THE THEORETICAL AND NUMERICAL TREATMENT OF THE "COLOR" EQUATION

by

GLEN WALLACE WITTEKIND

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Approved by:

Major Professor
Kane S. Yee

TABLE OF CONTENTS

Introduction

The purpose of this report is to demonstrate the practicality of making a theoretical analysis before attempting to numerically solve a partial differential equation (PDE). The report is divided into two parts. In part I, a theoretical analysis is made to find an appropriate finite difference scheme. In part II, actual calculations are made to affirm the results of the analysis.

We consider the PDE

$$u_t+cu_x=0$$

with initial conditions

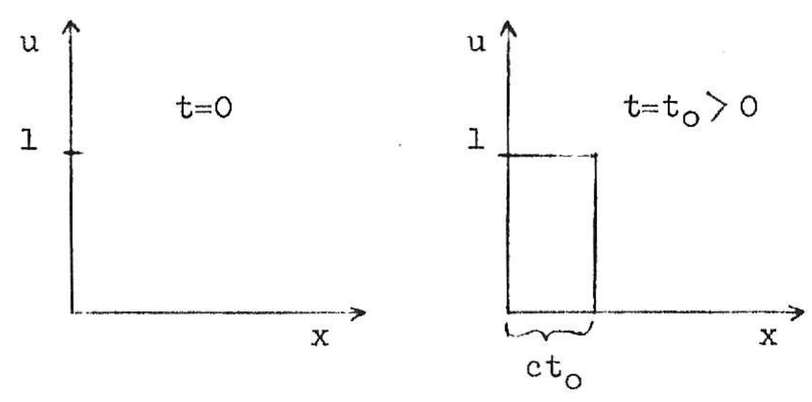
$$u(x,0)=0 \qquad x>0,$$

and boundary conditions

$$u(0,t)=\begin{cases}0 & t<0\\ 1 & t>0.\end{cases}$$

A finite difference scheme is developed which depends on a parameter $\alpha=\Delta x/(c\Delta t)$.

The exact solution to this problem is a positively traveling wave with velocity c, as shown in the figure.



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Part I.

A. Discussion of the equation

We shall first find the general solution to the so-called "color equation," $u_t+cu_x=0$, where x and t are real variables, and c is a positive constant. To this end, we introduce the one-to-one change of variables $\xi=x-ct$, and $\eta=x+ct$. Then

$$u_t=u_\xi \xi_t+u_\eta \eta_t=u_\xi(-c)+u_\eta(c), \text{ and}$$

$$u_x=u_\xi \xi_x+u_\eta \eta_x=u_\xi+u_\eta .$$

Hence the color equation becomes

$$-cu_\xi+cu_\eta+cu_\xi+cu_\eta=0, \text{ or}$$

$$2cu_\eta=0, \text{ or } u=f(\xi).$$

This last equation shows that the general solution must be of the form $u(x,t)=f(x-ct)$, where f is an arbitrary function which has a first derivative.

The solution is easy to interpret. It shows that u takes on a constant value along each straight line of slope $dt/dx=1/c$ (see Fig. 1). These lines are called the characteristic lines.

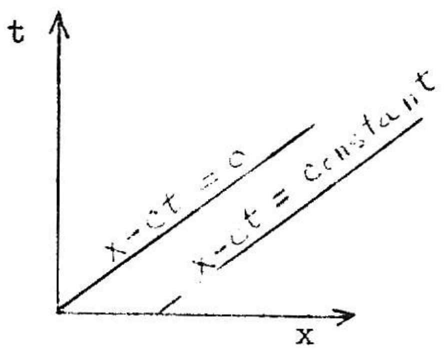

Fig. 1

B. Finite difference schemes

1. An "unstable" finite difference scheme

Consider a rectangle $0 \leq t \leq T$, and $0 \leq x \leq 1$ in the x-t plane, where T is a positive constant. We form a "grid" in the rectangle by specifying positive finite differences Δx and Δt. For simplicity, we'll take $\Delta x = 1/J$, where J is any given positive integer. To simplify the writing, we denote $u(j\Delta x, n\Delta t)$ by $(u)_j^n$ (or u_j^n), and similarly for u_x and u_t.

In order to solve the color equation numerically, we use the Taylor series to derive:

$$(u_t)_j^n = \frac{u_j^{n+1} - u_j^n}{\Delta t} + O(\Delta t), \text{ and}$$

$$(u_x)_j^n = \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + O((\Delta x)^2).$$

The latter is called a "center difference" scheme. We use a "forward difference" scheme for u_t since we want to go directly from level n to level n+1. Then the approximation to $u_t + cu_x = 0$ gives rise to the finite difference equation

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + c\frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} = 0, \text{ or}$$

$$U_j^{n+1} = U_j^n - \frac{c\Delta t}{2\Delta x}\left(U_{j+1}^n - U_{j-1}^n\right).$$

Observe that u is used for the exact solution, and U is used for the finite difference solution. If we set $\lambda = \frac{c\Delta t}{\Delta x}$, we have,

(1) $U_j^{n+1} = U_j^n - \frac{\lambda}{2}\left(U_{j+1}^n - U_{j-1}^n\right)$.

We now analyze equation (1). The simplest kind of analysis is the von Neumann analysis[1] where we assume periodicity in x (by assuming k is real) and we assume

$$U_j^n = e^{\beta t_n + ikx_j}$$

where $t_n = n\Delta t$, and $x_j = j\Delta x$. Then (1) becomes

$$e^{\beta t_{n+1}+ikx_j} = e^{\beta t_n+ikx_j} - \frac{\lambda}{2}\left(e^{\beta t_n+ikx_{j+1}} - e^{\beta t_n+ikx_{j-1}}\right) \text{, or}$$

$$e^{\beta t_n+\beta\Delta t+ikx_j} = e^{\beta t_n+ikx_j} - \frac{\lambda}{2}\left(e^{\beta t_n+ikx_j+ik\Delta x} - e^{\beta t_n+ikx_j-ik\Delta x}\right).$$

Hence,

$$e^{\beta\Delta t} = 1 - \frac{\lambda}{2}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right) = 1 - \lambda i \sin k\Delta x \text{, so}$$

$$\left|e^{\beta\Delta t}\right| = \sqrt{1+\lambda^2\sin^2 k\Delta x}\text{ . Thus,}$$

$$\left|U_j^n\right| = \left|\left(e^{\beta\Delta t}\right)^n e^{ikx_j}\right| = \left|e^{\beta\Delta t}\right|^n$$

$$= \left(1+\lambda^2\sin^2 k\Delta x\right)^{n/2}.$$

Our next aim is to introduce the notion of stability. To this end, for the PDE $u_t + cu_x = 0$, let continuous boundary

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[1] Isaacson, E., and Keller, H., Analysis of Numerical Methods, P. 523, John Wiley & Sons, Inc., N. Y., 1966.

conditions $u(0,t)=g(t)$, for $0 \leq t \leq T$, and continuous initial conditions $u(x,0)=f(x)$, for $0 \leq x \leq 1$, be given. (We require $g(0)=f(0)$.) Note that for the finite difference scheme, the initial conditions are $U_j^0=f(j\Delta x)$, and the boundary conditions are $U_0^n=g(n\Delta t)$.

Next, for any n such that $n\Delta t \leq T$, let U_j^n be obtained by some finite difference scheme, and define

$$\| U^n \| = \max_{0 \leq j \leq J} |U_j^n| ,$$

$$\| U^0 \| = \max_{0 \leq x \leq 1} |f(x)| , \text{ and}$$

$$\| U_0 \| = \max_{0 \leq t \leq T} |g(t)| . \text{ Then}$$

<u>Definition</u>: The finite difference scheme is stable if there are constants K_1 and K_2 such that

$$\|U^n\| \leq K_1 \| U^0 \| + K_2 \|U_0\| \quad \text{for all } n\Delta t \leq T.$$

The definition requires a continuous dependence upon the initial and boundary conditions, i.e., a small change in them means at most a small change in the computed solution.

The following claim will show that this scheme is unstable. In particular, we can find some n which violates the conditions of the definition.

<u>Claim</u>. Given any positive number N, any arbitrary upper bound $(\Delta x)_o$, and any $T > 0$, there is an n_o such that $n_o \Delta t \leq T$ and $|U_j^n| = |e^{\beta\Delta t}|^{n_o} > N$.

<u>Proof</u>:

(1) Find an n_o such that $(1+ \chi^2/2)^{n_o/2} > N$.

