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INVERSE PROBLEM FOR A HEAT EQUATION WITH PIECEWISE-CONSTANT CONDUCTIVITY

S. GUTMAN* AND A.G. RAMM

Abstract. We consider the inverse problem of the identification of a piecewise-constant conductivity in a bar given the extra information of the heat flux through one end of the bar. Our theoretical results show that such an identification is unique. This approach utilizes a "layer peeling" argument. A computational algorithm based on this method is proposed and implemented. The advantage of this algorithm is that it requires only 3D minimizations irrespective of the number of the unknown discontinuities. Its numerical effectiveness is investigated for several conductivities.

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1. Introduction

Let Π be the set of piecewise constant functions on $[0,1]$ with finitely many discontinuity points,

$$\Pi = \{a(x) : 0 < \nu \leq a(x) \leq \mu,\ a(x) = a_j,\ x \in [x_{j-1}, x_j),\ j = 1, 2, ..., n\} \quad (1.1)$$

with $x_0 = 0$ and $x_n = 1$.
Consider the following heat conduction problem in an inhomogeneous bar of the unit length with a conductivity $a \in \Pi$:

$$\begin{cases} u_t = (a(x)u_x)_x, & (x,t) \in Q = (0,1) \times (0,\infty), \\ u(0,t) = g(t),\ u(1,t) = 0, & t \in (0,\infty), \\ u(x,0) = 0, & x \in (0,1). \end{cases} \quad (1.2)$$

Suppose that the extra data $f(t) = a(0)u_x(0,t) \not\equiv 0$, i.e., the heat flux through the left end of the bar, is known. The inverse problem (IP) for (1.1)-(1.2) is:

IP: *Given $f(t)$ and $g(t)$ for all $t > 0$, find $a(x)$.*

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There exists an extensive literature on inverse problems for heat equations, see e.g. [1, 3, 4, 14] and the references therein. Fewer results are available for the *identifiability*, i.e., the unique identification of the conductivity, for such inverse problems.

Some identifiability results for smooth or constant conductivities were obtained previously, see [8, 17, 16]. These works show that one can identify a constant conductivity a in (1.2) from the measurement $z(t)$ taken at one point $p \in (0,1)$. These works also discuss problems more general than (1.2), including problems with a broad range of boundary conditions, non-zero forcing functions, as well as elliptic and hyperbolic problems. Additional identifiability results have been obtained by us in [11, 12] and [13].

In [6] and [7] we showed that one can uniquely identify certain piecewise-constant conductivities from measurements $z_m(t)$ of the heat conduction process (1.2) taken at finitely many points $p_m \in (0,1)$. The stability of this procedure was studied in [7].

In this paper we establish the identifiability for the IP. That is the piecewise-constant conductivity $a \in \Pi$ in (1.2) can be uniquely recovered from the observations of the heat flux taken at just one point $x = 0$. This result is proved in Section 2 following the arguments in [15]. The main idea of the proof is to apply a "layer peeling" argument. Suppose that two conductivities $a, b \in \Pi$ satisfy (1.2) with the same data $f(t)$ and $g(t)$ for $t > 0$. Let both a and b have no discontinuities on an interval $[0, y]$, $0 < y \leq 1$. Then we can show that $a(x) = b(x)$ for $x \in [0, y]$. A repeated application of this argument shows that $a = b$ on the entire interval $[0, 1]$. See [15] for further refinements of this result, in particular for the data f, g available only on a finite interval $(0, T)$.

A "layer peeling" algorithm consistent with our theoretical developments is presented in Section 4. The advantage of this method is that it requires only 3D minimizations irrespective of the number of the unknown discontinuities. In Section 3 the heat conduction problem is represented in terms of iteratively applied mappings E. In Section 4 the layer peeling algorithm is presented as a nonlinear minimization problem for an E in a 3D box. In Section 6 the algorithm is tested for conductivities with a small number of discontinuities.

2. IP identifiability

The main tool for the uniqueness proof is the Property C (completeness of the products of solutions for (1.2)). See [10] for details on the Property C for ODE, [9] and [14] for Property C for PDE.

We will use the following Property C result established in [15].

