APPLICATION OF MODERN CONTROL THEORY TO THE ENVIRONMENTAL CONTROL OF CONFINED SPACES

by

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TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

A life support system is a system for creating, maintaining, and controlling an environment so as to permit personnel to function efficiently. The control of temperature is perhaps the most important role of the life support system. This report contains the results of the investigation on the optimum temperature control of life support systems by means of the modern control theory.

The need for providing an automatic control system to an air-conditioning system has long been recognized [19, 28]. It is also a well known fact that use of automatic control is necessary for the life support system of a space cabin or submarine or underground shelter [29, 30]. It appears that analysis and synthesis of the control systems for the air-conditioning and life support systems have so far been carried out by the classical approach [19, 28, 29, 30]. This approach is essentially a trial and error procedure or a disturbance-response (or input-output) approach. Extensive use is made of the transform methods such as the Laplace transform (s-domain), Fourier transform (ω-domain), and z-transform (discrete time-domain). Even though mathematics is extensively used, the classical approach is essentially an empirical one [27].

In recent years, an approach to the analysis and synthesis of a control system, which is distinctly different from the classical one, has been developed. This modern approach is generally called the modern (optimal) control theory [2, 8, 21, 22, 24, 25, 27]. It is based on the state-space characterization of a system. The state space is the abstract space whose

coordinates are the state properties of the system or the variables which define the characteristics of the system [27]. This approach involves mainly maximization or minimization of an objective function (functional) which is a function of state and control variables which are in turn functions of time and/or distance coordinate. The objective function is specified, constraints are imposed on the state and decision variables, and an optimal control policy is determined by extremizing the objective function by means of mathematical techniques such as the calculus of variations, maximum principle, and dynamic programming [2, 9, 24]. This modern approach is entirely theoretical in the sense that no trial and error procedure is involved in "adjusting the controller".

There are reasons to believe that the classical approach suffices in the analyses and syntheses of the control systems for a majority of air-conditioning and life support systems because usually the requirements are not extremely critical and specifications are not very tight. It is, therefore, justifiable that most of the control and dynamic investigations be based on the classical approach. There is, however, a certain incentive in applying the modern approach to analysis and synthesis of automatic environmental control systems in spacecrafts, submarines, underground civil defense shelters and certain medical facilities. In these systems, very strigent requirements in the response time, control effort, etc are imposed. For example, the control system of a space-craft must have an extremely small response time and furthermore, the amount of energy required for the control must be very small because of the weight limitation imposed on the space-craft.

Modern control theory has been applied to life support systems in [10, 11, 12, 13, 14, 15]. In [11], the mathematical models representing several different systems have been derived. In [12], modern control theory has been applied to minimum time problems, where the heat disturbance is considered to have an impulse form. The resultant control is of bang-bang type. The maximum principle has been advantageously applied to evaluate the number of switching points of the bang-bang control policy via the switching function and adjoint vector. Bellman [3] has proved that the number of switching points is one less than the dimension of the problem. In [13] the optimal control of a system with equality state variable constraints imposed at the end of the control period is considered. The problems of controlling a system with constraints imposed on the state variable have been dealt with in [14]. In [15] some aspects of sensitivity analysis have been discussed.

The system considered in this report consists of a confined space subjected to a heat disturbance and a heat exchanger. In chapter 2, the performance equations which represent the dynamic characteristics of the system for a step function heat disturbance are derived. The model is simulated on a digital computer. In chapter 3, Pontryagin's maximum principle is applied to establish the optimal policy for this system when the objective function is to minimize the sum of integrated deviation of the state of the system from the desired one and the integrated effort required to maintain the system in the desired state over a specified control period. The optimal control in this case is of continuous type. In chapter 4, the maximum principle is applied to obtain optimum control policy when the heat disturbance has an impulse form. The objective

function to be minimized in this case is the integrated effort required to bring the system back from the deviated state to the desired one in the shortest time. The optimal control is again of continuous type.

CHAPTER II

ENVIRONMENTAL CONTROL SYSTEMS - MODELING AND SIMULATION

The basic model of a environmental control system consisting of a confined space subjected to step heat disturbance and of a heat exchanger is developed. The performance equations which represent the dynamic characteristics of the system and system components are derived. The procedure for deriving the performance equations is fairly general and can be extended to cases in which the heat disturbances have forms other than the step function. The model is tested by carrying out a simulation on a digital computer. The performance equations of a system in which the flow of air in the confined space can be characterized by the 2 CST's-in-series model are also derived. This model is also simulated.

2.1 MODELING

A control system usually consists of three elements: the feedback element, the control element, and the system proper [7]. The feedback element in a life support control system or an environmental control system may be composed of a thermostat, humidstat and pressure regulator, or any combination of these, depending on the purpose of control. The control element may include a heat exchanger, humidifier, dehumidifier, blower, portable air-conditioner, or any combination of these, depending on the objective of control. For instance, both the thermostat and heat exchanger are often used to control the air temperature inside a building. The system proper may be a confined space, e.g., an underground shelter, a space vehicle, a space suit, a submarine or a building.

The system considered here is shown schematically in Fig. 1. The confined space may be a typical office located in a multi-story building or the cabin of a space ship. Air or oxygen or a mixture of oxygen and nitrogen is circulated through the room or confined space via an air duct by mechanical means, e.g., a blower or a fan. Control of air temperature in the system is accomplished with a duct system. The thermostat in the system adjusts the position of the control valve of the heat exchanger in order to provide the desired temperature.

The performance equations of the system, which represent the dynamic characteristics of the system and system components are derived.

2.1.1 The System Proper

The following three main assumptions are made concerning the system proper:

i) Room or cabin air is well mixed, i.e., air temperature within the system proper is uniform throughout at any instant in time.

ii) The thermal capacitance of room walls, floors, ceiling, and window is neglected, as well as that of any furniture within the system proper.

iii) Heat loss through the walls and windows is negligible.

The performance equation of the system proper can be obtained by using the heat balance. Thus referring to Fig. 2, we have

$$[\text{heat in}] - [\text{heat out}] = [\text{heat accumulation}] \tag{1}$$

$$[\text{heat in}] = q_{i1} + q_{i2} + q_{ds} \tag{2}$$

$$[\text{heat out}] = q_{01} + q_{02} \tag{3}$$

$$[\text{heat accumulation}] = q_s \tag{4}$$

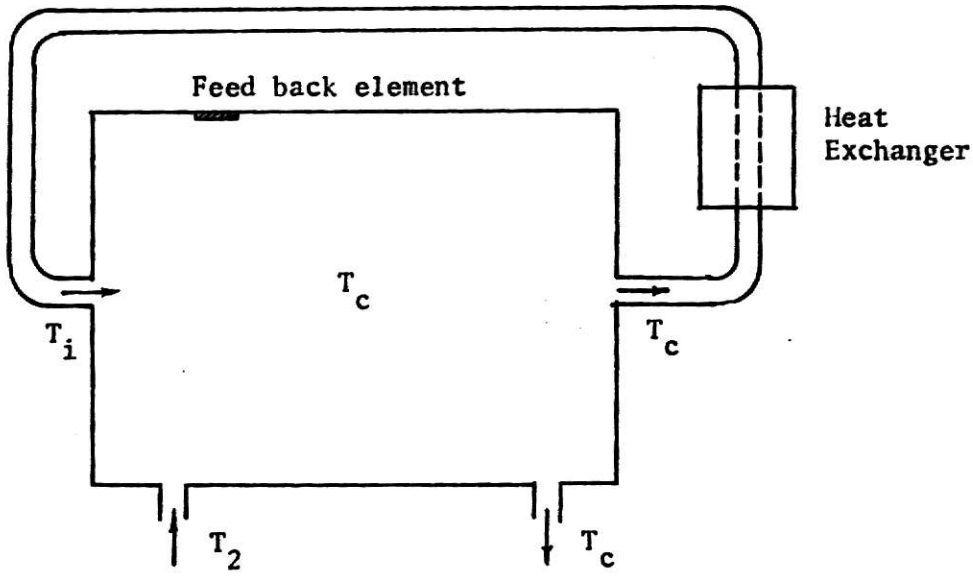


Fig. 1 The system, an air-conditioned room

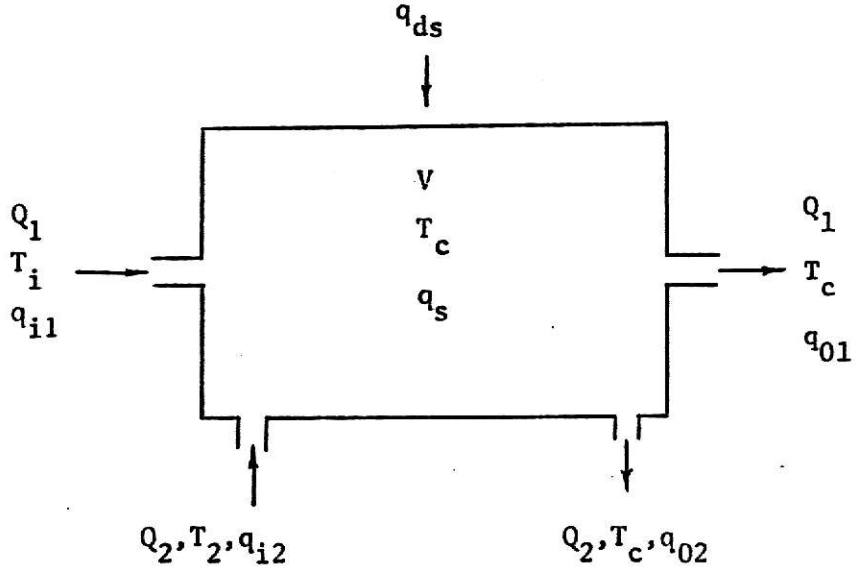


Fig. 2 The system proper: heat flow rates

where

q_{i1} = heat flow into the system proper by circulation air in Kcal/sec

q_{i2} = heat flow into the system proper by fresh air in Kcal/sec

q_{01} = heat flow out of the system proper by circulation air in Kcal/sec

q_{02} = heat flow out of the system proper by fresh air in Kcal/sec

q_{ds} = step heat disturbance rate in Kcal/sec

q_s = rate of heat stored inside the system proper in Kcal/sec

Inserting equations (2), (3) and (4) into equation (1) gives

$$[q_{i1} + q_{i2} + q_{ds}] - [q_{01} + q_{02}] = q_s \tag{5}$$

Assuming perfect mixing, the expressions for q_{i1}, q_{i2}, q_{01}, and q_{02} are

$$q_{i1} = Q_1 \rho C_p (t_i - t_a)$$

$$= Q_1 \rho C_p T_i \tag{6}$$

$$q_{i2} = Q_2 \rho C_p (t_2 - t_a)$$

$$= Q_2 \rho C_p T_2 \tag{7}$$

$$q_{01} = Q_1 \rho C_p (t_c - t_a)$$

$$= Q_1 \rho C_p T_c \tag{8}$$

$$q_{02} = Q_2 \rho C_p (t_c - t_a)$$

$$= Q_2 \rho C_p T_c \tag{9}$$

The heat disturbance is assumed to be a step input form such as

$$q_{ds} = (Q_1 + Q_2) \rho C_p (t_d - t_a) U_0(\alpha)$$

$$= (Q_1 + Q_2) \rho C_p T_d U_0(\alpha) \tag{10}$$

where $U_0(\alpha)$ is the unit step function. The rate at which heat energy is accumulated in the system proper can be expressed as

$$q_s = V_1 \rho C_p \frac{dt_c}{d\alpha}$$

$$= V_1 \rho C_p \frac{dT_c}{d\alpha} \qquad (11)$$

where

C_p = specific heat of air in Kcal/Kg ^oC

Q_1 = air flow rate by circulation in m^3/sec

Q_2 = flow rate of fresh air in m^3/sec

V_1 = room volume in m^3

t_a = arbitrary and suitable reference temperature in ^oC

t_c = room temperature in ^oC

t_d = disturbance temperature in ^oC

t_i = temperature of the circulation air into the system proper in ^oC

t_2 = outside air temperature in ^oC

$T_c = (t_c - t_a)$ in ^oC

$T_i = (t_i - t_a)$ in ^oC

$T_2 = (t_2 - t_a)$ in ^oC

$T_d = (t_d - t_a)$ in ^oC

α = time in sec

ρ = air density in Kg/m^3

The insertion of equations (6) through (11) into equation (5) yields

$$V_1 \rho C_p \frac{dT_c}{d\alpha} + [Q_1 \rho C_p + Q_2 \rho C_p] T_c = Q_1 \rho C_p T_i$$

$$+ Q_2 \rho C_p T_2 + (Q_1 + Q_2) \rho C_p T_d U_0(\alpha)$$

Q_1, Q_2, ρ, C_p, V_1 and T_d are considered to be constants. The rate of heat loss through walls and windows can be included in the q_{ds} term. The above equation can be simplified by dividing both sides of the equation by $(Q_1 + Q_2)\ \rho\ C_p$ as

$$\tau_1 \frac{dT_c}{d\alpha} + T_c = r_1 T_i + r_2 T_2 + T_d\ U_0(\alpha) \tag{12}$$

$$T_c = T_{c0} \text{ at } \alpha = 0$$

Note that $\alpha = 0^-$ is the time just before the heat disturbance occurs and $\alpha = 0^+$ is the time just after the heat disturbance has occured. When the heat disturbance has a form of a step function it is not necessary to distinguish the state variables between time $\alpha = 0^-$ and time $\alpha = 0^+$ because the values of the state are the same. However, in the case of impulse heat disturbance the state variables are changed between $\alpha = 0^-$ and $\alpha = 0^+$ and it is important to specify the time exactly.

Equation (12) can be rewritten in a dimensionless form as

$$\frac{dx_1}{dt} + x_1 = r_1 x_2 + r_2 K_\alpha + \sigma_s U_0(t) \tag{13}$$

$$x_1 = 1 \text{ at } t = 0$$

where

$$r_1 = \frac{Q_1}{Q_1 + Q_2}$$

$$r_2 = \frac{Q_2}{Q_1 + Q_2} = 1 - r_1$$

$$x_1 = \frac{T_c}{T_{c0}} = \frac{t_c - t_a}{t_{c0} - t_a}$$

$$x_2 = \frac{T_i}{T_{c0}} = \frac{t_i - t_a}{t_{c0} - t_a}$$

$$\sigma_s = \frac{T_d}{T_{c0}} = \frac{t_d - t_a}{t_{c0} - t_a}$$

$t = \frac{\alpha}{\tau_1}$ = dimensionless time

$$K_\alpha = \frac{T_2}{T_{c0}} = \frac{t_2 - t_a}{t_{c0} - t_a}$$

τ_1 = time constant of the system proper in sec which is equal to the mean resident time of air in the system proper

$$= \frac{V_1}{Q_1 + Q_2}$$

T_{c0} = room (system proper) temperature at $\alpha = 0$

2.1.2. The Control Element

The heat exchanger which is the control element of the system under consideration can perform its control function in various ways, for example, by changing the temperature or flow rate of the heat transfer medium, or changing both.

The performance equation of the control element of the system can be obtained again by employing the heat balance, which can be expressed as

$$[\text{heat in}] - [\text{heat out}] = [\text{heat accumulation}] \tag{14}$$

$$[\text{heat in}] = q_{mi1} + q_{mi2}$$

$$[\text{heat out}] = q_{m01} + q_{m02}$$

$$[\text{heat accumulation}] = q_{ms}$$

where

q_{mi1} = heat brought into the heat exchanger by circulation air in Kcal/sec

q_{mi2} = heat brought into the heat exchanger by cooling water in Kcal/sec

q_{m01} = heat flow out of the heat exchanger with circulation air in Kcal/sec

q_{m02} = heat flow out of the heat exchanger with cooling water in Kcal/sec

q_{ms} = rate of heat accumulation in the heat exchanger in Kcal/sec

Inserting these definitions into equation (14) gives

$$[q_{mi1} + q_{mi2}] - [q_{m01} + q_{m02}] = q_{ms} \tag{15}$$

Assuming perfect mixing of both air and the heat transfer medium in the heat exchanger, ignoring the heat loss through the shell and neglecting the thermal capacitance of the heat exchanger, the expressions for q_{mi1}, q_{mi2}, q_{m01}, and q_{m02} are as follows

$$q_{mi1} = Q_1 \rho C_p (t_c - t_a) \tag{16}$$

$$= Q_1 \rho C_p T_c$$

$$q_{mi2} = Q_w \rho_w C_{pw} (t_{wc} - t_a)$$
$$= Q_w \rho_w C_{pw} T_{wc} \quad (17)$$

$$q_{m01} = Q_1 \rho C_p (t_i - t_a)$$
$$= Q_1 \rho C_p T_i \quad (18)$$

$$q_{m02} = Q_w \rho_w C_{pw} (t_{wh} - t_a)$$
$$= Q_w \rho_w C_{pw} T_{wh} \quad (19)$$

The rate at which heat energy is stored in the heat exchanger can be expressed as

$$q_{ms} = V_2 \rho C_p \frac{dt_i}{d\alpha}$$
$$= V_2 \rho C_p \frac{dT_i}{d\alpha} \quad (20)$$

where

C_{pw} = specific heat of coolant in Kcal/Kg ^oC

Q_w = flow rate of coolant in m^3/sec

t_{wc} = inlet temperature of coolant in ^oC

t_{wh} = outlet temperature of coolant in ^oC

V_2 = volume of the heat exchanger occupied by air in m^3

ρ_w = density of coolant in Kg/m^3

Insertion of equations (16) through (20) into equation (15) gives

$$(Q_1 \rho C_p T_c + Q_w \rho_w C_{pw} T_{wc}) - (Q_1 \rho C_p T_i + Q_w \rho_w C_{pw} T_{wh})$$

$$dT.$$

or dividing by $Q_1 \rho C_p$

$$\tau_2 \frac{dT_i}{d\alpha} + T_i = T_c - \frac{Q_w \rho_w C_{pw} (T_{wh} - T_{wc})}{Q_1 \rho C_p} \tag{21}$$

where τ_2 is the mean residence time of air in (the time constant with respect to air fow of) the heat exchanger in sec. and is defined by

$$\tau_2 = V_2/Q_1$$

Note that $Q_w \rho_w C_{pw} (T_{wh} - T_{wc})$ is the amount of heat added or removed from the system which can be controlled by adjusting either Q_w, when ρ_w, C_{pw} and $(T_{wh} - T_{wc})$ are constant, or $(T_{wh} - T_{wc})$ when Q_w, ρ_w and C_{pw} are constants or both Q_w and $(T_{wh} - T_{wc})$ when ρ_w and C_{pw} are constant. Defining a hypothetical temperature T_r as

$$T_r = Q_w \rho_w C_{pw} (T_{wh} - T_{wc})/Q_1 \rho C_p$$

equation (21) can be written as

$$\tau_2 \frac{dT_i}{d\alpha} + T_i = T_c - T_r \tag{22}$$

or in dimensionless form

$$\frac{\tau_2}{\tau_1} \frac{dx_2}{dt} + x_2 = x_1 - (K_\beta \theta + K_\gamma) \tag{23}$$

where

$$K_\beta = \frac{1}{2T_{c0}} (T_{rmax} - T_{rmin})$$

$$K_\gamma = \frac{1}{2T_{c0}} (T_{rmax} + T_{rmin})$$

$$\theta = \frac{T_r - \frac{1}{2}(T_{rmax} + T_{rmin})}{T_{rmax} - \frac{1}{2}(T_{rmax} + T_{rmin})}$$

= control variable

$$\frac{T_r}{T_{c0}} = K_\beta \theta + K_\gamma$$

Note that $\theta = +1$ when $T_r = T_{rmax}$ and $\theta = -1$ when $T_r = T_{rmin}$. Also T_r is positive whenever heat is removed and negative whenever heat is added. Equation (23) is the performance equation of the heat exchanger which is shown schematically in Fig. 3.

2.1.3. The Feed-back Element - Thermostat

Here it is assumed that the sensing element measures the room temperature instantaneously and that there is no accumulation of heat in the element, or for simplicity, it will be assumed that the sensing element is the zero order element with its time constant, τ_3, equal to zero. A detailed discussion of the response of the thermostat can be found in [7].

2.2. SIMULATION

With the model a simulation should now be carried out by means of a digital computer or an analogue computer. The results should then be compared to the known characteristics of the system or to experimentally obtained data to determine the goodness of the model.

2.2.1. EXAMPLE 1. As a simple case consider a system in which the time constant of the heat exchanger is negligibly smaller as compared with the time constant of the system. For this system, equation (22) is simplified to

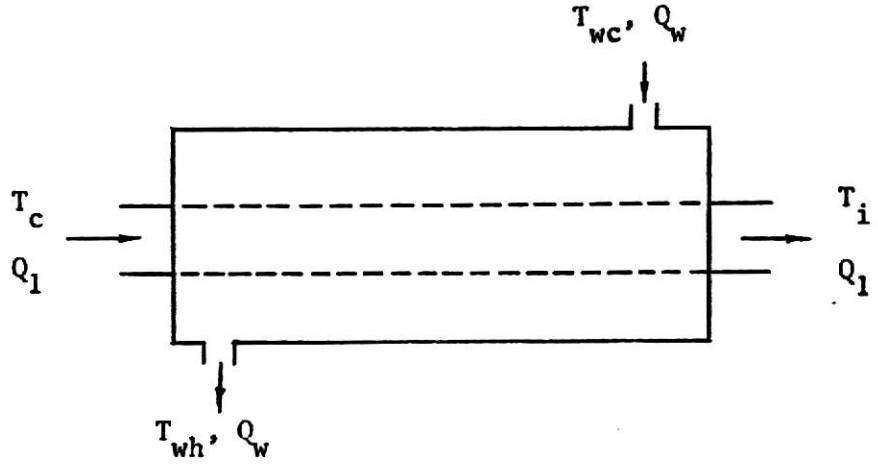


Fig. 3 Schematic diagram of the heat exchanger

$$T_i = T_c - T_r \tag{24}$$

Inserting equation (24) into equation (12) gives

$$\tau_1 \frac{dT_c}{d\alpha} + r_2 T_c = - r_1 T_r + r_2 T_2 + T_d U_0(\alpha)$$

$$T_c = T_{c0} \quad \text{at} \quad \alpha = 0 \tag{25}$$

The steady state value of T_r before disturbance can be evaluated by inserting

$$T_c = T_{c0},\ T_d = 0,\ \text{and}\ \frac{dT_c}{d\alpha} = 0$$

into equation (25). Hence

$$T_{r0} = \frac{r_2 (T_2 - T_{c0})}{r_1}, \qquad r_1 \neq 0$$

The final desired value of T_c is T_{c0}. The steady state value of the hypothetical temperature, which represents the capacity of the heat exchanger, denoted by T_{rf}, can be obtained by letting

$$T_c = T_{c0}, \quad \text{and} \quad \frac{dT_c}{d\alpha} = 0$$

into equation (25). This gives rise to

$$T_{rf} = \frac{1}{r_1} [T_d U_0(\alpha) + r_2(T_2 - T_{c0})] \tag{26}$$

Simulation of the model can be carried out when the form of T_r and the numerical values of the parameters are known. In case T_r is the step function, i.e., T_r remains constant after $\alpha = 0$, T_r can be adjusted to

the value T_{rf} given by equation (26) to maintain the system in the desired state.

NUMERICAL EXAMPLE

It is assumed that the volume of the system proper (room or cabin), V_1, is

$$V_1 = 3m \times 4m \times 5m$$

$$= 60\ m^3$$

The flow rate of air in the system proper, Q, is

$$Q = (\text{cross sectional area of the system}) \times (\text{air velocity in the system})$$

$$= (3m \times 4m)(.1m/sec)$$

$$= 1.2\ m^3/sec$$

and flow rates of circulation and fresh air are

$$Q_1 = 0.8Q = 0.96\ m^3/sec$$

$$Q_2 = 0.2Q = 0.24\ m^3/sec$$

The time constant of the system proper, τ_1, is

$$\tau_1 = \frac{V_1}{Q} = \frac{V_1}{Q_1 + Q_2} = \frac{60}{1.2} = 50\ sec$$

Other numerical values employed are

$$T_{c0} = 24^oC, \qquad T_2 = 36^oC$$

$$T_{rmin} = 0^oC$$

From equation (26) the steady-state heat removal capacity of the heat exchanger, T_{rf}, for a given disturbance temperature T_d, can be calculated as

$$T_{rf} = \frac{1}{0.8}\,[T_d + 0.2\,(36 - 24)]$$

or

$$T_{rf} = 1.25T_d + 3$$

The result is shown schematically in Fig. 4.

Similarly, it is possible to carry out the simulation using dimensionless form of the performance equation. The performance equation for the system can be obtained by combining equations (13) and (23) and setting $\tau_2 = 0$ as

$$\frac{dx_1}{dt} + r_2 x_1 = r_2 K_\alpha + \sigma_s - r_1 K_\beta \theta - r_1 K_\gamma \tag{27}$$

The initial conditions is

$$x_1(0) = 1 \qquad \text{at} \qquad t = 0$$

The required control θ can be obtained by setting

$$\frac{dx_1}{dt} = 0 \quad \text{and} \quad x_1 = 1$$

in equation (27). This gives

$$\theta = \frac{1}{r_1 K_\beta} [r_2 (K_\alpha - 1) - r_1 K_\gamma + \sigma_s]$$

Considering an example which corresponds to the example solved in dimensional form and further assuming that the upper limit for T_r, T_{rmax}, equal to 30^oC, the values of the constants K_α, K_β, and K_γ become

$$K_\alpha = \frac{T_c}{T_{c0}} = \frac{36}{24} = 1.5$$

$$K_\beta = \frac{1}{2T_{c0}} [T_{rmax} - T_{rmin}]$$

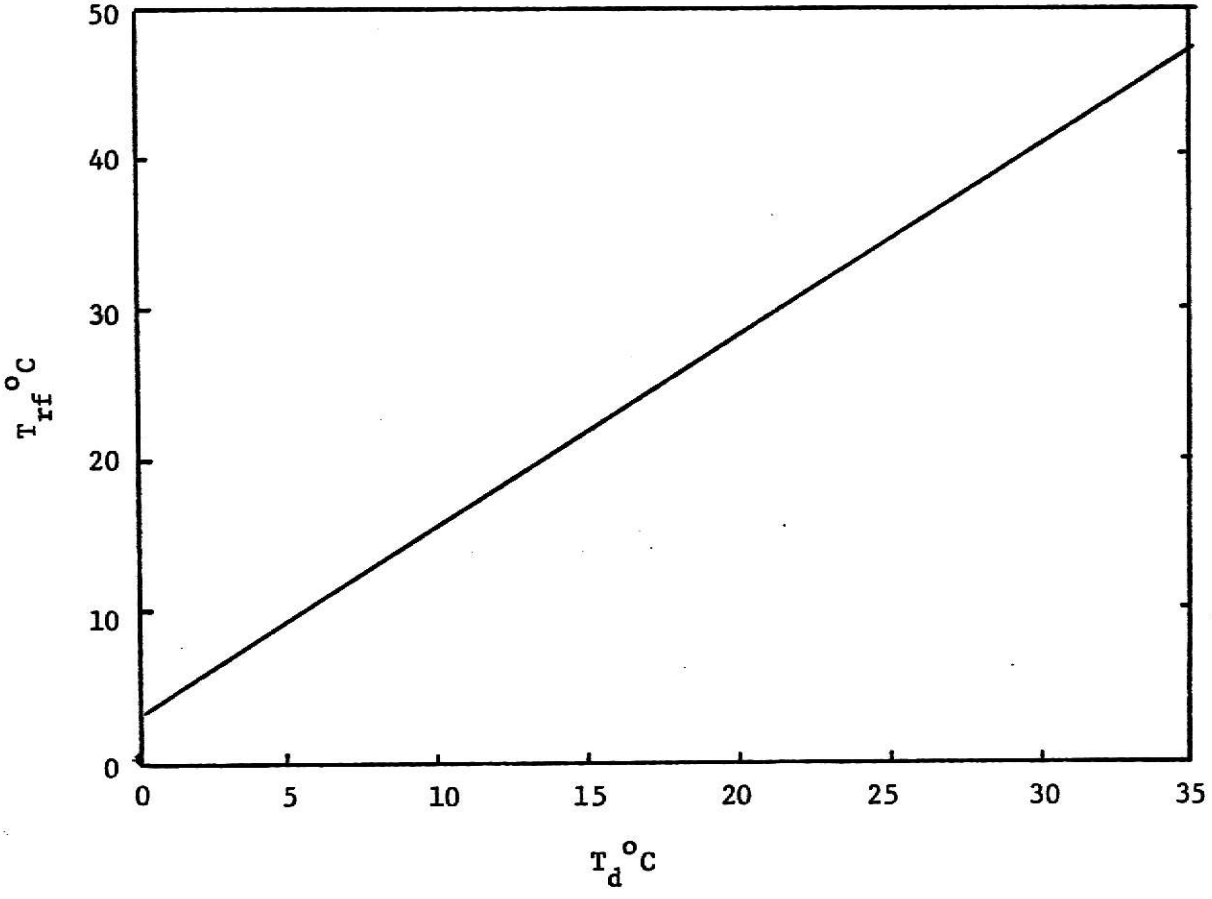


Fig. 4 The required heat removal capacity of heat exchanger vs the step heat disturbance at the steady-state condition

$$= \frac{1}{2 \times 24} [30 - 0]$$

$$= 0.625$$

$$K_\gamma = \frac{1}{2T_{c0}} [T_{rmax} + T_{rmin}]$$

$$= \frac{1}{2 \times 24} [30 + 0]$$

$$= 0.625$$

Substituting these numerical constants in equation (28) gives

$$\theta = 2\sigma_s - 0.8$$

The result is shown in Fig. 5. This result indicates that to maintain the room temperature, $x_1 = 1$, during the step heat input disturbance given by σ_s, the control, θ, of a heat exchanger with constants K_β and K_γ, can be taken as given by equation (28).

