Nonconvexity and compact containment of mean value sets for second order uniformly elliptic operators in divergence form

by

Niles Armstrong

B.S., Black Hills State University, 2010

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

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Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas

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Abstract

The mean value theorem for harmonic functions has historically been an important and powerful result. As such, a generalization of this theorem that was stated by Caffarelli in 1998 and later proved by Blank-Hao in 2015 is of immediate interest. However, in order to make more use of this new general mean value theorem, more information about the mean value sets that appear in the theorem is needed. We present here a few new results regarding properties of such mean value sets.

In the first chapter we study the mean values sets of the second order divergence form elliptic operator with principal coefficients defined as

$$a_k^{ij}(x) := \begin{cases} \alpha_k \delta^{ij}, & x_n > 0 \\ \beta_k \delta^{ij}, & x_n < 0. \end{cases}$$

In particular, we show that the mean value sets associated to such an operator need not be convex as α_k and β_k converge to 1. This result then leads to an example of nonconvex mean value sets for an operator where the $a^{ij}(x)$ are smooth.

In the second chapter we show that all points on the free boundary of an obstacle problem in some settings move immediately in response to varying data. Three applications of this result are given, and in particular, we show the following uniqueness result: For sufficiently smooth elliptic divergence form operators on domains in $\mathbb{R}^n$ and for the Laplace-Beltrami operator on a smooth manifold, the boundaries of distinct mean value sets which are centered at the same point do not intersect.

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Approved by:

Major Professor
Ivan Blank

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Dedication

To Bayleigh

Chapter 1

Introduction

1.1 Preliminary Results and Definitions

The mean value theorem has been one of the most important tools in the study of harmonic and/or weakly harmonic functions. (By Weyl's Lemma these sets of functions are identical.) We define such functions over a domain $\Omega \subset \mathbb{R}^n$. (For a list of commonly used notation we direct the reader to section 1.2.)

Definition 1.1.1. A function $u \in L^1_{loc}(\Omega)$ is said to be weakly superharmonic (subharmonic) in Ω if

$$\int_\Omega u\Delta\phi\, dx \leq 0\, (\geq 0)$$

for all nonnegative $\phi \in C_0^{1,1}(\Omega)$. We take this definition to be the meaning of the notation $\Delta u \leq 0$. Also, if a function is both weakly superharmonic and weakly subharmonic we say the function is weakly harmonic, denoted as $\Delta u = 0$.

The mean value theorem for these superharmonic functions can be stated as follows.

Theorem 1.1.2 (Mean Value Theorem)**.** *Let $u \in L^1_{loc}(\Omega)$ satisfy $\Delta u \leq 0$ in Ω. Then, for any $0 < r < s$ such that $\overline{B_s(x_0)} \subset \Omega$, we have*

$$u(x_0) \geq \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} u\, dx \geq \frac{1}{|B_s(x_0)|}\int_{B_s(x_0)} u\, dx. \tag{1.1}$$

If instead we have $\Delta u \geq 0$ the inequalities in Equation (1.1) are flipped.

Remark 1.1.3. The inequality in Equation (1.1) and the flipped version are referred to as the mean value inequalities. We also note here that these mean value inequalities hold if the averages over Euclidean balls are replaced by averages over their boundaries.

Theorem 1.1.2 can and has been used to prove many of the classical results for harmonic functions. For example the mean value theorem can be used to prove: the maximum principle, Harnack's Inequality, compactness results, regularity results, and interior derivative estimates. (See [14] and [15].)

Throughout this paper instead of working with the Laplacian we will work with operators that can be viewed as a generalization of the Laplacian. Specifically we will work with the operator L defined as,

$$L := \mathrm{div}(A(x)\nabla) = D_i(a^{ij}(x)D_j),$$

where $A(x) = (a^{ij}(x))$ is a matrix valued function with $a^{ij}(x)$ assumed to be bounded and measurable functions that satisfy:

$$a^{ij}(x) \equiv a^{ji}(x) \ \text{ and } \ 0 < \lambda|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \ \text{ for all } \ \xi \in \mathbb{R}^n, \ \xi \neq 0.$$

Such an operator appears in mathematics quite naturally. In fact, let

$$J(u) := \frac{1}{2}\int_\Omega a^{ij}(x)D_juD_iu\,dx$$

and

$$\mathcal{A} := \{u \in W^{1,2}(\Omega) \,|\, u - g \in W_0^{1,2}(\Omega)\}.$$

Then it is easy to show that a function w minimizes the functional J over the set $\mathcal{A}$ if and only if

$$\begin{cases} Lw = 0 & \text{for } x \in \Omega \\ w = g & \text{for } x \in \partial\Omega. \end{cases}$$

Furthermore, $Lw = 0$ is the associated Euler-Lagrange for the functional J. Outside of pure mathematics, operators of this form appear in the study of anisotropic and inhomogeneous materials. In this case the matrix $A(x)$ carries constitutive information about the material at the point x.

With this new operator we get a slightly different definition for weakly L-superharmonic functions.

Definition 1.1.4. A function $u \in W^{1,2}_{loc}(\Omega)$ is said to be weakly L-superharmonic (L-subharmonic) in Ω if

$$\int_\Omega a^{ij}(x) D_j u D_i \phi \, dx \geq 0 \, (\leq 0)$$

for all nonnegative $\phi \in W^{1,2}_0(\Omega)$. As before we take this definition to be the meaning of the notation $Lu \leq 0$. Also, if a function is both weakly L-superharmonic and weakly L-subharmonic we say the function is weakly L-harmonic, denoted as $Lu = 0$.

Similarly we define solutions for such an operator in the weak sense.

Definition 1.1.5. Given $f \in L^2(\Omega)$ and $u \in W^{1,2}(\Omega)$, we take "$Lu = f$" to mean

$$-\int_\Omega a^{ij}(x) D_j u D_i \phi \, dx = \int_\Omega f \phi \, dx \quad \text{for any } \phi \in W^{1,2}_0(\Omega).$$

Given the usefulness of the mean value theorem for harmonic functions, an analogous result that holds when the Laplacian is replaced by the above operator L is immediately of interest. One of the first such analogs can be found in [19], where a mean value theorem for such an operator was proved. Using Equation 8.3 in their paper you can derive the following theorem.

Theorem 1.1.6 (Littman-Stampacchia-Weinerger)**.** *Let u be an L-subharmonic function on Ω with $u = 0$ on $\partial\Omega$, define*

$$I(y; s) := \frac{s}{2} \int_{D_G(s)} u(x) a^{ij}(x) D_{x_i} G(x, y) D_{x_j} G(x, y) dx$$

where $G(x,y)$ is the Green's function for L on Ω and $D_G(s) := \left\{ x \in \mathbb{R}^n \,\middle|\, \frac{1}{s} \leq G(x,y) \leq \frac{3}{s} \right\}$. *Then we have*

$$u(y) \leq I(y;r) \leq I(y;s) \text{ for all } 0 < r < s.$$

While this formula does provide us with an integral definition of $u(y)$, like the mean value theorem for harmonic functions does, it also has a number of issues. First, it is not a simple average due to the presence of weights and indeed these weights involve derivatives of the Green's functions, which may be nontrivial to estimate. Second, the sets involved are level sets of the Green's function instead of some nice set containing the point y.

On the other hand, in his Fermi Lectures for the obstacle problem, Caffarelli gave a very elegant proof of Theorem 1.1.2, where he also stated that this proof would generalize to provide us with a mean value theorem for L-harmonic functions. It is worth recalling Caffarelli's proof here as it provides some intuition as to how the generalized mean value theorem for L-harmonic functions is proved (see Theorem 1.1.8) and in particular can illustrate where and why the obstacle problem appears in the statement of that theorem (see Equation (1.4).)

