THE RELATIONSHIP BETWEEN THE PRODUCT AND THE GEOMETRIC EFFECTS IN SYMMETRICAL FACTORIAL EXPERIMENTS

by

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CHAPTER I

INTRODUCTION

The nature of many agricultural, biological and industrial experiments necessitate the estimation of the effects of several different factors on the variable of interest to the researcher. Therefore, in these experiments, factorial designs are widely used as they allow several factors to be included simultaneously in the experiment.

Sometimes in factorial designs it may not be possible to apply all the treatments in one homogeneous set of experimental units. In this case the treatments are often divided into subsets and applied to smaller sets of homogeneous units (which are called 'blocks'). It is well known that some 'information' is sacrificed in this case, i.e., some of the factorial 'effects' are confounded with the block 'effects'.

As the number of factors and their levels increase, the size of factorial experiment enlarges rapidly and sometimes goes beyond the limits of available resources. In such situations, fractional factorial experiments (an experiment consisting of only a fraction of the complete set of treatments) may be considered. Again, some 'information' on the factorial 'effects' is sacrificed.

It is not generally known among experimenters that there is more than one way of defining factorial effects. In general, the definitions do not agree when there are at least two factors with more than two levels. Unfortunately, the definition used most often in designing 'fractional replicates of factorial experiments' and in designing experiments where some effects are 'confounded with blocks' does not give factorial effects with desirable physical interpretations. Thus some of the designs recommended in the standard reference books do not have desirable properties as far as interpretations are concerned.

1.1 Contents and Goal of the Report

A factorial effect is a linear combination of the 'true' yields of all the assemblies (treatment combinations) in the experiment. There are two types of definitions available for the factorial effects in an experiment (Srivastava 1969). When there are at least two factors with more than two levels, the two definitions differ in the coefficients of the assemblies in the linear combination. A lot of misconception exists in the literature as to the nature of the relationship between these two definitions, their practical importance to an applied statistician, their physical interpretations and their use in designing fractional factorial experiments or developing confounding plans for factorial experiments. In the following, if it is implied that the two definitions do not agree then it is to be assumed that there are at least two factors with more than two levels in the experiment.

Effects defined by the product definition are the easiest to interpret and consequently, are the most desirable from the experimenter's point of view. The geometric definition utilizes the concepts of hyperplanes in finite Euclidian Geometry and has desirable mathematical properties but no physical interpretations. The geometric definition is useful in construction of the fractional factorial designs and confounded designs but, in general, product effects are not estimable in such designs.

A design in which the variances of the estimates of the factorial effects of the same order, i.e., with the same number of factors, are equal is called a balanced design. If balance is achieved in a fractional factorial design with respect to the effects defined by the geometric definition, then usually it will not be balanced if the corresponding product effects are estimated. If the block size is less than the number of treatment combinations then often the experimenter will choose to partially confound some degrees of freedom of geometric effects. In such partially balanced designs, estimation of the product effects is not straight forward. The geometric effects which can be estimated with certain precision cannot be interpreted physically.

A discussion of the two types of definitions and a brief review of Finite Euclidian Geometry and Galois Fields is presented in Section 2.1. The relationship between the effects defined by these two definitions is investigated for symmetrical factorial experiments in Section 2.2. This will enable one to use these effects interchangeably in some designs.

A computational technique for estimating the product effects from geometric effects is given in Section 2.3. Three confounding plans, given in Cochran and Cox (1968), are considered in Chapter 3 to illustrate the theory. These plans are designed by partially confounding some geometric effects which are estimable from part of the design but the corresponding product effects are not directly estimable.

1.2 Notations

A general factorial experiment involving n factors with s_i levels of the ith factor ($i = 1,2,..,n$), is said to be a $s_1 x s_2 x..x s_n$ Factorial Experiment (FE).

Example 1.2.1 An agricultural experiment consisting of 3 factors (i) water at two levels (ii) nitrogen at three levels and (iii) potash at five levels is said to be a 2x3x5 factorial experiment.

If all s_i are equal ($s_1=s_2=..=s_n=s$) then the experiment is known as a symmetrical factorial experiment and is denoted by s^n, i.e., n factors each at s levels.

Example 1.2.2 In an experiment in which 3 factors potash, nitrogen and phosphate occur each at 2 levels, it is a 2^3 symmetrical factorial experiment.

If at least one s_i is different than the others then the experiment is an asymmetrical factorial experiment (AFE).

In the above experiment the set (say Ω) of all possible treatment combinations (or assemblies) consists of $s_1 x s_2 x .. x s_n$ elements. A general assembly can be denoted by $(a_1^{j_1} a_2^{j_2} .. a_n^{j_n})$, where factor a_i occurs at j_ith level for $i = 1,2,..,n$ and $j_i = 0,1,..,s_i-1$. If m of the j_i are non-zero the assembly is an m-factor assembly. The assembly with all $j_i = 0$ is denoted by $\emptyset$. The expected value $E(a_1^{j_1} a_2^{j_2} .. a_n^{j_n})$ denotes the 'true' yield when assembly $(a_1^{j_1} a_2^{j_2} .. a_n^{j_n})$ is applied to an experimental unit.

Example 1.2.3 In the experiment 2x3x5, the assembly $(a_1^1 a_2^0 a_3^3)$ is the treatment combination of 1st level of the first factor (water), 0th level of the second factor (nitrogen) and 3rd level of the third factor (potash). This assembly is equivalent to the assembly $(a_1^1 a_3^3)$ or (103) and is a 2-factor assembly. The expected value $E(a_1^1 a_2^0 a_3^3)$ denotes the 'true' yield of the assembly in which water is at 1st level, nitrogen at 0th level and potash at 3rd level.

The factorial effects are denoted by capital letters such as $A_1^{k_1} A_2^{k_2} .. A_n^{k_n}$. The effect $A_1^{k_1} A_2^{k_2} .. A_n^{k_n}$ is an m-factor interaction when m of the k_i are nonzero. When m = 1 it is called a main effect of the respective factor. When m = 0 it is the mean effect, denoted by μ. Technical definitions are given in Section 2.1.

<u>Example 1.2.4</u> In the 2x3x5 AFE, the effect $A_1^0 A_2^2 A_3$ (which is equivalent to $A_2^2 A_3$) is a 2-factor interaction of the second and the third factors. The factorial effect $A_1 A_2^0 A_3^0$ (= A_1) is the main effect of the first factor and $A_1^0 A_2^0 A_3^0$ is the mean effect μ.

CHAPTER II

THE RELATIONSHIP BETWEEN THE PRODUCT AND THE GEOMETRIC EFFECTS

2.0 Introduction

In this chapter the relationship between the two types of definitions is examined. The definitions are introduced in the first section. The product definition is essentially the same as used in Cochran and Cox (1968), Chapter 5. For the physical interpretation the reader is refered to the above book. Cochran and Cox also use the geometric definition implicitly in Chapter 6. The geometric definition is taken from an unpublished paper by Srivastava (1969).

2.1 Product and Geometric Definitions

2.1.1 Product Definition

Consider a $s_1 x s_2 x..x s_n$ factorial experiment (s_i not necessarily equal). The product definition of the factorial effect $A_1^{k_1} A_2^{k_2} .. A_m^{k_m}$, is given by

$$(2.1.1) \qquad A_1^{k_1} A_2^{k_2} .. A_m^{k_m} = \prod_{j=1}^{m} \left\{ \sum_{i=0}^{s_j-1} d_i^{(s_j)}(k_j)\, E(a_j^i) \right\}$$

where $E(a_j^i)$ is the sum of 'true' yields of all the assemblies in which jth factor occurs at level i. The constants $d_j^{(s)}(k)$ satisfy the following conditions:

$$(2.1.2) \qquad \sum_{j=0}^{s-1} d_j^{(s)}(k) = 0, \qquad 0 \leq k \leq s-1$$

which makes every effect a linear contrast,

$$(2.1.3) \qquad \sum_{j=0}^{s-1} \left\{d_j^{(s)}(k_1)\right\} \cdot \left\{d_j^{(s)}(k_2)\right\} = 0, \quad \text{for } k_1 \neq k_2 \text{ and}$$

$0 \leqslant k_1, k_2 \leqslant s-1$; so that any two effects are orthogonal and

$$(2.1.4) \qquad d_j^{(s)}(0) = 1, \quad \text{for all } j \text{ and } s$$

for defining the mean effect.

These d's depend upon the values of s, j and k. The conventional values are given in the Table 2.1.1, but the results in this report are valid for any choice of these constants so long as they satisfy Eqns. (2.1.2) and (2.1.3).

Table 2.1.1

Values of $d_j^{(s)}(k)$ for s = 2,3,4,5; k = 1,2,3,4; j = 0,1,..,s-1.

s		2		3			4				5				
j		0	1	0	1	2	0	1	2	3	0	1	2	3	4
k	1	−1	+1	−1	0	+1	−3	−1	+1	+3	−2	−1	0	+1	+2
	2			+1	−2	+1	+1	−1	−1	+1	+2	−1	−2	−1	+2
	3						−1	+3	−3	+1	−1	+2	0	−2	+1
	4										+1	−4	6	−4	+1

Equation (2.1.1) can also be written as

$$(2.1.5) \qquad A_1^{k_1} A_2^{k_2} \dots A_m^{k_m} = \sum_{r_1=0}^{s_1-1} \dots \sum_{r_m=0}^{s_m-1} \left[d_{k_1,\dots,k_m}^{r_1,\dots,r_m} \right] E(a_1^{r_1} \dots a_m^{r_m})$$

where $d_{k_1,\dots,k_m}^{r_1,\dots,r_m} = \left\{ d_{r_1}^{(s_1)}(k_1) \right\} \left\{ d_{r_2}^{(s_2)}(k_2) \right\} \dots \left\{ d_{r_m}^{(s_m)}(k_m) \right\}$ and $E(a_1^{r_1} a_2^{r_2} \dots a_m^{r_m})$ is the 'true' yield of the assembly $(a_1^{r_1} a_2^{r_2} \dots a_m^{r_m})$.