Theorem 1. *Let $PC[0,1]$ be the set of piecewise-constant functions on $[0,1]$. Let $q_1, q_2 \in PC[0,1]$ be two positive functions. Suppose that $\psi_1(x,k)$ and $\psi_2(x,k)$ satisfy*

$$-\psi_j''(x,k)+k^2 q_j^2(x)\psi_j(x,k) = 0, \quad \psi_j(1,k) = 1, \quad \psi_j'(1,k) = 0, \quad j = 1,2. \tag{2.1}$$

Then the set of products $\{\psi_1(x,k)\psi_2(x,k)\}_{k>0}$ *is dense in* $PC[0,1]$. *That is, if* $h \in PC[0,1]$ *and*

$$\int_0^1 h(x)\psi_1(x,k)\psi_2(x,k)dx = 0 \tag{2.2}$$

for any $k > 0$, *then* $h = 0$.

Now we prove our main identifiability result

Theorem 2. *Problem IP has at most one solution* $a \in \Pi$.

Proof. Following [15], problem (1.2) is restated in terms of the Laplace transform

$$v(x,s;a) = (\mathcal{L}u)(x,s;a) = \int_0^\infty u(x,t;a)e^{-st}dt, \quad s > 0.$$

Let $G(s) = \mathcal{L}(g(t))$ and $F(s) = \mathcal{L}(f(t))$. Thus (1.2) with the extra condition $a(0)u_x(0,t) = f(t)$ becomes

$$\begin{aligned} &(a(x)v')' - sv = 0, \quad 0 < x < 1, \\ &v(0,s;a) = G(s), \quad a(0)v'(0,s;a) = F(s) \\ &v(1,s;a) = 0. \end{aligned} \tag{2.3}$$

Let

$$k = \sqrt{s}, \quad \psi(x,k) = a(x)v'(x,s;a), \quad \text{and} \quad q(x) = \sqrt{\frac{1}{a(x)}}.$$

Then, using $k^2 v(x,s;a) = \psi'(x,k)$, system (2.3) is rewritten as

$$\begin{aligned} &-\psi''(x,k) + k^2 q^2(x)\psi(x,k) = 0, \quad 0 < x < 1, \\ &\psi(0,k) = F(k^2), \quad \psi'(0,k) = k^2 G(k^2), \quad \psi'(1,k) = 0. \end{aligned} \tag{2.4}$$

Let $\psi_1(x,k)$ and $\psi_2(x,k)$ be the solutions of (2.4) for two positive piecewise-constant functions $q_1(x)$ and $q_2(x)$ correspondingly. That is,

$$\begin{aligned} &-\psi_1''(x,k) + k^2 q_1^2(x)\psi_1(x,k) = 0, \quad 0 < x < 1, \\ &\psi_1(0,k) = F(k^2), \quad \psi_1'(0,k) = k^2 G(k^2), \quad \psi_1'(1,k) = 0, \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} &-\psi_2''(x,k) + k^2 q_2^2(x)\psi_2(x,k) = 0, \quad 0 < x < 1, \\ &\psi_2(0,k) = F(k^2), \quad \psi_2'(0,k) = k^2 G(k^2), \quad \psi_2'(1,k) = 0. \end{aligned} \tag{2.6}$$

Multiply equation (2.5) by $\psi_2(x,k)$ and integrate it over $[0,1]$. Then use an integration by parts and the boundary conditions in (2.5) and (2.6) to obtain

$$k^2 \int_0^1 q_1^2\psi_1\psi_2 dx = \psi_1'\psi_2|_0^1 - \int_0^1 \psi_1'\psi_2' dx = -k^2 G(k^2)F(k^2) - \int_0^1 \psi_1'\psi_2' dx. \tag{2.7}$$

Similarly,

$$k^2 \int_0^1 q_2^2\psi_1\psi_2 dx = -k^2 G(k^2)F(k^2) - \int_0^1 \psi_1'\psi_2' dx. \tag{2.8}$$

Subtracting (2.8) from (2.7) gives

$$\int_0^1 (q_1^2 - q_2^2)\psi_1\psi_2 dx = 0$$

for any $k > 0$.