Caffarelli's Proof of Theorem 1.1.2. Without loss of generality we will assume $x_0 = 0 \in \Omega$. The main idea in this proof is to create a special test function, ϕ, to use in Definition 1.1.1 that naturally leads to the desired inequality. In order to construct such a function we make use of the fundamental solution for the Laplacian centered at 0. More specifically we use

$$\Gamma(x) = \Gamma(|x|) := \begin{cases} -\frac{1}{2\pi}\ln(|x|) & n = 2 \\ \frac{1}{n(n-2)\omega_n}|x|^{2-n} & n > 2 \end{cases}$$

where ω_n is the volume of the n-dimensional unit ball. We now define,

$$P_r(x) := -\alpha(r)|x|^2 + \beta(r)$$

where $\alpha(r)$ and $\beta(r)$ are chosen so that $P_r(x) \leq \Gamma(x)$ and touches $\Gamma(x)$ tangentially on $\partial B_r(0)$. (Note that r is not equal to $|x|$ here.) Using these two functions we now construct the crucial function that will be the main ingredient in our test function. Let

$$\psi_r(x) := \begin{cases} \Gamma(x) & \text{for } x \in B_r^c(0) \\ P_r(x) & \text{for } x \in B_r(0). \end{cases}$$

Note that $\psi_r(x)$ is a nonnegative function in $C^{1,1}(\mathbb{R}^n)$,

$$\Delta\psi_r(x) = \begin{cases} 0 & \text{for } x \in B_r^c(0) \\ -2n\alpha(r) & \text{for } x \in B_r(0), \end{cases}$$

and that $\psi_s(x) \leq \psi_r(x)$ in $\mathbb{R}^n$ for $0 < r \leq s$. Hence, the function $\phi_{r,s}(x) := \psi_r(x) - \psi_s(x)$ is a nonnegative function in $C_0^{1,1}(\mathbb{R}^n)$ and thus, is an appropriate test function. Then, from Definition 1.1.1 we have

$$0 \geq \int_\Omega u\Delta\phi_{r,s}\,dx = \int_{B_r(0)} u\Delta\psi_r\,dx - \int_{B_s(0)} u\Delta\psi_s\,dx$$

$$= -2n\alpha(r)\int_{B_r(0)} u\,dx + 2n\alpha(s)\int_{B_s(0)} u\,dx.$$

Therefore,

$$2n\alpha(s)\int_{B_s(0)} u\,dx \leq 2n\alpha(r)\int_{B_r(0)} u\,dx$$

which gives us,

$$2n\alpha(s)\frac{|B_s(0)|}{|B_s(0)|}\int_{B_s(0)} u\,dx \leq 2n\alpha(r)\frac{|B_r(0)|}{|B_r(0)|}\int_{B_r(0)} u\,dx. \tag{1.2}$$

Now using the fact that $u \equiv 1$ is both weakly superharmonic and subharmonic and following the previous computation we obtain,

$$0 = \int_\Omega 1\Delta\phi_{r,s}\,dx = \int_{B_r(0)} \Delta\psi_r\,dx - \int_{B_s(0)} \Delta\psi_s\,dx = -2n\alpha(r)|B_r(0)| + 2n\alpha(s)|B_s(0)|.$$

Hence, $2n\alpha(s)|B_s(0)| = 2n\alpha(r)|B_r(0)|$ and so Equation (1.2) can be rewritten as,

$$\frac{1}{|B_s(0)|}\int_{B_s(0)} u\,dx \leq \frac{1}{|B_r(0)|}\int_{B_r(0)} u\,dx.$$

Finally, by using this equation centered at an arbitrary $x_0 \in \Omega$ for $0 < r' \leq r$ and letting $r' \to 0$ we obtain

$$u(x_0) \geq \frac{1}{B_r(x_0)}\int_{B_r(x_0)} u\,dx$$

for almost every x_0, using the Lebesgue Differentiation Theorem. □

Remark 1.1.7. Note that the crucial function, $\psi_r(x)$ in the previous proof, is the solution to the obstacle problem given in (1.4) when L is the Laplacian with $2n\alpha(r) = r^{-n}$. Hence, in the general case the solution of the equations given in (1.4) ends up being the appropriate replacement for $\psi_r(x)$ when creating the test function.

The details of generalizing this proof were later provided by Blank and Hao in [7] where they proved the following theorem.

Theorem 1.1.8. *Let L be as above. Then, for any $x_0 \in \mathbb{R}^n$, there exists an increasing family $D_r(x_0)$ which satisfies the following:*

1. *$B_{cr}(x_0) \subset D_r(x_0) \subset B_{Cr}(x_0)$, with c, C depending only on n, λ, and Λ.*

2. *For any L-subharmonic function v and $0 < r < s$, we have*

$$v(x_0) \leq \frac{1}{|D_r(x_0)|}\int_{D_r(x_0)} v\,dx \leq \frac{1}{|D_s(x_0)|}\int_{D_s(x_0)} v\,dx. \tag{1.3}$$

Finally, the sets $D_r(x_0)$ are noncontact sets of the following obstacle problem:

$$\begin{cases} Lu = -\chi_{\{u<G(\cdot,x_0)\}}r^{-n} & \text{in} \quad B_M(x_0) \\ u \leq G(\cdot,x_0) & \text{in} \quad B_M(x_0) \\ u = G(\cdot,x_0) & \text{on} \quad \partial B_M(x_0) \end{cases} \tag{1.4}$$

where $B_M(x_0) \subset \mathbb{R}^n$, $M > 0$ is sufficiently large, and $G(x, y)$ is the Green's function for L on $\mathbb{R}^n$.

Remark 1.1.9. The boundary condition in (1.4) has a small technical issue in that $u = G(\cdot, x_0)$ on $\partial B_M(x_0)$ is usually interpreted as $u - G(\cdot, x_0) \in W_0^{1,2}(B_M(x_0))$. However, $G(\cdot, x_0)$ is certainly not in $W^{1,2}(B_M(x_0))$. On the other hand, $G(\cdot, x_0)$ is in $W^{1,2}$ away from the singularity at x_0, which is away from where the boundary condition appears. Thus, we can follow [7] and truncate $G(\cdot, x_0)$ near x_0 in a way that does not change what is occurring near the boundary. (See [7] for full details about this technicality.)

Remark 1.1.10. Theorem 1.1.8 appears to assume $n \geq 3$, since there does not exist a Green's function on $\mathbb{R}^2$. However, the theorem still holds in this case by instead choosing $G(\cdot, x_0)$ to be the fundamental solution. This change works due to the fact that the nondegeneracy statement for the obstacle problem, Theorem 3.9 in [7], guarantees that the solution to (1.4) must be identically equal to $G(\cdot, x_0)$ far enough away from x_0. This fact is also why there is no dependence on M when chosen to be sufficiently large. In fact in [5] instead of using the Green's function for $\mathbb{R}^n$ they use the Green's function for a set S that is "big enough," meaning $\overline{\{u < G(\cdot, x_0)\}} \subset S$. (See Theorem 1.1.14.) While this avoids the technicality for the case $n = 2$ it appears to introduce a dependence on S. However, a simple exercise shows that if one solves (1.4) on two such sets S and S', then the Green's functions and solutions differ by the exact same L-harmonic function. Thus, the noncontact set in the two cases are the same set.

Definition 1.1.11. We defined the mean value sets associated to an operator L to be the sets $D_r(x_0)$ in Theorem 1.1.8. When it is clear we may simply refer to these sets as the mean value sets.

Remark 1.1.12. While we will refer to $D_r(x_0)$ as the mean value set associated to L, we do not have a uniqueness statement for these sets in general. To be specific, it is possible that there exists a set that satisfies the mean value theorem but is not a noncontact set for the obstacle problem in (1.4). However, in the case for the Laplacian the Euclidean balls are unique in some sense. (See [17] for details.)

Theorem 1.1.8 is a clear analogue of the classical mean value theorem for balls for the Laplacian, but here the role of the balls is replaced with the sets, $D_r(x_0) = \{u_r(x) < G(x, x_0)\}$ where u_r is the solution to (1.4). Of course, this theorem immediately leads to questions about exactly what can be said about these $D_r(x_0)$'s. Initially all that was known about these sets can be seen in Theorem 1.1.8 along with Corollary 3.10 in [7] which states that $\partial D_r(x_0)$ is porous, so in particular has Hausdorff dimension strictly less than n. More recently the following three properties of these sets were shown in [3]:

1. If $y_0 \neq x_0$ then there exists an $r > 0$ such that $y_0 \in \partial D_r(x_0)$.
2. $D_r(x_0)$ has uniformly positive density at every point on $\partial D_r(x_0)$.
3. $D_r(x_0)$ has exactly one component.

Moreover, it is rather trivial to prove the converse of Theorem 1.1.8

Theorem 1.1.13 (Converse MVT for Divergence Form Elliptic PDE). *If $\{D_r(x_0)\}_{\{r>0, x_0 \in \mathbb{R}^n\}}$ is a complete collection of sets obtained for a specific operator, L, of the form given in Theorem 1.1.8, and v is a function such that*

$$v(x_0) \equiv \frac{1}{|D_r(x_0)|} \int_{D_r(x_0)} v(x)\, dx \tag{1.5}$$

whenever $D_r(x_0) \subset \Omega$, then $Lv = 0$ in Ω.

The upshot is that all of the information contained in the operator must be contained within the collection of mean value sets and vice versa.

Soon after these properties were shown, Theorem 1.1.8 along with the above properties were extended to the Laplace-Beltrami operator, Δ_g, on Riemannian manifolds in [5], which we state here for convenience.

