J,I)/(2*CU)-Y (J,I**3)*W (J,I)/(2*CU*(P**2))
      CC5=2*CT*(AIM-X (J,I))
      CCR=2 (J,I)*CS-C*W (J,I)*3*W (J,I)/(CU*P)+W (J,I)/(2*CU*(P**2))
      1)=-W (J,I)**2)*Y (J,I)/(CU*P)+W (J,I)/(2*CU*(P**2))
      1 (J,I)/P
      R1(1)=(C*DTA-A-(P*Y (J,I))**2
      R2(1)=(C*DTA-CC1)**2
      R3(1)=(C*DTA-CC5)**2
      R4(1)=(C*DTA-CC8)**2
71 CONTINUE
      CALL SIMP(R1,R2,P2,R4,DT,PIX)
      C5=(Z (J,101)**2)*W (J,101)**2)
      DP1(NN)=PX(101)+C5
      WRTT(6,5) NN,ALPH,DP1(NN),PX(101),C5
45  FORMAT(1, 'NN=',14,3X, 'ALPH=',E15.5,3X, 'DP1(NN)',E15.5,3X, 'PX(1
1,101)',E15.5,3X, 'C5=',E15.5)
      IF(ALPH.EQ.C) GOTO R1

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IF(ALPH=1.) 83,82,177
21 AN=NN+1
   ALPH=1.
   GOTO 18C
23 IF(PP1(N).LE.PP1(1)) GOTO 177
   IF(NN.LT.9) GOTO 84
   IF(PP1(3).LE.PP1(2)) GOTO 998
   JJ=1
   ALPH=C.C625
   GOTO 18C
84 AN=N+1
   ALPH=C.3*ALPH
   GOTO 18C
998 JJ=1
   ALPH=0.3
   GOTO 18C

C      CALCULATION OF CONTROL VARIABLE AND PROFIT
C
C 177 CONTINUE=
   P1R=0.
   P2R=0.
   P3R=0.
   DO 200 I=1,101
     ADVT(I)=(K(J,I)/I2.*CU)*(Y(J,I)/P-1.)
     P1(I)=P1R+DT*(CS*Y(J,I))
     P1R=P1(I)
     P2(I)=P2R+DT*(CI*(I*P-X(J,I)**2))
     P2R=P2(I)
     P3(I)=P3R+DT*(CL*Y(J,I)*(ADVT(I)**2))
     P3R=P3(I)
     PRFT(I)=PR1(I)-PR2(I)-PR3(I)
200 CONTINUE
   WRITE(16,'3')
53 FORMAT(1,3X,'T',7X,'INVENTORY',6X,'PRODUCTION',4X,'ADVERTISEMENT',
   1X,'MULTIPLIER 1',4X,'MULTIPLIER 2')
   DO 75 I=1,101
75 WRITE(16,5) I,X(J,I),Y(J,I),ADVT(I),PRFT(I),Z(J,I),K(J,I)
55 FORMAT(1,13,2X,6E15.5)
500 CONTINUE
501 STOP
502 END

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```

      STOP
      SUBROUTINE RANGE
      DIMENSION E1(101),E2(101),E3(101),E4(101),F1(101),F2(101),F3(101),
      F4(101),G1(101),G2(101),G3(101),G4(101),H1(101),H2(101),H3(101),H4
      (101)
      COMMON X(20,101),Y(20,101),Z(20,101),W(20,101),E1(101),E2(101),E3(
      101),E4(101),A1(101),A2(101),A3(101),A4(101)
      COMMON A, B,C,CS,CU,C1,A1M,P,DT,J
      C=100 I=1,100
      TA=I-1
      T=DT*TA
      U=X(IJ-1,T)
      V=Y(IJ-1,T)
      C=Z(IJ-1,T)
      S=W(IJ-1,T)
      C1=(C*V)-C*(V**2)/P*(V**2)*S/(CU*P)-S/(2.*CU)-(V**3)*S/(2.*CU*
      (1+P**2))
      C2=C-(2.*C*V)/P*(2.*V*S)/(CU*P)-S/(2.*CU)-(3.*S*(V**2))/(2.*CU*(P*
      1+2))
      C3=(V**2)/(CU*P)-V/(2.*CL)-(V**3)/(2.*CU*(P**2))
      C4=-2.*CT
      C5=(2.*CT*(A1M-C))
      C6=3.*V*(S**2)/(2.*CU*(P**2))-(S**2)/(CU*P)+(2.*C*S)/P
      C7=-C+(3.*(V**2)*S)/(2.*CU*(P**2))-(2.*S*V)/(CU*P)+S/(2.*CU)*(2.*C
      1+V)/P
      CA=C*CS-(C*S)+3.*(S**2)*(V**2)/(4.*CU*(P**2))-(S**2)*V/(CU*P)+(S**
      2)/(4.*C1)+(2.*C*S*V)/P
      F1(I)=-A7(I)*CT
      F1(I)=(C7*A2(I)+C3*A4(I))*DT
      G1(I)=C*A1(I)*DT
      H1(I)=(C7*A2(I)+A3(I)+C7*A4(I))*CT
      F2(I)=-((A2(I)+F1(I))/2.)*CT
      F2(I)=(C7*(A2(I)+F1(I))/2.)*C3*(A4(I)+H1(I)/2.))*DT
      G2(I)=C*A1(I)+E1(I)/2.)*DT
      H2(I)=(C7*(A2(I)+F1(I))/2.)*(A3(I)+G1(I)/2.)+C7*(A4(I)+H1(I)/2.))*DT
      1T
      F3(I)=I-((A2(I)+F2(I))/2.)*DT
      F3(I)=(C7*(A2(I)+F2(I))/2.)*C3*(A4(I)+F2(I)/2.))*DT
      G3(I)=C*A1(I)+E2(I)/2.)*DT
      H3(I)=(C7*(A2(I)+F2(I))/2.)*(A3(I)+G2(I)/2.)+C7*(A4(I)+H2(I)/2.))*DT
      1T
      F4(I)=-((A2(I)+F3(I))*CT
      F4(I)=(C7*(A2(I)+F3(I))*C3*(A4(I)+H3(I))*CT
      H4(I)=(C7*(A2(I)+F3(I))*A3(I)+C3(I)+C7*(A4(I)+H3(I))*DT
      A1(I)=I-((I)+(I)+(I)+2.*E2(I)+2.*E3(I)+E4(I))/6.
      A2(I)=I-((I)+(I)+(I)+2.*E2(I)+2.*F3(I)+E4(I))/6.
      A3(I)=I-((I)+(I)+(I)+2.*C2(I)+2.*G3(I)+E4(I))/6.
      A4(I)=I-((I)+(I)+(I)+2.*F2(I)+2.*F3(I)+E4(I))/6.
      100 CONTINUE
      RETURN
      END

```

[illegible]