The set of products $\{\psi_1(x,k)\psi_2(x,k)\}_{k>0}$ is dense in $PC[0,1]$ by Theorem 1. Therefore $q_1 = q_2$. Thus (2.4) has unique solution $q \in PC[0,1]$. Consequently (2.3) has a unique solution $a \in \Pi$, and the Theorem is proved. □

3. Direct problem

Since the temperature $u(x,t)$ and the heat flow $a(x)u_x(x,t)$ are continuous in the bar, initial and boundary value problem (1.2) can be written as

$$\begin{cases} u_t(x,t) = a_i u_{xx}(x,t), \quad x \in (x_{i-1}, x_i), \quad t > 0 \\ u(0,t) = g(t), \ u(1,t) = 0, \quad t > 0 \\ u(x_i + 0, t) = u(x_i - 0, t), \\ a_{i+1}u_x(x_i + 0, t) = a_i u_x(x_i - 0, t), \\ u(x,0) = 0, \quad x \in (0,1) \end{cases} \tag{3.1}$$

for $i = 1, 2, ..., n-1$.

Consider the following Sturm-Liouville problem:

$$\begin{cases} a_i\psi''(x) = -\lambda\psi(x), \quad x \in (x_{i-1}, x_i), \\ \psi(0) = \psi(1) = 0, \\ \psi(x_i + 0) = \psi(x_i - 0), \\ a_{i+1}\psi'(x_i + 0) = a_i\psi'(x_i - 0), \end{cases} \tag{3.2}$$

$i = 1, 2, ..., n-1$, where $\psi'(x) = \frac{d\psi}{dx}$.

It follows from standard arguments, see, e.g., [5, 6], that (3.2) has infinitely many eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ satisfying

$$\nu\pi^2 k^2 \leq \lambda_k \leq \mu\pi^2 k^2. \tag{3.3}$$

The corresponding orthonormal set of eigenfunctions $\{\psi_k\}_{k=1}^{\infty}$ forms a basis in the Hilbert space $H = L^2[0,1]$. Denote the inner product in H by $< \cdot, \cdot >$ and the norm by $\| \cdot \|$. Let $a \in \Pi$, u be the solution of (1.2), and

$$\Phi(x;a) = -\frac{1}{\int_0^1 \frac{1}{a(\xi)}\, d\xi} \int_0^x \frac{1}{a(\xi)}\, d\xi + 1. \tag{3.4}$$

The compatibility condition for the data implies $g(0) = 0$. Thus, the function $\tilde{u}(x,t;a) = u(x,t;a) - \Phi(x;a)g(t)$ is the solution of the problem:

$$\begin{cases} \tilde{u}_t - (a\tilde{u}_x)_x = -g'(t)\Phi, \quad 0 < x < 1, \ t > 0, \\ \tilde{u}(0,t) = 0, \quad \tilde{u}(1,t) = 0, \quad t > 0, \\ \tilde{u}(x,0) = 0, \quad 0 < x < 1. \end{cases} \tag{3.5}$$

Let $\{\lambda_k, \psi_k\}_{k=1}^{\infty}$ be the eigenvalues and the eigenfunctions of (3.2), and $\phi_k(a) =< \Phi, \psi_k >$ for $k = 1, 2, \cdots$. Then the solution $u(x,t;a),\ t > 0$ of (1.2) is given by

$$u(x,t;a) = g(t)\Phi(x;a) + \sum_{k=1}^{\infty} B_k(t;a)\psi_k(x;a), \tag{3.6}$$

where

$$B_k(t;a) = -\left(g(t) - \lambda_k \int_0^t e^{-\lambda_k(t-\tau)} g(\tau)d\tau\right)\phi_k(a) \tag{3.7}$$

for $k = 1, 2, \ldots$

Using the Laplace transform $v = \mathcal{L}u$ system (3.1) becomes

$$\begin{aligned} &a_i v'' - sv = 0, \quad x \in (x_{i-1}, x_i), \quad v(0,s;a) = G(s), \quad a_1 v'(0,s;a) = F(s), \\ &v(x_i + 0, s; a) = v(x_i - 0, s; a), \quad a_{i+1}v'(x_i + 0, s; a) = a_i v'(x_i - 0, s; a). \end{aligned} \tag{3.8}$$