Theorem 1.1.14 (Mean Value Theorem on Riemannian manifolds). *Given a point x_0 in a complete Riemannian manifold $\mathcal{M}$ (possibly with boundary), there exists a maximal number $r_0 > 0$ (which is finite if $\mathcal{M}$ is compact) and a family of open sets $\{D_r(x_0)\}$ for $0 < r < r_0$, such that*

(A) $0 < r < s < r_0$ *implies,* $D_r(x_0) \subset D_s(x_0)$*, and*

(B) $\lim_{r\downarrow 0} \max \mathrm{dist}_{x_0}(\partial D_r(x_0)) = 0$*, and*

(C) if u *is a subsolution of the Laplace-Beltrami equation, then*

$$u(x_0) = \lim_{r\downarrow 0} \frac{1}{|D_r(x_0)|} \int_{D_r(x_0)} u(x)\, dx \ ,$$

and $0 < r < s < r_0$ *implies*

$$\frac{1}{|D_r(x_0)|} \int_{D_r(x_0)} u(x)\, dx \leq \frac{1}{|D_s(x_0)|} \int_{D_s(x_0)} u(x)\, dx \ .$$

Finally, if $r < r_0$*, then the set* $D_r(x_0)$ *is uniquely determined as the noncontact set of any one of a family of obstacle problems. In fact, as long as the set* $S \subset \mathcal{M}$ *is "big enough," then* $D_R(x_0)$ *is the noncontact set of the following obstacle problem:*

$$\begin{aligned} \Delta_g u = -\chi_{\{u<G\}} R^{-n} \quad & \textit{in} \ \ S \\ u = G(\cdot, x_0) \quad & \textit{on} \ \ \partial S \ , \end{aligned} \tag{1.6}$$

where G *is the Green's function for the Laplace-Beltrami operator on* S*.*

Besides the few properties stated above, the topology and geometry of these mean value sets was, and largely is, still unknown. As in [3] one can ask if such sets are convex, star-shaped, or homeomorphic to a ball. On the other hand, one may instead ask for what types of operators do the mean value sets have a given property. For example, by imposing extra smoothness or structural assumptions on $a^{ij}(x)$ do we get properties of our mean value set that are not true in general? This type of questioning is the focus of the second chapter where we study the mean value sets associated to operators with $a^{ij}(x)$ of a particular form and prove the following result.

Theorem 1.1.15. *Let* $D_{R;k}(x_0, y_0) \subset \mathbb{R}^2$ *be as in Theorem 1.1.8 where the principal coeffi-*

cients of L_{α_k,β_k} are defined as

$$a_k^{ij}(x,y) := \begin{cases} \alpha_k \delta^{ij} & y > 0 \\ \beta_k \delta^{ij} & y < 0 \end{cases}$$

with α_k and β_k converging to 1 as $k \to \infty$. Then, for almost every choice of y_0, such that $\partial D_{R;k}(x_0, y_0)$ eventually crosses the interface $\{y = 0\}$, there exists a constant $K > 0$ such that $D_{R;k}(x_0, y_0)$ is nonconvex for all $k > K$.

This result then leads to the existence of nonconvex mean value sets for operators with smooth coefficients $a^{ij}(x)$. Thus, suggesting that there is not a simple assumption one can make on the coefficients of an operator to ensure convex mean value sets.

In the third and final chapter we will primarily be concerned with a question of uniqueness in regards to property 1 in the previous list. If $y \in \partial D_r(x_0)$, then is it possible for there to exist an $\tilde{r} \neq r$ such that $y \in \partial D_{\tilde{r}}(x_0)$? One can restate this question as whether $r < s$ merely implies $D_r(x_0) \subset D_s(x_0)$, or does it imply the stronger result: $\overline{D_r(x_0)} \subset D_s(x_0)$? The main results of this chapter show that in the case of the Laplace-Beltrami operator on Riemannian manifolds, and for the case in $\mathbb{R}^n$ where the operator L has coefficients in $C^{1,1}$, the answer is yes. (When the coefficients are not $C^{1,1}$ the question is still open.) More specifically we prove the following theorems.

Theorem 1.1.16 (Compact Containment of Mean Value Sets, Part I)**.** *Under the assumptions of Theorem 1.1.8 along with the assumption that the a^{ij} belong to $C^{1,1}$ the family $\{D_r(x_0)\}$ is always strictly increasing in the sense that $r < s$ implies $\overline{D_r(x_0)} \subset D_s(x_0)$.*

Theorem 1.1.17 (Compact Containment of Mean Value Sets, Part II)**.** *Under the assumptions of Theorem 1.1.14 the family $\{D_r(x_0)\}$ is always strictly increasing in the sense that*

(A′) $0 < r < s < r_0$ *implies* $\overline{D_r(x_0)} \subset D_s(x_0)$.

1.2 Notation

We collect here some basic notation that will be used throughout the paper:

χ_D	the characteristic function of the set D
$\overline{D}$	the closure of the set D
$int(D)$	the interior of the set D
∂D	the boundary of the set D
D^c	the complement of the set D
x	$(x_1, x_2, \ldots, x_n)$
x'	$(x_1, x_2, \ldots, x_{n-1})$
$B_r(x)$	the open ball with radius r centered at the point x
B_r	$B_r(0)$
$\Omega(w)$	$\{w > 0\}$
$\Lambda(w)$	$\{w = 0\}$
$FB(w)$	$\partial\Omega(w) \cap \partial\Lambda(w)$
$\mathrm{Sing}(w)$	$\{x \in \mathrm{FB}(w) \,\vert\, x \text{ is a singular point}\}$
$\mathrm{Reg}(w)$	$\{x \in \mathrm{FB}(w) \,\vert\, x \text{ is a regular point}\}$
Δ	Laplacian or symmetric difference operator
	In the second case $A \,\Delta\, B := (A \setminus B) \cup (B \setminus A)$
$\Gamma(x, y)$	the fundamental solution for the Laplacian on $\mathbb{R}^n$ (See [14] and [15])
$L_{\alpha,\beta}, L_k$	operators found in Equations (2.1) and (2.4)
dx	used in integrals to denote integration with respect to the Lebesgue Measure
$d\mathcal{H}^{n-1}$	$(n-1)$-dimensional Hausdorff measure
$C^k(\Omega)$	set of functions having all derivatives of order $\leq k$ continuous in Ω where $k \geq 0$ or ∞

$C^{k,\alpha}(\Omega)$	set of functions whose k-th order partial derivatives are uniformly Hölder continuous with exponent α in Ω where $k \geq 0$ or ∞ and $0 < \alpha \leq 0$
$C^{\alpha}(\Omega)$	the function space $C^{0,\alpha}(\Omega)$
$C^{\omega}(\Omega)$	the set of functions that are real analytic on Ω
$L^p(\Omega)$	for $1 \leq p < \infty$ the set of measurable functions on Ω that are p-integrable
$W^{k,p}(\Omega)$	the set of functions whose weak partial derivatives up to order k exist and belong to $L^p(\Omega)$ where $k \geq 0$ and $1 \leq p \leq \infty$
$F(\Omega)_0$	the set of functions in $F(\Omega)$ with compact support in Ω
$F(\Omega)_{loc}$	the set of functions in $F(\Omega')$ for all $\Omega' \subset\subset \Omega$

We refer the reader to [14] for more detailed descriptions of the function spaces listed above.

Chapter 2

Properties of Mean Value Sets: Angle Conditions, Blowup Solutions, and Nonconvexity

2.1 Introduction

In this chapter we aim to study operators with a very particular form. Namely operators whose principal coefficients are defined as

$$a^{ij}(x) := \begin{cases} \alpha\delta^{ij} & x_n > 0 \\ \beta\delta^{ij} & x_n < 0 \end{cases}$$

for some α, $\beta > 0$. We have two main reasons for studying such operators. First these operators appear in the study of composite materials where one has two different constituent materials. (See [18].) Secondly these $a^{ij}(x)$ have, what we view to be, the simplest type of discontinuity for such coefficients. Indeed, we thought these operators to be the simplest in which computing the mean value sets would be nontrivial, but still possible. In fact if α is very close to β such an operator would be very close to the Laplacian. Hence, it is reasonable to assume the mean value sets would be very close to Euclidean balls. This intuition does in fact turn out to be true. In particular, in Theorem 2.3.3 we show that

$$|D_R(x_0)\,\Delta\,B_R(x_0)| \to 0 \text{ as } \alpha,\ \beta \to 1.$$

On the other hand, this fact makes our main result of this chapter, Theorem 1.1.15, somewhat surprising, where we show that as α tends towards β almost every R such that $\partial D_R(x_0)$ crosses the interface, $D_R(x_0)$ will be nonconvex. One may think that this behavior is due to the discontinuous nature of the principal coefficients, but this idea is incorrect. In fact we will prove the existence of nonconvex mean value sets corresponding to operators whose principal coefficients are smooth on all of $\mathbb{R}^n$. Moreover, this suggests that convexity may only be possible for very simple operators, such as the Laplacian after a linear change of variables.