Equation (27) can be integrated if the functional form of control, θ, is known. When the control is constant equation (27) can be integrated as

$$x_1 = Ae^{-r_2 t} + K \tag{29}$$

where A is the integration constant and K is the particular solution given by

$$K = \frac{r_2 K_\alpha + \sigma_s - r_1 K_\beta \theta - r_1 K_\gamma}{r_2}$$

The constant A in equation (29) can be determined by using the initial condition

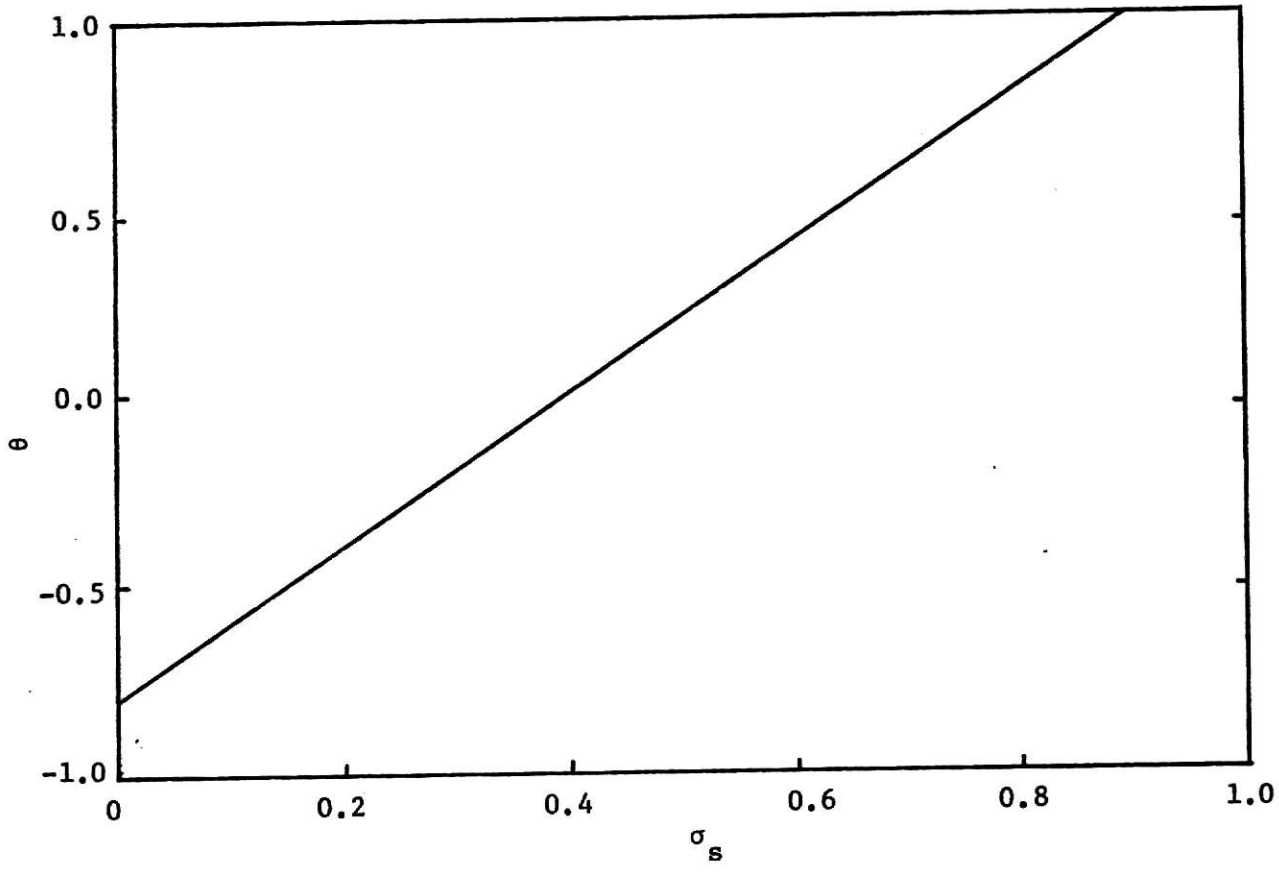


Fig. 5 The required control vs the step heat disturbance at the steady-state condition

$$A = 1 - K$$

Hence, the solution is

$$x_1 = (1 - K)e^{-r_2 t} + K \tag{29a}$$

The steady state value of x_1 can be obtained by substituting $t = \infty$ in equation (29a). This gives

$$x_1 \to K \quad \text{as} \quad t \to \infty$$

Figure 6 shows the behaviour of x_1 for different values of θ and σ_s.

2.2.2 EXAMPLE 2. Consider a system in which the time constant of the heat exchanger is not negligible, i.e., $\tau_2 \neq 0$. The performance equations for such a system have been derived as [equations (13) and (23)]:

$$\frac{dx_1}{dt} + x_1 = r_1 x_2 + r_2 K_\alpha + \sigma_s U_0(t) \tag{13}$$

$$\frac{dx_2}{dt} + rx_2 = rx_1 - rK_\beta\theta - rK_\gamma \tag{23a}$$

with the boundary conditions

$$x_1(0) = 1 \quad \text{at} \quad t = 0$$

$$x_2(0) = \frac{1 - r_2 K_\alpha}{r_1} \quad \text{at} \quad t = 0$$

where

$$r = \frac{\tau_1}{\tau_2}$$

The above equations can be simulated for different values of control θ.

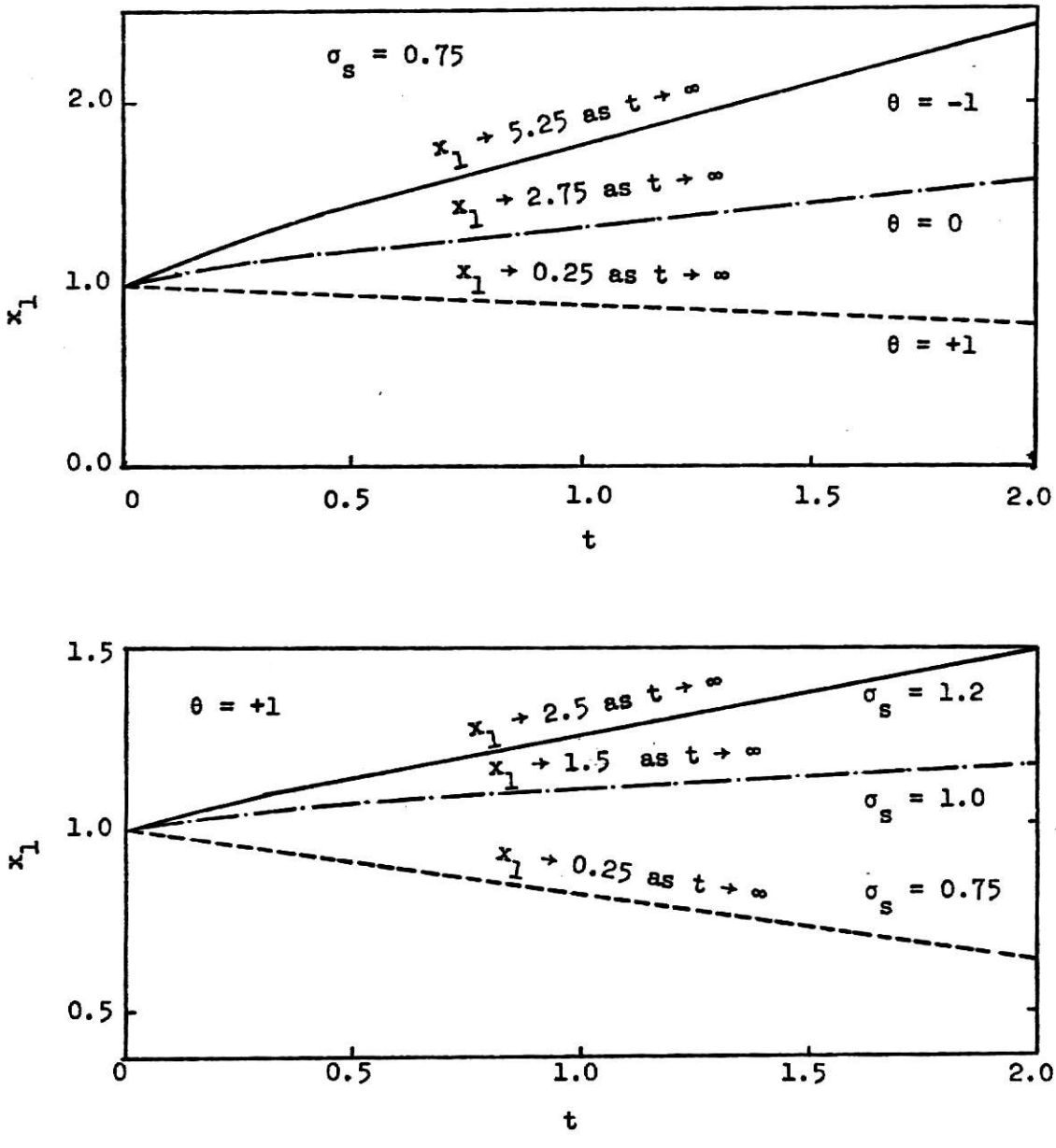


Fig. 6 Result of simulation of example 1

Thus eliminating x_2 from the two equations gives

$$\frac{d^2x_1}{dt^2} + (r + 1)\frac{dx_1}{dt} + (r - r_1 r)\, x_1 = rr_2 K_\alpha + r\sigma_s - rr_1 (K_\gamma + K_\beta\theta) \tag{30}$$

The solution of this equation is

$$x_1(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + K \tag{31}$$

where A_1 and A_2 are constants of integration. λ_1 and λ_2 are roots of the characteristic equation

$$\lambda^2 + (r + 1)\lambda + (r - r_1 r) = 0$$

and

$$K = \frac{rr_2K_\alpha + r\sigma_s - rr_1K_\beta\theta - rr_1K_\gamma}{r - r_1r}$$

$$= \frac{r_2K_\alpha + \sigma_s - r_1K_\beta\theta - r_1K_\gamma}{r_2}$$

Note that λ_1 and λ_2 are negative since $0 \leq r_1 \leq 1$. The solution of x_2 is obtained as

$$x_2(t) = \frac{1}{r_1}\left[(1+\lambda_1)A_1 e^{\lambda_1 t} + (1+\lambda_2)A_2 e^{\lambda_2 t} + K - r_2K_\alpha - \sigma_s\right] \tag{32}$$

The constants A_1 and A_2 can be determined by applying the initial conditions to equations (31) and (32) as

$$A_1 = \frac{1}{(\lambda_1 - \lambda_2)} (\sigma_s - \lambda_2 + K\lambda_2)$$

$$A_2 = 1 - A_1 - K$$

The steady state of the system can be obtained by substituting $t = \infty$ in equations (31) and (32). This gives

$$x_1 = K \qquad \text{as} \qquad t \to \infty$$

and

$$x_2 = \frac{K - r_2 K_\alpha - \sigma_s}{r_1} \qquad \text{as} \qquad t \to \infty$$

Figures 7 and 8 show the results of simulation of this example for two different sets of parameters. The results of the limiting case of $r = \infty$, i.e., when the time constant of the heat exchanger is negligible are also shown in these Figures. The numerical values used in Fig. 7 are

$$r_1 = 0.8, \qquad r_2 = 0.2$$

$$K_\alpha = 1.5, \qquad K_\beta = 0.625$$

$$K_\gamma = 0.625, \qquad \theta = 1$$

$$\sigma_s = 0.75$$

It should be noted that as $r \to \infty$, i.e., $\tau_2 \to 0$, the results are reduced to the results of example 1. The graphs of x_1 for $r = 100$ and $r = \infty$ in Fig. 7 are almost the same.

In Fig. 8 the following numerical values are used

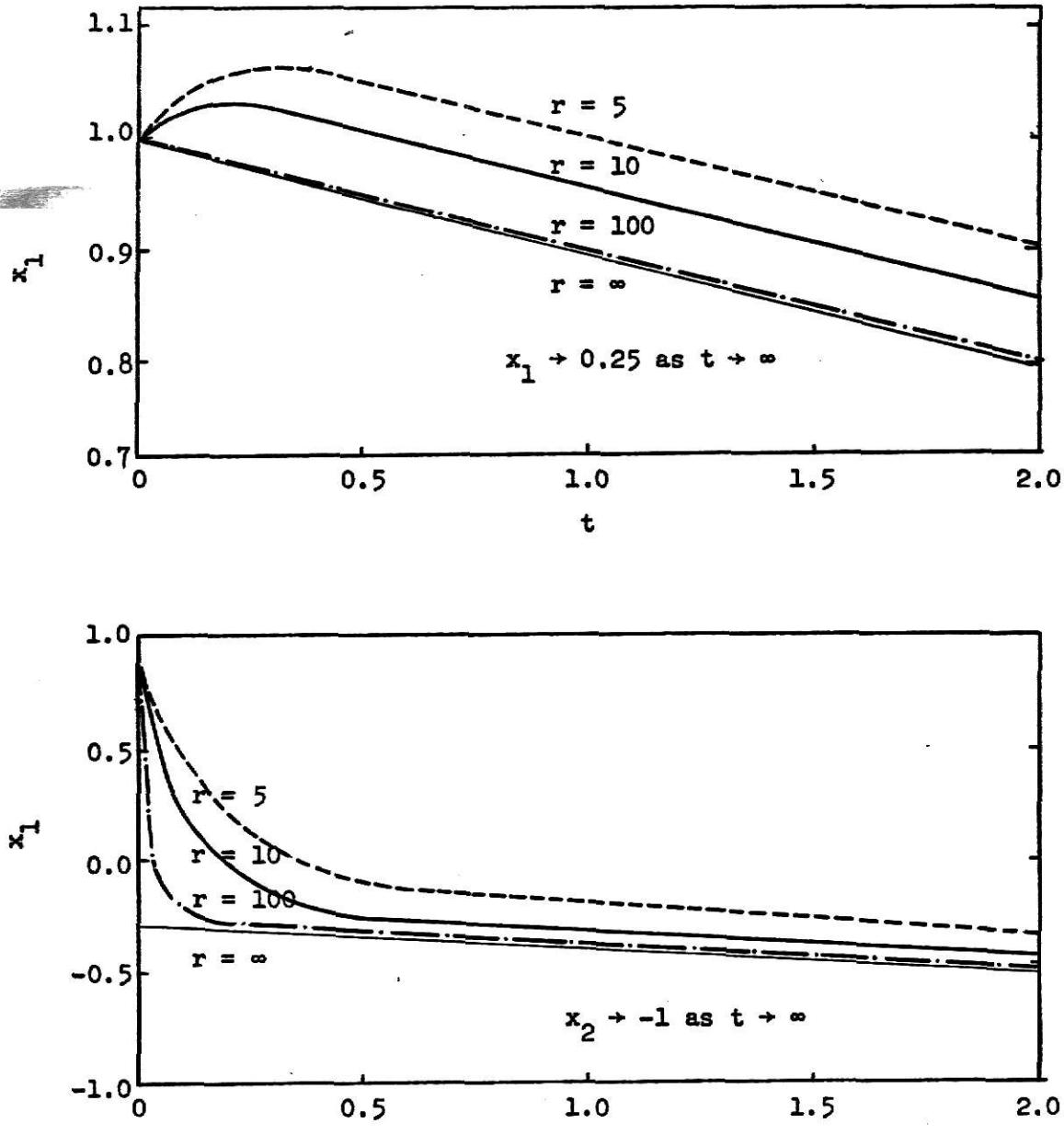
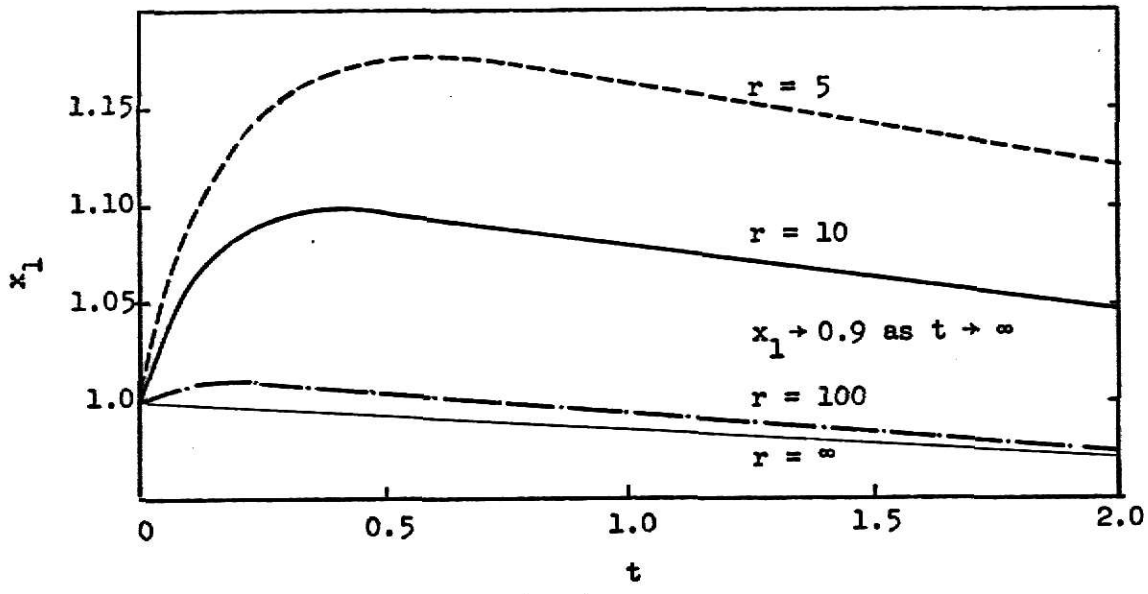


Fig. 7 Result of simulation of example 2, $\sigma_s = 0.75$, $K_\beta = K_\gamma = 0.625$



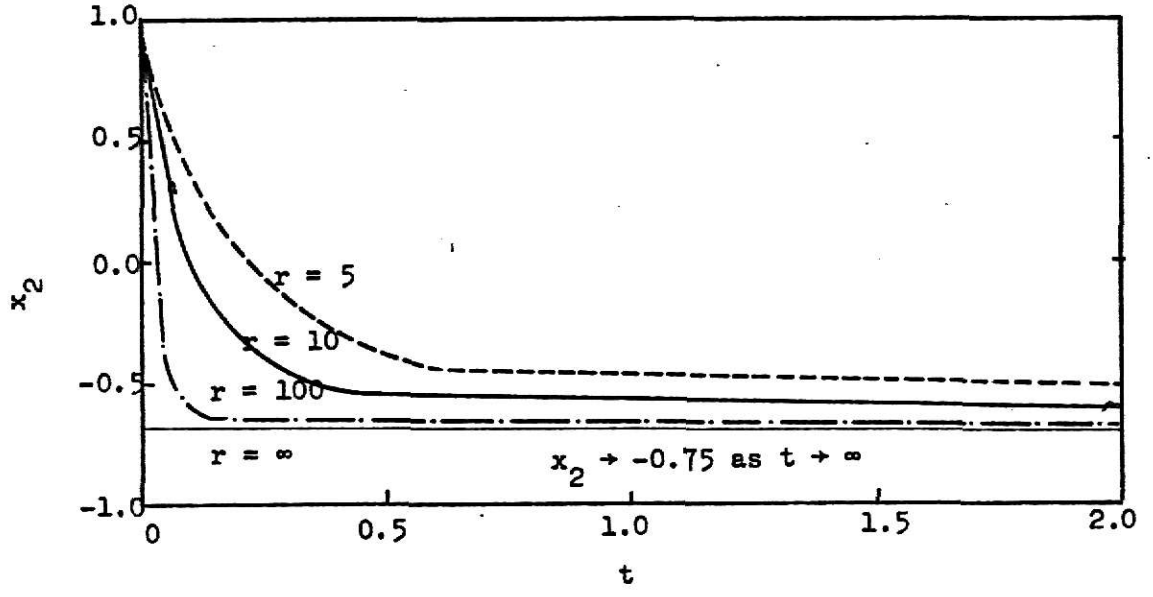


Fig. 8 Result of simulation of example 2, $\sigma_s = 1.2$, $K_\beta = K_\gamma = 0.825$

$r_1 = 0.8, \quad r_2 = 0.2$

$K_\alpha = 1.5, \quad K_\beta = 0.825$

$K_\gamma = 0.825, \quad \theta = 1$

$\sigma_s = 1.2$

From these figures the effect of the time constant of the heat exchanger upon the system responses x_1 and x_2 is evident. When the time constant of the heat exchanger is significantly large, which is equivalent to smaller value of r, the temperature of the room, x_1, increases for a short time and then decreases exponentially. The response of the heat exchanger, x_2, decreases slower when the time constant of the heat exchanger is significantly larger at the beginning. After t = 0.5 the response of the heat exchanger becomes stable and decreases very slowly.

2.3. PERFORMANCE EQUATIONS FOR TWO COMPARTMENTS MODEL

Consider now the case in which flow of air in the room can be characterized by two completely stirred compartments in series model (2 CST's-in-series model). The following assumptions are made for the system proper:

a) The room is divided into two well mixed compartments in series. Volume of each pool is denoted by V_{11} and V_{12}, and the temperature in each pool is denoted by T_{c1} and T_{c2}.

b) Backflow of air from the second compartment to the first compartment is negligible.

c) Disturbances are equally distributed over the system.

d) Fresh air comes into the room at a constant flow rate, while exhaust air is released from the second compartment at a constant flow rate.

The schematic diagram of the system is shown in Fig. 9. The performance equations for each pool can be obtained by using the heat balance around each compartment. Thus, for pool 1,

[heat in] - [heat out] = [heat accumulation]

or

$$[Q_1 T_i \rho C_p + Q_2 T_2 \rho C_p + \frac{V_{11}}{V_1} (Q_1 + Q_2) T_d \rho C_p U_0(\alpha)]$$

$$- [(Q_1 + Q_2) T_{c1} \rho C_p] = V_{11} \rho C_p \frac{dT_{c1}}{d\alpha} \tag{33}$$

Dividing the equation by $(Q_1 + Q_2) \rho C_p$ gives

$$\tau_{11} \frac{dT_{c1}}{d\alpha} + T_{c1} = r_1 T_i + r_2 T_2 + \frac{\tau_{11}}{\tau_1} T_d U_0(\alpha) \tag{34}$$

$$T_{c1} = T_{c10} \quad \text{at} \quad \alpha = 0$$

where τ_{11} is the time constant of pool 1 and is defined by

$$\tau_{11} = \frac{V_{11}}{Q_1 + Q_2} \tag{35}$$

Similarly, for pool 2 we have

$$[(Q_1 + Q_2) T_{c1} \rho C_p + \frac{V_{12}}{V_1} (Q_1 + Q_2) T_d \rho C_p U_0(\alpha)]$$

$$- [(Q_1 + Q_2) T_{c2} \rho C_p] = V_{12} \rho C_p \frac{dT_{c2}}{d\alpha}$$

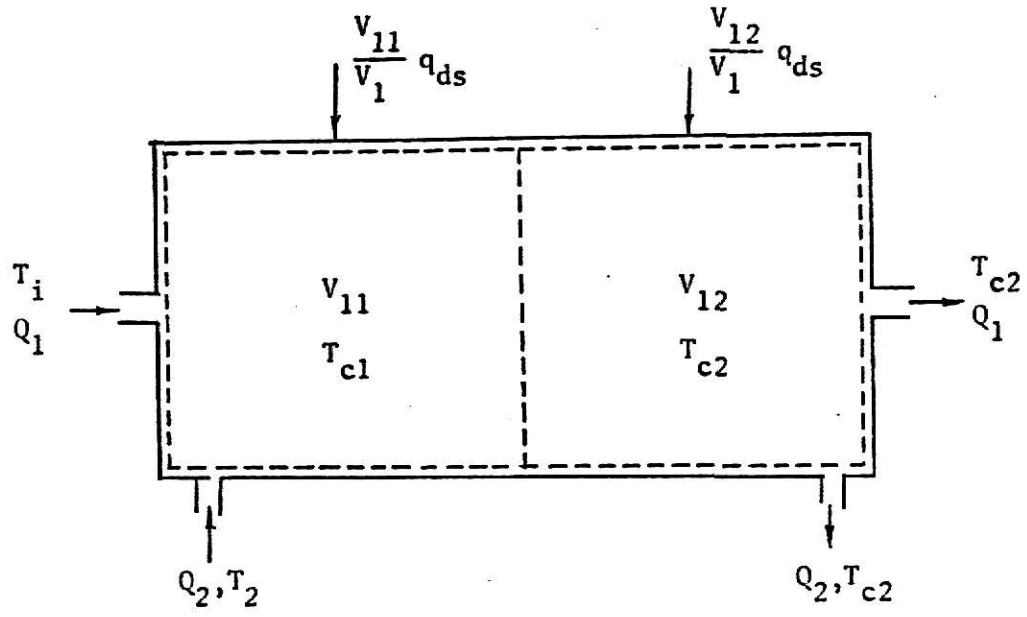


Fig. 9 Schematic representation of the 2 CST's-in series model

Again dividing by $(Q_1 + Q_2)\ \rho\ C_p$ yields

$$\tau_{12} \frac{dT_{c2}}{d\alpha} + T_{c2} = T_{c1} + \frac{\tau_{12}}{\tau_1} T_d\ U_0(\alpha) \tag{36}$$

$$T_{c2} = T_{c20} \quad \text{at} \quad \alpha = 0$$

where τ_{12} is the time constant of the second pool and is defined as

$$\tau_{12} = \frac{V_{12}}{Q_1 + Q_2} \tag{37}$$

For the heat exchanger the following equation has been derived as equation (21)

$$\tau_2 \frac{dT_i}{d\alpha} + T_i = T_{c2} - T_r \tag{38}$$

Equations (34), (36) and (38) can be written in dimensionless form as

$$\frac{dx_{11}}{dt} + r_{11}\ x_{11} = a_{11}\ x_2 + a_{12} + \sigma_s\ U_0(t) \tag{39}$$

$$\frac{dx_{12}}{dt} + r_{12}\ x_{12} = r_{12}\ x_{11} + \sigma_s\ U_0(t) \tag{40}$$

$$\frac{dx_2}{dt} + rx_2 = rx_{12} - a_5\theta - a_6 \tag{41}$$

with the initial conditions

$$x_{11}(0) = x_{12}(0) = 1 \quad \text{at} \quad t = 0$$

$$x_2(0) = \frac{1 - r_2 K_\alpha}{r_1} \quad \text{at} \quad t = 0 \tag{41}$$

where

$$x_{11} = \frac{T_{c1}}{T_{c10}}, \qquad K_{\alpha} = \frac{T_2}{T_{c10}}$$

$$x_{12} = \frac{T_{c2}}{T_{c10}}, \qquad K_{\beta} = \frac{1}{2T_{c10}} [T_{rmax} - T_{rmin}]$$

$$x_2 = \frac{T_i}{T_{c10}}, \qquad K = \frac{1}{2T_{c10}} [T_{rmax} + T_{rmin}]$$

$$a_{11} = r_1 r_{11}, \qquad a_{12} = r_{11} r_2 K_{\alpha}$$

$$a_5 = rK_{\beta}, \qquad r_{11} = \frac{\tau_1}{\tau_{11}}$$

$$a_6 = rK_{\gamma}, \qquad r_{12} = \frac{\tau_1}{\tau_{12}}$$

$$r = \frac{\tau_1}{\tau_2}, \qquad r_1 = \frac{Q_1}{Q_1 + Q_2}$$

$$\tau_1 = \frac{V_{11} + V_{12}}{Q_1 + Q_2}, \qquad r_2 = \frac{Q_2}{Q_1 + Q_2}$$

$$\tau_2 = \frac{V_2}{Q_1}, \qquad t = \frac{\alpha}{\tau_1}$$

$$\sigma_s = \frac{T_d}{T_{c10}}$$

2.4 SIMULATION OF THE 2 CST'S-IN-SERIES MODEL

The performance equations (39), (40) and (41) can be used to simulate the 2 CST's-in-series model. Thus if the time constant of the heat

exchanger is negligibly small, the performance equations of the system become

$$\frac{dx_{11}}{dt} + r_{11}x_{11} = a_{11}x_{12} - a_{11}K_{\beta}\theta - a_{11}K_{\gamma} + a_{12} + \sigma_s U_0(t) \tag{42}$$

$$\frac{dx_{12}}{dt} + r_{12}x_{12} = r_{12}x_{11} + \sigma_s U_0(t) \tag{43}$$

with the initial conditions

$$x_{11} = x_{12} = 1 \quad \text{at} \quad t = 0 \tag{44}$$

Now eliminating x_{11} from equations (42) and (43) gives rise to the following differential equation.

$$\frac{d^2x_{12}}{dt^2} + (r_{11} + r_{12})\frac{dx_{12}}{dt} + (r_{11}r_{12} - a_{11}r_{12})x_{12}$$

$$= a_{12}r_{12} + \sigma_s(r_{11}+r_{12}) - a_{11}r_{12}K_{\beta}\theta - a_{11}r_{12}K_{\gamma} \tag{45}$$

Solution of equation (45) can be obtained if the functional form of the control variable θ and the numerical values of the parameters are known. In case θ is a step function, equation (45) can be integrated as

$$x_{12} = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + K \tag{46}$$

where λ_1 and λ_2 are roots of the characteristic equation

$$\lambda^2 + (r_{11} + r_{12})\lambda + r_{11}r_{12} - a_{11}r_{12} = 0$$

and A_1 and A_2 are constants of integration. K is the particular solution of the differential equation and given by

$$K = \frac{a_{12}r_{12} + \sigma_s(r_{11} + r_{12}) - a_{11}r_{12}K_\beta\theta - a_{11}r_{12}K_\gamma}{r_{11}r_{12} - a_{11}r_{12}}$$

From equation (42) the solution of x_{11} can be written as

$$x_{11} = \frac{1}{r_{12}}\left[(r_{12} + \lambda_1)A_1e^{\lambda_1 t} + (r_{12} + \lambda_2)A_2e^{\lambda_2 t} + r_{12}K - \sigma_s\right] \quad (47)$$

The constants in equations (46) and (47) can be determined by applying the initial conditions which yield

$$A_1 + A_2 + K = 1$$

$$(r_{12}+\lambda_1)A_1 + (r_{12}+\lambda_2)A_2 + r_{12}K - \sigma_s = r_{12}$$

Thus the solution of these simultaneous equations yields

$$A_1 = \frac{\lambda_2 - \lambda_2 K - \sigma_s}{\lambda_2 - \lambda_1}$$

and

$$A_2 = 1 - K - A_1$$

The steady state values of the responses can be obtained by letting $t \to \infty$ in equations (46) and (47). Thus,

$$x_{12} = K \quad \text{as} \quad t \to \infty$$

and

$$x_{11} = K - \frac{\sigma_s}{r_{12}} \quad \text{as} \quad t \to \infty$$

Numerical Examples:

The following numerical values are used

$$r_1 = 0.8, \quad r_2 = 0.2$$

$$K_\alpha = 1.5, \quad \theta = 1$$

Figure 10 shows the results of simulation when the constants of the heat exchanger, K_β and K_γ, are both equal to 0.625 and the step heat disturbance σ_s is equal to 0.75. The result of the one CST model when the time constant of the heat exchanger is negligible is also shown for comparison. It is seen from the figure that as the time constant of the pool 1 decreases, i.e., r_{11} increases, the temperature of pool 1 decreases faster and to a lower value. The temperature of the pool 2, x_{12}, increases for a short time and then starts dropping down. The curves for x_{11} and x_{12} are approximately parallel for the different values of r_{11} after t = 0.5.

When $r_{11} \to 1$, the 2 CST's-in-series model approaches the one CST model. The results of the 2 CST's-in-series model, in this case, should be reduced to that of the one CST model. In Fig. 10, graphs of x_{11} for $r_{11} = 1.2$ and $r_{11} = 1.0$ have approximately the same form. As $r_{11} \to \infty$, the 2 CST's-in-series model again approaches the one CST model. In this case x_{12} should have approximately the same form as that of x_1 in Fig. 6 ($\theta = 1$, $\sigma_s = 0.75$).

Figure 11 contains the results of simulation for $K_\beta = 0.825$, $K_\gamma = 0.825$ and $\sigma_s = 1.2$. The results are similar to that of Fig. 10.