From (3.6)-(3.7), the solution $v(x,s)$ is given by

$$\begin{aligned} v(x,s;a) &= G(s)\Phi(x;a) - \sum_{k=1}^{\infty} \phi_k(a)\left(G(s) - \lambda_k \frac{G(s)}{s+\lambda_k}\right)\psi_k(x;a) \\ &= G(s)\left(\Phi(x;a) - \sum_{k=1}^{\infty} \frac{s\phi_k(a)}{s+\lambda_k}\psi_k(x;a)\right). \end{aligned} \tag{3.9}$$

Therefore

$$F(s) = a_1 v'(0,s;a) = -G(s)\left(\frac{1}{\int_0^1 \frac{1}{a(\xi)}\, d\xi} + a_1 \sum_{k=1}^{\infty} \frac{s\phi_k(a)}{s+\lambda_k}\psi_k'(0;a)\right). \tag{3.10}$$

Note that the solution $v(x,s;a)$ of (3.8) can also be described in a different way. Since $a(x) = a_1$ on $[0, x_1)$, we consider the initial value problem $a_1 v'' - sv = 0$ with $v(0,s;a) = G(s)$ and $a_1 v'(0,s;a) = F(s)$. Thus, for $x \in [0, x_1)$ we have

$$v(x,s;a) = G(s)\cosh\sqrt{\frac{s}{a_1}}x + \frac{F(s)}{\sqrt{a_1 s}}\sinh\sqrt{\frac{s}{a_1}}x \tag{3.11}$$

and

$$a_1 v'(x,s;a) = G(s)\sqrt{a_1 s}\sinh\sqrt{\frac{s}{a_1}}x + F(s)\cosh\sqrt{\frac{s}{a_1}}x. \tag{3.12}$$

Using the matching conditions $v(x_1 + 0, s; a) = v(x_1 - 0, s; a)$ and $a_2 v'(x_1 + 0, s; a) = a_1 v'(x_1 - 0, s; a)$ we conclude that

$$v(x_1 + 0, s; a) = G(s)\cosh\sqrt{\frac{s}{a_1}}x_1 + \frac{F(s)}{\sqrt{a_1 s}}\sinh\sqrt{\frac{s}{a_1}}x_1 \tag{3.13}$$

and

$$a_2 v'(x_1 + 0, s; a) = G(s)\sqrt{a_1 s}\sinh\sqrt{\frac{s}{a_1}}x_1 + F(s)\cosh\sqrt{\frac{s}{a_1}}x_1. \tag{3.14}$$

Equations (3.13) and (3.14) can be compactly rewritten as

$$\begin{pmatrix} v(x_1+0,s;a) \\ a_2 v'(x_1+0,s;a) \end{pmatrix} = E(s;0,x_1,a_1)\begin{pmatrix} G(s) \\ F(s) \end{pmatrix}$$
$$= \begin{pmatrix} \cosh\sqrt{\frac{s}{a_1}}x_1 & \frac{1}{\sqrt{a_1 s}}\sinh\sqrt{\frac{s}{a_1}}x_1 \\ \sqrt{a_1 s}\sinh\sqrt{\frac{s}{a_1}}x_1 & \cosh\sqrt{\frac{s}{a_1}}x_1 \end{pmatrix}\begin{pmatrix} G(s) \\ F(s) \end{pmatrix}. \quad (3.15)$$

Define

$$E(s;p,x,b) := \begin{pmatrix} \cosh\sqrt{\frac{s}{b}}(x-p) & \frac{1}{\sqrt{bs}}\sinh\sqrt{\frac{s}{b}}(x-p) \\ \sqrt{bs}\sinh\sqrt{\frac{s}{b}}(x-p) & \cosh\sqrt{\frac{s}{b}}(x-p) \end{pmatrix}. \quad (3.16)$$

Because of the requirement $v(1,s;a)=0$ system (2.3) takes the form

$$\prod_{i=n}^{1} E(s;x_{i-1},x_i,a_i)\begin{pmatrix} G(s) \\ F(s) \end{pmatrix} = \begin{pmatrix} 0 \\ . \end{pmatrix}, \quad (3.17)$$

where the second component of the right hand side of (3.17) can be any function of s.