(2) Find a Δt such that $n_o \Delta t \leq T$.

(3) Choose any $\Delta x < (\Delta x)_0$, satisfying $\Delta x = c\Delta t/\lambda$.

Then there is a k such that $|\sin k\Delta x|^2 \geq \frac{1}{2}$. For this k and n_0, the claim holds since

$$|e^{\beta\Delta t}| = \sqrt{1+\lambda^2\sin^2 k\Delta x} \geq (1+\ \ ^2/2)^{\frac{1}{2}}, \text{ and by (2)}$$

$$|U_j^{n_0}| = |e^{\beta\Delta t}|^{n_0} > N.$$

It should be remarked that although this analysis may not seem rigorous, it is simple and practical.

2. A finite difference scheme that is stable

We now turn our attention to a finite difference scheme that is stable for certain values of the ratio $\lambda = c\Delta t/\Delta x$. Specifically,

$$U_j^{n+1} - \frac{U_{j+1}^n + U_{j-1}^n}{2} + \frac{c\Delta t}{2\Delta x}\left(U_{j+1}^n - U_{j-1}^n\right) = 0, \text{ or}$$

(3) $$U_j^{n+1} = \tfrac{1}{2}(U_{j+1}^n + U_{j-1}^n) - \frac{\lambda}{2}(U_{j+1}^n - U_{j-1}^n).$$

This scheme is obtained from the previous one by replacing U_j^n by an average, namely $(U_{j+1}^n + U_{j-1}^n)/2$.

In order to apply the von Neumann analysis to (3), we again assume

$$U_j^n = e^{\beta t_n} e^{ikx_j} \quad \text{where } t_n = n\Delta t, \text{ and } x_j = j\Delta x. \text{ Substituting}$$

this into equation (3) gives

$$e^{\beta\Delta t} = \tfrac{1}{2}\left(e^{ik\Delta x} + e^{-ik\Delta x}\right) - \frac{\lambda}{2}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right)$$

$$= \cos k\Delta x - i\lambda\sin k\Delta x. \text{ Thus,}$$

(4) $$|e^{\beta\Delta t}| = \sqrt{\cos^2 k\Delta x + \lambda^2\sin^2 k\Delta x}.$$

We see that if $\lambda \leq 1$, then $|e^{\beta \Delta t}| \leq 1$ for all k, i.e., all modes. Hence, there is no unlimited growth in any mode. We call $M(k)=e^{\beta\Delta t}= \cos k\Delta x - i\lambda \sin k\Delta x$ the amplification factor because we can write

$U_j^n=(M(k))^n e^{ikx_j}$. It's evident that no mode can grow whenever $\lambda \leq 1$, for then $|M(k)| \leq 1$.

We are now in a position to formulate the algorithm for numerical computations using equation (3). We consider the following initial boundary value problem:

$u_t+cu_x=0$, $0 \leq x \leq 1$, $0 \leq t \leq T$, $u(x,0)=f(x)$ and

$u(0,t)=g(t)$. (Note that $f(x)$ and $g(t)$ must be compatible at $x=t=0$.) The x domain is divided into an integral number of zones, J.

The Algorithm

1. Use the initial condition to get $U_j^0=f_j$, $j=0,1,\ldots,J$, where $f_j=f(j\Delta x)$.

2. Use equation (3) for $j=1,2,\ldots,J-1$:

$$(3) \quad U_j^{n+1}=\tfrac{1}{2}(U_{j+1}^n+U_{j-1}^n) - \frac{\lambda}{2}(U_{j+1}^n-U_{j-1}^n).$$

3. Use the boundary condition at $x=0$ to get $U_0^n=g_n$ where $g_n=g(n\Delta t)$.

4. When we try to find U_J^1 (or U_J^n in general), (3) will not work since U_{J+1}^0 is unknown; nevertheless, the solution for this point is easy. One interpolates by using lines of slope $1/c$.

A diagram will be helpful in developing the general case (see Fig. 2). The line of slope $1/c$ is a characteristic,

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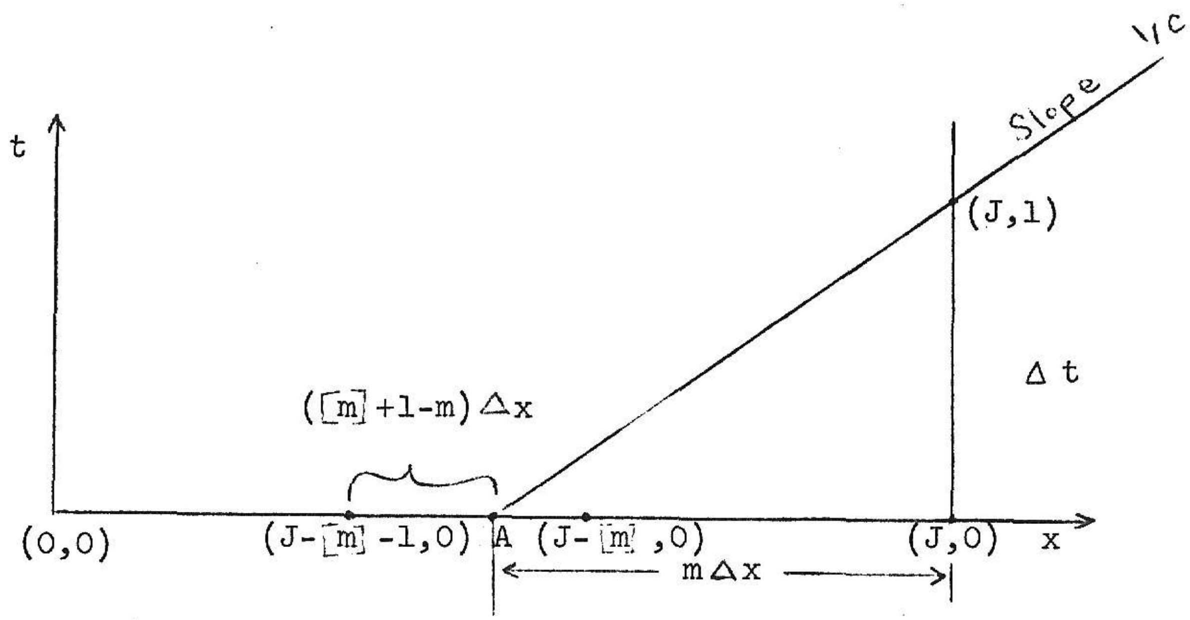


Notes:

1. The slope $1/c = \Delta t/(m\Delta x)$ gives $m = c\Delta t/\Delta x = \lambda$, the λ used throughout.

2. The brackets [ ] denote the greatest integer part of the quantity in the brackets.

Figure 2.

and (as noted earlier) the solution to our PDE is a constant along a characteristic. The linear interpolation formula is thus

$$U_J^1 = U(A) = ([m]+1-m)U^0_{J-[m]} + (m-[m])U^0_{J-[m]-1},$$

where $m = \lambda$ as noted in the figure.

The derivation shows that we must require $m < J$ so that $[m] \leq J-1$. This will assure us that the line of interpolation crosses the x-axis before it crosses the t-axis. But, as we've seen, $\lambda \leq 1$ must be used if we're interested in a stable scheme. Thus in general, we have

$$(5) \quad U_J^{n+1} = ([\lambda]+1-\lambda)U^n_{J-[\lambda]} + (\lambda-[\lambda])U^n_{J-[\lambda]-1}.$$

It should be remarked that if $\lambda=1$, then equation (3) becomes $U_j^{n+1} = U_{j-1}^n$, which is the exact solution since then $\Delta t/\Delta x = 1/c$, i.e., the grid points are the slope of the characteristics away from each other.

There are two more things that we can do that are appropriate to our discussion of equation (3). One is to apply another elementary analysis, and the other is to discuss convergence to the true solution. We first present the analysis. It will show that the computed solution has no unlimited growth for $\lambda \leq 1$, and is in fact bounded by the boundary and initial conditions.