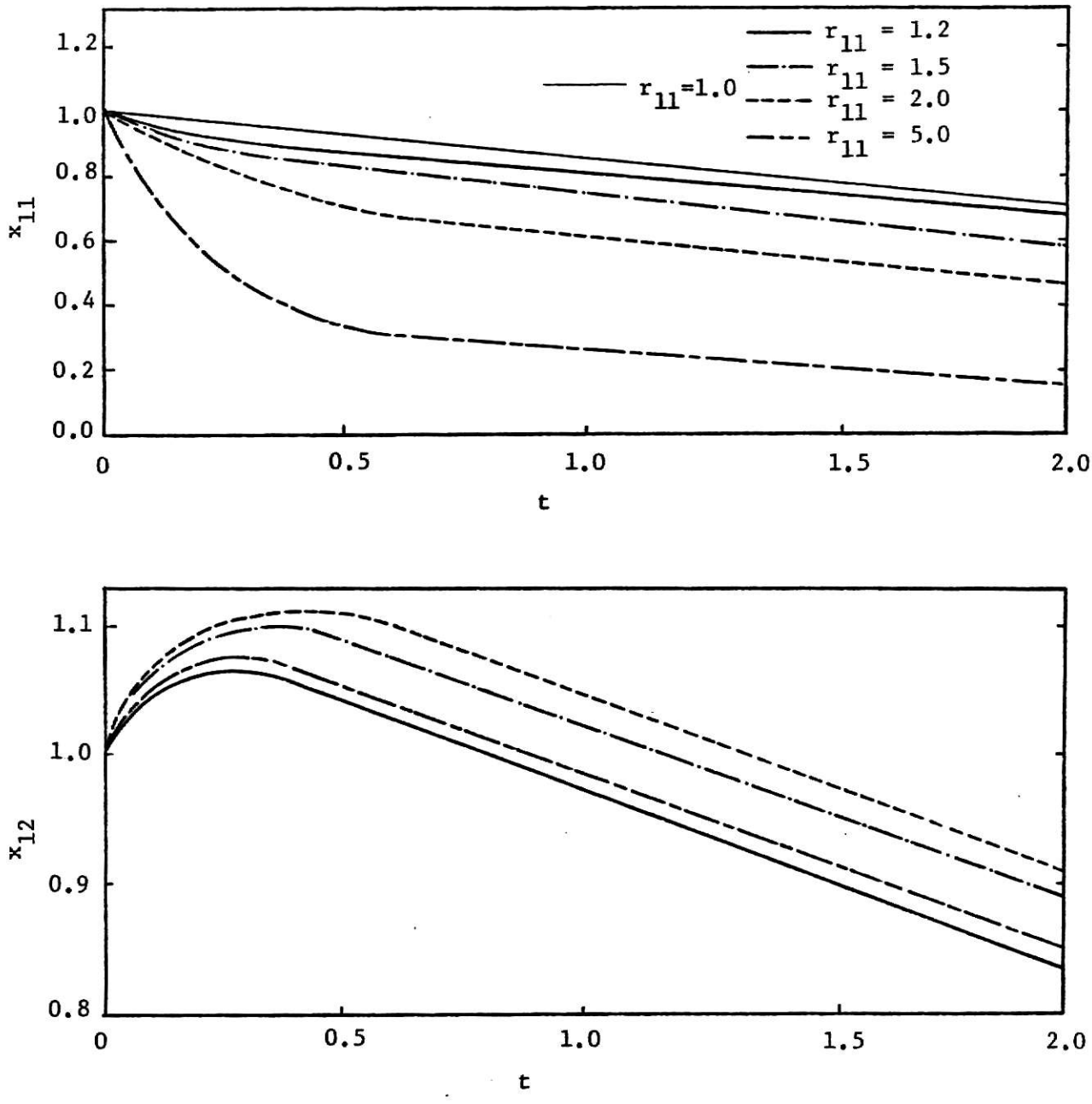


Fig. 10 Result of simulation of the 2 CST's-in-series model, $\sigma_s = 0.75$, $K_\beta = K_\gamma = 0.625$

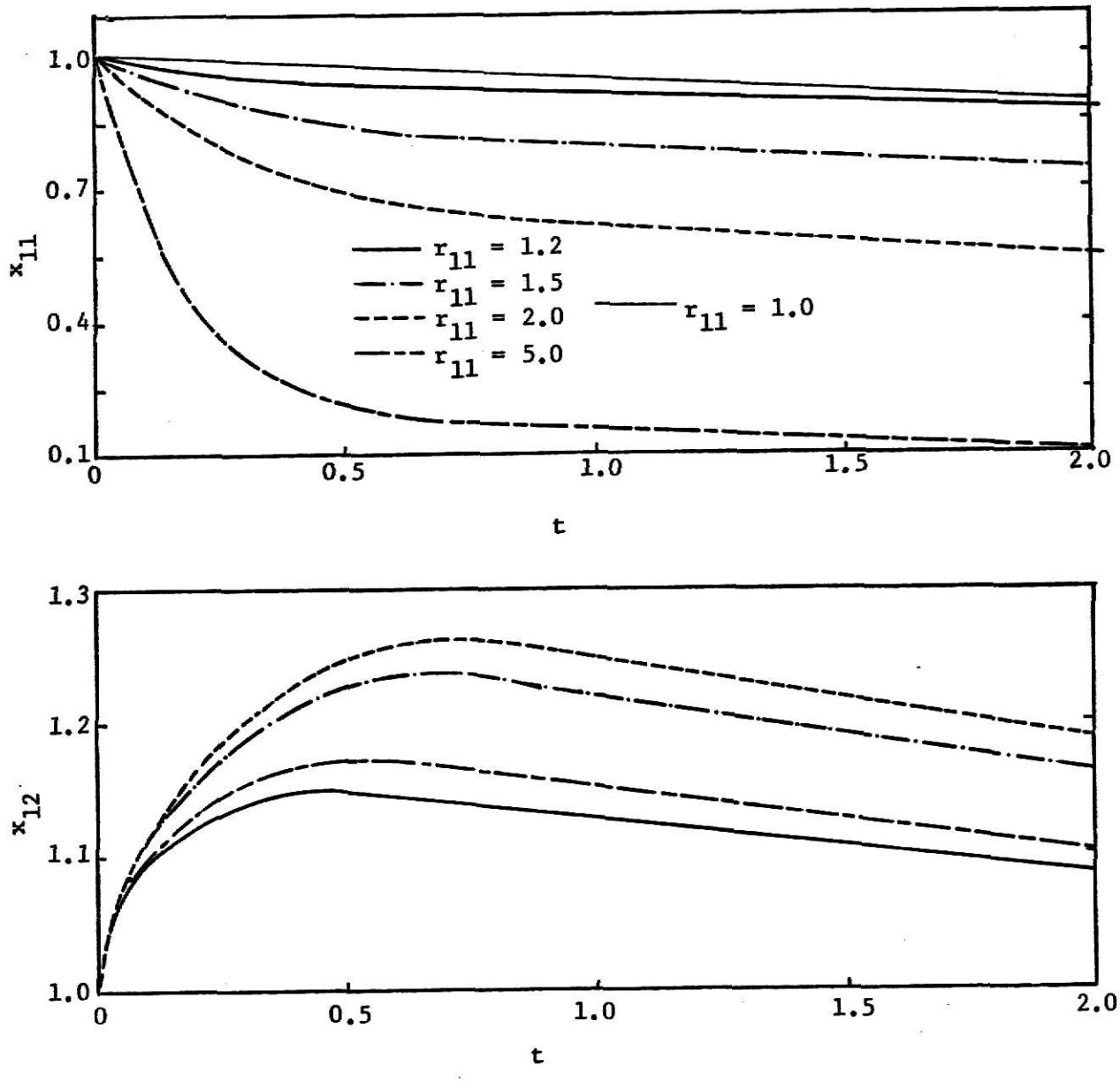


Fig. 11 Result of simulation of the 2 CST's-in-series model, $\sigma_s = 1.2$, $K_\beta = K_\gamma = 0.825$

CHAPTER III

APPLICATION OF PONTRYAGIN'S MAXIMUM PRINCIPLE TO OPTIMAL ENVIRONMENTAL CONTROL OF CONFINED SPACES WITH STEP HEAT DISTURBANCE

The basic form of Pontryagin's maximum principle which is a major part of the modern control theory is presented. The principle is used here to determine optimal temperature control policies of life support systems. The procedures and computational approaches employed for obtaining the optimal control policies are given in detail. Three concrete examples concerning the temperature control of a life support system consisting of an air-conditioned room or cabin subjected to step heat disturbance and of a heat exchanger are considered.

The step heat disturbance arises from various sources, such as opening a window or a door, a temperature change in the incoming air, and heat generated by the people or animals. Sudden start up of a machine or a process (furnace, engine etc.) in the confined space will also give rise to a step heat disturbance. In this type of disturbance the rate of heat input in the control period is constant.

3.1 INTRODUCTION

The mathematical models of a life support system have been established in the preceding chapter. In this chapter the application of the basic algorithm of Pontryagin's maximum principle for determining optimum control policies for such systems is illustrated. Application of the maximum principle provides only the necessary condition for the optimum control and almost always gives rise to a split-boundary value problem. However, the maximum principle still provides in most cases a practical approach

to process-systems optimization.

The basic form of Pontryagin's maximum principle is stated first and it is then applied to obtain optimal control policies for environmental control systems.

The problem is to maintain the system in a desired state over a fixed control period with minimum effort. Thus, the objective function to be minimized is the sum of the integrated deviation from the desired state of the system and the integrated effort required to maintain the desired state over a specified control period. Thus, the performance index to be minimized is

$$S = \int_0^T [b\theta^2 + c(x_1 - x_{1d})^2]dt$$

where b and c are suitable weighting factors and the desired state of the system is taken as $x_{1d} = 1$.

The desired state may be the comfort conditions in a life support system or as in the case of a biomedical process, the optimum temperature required for a chemical reaction. The effort is the amount of fuel required to achieve the desired control action. By giving appropriate values to the weighting factors it is possible to achieve the uniformity of units and the desired objective function. For example, if it is very important to maintain the system in a desired state, a high value of c will be used. On the other hand, if the fuel is scarce a high value will be given to b.

In the first example the time constant of the heat exchanger is considered to be negligibly small. In the second example the time constant of the heat exchanger is not neglected. In the third example the optimal policy of the two CST's-in-series model is derived, the time constant of the heat exchanger is again neglected.

3.2 STATEMENT OF ALGORITHM

Consider that the dynamic behavior of a controlled system can be represented by a set of differential equations

$$\frac{dx_i}{dt} = f_i[x_1(t), x_2(t), \ldots, x_s(t); \theta_1(t), \theta_2(t), \ldots, \theta_r(t)],$$

$$i = 1, 2, \ldots, s, \qquad t_0 \leq t \leq T \tag{1}$$

or in vector form

$$\frac{dx}{dt} = f[x(t), \theta(t)], \qquad t_0 \leq t \leq T \tag{1a}$$

where $x(t)$ is an s-dimensional vector function representing the state of the process at time t and $\theta(t)$ is an r-dimensional vector function representing the decision at time t [9, 24]. The functions f_i, $i = 1,2,\ldots,s$, are single valued, bounded, differentiable with respect to the x's with bounded first partial derivatives, and are continuous in the θ's on a product region $x\theta$, where x and θ are closed regions in the s-dimensional x-space and r-dimensional θ-space respectively. Note that we are dealing with the autonomous systems in which the right-hand side of the performance equation, equation (1), depends implicitly on time t. The non-autonomous systems are those in which the right-hand side of the performance equation, equation (1), depends explicitly on time t.

A typical optimization problem associated with such a process is to find a piecewise continuous decision vector function, $\theta(t)$ subject to the p-dimensional constraints

$$h_i[\theta(t)] \leq 0, \qquad i = 1, 2, \ldots, p \tag{2}$$

such that the performance index

$$S = \sum_{i=1}^{s} c_i \, x_i(T), \qquad c_i = \text{constant} \tag{3}$$

is minimum (or maximum) when the initial conditions

$$x_i(t_0) = x_{i0}, \qquad i = 1, 2, \ldots, s \tag{4}$$

are given. The duration of control, T, is specified and the final conditions of state variables are unfixed. This type of problem is often called the free right-end problem (with fixed T). The decision vector (or a collection of control variables) so chosen is called an optimal decision vector (or optimal control variables) and is denoted by $\bar{\theta}(t)$.

The procedure for solving the problem is to introduce an s-dimensional adjoint vector z(t) and a Hamiltonian function H which satisfy the following relations

$$H[x(t), \theta(t), z(t)] = \sum_{i=1}^{s} z_i(t) \, f_i[x(t), \theta(t)] \tag{5}$$

$$\frac{dz_i}{dt} = -\frac{\partial H}{\partial x_i} = -\sum_{j=1}^{s} z_j \frac{\partial f_j}{\partial x_i}, \qquad i = 1, 2, \ldots, s \tag{6}$$

$$z_i(T) = c_i, \qquad i = 1, 2, \ldots, s \tag{7}$$

The set of equations, equations (1), (4), (6) and (7), constitutes a two-point split boundary value problem, whose solution depends on $\theta(t)$. The optimal decision vector $\bar{\theta}(t)$ which makes S an extremum also makes the Hamiltonian an extremum for all t, i.e., $t_0 \leq t \leq T$ [9, 24, 25].

A necessary condition for S to be an extremum with respect to $\theta(t)$ is

$$\frac{\partial H}{\partial \theta_i} = 0, \qquad i = 1, 2, \ldots, r \tag{8}$$

if the optimal decision vector is interior to the set of admissible decision $\theta(t)$ [the set given by equation (2)]. If $\theta(t)$ is constrained, the optimal decision vector $\bar{\theta}(t)$ is determined either by solving equation (8) for $\theta(t)$ or by searching the boundary of the set. More specifically, the extremum value of Hamiltonian is maximum (or minimum) when the control variables are on the constraint boundary. Furthermore, the extremum value of the Hamiltonian is constant at every point of time under the optimal condition. It is worth noting that the final conditions of the adjoint variables, $z_i(T)$, are often given as $-c_i$ instead of c_i as shown in equation (7), in imploying the maximum principle of Pontryagin. The use of such final conditions of $z_i(t)$ gives rise to the condition that the Hamiltonian is maximum when the objective function is minimized, and minimum when the objective function is maximized as stated in the original version of the maximum principle of Pontryagin [9, 24].

If both the initial and final conditions of state variables are given, the problem is said to be a boundary value problem. The basic algorithm presented except the condition given by equation (7) is still applicable [9].

If optimization (usually minimization) of time t is involved in the objective function in a problem with an unfixed duration of control, T, the problem is then called a time optimal problem. In this case, the basic algorithm presented is still applicable with an additional condition that the extremal value of the Hamiltonian is not only a constant but also identical to zero. The simplest example of the time optimal control problem is one in which the performance index is of the form

$$S = \int_0^T dt \tag{9}$$

Such a problem is often called a minimum time problem.

3.3 EXAMPLE 1. Consider a life support system consisting of an air-conditioned room or cabin and a heat exchanger of negligibly small time constant ($\tau_2 = 0$) and subject to a step heat disturbance. The performance equation of such a system has been derived as (see equation (27) of Chapter 2)

$$\frac{dx_1}{dt} + r_2 x_1 = r_2 K_\alpha = r_1 K_\beta \theta - r_1 K_\gamma + \sigma_s$$

$$x_1 = 1 \quad \text{at} \quad t = 0 \tag{10}$$

The objective function to be minimized is the sum of the integrated control effort to maintain the state of the system in the desired state and the integrated deviation from the desired state over a specified control time and is given by

$$S = \int_0^T [b\theta^2 + c(x_1 - x_{1d})^2]dt \tag{11}$$

where b and c are weighting factors. The desired state, x_{1d}, is equal to one.

Introducing another state variable $x_2(t)$ such that

$$x_2(t) = \int_0^t [b\theta^2 + c(x_1-1)^2]dt$$

it follows that

$$\frac{dx_2}{dt} = b\theta^2 + c(x_1-1)^2, \qquad x_2(0) = 0 \tag{12}$$

The problem is thus transformed into that of minimizing $x_2(T)$.

The Hamiltonian is

$$H = z_1[-r_2x_1 + r_2K_\alpha - r_1K_\beta\theta - r_1K_\gamma + \sigma_s] + z_2[b\theta^2 + c(x_1-1)^2] \tag{13}$$

The adjoint variables are defined by

$$\frac{dz_1}{dt} = -\frac{\partial H}{\partial x_1} = r_2z_1 - 2z_2\, c(x_1-1), \qquad z_1(T) = 0 \tag{14}$$

$$\frac{dz_2}{dt} = -\frac{\partial H}{\partial x_2} = 0, \qquad z_2(T) = 1 \tag{15}$$

From equation (15) the solution of z_2 is obtained as

$$z_2(t) = 1, \qquad 0 \leq t \leq T \tag{16}$$

The Hamiltonian can now be rewritten as

$$H = z_1[-r_2x_1 + r_2K_\alpha - r_1K_\beta\theta - r_1K_\gamma + \sigma_s] + b\theta^2 + c(x_1-1)^2 \tag{17}$$

The variable portion of H that depends on θ, H^*, is

$$H^* = -r_1K_\beta z_1\theta + b\theta^2 \tag{18}$$

Inspection of H^* shows that the optimal control is of continuous type and is obtained from the following necessary condition for optimality [9]

$$\frac{\partial H^*}{\partial \theta} = 0 = -r_1K_\beta z_1 + 2b\theta \tag{19}$$

which gives

$$\theta = \frac{r_1K_\beta}{2b}\, z_1 \tag{20}$$

Using this relationship into equation (10) and eliminating x_1 from equation (10) and (14) give

$$\frac{d^2z_1}{dt^2} - [r_2^2 + \frac{c}{b}(r_1K_\beta)^2]z_1 = 2c(r_2 + r_2K_\alpha + r_1K_\gamma - \sigma_s)$$

The solution of this equation is

$$z_1 = A_1e^{\lambda t} + A_2e^{-\lambda t} + K \tag{21}$$

where

$$\lambda = \sqrt{r_2^2 + \frac{c}{b}(r_1K_\beta)^2}$$

$$K = \frac{2c(r_2K_\alpha - r_1K_\gamma + \sigma_s - r_2)}{r_2^2 + \frac{c}{b}(r_1K_\beta)^2}$$

and A_1 and A_2 are constants of integration.

The solution of x_1 can be obtained from equations (14) and (21) as

$$x_1(t) = \frac{1}{2c}[(r_2-\lambda)A_1e^{\lambda t} + (r_2+\lambda)A_2e^{-\lambda t} + r_2K] + 1 \tag{22}$$

Now employing the initial condition, $x_1(0) = 1$, in equation (22) yields

$$(r_2-\lambda)A_1 + (r_2+\lambda)A_2 + r_2K = 0 \tag{23}$$

Applying the final condition, $z_1(T) = 0$, to equation (21) yields

$$A_1e^{\lambda T} + A_2e^{-\lambda T} + K = 0 \tag{24}$$

The constants A_1 and A_2 can be determined from equations (23) and (24) by applying Cramer's rule as

$$A_1 = \frac{\begin{vmatrix} -r_2K & r_2+\lambda \\ -K & e^{-\lambda T} \end{vmatrix}}{\begin{vmatrix} r_2-\lambda & r_2+\lambda \\ e^{\lambda T} & e^{-\lambda T} \end{vmatrix}} = \frac{r_2K(1-e^{-\lambda T}) + \lambda K}{(r_2-\lambda)e^{-\lambda T} - (r_2+\lambda)e^{\lambda T}}$$

$$A_2 = \frac{\begin{vmatrix} r_2-\lambda & -r_2K \\ e^{\lambda T} & -K \end{vmatrix}}{(r_2-\lambda)e^{-\lambda T} - (r_2+\lambda)e^{\lambda T}} = \frac{r_2K(e^{\lambda T}-1) + \lambda K}{(r_2-\lambda)e^{-\lambda T} - (r_2+\lambda)e^{\lambda T}}$$

The optimal control policy can now be determined from equations (20) and (21) as

$$\theta(t) = \frac{r_1K_\beta}{2b} [A_1e^{\lambda t} + A_2e^{-\lambda t} + K] \tag{25}$$

The response of the system can be found from equation (22).

In the cases where σ_s is appreciably large compared to constants K_β and K_γ, the optimal control given by equation (25) will violate the constraint $|\theta| \leq 1$. Therefore the condition

$$\frac{dH^*}{d\theta} = 0$$

does not yield the admissible control action. The optimal control for $0 \leq t \leq t_s$ is

$$\theta = +1$$

where t_s is the time when the saturation period ends, which is to be determined. With $\theta = +1$ equation (10) can be integrated as

$$x_1(t) = B_1e^{-r_2t} + K', \qquad 0 \leq t \leq t_s \tag{26}$$

where

$$K' = \frac{r_2 K_\alpha - r_1 K_\beta - r_1 K_\gamma + \sigma_s}{r_2}$$

The constant of integration, B_1, can be determined by employing the initial condition in equation (26) which yields

$$B_1 = 1 - K'$$

With the solution of $x_1(t)$ given by equation (25), equation (14) can be integrated as

$$z_1(t) = B_2 e^{r_2 t} + \frac{cB_1}{r_2} e^{-r_2 t} + \frac{2c(K'-1)}{r_2}, \qquad 0 \leq t \leq t_s \tag{27}$$

where B_2 is the integration constant to be determined later.

After time t_s the control action will no longer be saturated and thus the condition

$$\frac{dH^*}{d\theta} = 0$$

can be used to determine optimal condition. Thus the optimal control, optimal state, and adjoint variable given by equations (25), (22), and (21) can be used for $t_s \leq t \leq T$. The constants A_1 and A_2 in equations (21) and (22) can be determined by using the fact that x_1 and z_1 are continuous with respect to t. Hence at $t = t_s$

$$z_1(t_s) = A_1 e^{\lambda t_s} + A_2 e^{-\lambda t_s} + K$$

$$= B_2 e^{r_2 t_s} + \frac{cB_1}{r_2} e^{-r_2 t_s} + \frac{2c(K'-1)}{r_2} \tag{28}$$

$$x_1(t_s) = \frac{1}{2c}\left[(r_2-\lambda)A_1 e^{\lambda t_s} + (r_2+\lambda)A_2 e^{-\lambda t_s} + r_2 K\right] + 1$$

$$= B_1 e^{-r_2 t_s} + K' \tag{29}$$

Also at $t = t_s$, $\theta = 1$. Hence from equation (20) and (21) we have

$$\frac{r_1 K_\beta}{2b}\left[A_1 e^{\lambda t_s} + A_2 e^{-\lambda t_s} + K\right] = 1 \tag{30}$$

Using the final condition in equation (21) yields

$$A_1 e^{\lambda T} + A_2 e^{-\lambda T} + K = 0 \tag{31}$$

There are four unknowns A_1, A_2, B_2 and t_s in equations (28), (29), (30), and (31). The constants A_1 and A_2 can be solved from equations (28) and (29) in terms of B_2 and t_s as

$$A_1 = \frac{e^{-\lambda t_s}}{2\lambda}\left\{(r_2+\lambda)\left[B_2 e^{r_2 t_s} + \frac{cB_1}{r_2} e^{-r_2 t_s}\right] - 2cB_1 e^{-r_2 t_s} + \left[\frac{2c(K'-1)}{r_2} - K\right]\right\} \tag{32}$$

$$A_2 = \frac{e^{\lambda t_s}}{2\lambda}\left\{(\lambda-r_2)\left[B_2 e^{r_2 t_s} + \frac{cB_1}{r_2} e^{-r_2 t_s}\right] - 2cB_1 e^{-r_2 t_s} + \lambda\left[\frac{2c(K'-1)}{r_2} - K\right]\right\} \tag{33}$$

Substitution of equations (32) and (33) into equation (30) and (31) leaves two equations with two unknowns B_2 and t_s. These equations can be solved simultaneously using a search technique to determine these constants. Then the optimal policy and response are summarized as follows:

$$\theta = +1, \qquad 0 \leq t \leq t_s \tag{33a}$$

$$x_1 = B_1 e^{-r_2 t_s} + K', \qquad 0 \leq t \leq t_s \tag{33b}$$

$$\theta = \frac{r_1 K_\beta}{2b} [A_1 e^{\lambda t} + A_2 e^{-\lambda t} + K], \qquad t_s \leq t \leq T \tag{33c}$$

$$x_1 = \frac{1}{2c} [(r_2-\lambda)A_1 e^{\lambda t} + (r_2+\lambda)A_2 e^{-\lambda t} + r_2 K] + 1, \qquad t_s \leq t \leq T \tag{33d}$$

The following four cases are considered in this example:

case 1: b = 1, c = 0.1

case 2: b = 1, c = 1

case 3: b = 1, c = 10

case 4: b = 1, c = 100

The results of this example for T = 1 and T = 5 are shown in Figs. 1, 2, 3 and 4. The numerical values of the system constants with which the model was simulated are used in this example.

In case 1, where more weight is given to the control effort, θ, than to the state deviation, the optimal control has a very small positive value. In case 4, where more weight is given to the deviation of x_1 from the desired value, the optimal control θ has an approximately constant value that is required to maintain the system in the desired state, i.e., $x_1 = 1$. This value of θ for a given σ_s can be found from Fig. 5 of Chapter 2. Cases 2 and 3 are intermediate between the two extreme cases 1 and 4.

It should be noted that the control θ in all the cases drops down to zero at the end of control period T.

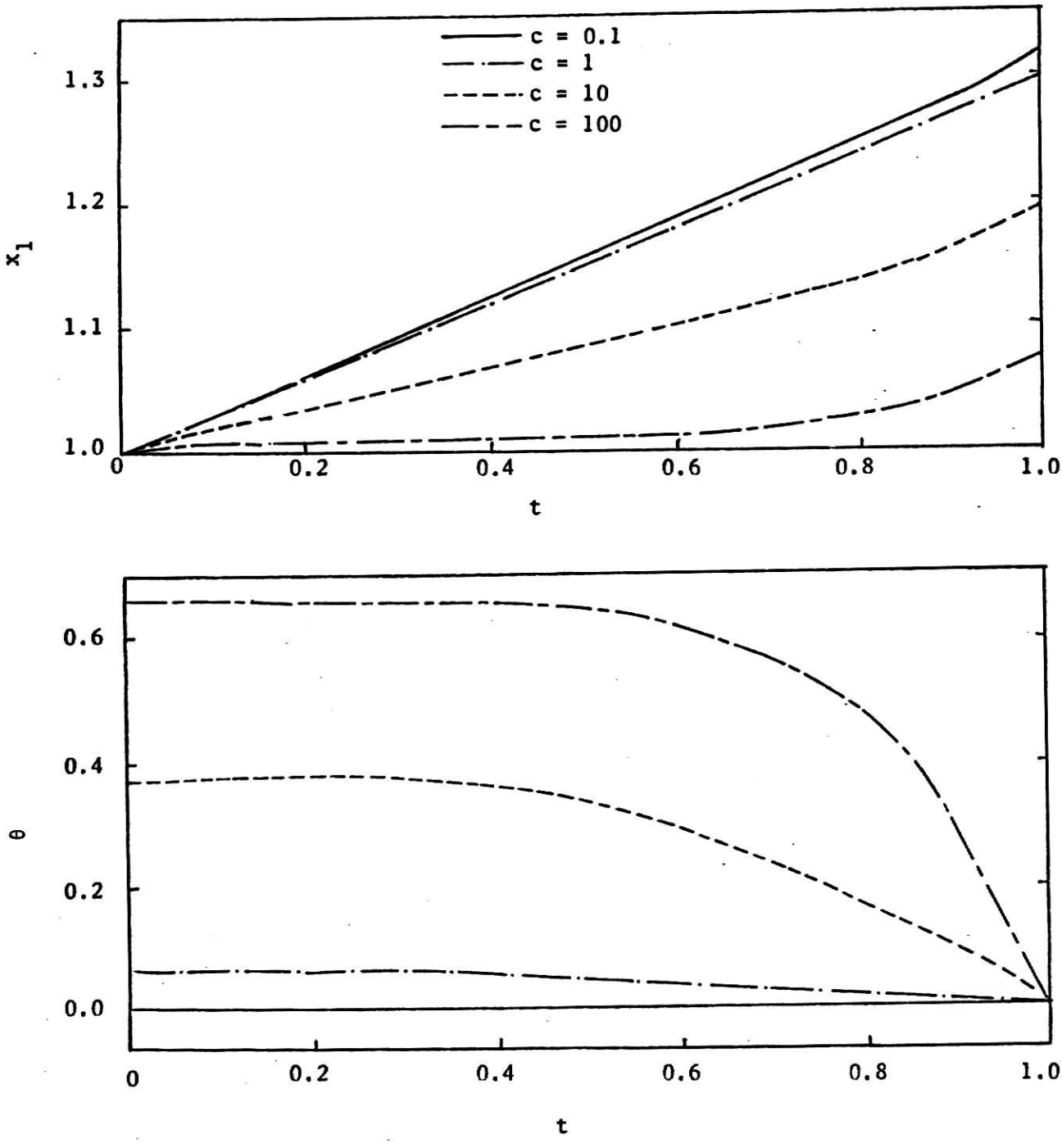


Fig. 1 Optimal control policy and system responses of the 1 CST model with $\tau_2 = 0$ (Ex. 1), $\sigma_s = 0.75$, $K_\beta = K_\gamma = 0.625$, $T = 1$