<u>Example 2.1.1</u> Consider a 3^4 FE, i.e., the 4 factors (A_1, A_2, A_3, A_4) with 3 levels each $(a_i^0, a_i^1, a_i^2;\ i = 1,2,3,4)$. A component of the two factor interaction between A_1 and A_2 is defined by

$$A_1^2 A_2 = \left\{ E(a_1^0) - 2E(a_1^1) + E(a_1^2) \right\} \cdot \left\{ - E(a_2^0) + E(a_2^2) \right\}$$

$$= - E(a_1^0 a_2^0) + E(a_1^0 a_2^2) + 2E(a_1^1 a_2^0) - 2E(a_1^1 a_2^2) - E(a_1^2 a_2^0)$$

$$+ E(a_1^2 a_2^2) .$$

Here $E(a_1^i a_2^j)$ is the sum of 'true' yields of all assemblies in which first factor (A_1) occurs at the level i and the second factor (A_2) occurs at the level j. This is the sum of 9 assemblies $\left\{ (a_1^i a_2^j a_3^k a_4^f);\ k,f = 0,1,2 \right\}$.

<u>Example 2.1.2</u> For an $2^3 \times 3^2$ AFE in which three factors (A_1, A_2, A_3) are each at 2 levels $(a_i^0, a_i^1;\ i = 1,2,3)$ and factors (A_4, A_5) are each at 3 levels $(a_j^0, a_j^1, a_j^2;\ j = 4,5)$, the interaction

$$A_1A_2A_4^2 = \left\{-E(a_1^0) + E(a_1^1)\right\} \cdot \left\{-E(a_2^0) + E(a_2^1)\right\} \cdot$$
$$\left\{E(a_4^0) - 2E(a_4^1) + E(a_4^2)\right\}$$
$$= E(a_1^0a_2^0a_4^0) - 2E(a_1^0a_2^0a_4^1) + E(a_1^0a_2^0a_4^2) - E(a_1^0a_2^1a_4^0)$$
$$+ 2E(a_1^0a_2^1a_4^1) - E(a_1^0a_2^1a_4^2) - E(a_1^1a_2^0a_4^0) + 2E(a_1^1a_2^0a_4^1)$$
$$- E(a_1^1a_2^0a_4^2) + E(a_1^1a_2^1a_4^0) - 2E(a_1^1a_2^1a_4^1) + E(a_1^1a_2^1a_4^2) .$$

In this effect each $(a_1^ia_2^ja_4^k)$ represents the sum of 'true' yields of the six assemblies $\left\{(a_1^ia_2^ja_3^fa_4^ka_5^m);\ f = 0,1;\ m = 0,1,2\right\}$.

2.1.2 Euclidian Geometry and Galois Fields

A field containing a finite number of elements (= s) is called a Galois Field or a Finite Field and is denoted by GF(s). A Galois Field can be always constructed when s is a prime power. When s is a prime number the elements of the field are the residue classes of modulo s, i.e., 0,1,..,s−1. The addition and multiplication operations are illustrated below in Table 2.1.2 for s = 3 and s = 5. The GF(3) has 0,1,2 as its three elements and GF(5) has 0,1,2,3,4 as its five elements.

When s is a prime power (nth power of a prime), however, the elements of Galois Field are polynomials in X of degree n which belong to and are irreducible in GF(s). For example, for $s = 4 = 2^2$, let $X^2 = X + 1$ and define addition and multiplication as in Table 2.1.3.

Table 2.1.3

Addition and Multiplication Tables for GF(4)

GF(4)

+	0	1	X	X+1
0	0	1	X	X+1
1	1	0	X+1	X
X	X	X+1	0	1
X+1	X+1	X	1	0

x	0	1	X	X+1
0	0	0	0	0
1	0	1	X	X+1
X	0	X	X+1	1
X+1	0	X+1	1	X

Consider the n-dimensional Euclidian Space EG(n,s) based on GF(s). There are s^n points in EG(n,s) and each point is represented by a n-vector over GF(s). The points satisfying the equation

$$(2.1.6) \qquad c_1X_1 + c_2X_2 + .. + c_nX_n = \alpha$$

where c_i, $i = 1,2,..,n$ and α are elements of GF(s), and not all c_i are zero, form a n-1 dimensional hyperplane which is denoted by $\Delta(c_1,c_2,..,c_n;\alpha)$ or by Δ_α.

As α is an element of GF(s) it has s choices which give s parallel n-1 dimensional hyperplanes which do not have any

Table 2.1.2

Addition and Multiplication Tables for GF(3) and GF(5)

GF(3)

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

x	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

GF(5)

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

x	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

point in common. Each of these planes consists of s^{n-1} points which are the solutions of Eqn. (2.1.6).

These hyperplanes are used to define the geometric effects.

2.1.3 Geometric Definition

Geometrically, a factorial effect, in a symmetrical factorial experiment s^n, i.e., n factors each with s levels, where s is a prime power, is defined as

$$(2.1.7) \qquad A_1^{k_1} A_2^{k_2} \ldots A_n^{k_n} = \sum_{j=0}^{s-1} d_j^{(s)}(k_r) \quad \left\{\Delta(0,\ldots,0,1,k_{r+1},\ldots,k_n;\alpha_j)\right\}$$

where $d_j^{(s)}(k_r)$ are the same constants as defined in the product definition, r is the least integer such that $k_r \neq 0$, $\left\{\Delta(0,\ldots,0,1,k_{r+1},\ldots,k_n;\alpha_j)\right\}$ is the sum of the 'true' yields of the s^{n-1} assemblies $(a_1^{j_1} a_2^{j_2} \ldots a_n^{j_n})$ lying on the hyperplane or satisfying the equation

$$(2.1.8) \qquad j_r + k_{r+1} j_{r+1} + \ldots + k_n j_n = \alpha_j$$

α_j being an element of GF(s).

<u>Example 2.1.3</u> In a 3^4 FE, the geometric effect $A_1^2 A_2$ is defined as

$$A_1^2 A_2 = \sum_{j=0}^{3-1} d_j^{(3)}(2) \quad \left\{\Delta(2,1,0,0;\Delta_j)\right\}$$

$$= (X_1 + X_2 = 0) - 2(X_1 + X_2 = 1) + (X_1 + X_2 = 2)$$

$$= \left\{E(a_1^0 a_2^0) + E(a_1^1 a_2^2) + E(a_1^2 a_2^1)\right\} - 2\left\{E(a_1^0 a_2^1) + E(a_1^1 a_2^0) + E(a_1^2 a_2^2)\right\} + \left\{E(a_1^0 a_2^2) + E(a_1^1 a_2^1) + E(a_1^2 a_2^0)\right\}.$$

Again, $E(a_1^i a_2^j)$ stands for the sum of 'true' yields of the 9 assemblies in $\left\{(a_1^i a_2^j a_3^k a_4^f);\ k,f = 0,1,2\right\}$.

2.1.4 Product-Geometric Definition for Asymmetrical Case

Consider a $(s_1^m x s_2^n)$ AFE, where s_1 and s_2 are prime powers. If $A_1, A_2, .., A_m$ and $B_1, B_2, .., B_n$ denote the factors at s_1 and s_2 levels respectively, the geometric effects $A_1^{k_1} A_2^{k_2} .. A_m^{k_m}$ and $B_1^{f_1} B_2^{f_2} .. B_n^{f_n}$ are defined as in symmetrical cases s_1^m and s_2^n respectively. These are called pure A-effect and pure B-effect.

The mixed effect of the type $A_1^{k_1} .. A_m^{k_m} B_1^{f_1} .. B_n^{f_n}$ may be defined by product-geometric definition (which is a product of two geometric effects) as

$$A_1^{k_1} .. A_m^{k_m} B_1^{f_1} .. B_n^{f_n} = \sum_{j_1=0}^{s_1-1} \sum_{j_2=0}^{s_2-1} d_{j_1}^{(s_1)}(k_1)\, d_{j_2}^{(s_2)}(k_2) \left\{ \Delta^A_{\alpha_{j_1}} \times \Delta^B_{\beta_{j_2}} \right\} \tag{2.1.9}$$

$$= \left\{ \sum_{j_1=0}^{s_1-1} d_{j_1}^{(s_1)}(k_1)\ \Delta^A_{\alpha_{j_1}} \right\} \times \left\{ \sum_{j_2=0}^{s_2-1} d_{j_2}^{(s_2)}(k_2)\ \Delta^B_{\beta_{j_2}} \right\}$$

where α_{j_1} and β_{j_2} vary over all the elements of $GF(s_1)$ and $GF(s_2)$ respectively. Here $\Delta^A_{\alpha_{j_1}}$ and $\Delta^B_{\beta_{j_2}}$ are the same hyperplanes as defined in symmetrical experiments s_1^m and s_2^n respectively. The product x denotes the product of the assemblies lying on $\Delta^A_{\alpha_{j_1}}$ and $\Delta^B_{\beta_{j_2}}$; and is the same as the product of the assemblies as used in the product definition.

<u>Example 2.1.4</u> In a $2^3 x 3^2$ AFE, where factors (A_1, A_2, A_3) are at 2 levels each and factors (B_1, B_2) are at 3 levels each, a mixed effect can be defined as

$$A_1A_2B_1^2 = \sum_{j_1=0}^{1} \sum_{j_2=0}^{2} d_{j_1}^{(2)}(1)\, d_{j_2}^{(3)}(2) \left\{\Delta(1,1,0;\alpha_{j_1}) \times \Delta(2,0;\beta_{j_2})\right\}$$

$$= \left\{-(X_1+X_2=0)+(X_1+X_2=1)\right\} \times \left\{(Y_1=0) - 2(Y_1=1)+(Y_1=2)\right\}$$

$$= \left\{-E(a_1^0a_2^0) - E(a_1^1a_2^1) + E(a_1^0a_2^1) + E(a_1^1a_2^0)\right\} \times \left\{E(b_1^0) - 2E(b_1^1) + E(b_1^2)\right\}$$

$$= -E(a_1^0a_2^0b_1^0) + 2E(a_1^0a_2^0b_1^1) - E(a_1^0a_2^0b_1^2) - E(a_1^1a_2^1b_1^0) + 2E(a_1^1a_2^1b_1^1) - E(a_1^1a_2^1b_1^2) + E(a_1^0a_2^1b_1^0) - 2E(a_1^0a_2^1b_1^1) + E(a_1^0a_2^1b_1^2) + E(a_1^1a_2^0b_1^0) - 2E(a_1^1a_2^0b_1^1) + E(a_1^1a_2^0b_1^2).$$

In this case $E(a_1^i a_2^j b_1^k)$ stands for the sum of 'true' yields of the six assemblies $\left\{(a_1^i a_2^j a_3^f b_1^k b_2^m);\ f = 0,1 \text{ and } m = 0,1,2\right\}$.

2.2 The Relationship between the Product and Geometric Effects

Consider a s^m symmetrical factorial experiment. We will state and prove a lemma which will be used to derive the relationship between the effects defined by the product definition and the effects defined by the geometric definition.