We introduce $U^n = (U_0^n, U_1^n, \ldots, U_J^n)$ with a norm $\|U^n\| = \sup_{0 \leq j \leq J} |U_j^n|$. For $\lambda \leq 1$, we have by (3)

$$U_j^{n+1} = (\tfrac{1}{2} - \lambda/2)U^n_{j+1} + (\tfrac{1}{2} + \lambda/2)U^n_{j-1} \text{ or,}$$

$$U_j^{n+1} \quad (\tfrac{1}{2} - \lambda/2)\|U^n\| + (\tfrac{1}{2} + \lambda/2)\|U^n\| \quad \text{or,}$$

$|U_j^{n+1}| \le \|U^n\|$, $j=1,2, \ldots, J-1$. Furthermore,

$|U_0^{n+1}| = |g_{n+1}|$, and equation (5) gives

$|U_J^{n+1}| \le \|U^n\|$. Thus

$$\begin{aligned}
\|U^{n+1}\| &\le \max(|g_{n+1}|, \|U^n\|) \\
&\le \max(|g_{n+1}|, |g_n|, \|U^{n-1}\|) \\
&\le \ldots \\
&\le \max(|g_{n+1}|, |g_n|, \ldots, |g_1|, \|U^0\|).
\end{aligned}$$

Now assume $(n+1)\Delta t \le T$, and let $\|g\| = \sup_{0 \le t \le T} |g(t)|$, and $\|f\| = \sup_{0 \le x \le 1} |f(x)|$. Then

$$\|U^{n+1}\| \le \max(\|f\|, \|g\|),$$

as was to be shown. Thus, by a previous definition, the scheme is stable.

Finally, it is important to discuss convergence. In this discussion, we assume that as many partial derivatives exist as are necessary, and that they are bounded.

Denote by $u(x,t)$ the actual solution to our problem, and by U_j^n the finite difference solution. Then for $j=1,2,\ldots,J-1$ we have

(6) $U_j^{n+1} = \frac{1}{2}(U_{j+1}^n + U_{j-1}^n) - \frac{\lambda}{2}(U_{j+1}^n - U_{j-1}^n)$, and

(7) $u_j^{n+1} = \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) - \frac{\lambda}{2}(u_{j+1}^n - u_{j-1}^n) + \Delta t\, \tau_j^n$ where τ_j^n is the truncation error, which will be determined, and $\lambda = c\Delta t/\Delta x$.

In order to determine τ_j^n, in (7) we use

$$u_j^{n+1} = u_j^n + (u_t)_j^n \Delta t + (u_{tt})_j^{n+} \frac{(\Delta t)^2}{2}, \text{ where } n < n+ < n+1,$$

and similar expressions for u_{j+1}^n and u_{j-1}^n. We also assume that there are constants C_1, C_2, and C_3 such that $|u_{tt}| \le C_1$, $|u_{xx}| \le C_2$, and $|u_{xxx}| \le C_3$ on $0 \le x \le 1$, $0 \le t \le T$, where T is the maximum time we allow.

Thus (7) gives

$$u_j^n+(u_t)_j^n\Delta t+(u_{tt})_j^{n+}\frac{(\Delta t)^2}{2}=\tfrac{1}{2}(u_j^n+(u_x)_j^n\Delta x+(u_{xx})_{j+}^n\frac{(\Delta x)^2}{2}$$
$$+u_j^n-(u_x)_j^n\Delta x+(u_{xx})_{j-}^n\frac{(\Delta x)^2}{2}) - \frac{\lambda}{2}(u_j^n+(u_x)_j^n\Delta x+(u_{xx})_{j+}^n\frac{\Delta x^2}{2}$$
$$+(u_{xxx})_{j+}^n\frac{(\Delta x)^3}{6}-u_j^n+(u_x)_j^n\Delta x-(u_{xx})_j^n\frac{(\Delta x)^2}{2}+(u_{xxx})_{j-}^n\frac{(\Delta x)^3}{6})$$
$$+\Delta t\,\tau_j^n, \text{ or}$$

$$(u_t)_j^n\ \ t+(u_{tt})_j^{n+}\frac{(\Delta t)^2}{2}=\tfrac{1}{2}((u_{xx})_{j+}^n\frac{(\Delta x)^2}{2}+(u_{xx})_{j-}^n\frac{(\Delta x)^2}{2})$$
$$-\frac{c\Delta t}{2\Delta x}((u_x)_j^n\Delta x+(u_{xxx})_{j+}^n\frac{(\Delta x)^3}{6}+(u_x)_j^n\Delta x+(u_{xxx})_{j-}^n\frac{(\Delta x)^3}{6})$$
$$+\Delta t\,\tau_j^n.$$

The three first derivative terms in the last expression drop out because u satisfies $(u_t)_j^n=-c(u_x)_j^n$ by the assumption that it satisfies the PDE $u_t+cu_x=0$. Further, since $\lambda=c\Delta t/\Delta x$, we have $(\Delta x)^2=\frac{c^2(\Delta t)^2}{\lambda^2}$. Using these facts, the above becomes

$$(u_{tt})_j^{n+}\frac{(\Delta t)^2}{2}-\tfrac{1}{2}((u_{xx})_{j+}^n+(u_{xx})_{j-}^n)\frac{c^2(\Delta t)^2}{2\lambda^2}$$
$$+\frac{c\Delta t}{2\Delta x}((u_{xxx})_{j+}^n\frac{(\Delta x)^3}{6}+(u_{xxx})_{j-}^n\frac{(\Delta x)^3}{6}) = \Delta t\,\tau_j^n.$$

Now recall that u_{tt}, u_{xx}, and u_{xxx} are bounded by C_1, C_2, and C_3 respectively. Then, by the last equation, we find that there exist constants K_1 and K_2 such that

$|\tau_j^n|\leq K_1\Delta t+K_2(\Delta x)^2$ for j=1, 2, ..., J-1 and all n such that $n\Delta t\leq T$. Further, considering λ to be fixed, by the definition of λ we have

$\lambda\Delta x=c\Delta t$, or $\Delta x=(c\Delta t)/\lambda$. Thus

$|\tau_j^n|\leq K_1\Delta t+K_2c^2\frac{(\Delta t)^2}{\lambda^2}$. Finally, let $(\Delta t)_0$ be so small that $\Delta t\leq(\Delta t)_0$ implies

$\left(\frac{1}{\lambda^2}\right)K_2c^2(\Delta t)^2\leq K_1\Delta t$. Then for any $\Delta t\leq(\Delta t)_0$, we have

$\tau_j^n\leq 2K_1\Delta t$. We say $|\tau_j^n|=O(\Delta t)$.

We also need to recall $u_0^{n+1}=g_{n+1}$, and (by Fig. 2)

$u_J^{n+1}=u_{J-\lambda}^n=u((J-\lambda)\Delta x,n\Delta t)$.

If we use (5), the linear interpolation formula, then

$$u_J^{n+1}=u_{J-\lambda}^n=([\lambda]+1-\lambda)u_{J-[\lambda]}^n+(\lambda-[\lambda])u_{J-[\lambda]-1}^n+K_3(\Delta x)^2$$

since the correction to a linear interpolation involves $(\Delta x)^2$ (and higher orders which we neglect).

Define the error to be $e_j^n=u_j^n-U_j^n$. Then equation (7) minus equation (6) gives

$$(8)\quad e_j^{n+1}=\tfrac{1}{2}(e_{j+1}^n+e_{j-1}^n)-\frac{\lambda}{2}(e_{j+1}^n-e_{j-1}^n)+\Delta t\,\tau_j^n \text{ for } j=1,2,\ \ldots,\ J-1.$$

Furthermore, at the boundaries we have

$$(9)\quad e_0^{n+1}=u_0^{n+1}-U_0^{n+1}=g_{n+1}-g_{n+1}=0, \text{ and}$$

$$(10)\quad e_J^{n+1}=([\lambda]+1-\lambda)e_{J-[\lambda]}^n+(\lambda-[\lambda])e_{J-[\lambda]-1}^n+K_3(\Delta x)^2.$$

Let $\tau_J^n=K_3\dfrac{c\Delta x}{\lambda}=K_3\dfrac{c^2\Delta t}{\lambda^2}$. Define $\|\tau^n\|=\sup\limits_{1\leq j\leq J}|\tau_j^n|$ and $\|e^n\|=\max\limits_{0\leq j\leq J}|e_j^n|$. Now require $\lambda\leq 1$ (and fixed).