The remainder of the chapter will be organized in the following way. In section 2.2 we will state a few preliminary results that will be needed throughout the chapter and define some specific notation we will be using. Section 2.3 is devoted to proving a measure stability result, *(4)* in Theorem 2.3.3, which states that our mean value sets are close to Euclidean balls in measure when the operator L is close to the Laplacian. For section 2.4 we prove a Weiss' type monotonicity formula, Theorem 2.4.1, that leads to homogeneous of degree two blowup solutions at free boundary points on the interface. We note here that other such quasi-monotonicity formulas have been derived for rather general $a^{ij}(x)'s$, which lead to many of the same results to that of a full monotonicity formula. In [12] Theorem 3.7 provides such a formula where $a^{ij}(x)$ are Lipschitz continuous and in [13] Theorem 1.1 provides such a formula where $a^{ij}(x)$ are in a fractional Sobolev space. While these formulas are shown with few structural assumptions on the $a^{ij}(x)'s$ the mild regularity that is required excludes the $a^{ij}(x)'s$ we will look at here. In section 2.5 we prove an angle condition for blowup solutions at free boundary points on the interface, Lemma 2.5.1. This condition then leads to the proof of Theorem 1.1.15. Finally, in section 2.6 we show an analog of Theorem 1.1.15 where the principal coefficients of the operator are no longer discontinuous. This is done by convolving the discontinuous $a^{ij}(x)'s$ with a mollifier and showing that the convergence results found in section 2.3 still hold in this case.

2.2 Preliminaries and Terminology

The following notation is specific to this chapter:

$B_r^+(x)$, $B_r^-(x)$	$\{y \in B_r(x) \mid y_n > 0\}$, $\{y \in B_r(x) \mid y_n < 0\}$
$\partial^+ B_r(x)$, $\partial^- B_r(x)$	$\{y \in \partial B_r(x) \mid y_n > 0\}$, $\{y \in \partial B_r(x) \mid y_n < 0\}$
$\mathbb{R}^n_+$, $\mathbb{R}^n_-$	$\{x \in \mathbb{R}^n \mid x_n \geq 0\}$, $\{x \in \mathbb{R}^n \mid x_n \leq 0\}$

In the entirety of this chapter we will work with a divergence form elliptic operator

$$L_{\alpha,\beta} := D_j a^{ij}(x) D_i \tag{2.1}$$

with the $a^{ij}(x)$ having the following structure

$$a^{ij}(x) := \begin{cases} \alpha\delta^{ij} & x_n > 0 \\ \beta\delta^{ij} & x_n < 0 \end{cases} \quad \text{for} \qquad 0 < \lambda \leq \alpha, \beta \leq \Lambda \tag{2.2}$$

where δ^{ij} is the Kronecker delta function. Throughout the chapter we will refer to the set $\{x_n = 0\}$ as the interface. For such operators the Green's function on $\mathbb{R}^n$ have been computed explicitly in [1]; we will restate them here for the convenience of the reader. Let $\widetilde{\Gamma}(x,y) := \Gamma((x', -x_n), y)$. Then the Green's function for $L_{\alpha,\beta}$ on $\mathbb{R}^n$ is as follows

$$G(x,y) := \begin{cases} G_h(x,y) & \text{when } y_n > 0 \\ G_l(x,y) & \text{when } y_n < 0 \end{cases}$$

where

$$G_h(x,y) := \frac{1}{\alpha}\Gamma(x,y) + \begin{cases} \frac{\alpha-\beta}{\alpha(\alpha+\beta)}\widetilde{\Gamma}(x,y) & \text{when } x_n \leq 0 \\ \frac{\alpha-\beta}{\alpha(\alpha+\beta)}\Gamma(x,y) & \text{when } x_n \geq 0 \end{cases}$$

and

$$G_l(x,y) := \frac{1}{\beta}\Gamma(x,y) + \begin{cases} \frac{\beta-\alpha}{\beta(\alpha+\beta)}\Gamma(x,y) & \text{when } x_n \leq 0 \\ \frac{\beta-\alpha}{\beta(\alpha+\beta)}\widetilde{\Gamma}(x,y) & \text{when } x_n \geq 0. \end{cases}$$

When $n = 2$ we will abuse the vocabulary slightly and still refer to such a $G(x,y)$ as the Green's function, where in this case the limit at ∞ is $-\infty$ instead of 0. We will also need a well known transmission condition for $L_{\alpha,\beta}$-harmonic functions in this setting, which we state here.

Lemma 2.2.1 (Transmission Conditions)**.** *With $a^{ij}(x)$ and $L_{\alpha,\beta}$ definded as above, if $L_{\alpha,\beta}w = 0$ in $\mathbb{R}^n$, then $w \in C^{\omega}(\mathbb{R}^n_+) \cap C^{\omega}(\mathbb{R}^n_-)$ and for any $x' \in \mathbb{R}^{n-1}$ we have*

$$\alpha \lim_{s\downarrow 0} \frac{\partial}{\partial x_n} w(x',s) = \beta \lim_{t\uparrow 0} \frac{\partial}{\partial x_n} w(x',t). \tag{2.3}$$

The regularity of the solution w in this lemma was shown by Li and Vogelius in [18], whereas the condition in (2.3) can be derived by doing an integration by parts with standard arguments from calculus of variations. Note that Lemma 2.2.1 gives us solutions that are real analytic up to and including the interface, but not across the interface.

2.3 Measure Stability

This section will show the convergence of solutions to obstacle problems of the form (1.4), with the operator $L_{\alpha,\beta}$, as α and β converge to 1. This fact is then used to show a type of measure convergence for the mean value sets associated to the operator, $L_{\alpha,\beta}$, to those of the Laplacian, i.e. Euclidean balls. In this sense the mean value sets do begin to act like Euclidean balls.

Now we fix some $x_0 \in \mathbb{R}^n$ and consider $w_k \in W^{1,2}(B_{16M}(x_0))$ to be the solutions to

$$\begin{cases} L_k u := D_i(a_k^{ij}(x) D_j u) = -R^{-n} \chi_{\{u < G_k(\cdot, x_0)\}} & \text{in } B_{16M}(x_0) \\ u \leq G_k(\cdot, x_0) & \text{in } B_{16M}(x_0) \\ u = G_k(\cdot, x_0) & \text{on } \partial B_{16M}(x_0) \end{cases} \tag{2.4}$$

where $L_k := L_{\alpha_k, \beta_k}$ with ellipticity constants $0 < \lambda < \Lambda$ for all k. We will choose $M > 0$ large enough so that the mean value sets are independent of M, see [7] for details. In fact, we will choose M large enough to ensure that $\{w_k < G_k(\cdot, x_0)\} \subset B_M(x_0)$ for all k.

We now wish to show two compactness results for the solutions w_k. These results are very similar in proof and statement to that found in [8] Lemma 3.1 and Lemma 3.2. A key difference being we can not assume $0 \in FB(w_k)$ for all k. To accommodate this change we first prove a uniform interior L^∞ bound on the solutions. Another discrepancy here is that we will not be able to utilize the height function $G_k(\cdot, x_0) - w_k$ due to the singularity from the Green's function.

Lemma 2.3.1 (Uniform Bound)**.** *There exists a constant C such that $w_k \leq C$ in $B_{4M}(x_0)$ for all k.*

Proof. By the proof of Lemma 4.4 from [7] we have

$$w_k \leq b_k + K\Big(b_k + \frac{R^{-n} M^2}{4n}\Big) \text{ in } B_{4M}(x_0) \text{ where } b_k := \max_{\partial B_{16M}(x_0)} G_k(\cdot, x_0).$$

Using $G_k(\cdot, x_0) \leq \frac{1}{\lambda}\Gamma(\cdot, x_0)$ gives

$$b_k \leq b := \max_{\partial B_{16M}(x_0)} \frac{1}{\lambda}\Gamma(\cdot, x_0).$$

Hence, $w_k \leq b + K\Big(b + \frac{R^{-n}M^2}{4n}\Big)$ for all k. □

Lemma 2.3.2 (Compactness)**.** *There exists a function $w \in W^{1,2}(B_{2M}(x_0))$ and a subsequence of $\{w_k\}$ such that along this subsequence we have uniform convergence of w_k to w,* *and weak convergence in $W^{1,2}$.*

Proof. Lemma 2.3.1 along with De Giorgi-Nash-Moser theorem implies there exists an $\alpha \in (0,1)$ and a constant C such that $\{w_k\} \subset C^{\alpha}(\overline{B_{2M}(x_0)})$ with $||w_k||_{C^{\alpha}(\overline{B_{2M}(x_0)})} \leq C$. Then by Arzela-Ascoli there exists a subsequence of $\{w_k\}$ that converges uniformly, to $w \in C^0(\overline{B_{2M}(x_0)})$. Also, standard elliptic regularity plus the uniform L^2 bound on $G_k(\cdot, x_0)$ implies a uniform $W^{1,2}$ bound on w_k. Then by standard functional analysis there exists a subsequence such that $w_k \rightharpoonup w$ in $W^{1,2}$. □

Theorem 2.3.3. *If we now assume that*

$$\alpha_k \textit{ and } \beta_k \to 1 \qquad \textit{in } B_M(x_0)$$

we then have

(1) $G_k(\cdot, x_0) \to \Gamma(\cdot, x_0)$ *uniformly in compact subsets of* $\mathbb{R}^n \setminus \{x_0\}$

(2) $\chi_{\{w_k < G_k(\cdot,x_0)\}} \to \chi_{\{w<\Gamma(\cdot,x_0)\}}$ *almost everywhere in* $B_M(x_0)$

(3) w *satisfies*

$$\begin{cases} \Delta w = -R^{-n}\chi_{\{w<\Gamma(\cdot,x_0)\}} & \textit{in } B_M(x_0) \\ w \leq \Gamma(\cdot, x_0) & \textit{in } B_M(x_0) \\ w = \Gamma(\cdot, x_0) & \textit{on } \partial B_M(x_0) \end{cases}$$

(4) $|\{w_k < G_k(\cdot, x_0)\} \,\Delta\, \{w < \Gamma(\cdot, x_0\}| \to 0$

where w is the limiting function from the previous lemma.