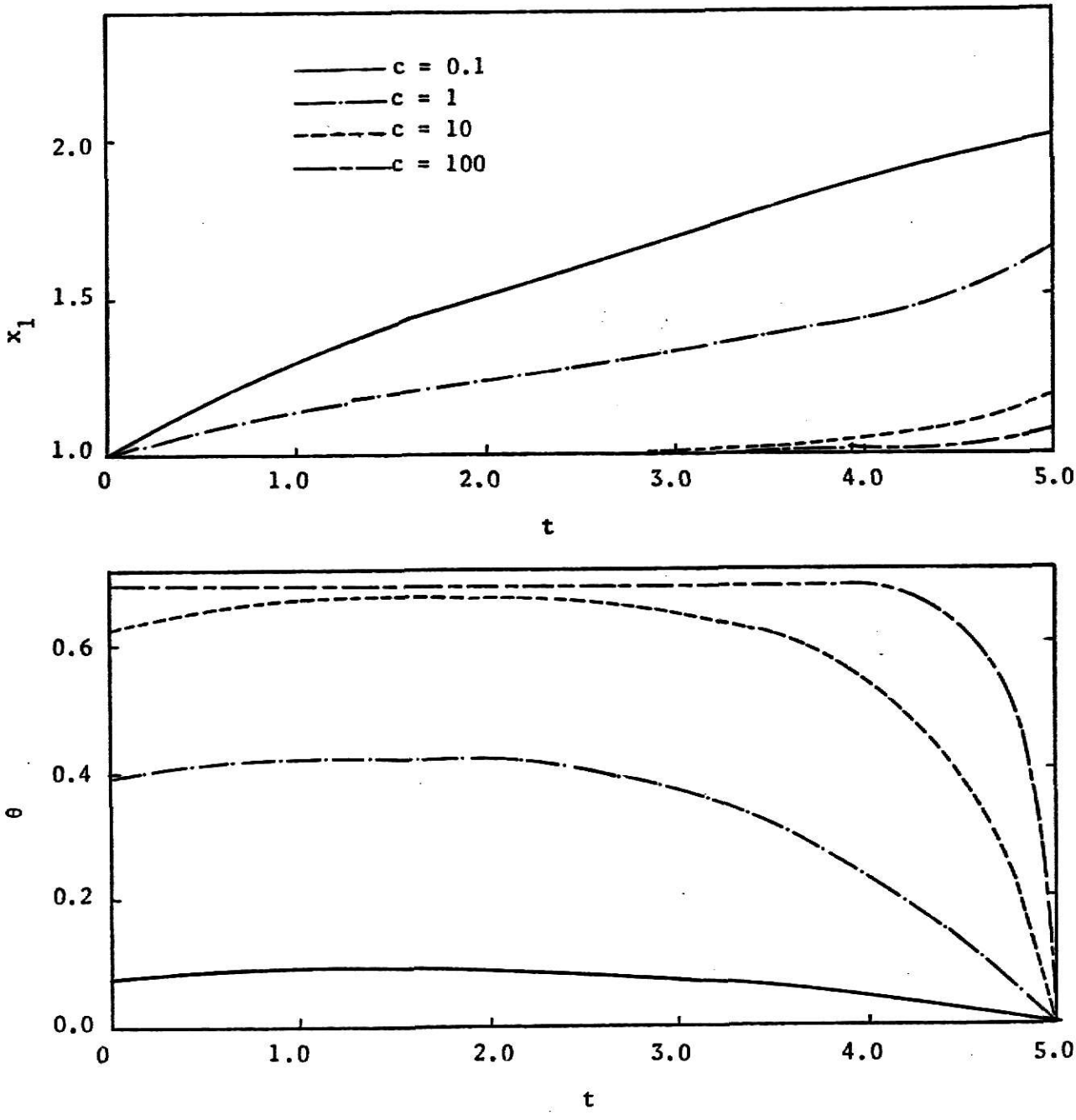


Fig. 2 Optimal control policy and system response of the one CST model with $\tau_2 = 0$ (Ex. 1), $\sigma_s = 0.75$, $K_\beta = K_\gamma = 0.625$, $T = 5$

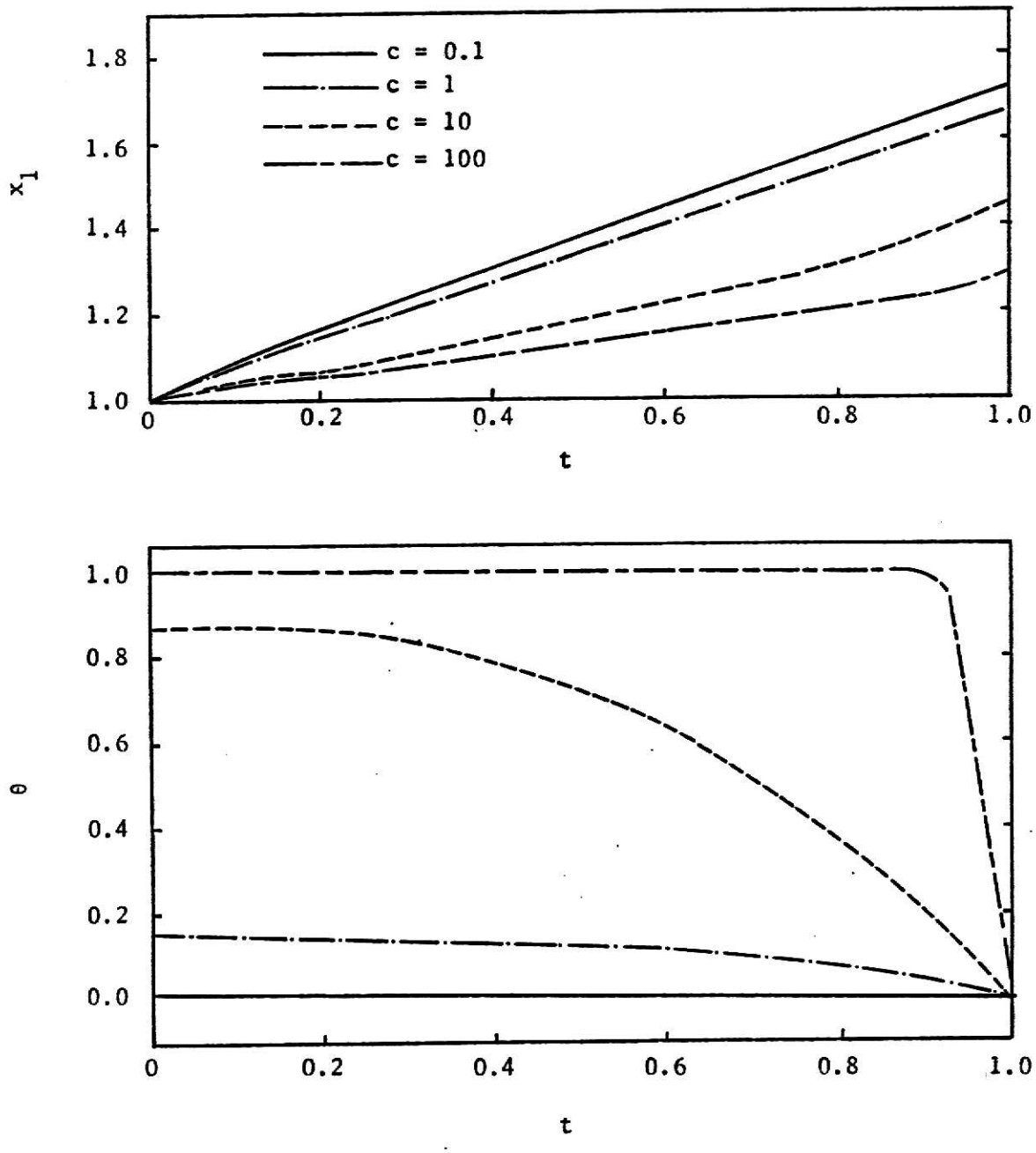


Fig. 3 Optimal control policy and system response of the one CST model with $\tau_2 = 0$ (Ex. 1), $\sigma_s = 1.2$, $K_\beta = K_\gamma = 0.625$, $T = 1$

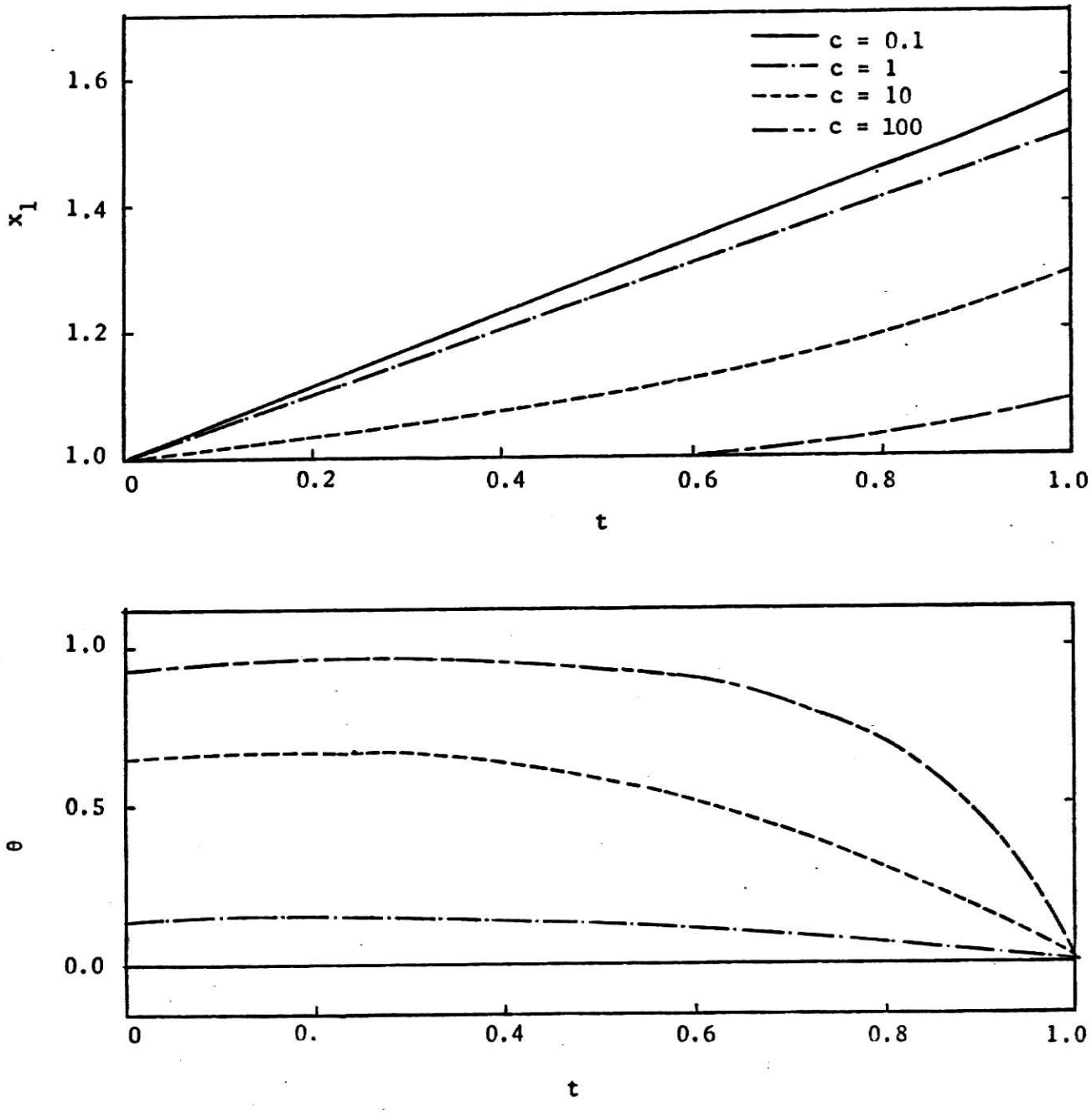


Fig. 4 Optimal control policy and system response of the one CST model with $\tau_2 = 0$ (Ex. 1), $\sigma_s = 1.2$, $K_\beta = K_\gamma = 0.825$, $T = 1$

In Fig. 3 for case 4 the optimal control policy corresponds to the optimal policy given by equations (33a) and (33c). The optimal control is at the upper limit, $\theta = +1$, in the period $(0, t_s)$. The time t_s at which the saturation period ends is 0.92. After the saturation period the control switches to the continuous control given by equation (33c) and drops steeply to zero at the end of the control period T.

3.4 EXAMPLE 2. Consider a life support system consisting of an air-conditioned room and a heat exchanger and subject to a step heat disturbance as in the first example, the time constant of the heat exchanger, however, being not negligible. The performance equations for such a system have been derived as [equations (13) and (23a) in Chapter 2]

$$\frac{dx_1}{dt} + x_1 = r_1 x_2 + r_2 K_\alpha + \sigma_s \tag{34}$$

$$\frac{dx_2}{dt} + r x_2 = r x_1 - r K_\beta \theta - r K_\gamma \tag{35}$$

with

$$x_1 = 1 \quad \text{at} \quad t = 0 \tag{36}$$

$$x_2 = \frac{1 - r_2 K_\alpha}{r} \quad \text{at} \quad t = 0$$

The objective function to be minimized is

$$S = \int_0^T [b\theta^2 + c(x_1 - 1)^2]dt$$

where T is the specified final control time. Introducing an additional state variable x_3 such that

$$x_3(t) = \int_0^t [b\theta^2 + c(x_1-1)^2]dt$$

it follows that

$$\frac{dx_3}{dt} = b\theta^2 + c(x_1-1)^2, \qquad x_3(0) = 0 \tag{37}$$

The problem is thus transformed into that of minimizing $x_3(T)$.

The Hamiltonian is

$$H[z(t), x(t), \theta(t)] = z_1 \frac{dx_1}{dt} + z_2 \frac{dx_2}{dt} + z_3 \frac{dx_3}{dt}$$

$$= z_1[-x_1 + r_1x_2 + r_2K_\alpha + \sigma_s] + z_2[-rx_2 + rx_1 - rK_\beta\theta - rK_\gamma]$$

$$+ z_3[b\theta^2 + c(x-1)^2] \tag{38}$$

The adjoint variables are defined as

$$\frac{dz_1}{dt} = -\frac{\partial H}{\partial x_1} = z_1 - rz_2 - 2cz_3(x_1-1) \tag{39}$$

$$\frac{dz_2}{dt} = -\frac{\partial H}{\partial x_2} = -r_1z_1 + rz_2 \tag{40}$$

$$\frac{dz_3}{dt} = -\frac{\partial H}{\partial x_3} = 0, \qquad z_3(T) = 1 \tag{41}$$

From equation (41) the solution of z_3 is obtained as

$$z_3(t) = 1, \qquad -\leq t \leq T \tag{42}$$

The Hamiltonian can now be rewritten as

$$H = z_1[-x_1 + r_1x_2 + r_2K_\alpha + \sigma_s] + z_2[-rx_2 + rx_1 - rK_\beta\theta - rK_\gamma]$$

$$+ b\theta^2 + c(x_1-1)^2 \tag{43}$$

The portion of H that depends on θ, H^*, is

$$H^* = -rK_\beta z_2\theta + b\theta^2 \tag{44}$$

Inspection of H^* shows that the optimum control should be of the continuous type. The optimum control is obtained by the necessary condition of optimality

$$\frac{\partial H}{\partial \theta} = \frac{dH^*}{d\theta} = 0 = -rK_\beta z_1 + 2b\theta$$

as

$$\theta = \frac{rK_\beta}{2b} z_2 \tag{45}$$

The boundary conditions for equations (39), and (40) are

$$z_1 = 0, \qquad z_2 = 0 \qquad \text{at} \qquad t = T \tag{46}$$

The Hamiltonian must remain constant at every point of its response under the optimal conditions. Substituting equation (45) into equation (35) and eliminating x_2 from equations (34), and (35) gives

$$\frac{d^2x_1}{dt^2} + (r+1)\frac{dx_1}{dt} + (r-r_1r)x_1 = rr_2K_\alpha + r\sigma_s$$

$$- rr_1K_\gamma - \frac{r_1(rK_\beta)^2}{2b} z_2 \tag{47}$$

Now eliminating z_1 from equations (39) and (40) gives rise to

$$z_2 = \frac{2cr_1(x_1-1)}{D^2 - (r+1)D + (r-r_1r)} \tag{48}$$

Inserting equation (48) into equation (47) gives

$$\frac{d^4x_1}{dt^4} - (2r_1r + r^2 + 1)\frac{d^2x_1}{dt^2} + r^2\left(r_2^2 + \frac{(r_1K_\beta)^2c}{b}\right)$$

$$= (r_2r)^2 K_\alpha + r^2r_2\sigma_s - r^2r_1r_2K_\gamma + \frac{(r^2r_1^2K_\beta^2)c}{b}$$

The solution of this equation is

$$x_1(t) = A_1e^{\lambda_1 t} + A_2e^{\lambda_2 t} + B_1e^{-\lambda_1 t} + B_2e^{-\lambda_2 t} + K \tag{49}$$

where λ_1, λ_2, $-\lambda_1$, $-\lambda_2$ are roots of the following characteristic equation

$$\lambda^4 - (2r_1r + r^2 + 1)\lambda^2 + r^2\left(r_2^2 + \frac{r_1^2K_\beta^2c}{b}\right) = 0$$

A_1, A_2, B_1, and B_2 are constants of integration. The particular solution, K, is

$$K = \frac{r_2^2K_\alpha + r_2\sigma_s - r_1r_2K_\gamma + \frac{r_1^2K_\beta^2c}{b}}{r_2^2 + \frac{r_1^2K_\beta^2c}{b}}$$

The solution of x_2 can be obtained from equation (34) as

$$x_2 = \frac{1}{r_1}\,[1+\lambda_1)A_1e^{\lambda_1 t} + (1+\lambda_2)A_2e^{\lambda_2 t} + (1-\lambda_1)B_1e^{-\lambda_1 t}$$

$$+ (1-\lambda_2)B_2e^{-\lambda_2 t} + K - r_2K_\alpha - \sigma_s] \tag{50}$$

From equations (30), (31), (36) and (26) the solution of z_2 becomes

$$z_2 = -\frac{2b}{(rK_\beta)^2 r_1}\left\{[\lambda_1^2 + (r+1)\lambda_1 + r_2r]A_1e^{\lambda 1 t} + [\lambda_2^2 + (r+1) + r_2r]\,A_2e^{\lambda_2 t}\right.$$

$$+ [\lambda_1^2 - (r+1)\lambda_1 + r_2r]B_1e^{-\lambda_1 t} + [\lambda_2^2 - (r+1)\lambda_2 + r_2r]\,B_2e^{-\lambda_2 t}$$

$$\left. - rr_2K_\alpha - r\sigma_s + rr_1K_\gamma + rr_2K\right\} \tag{51}$$

And now from equations (51) and (40)

$$z_1 = \frac{2b}{(r_1rK_\beta)^2}\left\{(\lambda_1^3 + \lambda_1^2 - r\lambda_1(r_1 + r) - r^2r_2]A_1e^{\lambda_1 t}\right.$$

$$+ [\lambda_2^3 + \lambda_2^2 - r\lambda_2(r_1 + r) - r^2r_2]A_2e^{\lambda_2 t}$$

$$+ [-\lambda_1^3 + \lambda_1^2 + r\lambda_1(r_1 + r) - r^2r_2]B_1e^{-\lambda_1 t} + [-\lambda_2^3 + \lambda_2^2 + r\lambda_2\,(r_1 + r) - r^2r_2$$

$$\left. B_2e^{-\lambda_2 t} + r^2r_2K_\alpha + r^2\sigma_s - r^2r_1K_\gamma] + r^2r_2K\right\} \tag{52}$$

Now using the boundary conditions, equation (36), equations (49), (50, (51) and (52) give

$$A_1 + A_2 + B_1 + B_2 + K = 1 \tag{53}$$

$$(1+\lambda_1)A_1 + (1+\lambda_2)A_2 + (1-\lambda_1)B_1 + (1-\lambda_2)B_2 + K = 1 + \sigma_s \tag{54}$$

$$[\lambda_1^2 + (r+1)\lambda_1 + r_2 r]A_1 e^{\lambda_1 T} + [\lambda_2^2 + (r+1)\lambda_2 + r_2 r]A_2 e^{\lambda_2 T} + [\lambda_2^2 - (r+1)\lambda_1 + r_2 r]B_1 e^{-\lambda_1 T} + [\lambda_2^2 - (r+1)\lambda_2 + r_2 r]B_2 e^{-\lambda_2 T} - r r_2 K_\alpha - r\sigma_s + r r_1 K_\gamma + r r_2 K = 0 \tag{55}$$

$$[\lambda_1^3 + \lambda_1^2 - r\lambda_1(r_1+r) - r^2 r_2]A_1 e^{\lambda_1 T} + [\lambda_2^3 + \lambda_2^2 - r\lambda_2(r_1 + r) - r^2 r_2]A_2 e^{\lambda_2 T} + [-\lambda_1^3 + \lambda_1^2 + r\lambda_1(r_1+r) - r^2 r_2]B_1 e^{-\lambda_1 T} + [-\lambda_2^3 + \lambda_2^2 + r\lambda_2(r_1+r) - r^2 r_2]B_2 e^{-\lambda_2 T} + r^2 r_2 K_\alpha + r^2\sigma_s - r^2 r_1 K_\gamma + r^2 r_2 K = 0 \tag{56}$$

Subtracting equation (53) from equation (54) and r times equation (55) from equation (56) gives

$$\lambda_1 A_1 + \lambda_2 A_2 - \lambda_1 B_1 - \lambda_2 B_2 = \sigma_s \tag{57}$$

$$\lambda_1[\lambda_1^2 + (r+1)\lambda_1 + r_2 r]A_1 e^{\lambda_1 T} + \lambda_2[\lambda_2^2 + (r+1)\lambda_2 + r_2 r]A_2 e^{\lambda_2 T} - \lambda_1[\lambda_1^2 - (r+1)\lambda_1 + r_2 r]B_1 e^{-\lambda_1 T} - \lambda_2[\lambda_2^2 - (r+1)\lambda_2 + r_2 r]B_2 e^{\lambda_2 T} = 0 \tag{58}$$

The set of equations (53), (55), (57), and (58) can be solved simultaneously to determine the constants A_1, A_2, B_1 and B_2. The optimal policy θ and system responses x_1 and x_2 can then be determined from equations (45), (51) and equations (49) and (50) respectively. The solutions for three different values of r are shown in Figs. 5, 6, and 7. The results for x_1 and θ are similar to the results of example 1. The response x_2 drops steeply downwards from the initial state to t = 0.1 and then increases slowly and approximately linearly to the final time t = T in Figs. 6 and 7. When the time constant of the heat exchanger becomes negligible as compared to the time constant of the system proper, i.e, for large values of r, the results of this example approach the results of example 1. Thus, in Fig. 17 the results of x_1 and θ are approximately the same as that of Fig. 1.

3.4 EXAMPLE 3: Consider a life support system consisting of an air-conditioned room and a heat exchanger as in the previous two examples, except that the flow of air in the room is characterized by the two CST's-in series model. The performance equations for such a system have been derived [equations (39), (40) and (41)] of Chapter 2. Now assuming that the heat exchanger has negligibly small time constant, the performance equations become:

$$\frac{dx_{11}}{dt} + r_{11}x_{11} = a_{11}x_{12} - a_{11}K_{\beta}\theta - a_{11}K_{\gamma} + a_{12} + \sigma_s U_0(t) \qquad (59)$$

$$\frac{dx_{12}}{dt} + r_{12}x_{12} = r_{12}x_{11} + \sigma_s U_0(t) \qquad (60)$$

with the initial conditions

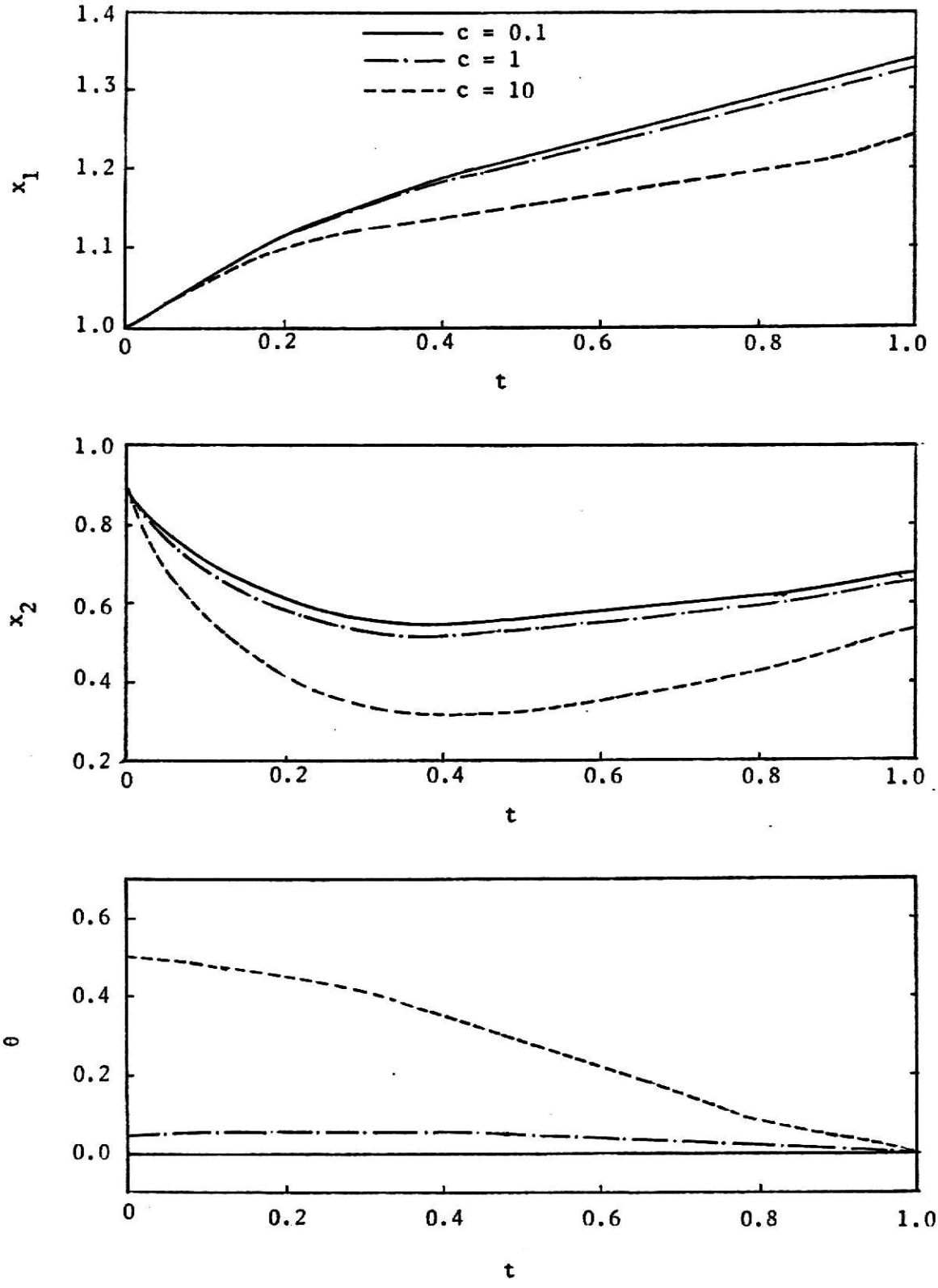


Fig. 5 Optimal control policy and system responses of the one CST model (Ex. 2), r = 5

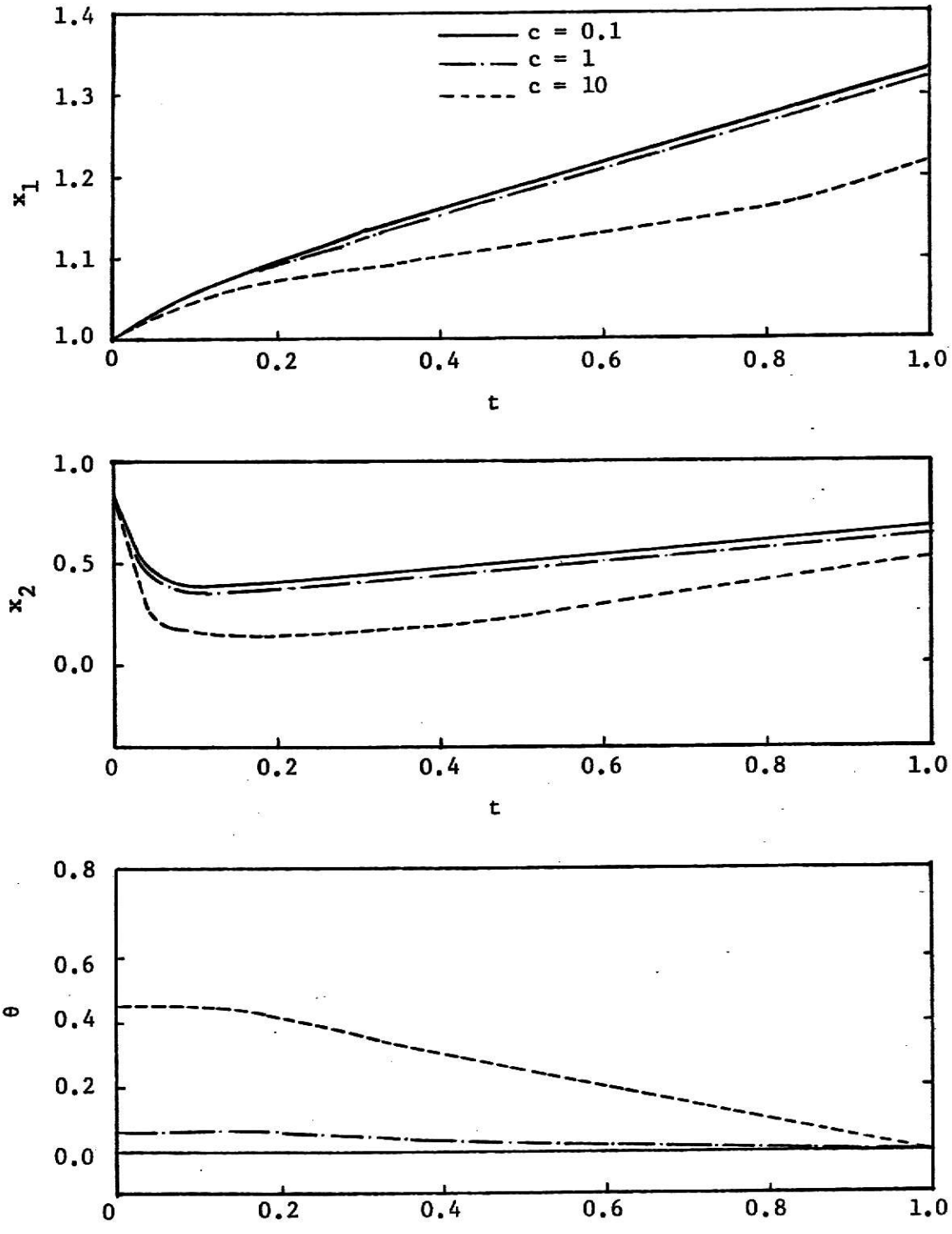


Fig. 6 Optimal control policy and system responses of the one CST model (Ex. 2), r = 10

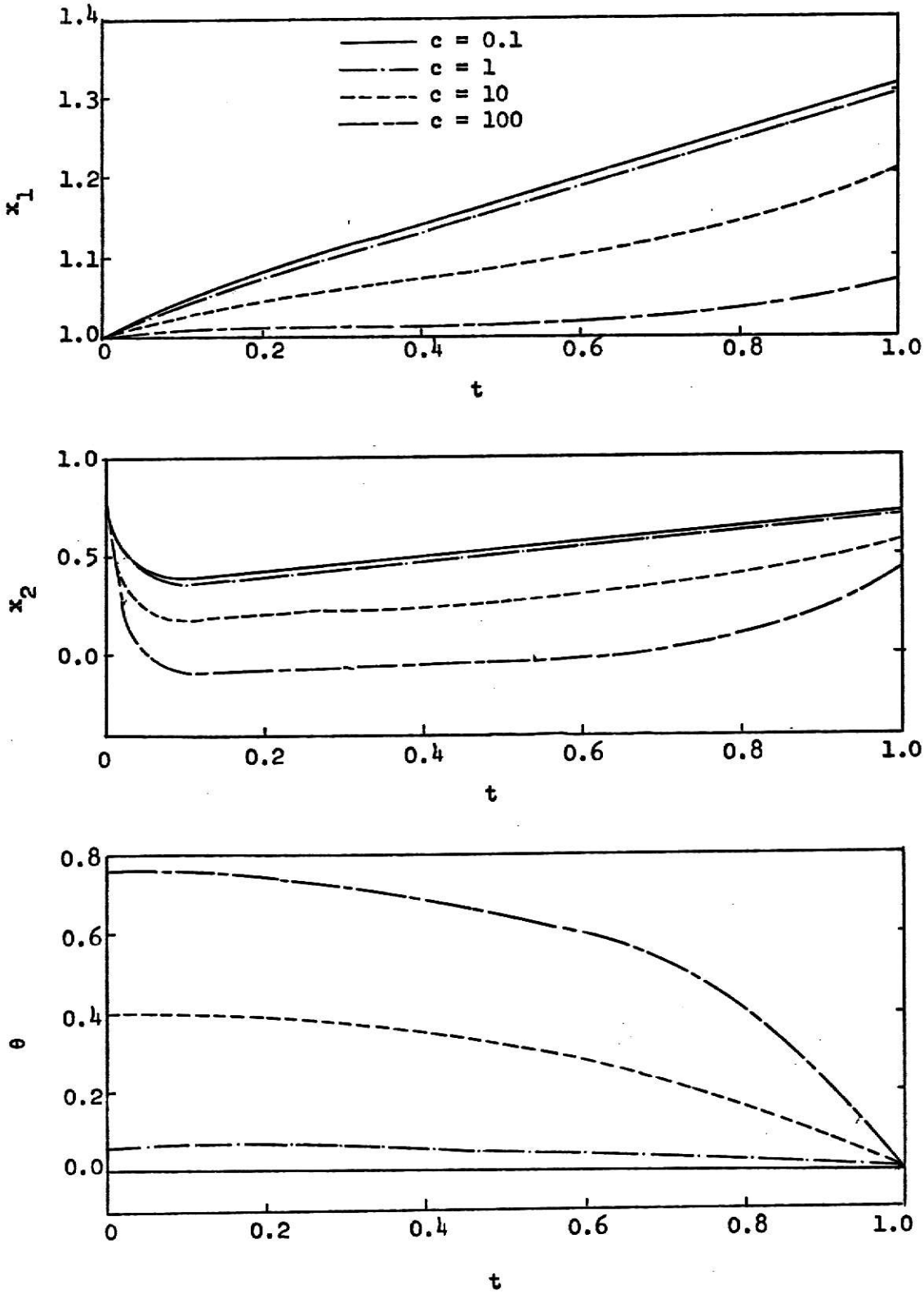


Fig. 7 Optimal control policy and system responses of the one CST model (Ex. 2), r = 50

$$x_{11} = x_{12} = 1 \quad \text{at} \quad t = 0 \tag{61}$$

The problem is to minimize

$$S = \int_0^T [b\theta^2 + c(x_{11}-1)^2 + c(x_{12}-1)^2]dt$$

where T is the specified final control time and b and c are constants.