Then $\tau_J^n=O(\Delta t)$, and from (8), we get

$$|e_j^{n+1}|\leq\|e^n\|+\Delta t\|\tau^n\|,\ j=1,\ 2,\ \ldots,\ J-1.$$

From (9) and (10),

$$|e_0^{n+1}|=0, \text{ and}$$

$$\begin{aligned}|e_J^{n+1}|&\leq\|e^n\|+(\Delta x)^2K_3=\|e^n\|+\Delta t(c\Delta xK_3)/\lambda\\&=\|e^n\|+\Delta t\,\tau_J^n\leq\|e^n\|+\Delta t\|\tau^n\|.\end{aligned}$$

Thus

$$\begin{aligned}\|e^{n+1}\|&\leq\|e^n\|+\Delta t\|\tau^n\|\\&\leq\|e^{n-1}\|+\Delta t\|\tau^{n-1}\|+\Delta t\|\tau^n\|\\&\leq\|e^0\|+\Delta t\|\tau^0\|+\ \ldots\ +\Delta t\|\tau^n\|\\&\leq(n+1)\Delta t\,\tau\leq T\tau=O(\Delta t)\end{aligned}$$

since $\|e^0\|=0$, and τ, defined by $\tau=\max\limits_{(n+1)\Delta t\leq T}\|\tau^n\|$

$$=\max_{(n+1)\Delta t\leq T}\ \max_{1\leq j\leq J}|\tau_j^n|,$$ is of $O(\Delta t)$ since all τ_j^n

are of $O(\Delta t)$. Notice that n is restricted by $(n+1)\Delta t \leq T$. Hence as Δt goes to zero, we see $\|e^n\|$ goes to zero. This fact implies that the finite difference solution converges to the true solution provided that the initial and boundary data are smooth.

Define $\alpha = 1/\lambda$. We have seen that the finite difference scheme converges for all values of $\lambda \leq 1$, or $\alpha \geq 1$, with $\lambda = 1$ giving the exact solution.

Part II. Actual Calculations

This part is done in three sections: (1) statement of the problem, (2) flow chart and explaination, and (3) the program and results.

(1) Statement of the problem

In the previously considered equation, we may take the constant c to be one since it may be made thus by using a suitable time (or distance) scale. Hence we investigate the PDE

$u_t+u_x=0$, with initial conditions

$u(x,0)=0, \quad x \geqslant 0$, and boundary conditions

$$u(0,t)=\begin{cases} 0 & t<0 \\ 1 & t>0. \end{cases}$$

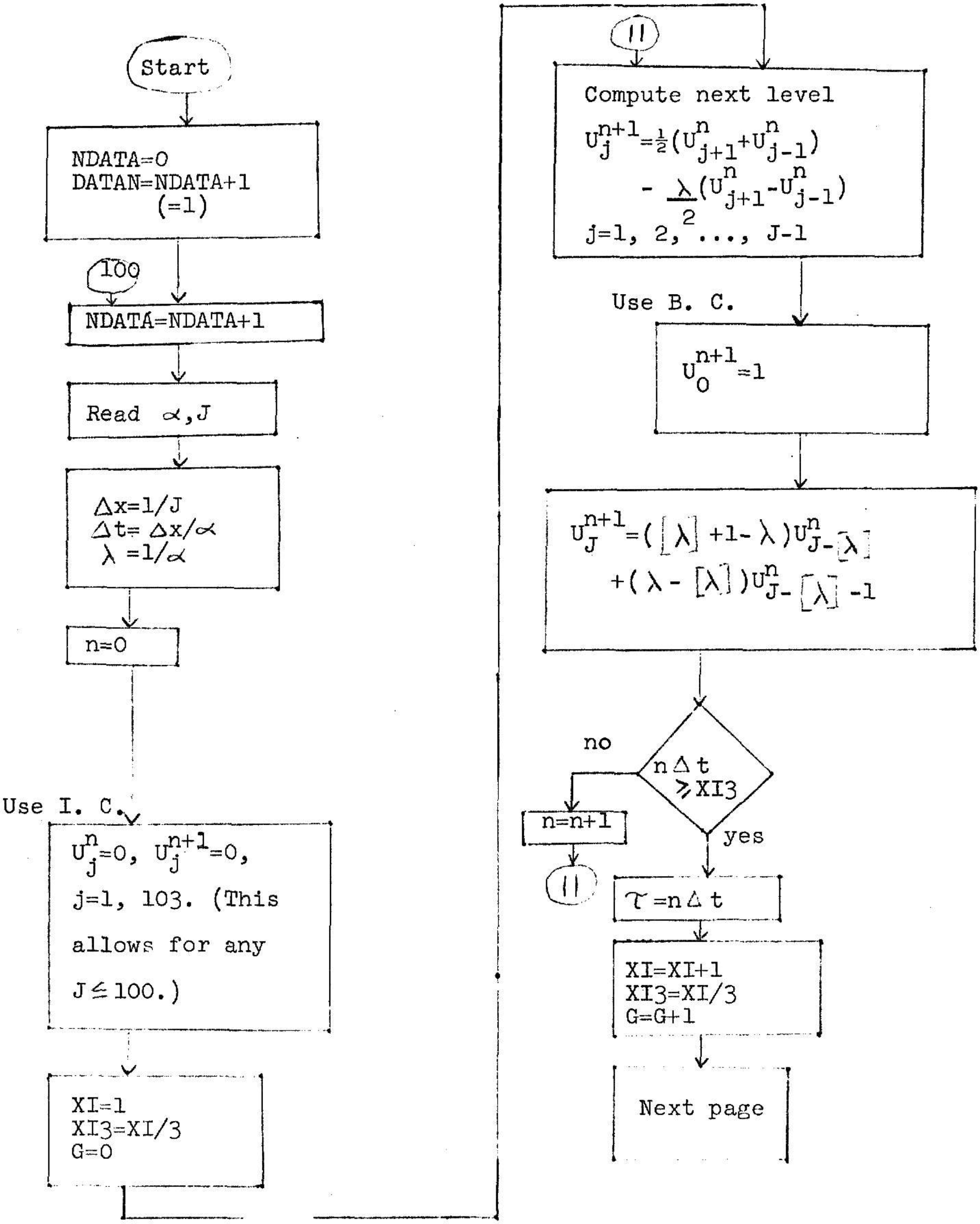
The ideal solution is a positively traveling wave. It can be noted that these boundary conditions are not as differentiable as those needed in the convergence proof.

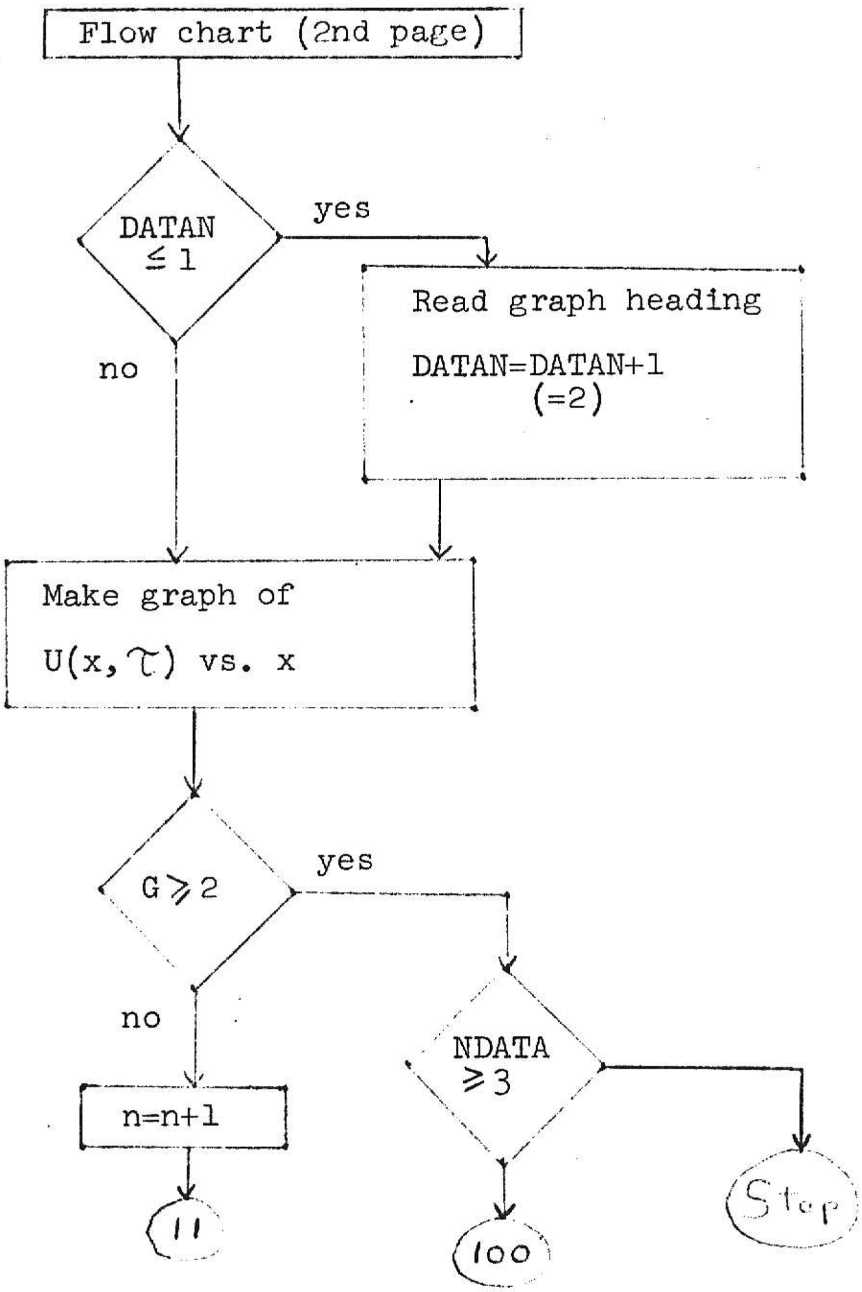
The calculations are done for a few values of $\alpha=\Delta x/\Delta t$. For each α, graphs of $u(x,t)$ vs. x are plotted for $t=1/3$ and $t=2/3$.