4. Inverse problem

Using the notation of the previous section, the Inverse Problem IP can be stated as:

Given $G(s)$ and $F(s)$, find $a \in \Pi$ such that vector equation (3.17) *is satisfied for its first component.*

Consistent with the "layer peeling" arguments of [15], we propose an algorithm that determines the sought conductivity $a^* \in \Pi$ by determining its values a_i and the discontinuity points x_i one by one starting at the left end of the interval $[0,1]$ and iteratively advancing to its right end.

First, we present an informal description of the algorithm. Let $G(s)$ and $F(s)$ be known. Choose a point $p \in (0,1)$, and $\nu \le b, c \le \mu$. This choice corresponds to the heat conduction problem (1.1)-(1.2) with $a(x)=b$ for $0 \le x \le p$ and $a=c$ for $p \le x \le 1$. Let the extra condition for the heat flow at $x=0$ be $bu_x(0,t)=f(t)$ with $(\mathcal{L}f)(s)=F(s)$. Then $E(s;0,p,b)\binom{G(s)}{F(s)}$ gives the temperature and the heat flow in the bar at the point $x=p$. Furthermore

$$\begin{pmatrix} H(s) \\ . \end{pmatrix} = E(s;p,1,c)E(s;0,p,b)\begin{pmatrix} G(s) \\ F(s) \end{pmatrix}$$

is the temperature and the heat flow at $x=1$.

Because of the condition $u(1,t)=0$, we would like to have $H(s)=0$. So, in the first iteration, one minimizes $J=\|H(s)\|^2$ with respect to the variables p,b,c. Let the resulting minimizer be $(a_1,x_1,\tilde{c})$. We take x_1 and a_1 as the first discontinuity point and the value of the sought conductivity a^* for $0 \le x < x_1$.

Next iteration works with the remaining interval $[x_1, 1]$. It involves the minimization with respect to p, b, c of $J = \|H(s)\|^2$ now defined by

$$\begin{pmatrix} H(s) \\ \cdot \end{pmatrix} = E(s; p, 1, c)E(s; x_1, p, b)E(s; 0, x_1, a_1)\begin{pmatrix} G(s) \\ F(s) \end{pmatrix}.$$

Let the minimizer of J be $(x_2, a_2, \tilde{c})$. Then x_2 is the second discontinuity point of a^* and $a^*(x) = a_2$ for $x_1 \le x < x_2$. The iterations continue until the desired tolerance *tol* is reached, or the allowable number of iterations N_{max} has been exceeded. Note that the values of the discrepancy J are decreasing with each iteration.

Layer Peeling (LP) Algorithm

Let $g(t)$ be the temperature, and $f(t)$ be the heat flux at $x = 0$ for the heat conduction described by (1.1)-(1.2) with $\hat{a} \in \Pi$.

Let the Laplace transforms $(\mathcal{L}g)(s) = G(s)$ and $(\mathcal{L}f)(s) = F(s)$ be known for $s_1 \le s \le s_2$.

Choose $tol > 0$ for the tolerance, and N_{max} for the maximal number of iterations.

Let $x_0 = 0$ and $k = 1$.

Iteration.

For $k = 1$ define $H(s; p, b, c)$ by

$$\begin{pmatrix} H(s; p, b, c) \\ \cdot \end{pmatrix} = E(s; p, 1, c)E(s; 0, p, b)\begin{pmatrix} G(s) \\ F(s) \end{pmatrix}.$$

For $k \ge 2$ with the discontinuity points $\{x_1, ..., x_{k-1}\}$ and the values $\{a_1, ..., a_{k-1}\}$ known from the previous iterations, define $H(s; p, b, c)$ by

$$\begin{pmatrix} H(s; p, b, c) \\ \cdot \end{pmatrix} = E(s; p, 1, c)E(s; x_{k-1}, p, b)\prod_{i=k-1}^{1} E(s; x_{i-1}, x_i, a_i)\begin{pmatrix} G(s) \\ F(s) \end{pmatrix}.$$

Let

$$J(p, b, c) = \int_{s_1}^{s_2} |H(s; p, b, c)|^2 ds$$

and

$$(x_k, a_k, \tilde{c}) = argmin\ J(p, b, c), \quad x_{k-1} \le p \le 1, \quad \nu \le b, c \le \mu.$$

Stopping Criterion.

If $J(x_k, a_k, \tilde{c}) > tol$ and $k < N_{max}$ then let $k := k+1$ and do another *Iteration.*

Otherwise let $n = k + 1,\ x_n = 1,\ a_n = \tilde{c}$. Stop.