Proof of (1). Let

$$\Psi_k(\cdot, x_0) := \begin{cases} \frac{1}{\alpha_k}\Gamma(\cdot, x_0) & \text{when } x_n > 0 \\ \frac{1}{\beta_k}\Gamma(\cdot, x_0) & \text{when } x_n < 0 \end{cases}$$

Then we have

$$|G_k(\cdot, x_0) - \Gamma(\cdot, x_0)| \leq |G_k(\cdot, x_0) - \Psi_k(\cdot, x_0)| + |\Psi_k(\cdot, x_0) - \Gamma(\cdot, x_0)|$$

which converges to 0 uniformly on any compact subset of $\mathbb{R}^n \setminus \{x_0\}$.

Proof of (2). Note that from *(1)* and Lemma 2.3.2 we have

$$(G_k(\cdot, x_0) - w_k) \to (\Gamma(\cdot, x_0) - w) \text{ in } B_M(x_0) \setminus \{x_0\}.$$

Hence, in the interior of $\{\Gamma(\cdot, x_0) - w > 0\}$ we have

$$\chi_{\{w_k < G_k(\cdot, x_0)\}} \to \chi_{\{w < \Gamma(\cdot, x_0)\}} \text{ on } B_M(x_0) \setminus \{x_0\}.$$

The same is true in the interior of $\{\Gamma(\cdot, x_0) - w = 0\}$ by using the nondegeneracy of our solutions, which can be found in [7]. Finally, we note that $\partial\{\Gamma(\cdot, x_0) - w = 0\}$ has Lebesgue measure zero from [7].

Proof of (3). For any $\phi \in W_0^{1,2}(B_M(x_0))$,

$$\int_{B_M(x_0)} a_k^{ij} D_i w_k D_j \phi = \int_{B_M(x_0)} (a_k^{ij} - \delta^{ij})(D_i w_k - D_i w) D_j \phi$$

$$+ \int_{B_M(x_0)} a_k^{ij} D_i w D_j \phi + \int_{B_M(x_0)} \delta^{ij}(D_i w_k - D_i w) D_j \phi =: I + II + III.$$

Since $a_k^{ij} \to \delta^{ij}$ and $D_i w_k \rightharpoonup D_i w$, we have

$$I = \int_{B_M(x_0)} (a_k^{ij} - \delta^{ij})(D_i w_k - D_i w) D_j \phi \to 0$$

$$II = \int_{B_M(x_0)} a_k^{ij} D_i w D_j \phi \to \int_{B_M(x_0)} \delta^{ij} D_i w D_j \phi$$

and

$$III = \int_{B_M(x_0)} \delta^{ij}(D_i w_k - D_i w) D_j \phi \to 0.$$

Hence,

$$\int_{B_M(x_0)} a_k^{ij} D_i w_k D_j \phi \to \int_{B_M(x_0)} \delta^{ij} D_i w D_j \phi.$$

Together with *(2)*, we proved

$$\Delta w = \chi_{\{w<\Gamma(\cdot,x_0)\}} \quad \text{in } B_M(x_0).$$

From *(1)* and Lemma 2.3.2 it is clear that $w \leq \Gamma(\cdot, x_0)$ in $B_M(x_0)$. Finally, recall we have $w_k = G_k(\cdot, x_0)$ on $\partial B_M(x_0)$ then again from *(1)* and Lemma 2.3.2 we have $w = \Gamma(\cdot, x_0)$ on $\partial B_M(x_0)$.

Proof of (4). This immediately follows from *(2)*. □

Remark 2.3.4. Due to the result in [17] we know $B_R(x_0) = \{w < \Gamma(\cdot, x_0)\}$. Hence, *(4)* from the previous theorem can be rewritten to say

$$|\{w_k < G_k(\cdot, x_0)\} \,\Delta\, B_R(x_0)| \to 0.$$

In this sense we have shown our mean value sets converge to Euclidean balls.

2.4 Monotonicity Formula

In this section we aim to adapt Weiss' monotonicity formula, from [22], to solutions of the following problem

$$\begin{cases} L_{\alpha,\beta} u = \frac{1}{2}\chi_{\{u>0\}} & \text{in } B_1 \\ u \geq 0 & \text{in } B_1 \\ 0 \in FB(u). \end{cases} \tag{2.5}$$

Theorem 2.4.1. *Let w be a solution to (2.5), $f(x) := \alpha\chi_{\{x_n>0\}} + \beta\chi_{\{x_n<0\}}$, and $x_0 \in \{x_n = 0\}$. Then the function*

$$\Phi_{x_0}(r) := r^{-n-2} \int_{B_r(x_0)} (f|\nabla w|^2 + w)dx - 2r^{-n-3} \int_{\partial B_r(x_0)} fw^2 d\mathcal{H}^{n-1}$$

defined in $(0, dist(x_0, \partial B_1)$, *satisfies the monotonicity formula*

$$\Phi_{x_0}(\sigma) - \Phi_{x_0}(\rho) = \int_\rho^\sigma r^{-n-2} \int_{\partial B_r(x_0)} 2\Big(\nabla w \cdot \nu - 2\frac{w}{r}\Big)^2 f \, d\mathcal{H}^{n-1} dr \geq 0$$

for all $0 < \rho < \sigma < 1$.

Remark 2.4.2. Note that this theorem requires $x_0 \in \{x_n = 0\}$. While this may seem like a big restriction, realize that off of the interface Weiss' original monotonicity formula would hold. Also, the key difference in our monotonicity formula from Weiss' original is the necessity of the function f in the definition of Φ. This change is needed to reflect that our solution w minimizes

$$\int_D (f|\nabla u|^2 + u)dx \quad \text{instead of} \quad \int_D (|\nabla u|^2 + u)dx$$

among all functions $u \in W^{1,2}(D)$ with $u \geq 0$.

Proof. We will omit the measures dx and $d\mathcal{H}^{n-1}$ throughout the proof. It is to be understood that the measure is $d\mathcal{H}^{n-1}$ when integrating over the boundary of a set and is otherwise dx, which is the Lebesgue n-dimensional measure.

Let $w_r(x) := \frac{w(x_0+rx)}{r^2}$ and observe that

$$\Phi(r) = \int_{B_1} (f|\nabla w_r|^2 + w_r) - 2\int_{\partial B_1} fw_r^2.$$

Since we have $w \in C^\omega(\mathbb{R}^n_+) \cap C^\omega(\mathbb{R}^n_-)$ from Lemma 2.2.1, it is convenient to work with Φ in the form

$$\Phi(r) = \int_{B_1^+} (\alpha|\nabla w_r|^2 + w_r) + \int_{B_1^-} (\beta|\nabla w_r|^2 + w_r)$$

$$-2\int_{\partial^+ B_1} \alpha w_r^2 - 2\int_{\partial^- B_1} \beta w_r^2.$$

Now compute

$$\Phi'(r) = \int_{B_1^+} 2\alpha \nabla w_r \cdot \nabla\left(\frac{x\cdot\nabla w(x_0+xr)}{r^2} - \frac{2}{r}\frac{w(x_0+rx)}{r^2}\right)$$

$$+\int_{B_1^-} 2\beta \nabla w_r \cdot \nabla\left(\frac{x\cdot\nabla w(x_0+xr)}{r^2} - \frac{2}{r}\frac{w(x_0+rx)}{r^2}\right)$$

$$+\int_{B_1^+}\left(\frac{x\cdot\nabla w(x_0+xr)}{r^2} - \frac{2}{r}\frac{w(x_0+rx)}{r^2}\right)$$

$$+\int_{B_1^-}\left(\frac{x\cdot\nabla w(x_0+xr)}{r^2} - \frac{2}{r}\frac{w(x_0+rx)}{r^2}\right)$$

$$-2\int_{\partial^+ B_1} 2\alpha w_r\left(\frac{x\cdot\nabla w(x_0+xr)}{r^2} - \frac{2}{r}\frac{w(x_0+rx)}{r^2}\right)$$

$$-2\int_{\partial^- B_1} 2\beta w_r\left(\frac{x\cdot\nabla w(x_0+xr)}{r^2} - \frac{2}{r}\frac{w(x_0+rx)}{r^2}\right).$$