(2) Flow chart

We now pass on to presenting a flow chart made to aid in writing the Fortran program. The algorithm of part I is used. The computations of $u(x,t)$ are limited to $0\leq x\leq 1$ and $0\leq t\leq T=2/3$.

Flow Chart:


Start
NDATA=0
DATAN=NDATA+1
(=1)
100
NDATA=NDATA+1
Read α,J
$\Delta x=1/J$
$\Delta t=\Delta x/\alpha$
$\lambda=1/\alpha$
n=0
Use I. C.
$U_j^n=0$, $U_j^{n+1}=0$,
j=1, 103. (This
allows for any
$J\leq 100$.)
XI=1
XI3=XI/3
G=0
11
Compute next level
$U_j^{n+1}=\frac{1}{2}(U_{j+1}^n+U_{j-1}^n)$
$-\frac{\lambda}{2}(U_{j+1}^n-U_{j-1}^n)$
j=1, 2, ..., J-1
Use B. C.
$U_0^{n+1}=1$
$U_J^{n+1}=([\lambda]+1-\lambda)U^n_{J-[\lambda]}$
$+(\lambda-[\lambda])U^n_{J-[\lambda]-1}$
no
$n\Delta t$
$\geqslant$XI3
yes
n=n+1
11
$\tau=n\Delta t$
XI=XI+1
XI3=XI/3
G=G+1
Next page


Flow chart (2nd page)
DATAN ≤ 1
yes
no
Read graph heading
DATAN=DATAN+1
(=2)
Make graph of
U(x, τ) vs. x
G ≥ 2
yes
no
NDATA ≥ 3
n=n+1
11
100
Stop

Flow Chart (continued)

This flow program reads three sets of   and J, and makes graphs of $u(x,\tau)$ vs. x for $\tau = 1/3$ and $\tau = 2/3$ for each set, i.e., two graphs (G=2) for each set of data.

The Variables

J = number of x grid zones

α = $\Delta x/\Delta t = 1/\lambda$

NDATA = number of data sets read in

G = number of graphs made per data set

N = time level

The Indicators

DATAN tells whether graph heading is to be read (≤ 1) or not (> 1).

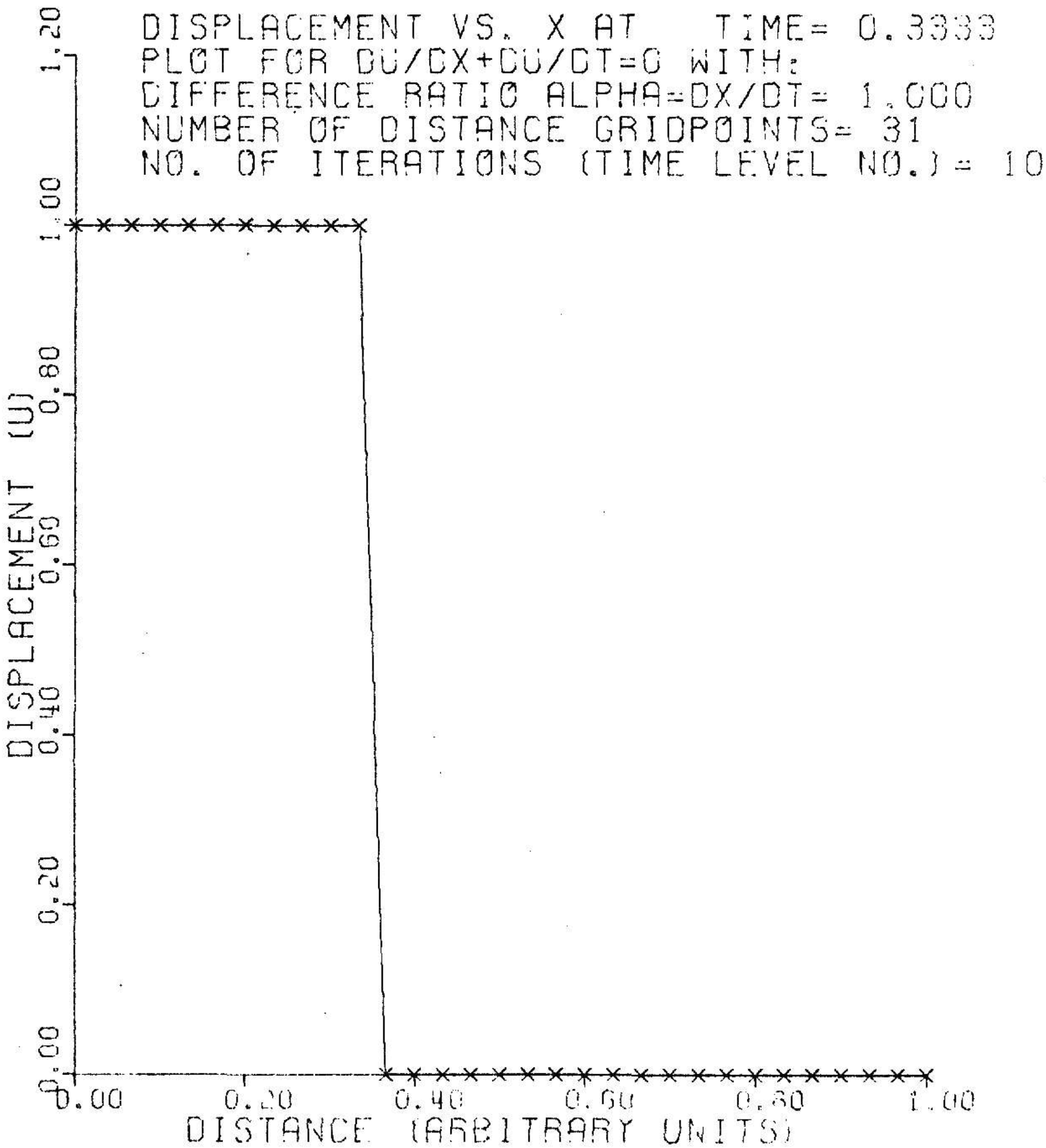
XI(XI3) tells whether graphs are to be made or not.