The sought conductivity $a^* \in \Pi$ has discontinuities at $x_1, x_2, ..., x_{n-1}$ and the values $a^*(x) = a_i$ for $x_{i-1} \le x < x_i,\ i = 1, 2, ..., n$.

5. Computational algorithms

To test the effectiveness of the LP algorithm, let us assume that a conductivity $\hat{a} \in \Pi$ is given together with the temperature $g(t) = u(0, t; \hat{a}),\ t > 0$ satisfying $g(0) = 0$. Let $G(s)$ be the Laplace transform of $g(t)$. Then the heat flux $F(s)$ can

be computed by (3.10). This computation requires the eigenvalues and the eigenfunctions of the Sturm-Liouville problem (3.2) with $a = \hat{a}$. They are computed as follows.

Shooting Method for Eigenvalues and Eigenfunctions. Let $a \in \Pi$. Then there exists a partition $0 < x_1 < x_2 < \cdots < x_{n-1} < 1$ of the interval $[0, 1]$, and a sequence $a_1, a_2, \ldots, a_n$ such that $a(x) = a_i$ for $x_{i-1} < x < x_i,\ i = 1, 2, ..., n$.

Fix a $\lambda > 0$ and solve the sequence of the initial value problems

$$\begin{aligned} &(a(x)v'(x))' = -\lambda v(x), \quad x \neq x_i, \\ &v(0) = 0, \quad v'(0) = 1, \\ &v(x_i + 0) = v(x_i - 0), \\ &a_{i+1}v'(x_i + 0) = a_i v'(x_i - 0), \end{aligned} \tag{5.1}$$

on intervals $[0, x_1],\ [x_1, x_2], ..., [x_{n-1}, 1]$.

Let $F(\lambda) = v(1; \lambda)$, that is $F(\lambda)$ is the value of the solution $v(x)$ of (5.1) at $x = 1$. Let $h_\lambda > 0$. We used $h_\lambda = (\mu\pi^2 - \nu\pi^2)/20$ utilizing the bounds for the first eigenvalue from (3.3). Let $\lambda_l = \nu\pi^2$ and $\lambda_r = \lambda_l + h_\lambda$. Compute $F(\lambda_l)$ and $F(\lambda_r)$. If these two values have a different sign, then use a bisection (or other root finding method, see [18]) in λ to find the value λ_1 for which $F(\lambda_1) = 0$ within a prescribed tolerance. If $F(\lambda_l)$ and $F(\lambda_r)$ have the same sign, then translate the interval $[\lambda_l,\ \lambda_r]$ by h_λ and repeat the procedure until all the required eigenvalues are found.

Let $v(x; \lambda_1)$ be the solution of (5.1) with $\lambda = \lambda_1$. This solution is normalized to obtain the first eigenfunction $\psi_1(x; a)$. The process is repeated for all the other eigenvalues.

The central part of the LP algorithm, described in section 4, is the nonlinear minimization procedure for the discrepancy $J(p, b, c)$ in its *Iteration* step. The variables p, b, c appear only in the product $E(s; p, 1, c)E(s; x_{k-1}, p, b)$. The definition (3.16) of the operator E allows one to compute the partial derivatives of $J(p, b, c)$ explicitly. Thus various gradient type minimization methods can be applied, see [18].

6. Numerical results

In all our numerical experiments we let $\nu = 0.1,\ \mu = 1.0,\ tol = 0.0001$ and $N_{max} = 3$. The temperature $u(0, t)$ was taken to be $g(t) = \sin t,\ t > 0$. Thus $G(s) = 1/(s^2 + 1)$. Given a conductivity $\hat{a} \in \Pi, M = 250$ eigenvalues and the eigenfunctions of the associated Sturm-Liouville problem (3.2) were computed by the Shooting Method. It was found experimentally that the Fourier series approximation $\sum_{k=1}^{M} \phi_k(\hat{a})\psi_k(x; \hat{a})$ for $\Phi(x; \hat{a})$ with a significantly smaller M had a detrimental effect on the quality of the identification. Then the Laplace transform $F(s)$ of the heat flux $f(t)$ was computed using (3.10) at $J = 400$ equidistant points $s_j,\ j = 1, ..., J$ spanning the interval $(0, 80]$.