An integration by parts on the first two terms yields:

$$= -\frac{2}{r}\int_{B_1^+} \alpha \Delta w_r (x\cdot\nabla w_r - 2w_r) - \frac{2}{r}\int_{B_1^-} \beta \Delta w_r (x\cdot\nabla w_r - 2w_r)$$

$$+\frac{2\alpha}{r}\int_{\partial^+ B_1} x\cdot\nabla w_r(x\cdot\nabla w_r - 2w_r) + \frac{2\beta}{r}\int_{\partial^- B_1} x\cdot\nabla w_r(x\cdot\nabla w_r - 2w_r)$$

$$+\frac{2\alpha}{r}\int_{B_1\cap\{x_n=0\}} (-e_n)\cdot\nabla w_r(x\cdot\nabla w_r - 2w_r)$$

$$+\frac{2\beta}{r}\int_{B_1\cap\{x_n=0\}} (e_n)\cdot\nabla w_r(x\cdot\nabla w_r - 2w_r)$$

$$+\frac{1}{r}\int_{B_1^+}(x\cdot\nabla w_r-2w_r)+\frac{1}{r}\int_{B_1^-}(x\cdot\nabla w_r-2w_r)$$

$$-\frac{2\alpha}{r}\int_{\partial^+B_1}2w_r(x\cdot\nabla w_r-2w_r)-\frac{2\beta}{r}\int_{\partial^-B_1}2w_r(x\cdot\nabla w_r-2w_r)$$

$$=\frac{2\alpha}{r}\int_{\partial^+B_1}(x\cdot\nabla w_r-2w_r)^2+\frac{2\beta}{r}\int_{\partial^-B_1}(x\cdot\nabla w_r-2w_r)^2$$

$$+\frac{2\alpha}{r}\int_{B_1\cap\{x_n=0\}}(-e_n)\cdot\nabla w_r(x\cdot\nabla w_r-2w_r)$$

$$+\frac{2\beta}{r}\int_{B_1\cap\{x_n=0\}}(e_n)\cdot\nabla w_r(x\cdot\nabla w_r-2w_r).$$

The last two terms cancel due to the transmission condition Lemma 2.2.1. Hence, we obtain

$$=\frac{2}{r}\int_{\partial B_1}f(x\cdot\nabla w_r-2w_r)^2d\mathcal{H}^{n-1}.$$

After rescaling we arrive at the desired result

$$\Phi'_{x_0}(r)=2r^{-n-2}\int_{\partial B_r(x_0)}f\left(\nu\cdot\nabla w-2\frac{w}{r}\right)^2d\mathcal{H}^{n-1}\geq 0$$

for $r\in(0,1)$. □

Similar to Weiss' original paper [22] we now use our monotonicity formula to prove blowup solutions of (2.5) at free boundary points are homogeneous of degree two.

Proposition 2.4.3. *Let w be a solution to (2.5) then we have*

(1) For all $x_0\in B_1\cap\{w=0\}\cap\{x_n=0\}$ the function $\Phi_{x_0}(r)$ has a real right limit $\Phi_{x_0}(0^+)$

(2) Let $x_0\in B_1\cap\{w=0\}\cap\{x_n=0\}$ and $0<\rho_m\longrightarrow 0$ be a sequence such that the blow-up sequence $w_m(x):=\frac{w(x_0+\rho_m x)}{\rho_m^2}$ converges a.e. in $\mathbb{R}^n$ to a blow-up limit w_0. Then w_0 is a homogeneous function of degree 2.

(3) $\Phi_{x_0}(r) \geq 0$ *for every* $x_0 \in B_1 \cap \{w = 0\} \cap \{x_n = 0\}$ *and every* $0 \leq r < dist(x_0, \partial B_1)$. *Equality holds if and only if* $w = 0$ *in* $B_r(x_0)$.

(4) The function $x_0 \longmapsto \Phi_{x_0}(0^+)$ *restricted to the set* $\{x_n = 0\}$ *is upper semi-continuous.*

Proof of (1). Since $w \in C^\omega(\mathbb{R}^n_+) \cap C^\omega(\mathbb{R}^n_-)$ we get that $\Phi_{x_0}(r)$ is bounded and non-decreasing from Theorem 2.4.1, assuming that r is sufficiently small.

Proof of (2). First we may assume ρ_m's are sufficiently small, depending on x_0 so that Theorem 2.4.1 holds. Then for all $0 < R < S < \frac{1}{\rho_m}$ we have

$$\int_R^S r^{-n-2} \int_{\partial B_r} 2\left(\nabla w_m \cdot \nu - 2\frac{w_m}{r}\right)^2 f \, d\mathcal{H}^{n-1} dr$$

$$= \int_{R\rho_m}^{S\rho_m} \delta^{-n-2} \int_{\partial B_\delta} 2\left(\nabla w \cdot \nu - 2\frac{w}{\delta}\right)^2 f \, d\mathcal{H}^{n-1} d\delta = \Phi_0(S\rho_m) - \Phi_0(R\rho_m)$$

where $\delta = r\rho_m$. From *(1)* we have $\Phi_{x_0}(S\rho_m) - \Phi_{x_0}(R\rho_m) \longrightarrow 0$. This together with the boundedness of $\{w_m\} \subset C^{1,1}_{loc}(\mathbb{R}^n_+) \cap C^{1,1}_{loc}(\mathbb{R}^n_-)$ implies w_0 is homogeneous of degree 2.

Proof of (3). First suppose $\Phi_{x_0}(0^+) < 0$ for some $x_0 \in \{x_n = 0\} \cap \{w = 0\}$. Then $\exists$ a sequence of positive real values $\rho_m \longrightarrow 0$ and $w_m := \frac{w(x_0 + \rho_m x)}{\rho_m^2} \longrightarrow w_0$ in $C^{1,\alpha}_{loc}(\mathbb{R}^n_+) \cap C^{1,\alpha}_{loc}(\mathbb{R}^n_-)$ such that

$$0 > \lim_{m \to \infty} \Phi_{x_0}(\rho_m)$$

$$= \lim_{m \to \infty} \left[\int_{B_1^+} (\alpha |\nabla w_m|^2 + w_m) + \int_{B_1^-} (\beta |\nabla w_m|^2 + w_m) \right.$$

$$\left. -2 \int_{\partial^+ B_1} \alpha w_m^2 d\mathcal{H}^{n-1} - 2 \int_{\partial^- B_1} \beta w_m^2 d\mathcal{H}^{n-1} \right]$$

$$= \int_{B_1^+} (\alpha|\nabla w_0|^2 + \alpha w_0) + \int_{B_1^-} (\beta|\nabla w_0|^2 + \beta w_0)$$

$$-2 \int_{\partial^+ B_1} \alpha w_0^2 d\mathcal{H}^{n-1} - 2 \int_{\partial^- B_1} \beta w_0^2 d\mathcal{H}^{n-1}.$$

From *(2)* we know that w_0 is homogeneous of degree 2. An integration by parts yields

$$0 > \int_{B_1^+} w_0(1 - \alpha\Delta w_0) + \int_{B_1^-} w_0(1 - \beta\Delta w_0)$$

$$+ \int_{\partial^+ B_1} \alpha w_0(\nabla w_0 \cdot x - 2w_0) d\mathcal{H}^{n-1} + \int_{\partial^- B_1} \beta w_0(\nabla w_0 \cdot x - 2w_0) d\mathcal{H}^{n-1}$$

$$= \frac{1}{2} \int_{B_1} w_0 \geq 0,$$

a contradiction. Hence, $0 \leq \Phi_{x_0}(0^+) \leq \Phi_{x_0}(r)$. Finally if $w = 0$ in $B_r(x_0)$ then obviously $\Phi_{x_0}(r) = 0$. On the other hand if $\Phi_{x_0}(r) = 0$ then $\Phi'_{x_0}(r) = 0$, which implies w is homogeneous of degree 2. Then following as before

$$0 = \Phi_{x_0}(r) = \frac{1}{2} r^{-n-2} \int_{B_r(x_0)} w \geq 0.$$

Hence, $w = 0$ in $B_r(x_0)$.