Lemma 2.2.1 The k-factor interactions as defined by product (geometric) definition are orthogonal to all interactions as defined by the geometric (product) definition except those k-factor interactions involving the same k factors.

Proof: We will consider a 'p' factor interaction as defined by product definition and a 'g' factor interaction as defined by geometric definition. Without loss of generality we will take the 'p' factor product effect involving the first 'p' factors $(A_1, A_2, \ldots, A_p)$ and the 'g' factor geometric effect involving the last 'g' factors $(A_{h+1}, A_{h+2}, \ldots, A_{h+g} = A_m)$ so that $h + g = m$.

The product effect can be given as

$$(2.2.1) \qquad A_1^{k_1} A_2^{k_2} \ldots A_p^{k_p} = \prod_{j=1}^{p} \left\{ \sum_{i=0}^{s-1} d_i^{(s)}(k_j)\, E(a_j^i) \right\}$$

which is also equal to

$$(2.2.2) \qquad A_1^{k_1} A_2^{k_2} \cdot\cdot A_p^{k_p} = \sum_{r_1,\ldots,r_p=0}^{s-1} (d_{k_1,\ldots,k_p}^{r_1,\ldots,r_p})\, E(a_1^{r_1} \cdot\cdot a_p^{r_p})$$

where $d_{k_1,\ldots,k_p}^{r_1,\ldots,r_p} = d_{r_1}^{(s)}(k_1) \cdot\cdot d_{r_p}^{(s)}(k_p)$.

This product effect is a linear combination of s^p terms of the form $E(a_1^{r_1} \cdot\cdot a_p^{r_p})$ with coefficient $\{d_{r_1}^{(s)}(k_1)\} \cdot\cdot \{d_{r_p}^{(s)}(k_p)\}$ such that $r_1,\ldots,r_p = 0,1,\ldots,s-1$. These terms are summed over all the levels of remaining $(m-p)$ factors in the experiment. That is, each term $E(a_1^{r_1} \cdot\cdot a_p^{r_p})$ is the sum of the 'true' yields of s^{m-p} assemblies which are all the possible combinations of the s levels of $(m-p)$ factors combined with $(a_1^{r_1} \cdot\cdot a_p^{r_p})$. Thus the s^p terms of this effect account for all the s^m assemblies in the experiment.

The geometric effect of the last 'g' factors can be defined as

$$(2.2.3) \qquad A_{h+1}^{f_{h+1}} \cdot\cdot A_m^{f_m}$$

$$= \sum_{u=0}^{s-1} d_u^{(s)}(f_{h+1}) \;\; \{\Delta(0,\ldots,0,1,f_{h+2},\ldots,f_m;\alpha_u)\}$$

where α_u are the elements of GF(s), $f_{h+1} \neq 0$; and $\{\Delta(0,\ldots,0,1,f_{h+2},\ldots,f_m;\alpha_u)\}$ is the sum of 'true' yields of s^{m-1} assemblies lying on the hyperplane $\Delta(0,\ldots,0,1,f_{h+2},\ldots,f_m;\alpha_u)$, that is, the assemblies $(a_1^{r_1} \cdot\cdot a_p^{r_p})$ satisfying the equation

$$(2.2.4) \qquad r_{h+1} + f_{h+2}r_{h+2} + \dots + f_m r_m = \alpha_u .$$

The geometric effect is a linear combination of 's' terms of the form $\Delta(0,\dots,0,1,f_{h+2},\dots,f_m;\alpha_u)$ denoted by Δ_{α_u} with the coefficients $d_u^{(s)}(f_{h+1})$. Each of these terms (which corresponds to a hyperplane in EG(m,s)) is the sum of 'true' yields of s^{m-1} assemblies which satisfy the Eqn. (2.2.4). So the above s terms (hyperplanes) also account for all the s^m assemblies in the experiment.

The lemma is proved in two parts, (i) both the product and the geometric effects have some factors common, and (ii) the two effects do not have any factor in common. When they have all the factors common the two effects are of same order involving same factors in which case they are not orthogonal.

Orthogonality is proved by showing that the sum of products of corresponding coefficients of the assemblies in Eqn. (2.2.2) and Eqn. (2.2.3) is zero.

Case(i) Let there be 't' factors common in both the product effect of Eqn. (2.2.2) and the geometric effect of Eqn. (2.2.3). For this case consider the 'p' factors of the product effect and the 'g' factors of the geometric effect to be overlapping so that $A_{h+1}, A_{h+2},\dots,A_{h+t}=A_p$ are the 't' common factors.

Let c be a fixed value from the set $0,1,\dots,s-1$. The hyperplane Δ_{α_c} has s^{m-1} assemblies all with coefficient

$d_{\alpha_c}^{(s)}(f_{h+1})$ in the geometric effect. Consider s^h of them, which can be denoted by

$$\Delta^*(c_{h+1},\dots,c_m) = \left\{a_1^{c_1}\dots a_h^{c_h} a_{h+1}^{c_{h+1}}\dots a_m^{c_m};\ c_1,\dots,c_h = 0,1,\dots,s-1\right\}$$

for one fixed choice of $c_{h+1},\dots,c_m$. By varying $c_{h+1},\dots,c_m$ over all solutions of Eqn. (2.2.4) we get s^{m-h-1} such subsets each with s^h assemblies.

The coefficients of the s^h assemblies in Δ^* are

$\left\{d_{c_1}^{(s)}(k_1)\dots d_{c_h}^{(s)}(k_h)\right\}$ where $c_i = 0,1,\dots,s-1;\ i = 1,2,\dots,h.$

These numbers are the coefficients of the *h*-factor product effect $A_1^{k_1}A_2^{k_2}\dots A_h^{k_h}$ in an s^h factorial experiment. Hence, since all product effects are contrasts, we have

$$(2.2.5) \qquad \sum_{c_1,\dots,c_h = 0}^{s-1} d_{c_1}^{(s)}(k_1)\dots d_{c_h}^{(s)}(k_h) = 0 .$$

Thus for these s^h terms the sum of products of the corresponding coefficients in the product and the geometric effects is

$$\left\{d_{\alpha_c}(f_{h+1})\right\}\cdot 0 = 0 .$$

The same is true for all the s^{m-h-1} subsets and thus for Δ_{α_c}. By letting c vary over GF(s), all assemblies are considered. This completes the proof of Case (i).

Example 2.2.1 Consider a 3^3 FE, i.e., three factors (A_1,A_2,A_3) at 3 levels each $\{(a_i^0,a_i^1,a_i^2);\ i = 1,2,3\}$. Let $(A_1^2A_2)_p$ be the 2-factor effect by product definition and let $(A_1A_3^2)_g$ be a 2-factor effect defined by geometric definition. Both these effects have one factor (A_1) in common.

Let c = 1, i.e., Δ_1 is the hyperplane defined by the equation

$$X_1 + 2X_3 = 1.$$

Here, t = 1 and h = 1 and so the $3^{3-1} = 9$ assemblies lying on this hyperplane are $\{002,\ 012,\ 022,\ 100,\ 110,\ 120,\ 201,\ 211,\ 221\}$. They have the same coefficient in the geometric effect $(A_1A_3^2)_g$, namely $d_1^{(3)}(1) = -1$.

Let c_1 be the level of the factor A_1, which is common to both the product and the geometric effect, then for $c_1 = 0,1,2$ the above 9 assemblies form $3^{3-1-1} = 3$ subsets each with $3^1 = 3$ assemblies,

i.e., $\{002,\ 012,\ 022\}$ for $c_1 = 0$

$\{100,\ 110,\ 120\}$ for $c_1 = 1$

$\{201,\ 211,\ 221\}$ for $c_2 = 2$.

Consider the first subset $\{002,\ 012,\ 022\}$. Their coefficients in product effect $(A_1^2A_2)_p$ are (−1, 0, 1) respectively which sum to zero.

Similarly, for other two subsets the sum of their coefficients in the product effect is zero and hence the sum of their corresponding coefficients in the product effect $(A_1^2A_2)_p$ and in the geometric effect $(A_1A_3^2)_g$ is $(-1)\cdot 0 = 0$.

This is also true for the other hyperplanes Δ_0 and Δ_2.

Thus the two effects are orthogonal.

<u>Case (ii)</u> There are no common factors in the 'p' factor product effect and the 'g' factor geometric effect. For this case assume that the 'p' factors of the product effect and the 'g' factors of the geometric effect are not overlapping and they do not have any factor in common.

Consider one general term of the product effect, say $E(a_1^{c_1} \ldots a_p^{c_p})$ where $c_1, c_2, \ldots, c_p$ are constant values belonging to GF(s). This term has the coefficient $d_{k_1,\ldots,k_p}^{c_1,\ldots,c_p}$ in the product effect and it represents the total of the 'true' yields of s^{m-p} assemblies $(a_1^{c_1} \ldots a_p^{c_p} a_{p+1}^{c_{p+1}} \ldots a_m^{c_m})$ such that $i_{p+1}, \ldots, i_m$ vary over GF(s). These s^{m-p} assemblies form s groups according to α_c in the equation

$$(2.2.6) \qquad i_{h+1} + i_{h+2} f_{h+2} + \ldots + i_m h_m = \alpha_c \, .$$

The s groups consist of an equal number $(= s^{m-p-1})$ of assemblies each, and lie on the s hyperplanes of Eqn. (2.2.3) such that one group lies on one hyperplane.

Thus the s^{m-p} assemblies considered above are equally distributed on the s hyperplanes of Eqn. (2.2.3). The sum of the products of the corresponding coefficients of these s^{m-p} assemblies in the two effect is

$$(2.2.7) \qquad \left[d_{k_1,\ldots,k_p}^{c_1,\ldots,c_p} \right] \sum_{\alpha_c=0}^{s-1} d_{\alpha_c}(f_{h+1})$$

which is equal to zero since $\sum_{\alpha_c=0}^{s-1} d_{\alpha_c}(f_{h+1}) = 0$.

The samething is true for all the other terms in Eqn. (2.2.2). Thus the two effects are orthogonal.