```

$PARM SURROUTINE SIMP(R1,R2,R3,R4,DT,C)
DIMENSION R1(101),R2(101),R3(101),R4(101),P1(101),P2(101),P3(101),
1 P4(101),Q(101)
Q(1)=C.
DO 80 I=2,100,Z
P1(I)=(DT/3.)*(R1(I-1)+4.*R1(I)+R1(I+1))
P2(I)=(DT/3.)*(R2(I-1)+4.*R2(I)+R2(I+1))
P3(I)=(DT/3.)*(R3(I-1)+4.*R3(I)+R3(I+1))
P4(I)=(DT/3.)*(R4(I-1)+4.*R4(I)+R4(I+1))
Q(I+1)=P1(I)+P2(I)+P3(I)+P4(I)+C(I-1)
80 CONTINUE
RETURN
END

```

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ACKNOWLEDGEMENT

The author wishes to express his deep sense of appreciation to his major professor, Dr. E. S. Lee for his guidance, constructive criticism, helpful suggestions taken in the preparation of this Master's thesis. This work is partially supported by Grant GK-26212X, National Science Foundation.

APPLICATION OF THE METHOD OF PARTICULAR
SOLUTIONS TO INDUSTRIAL MANAGEMENT SYSTEMS

by

SHIH-TSU SUN

B.S.(Ind.Mgt.), Cheng Kung University
Tainan, Taiwan, China, 1966

AN ABSTRACT OF MASTER'S THESIS
submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Industrial Engineering

KANSAS STATE UNIVERSITY

Manhattan, Kansas

1972

The method of particular solutions due to Miele and his coworkers is essentially a revised quasilinearization method. It can be applied to both linear or nonlinear two-point boundary value problems. The method consists of combining linearly several sets of particular solutions of the original differential system. The performance index P , which estimates the cumulative error in the differential system, was employed as a guide during the progression of the algorithm. There are two computational approaches, the main difference of these two approaches lies in the selection of the initial approximations.

The purpose of this work is to investigate the effectiveness of the computational procedures of this recently developed tool in solving two industrial management problems. First, the method was applied to an advertisement problem with two state variables and one control variable. Then, the method was applied to an advertisement and production scheduling problem with six state variables and three control variables.

Advantages of this method are: (1) it employs only the original differential system, (2) it employs stepsize control to speed the convergence rate, and (3) in physical problems, each particular solution represents a possible trajectory even if it satisfies only the initial conditions. The disadvantage is that it requires the determination of more undetermined constants than the method of quasilinearization.

One approach of the particular solutions method is to use the initial approximation $X(t)$, $t_0 \leq t \leq t_f$, which satisfies the given initial conditions. Another approach which gives a better initial approximations can be obtained by integrating the original system with the initial

point $X(t_0)$. For the advertisement problem, it appears that when the first approach was used, 13 iterations were required to obtain 5 digit accuracy. However, only 4 iterations were required for the same degree of accuracy by the quasilinearization method. When the second approach was used, 6 iterations were required for convergence; with the help of stepsize control, the convergence rate was reduced from 6 iterations to 4 iterations. For the advertisement and production scheduling problem, the quasilinearization method also has a better result than the particular solutions method. For this particular problem, the second approach fails when we try to obtain the control variables T_1 and T_2 by the Newton-Raphson method.