Proof of (4). Let $\epsilon > 0$ and $x \in \{x_n = 0\}$. If $\Phi_{x_0}(0^+) > -\infty$ then there exists ρ such that $\Phi_{x_0}(\rho) + \frac{\epsilon}{2} \leq \Phi_{x_0}(0^+) + \epsilon$. Otherwise, if $\Phi_{x_0}(0^+) = -\infty$ then $\exists\ m < \infty$ such that $\Phi_{x_0}(\rho) + \frac{\epsilon}{2} \leq -m$. Now, considering ρ to be fixed, if $|x - x_0|$ is sufficiently small we have $\Phi_x(\rho) \leq \Phi_{x_0}(\rho) + \frac{\epsilon}{2}$. Also, from Theorem 2.4.1 we have $\Phi_x(0^+) \leq \Phi_x(\rho)$. Hence we have,

$$\Phi_x(0^+) \leq \begin{cases} \Phi_{x_0}(0^+) + \epsilon, & \Phi_{x_0}(0^+) > -\infty \\ -m, & \Phi_{x_0}(0^+) = -\infty \end{cases}$$

by first choosing ρ then $|x - x_0|$ small enough. $\square$

2.5 Angle Conditions and Nonconvexity

In this section, working in $\mathbb{R}^2$, we will explicitly construct the blowup solutions at free boundary points on the interface to (2.5). This classification will lead to an interesting condition on how the free boundary can cross the interface. In particular it will provide an angle condition that depends only on the given α and β from the operator $L_{\alpha,\beta}$. In other words, the free boundary must cross the interface in a very particular way. Indeed, we expected the angle of incidence and angle of reflection to be related, but that one angle would be free. However, we find that both angles are completely determined by the given α and β. Using this angle condition and the measure convergence in Theorem 2.3.3 we will be able to construct examples of mean value sets that must be nonconvex.

Note that if w is a solution to (2.4) then locally, away from the singularity, $G_k(x,y) - w$ is a solution to (2.5). Hence, proposition 2.4.3 implies that blowup solutions to $G_k(x,y) - w$ at free boundary points are homogeneous of degree two. If we then restrict to working in $\mathbb{R}^2$ these blowup solutions have the form $r^2 g(\theta)$. Using this fact the next lemma computes such blowup solutions explicitly, in doing so we derive an angle condition for mean value sets as they cross the interface, $\{x_n = 0\}$.

Lemma 2.5.1 (Angle Condition Across the Interface)**.** *Let $v_0 = r^2 g(\theta)$ be a blow-up solution to (2.5) at the origin and assume $\exists\ \theta_1,\ \theta_2 \in (0,\pi)$ such that $g(\theta) = 0\ \forall\ \theta \in [0, \pi - \theta_1] \cup [\pi + \theta_2, 2\pi]$.*

$$g(\theta) := \begin{cases} \frac{1}{8\alpha}[1 - \cos(2\theta_1)\cos(2\theta) + \sin(2\theta_1)\sin(2\theta)] & \pi - \theta_1 \le \theta \le \pi \\ \frac{1}{8\beta}[1 - \cos(2\theta_2)\cos(2\theta) - \sin(2\theta_2)\sin(2\theta)] & \pi \le \theta \le \pi + \theta_2 \\ 0 & \textit{otherwise} \end{cases} \tag{2.6}$$

$$\theta_2 = \begin{cases} \theta_1 + \frac{\pi}{2} & 0 < \theta_1 < \frac{\pi}{2} \\ \theta_1 - \frac{\pi}{2} & \frac{\pi}{2} < \theta_1 < \pi \end{cases} \qquad \textit{and} \qquad \cos(2\theta_1) = \frac{\beta - \alpha}{\beta + \alpha}$$

Proof. Such a g must satisfy the following conditions:

1. $g''(\theta) + 4g(\theta) = \frac{1}{2\alpha}$ for $\theta \in (\pi - \theta_1, \pi)$
2. $g''(\theta) + 4g(\theta) = \frac{1}{2\beta}$ for $\theta \in (\pi, \pi + \theta_2)$
3. $g(\pi - \theta_1) = g(\pi + \theta_2) = 0$
4. $g'(\pi - \theta_1) = g'(\pi + \theta_2) = 0$
5. $\alpha g'(\pi^-) = \beta g'(\pi^+)$
6. $g(\pi^-) = g(\pi^+)$

Conditions 1 and 2 give us

$$g(\theta) := \begin{cases} \frac{1}{8\alpha} + C_1 \cos(2\theta) + C_2 \sin(2\theta) & \pi - \theta_1 < \theta < \pi \\ \frac{1}{8\beta} + D_1 \cos(2\theta) + D_2 \sin(2\theta) & \pi < \theta < \pi + \theta_2 \end{cases}$$

Then conditions 3 and 4 give

$$C_1 = \frac{-\cos(2\theta_1)}{8\alpha} \quad C_2 = \frac{\sin(2\theta_1)}{8\alpha} \quad D_1 = \frac{-\cos(2\theta_2)}{8\beta} \quad D_2 = \frac{-\sin(2\theta_2)}{8\beta}$$

Now condition 5 implies

$$-\sin(2\theta_2) = \sin(2\theta_1)$$

which then gives

$$\theta_2 = \begin{cases} \theta_1 + \frac{\pi}{2} & 0 < \theta_1 < \frac{\pi}{2} \\ \theta_1 - \frac{\pi}{2} & \frac{\pi}{2} < \theta_1 < \pi \end{cases}.$$

Finally using condition 6 we get

$$\frac{1-\cos(2\theta_2)}{\beta} = \frac{1-\cos(2\theta_1)}{\alpha}$$

Hence we have

$$\cos(2\theta_1) = \frac{\beta-\alpha}{\beta+\alpha}$$

. $\square$

Remark 2.5.2. If in the previous lemma we instead assumed that $\Lambda(v_0)$ was on the "left" side we get very much the same result. As in

$$g(\theta) := \begin{cases} \frac{1}{8\alpha}[1-\cos(2\theta_1)\cos(2\theta)+\sin(2\theta_1)\sin(2\theta)] & 0 \le \theta \le \pi-\theta_1 \\ \frac{1}{8\beta}[1-\cos(2\theta_2)\cos(2\theta)-\sin(2\theta_2)\sin(2\theta)] & \pi+\theta_2 \le \theta \le 2\pi \\ 0 & \text{otherwise} \end{cases} \tag{2.7}$$

with the same definitions for θ_1 and θ_2. Hence, assuming $\Lambda(v_0)$ crosses the interface, we only have nine distinct cases. Four possible blowup solutions are as in equations (2.6) and (2.7). (See figures 2.1a, 2.1b, 2.1d, and 2.1e.) Then by allowing the free boundary to be tangent to the interface we find the standard half plane and whole plane solutions. (See figures 2.1g, 2.1h, and 2.1i.) Finally we obtain two more whole plane solutions by summing solutions as in equation (2.6) and (2.7). (See figures 2.1c and 2.1f.)

At this point it is very easy to believe that convexity of our mean value sets will not always be possible. The angle condition in Lemma 2.5.1 depends only on α and β and as α converges to β we have (θ_1, θ_2) converging to either $(\frac{\pi}{4}, \frac{3\pi}{4})$ or $(\frac{3\pi}{4}, \frac{\pi}{4})$. Thus, the boundaries of our mean values sets must satisfy such an angle condition as they cross the interface or become tangent to the interface, but we also have from Theorem 2.3.3 that these mean value sets converge in measure to Euclidean balls as α and β converge to one. Then by picking an appropriate center for the Euclidean balls they will cross the interface with any angle we desire. This angle condition with the measure convergence statement in Theorem 2.3.3