If an additional state variable is introduced as

$$x_3(t) = \int_0^t [b\theta^2 + c(x_{11}-1)^2 + c(x_{12}-1)^2]dt$$

it follows that

$$\frac{dx_3}{dt} = b\theta^2 + c(x_{11}-1)^2 + c(x_{12}-1)^2, \quad x_3(0) = 0 \tag{62}$$

and

$$S = \int_0^T [b\theta^2 + c(x_{11}-1)^2 + c(x_{12}-1)^2]dt = x_3(T)$$

The problem is thus transformed into that of minimizing $x_3(T)$.

The Hamiltonian is

$$H[z,x,\theta] = z_{11}\frac{dx_{11}}{dt} + z_{12}\frac{dx_{12}}{dt} + z_{13}\frac{dx_3}{dt}$$

$$= z_{11}[-r_{11}x_{11} + a_{11}x_{12} - a_{11}K_\beta\theta - a_{11}K_\gamma + a_{12} + \sigma_s]$$

$$+ z_{12}[-r_{12}x_{12} + r_{12}x_{11} + \sigma_s] + z_3[b\theta^2 + c(x_{11}-1)^2 + c(x_{12}-1)^2] \tag{63}$$

The adjoint variables are defined by the following differential equations.

$$\frac{dz_{11}}{dt} = -\frac{\partial H}{\partial x_{11}} = r_{11}z_{11} - r_{12}z_{12} - 2z_3c(x_{11}-1), \quad z_{11}(T) = 0 \tag{64}$$

$$z_{11} = 0$$

$$\frac{dz_{12}}{dt} = -\frac{\partial H}{\partial x_{12}} = r_{12}\, z_{11} - a_{11}\, z_{11} - 2z_3 c(x_{12}-1), \quad z_{12}(T) = 0 \tag{65}$$

$$z_{12} = 0$$

$$\frac{dz_3}{dt} = -\frac{H}{x_3} = 0, \quad z_3(T) = 1 \tag{66}$$

The solution of equation (66) is

$$z_3(t) = 1, \qquad 0 \leq t \leq T \tag{67}$$

The Hamiltonian can be rewritten as

$$H = z_{11}[-r_{11}x_{11} + a_{11}x_{12} - a_{11}K_\beta\theta - a_{11}K_\gamma + a_{12} + a_{12} + \sigma_s]$$

$$+ z_{12}[-r_{12}x_{12} + r_{12}x_{11} + \sigma_s] + b\theta^2 + c(x_{11}-1)^2 + c(x_{12}-1)^2 \tag{68}$$

The portion of H that depends on θ, H^*, is

$$H^* = -a_{11}K_\beta z_{11}\theta + b\theta^2 \tag{69}$$

An inspection of H^* shows that the optimum control should be of the continuous type and is found from the condition of optimality:

$$\frac{\partial H}{\partial \theta} = \frac{dH^*}{d\theta} = 0 = -a_{11}K_\beta z_{11} + 2b\theta$$

as

$$\theta = \frac{a_{11}K_\beta}{2b}\, z_{11} \tag{70}$$

The maximum principle requires that the system equations and the adjoint variables, equations (59), (60), (64) and (65) are integrated in such a manner that the two-point boundary conditions

$$x_{11}(0) = x_{12}(0) = 1$$

$$x_{11}(T) = z_{12}(T) = 0$$

are satisfied. Meanwhile, the Hamiltonian must remain constant at every point of its response under the optimal condition. Thus eliminating z_{12} from equations (64) and (65) gives

$$[D^2 - (r_{11} + r_{12})\, D + (r_{11}r_{12} = a_{11}r_{12})]z_{11}$$

$$= 2cr_{12}(x_{12}-1) - 2c(D-r_{12})(x_{11}-1) \tag{71}$$

From equation (60)

$$x_{11} = \frac{1}{r_{12}} [(D+r_{12})x_{12}-\sigma_s] \tag{72}$$

Substituting equation (72) into equation (64) and eliminating z_{12} from equations (64) and (65) gives

$$z_{11} = \frac{2c\{(-\frac{D^2}{r_{12}} - 2r_{12})x_{12} - 2r_{12} - \sigma_s\}}{D^2 - (r_{11}+r_{12})D + (r_{11}r_{12}-a_{11}r_{12})} \tag{73}$$

Also inserting equations (72) and (73) into equation (71) gives

$$[D^2 + (r_{11}+r_{12})D + (r_{11}r_{12}-a_{11}r_{12})]x_{12}$$

$$= -a_{11}r_{12}K_\gamma + a_{12}r_{12} + \sigma_s(r_{12}+r_{11}) + \frac{(a_{11}K_\beta)^2 c}{b}$$

$$\frac{[(D^2 - 2r_{12}^2)x_{12} + 2r_{12}^2 + \sigma_s r_{12}]}{D^2 = (r_{11}+r_{12})D + (r_{11}r_{12}-a_{11}r_{12})}$$

or

$$\left\{D^4 + [-(r_{11}+r_{12})^2 + 2(r_{11}r_{12} - a_{11}r_{12}) - \frac{c}{b}(a_{11}K_\beta)^2]D^2\right.$$

$$\left. + [(r_{11}r_{12} - a_{11}r_{12})^2 + 2r_{12}^2 \frac{c}{b}(a_{11}K_\beta)^2]\right\}x_{12}$$

$$= (r_{11}r_{12} - a_{11}r_{12})[-a_{11}r_{12}K_\gamma + a_{12}r_{12} + \sigma_s(r_{11}+r_{12})]$$

$$+ \frac{(a_{11}K_\beta)^2 c}{b}\{2r_{12}^2 + \sigma_s r_{12}\} \qquad (74)$$

From equation (74) the solution of x_{12} can be written as

$$x_{12} = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + B_1 e^{-\lambda_1 t} + B_2 e^{-\lambda_2 t} + K \qquad (75)$$

where A_1, A_2, B_1, and B_2 are integration constants, and λ_1, λ_2, $-\lambda_1$ and $-\lambda_2$ are roots of the following characteristic equation

$$\lambda^4 - [(r_{11} + r_{12})^2 + \frac{c}{b}(a_{11}K_\beta)^2 - 2(r_{11}r_{12} - a_{11}r_{12})]\lambda^2$$

$$+ [(r_{11}r_{12} - a_{11}r_{12})^2 + 2r_{12}^2 \frac{c}{b}(a_{11}K_\beta)^2] = 0$$

The particular solution, K, is given by

$$K = \frac{(r_{11}r_{12}-a_{11}r_{12})[-a_{11}r_{12}K_\gamma + a_{12}r_{12} + \sigma_s(r_{11}+r_{12})] + (a_{11}K_\beta)^2 \frac{c}{b}[2r_{12}^2 + \sigma_s r_{12}]}{(r_{11}r_{12} - a_{11}r_{12})^2 + 2r_{12}^2 \frac{c}{b}(a_{11}K_\beta)^2}$$

The solution of x_{11} can now be written from equation (60) as

$$x_{11} = \frac{1}{r_{12}} [(r_{12} + \lambda_1)A_1 e^{\lambda_1 t} + (r_{12} + \lambda_2)A_2 e^{\lambda_2 t}$$

$$+ (r_{12}-\lambda_1)B_1 e^{\lambda_1 t} + (r_{12}-\lambda_2)B_2 e^{\lambda_2 t} + r_{12}K - \sigma_s] \quad (76)$$

Substituting equations (75) and (76) into equation (59) the solution of z_{11} can be written as

$$z_{11} = \frac{2b}{(a_{11}K_\beta)^2 r_{12}} \{[a_{11}r_{12} - (r_{12}+\lambda_1)(r_{11}+\lambda_1)]A_1 e^{\lambda_1 t}$$

$$+ [a_{11}r_{12} = (r_{12}+\lambda_2)(r_{11}+\lambda_2)]A_2 e^{\lambda_2 t} + [a_{11}r_{12} -$$

$$(r_{12}-\lambda_1)(r_{11}-\lambda_1)]B_1 e^{\lambda_1 t} + [a_{11}r_{12} - (r_{12}-\lambda_2)(r_{11}-\lambda_2)]B_2 e^{\lambda_2 t}$$

$$+ (r_{11}+r_{12})\sigma_s + a_{12}r_{12} = a_{11}r_{12}K_\gamma - (r_{11}r_{12}-r_{12}a_{11})K\} \quad (77)$$

And from equation (64) the solution of z_{12} can be written as

$$z_{12} = \left\{\frac{2b(r_{11}-\lambda_1)}{(a_{11}K_\beta r_{12})^2} [a_{11}r_{12} - (r_{12}+\lambda_1)(r_{11}+\lambda_1)] - \frac{2c}{r_{12}^2}(r_{12}+\lambda_1)\right\}$$

$$A_1 e^{\lambda_1 t} + \left\{\frac{2b(r_{11}-\lambda_2)}{(a_{11}K_\beta r_{12})^2} [a_{11}r_{12} - (r_{12}+\lambda_2)(r_{11}+\lambda_2)]\right.$$

$$\left. - \frac{2c}{r_{12}^2}(r_{12}+\lambda_2)\right\} A_2 e^{\lambda_2 t} + \frac{2b(r_{11}+\lambda_1)}{(a_{11}K_\beta r_{12})^2} [a_{11}r_{12} - (r_{12} - \lambda_1)(r_{11}-\lambda_1)]$$

$$\left. - \frac{2c}{r_{12}^2}(r_{12}-\lambda_1)\right\} B_1 e^{\lambda_1 t} + \left\{\frac{2b(r_{11}+\lambda_2)}{(a_{11}K_\beta r_{12})^2} [a_{11}r_{12} - (r_{12}-\lambda_2)(r_{11}-\lambda_2)]\right.$$

$$- \frac{2c}{r_{12}^2}(r_{12}-\lambda_2)\Big\} B_2 e^{\lambda_2 t} + \frac{r_{11}}{r_{12}}(r_{11}=r_{12})\sigma_s + a_{12}r_{11} - a_{11}r_{11}K_\gamma$$

$$- (r_{11}^2 - a_{11}r_{11})K - \frac{2c}{r_{12}^2}[K - 1 - \frac{\sigma_s}{r_{12}}] \tag{78}$$

The constants A_1, A_2, B_1, and B_2 in equations (75) through (78) can be determined by applying the boundary conditions. Thus, at t = 0, from equations (75) and (76)

$$A_1 + A_2 + B_1 + B_2 + K = 1 \tag{79}$$

$$(r_{12}+\lambda_1)A_1 + (r_{12}+\lambda_2)A_2 + (r_{12}-\lambda_1)B_1 + (r_{12}-\lambda_2)B_2 + r_{12}K$$

$$= r_{12} \tag{80}$$

or subtracting r_{12} times equation (79) from equation (80)

$$\lambda_1 A_1 + \lambda_2 A_2 - \lambda_1 B_1 - \lambda_2 B_2 = 0 \tag{81}$$

and at t = T from equations (77) and (78)

$$[a_{11}r_{12} - (r_{12}+\lambda_1)(r_{11}+\lambda_1)]A_1 e^{\lambda_1 T} + [a_{11}r_{12} - (r_{12}+\lambda_2)(r_{11}+\lambda_2)]$$

$$A_2 e^{\lambda_2 T} + [a_{11}r_{11}-(r_{12}-\lambda_1)(r_{11}-\lambda_1)]B_1 e^{-\lambda_1 T} + [a_{11}r_{12}-$$

$$(r_{12}-\lambda_2)(r_{11}-\lambda_2)]B_2 e^{-\lambda_2 T} + (r_{11}+r_{12})\sigma_s + a_{12}r_{12} - a_{11}r_{12}K_\gamma$$

$$- (r_{11}r_{12} - r_{12}a_{11})K = 0 \tag{82}$$

$$\left\{\frac{2b(r_{11}-\lambda_1)}{(a_{11}K_\beta r_{12})^2}\,[a_{11}r_{12} - (r_{12}+\lambda_1)[(r_{11}+\lambda_1)] - \frac{2c}{r_{12}^2}\,(r_{12}+\lambda_1)\right\}$$

$$A_1 e^{\lambda_1 T} + \left\{\frac{2b(r_{11}-\lambda_2)}{(a_{11}K_\beta r_{12})^2}\,[a_{11}r_{12} - (r_{12}+\lambda_2)(r_{11}+\lambda_2)] -\right.$$

$$\left.\frac{2c}{r_{12}^2}\,(r_{12}+\lambda_2)\right\} A_2 e^{\lambda_2 T} + \frac{2b(r_{11}+\lambda_1)}{(a_{11}K_\beta r_{12})^2}\,[a_{11}r_{12} - (r_{12}-\lambda_1)$$

$$(r_{11}-\lambda_1)] - \frac{2c}{r_{12}^2}\,(r_{12}-\lambda_1)\Big\} B_1 e^{\lambda_1 T} + \frac{2b(r_{11}+\lambda_2)}{(a_{11}K_\beta r_{12})^2}\,[a_{11}r_{12}$$

$$-(r_{12}-\lambda_2)(r_{11}-\lambda_2)] - \frac{2c}{r_{12}^2}\,(r_{12}-\lambda_2)\,B_2 e^{\lambda_2 T} + \frac{r_{11}}{r_{12}}\,(r_{12}+r_{11})\sigma_s$$

$$+ a_{12}r_{11} - a_{11}r_{11}K_\gamma - \left(r_{11}^2 - a_{11}r_{11} + \frac{2c}{r_{12}}\right) K + \frac{2c}{r_{12}}\left(1 + \frac{\sigma_s}{r_{12}}\right) = 0 \tag{83}$$

With the values of A_1, A_2, B_1, and B_2 determined, the optimal policy can be established from equations (70) and equation (77). The system responses x_{11} and x_{12} at any time, $0 \leq t \leq T$, can be found out from equations (75) and (76) respectively.

The results of this example are shown in Figs. 8, 9, 10 and 11. Again four cases are considered

case 1: $b = 1$, $c = 0.1$

case 2: $b = 1$, $c = 1$

case 3: $b = 1$, $c = 10$

case 4: $b = 1$, $c = 100$

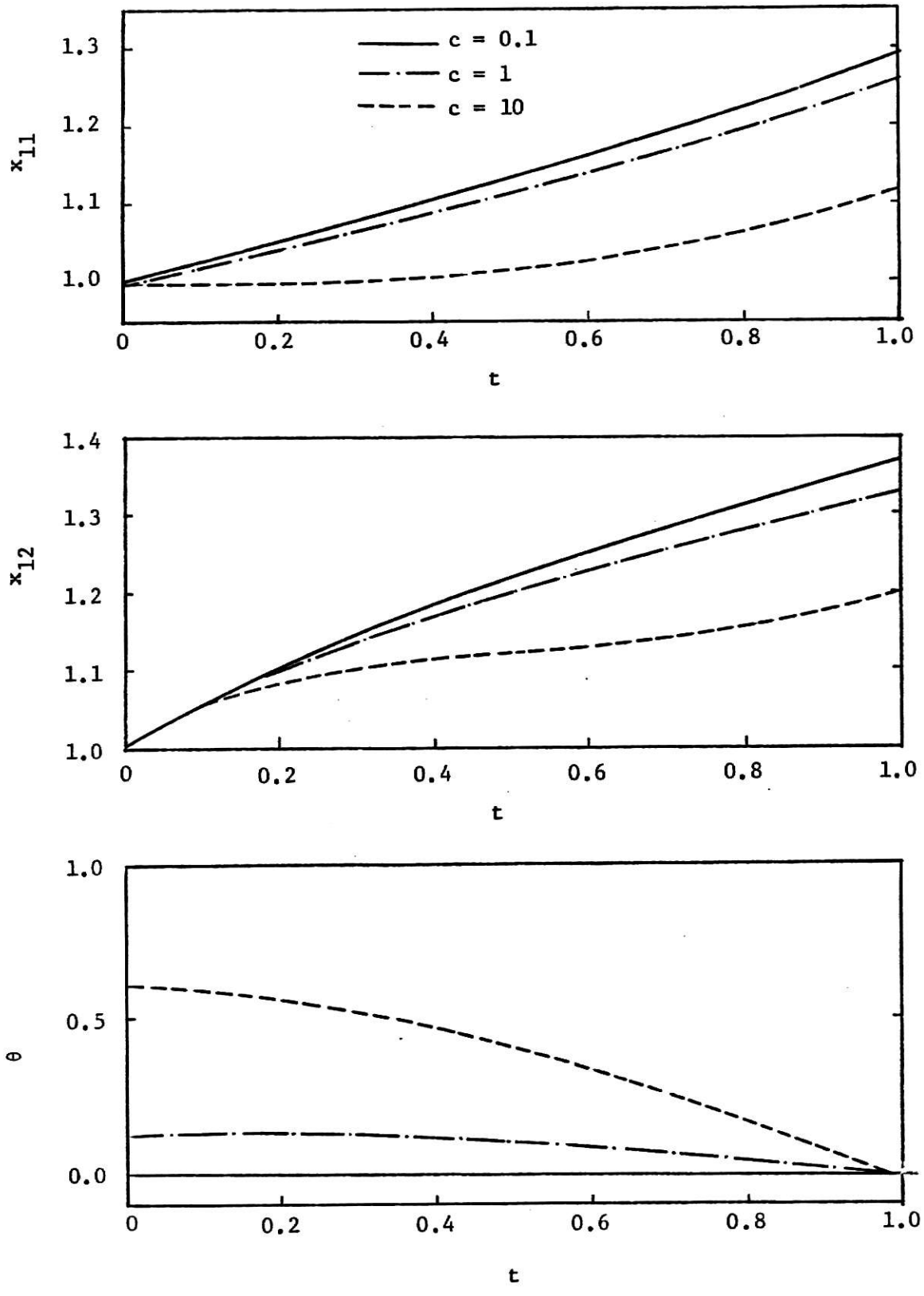


Fig. 8 Optimal control policy and system responses of the 2 CST's-in-series model (Ex. 3), $r_{11} = 1.2$

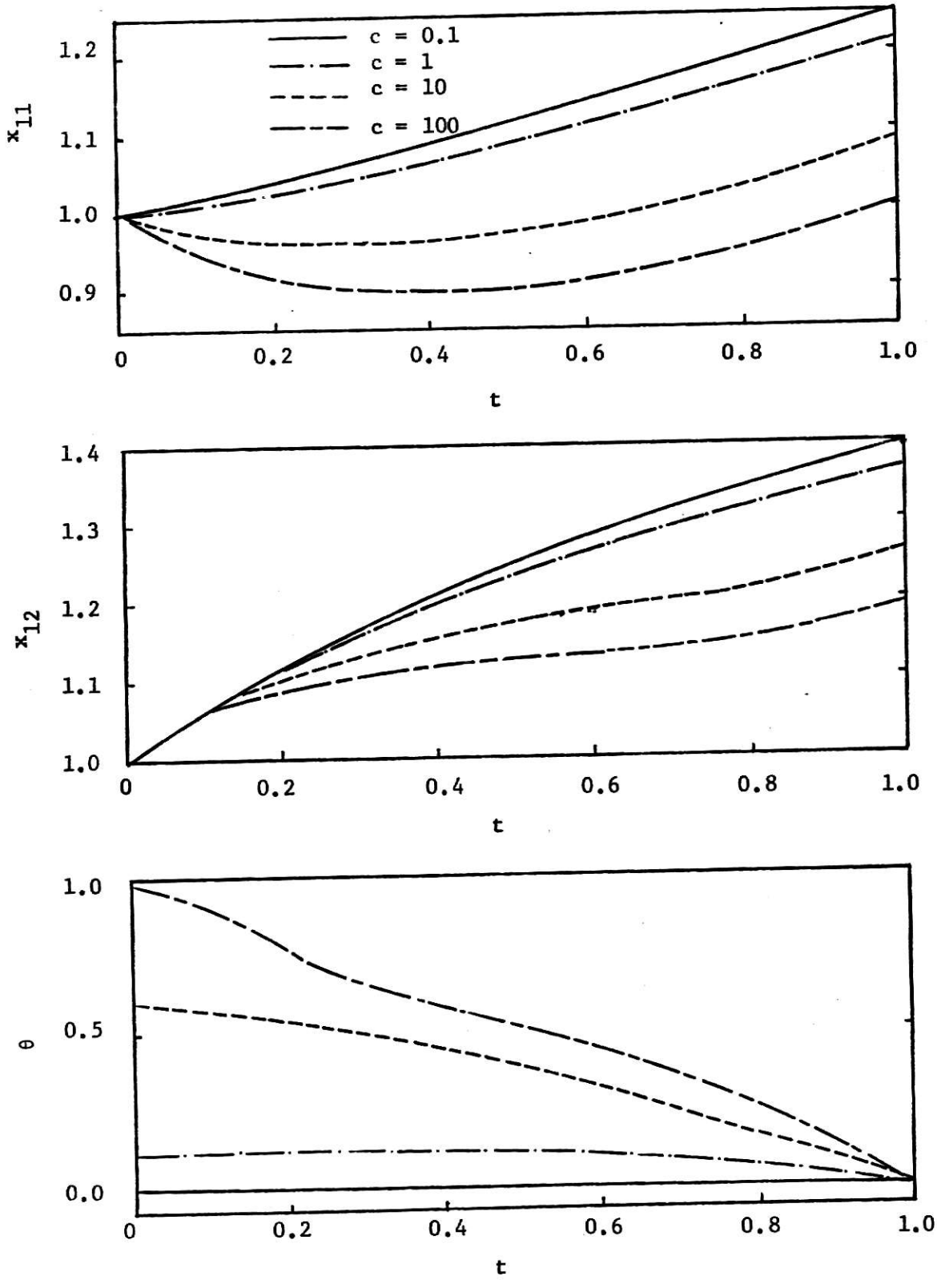


Fig. 9 Optimal control policy and system responses of the 2 CST's-in-series model (Ex. 3), $r_{11} = 1.5$

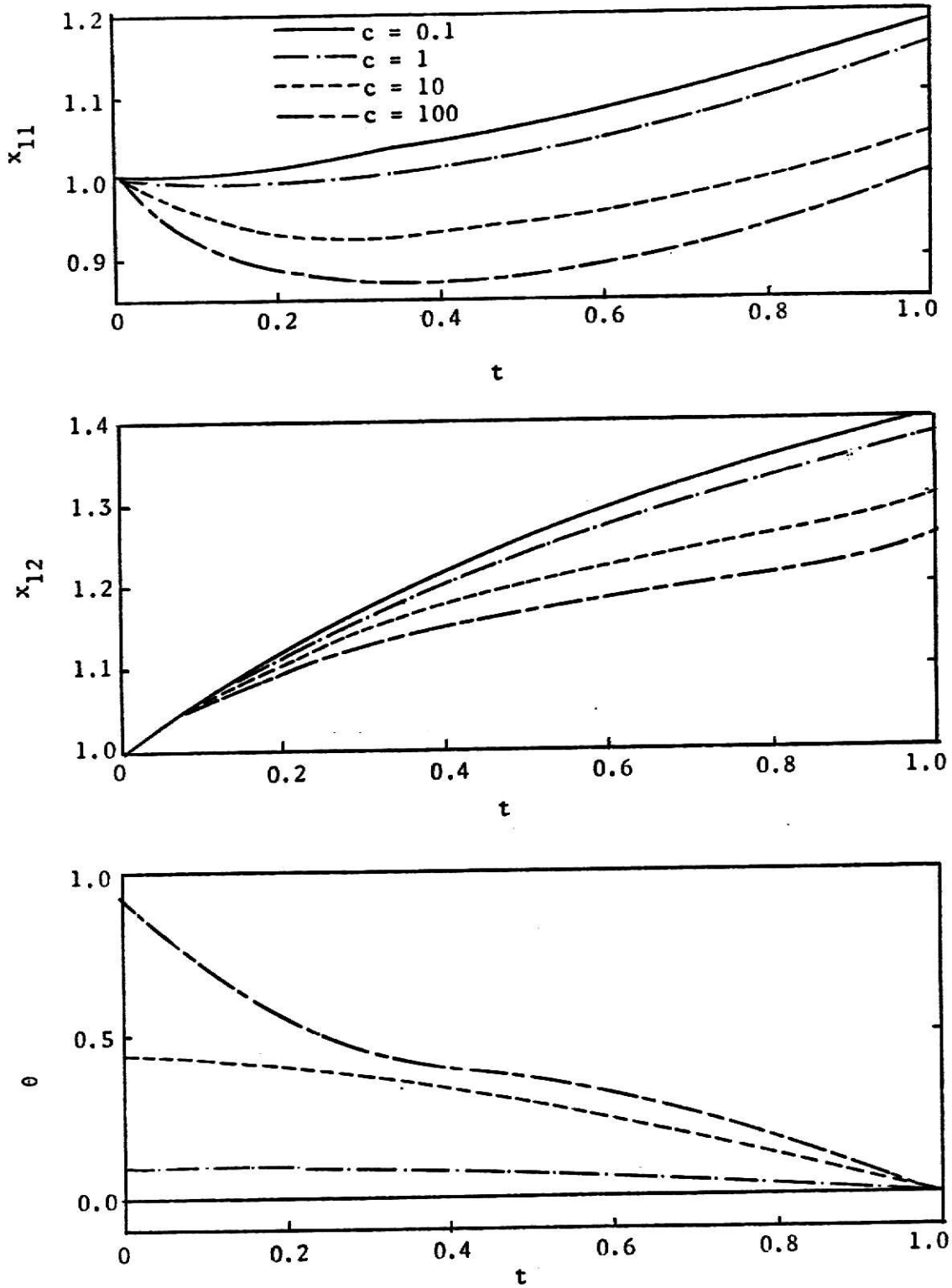


Fig. 10 Optimal control policy and system responses of the 2 CST's-in-series model (Ex. 3), $r_{11} = 2.0$

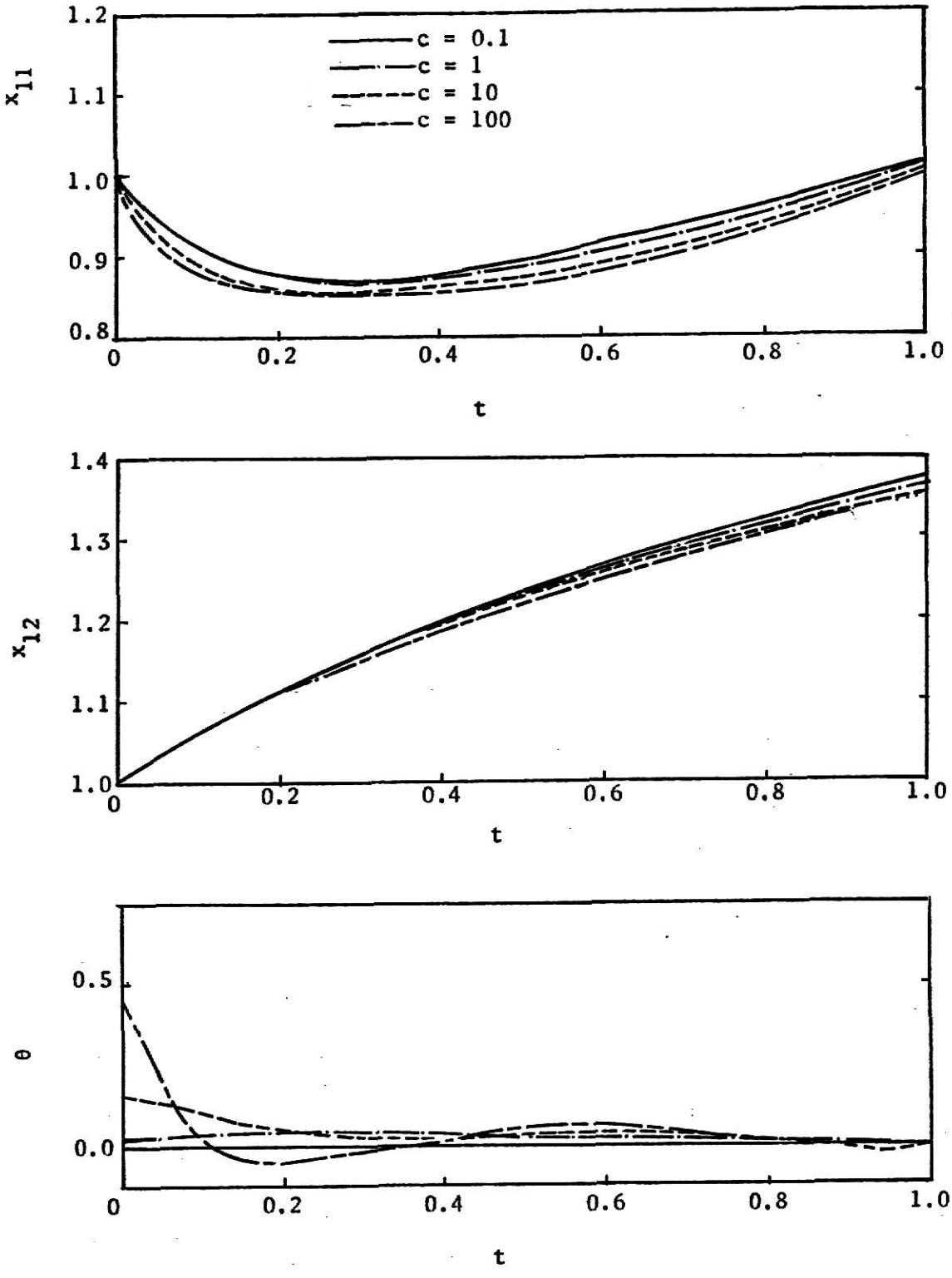


Fig. 11 Optimal control policy and system responses of the 2 CST's-in-series model (Ex. 3), $r_{11} = 5.0$

In case 1, where more weight is given to the control effort, θ, has a very small positive value. In case 4, where more weight is given to the deviation of x_{11} and x_{12} from the desired states, θ has a high value. Cases 2 and 3 are intermediate between cases 1 and 4. In all the cases θ drops to zero at the final time T.