<u>Example 2.2.2</u> Consider a 3^4 FE with four factors (A_1, A_2, A_3, A_4) each at three levels. The product effect $(A_1^2 A_2)_p$ and the geometric effect $(A_3^2 A_4)_g$ do not have any factor common. We have

$$\begin{aligned}(A_1^2 A_2)_p &= \left\{E(a_1^0) - 2E(a_1^1) + E(a_1^2)\right\} \left\{- E(a_2^0) + E(a_2^2)\right\} \\ &= - E(a_1^0 a_2^0) + E(a_1^0 a_2^2) + 2E(a_1^1 a_2^0) - 2E(a_1^1 a_2^2) \\ &\quad - E(a_1^2 a_2^0) + E(a_1^2 a_2^2) .\end{aligned}$$

In this effect the term $E(a_1^0 a_2^0)$ accounts for the 'true' yields of 9 assemblies:

0000	0001	0002
0010	0011	0012
0020	0021	0022 ,

and so do all the other 8 terms.

The geometric effect is

$$(A_3^2A_4)_g = (X_3+ X_4 = 0) - 2(X_3+ X_4 = 1) + (X_3+ X_4 = 2).$$

Out of the 9 assemblies accounted for by $E(a_1^0a_2^0)$ in the product effect, 3 lie on each of the above 3 hyperplanes, i.e.,

0000, 0012, 0021 lie on $(X_3+ X_4 = 0)$,

0001, 0010, 0022 lie on $(X_3+ X_4 = 1)$,

and 0002, 0011, 0020 lie on $(X_3+ X_4 = 2)$.

Hence the sum of the corresponding coefficients of the assemblies in the two effects is zero and the two effects are orthogonal.

Thus a product (geometric) effect is orthogonal to all the geometric (product) effects except when both the effects have all the factors common.

Example 2.2.3 We will consider a 3^4 FE to show that the product and the geometric effects are not orthogonal when the two effects have all the factors common. The product effect $(A_1^2A_2)_p$ and the geometric effect $(A_1A_2^2)_g$ are 2-factor interactions involving the same two factors.

We have

$$(A_1^2A_2)_p = - E(a_1^0a_2^0) + E(a_1^0a_2^2) + 2E(a_1^1a_2^0) - 2E(a_1^1a_2^2)$$
$$- E(a_1^2a_2^0) + E(a_1^2a_2^2)$$

and

$$(A_1A_2^2)_g = - (X_1 + 2X_2 = 0) + (X_1 + 2X_2 = 2)$$
$$= - \left\{E(a_1^0a_2^0) + E(a_1^1a_2^1) + E(a_1^2a_2^2)\right\} + \left\{E(a_1^0a_2^1)\right.$$
$$\left. + E(a_1^1a_2^2) + E(a_1^2a_2^0)\right\}.$$

These two effects have both the factors common and the sum of the corresponding coefficients of the assemblies in the two effects is equal to -3. Hence the above two effects are not orthogonal.

In a s^m factorial experiment, let $\left\{(a_1^{j_1}a_2^{j_2}..a_m^{j_m});\ 0 \leqslant j_r \leqslant s-1;\ r = 1,2,..,m\right\}$ denote the treatment combinations or assemblies. Let $(A_1^{k_1}A_2^{k_2}..A_m^{k_m})_p$ denote an interaction with product definition, which is a linear combination of all the assemblies as

$$(2.2.8) \qquad (A_1^{k_1}..A_m^{k_m})_p = \sum_{r_1,\dots,r_m=0}^{s-1} \left[d_{k_1,\dots,k_m}^{r_1,\dots,r_m}\right] E(a_1^{r_1}..a_m^{r_m})$$

where $d_{k_1,\dots,k_m}^{r_1,\dots,r_m} = \left\{d_{r_1}^{(s)}(k_1)\right\}..\left\{d_{r_m}^{(s)}(k_m)\right\}.$

Let $\underline{a}$ denote a column vector of all assemblies in the following arbitrary but fixed order:

$$\left[\emptyset,\ a_1,\ a_1^2,\ \ldots,\ a_1^{s-1},\ a_2,\ \ldots\ldots,\ a_m^{s-1},\ a_1a_2,\ a_1^2a_2,\ \ldots\ldots,\ a_1^{s-1}a_2^{s-1},\ a_1a_3,\ \ldots\ldots,\ a_{m-1}^{s-1}a_m^{s-1},\ a_1a_2a_3,\ \ldots\ldots,\ a_1^{s-1}\ldots a_m^{s-1}\right].$$

Let $\underline{A}_p$ denote the column vector effects in the same order as $\underline{a}$. Then Eqn. (2.2.8) can be written as

$$(2.2.9) \qquad \underline{A}_p = \underline{D}_p\, E(\underline{a}),$$

where $\underline{D}_p$ is a $s^m x s^m$ matrix of the coefficients $d_{k_1,\ldots,k_m}^{r_1,\ldots,r_m}$. As these contrasts are mutually orthogonal, the sum of products of corresponding elements in any two rows of $\underline{D}_p$ is zero. Let $(\delta_i^2)_p$ be the sum of squares of the elements in ith row of $\underline{D}_p$. To make $\underline{D}_p$ orthonormal divide the ith row of $\underline{D}_p$ by $(\delta_i)_p$. Let $\underline{\Delta}_p$ be a diagonal matrix of order s^m with $(\delta_i)_p^{-1}$ in the (i,i) cell. Let $\underline{C}_p = \underline{\Delta}_p\, \underline{D}_p$ and so

$$\underline{\Delta}_p\, \underline{A}_p = \underline{C}_p\, E(\underline{a})$$

or

$$(2.2.10) \qquad E(\underline{a}) = \underline{C}_p'\, \underline{\Delta}_p\, \underline{A}_p\ .$$

Similarly by geometric definition

$$(2.2.11) \qquad (A_1^{k_1} \cdots A_m^{k_m})_g = \sum_{j=0}^{s-1} d_j^{(s)}(k_r) \{\Delta(0,\ldots,0,1,k_{r+1},\ldots,k_m;\alpha_j)\}$$

If $\underline{A}_g$ is the column vector of these effects in the same order as $\underline{a}$, then

$$(2.1.12) \qquad \underline{A}_g = \underline{D}_g \, E(\underline{a}) \; .$$

An expression similar to Eqn. (2.2.10) can be obtained in terms of $\underline{A}_g$, which gives

$$(2.2.13) \qquad E(\underline{a}) = \underline{C}_g' \, \underline{\Delta}_g \, \underline{A}_g \; .$$

From Eqn. (2.2.10) and Eqn. (2.2.13), we get

$$\underline{C}_p' \, \underline{\Delta}_p \, \underline{A}_p = \underline{C}_g' \, \underline{\Delta}_g \, \underline{A}_g$$

or

$$\begin{aligned} \underline{A}_p &= \underline{\Delta}_p^{-1} \, \underline{C}_p \, \underline{C}_g' \, \underline{\Delta}_g \, \underline{A}_g \\ &= \underline{D}_p \, \underline{D}_g' \, \underline{\Delta}_g^2 \, \underline{A}_g \qquad = \underline{B} \, \underline{A}_g \end{aligned}$$

where $\underline{B} = \underline{D}_p \, \underline{D}_g' \, \underline{\Delta}_g^2$.

By the previous lemma, then, this matrix $\underline{B}$ will have a pattern as follows:

$$\underline{B} = \begin{bmatrix} \underline{B}_0 & & & & \\ & \underline{B}_1 & & & \underline{0} \\ & & \underline{B}_2 & & \\ \underline{0} & & & \ddots & \\ & & & & \underline{B}_{m-1} \end{bmatrix}$$

where $\underline{B}_0$ is a matrix of order $(s-1)^m+1$ corresponding to the mean and main effects and $\underline{B}_1$, $\underline{B}_2$, .. , $\underline{B}_{m-1}$ are the matrices corresponding to the interactions.

<u>Theorem 2.2.1</u> In a s^m FE, where s is a prime power, the k-factor interactions, as defined by the product (geometric) definition, depend only on (are linear combinations of only) the k-factor interactions involving the same factors, as defined by geometric (product) definition.

Proof: The proof follows immediately from the Lemma 2.2.1 and the fact that the matrix $\underline{B}$ in Eqn. (2.2.15) is a matrix of real numbers.

<u>Lemma 2.2.2</u> If $\hat{\underline{A}}_g$ is the BLUE of $\underline{A}_g$ then $\hat{\underline{A}}_p = \underline{B}\,\hat{\underline{A}}_g$ is the BLUE of $\underline{A}_p$.

<u>Corollary 2.2.1</u> In a 2^m FE, the product effects and the geometric effects are identical (except possibly for sign).

Proof: Consider a 2^m FE, involving m factors $(A_1, A_2, .., A_m)$ at 2 levels each $\{(a_i^0, a_i^1); \ i = 1,2,..,m\}$. A k-factor interaction, involving without loss of generality the first k factors, by product definition is

$$(2.2.16) \qquad A_1 A_2 .. A_k = \prod_{i=1}^{k} \{- E(a_i^0) + E(a_i^1)\}$$
$$= (-1)^k \ (\theta_0) + (-1)^{k-1} \ (\theta_1)$$

where θ_0 is the sum of the 'true' yields of the assemblies $(a_1^{r_1} a_2^{r_2} .. a_k^{r_k})$ in which $\sum_{i=1}^{k} r_i = 0$ and θ_1 is the sum of 'true' yields of the assemblies $(a_1^{r_1} a_2^{r_2} .. a_k^{r_k})$ in which $\sum_{i=1}^{k} r_i = 1$.

The same effect by geometric definition is given by

$$(2.2.17) \qquad A_1 A_2 .. A_k = \sum_{u=0}^{1} d_u^{(2)}(1) \ \{\Delta(1,1,..,1,0,..,0;\alpha_u)\}$$
$$= - (\Delta_0) + (\Delta_1)$$

where Δ_c is the sum of the 'true' yields of the assemblies $(a_1^{r_1} a_2^{r_2} .. a_k^{r_k})$ satisfying the equation

$$(2.2.18) \qquad r_1 + r_2 + .. + r_k = c \ .$$

It can be seen that θ_c and Δ_c represent the same assemblies and so if k is odd the two effects are identical. If k is even, then the two effects are identical in magnitude but opposite in sign.

Corollary 2.2.2 In a s^m FE, the mean and main effects defined by the two definitions are the same; and hence $\underline{B}_0 = \underline{I}$, in matrix $\underline{B}$ of Eqn. (2.2.15).

Proof: The mean in both definitions is the sum of all the assemblies as $d_j^{(s)}(0) = 1$.

For main effects let us consider a main effect $A_1^{k_1}$. By product definition it can be given as

$$(2.2.19) \qquad A_1^{k_1} = \sum_{i=0}^{s-1} d_i^{(s)}(k_1) \ E(a_1^i)$$

where $E(a_1^i)$ is the sum of the 'true' yields of all assemblies in which the factor A_1 occurs at level i.