(3) THE FINAL FORTRAN PROGRAM. THE JOB (ACCOUNT NUMBER) CARD OCCUPIES THIS LIN

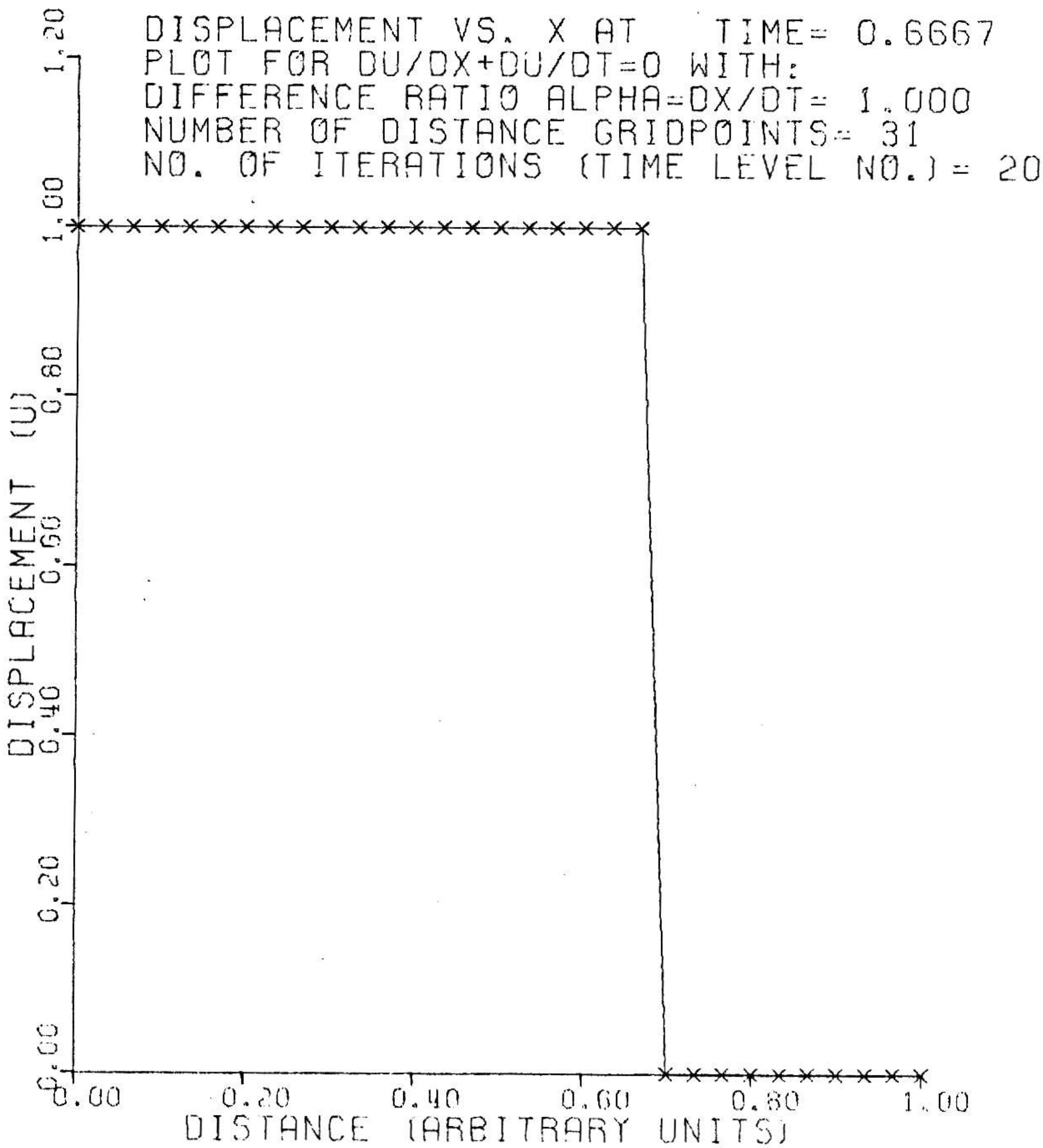
```
//*TAPE9
//            EXEC FORTGCLG,PARM=PLOT
//FORT.SYSIN DD *
      DIMENSION X(105),BUF(8000),Y(105),TITLE(50)
      DOUBLE PRECISION U(103),UNP1(103)
      CALL PLOTS(BUF,8000)
      CALL PLOT(0.0,0.5,-3)
C STORE SPACE FOR GRIDPOINT VALUES OF U AND X. JCAP<100.
C JCAP=NUMBER OF GRID ZONES FOR X.
C NDATA =NUMBER OF DATA PAIRS READ IN.
      NDATA=0
      DATAN=NDATA+1
  100 NDATA=NDATA+1
      READ (5,101)ALFA,JCAP
  101 FORMAT(F10.4,I4)
      XJ=JCAP
      DX=1./XJ
      DT=(DX/ALFA)
      XL=1./ALFA
C CALCULATE X AND T INCREMENTS AND LLAMDA (=XL).
C INITIALIZATION. NOTE THAT SUBSCRIPTS MUST START AT ONE.
      N=0
      DO 1 JJ=1,103
      U(JJ)=0.
    1 UNP1(JJ)=0.
C PRESCRIBE INITIAL CONDITIONS, IF THEY ARE NON ZERO.
      XI=1.
      XI3=XI/3
      G=0.
C G=NUMBER OF GRAPHS.
      JCAPM1=JCAP-1
      JCAPP1=JCAP+1
      IXL=XL
C IXL= THE GREATEST INTEGER IN LLAMDA.
      F=IXL+1.-XL
      FF=XL-IXL
   11 DO 2 J=1,JCAPM1
    2 UNP1(J+1)=.5*((U(J+2)+U(J))-XL*(U(J+2)-U(J)))
      UNP1(1)=1.
      UNP1(JCAPP1)=F*U(JCAPP1-IXL)+FF*U(JCAP-IXL)
      DO 3 JJ=1,JCAPP1
    3 U(JJ)=UNP1(JJ)
      IF(N*DT-XI3   )4,5,5
    4 N=N+1
      GO TO 11
    5 TAU=N*DT
      XI=XI+1.
      XI3=XI/3
      G=G+1.
C MAKE GRAPH.
      DO 51 JJ=1,JCAPP1
      X(JJ)=(JJ-1)*DX
   51 CONTINUE
      DO 52 J=1,105
   52 Y(J)=U(J)
      T=TAU
      IF(DATAN-1)90,90,91
   90 CONTINUE
      DATAN=DATAN+1
```

```
C A CARD SUCH AS   READ(5,102)TITLE   ASSUMES THAT THE FULL DIMENSION OF TITL
C IS TO BE READ IN.
      READ (5,102)(TITLE(I),I=1,8)
      READ (5,102)(TITLE(I),I=9,16)
      READ (5,102)(TITLE(I),I=17,24)
      READ (5,102)(TITLE(I),I=25,32)
  102 FORMAT(8A4)
      READ (5,104)(TITLE(I),I=33,41)
  104 FORMAT(9A4)
   91 CONTINUE
      NPTS=JCAPP1
C NPTS IS THE NUMBER OF DATA POINTS.
      CALL SCALE(X,5.0,NPTS,1)
      CALL SCALE(Y,6.0,NPTS,1)
C THE AXIS TITLES CAN POSSIBLY BE READ IN THE WAY THE TITLE IS HANDLED.
      CALL AXIS(0.0,0.0,26HDISTANCE (ARBITRARY UNITS),-26,5.0,0.0,
     1X(NPTS+1),X(NPTS+2))
      CALL AXIS(0.0,0.0,16HDISPLACEMENT (U),16,6.0,90.0,Y(NPTS+1),
     1 Y(NPTS+2))
      CALL LINE(X,Y,NPTS,1,1,4)
      CALL SYMBOL(0.4,6.1,0.14,TITLE(1),0.0,30)
      CALL NUMBER(999.0,999.0,0.14,T,0.0,4)
      CALL SYMBOL(0.4,5.9,0.14,TITLE(9),0.0,28)
      CALL SYMBOL(0.4,5.7,0.14,TITLE(17),0.0,30)
      CALL NUMBER(999.0,999.0,0.14,ALFA,0.0,3)
      CALL SYMBOL(0.4,5.5,0.14,TITLE(25),0.0,31)
      XJP1=JCAPP1
      CALL NUMBER(999.0,999.0,0.14,XJP1,0.0,-1)
      CALL SYMBOL(0.4,5.3,0.14,TITLE(33),0.0,36)
      XN=N
      CALL NUMBER(999.0,999.0,0.14,XN,0.0,-1)
      CALL PLOT(6.4,0.0,-3)
      NG=G
      IF(NG-2)6,7,7
    6 N=N+1
      GO TO 11
    7 CONTINUE
      WRITE(6,701)NDATA,G
  701 FORMAT('NDATA= ',I4,'  GRAPHS MADE= ',F9.1)
      IF(NDATA-3)100,8,8
    8 CONTINUE
      CALL PLOT( 9.0,0.0,999)
      STOP
      END
//GO.PLOTTAPE DD UNIT=TAPE9,DISP=(,KEEP),VOL=PRIVATE
//GO.SYSIN DD *
    1.0      30
DISPLACEMENT VS. X AT   TIME=
PLOT FOR DU/DX+DU/DT=0 WITH:
DIFFERENCE RATIO ALPHA=DX/DT=
NUMBER OF DISTANCE GRIDPOINTS=
NO. OF ITERATIONS (TIME LEVEL NO.)=
    0.5      30
    1.5      30
$STOP
/*
```