From the figures it can be seen that as r_{11} increases, i.e., volume of pool 1 decreases, the rise in temperature of pool 1, x_{11}, is smaller; however the rise in temperature of pool 2, x_{12}, is higher at the end of the control period T. In cases 3 and 4, the temperature of pool 1, x_{11}, is seen to drop for a short time and then rise.

As $r_{11} \to 1$ the 2 CST's-in-series model approaches the one CST model. Thus, x_{11} and θ in Fig. 8 have approximately the same form as that of x_1 and θ in Fig. 1. Also as $r_{11} \to \infty$ the 2 CST's-in-series model again approaches the one CST model. In this case the response x_{12} and control θ should have the same form as that of x_1 and θ in Fig. 1.

CHAPTER IV

APPLICATION OF MODERN CONTROL THEORY TO ENVIRONMENTAL CONTROL OF CONFINED SPACE WITH IMPULSE HEAT DISTURBANCE

The performance equations for a life support system consisting of a confined space subjected to an impulse heat disturbance are presented. A different form of dimensionless forms of the performance equations have been derived in [11]. These performance equations are used in this chapter to illustrate the application of modern control theory in the determination of optimal control of environmental control systems.

4.1 INTRODUCTION

In [12] the modern control theory has been used to establish the optimal control for the minimum time problems. The resulting optimal control is of bang-bang type. In this chapter the application of modern control theory to problems in which the optimal control is continuous is illustrated.

The problem concerned in this chapter is to determine $\theta(t)$, subject to the constraint $-1 \leq \theta(t) \leq 1$, so that the response of the system can be brought to the desired state in a minimum period of time and with minimum effort, that is to minimize

$$S = \int_0^T (a + b\theta^2)dt$$

where a and b are suitable weighting factors. By giving appropriate values to these weighting factors the units of the objective function can be unified.

The three examples used in this chapter to illustrate the application of modern control theory are similar to the examples of chapter III.

4.2 PERFORMANCE EQUATIONS OF THE SYSTEM BASED ON THE DIMENSIONLESS PARAMETERS DEFINED IN CHAPTER III

In chapter III, the performance equations have been derived for the cases in which the disturbance is of the form of step function. However, the procedure for deriving the performance equations is fairly general and can be extended to cases in which the disturbances are of the form other than the step function.

The impulse heat disturbance has the form

$$q_{di} = V_1 \rho c_p T_d \delta(\alpha)$$

Replacing the step heat disturbance term by this term, the performance equations of the system proper, equations (12) and (13) of chapter 3, become

$$\tau_1 \frac{dT_c}{d\alpha} + T_c = r_2 T_2 + r_1 T_i + \tau_1 T_d \delta(\alpha) \qquad (1)$$

$$T_c = T_{c0^-} \quad \text{at} \quad \alpha = 0^-$$

or in dimensionless form

$$\frac{dx_1}{dt} + x_1 = r_2 K_\alpha + r_1 x_2 + \sigma_i \delta(\alpha) \qquad (2)$$

$$x_1 = 1 \quad \text{at} \quad t = 0^- \quad \text{or} \quad x_1(0^-) = 1$$

where

$$\sigma_i = \frac{T_d}{T_{c0^-}} \quad \text{for impulse heat disturbance.}$$

Note again that $t = 0^-$ is the time just before the disturbance occurs and $t = 0^+$ is the time just after the disturbance has occurred. This distinction

is very important in the case of impulse heat disturbance

$$\frac{dx_2}{dt} + rx_2 = rx_1 - rK_\beta\theta - rK_\gamma \tag{3}$$

$$x_2(0) = \frac{1 - r_2K_\alpha}{r_1} \quad \text{at} \quad t = 0^-$$

The performance equations of the two CST's-in-series model can similarly be derived as

$$\frac{dx_{11}}{dt} + r_{11}x_{11} = a_{11}x_{12} + a_{12} + \sigma_i\delta(t) \tag{4}$$

$$\frac{dx_{12}}{dt} + r_{12}x_{12} = r_{12}x_{11} + \sigma_i\delta(t) \tag{5}$$

$$\frac{dx_2}{dt} + rx_2 = rx_{12} - a_5\theta - a_6 \tag{6}$$

with

$$x_{11} = x_{12} = 1 \quad \text{at} \quad t = 0^- \quad \text{or} \quad x_{11}(0^-) = x_{12}(0^-) = 1$$

$$x_2(0^-) = \frac{1 - r_2K_\alpha}{r_1}$$

It is possible to rewrite equation (2) in another form in which the effect of the heat disturbance is taken into account in the initial condition immediately after the onset of the process:

$$\frac{dx_1}{dt} + x_1 = r_1x_2 + r_2K_\alpha$$

$$x_1 = 1 + \sigma_i \quad \text{at} \quad t = 0^+ \tag{7}$$

Similarly for the 2 CST's-in-series model, equations (4) and (5) can be rewritten as

$$\frac{dx_{11}}{dt} + r_{11}x_{11} = a_{11}x_{12} + a_{12} \tag{8}$$

$$\frac{dx_{12}}{dt} + r_{12}x_{12} = r_{12}x_{11} \tag{9}$$

with

$$x_{11} = x_{12} = 1 + \sigma_i \qquad \text{at} \qquad t = 0^+$$

4.3 PERFORMANCE EQUATIONS BASED ON DIMENSIONLESS PARAMETERS DEFINED IN [11]

The performance equations of the system can be written in a variety of ways by defining the state variables x_1 and x_2 in different ways. A different set of performance equations in dimensionless form are given in [11]. The set is reproduced below for reference

$$\frac{dx_1}{dt} + x_1 = \frac{r_1K_1x_2}{K_4} + r_2K_1 + K_1\sigma\delta(t) \tag{10}$$

$$\frac{dx_2}{dt} + rx_2 = rx_1 - a_5\theta - a_6 \tag{11}$$

with

$$\left.\begin{aligned} x_1 &= 0 \\ x_2 &= 1 \end{aligned}\right\} \text{ at } \quad t = 0^-$$

where

$$x_1 = \frac{T_c}{T_{c0}} = \frac{K_1T_c}{T_2}, \qquad r_1 = \frac{Q_1}{Q_1 + Q_2}$$

$$x_2 = \frac{T_i}{T_{i0}} = \frac{K_4T_i}{T_2}, \qquad r_2 = \frac{Q_2}{Q_1 + Q_2}$$

$$\sigma = \frac{T_d}{T_2}, \qquad \tau_1 = \frac{V_1}{Q_1 + Q_2}$$

$$K_1 = \frac{T_2}{T_{c0}}, \qquad K_2 = \frac{1}{2T_2}\,[T_{rmax} - T_{rmin}]$$

$$K_4 = \frac{T_2}{T_{i0}}, \qquad K_3 = \frac{1}{2T_2}\,[T_{rmax} + T_{rmin}]$$

$$\tau_2 = \frac{V_2}{Q_1}, \qquad r = \frac{\tau_1}{\tau_2}$$

$$a_5 = rK_2K_4, \qquad a_6 = rK_3K_4$$

T_{c0} = room temperature at $\alpha = 0^+$

T_{i0} = temperature of the circulation air into the system at $\alpha = 0^+$

Note that the initial condition $x_1(0^-) = 0$ means that $T_c = 0$ at $\alpha = 0^-$, i.e., the reference temperature is the room temperature before the heat disturbance, that is, $\alpha = 0^-$. In the case of heat disturbances other than the impulse heat disturbance when $T_c(0^-) = T_{c0} = 0$ this set of performance equations cannot be used since x_1 and K_1 can not be defined. In these cases the reference temperature should be chosen in such a way that T_{c0} is not equal to zero. This means that the reference temperature should be other than the room temperature at $\alpha = 0^-$.

Equation (10) can be written in another form in which the effect of the disturbance is taken into account in the initial condition as

$$\frac{dx_1}{dt} + x_1 = \frac{r_1K_1x_2}{K_4} + r_2K_1$$

$$x_1 = 1 \quad \text{at} \quad t = 0^+ \tag{12}$$

The performance equations for the 2 CST's-in-series model are [11]

$$\frac{dx_{11}}{dt} + r_{11}x_{11} = a_{11}x_2 + a_{12} + a_{12}\delta(t) \tag{13}$$

$$\frac{dx_{12}}{dt} + r_{12}x_{12} = a_{21}x_{11} + a_{23}\delta(t) \tag{14}$$

$$\frac{dx_2}{dt} + rx_2 = a_{42}x_{12} - a_5\theta - a_6 \tag{15}$$

where

$$x_{11} = \frac{T_{c1}}{T_{c10}} = \frac{K_{11}T_{c1}}{T_2}, \qquad K_{11} = \frac{T_2}{T_{c10}}$$

$$x_{11} = \frac{T_{c2}}{T_{c20}} = \frac{K_{12}\,T_{c2}}{T_2}, \qquad K_{12} = \frac{T_2}{T_{c20}}$$

$$x_2 = \frac{T_i}{T_{i0}} = \frac{K_4T_i}{T_2}$$

$$a_{11} = r_1K_{11}r_{11}/K_4, \qquad a_{12} = r_{11}r_2K_{11}$$

$$a_{13} = T_dK_{11}/T_2r_{11}, \qquad a_{21} = r_{12}K_{12}/K_{11}$$

$$a_{23} = T_dK_{12}/T_2r_{12}, \qquad a_{42} = rK_4/K_{12}$$

$$r_{11} = \frac{\tau_1}{\tau_{11}}, \qquad r_{12} = \frac{\tau_1}{\tau_{12}}$$

If the time constant of the heat exchanger is negligible then equation (11) becomes

$$x_2 = x_1 - K_2K_4\theta - K_3K_4$$

and equation (10) can be rewritten as

$$\frac{dx_1}{dt} + r_2x_1 = r_1K_1K_2\theta - r_1K_1K_3 + r_2K_1 \tag{16}$$

Also the performance equations of the 2 CST's-in-series model, equations (13) and (14), become

$$\frac{dx_{11}}{dt} + r_{11}x_{11} = a_{11}a'_{42}x_{12} - a_{11}a'_5\theta - a_{11}a'_6 + a_{12} \tag{17}$$

$$\frac{dx_{12}}{dt} + r_{12}x_{12} = a_{21}x_{11} \tag{18}$$

where

$$a'_{42} = K_4/K_{12}$$

$$a'_5 = K_2K_4$$

$$a'_6 = K_3K_4$$

4.4 EXAMPLE 1. Suppose that the dynamic behaviour of a life support system consisting of an air-conditioned room or cabin subject to the impulse heat disturbance and a heat exchanger of negligibly small time constant ($\tau_2 \to 0$), can be represented by the following equation [equation (16)].

$$\frac{dx_1}{dt} + r_2x_1 = r_2K_1 - r_1K_1K_2\theta - r_1K_1K_3 \tag{19}$$

with

$$x_1(0) = 1 \quad \text{at} \quad t = 0^+$$

$$x_1(T) = 0 \quad \text{at} \quad t = T$$

where T is the unspecified final control time. It is desired to determine $\theta(t)$, subject to the constraint $-1 \leq \theta(t) \leq 1$, so that the response of the system can be brought to its desired state in a minimum period of time and with minimum effort, that is to minimize

$$S = \int_0^T (a + b\theta^2)dt \tag{20}$$

where a and b are suitable weighting factors.

Introducing an additional state variable as

$$x_2(t) = \int_0^t (a + b\theta^2)dt$$

it follows that

$$\frac{dx_2}{dt} = a + b\theta^2, \qquad x_2(0) = 0 \tag{21}$$

The problem is thus transformed into that of minimizing $x_2(T)$.

The Hamiltonian is

$$H[z(t), x(t), \theta(t)]$$

$$= z_1 \frac{dx_1}{dt} + z_2 \frac{dx_2}{dt}$$

$$= z_1(-r_2x_1 + r_2K_1 - r_1K_1K_2\theta - r_1K_1K_3) + z_2(a + b\theta^2) \tag{22}$$

The adjoint variables are defined by

$$\frac{dz_1}{dt} = -\frac{\partial H}{\partial x_1} = r_2z_1 \tag{23}$$

$$\frac{dz_2}{dt} = -\frac{\partial H}{\partial x_2} = 0, \qquad z_2(T) = 1 \tag{24}$$

Solutions of equations (23) and (24) are

$$z_1(t) = Ae^{r_2 t} \tag{25}$$

$$z_2(t) = 1, \qquad 0^+ \leq t \leq T \tag{26}$$

where A is the integration constant. Substitution of equation (26) into equation (24) gives

$$H = z_1(-r_2 x_1 + r_2 K_1 - r_1 K_1 K_2 \theta - r_1 K_1 K_3) + a + b\theta^2 \tag{27}$$

Therefore H^*, the portion of H which depends on θ, is

$$H^* = - r_1 K_1 K_2 z_1 \theta + b\theta^2 \tag{28}$$

Hence the optimum control is continuous and is found from the necessary condition for optimality as

$$\frac{\partial H}{\partial \theta} = \frac{\partial H^*}{\partial \theta} = 0 = -r_1 K_1 K_2 z_1 + 2b\theta$$

that is,

$$\theta = \frac{r_1 K_1 K_2}{2b} z_1 \tag{29}$$

The integration constant A in equation (25) can be determined using the condition that minimum H is zero at all the process time. At $t = 0^+$, from equations (25), (27), (29) and the initial condition

$$x_1(0) = 1 \quad \text{at} \quad t = 0^+$$

give

$$A(-r_2 + r_2K_1 - \frac{(r_1K_1K_2)^2}{2b} A - r_1K_1K_3) + a + \frac{(r_1K_1K_2)^2}{4b} A^2$$

$$= 0$$

that is

$$A^2 \frac{(r_1K_1K_2)^2}{4b} + A\,(r_2 - r_2K_1 + r_1K_1K_3) - a = 0 \tag{30}$$

The solution of equation (30) gives a positive value for A as

$$A = \frac{-(r_2 - r_2K_1 + r_1K_1K_3) + \sqrt{(r_2 - r_2K_1 + r_1K_1K_3)^2 + \frac{a}{b}(r_1K_1K_2)^2}}{\frac{(r_1K_1K_2)^2}{2b}} \tag{31}$$

Now substituting equation (31) into equation (19), $x_1(t)$ can be integrated as

$$x_1(t) = A_1 e^{-r_2 t} + \frac{r_2K_1 - r_1K_1K_3}{r_2} - \frac{(r_1K_1K_2)^2}{4br_2} Ae^{r_2 t} \tag{32}$$

The constant of integration A_1 can be determined using the initial condition that $x_1(0) = 1$, at $t = 0^+$ as

$$A_1 = 1 - \frac{1}{r_2}(r_2K_1K_3) + \frac{(r_1K_1K_2)^2}{4br_2} A \tag{33}$$

The final time T can be determined by letting $x_1(T) = 0$ in equation (32)

$$A_1 e^{-r_2 t} + \frac{1}{r_2}(r_2K_1 - r_1K_1K_3) + \frac{(r_1K_1K_2)^2}{4br_2} Ae^{r_2 T} = 0$$

or

$$\frac{(r_1K_1K_2)^2}{4br_2} A\left(e^{r_2 T}\right)^2 - \frac{1}{r_2}(r_2K_1 - r_1K_1K_3)e^{r_2 T} - A_1 = 0$$

This gives

$$e^{r_2 T} = \frac{\frac{1}{r_2}(r_2K_1 - r_1K_1K_3) + \sqrt{\frac{1}{r_2^2}(r_2K_1 - r_1K_1K_3)^2 + \frac{(r_1K_1K_2)^2}{br_2} A_1 A}}{\frac{(r_1K_1K_2)^2}{2br_2} A} \tag{34}$$

NUMERICAL EXAMPLES:

(1) The following values for the various constants are used.

$r_1 = 0.8$ $\quad$ $r_2 = 0.2$

$K_1 = 0.5$ $\quad$ $K_2 = 1.5$

$K_3 = 1.5$ $\quad$ $\sigma = 2$

$a = 1$ $\quad$ $b = 1$

Substituting the above numerical values into equation (30) yields

$$0.09A^2 + 0.7A - 1 = 0 \tag{35}$$

The solution of equation (35) gives

$$A = 1.258$$

Hence

$$z_1(t) = 1.258e^{0.2t} \tag{36}$$

$$\theta(t) = 0.3774e^{0.2t} \tag{37}$$

From equation (33), we have

$$A_1 = 4.055$$

substituting into equation (32) gives

$$x_1(t) = -2.5 - 0.555e^{0.2t} + 4.055e^{-0.2t} \tag{38}$$

At t = T from equation (38), we obtain

$$e^{0.2T} = \frac{-2.5 \pm \sqrt{(2.5)^2 + 8.999}}{2 \times 0.555}$$

$$= 1.265$$

This gives final time T = 1.18

(2) Again taking the same values for the various constants consider the case for which a = 1, b = 10.

From equation (30)

$$0.009A^2 + 0.7A - 1 = 0$$

This gives A = 1.432

Hence

$$z_1(t) = 1.432e^{0.2t}$$

$$\theta(t) = 0.04295e^{0.2t} \tag{44}$$

From equation (33) we have

$$A_1 = 3.563$$

Then equation (32) becomes

$$x_1(t) = -2.5 - 0.063e^{0.2t} + 3.563e^{-0.2t} \tag{45}$$

At t = T from equation (34)

$$e^{0.2T} = \frac{-2.5 \pm \sqrt{(2.5)^2 + 0.898}}{2 \times 0.063}$$

$$= 1.38$$

This gives final time T = 1.60

(3) Taking same values for the various constants as in (1) and (2) and changing b = 0.1, from equation (30) we have

$$0.9A^2 + 0.7A - 1 = 0 \tag{39}$$

From which

$$A = 0.736$$

Hence

$$z_1(t) = 0.736e^{0.2t} \tag{40}$$

$$\theta(t) = 2.208e^{0.2t} \tag{41}$$

Since $\theta(0) > 1$ in equation (41), we take $\theta(0) = 1$. Now the control is of bang-bang type and $\theta(t) = 1$ for $0 \leq t \leq T$. Substituting in equation (32) and solving it we have

$$x_1(t) = -5.5 - 6.5e^{-0.2t} \tag{42}$$

At t = T from equation (42), we obtain

$$e^{-0.2T} = \frac{5.5}{6.5}$$

and

$$T = 0.8353$$

The results of this example are shown in Fig. 1 and Table 1. It should be noted that for the case a = 1, b = 0.1 the optimal control is of bang-bang type which has the bang part only. For cases (2) and (3) the optimum control approximately linearly increases upto the final time T.

Table 1

Optimal solutions of the one CST model together with $\tau_2 = 0$ (EXAMPLE 1)

Case Number	Weighting Factor	Final Time T
1	a = 1, b = 1	1.1800
2	a = 1, b = 10	1.6001
3	a = 1, b = 0.1	0.8353

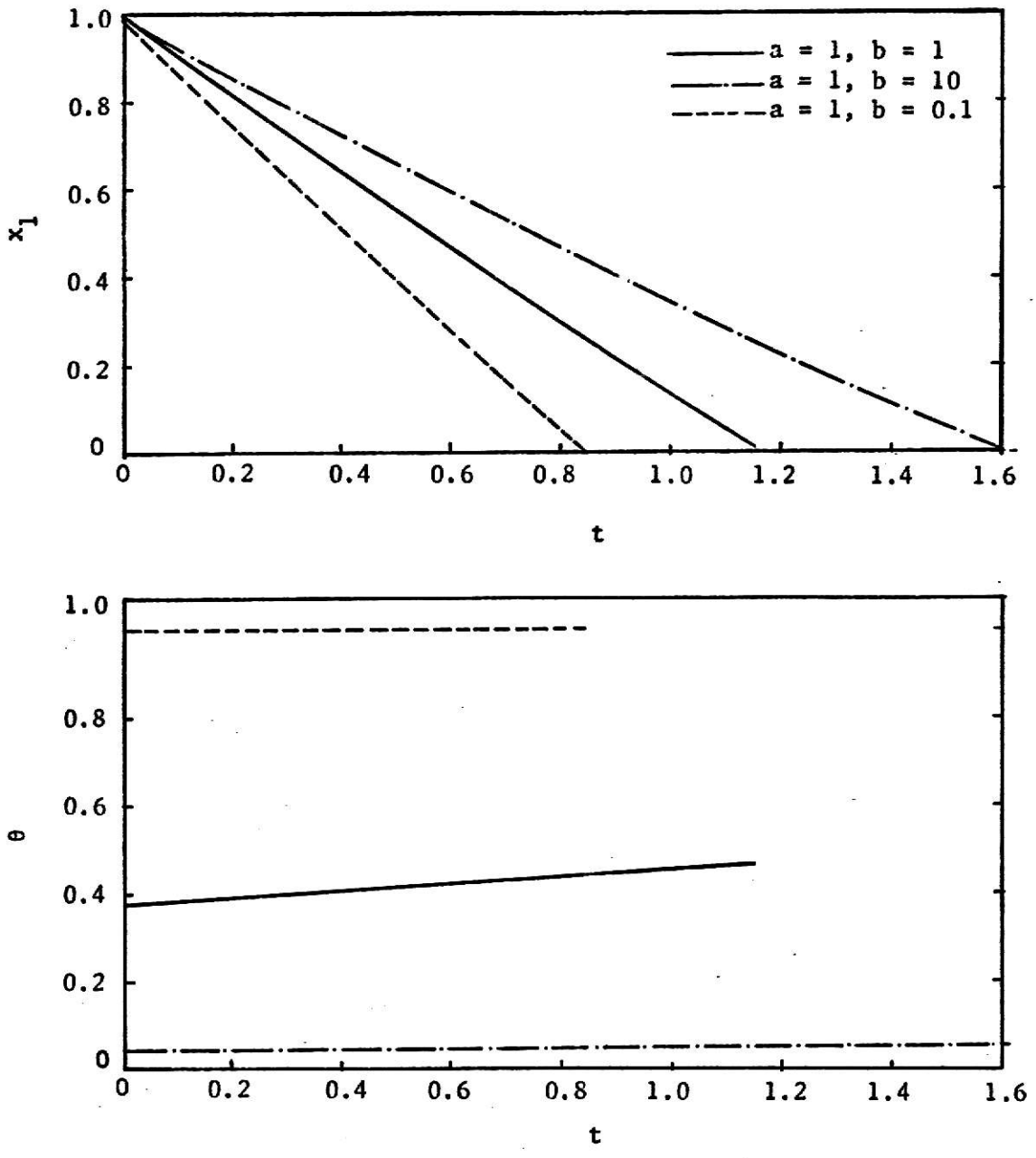


Fig. 1 Optimal control policy and system response of the one CST model with $\tau_2 = 0$ (Example 1)

4.5 EXAMPLE 2. Generally, responses of the heat exchanger as well as the cabin are not always instantaneous. Suppose that for the system considered in the first example, the time constant of the heat exchanger, τ_2, is not so small as to be ignored. The performance equations for such a system are

$$\frac{dx_1}{dt} + x_1 = a_1 x_2 + a_2 \tag{46}$$

$$\frac{dx_2}{dt} + rx_2 = a_4 x_1 - a_5\theta - a_6 \tag{47}$$

with the initial conditions

$$x_1(0) = 1, \quad x_2(0) = 1 \quad \text{at} \quad t = 0^+$$

and the final desired state

$$x_1(T) = 0, \quad x_2(T) = 1 \quad \text{at} \quad t = T$$

The objective function to be minimized is

$$S = \int_0^T (a + b\theta^2)dt \tag{48}$$

Introducing another state variable as

$$x_3(t) = \int_0^t (a + b\theta^2)dt$$

it follows that

$$\frac{dx_3}{dt} = a + b\theta^2, \qquad x_3(0) = 0 \tag{49}$$

and

$$x_3(T) = \int_0^T (a + b\theta^2)dt = S$$

The problem is thus transformed into that of minimizing $x_3(T)$

The Hamiltonian is

$$H(z(t), x(t), \theta(t))$$

$$= z_1 \frac{dx_1}{dt} + z_2 \frac{dx_2}{dt} + z_3 \frac{dx_3}{dt}$$

$$= z_1[- x_1 + a_1 x_2 + a_2] + z_2[- r x_2 + a_4 x_1 - a_5 \theta - a_6]$$

$$+ z_3[a + b\theta^2] \tag{50}$$

The adjoint variables are defined by

$$\frac{dz_1}{dt} = z_1 - a_4 z_2 \tag{51}$$

$$\frac{dz_2}{dt} = - a_1 z_1 + r z_2 \tag{52}$$

$$\frac{dz_3}{dt} = 0, \qquad z_3(T) = 1 \tag{53}$$

From equation (53) the solution of z_3 is

$$z_3(t) = 1 \qquad 0^+ \leq t \leq T \tag{54}$$

Hence, the Hamiltonian can be rewritten as

$$H = z_1(- x_1 + a_1 x_2 + a_2) + z_2(- r x_2 + a_4 x_1 - a_5 \theta - a_6)$$

$$+ a + b\theta^2 \tag{55}$$

and H^*, the portion of Hamiltonian which depends on θ, is

$$H^* = -a_5 z_2 \theta + b\theta^2 \tag{56}$$

Inspection of H^* shows that the optimal control is a continuous one and is given by the necessary condition that

$$\frac{\partial H}{\partial \theta} = \frac{\partial H^*}{\partial \theta} = 0 = - a_5 z_2 + 2b\theta$$

This gives

$$\theta = \frac{a_5 z_2}{2b} \tag{57}$$

Now the maximum principle requires that the system equations and adjoint variable equations, equations (46), (47), (51), (52) and (53) be integrated simultaneously with the split boundary conditions

$$x_1(0) = 1 \qquad x_1(T) = 0$$

$$x_2(0) = 1 \qquad x_2(T) = 1$$

$$x_3(0) = 0 \qquad x_3(T) = \text{undetermined}$$

$$z_1(0) = \text{undetermined} \qquad z_1(T) = \text{undetermined}$$

$$z_2(0) = \text{undetermined} \qquad z_2(T) = \text{undetermined}$$

Also the Hamiltonian must remain at zero at every point of its response under the optimal condition.

Eliminating z_1 from equations (51) and (52) yields

$$\frac{d^2 z_2}{dt^2} - (r + 1) \frac{dz_2}{dt} + (r - a_1 a_4) z_2 = 0 \tag{58}$$

The solution of this equation is

$$z_2(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}, \qquad 0 \leq t \leq T \tag{59}$$

where A_1, and A_2 are integration constants, and λ_1 and λ_2 are roots of the characteristic equation

$$\lambda^2 - (1 + r)\lambda + (r - a_1 a_4) = 0$$

From equation (53), the solution of $z_1(t)$ is

$$z_1(t) = \frac{1}{a_1} [(r - \lambda_1) A_1 e^{\lambda_1 t} + (r - \lambda_2) A_2 e^{\lambda_2 t}] \tag{60}$$

Also from equation (57), the optimum control can be written as

$$\theta(t) = \frac{a_5}{2b} [A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}] \tag{61}$$

Since the minimum H must remain at zero for all process times, at $t = 0^+$, from equation (55), we have

$$\frac{1}{a_1} [(r - \lambda_1) A_1 + (r - \lambda_2) A_2](-1 + a_1 + a_2)$$

$$+ (A_1 + A_2)[-r + a_4 - \frac{a_5^2}{2b} (A_1 - A_2) - a_6] + a + \frac{a_5^2}{4b} (A_1 + A_2)^2 = 0$$

or simplifying

$$\frac{a_5^2}{4b} (A_1 + A_2)^2 + (A_1 + A_2)(- a_4 + a_6 + \frac{1 - a_2}{a_1})$$

$$+ \frac{1}{a_1} (- 1 + a_1 + a_2)(\lambda_1 A_1 + \lambda_2 A_2) - a = 0 \tag{62}$$

Eliminating x_2 from equations (46) and (47) and substituting equation (61) gives

$$\frac{d^2x_1}{dt^2} + (r+1)\frac{dx_1}{dt} + (r-a_1a_4) = ra_2 - a_1a_6 - \frac{a_1a_5^2}{2b}(A_1e^{\lambda_1 t} + A_2e^{\lambda_2 t}) \tag{63}$$

The solution of this equation is

$$x_1(t) = B_1e^{-\lambda_1 t} + B_2e^{-\lambda_2 t} + \frac{ra_2 - a_1a_6}{r - a_1a_4}$$

$$- \frac{a_1a_5^2}{2b}\left[\frac{A_1e^{\lambda_1 t}}{2\lambda_1(r+1)} + \frac{A_2e^{\lambda_2 t}}{2\lambda_2(r+1)}\right] \tag{64}$$

since $-\lambda_1$ and $-\lambda_2$ are roots of the characteristic equation

$$\lambda^2 + (r+1)\lambda + (r-a_1a_4) = 0$$

Writing

$$K = \frac{ra_2 - a_1a_6}{r - a_1a_4}$$

$$E_1 = \frac{a_1a_5^2 A_1}{4b\lambda_1(r+1)}$$

$$E_2 = \frac{a_1a_5^2 A_2}{4b\lambda_2(r+1)},$$

equation (64) can be rewritten as

$$x_1(t) = B_1e^{-\lambda_1 t} + B_2e^{-\lambda_2 t} + K - E_1e^{\lambda_1 t} - E_2e^{\lambda_2 t} \tag{65}$$

Also from equation (48)

$$x_2(t) = \frac{1}{a_1}\left[(1-\lambda_1)B_1 e^{-\lambda_1 t} + (1-\lambda_2)B_2 e^{-\lambda_2 t}\right.$$

$$\left. - (1+\lambda_1)E_1 e^{\lambda_1 t} - (1+\lambda_2)E_2 e^{\lambda_2 t} + K - a_2\right] \tag{66}$$

Applying the initial conditions to equations (65) and (66) gives

$$A_1 + B_2 + K - E_1 - E_2 = 1 \tag{67}$$

$$(1-\lambda_1)B_1 + (1-\lambda_2)B_2 - (1+\lambda_1)E_1 - (1+\lambda_2)E_2 + K - a_2 = a_1 \tag{68}$$

Now employing the final conditions at $t = T$, equations (65) and (66) give

$$B_1 e^{-\lambda_1 T} + B_2 e^{-\lambda_2 T} + K - E_1 e^{\lambda_1 T} - E_2 e^{\lambda_2 T} = 0 \tag{69}$$

$$(1-\lambda_1)B_1 e^{-\lambda_1 T} + (1-\lambda_2)B_2 e^{-\lambda_2 T} - (1+\lambda_1)E_1 e^{\lambda_1 T}$$

$$-(1+\lambda_2)E_2 e^{-\lambda_2 T} + K - a_2 = a_1 \tag{70}$$

Subtracting equation (69) from equation (70) yields

$$-\lambda_1 B_1 e^{-\lambda_1 T} - \lambda_2 B_2 e^{-\lambda_2 T} - \lambda_1 E_1 e^{\lambda_1 T} - \lambda_2 E_2 e^{\lambda_2 T} - a_2 = a_1 \tag{71}$$

The five unknowns A_1, A_2, B_1, B_2 and T can now be determined by solving equations (67), (68), (69), (71) and (62) simultaneously with the help of a search technique. The optimal control $\theta(t)$ in equation (61) and the responses $x_1(t)$ and $x_2(t)$ of the system in equations (65) and (66) can be calculated.