By geometric definition, the same effect is given by

$$(2.2.20) \qquad A_1^{k_1} = \sum_{j=0}^{s-1} d_j^{(s)}(k_1) \ \{\Delta(1,0,\ldots,0;\alpha_j)\}$$

$$= \sum_{j=0}^{s-1} d_j^{(s)}(k_1) \ (\Delta_{\alpha_j})$$

where Δ_{α_j} is the hyperplane formed by the assemblies $(a_1^{j_1} \ldots a_m^{j_m})$ satisfying $j_1 = \alpha_j$, α_j being an element of GF(s).

It is easily seen that the assemblies represented by $E(a_1^c)$ are the same assemblies that are represented by Δ_c and hence the main effects given by the two definitions are identical.

Corollary 2.2.3 In a 2^mxs AFE, where s is a prime power, both definitions of factorial effects are same except possibly for sign.

Proof: Let factors $(A_1, A_2, .., A_m)$ occur at 2 levels each $\{(a_i^0, a_i^1);\ i = 1,2,..,m\}$ and let factor B occur at s levels $(b^0, b^1, ..., b^{s-1})$.

From Cor. 2.2.1 and Cor. 2.2.2 we can see that the main effects and k-factor pure-A effects will be identical.

A k+1 factor mixed effect containing factors $A_1, A_2, ..., A_k$ and the factor B can be defined by product effect as

$$(2.2.21) \qquad A_1A_2..A_kB^f = \left[\prod_{i=1}^{k} \{E(a_i^1) - E(a_i^0)\}\right] \left[\sum_{j=0}^{s-1} d_j^{(s)}(f)\ E(b^j)\right]$$

$$= \left[(-1)^k\ (\theta_0) + (-1)^{k-1}\ (\theta_1)\right] \left[\sum_{j=0}^{s-1} d_j^{(s)}(f)\ E(b^j)\right]$$

and the same effect by geometric definition is

$$(2.2.22) \qquad A_1A_2..A_kB^f = \left[\sum_{u=0}^{1} d_u^{(2)}(1)\{\Delta(1,...,1;\alpha_u)\}\right] \times \left[\sum_{j=0}^{s-1} d_j^{(s)}(f)\ \Delta^*_{\alpha_j}\right]$$

$$= \left[-(\Delta_0) + (\Delta_1)\right] \times \left[\sum_{j=0}^{s-1} d_j^{(s)}(f)\,\Delta^*_{\alpha_j}\right]$$

where Δ_c is the hyperplane formed by the assemblies $(a_1^{r_1} .. a_k^{r_k})$ satisfying the Eqn. (2.2.18) and $\Delta^*_{\alpha_j}$ is the hyperplane formed by the assemblies $(a_1^{r_1}..a_m^{r_m}b^t)$ satisfying $t = \alpha_j$.

From Cor. 2.2.1 and Cor. 2.2.2 these two effects are identical when k is odd. When k is even the effects are identical in magnitude but opposite in sign.

2.3 Finding the Elements of the Matrix '$\underline{B}$'

Let the vector $\underline{a}$ and the matrices $\underline{A}_p$, $\underline{A}_g$, $\underline{D}_p$, $\underline{D}_g$, $\underline{\Delta}_p$ and $\underline{\Delta}_g$ be the same as defined in Section 2.2.1. In this section we will derive a computational technique for finding the elements of the matrix $\underline{B}$. This will be useful to estimate the product effects from the designs where geometric effects are estimable.

We introduce a linear operator Φ(which is similar to the λ-operator in Bose and Srivastava (1964)), and a special product which we shall call 'Assembly Product'.

Let $\Phi_{i_1,\dots,i_r}^{j_1,\dots,j_r}$ be the number of times the sub-assembly $(a_{i_1}^{j_1}..a_{i_r}^{j_r})$ occurs in a complete replication of the factorial experiment. Define the linear operator by the following equations:

$$(2.3.1) \qquad \Phi(a_{i_1}^{j_1}..a_{i_r}^{j_r}) = \Phi_{i_1,\dots,i_r}^{j_1,\dots,j_r}$$

and

$$(2.3.2)\qquad \Phi\left[\beta_1(a_{i_1}^{j_1}..a_{i_r}^{j_r}) + \beta_2(a_{k_1}^{f_1}..a_{k_u}^{f_u})\right]$$

$$= \beta_1\Phi_{i_1,\ldots,i_r}^{j_1,\ldots,j_r} + \beta_2\Phi_{k_1,\ldots,k_u}^{f_1,\ldots,f_u}$$

where β_1 and β_2 are constants.

The linear operator, Φ, can be evaluated with the following formulae for any assembly in symmetrical and asymmetrical cases.

Let Φ^k be the number of times a k-factor sub-assembly occurs in a complete replication. Then, in s^m symmetrical factorial experiment

$$(2.3.3)\qquad \Phi^k = s^{m-k}, \quad \text{for } 1 \leq k \leq m$$
$$= 1 \quad , \quad \text{for } k = 0 \quad .$$

In a $s_1^m x s_2^n$ asymmetrical factorial experiment let $(A_i;\ i = 1,2,..,m)$ and $(B_j;\ j = 1,2,..,n)$ denote the factors at s_1 and s_2 levels respectively and let Φ_A^k and Φ_B^k be the number of times a k-factor pure-A and pure-B sub-assembly occurs in a complete replication, then

$$(2.3.4)\qquad \Phi_A^k = s_1^{m-k}s_2^n\ , \quad \text{for } 1 \leq k \leq m$$

and

$$(2.3.5)\qquad \Phi_B^k = s_1^m s_2^{n-k}\ , \quad \text{for } 1 \leq k \leq n\ .$$

For a mixed k-factor sub-assembly consisting of 'k_1' A-factors and 'k_2' B-factors, ϕ_{AB}^{k} is the number of times this sub-assembly occurs in a complete replication and is given by

$$(2.3.6) \qquad \phi_{AB}^{k} = s_1^{m-k_1} s_2^{n-k_2} \quad \text{for } k = k_1 + k_2,\ 1 \leqslant k_1 \leqslant m \text{ and } 1 \leqslant k_2 \leqslant n \ .$$

<u>Definition</u> The 'Assembly Product' of the coefficients of two assemblies is the product of the corresponding coefficients if the two assemblies are the same and zero otherwise.

If d_1 and d_2 are two coefficients and $(a_1^{j_1} .. a_m^{j_m})$ and $(a_1^{k_1} .. a_m^{k_m})$ are the two assemblies then their 'Assembly Product' can be represented by

$$(2.3.7) \qquad d_1 \ (a_1^{j_1} .. a_m^{j_m}) \otimes d_2 \ (a_1^{k_1} .. a_m^{k_m})$$

$$= (d_1)(d_2) \ (a_1^{j_1} .. a_m^{j_m}) \quad \text{if } j_i = k_i \text{ for all } i$$

$$= 0 \quad \text{if } j_i \neq k_i \text{ for at least one } i \ .$$

Also an assembly and its coefficient can be factored in a similar manner. The symbol $d(a_1^{j_1} .. a_m^{j_m})$ can be factored as

$$(2.3.8) \qquad d \ (a_1^{j_1} .. a_m^{j_m}) = d_1 \ (a_1^{j_1} .. a_m^{j_m}) \otimes d_2 \ (a_1^{j_1} .. a_m^{j_m})$$

where $d = (d_1)(d_2)$.

In a s^m factorial experiment, from Eqn. (2.2.15)

$$\underline{B} = \underline{D}_p \underline{D}_g' \underline{\Delta}_g^2 .$$

Let $\underline{B}^* = \underline{D}_p \underline{D}_g'$ then $\underline{B} = \underline{B}^* \underline{\Delta}_g^2$.

An element of the matrix $\underline{B}^*$ can be expressed in terms of the ϕ-operator. The element $\epsilon(i,j)$ in (i,j)th cell of the matrix $\underline{B}^*$ corresponds to ith row of $\underline{D}_p$ and jth row of $\underline{D}_g$, i.e., to the ith element of $\underline{A}_p$ and jth element of $\underline{A}_g$.

From the results of Lemma 2.2.1 and Theorem 2.2.1 it follows that the element $\epsilon(A_{i_1}^{k_1} \ldots A_{i_m}^{k_m} , A_{j_1}^{f_1} \ldots A_{j_m}^{f_m})$ of $\underline{B}^*$ is zero unless the two effects are of the same order and involve exactly the same factors. Hence we will find only those elements of the matrix $\underline{B}^*$ which correspond to the same order effects involving the same factors.

<u>Theorem 2.3.1</u> If $\epsilon(A_{i_1}^{j_1} \ldots A_{i_m}^{j_m} , A_{i_1}^{k_1} \ldots A_{i_m}^{k_m})$ is an element of $\underline{B}^*$ at the intersection of row corresponding to $A_{i_1}^{j_1} \ldots A_{i_m}^{j_m}$ and column corresponding to $A_{i_1}^{k_1} \ldots A_{i_m}^{k_m}$ then it is given by

(2.3.9) $$\epsilon(A_{i_1}^{j_1} \ldots A_{i_m}^{j_m} , A_{i_1}^{k_1} \ldots A_{i_m}^{k_m})$$

$$= \Phi\left[\left\{\prod_{r=1}^{m} \left\{\sum_{k=0}^{s-1} d_k^{(s)}(j_r) \; (a_{i_r}^{k})\right\}\right\} \otimes \left\{\sum_{\alpha=0}^{s-1} d_\alpha^{(s)}(k_1) \; \Delta_\alpha\right\}\right]$$

where $i_1,\ldots,i_m = 1,2,\ldots,m$; $j_1,\ldots,j_m,k_1,\ldots,k_m = 0,1,\ldots,s-1$; Δ_α is the total of the assemblies lying on the hyperplane $X_{i_1} + k_2 X_{i_2} + \ldots + k_m X_{i_m} = \alpha$; the product $\otimes$ denotes the 'assembly product'; and $(a_{i_r}^k)$ is the total of assemblies with factor A_{i_r} at level k.

Proof: We will prove this theorem for a case of 2-factor effects. The general case is a simple extension of this proof.

Consider the element $\in(A_{i_1}^{j_1} A_{i_2}^{j_2}, A_{i_1}^{k_1} A_{i_2}^{k_2})$. This is the innerproduct of the row R_p (of $\underline{D}_p$) corresponding to the element $A_{i_1}^{j_1} A_{i_2}^{j_2}$ in $\underline{A}_p$ and the row R_g (of $\underline{D}_g$) corresponding to the element $A_{i_1}^{k_1} A_{i_2}^{k_2}$ in $\underline{A}_g$.