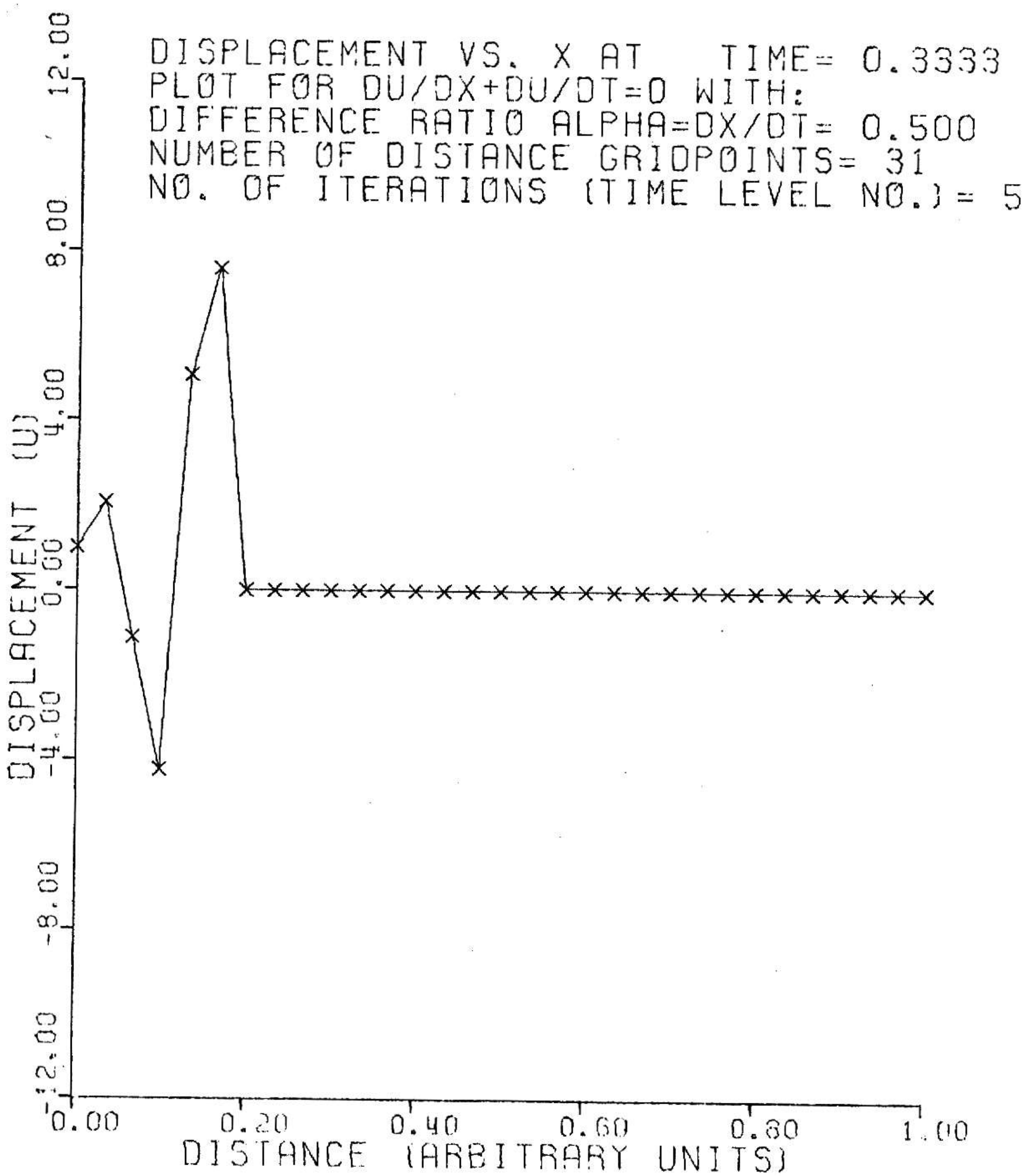
Exact solution



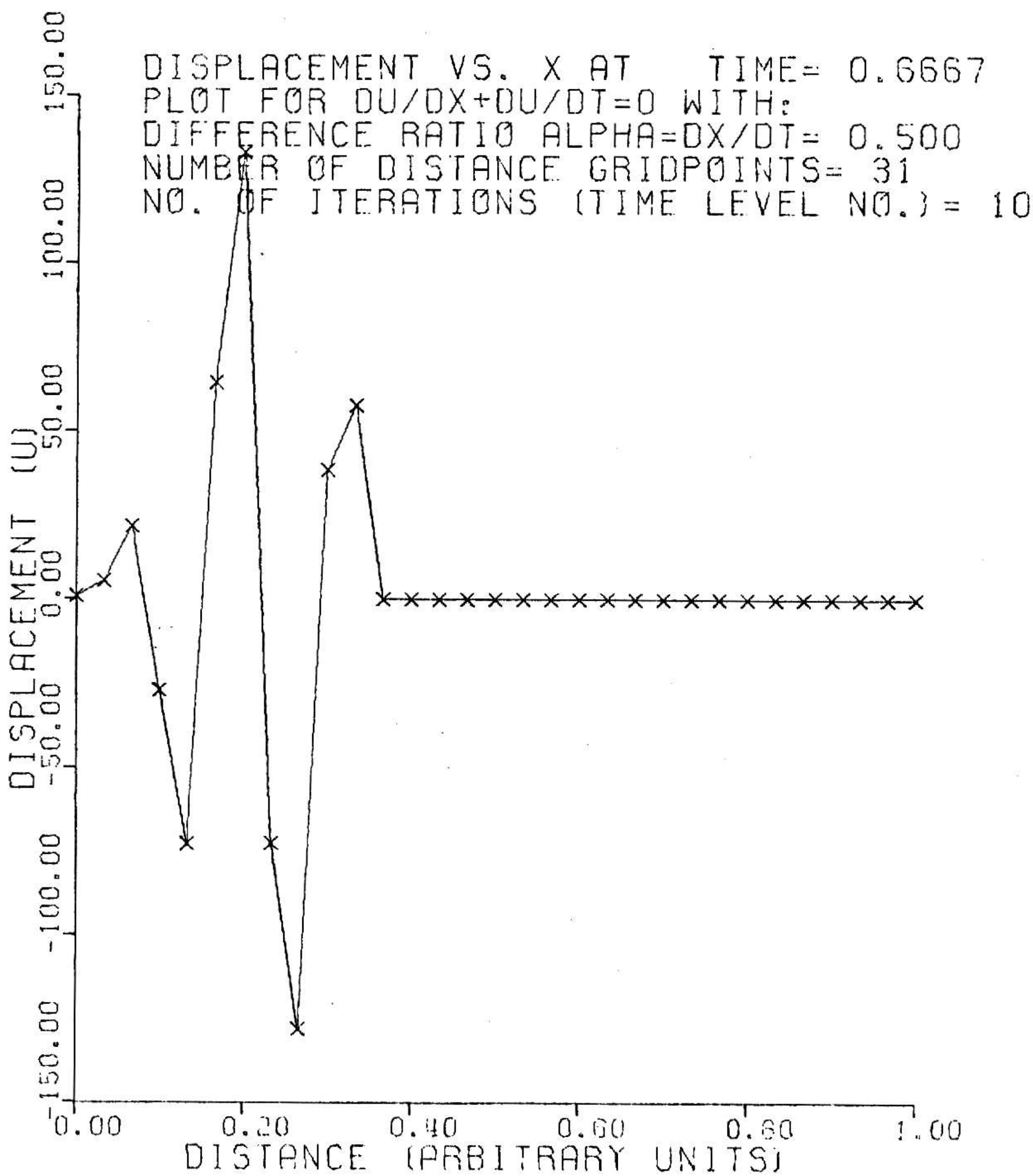
Exact solution



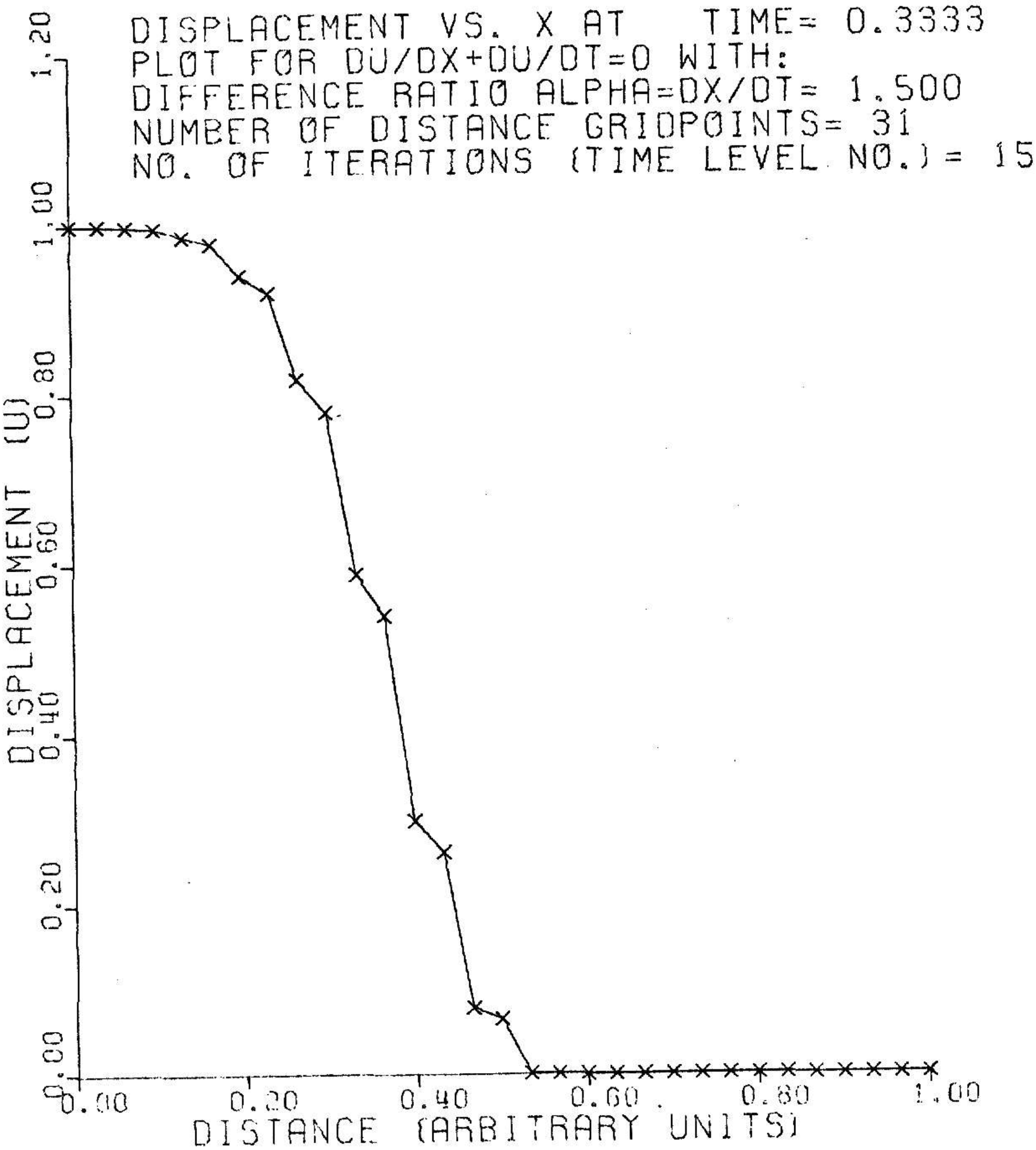
Instability



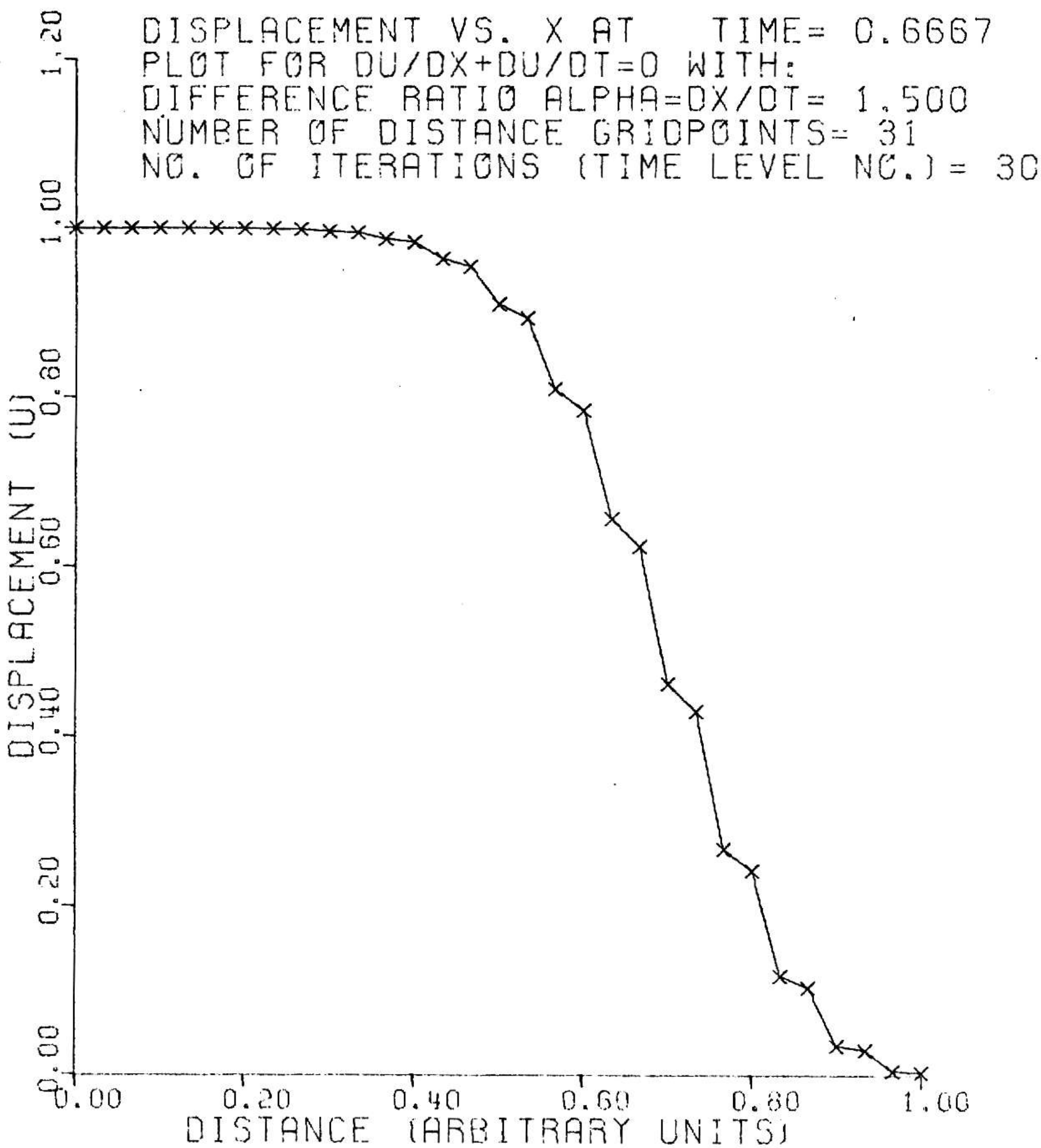
Instability



Numerical dispersion



Numerical dispersion



Conclusion

The graphs clearly show instability, i.e., the computed solution increasing without bound, numerical dispersion (due to the built-in interpolation in the finite difference scheme), and the exact solution, a step pulse. It was found that the numerical dispersion, or averaging, was not changed when the program was converted to double precision arithmetic.

Thus we have seen how an a priori analysis is necessary if one is to find a finite difference scheme that converges (as Δt and Δx tend to zero) to the desired solution of a PDE. The techniques and implementation of a stability analysis are not always this simple for other equations, and this report is meant only to give some insight into the problem.

THE THEORETICAL AND NUMERICAL
TREATMENT OF THE "COLOR" EQUATION

by

GLEN WALLACE WITTEKIND

B. A., University of Nebraska at Omaha, 1971

AN ABSTRACT OF A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mathematics

KANSAS STATE UNIVERSITY
Manhattan, Kansas

Abstract:

In this report, we consider the partial differential equation:

$u_t + cu_x = 0$, where c is a real constant, with initial conditions

$u(x,0)=0 \quad x > 0$, and boundary conditions

$$u(0,t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0. \end{cases}$$

Two finite difference schemes are developed, one unstable and one stable (it converges to the true solution) for certain values of the parameter $\alpha = \Delta x/(c \Delta t)$.

Finally, for the second scheme, numerical computations are made which demonstrate instability in a finite difference scheme, numerical dispersion, and the exact solution to the problem posed.