In the numerical examples it is found that the optimum control variable θ has a value less than -1 at $t = T$. Thus, there has to be a switching from the continuous control to a constant control at $\theta = -1$. Equations (61), (65) and (66) are applicable for $0 \leq t \leq t_s$, where t_s is the time at which switching occurs. Also equations (69) and (70) are not applicable in this case.

For $t_s \leq t \leq T$ the optimum control θ is equal to -1. Substituting this value of θ into equation (47) and eliminating x_2 from equations (46) and (47) gives

$$\frac{d^2x_1}{dt^2} + (r+1)\frac{dx_1}{dt} + (r-a_1a_4) = ra_2 + a_1a_5 - a_1a_6$$

The solution of this equation is

$$x_1 = D_1e^{-\lambda_1 t} + D_2e^{-\lambda_2 t} + K' \tag{72}$$

where

$$K' = \frac{ra_2 + a_1a_5 - a_1a_6}{ra_2 - a_1a_4}$$

and D_1 and D_2 are constants. The solution of x_2 can be obtained from equations (46) and (72) as

$$x_2 = \frac{1}{a_1}\left[(1-\lambda_1)D_1e^{-\lambda_1 t} + (1-\lambda_2)D_2e^{-\lambda_2 t} + K' - a_2\right] \tag{73}$$

The constants D_1 and D_2 can be determined by noting that x_1 and x_2 are continuous with respect to t. Thus, at $t = t_s$, from equations (65), (66), (72) and (73)

$$x_1(t_s) = D_1 e^{-\lambda_1 t_s} + D_2 e^{-\lambda_2 t_s} + K'$$

$$= B_1 e^{-\lambda_1 t_s} + B_2 e^{-\lambda_2 t_s} + K - E_1 e^{\lambda_1 t_s} - E_2 e^{\lambda_2 t_s} \tag{74}$$

$$x_2(t_s) = \frac{1}{a_1} [(1-\lambda_1) D_1 e^{-\lambda_1 t_s} + (1-\lambda_2) D_2 e^{-\lambda_2 t_s} + K' - a_2]$$

$$= \frac{1}{a_1} [(1-\lambda_1) B_1 e^{-\lambda_1 t_s} + (1-\lambda_2) B_2 e^{-\lambda_2 t_s} + K -$$

$$E_1 e^{\lambda_1 t_s} - E_2 e^{\lambda_2 t_s} - a_2] \tag{75}$$

Solving for D_1 and D_2 from these equations yields

$$D_1 = B_1 + \frac{\lambda_2(K-K') + (\lambda_1+\lambda_2) E_1 e^{\lambda_1 t_s} + 2\lambda_2 E_2 e^{\lambda_2 t_s}}{(\lambda_2-\lambda_1)} e^{\lambda_1 t_s}$$

$$D_2 = B_2 + \frac{\lambda_1(K-K') + 2\lambda_1 E_1 e^{\lambda_1 t_s} + (\lambda_1+\lambda_2) E_2 e^{\lambda_2 t_s}}{(\lambda_1-\lambda_2)} e^{\lambda_2 t_s}$$

Also at $t = t_s$, $\theta = -1$, hence

$$\frac{a_5}{2b} [A_1 e^{\lambda_1 t_s} + A_2 e^{\lambda_2 t_s}] = -1 \tag{76}$$

Employing final conditions in equations (72), and (73) gives

$$D_1 e^{-\lambda_1 T} + D_2 e^{-\lambda_2 T} + K' = 0 \tag{77}$$

$$(1-\lambda_1)D_1e^{-\lambda_1 T} + (1-\lambda_2)D_2e^{-\lambda_2 T} + K' - a_2 = a_1 \tag{78}$$

Subtracting equation (78) from equation (77) gives

$$\lambda_1 D_1 e^{-\lambda_1 T} + \lambda_2 D_2 e^{-\lambda_2 T} + a_2 = -a_1 \tag{79}$$

There are six unknowns B_1, B_2, A_1, A_2, t_s and T in equations (62), (67), (68), (76), (77), and (79). These equations can be solved simultaneously to determine these constants. The optimal policy can be summarized as

$$\theta(t) = \frac{a_5}{2b}\left[A_1e^{\lambda_1 t} + A_2e^{\lambda_2 t}\right] \qquad 0 \leq t \leq t_s$$

$$= -1 \qquad t_s \leq t \leq T$$

The system responses are

$$x_1(t) = B_1e^{-\lambda_1 t} + B_2e^{-\lambda_2 t} + K - E_1e^{\lambda_1 t} - E_2e^{\lambda_2 t} \qquad 0 \leq t \leq t_s$$

$$= D_1e^{-\lambda_1 t} + D_2e^{-\lambda_2 t} + K' \qquad t_s \leq t \leq T$$

and

$$x_2(t) = \frac{1}{a_1}\left[(1-\lambda_1)B_1e^{-\lambda_1 t} + (1-\lambda_2)B_2e^{-\lambda_2 t} + K\right.$$

$$\left.-(1+\lambda_1)E_1e^{\lambda_1 t} - (1+\lambda_2)E_2e^{\lambda_2 t}\right] \qquad 0 \leq t \leq t_s$$

$$= \frac{1}{a_1}\left[(1-\lambda_1)D_1e^{-\lambda_1 t} + (1-\lambda_2)D_2e^{-\lambda_2 t} + K'\right] \qquad t_s \leq t \leq T$$

The results of this example for various values of r are shown in Figs. 2, 3 and 4 and are tabulated in Table 2. Hooke and Jeeves pattern search was used to determine the unknowns B_1, B_2, A_1, A_2, t_s and T. Three different cases, as in example 1, are considered:

case 1 $a = 1$, $b = 0.1$

case 2 $a = 1$, $b = 1$

case 3 $a = 1$, $b = 10$

The optimal control is approximately constant at the beginning and then decreases to a value -1, and remains constant at this value upto the final time T. The value of the final time is larger for smaller value of r because of the time lag of the heat exchanger.

For case 1, the response x_2 increases and then drops down to the final value of 1. For cases 2 and 3, the response x_2 decreases for a short time at the beginning and then increases approximately linearly and then drops to the final value of 1.

As the time constant of the heat exchanger decreases, i.e., r increases, x_2 responds more quickly. As the time constant of the heat exchanger becomes negligible as compared to the time constant of the system proper the results of this problem should be reduced to the results of example 1. In Fig. 4, the optimum control has approached to the bang-bang type control.

4.6 EXAMPLE 3. Suppose that a life support system consists of an air-conditioned room and a heat exchanger of negligibly small time constant ($\tau_2 \to 0$) as in example 1. However, the flow of air in the room can be characterized by the two-CST's in series model. The performance equations of such a system are

Table 2

Optimal solutions of the one CST model together with $\tau_2 \neq 0$ (EXAMPLE 2)

Case No.	Weighting Factors	r	Switching Time T_s	Final Time T
1	a = 1, b = 1	5	1.0932	1.4237
		10	1.1663	1.3200
		50	1.1826	1.2254
2	a = 1, b = 10	5	1.6323	1.8621
		10	1.5864	1.7621
		50	1.5663	1.6035
3	a = 1, b = 0.1	5	0.7884	1.2053
		10	0.9678	1.0978
		50	0.9006	0.9423

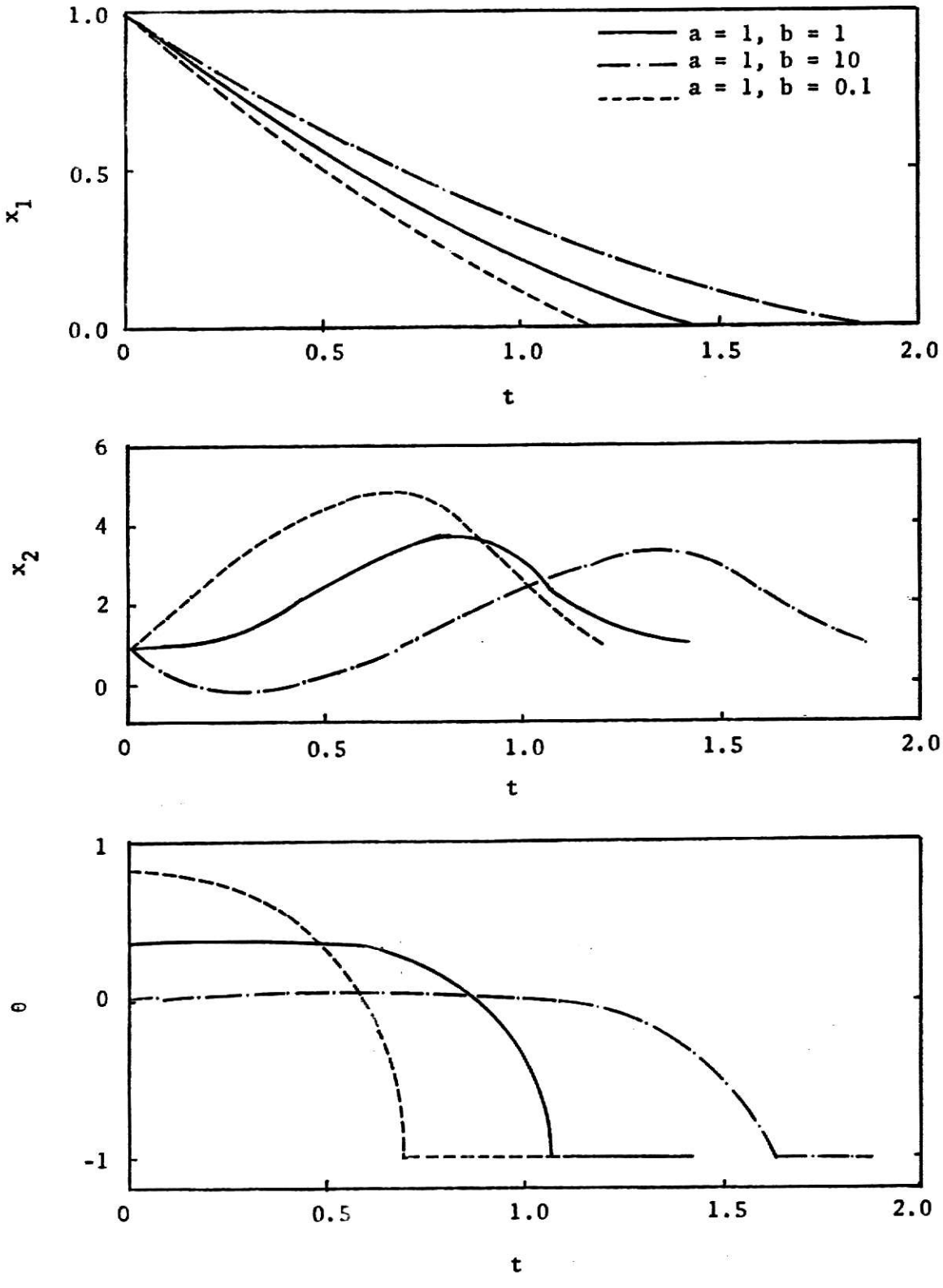


Fig. 2 Optimal control policy and system responses of the one CST model (Ex. 2), r = 5

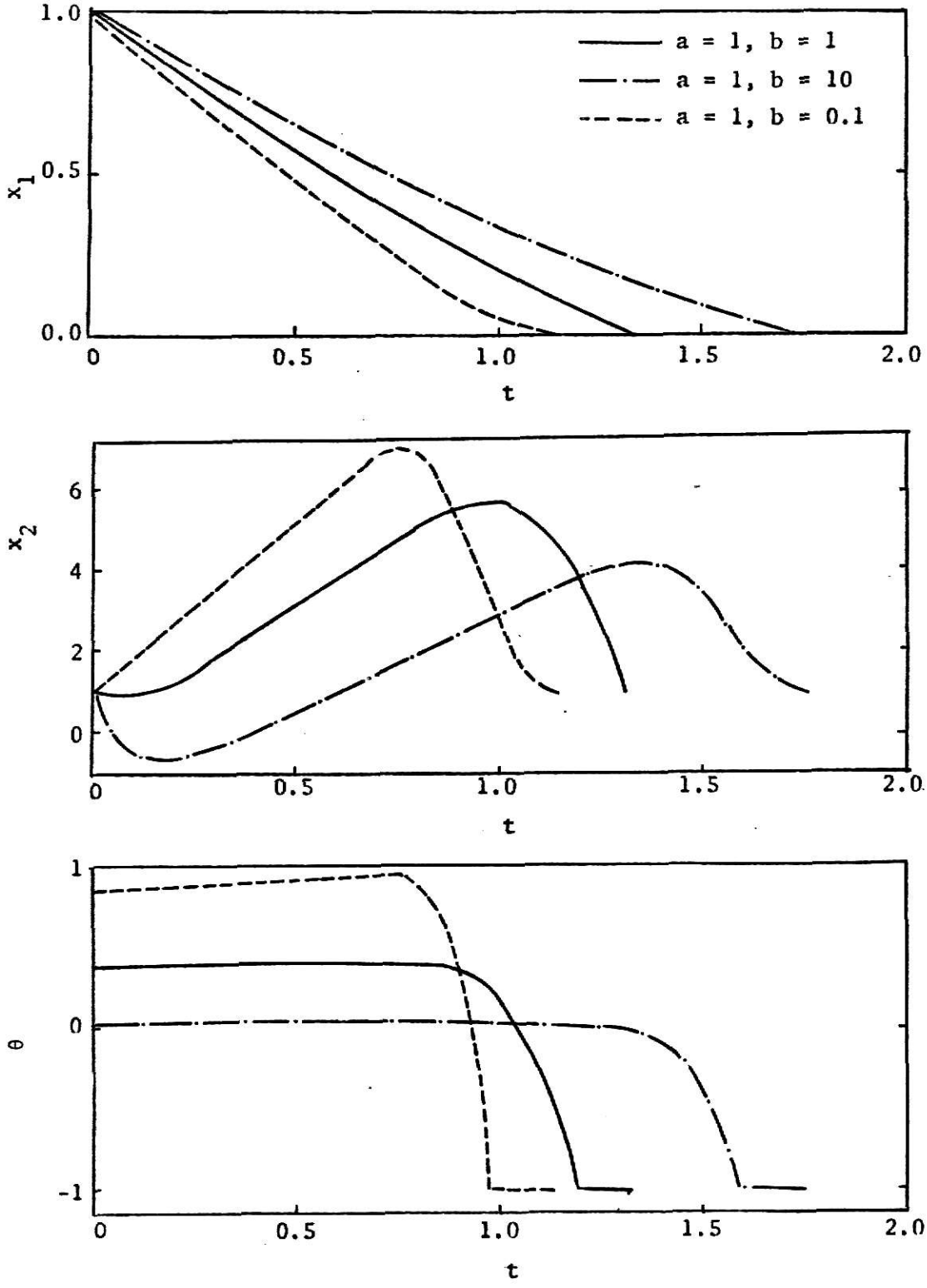


Fig. 3 Optimal control policy and system responses of the one CST model (Ex. 2), r = 10

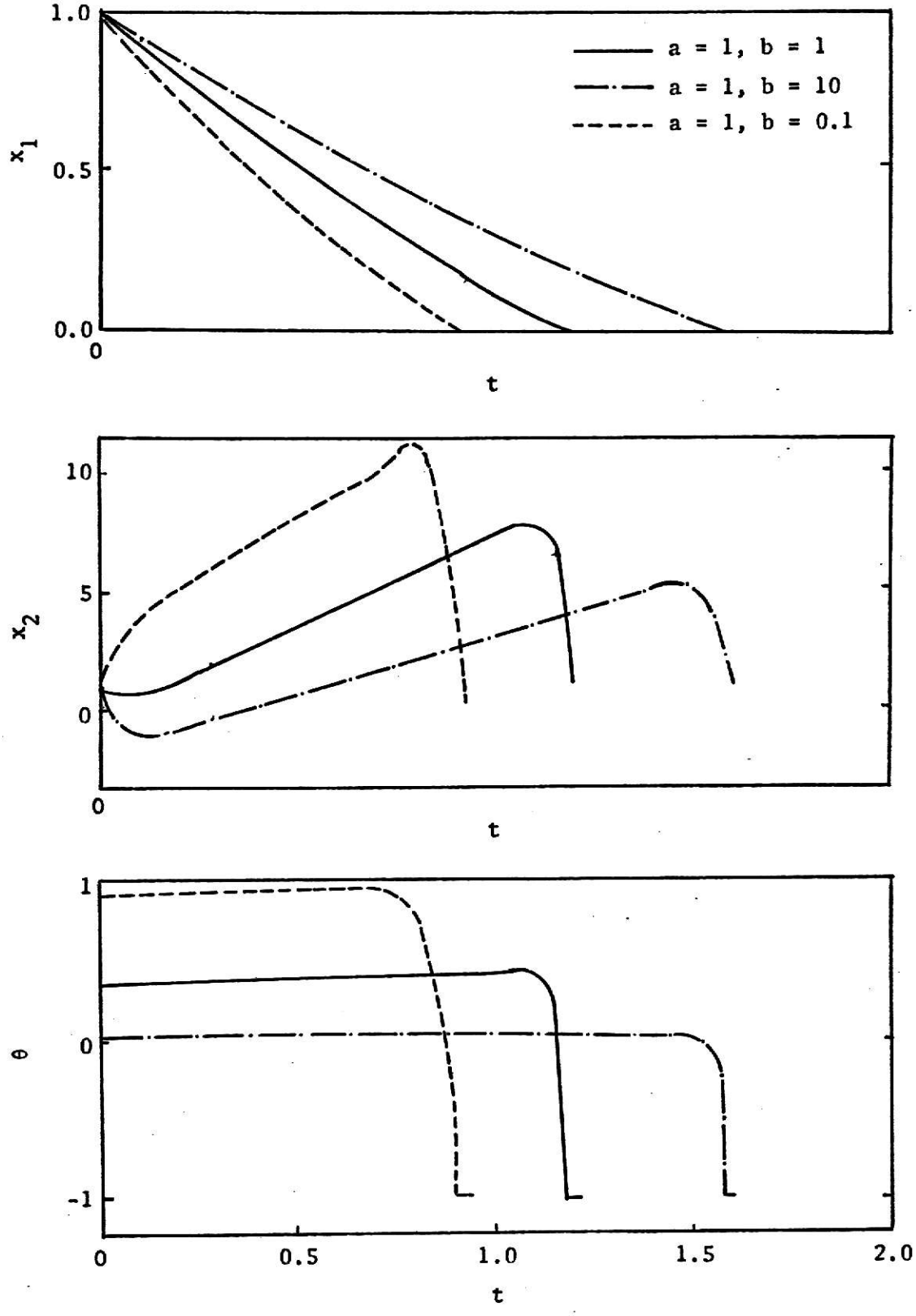


Fig. 4 Optimal control policy and system responses of the one CST model (Ex. 2), r = 50

$$\frac{dx_{11}}{dt} + r_{11}\, x_{11} = a_{11}\, a_{42}'\, x_{12} - a_{11}\, a_5'\, \theta - a_{11}\, a_6' + a_{12} \tag{80}$$

$$\frac{dx_{12}}{dt} + r_{12}\, x_{12} = a_{21}\, x_{11} \tag{81}$$

The initial and final conditions are

$$x_{11}(0) = x_{12}(0) = 1 \quad \text{at} \quad t = 0^+$$

$$x_{11}(T) = x_{12}(T) = 0 \quad \text{at} \quad t = T \tag{82}$$

where final time T is unspecified. The objective function to be minimized is

$$S = \int_0^T (a + b\theta^2)dt \tag{83}$$

Introducing an additional state variable

$$x_3(t) = \int_0^t (a + b\theta^2)dt$$

it follows that

$$\frac{dx_3}{dt} = a + b\theta^2 \qquad x_3(0) = 0 \tag{84}$$

The problem is thus transformed into that of minimizing $x_3(T)$

The Hamiltonian is

$$H(z(t), x(t), \theta(t)]$$

$$= z_{11}(-\, r_{11}\, x_{11} + a_{11}\, a_{42}'\, x_{12} - a_{11}\, a_5'\theta - a_{11}\, a_6' + a_{12})$$

$$+ z_{12}(-\, r_{11}\, x_{12} + a_{21}\, x_{11}) + z_3(a + b\theta^2) \tag{85}$$

The adjoint variables are defined by

$$\frac{dz_{11}}{dt} = - \frac{\partial H}{\partial x_{11}} = r_{11}\, z_{11} - a_{21}\, z_{12} \tag{86}$$

$$\frac{dz_{12}}{dt} = - \frac{\partial H}{\partial x_{12}} = - a_{11}\, a_{42}'\, z_{11} + r_{12}\, z_{12} \tag{87}$$

$$\frac{dz_3}{dt} = - \frac{\partial H}{\partial x_3} = 0, \qquad z_3(T) = 1 \tag{88}$$

The solution of z_3 from equation (88) is

$$z_3(t) = 1, \qquad 0 \leq t \leq T \tag{89}$$

Hence equation (85) can be rewritten as

$$H(z, x, \theta) = z_{11}(- r_{11}\, x_{11} + a_{11}\, a_{42}'\, x_{12} - a_{11}\, a_5'\, \theta - a_{11}\, a_6' + a_{12})$$

$$+ z_{12}(- r_{12}\, x_{12} + a_{21}\, x_{11}) + a + b\theta^2 \tag{90}$$

Therefore, H^*, the portion of H which depends on θ, is

$$H^* = - a_{11}\, a_5'\, z_{11}\, \theta + b\theta^2 \tag{91}$$

Hence the optimum control should be of continuous type and is found from the necessary condition

$$\frac{\partial H}{\partial \theta} = \frac{\partial H^*}{\partial \theta} = 0 = - a_{11}\, a_5'\, z_{11} + 2b\theta$$

or

$$\theta = \frac{a_{11}\, a_5'}{2b}\, z_{11} \tag{91}$$

Eliminating z_{12} from equations (86) and (87) gives

$$\frac{d^2 z_{11}}{dt^2} - (r_{11} + r_{12}) \frac{dz_{11}}{dt} + (r_{11}\, r_{12} - a_{11}\, a_{42}'\, a_{21})\, z_{11} = 0 \tag{92}$$

The solution of this equation is

$$z_{11}(t) = A_1 e^{\lambda_{11} t} + A_2 e^{\lambda_{12} t}, \qquad 0 \leq t \leq T \tag{93}$$

where λ_{11} and λ_{12} are roots of the characteristic equation

$$\lambda^2 - (r_{11} + r_{12})\lambda + (r_{11}\, r_{12} - a_{11}\, a'_{42}\, a_{21}) = 0 \tag{94}$$

and A_1, A_2 are constants of integration to be determined later.

From equation (86) the solution of $z_{12}(t)$ can be written as

$$z_{12}(t) = \frac{1}{a_{21}} [(r_{11} - \lambda_{11})\, A_1 e^{\lambda_{11} t} + (r_{12} - \lambda_{12})\, A_2 e^{\lambda_{12} t}] \tag{95}$$

Also from equation (91) the optimal control is

$$\theta(t) = \frac{a_{11}\, a'_5}{2b} [A_1 e^{\lambda_{11} t} + A_2\, e^{\lambda_{12} t}] \tag{96}$$

Now eliminating x_{11} from equations (90) and (91) gives rise to the differential equation

$$\frac{d^2 x_{12}}{dt^2} + (r_{11} + r_{12}) \frac{dx_{12}}{dt} + (r_{11}\, r_{12} - a_{11}\, a'_{42}\, a_{21})\, x_{12}$$

$$= - \frac{(a_{11}\, a'_5)^2\, a_{21}}{2b} [A_1 e^{\lambda_{11} t} + A_2 e^{\lambda_{12} t}] - a_{11}\, a_{21}\, a'_6$$

$$+ a_{12}\, a_{21} \tag{97}$$

The solution of this equation is

$$x_{12}(t) = B_1 e^{-\lambda_{11} t} + B_2 e^{-\lambda_{12} t} + K - E_1 e^{\lambda_{11} t} + E_2 e^{\lambda_{12} t} \tag{98}$$

where

$$K = \frac{a_{12}\, a_{21} - a_{11}\, a_{21}\, a_6'}{r_{11}\, r_{12} - a_{11}\, a_{21}\, a_{42}'}$$

$$E_1 = \frac{(a_{11}\, a_5')^2\, a_{21}\, A_1}{4b\lambda_{11}(r_{11} + r_{12})}$$

$$E_2 = \frac{(a_{11}\, a_5')^2\, a_{21}\, A_2}{4b\lambda_{12}(r_{11} + r_{12})}$$

and B_1, B_2 are constants of integration. From equation (91)

$$x_{11}(t) = \frac{1}{a_{21}} [(r_{12} - \lambda_{11})\, B_1 e^{-\lambda_{11} t} + (r_{12} - \lambda_{12})\, B_2 e^{-\lambda_{12} t}$$

$$+ r_{12}K - (r_{12} + \lambda_{11})E_1 e^{\lambda_{11} t} - (r_{12} + \lambda_{12})\, E_2 e^{\lambda_{12} t} \qquad (99)$$

Now employing the initial and final conditions in equations (98) and (99) give

$$B_1 + B_2 + K - E_1 - E_2 = 1 \qquad (100)$$

$$(r_{12} - \lambda_{11})B_1 + (r_{12} - \lambda_{12})\, B_2 + r_{12}K - (r_{12} + \lambda_{11})E_1$$

$$- (r_{12} + \lambda_{12})E_2 = a_{21} \qquad (101)$$

$$B_1 e^{-\lambda_{11} T} + B_2 e^{-\lambda_{12} T} + K - E_1 e^{\lambda_{11} T} - E_2 e^{\lambda_{12} T} = 0 \qquad (102)$$

$$(r_{12} - \lambda_{11})\, B_1 e^{-\lambda_{11} T} + (r_{12} - \lambda_{12})\, B_2 e^{-\lambda_{12} T} + r_{12}K$$

$$- (r_{12} + \lambda_{11})E_1 e^{\lambda_{11} T} - (r_{12} + \lambda_{12})E_2 e^{\lambda_{12} T} = 0 \qquad (103)$$

Since minimum H must remain at zero for all process times, at $t = 0^+$, from equation (96)

$$(A_1 + A_2)[- r_{11} + a_{11}\, a'_{42} - (a_{11}\, a'_5)^2 \frac{(A_1 + A_2)}{2b} - a_{11}\, a'_6 + a_{12}]$$

$$+ \frac{1}{a_{21}} [(r_{11} - \lambda_{11})\, A_1 + (r_{11} - \lambda_{12})\, A_2][- r_{12} + a_{21}] + a + \frac{(a_{11}\, a'_5)^2}{4b} (A_1+A_2)^2 = 0$$

or

$$\frac{(a_{11}\, a'_5)^2}{4b} (A_1 + A_2)^2 - (A_1 + A_2)\, [+ a_{11}\, a'_{42} - a_{11}\, a'_6 + a_{12}$$

$$- \frac{r_{11}\, r_{12}}{a_2}] + \frac{1}{a_{21}} (\lambda_{11}\, A_1 + \lambda_{12}\, A_2)\, [- r_{12} + a_{21}] - a = 0 \qquad (104)$$

There are five unknowns A_1, A_2, B_1, B_2 and the final time, T, in five equations (100) through (104). These equations can be solved simultaneously to determine these unknowns. These values can then be substituted into equations (96), (98) and (99) to give the optimal control policy $\theta(t)$ and the responses $x_{11}(t)$ and $x_{12}(t)$.

In the numerical examples it is found that there has to be a switch from the continuous type of control to a constant control at $\theta = -1$. In this case, equations (96), (98), and (99) are applicable for $0 \leq t \leq t_s$ and equations (102) and (103) no longer hold.