The elements in these rows corresponding to a general element $(a_1^{f_1} \ldots a_{i_1}^{f_{i_1}} \ldots a_{i_2}^{f_{i_2}} \ldots a_m^{f_m})$ in the vector $\underline{a}$ are $\{d_{f_{i_1}}(j_1)\}\{d_{f_{i_2}}(j_2)\}$ and $d_{\alpha(f_{i_1},f_{i_2})}(k_1)$ where

$$f_{i_1} + k_2 f_{i_2} = \alpha(f_{i_1},f_{i_2}) \; . \tag{2.3.10}$$

The product of these is δ where

$$\delta = d_{f_{i_1}}(j_1)\, d_{f_{i_2}}(j_2)\, d_{\alpha(f_{i_1},f_{i_2})}(k_1) \tag{2.3.11}$$

We get the same δ as many times as the sub-assembly $(a_{i_1}^{l_{i_1}} a_{i_2}^{l_{i_2}})$ occurs in the vector $\underline{a}$, i.e., $(\Phi_{i_1,i_2}^{l_{i_1},l_{i_2}})$ times. Let the symbol $\alpha(f_{i_1}, f_{i_2})$, which is a function of f_{i_1} and f_{i_2}, be denoted by $\alpha(f)$.

Hence we can write

$$(2.3.12) \qquad \epsilon(A_{i_1}^{j_1} A_{i_2}^{j_2} , A_{i_1}^{k_1} A_{i_2}^{k_2})$$

$$= \sum_{f_{i_1}, f_{i_2}=0}^{s-1} d_{f_{i_1}}(j_1)\, d_{f_{i_2}}(j_2)\, d_{\alpha(f)}(k_1)\, (\Phi_{i_1,i_2}^{l_{i_1},l_{i_2}})$$

$$= \Phi\left\{\sum_{f_{i_1}, f_{i_2}=0}^{s-1} d_{f_{i_1}}(j_1)\, d_{f_{i_2}}(j_2)\, d_{\alpha(f)}(k_1)\, (a_{i_1}^{l_{i_1}} a_{i_2}^{l_{i_2}})\right\}$$

$$= \Phi\left[\left\{\sum_{f_{i_1}, f_{i_2}=0}^{s-1} d_{f_{i_1}}(j_1)\, d_{f_{i_2}}(j_2)\, (a_{i_1}^{l_{i_1}} a_{i_2}^{l_{i_2}})\right\}\right.$$

$$\left.\otimes \left\{\sum_{f_{i_1}, f_{i_2}=0}^{s-1} d_{\alpha(f)}(k_1)\, (a_{i_1}^{l_{i_1}} a_{i_2}^{l_{i_2}})\right\}\right]$$

The terms in the second curly brackets of the right-hand-side of the above equation can be rearranged so that the assemblies which satisfy the Eqn. (2.3.10) with $\alpha = 0$ occur first, those with $\alpha = 1$ occur next and so on. Hence

$$(2.3.13)\qquad \epsilon(A_{i_1}^{j_1}A_{i_2}^{j_2}\ ,\ A_{i_1}^{k_1}A_{i_2}^{k_2})$$

$$= \Phi\left[\left\{\prod_{r=1}^{2}\left\{\sum_{k=0}^{s-1} d_k(j_r)\ (a_{i_r}^{k})\right\}\right\} \otimes \left\{\sum_{\alpha(f)=0}^{s-1} d_{\alpha(f)}(k_1)\ \Delta_{\alpha(f)}\right\}\right]$$

Lemma 2.3.1 An element of the matrix $\underline{B}$ corresponding to the element $\epsilon(A_{i_1}^{j_1}A_{i_2}^{j_2}\ ,\ A_{i_1}^{k_1}A_{i_2}^{k_2})$ (say ϵ) of $\underline{B}^*$ is $(\epsilon)(\Upsilon)$ where $\Upsilon = \left[s^{m-1}\left\{\sum_{k=0}^{s-1}\left\{d_k^{(s)}(k_1)\right\}^2\right\}\right]^{-1}$

Proof: Since the matrix $\underline{B} = \underline{B}^*.\underline{\Delta}_g^2$ and $\underline{\Delta}_g^2$ is a diagonal matrix, the element (i,j) of $\underline{B}$ can be obtained by multiplying the element ϵ in (i,j)th cell of $\underline{B}^*$ by the jth diagonal element Υ of $\underline{\Delta}_g^2$. The jth diagonal element of $\underline{\Delta}_g^2$ is $(\delta_j^2)_g^{-1}$ where $(\delta_j^2)_g$ is the sum of squares of the elements in jth row of $\underline{D}_g$.

For the element of $\underline{B}$ corresponding to the element $\epsilon(A_{i_1}^{j_1}A_{i_2}^{j_2}\ ,\ A_{i_1}^{k_1}A_{i_2}^{k_2})$ of $\underline{B}^*$ the jth row of $\underline{D}_g$ corresponds to the geometric effect $A_{i_1}^{k_1}A_{i_2}^{k_2}$. The elements in this row of $\underline{D}_g$ are $d_k^{(s)}(k_1)$; $k = 0,1,..,s-1$ where each is repeated s^{m-1} times.

Hence

$$(2.3.14)\qquad (\delta_j^2)_g \quad = \quad s^{m-1}\left[\sum_{k=0}^{s-1}\left\{d_k^{(s)}(k_1)\right\}^2\right]$$

and the element of $\underline{B}$ is

$$(2.3.15)\qquad (\epsilon)(\delta_j^2)_g^{-1} \quad = \quad (\epsilon)(\Upsilon) \quad .$$

Example 2.3.1 Consider a 3^2 factorial experiment. We have

$$\underline{B} = \begin{bmatrix} \underline{I} & \underline{0} \\ \underline{0} & \underline{B}_1 \end{bmatrix}$$

For the 3^2 FE, $\underline{I}$ is of order 5 and $\underline{B}_1$ is a square matrix of order 4. Let A_1 and A_2 be the two factors, then

$$\underline{a} = \left[a_1^0 a_2^0,\ a_1^1 a_2^0,\ a_1^2 a_2^0,\ a_1^0 a_2^1,\ a_1^0 a_2^2,\ a_1^1 a_2^1,\ a_1^2 a_2^1,\ a_1^1 a_2^2,\ a_1^2 a_2^2 \right]$$

and in the same order $\underline{A}_p$ and $\underline{A}_g$ are the vectors of factorial effects.

We have $d^{(3)}(1) = -1,\ 0,\ 1$ and $d^{(3)}(2) = 1,\ -2,\ 1$, then

$$\begin{aligned} \epsilon(A_1A_2, A_1A_2) &= \Phi\Big[\big\{(-\ a_1^0 + a_1^2)\ (-\ a_2^0 + a_2^2)\big\} \\ &\qquad \otimes \big\{-\ (X_1 + X_2 = 0) + (X_1 + X_2 = 2)\big\}\Big] \\ &= \Phi\Big[\big\{a_1^0a_2^0 - a_1^0a_2^2 - a_1^2a_2^0 + a_1^2a_2^2\big\} \otimes \big\{-\ (a_1^0a_2^0 \\ &\qquad + a_1^1a_2^2 + a_1^2a_2^1) + (a_1^0a_2^2 + a_1^1a_2^1 + a_1^2a_2^0)\big\}\Big] \\ &= \Phi\left\{-\ a_1^0a_2^0 - a_1^0a_2^2 - a_1^2a_2^0\right\} \\ &= -\left\{\Phi_{1,2}^{0,0} + \Phi_{1,2}^{0,2} + \Phi_{1,2}^{2,0}\right\} \\ &= -\ 3\ . \end{aligned}$$

The corresponding element of the matrix of $\underline{B}$ is $-3(\gamma)$ where

$$\gamma = \left[3\left\{\sum_{k=0}^{2}\left\{d_k^{(3)}(1)\right\}^2\right\}\right]^{-1} = 1/6$$

or $(-1/2)$.

Another element can be given as

$$\begin{aligned}\epsilon(A_1A_2, A_1^2A_2) &= \phi\Big[\left\{a_1^0a_2^0 - a_1^0a_2^2 - a_1^2a_2^0 + a_1^2a_2^2\right\} \otimes \\ &\qquad \left\{(X_1+X_2=0) - 2(X_1+X_2=1) + (X_1+X_2=2)\right\}\Big] \\ &= \phi\Big[\left\{a_1^0a_2^0 - a_1^0a_2^2 - a_1^2a_2^0 + a_1^2a_2^2\right\} \otimes \\ &\qquad \left\{(a_1^0a_2^0 + a_1^2a_2^0 + a_1^0a_2^2 + a_1^1a_2^1 + a_1^1a_2^2 + a_1^2a_2^1)\right. \\ &\qquad \left.- 2(a_1^1a_2^0 + a_1^0a_2^1 + a_1^2a_2^2)\right\}\Big] \\ &= \phi_{1,2}^{0,0} - \phi_{1,2}^{0,2} - \phi_{1,2}^{2,0} - 2\phi_{1,2}^{2,2} \\ &= -3 .\end{aligned}$$

The corresponding element of $\underline{B}$ is $-3(\gamma)$ where

$$\gamma = \left[3\left\{\sum_{k=0}^{2}\left\{d_k^{(3)}(2)\right\}^2\right\}\right]^{-1} = 1/18$$

or $(-1/6)$.

The element of $\underline{B}^*$ corresponding to two quadratic effects is

$$\begin{aligned}\epsilon(A_1^2A_2, A_1^2A_2^2) &= \phi\Big[\left\{(a_1^0 - 2a_1^1 + a_1^2)(-a_2^0 + a_2^2)\right\} \otimes \\ &\qquad \left\{(X_1+2X_2=0) - 2(X_1+2X_2=1) + (X_1+2X_2=2)\right\}\Big]\end{aligned}$$

$$= \phi\left[\left\{- a_1^0a_2^0 + a_1^0a_2^2 + 2a_1^1a_2^0 - 2a_1^1a_2^2 - a_1^2a_2^0 + a_1^2a_2^2\right\} \otimes \left\{a_1^0a_2^0 + a_1^2a_2^0 + a_1^0a_2^1 + a_1^1a_2^1 + a_1^1a_2^2 + a_1^2a_2^2 - 2(a_1^1a_2^0 + a_1^0a_2^2 + a_1^2a_2^1)\right\}\right]$$

$$= - \phi_{1,2}^{0,0} - 2\phi_{1,2}^{0,2} - 4\phi_{1,2}^{1,0} - 2\phi_{1,2}^{1,2} - \phi_{1,2}^{2,0} + \phi_{1,2}^{2,2}$$

$$= - 9 .$$

The corresponding element in $\underline{B}$ is $(- 9/18) = (- 1/2)$.