The following three numerical experiments illustrate the typical performance of the LP algorithm.

Let

$$\hat{a}_{(1)}(x) = \begin{cases} 0.8, & 0 \le x < 0.2 \\ 0.2, & 0.2 \le 1 \end{cases} \tag{6.1}$$

The identified conductivity was

$$a^*_{(1)}(x) = \begin{cases} 0.784, & 0 \le x < 0.1500 \\ 0.388, & 0.1500 \le x < 0.8385 \\ 0.622, & 0.8385 \le x < 0.8401 \\ 1.000, & 0.8401 \le 1. \end{cases} \tag{6.2}$$

In the other two experiments we had

$$\hat{a}_{(2)}(x) = \begin{cases} 0.2, & 0 \le x < 0.2 \\ 0.8, & 0.2 \le x < 0.5 \\ 0.2, & 0.5 \le 1 \end{cases} \tag{6.3}$$

with

$$a^*_{(2)}(x) = \begin{cases} 0.190, & 0 \le x < 0.2100 \\ 1.000, & 0.2100 \le x < 0.2416 \\ 1.000, & 0.2416 \le x < 0.3857 \\ 1.000, & 0.3857 \le 1, \end{cases} \tag{6.4}$$

and

$$\hat{a}_{(3)}(x) = \begin{cases} 0.8, & 0 \le x < 0.2 \\ 0.2, & 0.2 \le x < 0.5 \\ 0.8, & 0.5 \le 1 \end{cases} \tag{6.5}$$

with the identified conductivity

$$a^*_{(3)}(x) = \begin{cases} 0.784, & 0 \le x < 0.1700 \\ 0.3520, & 0.1700 \le x < 0.8506 \\ 1.000, & 0.8506 \le x < 0.8670 \\ 1.000, & 0.8670 \le 1. \end{cases} \tag{6.6}$$

The results of these experiments are shown on Figure 1. These typical results show that the method achieves good identification for the first (the closest to the left end of the interval $[0, 1]$) discontinuity. However, the second discontinuity cannot be reliably identified. A closer examination shows that the objective function J introduced in Section 4 is becoming practically independent of additional discontinuity points after a couple of iterations of the LP algorithm. Thus the method is unable to "see" the additional discontinuities without an increase in precision in the data and the computations.

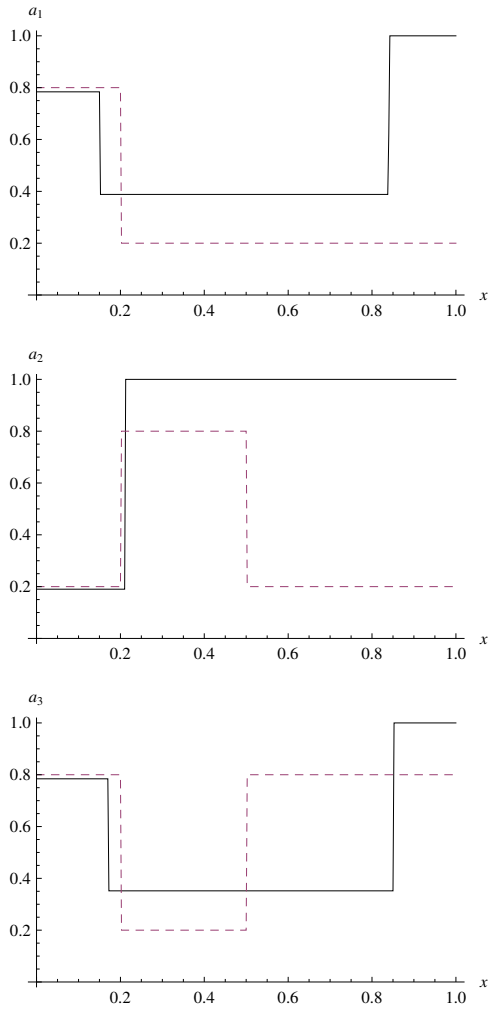

FIGURE 1. Original $\hat{a}_{(i)}$ (dashed line) and the identified conductivity $a^*_{(i)}$ (solid line), $i = 1, 2, 3$.

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