When the control is constant at $\theta = -1$ x_{11} and x_{12} can be integrated as

$$x_{12} = D_1 e^{-\lambda_{11} t} + D_2 e^{-\lambda_2 t} + K' \qquad t_s \leq t \leq T \qquad (105)$$

$$x_{11} = \frac{1}{a_{21}} \left[(r_{12}-\lambda_1) D_1 e^{-\lambda_{11} t} + (r_{12}-\lambda_2) D_2 e^{-\lambda_{12} t} + r_{12} K'\right] \qquad (106)$$

$$t_s \leq t \leq T$$

where

$$K' = \frac{a_{12}a_{21} + a_{11}a_{21}a_5' - a_{11}a_{21}a_6'}{r_{11}r_{12} - a_{11}a_{21}a_{42}'}$$

The constants of integration D_1 and D_2 can be determined by noting that x_{11} and x_{12} are continuous with respect to t. Thus, at $t = t_s$

$$x_{12}(t_s) = D_1 e^{-\lambda_{11}t_s} + D_2 e^{-\lambda_{12}t_s} + K'$$

$$= B_1 e^{-\lambda_{11}t_s} + B_2 e^{-\lambda_{12}t_s} + K - E_1 e^{\lambda_{11}t_s} - E_2 e^{\lambda_{12}t_s} \tag{107}$$

$$x_{11}(t_s) = \frac{1}{a_{21}} \left[(r_{12}-\lambda_{11})\, D_1 e^{-\lambda_{11}t_s} + (r_{12}-\lambda_{12})D_2 e^{-\lambda_{12}t_s} + r_{12}K'\right]$$

$$= \frac{1}{a_{21}} \left[(r_{12}-\lambda_{11})B_1 e^{-\lambda_{11}t_s} + (r_{12}-\lambda_{12})B_2 e^{-\lambda_{11}t_s} + r_{12}K\right.$$

$$\left. + (r_{12}+\lambda_{11})E_1 e^{\lambda_{11}t_s} - (r_{12}+\lambda_{12})E_2 e^{\lambda_{12}t_s}\right] \tag{108}$$

$$\theta(t_s) = \frac{a_{11}a_5'}{2b} \left[A_1 e^{\lambda_{11}t_s} + A_2 e^{\lambda_{12}t_s}\right] = -1 \tag{109}$$

Solving for D_1 and D_2 from equations (107) and (108) we have

$$D_1 = B_1 + \frac{\lambda_{12}(K-K') - (\lambda_{11}+\lambda_{12})E_1 e^{\lambda_{11}t_s} - 2\lambda_2 E_2 e^{\lambda_{12}t_s}}{(\lambda_{12}-\lambda_{11})} e^{\lambda_{11}t_s}$$

$$D_2 = B_2 + \frac{\lambda_{11}(K-K') - 2\lambda_{11}E_1 e^{\lambda_{11}t_s} - (\lambda_{11}+\lambda_{12})E_2 e^{\lambda_{12}t_s}}{(\lambda_{11}-\lambda_{12})} e^{\lambda_{12}t_s}$$

Using the final conditions in equations (107) and (108)

$$D_1 e^{-\lambda_{11}T} + D_2 e^{-\lambda_{12}T} + K' = 0 \tag{110}$$

$$(r_{12}-\lambda_{11})D_1 e^{-\lambda_{11}T} + (r_{12}-\lambda_{12})D_2 e^{-\lambda_{12}T} + r_{12}K' = 0 \tag{111}$$

Subtracting r_{12} times equation (111) from equation (110) yields

$$\lambda_{11}D_1 e^{-\lambda_{11}T} + \lambda_{12}D_2 e^{-\lambda_{12}T} = 0 \tag{112}$$

There are six unknowns A_1, A_2, B_1, B_2, t_s and T in equations (100), (101), (104), (109), (110) and (112). These equations can be solved simultaneously to determine these constants. The optimal policy and system responses can be determined as

$$\theta(t) = \frac{a_{11}a'_5}{2b} A_1 e^{\lambda_{11}t} + A_2 e^{\lambda_{12}t} \qquad 0 \le t \le t_s$$

$$= -1 \qquad t_s \le t \le T \tag{113a}$$

$$x_{12}(t) = B_1 e^{-\lambda_{11}t} + B_2 e^{-\lambda_{12}t} + K - E_1 e^{\lambda_{11}t} - E_2 e^{\lambda_{12}t} \qquad 0 \le t \le t_s$$

$$= D_1 e^{-\lambda_{11}t} + D_2 e^{-\lambda_{12}t} + K' \qquad t_s \le t \le T \tag{113b}$$

$$x_{11}(t) = \frac{1}{a_{21}} \Big[(r_{12}-\lambda_{11})B_1 e^{-\lambda_{11}t} + (r_{12}-\lambda_{12})B_2 e^{-\lambda_{12}t} + r_{12}K$$

$$- (r_{12}+\lambda_{11})E_1 e^{\lambda_{11}t} - (r_{12}+\lambda_{12})E_2 e^{\lambda_{12}t} \Big\} \qquad 0 \le t \le t_s$$

$$= \frac{1}{a_{21}} \Big\{ (r_{12}-\lambda_{11})D_1 e^{-\lambda_{11}t} + (r_{12}-\lambda_{12})D_2 e^{-\lambda_{12}t} + r_{12}K' \Big\}$$

$$t_s \le t \le T \tag{113c}$$

The results of this example for different values of parameter r_{11} are shown in Figs. 5, 6 and 7 are tabulated in Table 3. Again three different cases are considered:

Case 1: $a = 1, \quad b = 1$

Case 2: $a = 1, \quad b = 10$

Case 3: $a = 1, \quad b = 0.1$

The Hooke & Jeeves pattern search was used to determine the unknowns. The results are similar to the results of the preceeding examples. The dimensionless temperature of the room can become negative in this problem. For $r_{11} = 2.0$ the control time is maximum. When $r_{11} \to 1.$ and $r_{11} \to \infty$ the 2 CST's-in-series model approaches the one CST model. Thus, for $r_{11} = 1.2$ and $r_{11} = 10$ the results are approaching the results of example 1.

Table 3

Optimal solutions of the 2 CST's-in-series model with τ_2 = (EXAMPLE 3)

Case No.	Weighting Factors	r_{11}	Switching Time t_s	Final Time T
1	a = 1, b = 1	1.2	1.0418	1.3362
		2.0	1.1629	1.6935
		5.0	1.2075	1.4934
		10.0	1.1770	1.3278
2	a = 1, b = 10	1.2	1.5904	1.7828
		2.0	1.6583	2.0327
		5.0	1.6113	1.8346
		10.0	1.5917	1.7630
3	a = 1, b = 0.1	1.2	1.0981	1.3285
		2.0	1.0647	1.6761
		5.0	1.0296	1.3315
		10.0	1.1103	1.2823

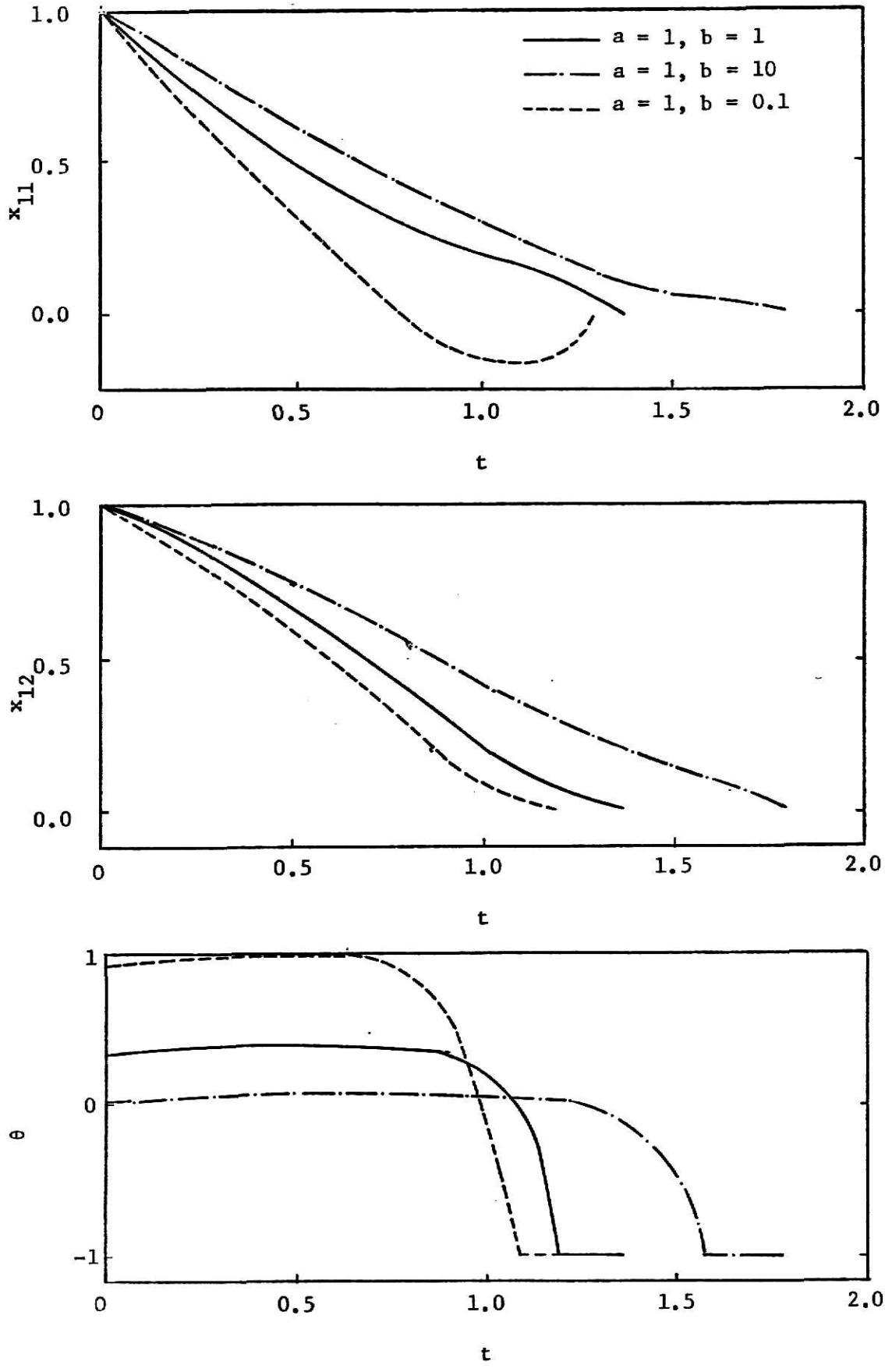


Fig. 5 Optimal policy and system responses of the 2 CST's-in-series model (Ex. 3), $r_{11} = 1.2$

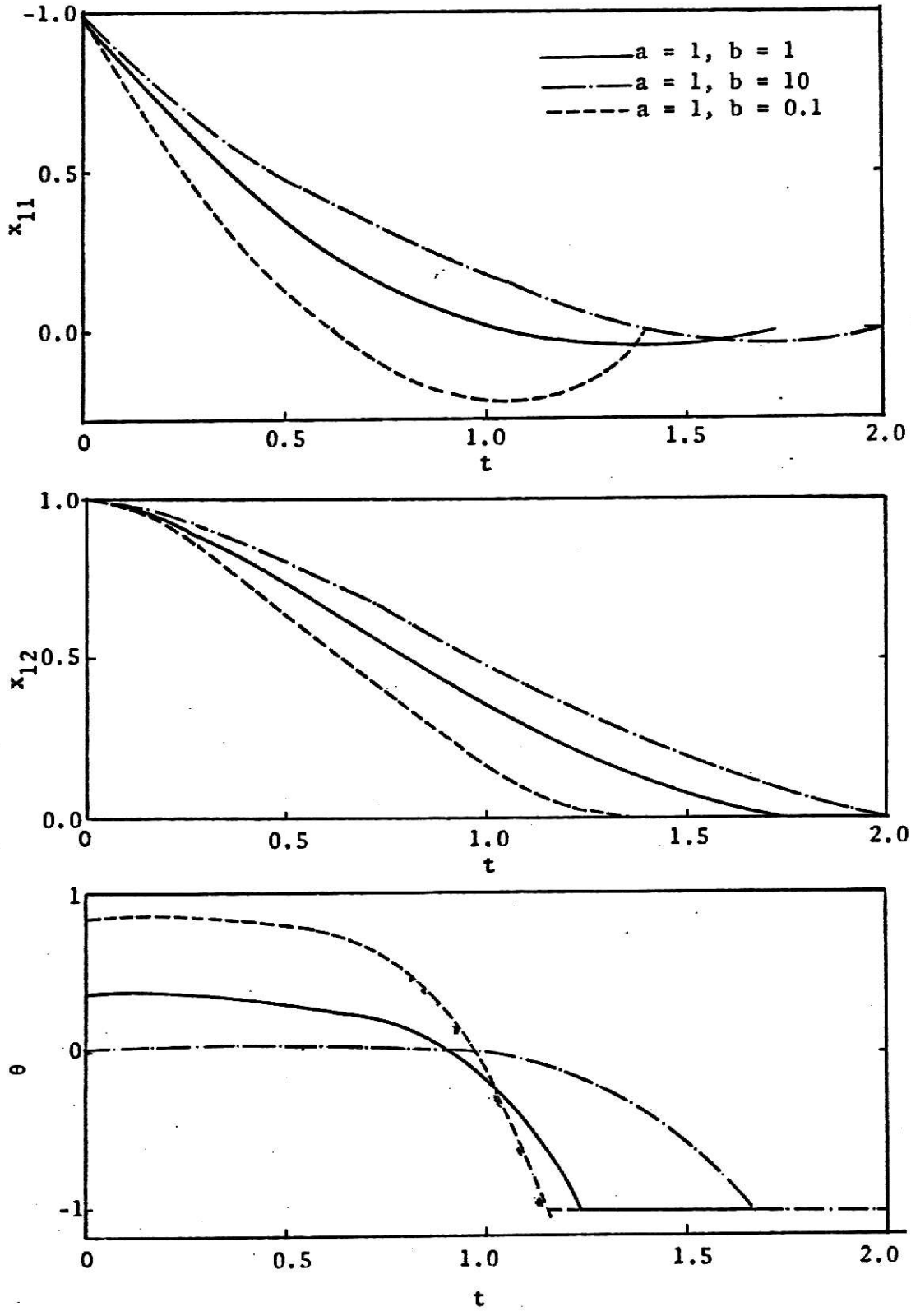


Fig. 6 Optimal control policy and system responses of the 2 CST's-in-series model (Ex. 3), $r_{11} = 2.0$

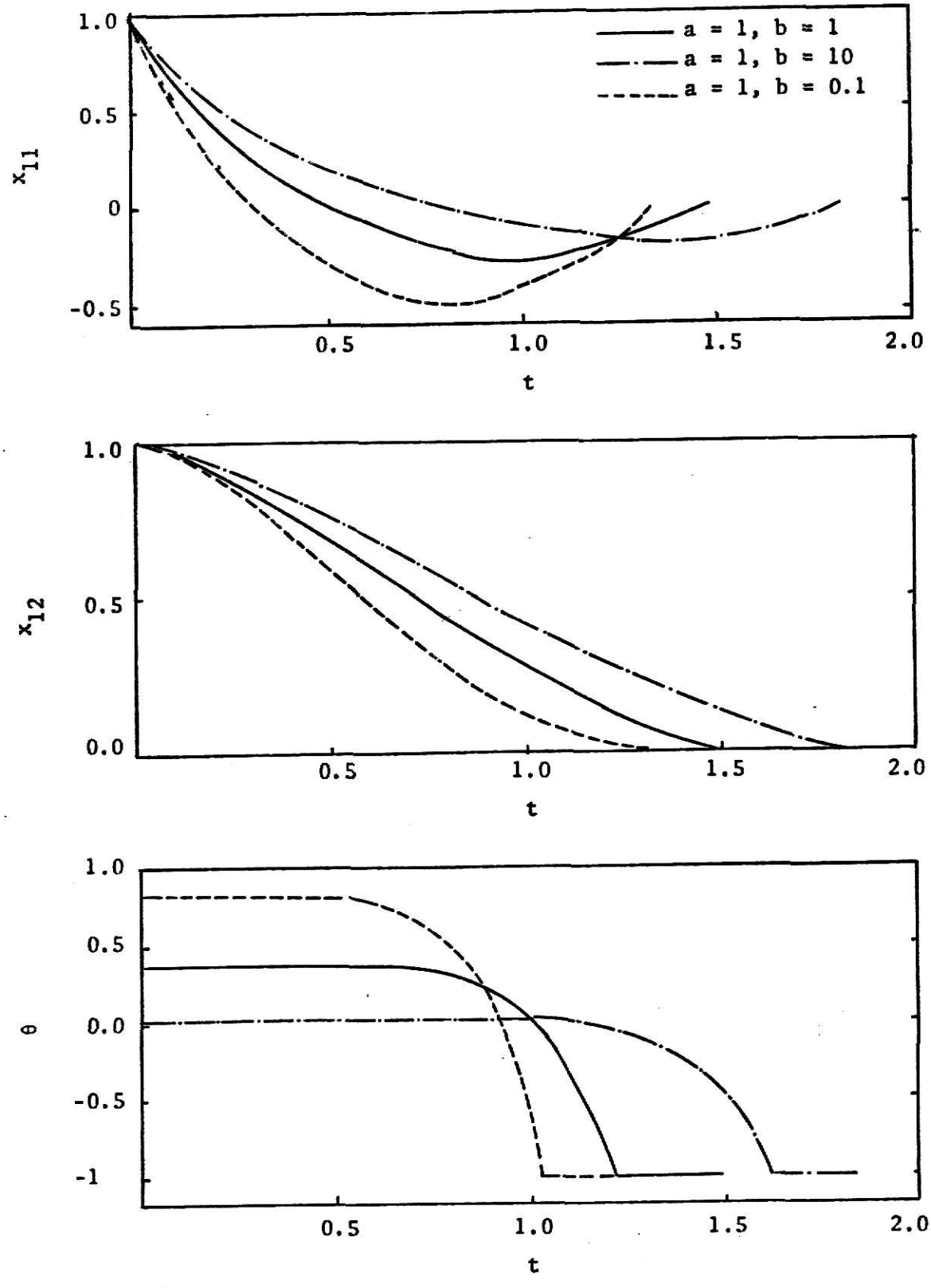


Fig. 7 Optimal control policy and system responses of the 2 CST's-in-series model (Ex. 3), $r_{11} = 5.0$

CHAPTER V

CONCLUSIONS AND SUGGESTIONS FOR FUTURE WORK

The methodology and procedure used in this report can also be employed for constructing and simulating models for other systems in which mass and momentum transfer take place in addition to heat transfer. The modern control theory can be applied to establish optimal control policy for such systems and to the environmental control of life support systems in general, including controls of humidity, purity and noise. The maximum principle has a certain advantage over other modern optimal techniques in that it can be applied not only to the system with linear performance equations but also to those with non-linear performance equations. The maximum principle can handle constraints on the state variables. Thus, any environmental control problem in which the temperature of the confined space has to be higher than a certain temperature - e.g. a biomedical process can be solved by means of the maximum principle [12].

The objective function can have terms other than those considered in this report. For example

$$S = \int_0^T dt$$

$$S = \int_0^T [a + b_1(x_1)^2]dt$$

$$S = \int_0^T [\theta]^2 dt$$

$$S = \int_0^T [b_1(x_1)^2 + c(\theta)^2]dt$$

$$S = \int_0^T [a + b_1(x_1)^2 + c(\theta)^2]dt$$

$$S = \int_0^T |\theta| dt$$

The objective functions have different physical significance [29, 30].

Some problems which are extension of the environmental control systems treated in chapters 3 and 4 are listed below. The optimal policies of these problems can be obtained by the similar approach presented in the preceding chapters.

Problem 1: This proposed problem is extension of Example 1 in chapter 3. The life support system consisting of an air-conditioned room and a heat exchanger of negligibly small time constant and subject to a step heat disturbance. The performance equation, the initial condition and the objective function are the same, however, the end condition of the state variable, the temperature, is fixed.

The performance equation then becomes

$$\frac{dx_1}{dt} + r_2 x_1 = r_2 K_\alpha - r_1 K_\beta \theta - r_1 K_\gamma + \sigma_s$$

with

$$x_1(0) = 1 \quad \text{at} \quad t = 0^-$$

$$x_1(T) = 1 \quad \text{at} \quad t = T$$

The objective function to be minimized is

$$S = \int_0^T [b\theta^2 + c(x_1 - x_{1d})^2] dt$$

where the final time, T, may or may not be specified.

The problem is to find an optimal control, $\bar{\theta}(t)$, which minimizes the objective function, S, subject to the performance equation and satisfy the initial and end conditions.

Problem 2: The system concerned is the same as in problem 1, however, the objective function is to include the optimal time term, i.e.,

$$S = \int_0^T [a + b\theta^2 + c(x_1 - x_{1d})^2]dt$$

where the final time, T, is not specified. The initial condition and the end condition of the state variable are given.

Problem 3: The system is the same as in problems 1 and 2, however, the objective function is of the form

$$S = \int_0^T [a + c(x_1 - 1)^2]dt$$

where the final time, T, is not specified. The initial and the end condition of the state variable are given.

Problem 4: If in problems 1, 2, and 3 and also in examples 1, 2, and 3 of chapter 3 an inequality constraint on the state variable that the temperature of the confined space should not exceed a given upper limit such as

$$x_1(t) \leq m \qquad 0 \leq t \leq T$$

is imposed, these problems will be transformed into a new set of problems. The maximum principle, with a slightly modified algorithm, can handle constraints on state variables [14].

Problem 5: In the previous problems it is assumed that initially the system is in the desired state, however, this may not be the case in some practical problems. In this example the system is the same as in the previous problems. The performance equation of the system (assuming the

time constant of the heat exchanger as negligible is

$$\frac{dx_1}{dt} + r_2 x_1 = r_2 K_\alpha - r_1 K_\beta \theta - r_1 K_\gamma + \sigma_s$$

The initial and the final conditions are

$$x_1(0) = \alpha_0$$

$$x_1(T) = 1$$

The problem is to find optimal controls, $\bar{\theta}(t)$, which brings the state from the initial, α_0, to the final desired state, 1, and minimizes the following objectives.

$$S = \int_0^T [b\theta^2 + c(x_1 - 1)^2]dt$$

$$S = \int_0^T [a + b\theta^2 + c(x_1 - 1)^2]dt$$

$$S = \int_0^T dt$$

where T may or may not be specified in the first objective function, but T are not specified for the second and the third objective functions.

Problem 6: Examples 1 and 2 of chapter 4 can be extended to include objective functions of the form

(i) $$S = \int_0^T [a + b_1(x_1 - x_{1d})^2]dt$$

(ii) $$S = \int_0^T [b_1(x_1 - x_{1d})^2 + c(\theta)^2]dt$$

(iii) $$S = \int_0^T [b_1(x_1 - x_{1d})^2 + c(\theta)^2]dt$$

(iv) $$S = \int_0^T |\theta| dt$$

(v) $$S = \int_0^T dt$$

The life support system in this problem consists of an air-conditioned room subjected to an impulse heat disturbance and a heat exchanger. The performance equations of this system are

$$\frac{dx_1}{dt} + x_1 = a_1 x_2 + a_2$$

$$\frac{dx_2}{dt} + rx_2 = a_4 x_1 - a_5\theta - a_6$$

with the boundary conditions

$$x_1(0) = 1, \quad x_2(0) = 1 \quad \text{at} \quad t = 0^+$$

$$x_1(T) = 0, \quad x_2(T) = 1 \quad \text{at} \quad t = T$$

Problem 7: In the 2 CST's-in-series model, the case in which the heat disturbance is distributed only on one pool can also be considered. Methodology and procedure used in this report can be employed in this case to derive the performance equations and to apply maximum principle for the solution of optimal policy.

In a summary, the variations of these proposed extension problems can be classified into three general types: (1) variation in the specification of initial or final conditions; (2) variations in the form of objective function; and (3) variation in the constraints on the state variables.

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NOMENCLATURE

a = Weighting factor in the objective function in Chapter 4

$a_5 = r K_2 K_4$ = rK_β in Chapters 2 and 3

$a_5' = K_2 K_4$

$a_6 = r K_3 K_4$ in Chapter 4, = rK_γ in Chapters 2 and 3

$a_6' = K_3 K_4$

$a_{11} = r_1 K_{11} r_{11} / K_4$ in Chapter 4, = $r_{11} r_1$ in Chapters 2 and 3

$a_{12} = r_1 r_2 K_{11}$ in Chapter 4, = $r_{11} r_2 K_\alpha$ in Chapters 2 and 3

$a_{13} = T_d K_{11} / T_2 r_{11}$

$a_{21} = r_{12} K_{12} / K_{11}$

$a_{23} = T_d K_{12} / T_2 r_{12}$

$a_{42} = r K_4 / K_{12}$

$a_{42}' = K_4 / K_{12}$

A_1 = Integration constant

A_2 = Integration constant

b = Weighting factor in the objective function

B_1 = Integration constant

B_2 = Integration constant

c = Weighting factor in the objective function

c_i = Constants defined in equation (3) of Chapter 2, i = 1, 2, ..., s

c_p = Specific heat of air in Kcal/Kg ^oC

c_{pw} = Specific heat of coolant in Kcal/Kg ^oC

D_1 = Integration constant

D_2 = Integration constant

E_1 = A constant

E_2 = A constant

$h_i[x(t)]$ = Function defined in equation [2] of Chapter 3, i = 1, 2, ..., p

$H[z(t),x(t),\theta(t)$ = Hamiltonian function defined in equation (5) of Chapter 3

K = A constant

K' = A constant

$K_1 = \frac{T_2}{T_{c0}}$

$K_2 = \frac{1}{2T_2}(T_{r\ max} - T_{r\ min})$

$K_3 = \frac{1}{2T_2}(T_{r\ max} + T_{r\ min})$

$K_4 = \frac{T_2}{T_{i0}}$

$K_{11} = \frac{T_2}{T_{1c0}}$

$K_{12} = \frac{T_2}{T_{2c0}}$

$K_\alpha = \frac{T_2}{T_{c0}}$

$K_\beta = \frac{1}{2T_{c0}}[T_{r\ max} - T_{r\ min}]$

$K_\gamma = \frac{1}{2T_{c0}}[T_{r\ max} - T_{r\ min}]$

$M_i = v_1\ \rho\ c_p\ T_d$

q_{di} = Impulse heat disturbance in impulse form in Kcal/sec

q_{ds} = Step heat disturbance in Kcal/sec

q_{i1} = Heat flow into the system proper by circulation air in Kcal/sec

q_{i2} = Heat flow into the system proper by fresh air in Kcal/sec

q_{mi1} = Heat flow into the heat exchanger by circulation air in Kcal/sec

q_{mi2} = Heat flow into the heat exchanger by cooling water in Kcal/sec

q_{m01} = Heat flow out of the heat exchanger by circulation air in Kcal/sec

q_{m02} = Heat flow out of the heat exchanger by cooling water in Kcal/sec

q_{ms} = Heat stored in the heat exchanger in Kcal/sec

q_{01} = Heat flow out of the system proper by circulation air in Kcal/sec

q_{02} = Heat flow out of the system proper by fresh air in Kcal/sec

q_s = Rate of heat accumulation in the system proper

Q = $Q_1 + Q_2$, flow rate of air in the system proper in m^3/sec

Q_1 = Air flow rate by circulation air in m^3/sec

Q_2 = Flow rate of fresh air in m^3/sec

Q_w = Flow rate of coolant in m^3/sec

r = $\frac{\tau_1}{\tau_2}$, the ratio of time constant of system proper to that of heat exchanger

r_1 = $\frac{Q_1}{Q_1 + Q_2}$, the fraction of circulation air

r_2 = $\frac{Q_2}{Q_1 + Q_2}$, the fraction of fresh air

r_{11} = $\frac{\tau_1}{\tau_{11}}$

r_{12} = $\frac{\tau_2}{\tau_{12}}$

t = $\frac{\alpha}{\tau_1}$, dimensionless time

t_a = Reference temperature in ^oC

t_c = Room temperature in ^oC

t_d = Disturbance temperature in ^oC

t_i = Temperature of incoming circulation air in ^oC

t_{wc} = Inlet temperature of coolant in ^oC

t_{wh} = Outlet temperature of coolant in ^oC

t_2 = Outside air temperature in ^oC

T = Final time, dimensionless

T_c = $(t_c - t_a)$, room temperature in ^oC

T_{c0} = Room temperature at $\alpha = 0^+$ in ^oC

T_{c1} = Temperature of pool 1 in ^oC

T_{c10} = Temperature of pool 1 at $\alpha = 0^+$ in ^oC

T_{c2} = Temperature of pool 2 in ^oC

T_{c20} = Temperature of pool 2 at $\alpha = 0^+$

T_d = $(t_d - t_a)$, disturbance temperature in ^oC

T_i = $(t_i - t_a)$, temperature of the circulation air into the system proper, in ^oC

T_{i0} = Temperature of the circulation air into the system proper at $\alpha = 0^+$ in ^oC

T_r = $\dfrac{Q_w \rho_w c_{pw} (T_{wh} - T_{wc})}{Q_1 \rho c_p}$, hypothetical temperature

T_{rf} = Final steady state value of T_r

$T_{r\,max}$ = Upper bound of T_r in ^oC

$T_{r\,min}$ = Lower bound of T_r in ^oC

T_{r0} = Value of T_r at $\alpha = 0$ in ^oC

T_{wc} = $t_{wc} - t_a$ in ^oC

T_{wh} = $t_{wh} - t_a$ in ^oC

$U_0(t)$ = Step heat disturbance function

V_1 = Volume of room in m^3

V_{11} = Volume of pool 1 of two completely stirred tanks in series model in m^3

V_2 = Volume of heat exchanger in m^3

V_{12} = Volume of pool 2 of two completely stirred tanks in series model in m^3

$x_1(t)$ = $\dfrac{T_c}{T_{c0}}$, dimensionless room temperature

x_{1d} = Desired value of x_1

$x_2(t)$ = Dimensionless temperature of the circulation air, = $\frac{T_i}{T_{c0}}$ in Chapters 2 and 3, = $\frac{T_i}{T_{i0}}$ in Chapter 4

x_{11} = $\frac{T_{c1}}{T_{c10}}$, dimensionless temperature of pool 1

x_{12} = Dimensionless temperature of pool 2, = $\frac{T_{c2}}{T_{c10}}$ in Chapters 2 and 3, = $\frac{T_{c2}}{T_{c20}}$ in Chapter 4

z_i = Adjoint variables defined in equation ( ) of Chapter 3

GREEK LETTERS

α = Time in sec.

$\delta(\alpha)$ = Impulse heat disturbance function, sec^{-1}

ρ = Air density in Kg/m^3

ρ_w = Density of coolant in Kg/m^3

σ = $\frac{T_d}{T_2}$, dimensionless disturbance temperature in Chapter 4

σ_s = $\frac{T_d}{T_{c0}}$, dimensionless disturbance temperature in Chapters 2 and 3

τ_1 = $\frac{V_1}{Q_1 + Q_2}$, time constant of the system proper in sec

τ_{11} = $\frac{V_{11}}{Q_1 + Q_2}$, time constant of pool 1 in sec

τ_2 = $\frac{V_2}{Q_1}$, time constant of heat exchanger in sec

τ_{12} = $\frac{V_{12}}{Q_1 + Q_2}$, time constant of pool 2 in sec

θ = $\frac{T_r - \frac{1}{2}(T_{r\,max} + T_{r\,min})}{T_{r\,max} - \frac{1}{2}(T_{r\,max} + T_{r\,min})}$, control variable

$$= \begin{cases} +1 & \text{at } T_r = T_{r\,max} \\ -1 & \text{at } T_r = T_{r\,min} \end{cases}$$

$\bar{\theta}(t)$ = Optimum value of $\theta(t)$

λ = Root of characteristic equation

λ_i = Roots of characteristic equation

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APPLICATION OF MODERN CONTROL THEORY TO THE ENVIRONMENTAL CONTROL OF CONFINED SPACES

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ABSTRACT

Mathematical models of an environmental control system which consists of a confined space or cabin, a heat exchanger and a feedback element such as a thermostat are presented. The performance equations representing the dynamic behaviour of the system are derived. In the basic model, the air in the confined space is considered to be in a state of complete mixing. To determine the goodness of the system model a computer simulation has been carried out and the results are compared with the known characteristics of the system. The performance equations for a life support system in which flow of air in the confined space is characterized by the two completely stirred tanks-in-series (2 CST's-in-series) model have also been derived.

The report illustrates the application of Pontryagin's maximum principle to obtain the optimal control policies of environmental control systems. The problem concerned in this report is to establish an optimal control policy that will minimize the sum of integrated deviations of the state of the system and the integrated control effort required to maintain the system in the desired state over a specified control period when the system is subjected to a step heat disturbance. Three concrete examples are considered. The first example treats the case in which the time constant of the heat exchanger is negligible. The second example considers the case in which the time constant of the heat exchanger is not ignored. In the third example the 2 CST's-in-series model is studied. The time constant of the heat exchanger is again neglected.

The case of an impulse heat disturbance is also considered. The objective function to be minimized in this case is the integrated effort

required to bring the system back from a deviated state to the desired one in the shortest time. The optimal control policy is established by again applying Pontryagin's maximum principle for three concrete examples.

The procedures and computational approaches used for obtaining optimal control policies of the environmental control system are given in detail. The modern control theory can advantageously be applied to automatic environmental control systems in space crafts, submarines, underground civil defence shelters and certain medical facilities. In these systems very stringent requirements on the response time, control effort, etc. are imposed.