Similarly we can find the other elements of the matrix $\underline{B}_1$, which can be given as

$$\underline{B}_1 = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{6} & -\frac{1}{2} & \frac{1}{6} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

For a 3^2 FE, this matrix $\underline{B}_1$ is repeated thrice on the diagonal and off-diagonal elements will be zero. For a 3^n FE, this matrix $\underline{B}_1$ will be repeated $\binom{n}{2}$ times on the diagonal. Thus if the 2-factor interactions of two fixed factors as defined by the geometric definition are estimable in a design, the 2-factors product effects involving the same factors can be estimated. For example if the geometric effects $(A_1A_2)_g$, $(A_1^2A_2)_g$, $(A_1A_2^2)_g$ and $(A_1^2A_2^2)_g$ are estimable from the design then $(A_1A_2)_p$ can be estimated by

$$\begin{aligned}(A_1A_2)_p &= -\tfrac{1}{2}(A_1A_2)_g - \tfrac{1}{6}(A_1^2A_2)_g - \tfrac{1}{2}(A_1A_2^2)_g + \tfrac{1}{6}(A_1^2A_2^2)_g \\ &= -\tfrac{1}{2}\left\{-E(00) - E(12) - E(21) + E(02) + E(11) + E(20)\right\} \\ &\quad - \tfrac{1}{6}\Big\{E(00) + E(20) + E(02) + E(11) + E(12) + E(21) \\ &\quad - 2\left[E(10) + E(01) + E(22)\right]\Big\} - \tfrac{1}{2}\Big\{-E(00) - E(11) \\ &\quad - E(22) + E(01) + E(12) + E(20)\Big\} - \tfrac{1}{6}\Big\{E(00) + E(20) \\ &\quad + E(01) + E(11) + E(12) + E(22) - 2\big[E(10) + E(02) \\ &\quad + E(21)\big]\Big\}.\end{aligned}$$

2.4 Balanced Designs

When a design is balanced the variances of all the estimates of the same order factorial effects are equal. Consider a s^m FE balanced with respect to geometric effects. Then the variance-covariance matrix of the estimates of the geometric factorial effects will have the following pattern.

$$\underline{V}(A_g) = \sigma^2 \begin{bmatrix} a & b\underline{j} & c\underline{j} & \cdots \\ b\underline{j} & ((d-e)\underline{I}+e\underline{J}) & f\underline{J} & \\ c\underline{j} & f\underline{J} & ((g-h)\underline{I}+h\underline{J}) & \cdots \\ \vdots & & \vdots & \end{bmatrix}$$

where $\underline{j}$ is a column vector of 1's, $\underline{J}$ is a matrix of 1's, $\underline{I}$ is the identity and a, b, c, d, e, f, g and h are constants.

Then the variance-covariance matrix of the estimates of the product effects will be

$$(2.4.1) \qquad \underline{V}(A_p) = \underline{B}\,\underline{V}(A_g)\,\underline{B}' .$$

In a 2^m FE and 2^mxs AFE, the matrix $\underline{B}$ is a diagonal with $\pm$ 1 on the diagonal and hence $\underline{V}(A_p)$ will have the same pattern as $\underline{V}(A_g)$ and thus the design is also balanced with respect to product effects.

However, in the general s^m FE or $s_1^m x s_2^n$ AFE, the above is necessarily true only for the mean and main effects. The variances for the interactions of same order under the product definition are not necessarily equal even when the design is balanced with respect to the geometric effects.

<u>Example 2.3.2</u> Consider a 3^2 FE, balanced with respect to geometric effects. Then the variance-covariance matrix of the estimates of the geometric effects will be of the form:

$$\underline{V}(A_g) = \begin{bmatrix} a & b & b & b & b & c & c & c & c \\ b & d & e & e & e & f & f & f & f \\ b & e & d & e & e & f & f & f & f \\ b & e & e & d & e & f & f & f & f \\ b & e & e & e & d & f & f & f & f \\ c & f & f & f & f & g & h & h & h \\ c & f & f & f & f & h & g & h & h \\ c & f & f & f & f & h & h & g & h \\ c & f & f & f & f & h & h & h & g \end{bmatrix}$$

The product effects can be estimated by multiplying the vector of geometric effects by the matrix $\underline{B}$ obtained in Example 2.3.1. The variance-covariance matrix of these product effects is given by

$$\underline{V}(A_p) = \underline{B}\,\underline{V}(A_g)\,\underline{B}'$$

$$= \begin{bmatrix} a & b & b & b & b & -c & -c & c & c \\ b & d & e & e & e & -f & -f & f & f \\ b & e & d & e & e & -f & -f & f & f \\ b & e & e & d & e & -f & -f & f & f \\ b & e & e & e & d & -f & -f & f & f \\ -c & -f & -f & -f & -f & \frac{5}{9}g+\frac{4}{9}h & h & -(\frac{1}{3}g+\frac{2}{3}h) & -h \\ -c & -f & -f & -f & -f & h & g & -h & g-2h \\ c & f & f & f & f & -(\frac{1}{3}g+\frac{2}{3}h) & -h & g & h \\ c & f & f & f & f & -h & g-2h & h & 5g-4h \end{bmatrix}$$

Thus the design balanced with respect to geometric effects may not be balanced with respect to product effects.

CHAPTER III

EXAMPLES ILLUSTRATING THE THEORY

In this chapter some of the confounding plans suggested by Cochran and Cox are considered. These plans are for s^n symmetrical factorial experiments where $s \geqslant 3$ and $s_1 x s_2 x..x s_n$ asymmetrical factorial experiment. The reader is led to believe that these plans are partially balanced so that the degrees of freedom confounded in one replication can be estimated from the other replications. These plans are designed by partially confounding certain geometric effects. However, the corresponding effects defined by product definition are not directly estimable in these designs. Furthermore, even though the design is partially balanced with respect to geometric effects, it is not partially balanced with respect to product effects.

The first plan (plan 6.7 Cochran and Cox 1968) considered is a 3^3 symmetrical design, with four replications each of which consists of 3 blocks of 9 units. Two degrees of freedom of the 3-factor geometric effect ABC are confounded in each replication. The assembly 021 denotes the treatment combination of the factor A at level 0, the factor B at level 2 and the factor C at level 1. The plan is given below. The estimibility of the effects is tested in replication 1 but the other replications can be tested similarly.

Plan 6.7 Cochran and Cox

Rep. 1			Rep. 2			Rep. 3			Rep. 4		
1	2	3	4	5	6	7	8	9	10	11	12
000	100	200	000	100	200	000	100	200	000	100	200
110	210	010	110	210	010	210	010	110	210	010	110
220	020	120	220	020	120	120	220	020	120	220	020
101	201	001	201	001	101	101	201	001	201	001	101
211	011	111	011	111	211	011	111	211	111	211	011
021	121	221	121	221	021	221	021	121	021	121	221
202	002	102	102	202	002	202	002	102	102	202	002
012	112	212	212	012	112	112	212	012	012	112	212
122	222	022	022	122	222	022	122	222	222	022	122

In replication 1, two degrees of freedom of geometric effect AB^2C^2, i.e., AB^2C^2 and $A^2B^2C^2$ are confounded. The other 6 degrees of freedom, 2 each of geometric effects ABC, ABC^2 and AB^2C are estimable from this replication. Let $(ABC)_g$ and $(ABC)_p$ denote the respective geometric and product effects.

Denote the observation from the above plan by $y_{k(i_1 i_2 i_3)}$ where k = 1,2,..,12 denote the block and $(i_1 i_2 i_3)$ denote the assembly. Assume

$$(3.1.1) \qquad E(y_{k(i_1 i_2 i_3)}) = \beta_k + E(i_1 i_2 i_3)$$

where β_k denotes the effect of the kth block and $E(i_1 i_2 i_3)$ denotes as before the 'true' yield of an experimental unit receiving the assembly $(i_1 i_2 i_3)$.

Then,

(3.1.2) $\quad E(\text{est. value } (ABC)_9)$

$$= E\left\{-(X_1+X_2+X_3=0)+(X_1+X_2+X_3=2)\right\}$$

$$= E\Big[-\{y_{1(000)}+y_{1(012)}+y_{1(021)}+y_{3(120)}+y_{3(102)}$$

$$+y_{3(111)}+y_{2(210)}+y_{2(201)}+y_{2(222)}\}+\{y_{2(002)}$$

$$+y_{2(020)}+y_{2(011)}+y_{1(110)}+y_{1(101)}+y_{1(122)}$$

$$+y_{3(200)}+y_{3(212)}+y_{3(221)}\}\Big]$$

$$= -\{E(000)+\beta_1+E(012)+\beta_1+E(021)+\beta_1+E(120)$$

$$\beta_3+E(102)+\beta_3+E(111)+\beta_3+E(210)+\beta_2+E(201)$$

$$\beta_2+E(222)+\beta_2\}+\{E(002)+\beta_2+E(020)+\beta_2+$$

$$E(011)+\beta_2+E(110)+\beta_1+E(101)+\beta_1+E(122)+\beta_1$$

$$+E(200)+\beta_3+E(212)+\beta_3+E(221)+\beta_3\}$$

$$= (ABC)_9$$

But the product effect ABC is not estimable from this replication as it is confounded with block effects.

(3.1.3) $\quad E(\text{est. value } (ABC)_p)$

$$= \{-E(a^0) + E(a^2)\}\{-E(b^0) + E(b^2)\}\{-E(c^0) + E(c^2)\}$$

$$= -\{E(000) + \beta_1 + E(022) + \beta_3 + E(202) + \beta_1 + E(220) + \beta_1\} + \{E(002) + \beta_2 + E(020) + \beta_2 + E(200) + \beta_3 + E(222) + \beta_2\}$$

$$= (ABC)_p + 3\beta_2 - 3\beta_1$$

Similarly, $(A^2BC^2)_g$ is estimable from this replication but $(A^2BC^2)_p$ is confounded. Since

(3.1.4) $\quad E(\text{est. value } (A^2BC^2)_g)$

$$= E\{(X_1 + X_2 + X_3 = 0) - 2(X_1 + X_2 + X_3 = 1) + (X_1 + X_2 + X_3 = 2)\}$$

$$= \{E(000) + \beta_1 + E(011) + \beta_2 + E(022) + \beta_3 + E(101) + \beta_1 + E(112) + \beta_2 + E(120) + \beta_3 + E(202) + \beta_1 + E(210) + \beta_2 + E(221) + \beta_3\} - 2\{E(021) + \beta_1 + E(002) + \beta_2 + E(010) + \beta_3 + E(122) + \beta_1 + E(100) + \beta_2 + E(111) + \beta_3 + E(220) + \beta_1 + E(201) + \beta_2 + E(212) + \beta_3\} + \{E(012) + \beta_1 + E(020) + \beta_2 + E(001) + \beta_3 + E(110) + \beta_1 + E(121) + \beta_2 + E(102) + \beta_3 + E(211) + \beta_1 + E(222) + \beta_2 + E(200) + \beta_3\}$$

$$= (A^2BC^2)_g$$

but

(3.1.5) $\quad E(\text{est. value } (A^2BC^2)_p)$

$$= \{E(a^0) - 2E(a^1) + E(a^2)\}\{-E(b^0) + E(b^2)\}$$

$$\{E(c^0) - 2E(c^1) + E(c^2)\}$$

$$= \{2E(001) + 2\beta_3 + E(010) + \beta_3 + E(012) + \beta_1 + 2E(100)$$

$$+ 2\beta_2 + 2E(102) + 2\beta_3 + 4E(111) + 4\beta_3 + 2E(201) + 2\beta_2$$

$$+ E(210) + \beta_2 + E(212) + \beta_3\} - \{E(000) + \beta_1 + E(002)$$

$$+ \beta_2 + 2E(011) + 2\beta_2 + 4E(101) + 4\beta_1 + 2E(110) + 2\beta_1 +$$

$$2E(112) + 2\beta_2 + E(200) + \beta_3 + E(202) + \beta_1 + 2E(211) + 2\beta_1\}$$

$$= (A^2BC^2)_p + 9\beta_3 - 9\beta_1 \;.$$

In a similar way it can be shown that none of the six degrees of freedom of product effect is estimable in this replication. It is further noted that these 6 degrees of freedom, two in each of the other three replications are confounded. The unconfounded geometric effects are estimable in all the replications, but all the degrees of freedom of the product effect are completely confounded.

Thus the confounded geometric effect is estimable with 3/4th precision and even though the product effect is not directly estimable from the design, it can be estimated from the geometric effects if the elements of the matrix $\underline{B}$ are known.

The second plan (plan 6.8 Cochran and Cox 1968) considered is a 3^4 symmetrical factorial experiment. The complete design consists of 4 replications each with 9 blocks. Consider the first replication in which 2 degrees of freedom of each of the geometric effects AB^2C^2, ABD^2, ACD and BC^2D are confounded.

Rep. 1 Plan 6.8 Cochran and Cox

1	2	3	4	5	6	7	8	9
0000	0011	0022	0012	0020	0001	0021	0002	0010
1011	1022	1000	1020	1001	1012	1002	1010	1021
2022	2000	2011	2001	2012	2020	2010	2021	2002
0121	0102	0110	0100	0111	0122	0112	0120	0101
1102	1110	1121	1111	1122	1100	1120	1101	1112
2110	2121	2102	2122	2100	2111	2101	2112	2120
0212	0220	0201	0221	0202	0210	0200	0211	0222
1220	1201	1212	1202	1210	1221	1211	1222	1200
2201	2212	2220	2210	2221	2202	2222	2200	2211

The definitions of the three factor effects are similar to the 3-factor effects in the previous case except that the assembly (021) now stands for the sum of three assemblies (0210), (0211) and (0212).

Thus in a similar manner as in plan 6.7 we can show that the 3-factor geometric effects (ABC) and (BCD) are estimable but the corresponding product effects are not directly estimable from this replication since

(3.1.6) E(est. value $(ABC)_p$)

$$= (ABC)_p + 3(\beta_3 + \beta_5 + \beta_7 - \beta_1 - \beta_6 - \beta_8)$$

and

(3.1.7) E(est. value $(BCD)_p$)

$$= (BCD)_p + 3(\beta_7 + \beta_8 + \beta_9 - \beta_1 - \beta_2 - \beta_3)$$

Also in this design the geometric effects are partially confounded in the four replications but product effects are completely confounded in each replication but can be estimated from the geometric effects and the matrix $\underline{B}$.

The third plan (plan 6.11 Cochran and Cox 1968) is a 3^2x2 asymmetrical factorial experiment. The complete design which is partially confounded with respect to geometric effects AB and ABC needs four replications. Consider the first replication:

Rep. 1 Plan 6.11 Cochran and Cox

1	2	3
100	200	000
210	010	110
020	120	220
201	001	101
011	111	211
121	221	021

In this replication, 2 degrees of freedom of the geometric effect AB^2 and the product-geometric effect AB^2C are confounded.

These effects are also confounded in the 2nd replication but are estimable from replication 3 and 4. The geometric effect AB and the product-geometric effect ABC are confounded in replications 3 and 4 but are estimable in the 1st and the 2nd replications. Since in the first replication

(3.1.8) $\qquad E(\text{est. value } (AB)_g)$

$$
\begin{aligned}
&= E\left\{-(X_1 + X_2 = 0) + (X_1 + X_2 = 2)\right\} \\
&= -\left\{E(00) + \beta_2 + \beta_3 + E(12) + \beta_1 + \beta_2 + E(21) + \beta_1 + \beta_3\right\} \\
&\quad + \left\{E(02) + \beta_1 + \beta_3 + E(20) + \beta_1 + \beta_2 + E(11) + \beta_2 + \beta_3\right\} \\
&= (AB)_g
\end{aligned}
$$

and

(3.1.9) $\qquad E(\text{est. value } (ABC)_g)$

$$
\begin{aligned}
&= E\left[\left\{-(X_1 + X_2 = 0) + (X_1 + X_2 = 2)\right\}\right. \\
&\qquad \left. \times \left\{-(X_3 = 0) + (X_3 = 2)\right\}\right] \\
&= -\left\{E(000) + \beta_3 + E(120) + \beta_2 + E(210) + \beta_1 + E(021)\right. \\
&\quad \left. + \beta_3 + E(201) + \beta_1 + E(111) + \beta_2\right\} + \left\{E(020) + \beta_1 + \right. \\
&\quad E(200) + \beta_2 + E(110) + \beta_3 + E(001) + \beta_2 + E(121) + \\
&\quad \left. \beta_1 + E(211) + \beta_3\right\} \\
&= (ABC)_g
\end{aligned}
$$

The corresponding product effects can be shown to be confounded with block effects as

(3.1.10) $\quad E(\text{est. value } (AB)_p)$

$$= \left\{- E(a^0) + E(a^2)\right\} \left\{- E(b^0) + E(b^2)\right\}$$

$$= \left\{E(00) + \beta_2 + \beta_3 + E(22) + \beta_2 + \beta_3\right\} - \left\{E(02) + \beta_1 + \beta_3 + E(20) + \beta_1 + \beta_2\right\}$$

$$= (AB)_p + \beta_2 + \beta_3 - 2\beta_1$$

and

(3.1.11) $\quad E(\text{est. value } (ABC)_p)$

$$= \left\{- E(a^0) + E(a^2)\right\} \left\{- E(b^0) + E(b^2)\right\} \left\{- E(c^0) + E(c^1)\right\}$$

$$= - \left\{E(000) + \beta_3 + E(220) + \beta_3 + E(021) + \beta_3 + E(201) + \beta_1\right\} + \left\{E(020) + \beta_1 + E(200) + \beta_2 + E(001) + \beta_2 + E(221) + \beta_2\right\}$$

$$= (ABC)_p + 3\beta_2 - 3\beta_3$$

The relationship between the product effects and the effects defined by product-geometric definition in asymmetrical factorial experiments has not been considered. However, it is conjectured that results similar to those for the symmetrical case given in Chapter 2 can be derived at which time estimates of the product effects in the above plan can be proposed.

CHAPTER IV

CONCLUDING REMARKS

Several problems arise from this study of the relationship between the two types of definitions of factorial effects. Some are outlined here in brief.

Most fractions of symmetrical factorial plans are constructed so that they are balanced with respect to the geometric effects. It has been shown in this report that such fractions are usually not balanced with respect to the product effects. A new theory needs to be developed for construction of fractional replicates which are balanced with respect to the product effects.

In this study the relationship between the product and the geometric effects was considered only for the symmetrical factorial experiments. Further research on the relationship between the product effects and the product-geometric effects for the asymmetrical factorial experiments could be undertaken.

The geometric definition has always been used in the construction of partially confounded symmetrical factorial plans. Such designs are partially balanced with respect to the geometric effects. Using the relationship obtained in this report it may be possible to construct designs so that they will be partially balanced with respect to the product effects but not necessarily balanced with respect to the geometric effects.

REFERENCES

1. Bose, R. C. and Srivastava, J. N. (1964). "Analysis of Irregular Factorial Fractions", Sankhya, A 26, 117-144.
2. Chakrabarti, M. C. (1962). Mathematics of Design and Analysis of Experiments. Asia Publishing House, New York.
3. Cochran, W. G. and Cox, G. M. (1968). Experimental Design. Wiley, New York.
4. Kempthorne, Oscar. (1952). Design and Analysis of Experiments. Wiley, New York.
5. Srivastava, J. N. (1969). Unpublished notes on Experimental Design, Colorado State University.

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THE RELATIONSHIP BETWEEN THE PRODUCT AND THE GEOMETRIC EFFECTS IN SYMMETRICAL FACTORIAL EXPERIMENTS

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ABSTRACT

The factorial experiments are widely used in experimental situations where several factors are to be examined simultaneously. In these experiments estimates of the factorial effects are of interest to the experimenters. There are two definitions available for the factorial effects and, in general, they do not agree when there are at least two factors with more than two levels. The effects defined by the geometric definition are often used to design confounding plans and fractional replicates but are difficult to interpret physically. The effects defined by the product definition, which can be interpreted easily, often cannot be directly estimated from designs constructed by confounding geometric effects or from fractional replicates constructed by aliasing geometric effects.

In this report, the relationship between the effects defined by the two definitions is given for the symmetrical factorial experiments. A computational technique is presented to estimate the product effects from the geometric effects. This relationship is illustrated with some examples. Finally, some directions for future research in this area are given.