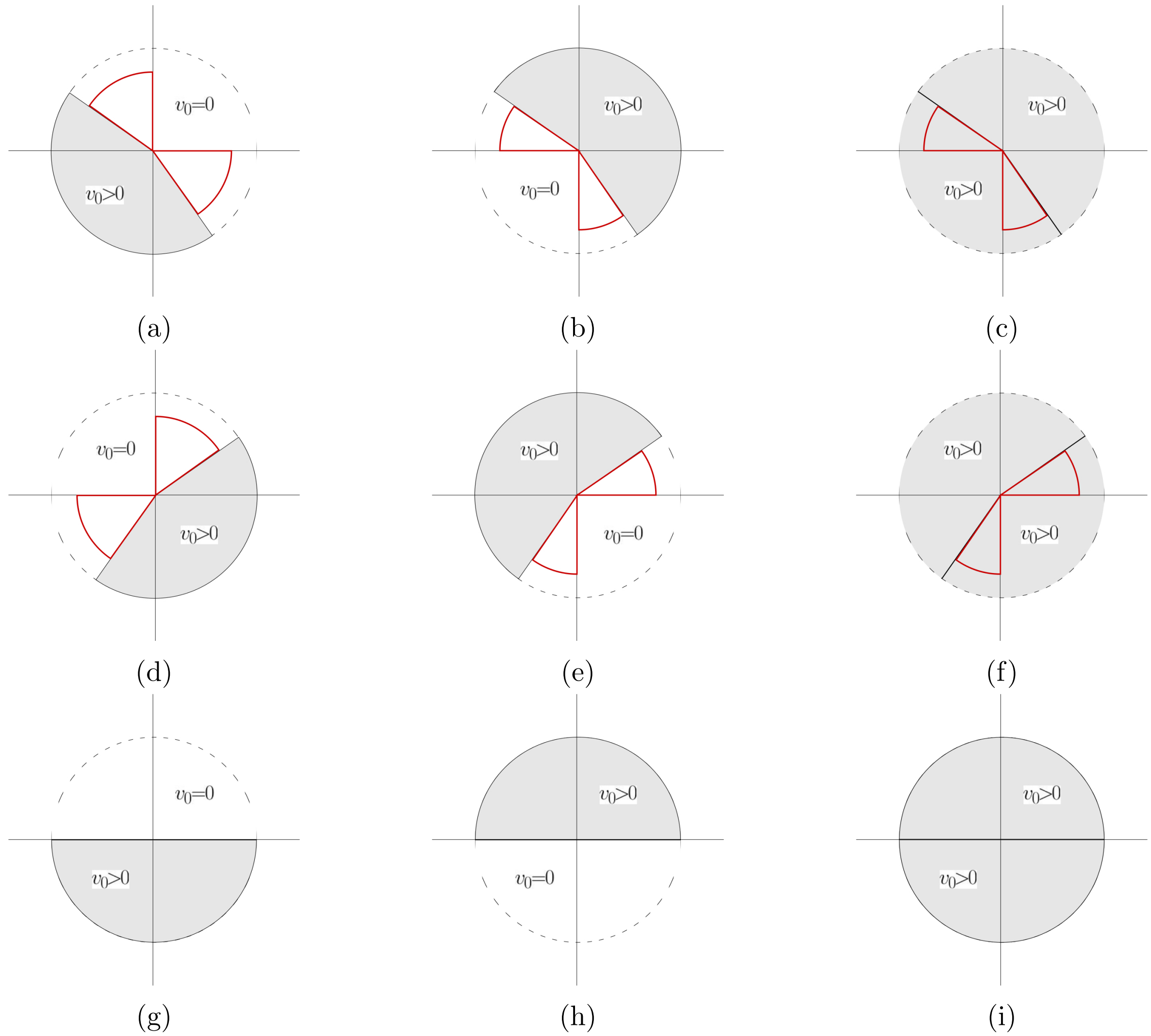

Figure 2.1: The nine possible blowup limits at a free boundary point for fixed α and β, assuming the positivity set crosses the interface

will force ∂D_R to curve rapidly back towards ∂B_R forcing D_R to be nonconvex. Before we can prove this we first need a small lemma to ensure the convergence of the contact and noncontact set as we rescale our solutions.

Lemma 2.5.3. *Let v be a solution to equation 2.5 and $0 < \rho_m \to 0$ be a sequence such that the blowup sequence $v_m(x) := \frac{v(\rho_m x)}{\rho_m^2}$ converges a.e. in $\mathbb{R}^n$ to a blowup limit v_0. Then for every $\epsilon > 0$ there exists an M such that for any point $p \in \Omega(v_0) \cap B_1$ that is at least ϵ away from $FB(v_0)$ we have $p \in \Omega(v_m) \cap B_1$ for all $m \geq M$. Furthermore, for any point $q \in \Lambda(v_0) \cap B_1$ that is at least ϵ away from $FB(v_0)$ we have $q \in \Lambda(v_m) \cap B_1$ for all $m \geq M$.*

Proof. Note that there exists a subsequence of v_m, which we again call v_m, that converges to v_0 uniformly on compact sets. Also, we have v_m converging to v_0 in $C^{1,\alpha}_{loc}(\mathbb{R}^n_+)$ and in $C^{1,\alpha}_{loc}(\mathbb{R}^n_-)$.

Denote the ϵ-neighborhood of a set S as $N_\epsilon(S)$. We first prove the statement for points in $\Omega(v_0)$. Outside of $N_\epsilon(FB(v_0) \cap B_1)$ in $\Omega(v_0)$ we have $v_0 > M$ for some constant $M > 0$. Since v_m converges uniformly to v_0 in B_1 there exists a K such that $|v_m - v_0| < \delta < M$ in B_1. Hence, $FB(v_m) \cap B_1$ can not be outside of the ϵ-neighborhood of $FB(v_0) \cap B_1$ in $\Omega(v_0)$.

Now we prove the statement for points in $\Lambda(v_0)$. If we suppose the statement is false, then there exists an $\epsilon > 0$ so that for any m there is a point $p_m \in FB(v_m) \cap \Lambda(v_0) \cap B_1$ such that it is at least ϵ away from $FB(v_0)$. However, nondegeneracy implies

$$C\left(\frac{\epsilon}{2}\right)^2 \leq \sup_{x \in B_{\epsilon/2}(p_m)} v_m.$$

Hence, for any m there exists a point $q_m \in \Lambda(v_0)$ such that $v_m(q_m) \geq C(\frac{\epsilon}{2})^2$ which contradicts uniform convergence. □

Proof of Theorem 1.1.15. Without loss of generality we will assume $x_0 = 0$. Also, for simplicity we will assume $y_0 = 0$. This assumption is only to ensure $B_R(0, y_0)$ does not satisfy the angle condition in Lemma 2.5.1 and to ensure the mean value sets cross the interface. It should be clear from the proof how to adapt this to any $y_0 \neq 0$ as long as ∂B_R does not satisfy the angle condition as it crosses the interface.

We wish to show that there exists a point $q_k := (\tilde{q}_k, 0) \in \partial D_{R;k}(0,0)$ such that $|-R-\tilde{q}_k| \to 0$ as $k \to \infty$. Note that we know such a point $q_k \in \partial D_{R;k}(0,0)$ exists due to Theorem 2.3.3 and the fact that $D_{R;k}(0,0)$ has exactly one component. (See [AB] Lemma 2.5.) If we assume that $|-R-\tilde{q}_k| \nrightarrow 0$ then since all of the $D_{R;k}(0,0)$ are contained in one fixed large ball we can extract a convergent subsequence of the q_k. Calling the new subsequence $\{q_k\}$ still, we must have that the density of $D_{R;k}(0,0)$ at q_k converges to either 1 or 0 as $k \to \infty$ according to whether the q_k converge to a point inside or outside of B_R respectively. In either case, it is clear that the resulting set cannot remain convex while approaching B_R in measure. See

Figure 2.2 where a dotted line is drawn in the zoomed picture which must start and end within $D_{R;k}(0,0)$, but which (after a slight adjustment if needed) will necessarily contain points of the complement in order to not violate the measure stability theorem. Thus, we may assume that $|-R-\tilde{q}_k| \to 0$.

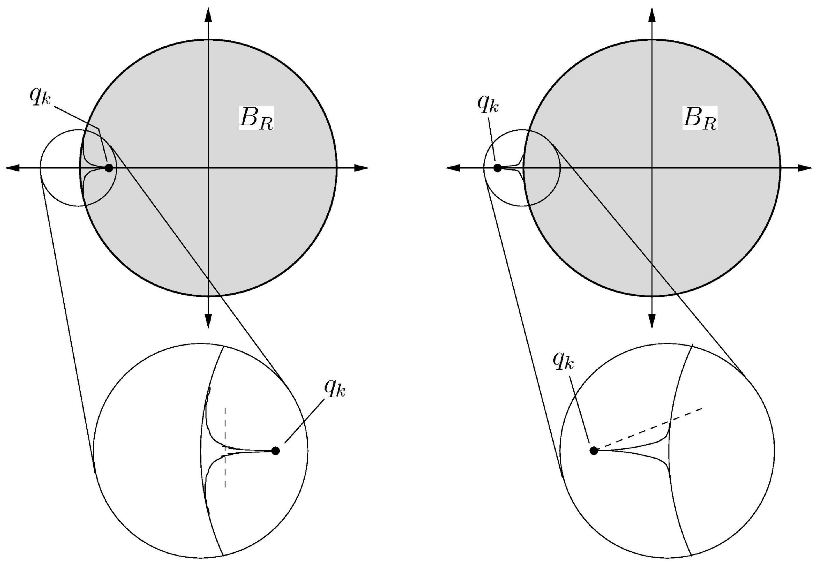

Figure 2.2: Cusps when q_k does not converge to $-R$

By the argument above, for any arbitrarily small $\gamma > 0$, there exists a $K > 0$ such that

$$k \geq K \Rightarrow |-R-\tilde{q}_k| < \gamma.$$

Zooming in at q_k we can be sure that our solution converges to one of the blowup limits in Figure 2.1. Note that blowup solutions as in Figure 2.1c, 2.1f, and 2.1i immediately lead to the nonconvexity of $D_{R;k}(0,0)$. The remainder of these cases can be dealt with similarly, so we will assume that there is a subsequence converging to a blowup limit of the variety in Figure 2.1d and leave the other cases as an exercise for the interested reader. Now within this setting, we observe that k can be chosen large enough so that the following quantities are as small as we like:

1. $|-R-\tilde{q}_k|$,

2. $|\theta_1 - \frac{\pi}{4}|$, and

3. $|D_{R;k}(0,0)\,\Delta\,B_R|$.

After fixing k sufficiently large to make the three quantities above adequately small, we can then use Lemma 2.5.3 so that the distance from the free boundary within $B_\rho(q_k)$ to the line with slope one through q_k is shrinking faster than ρ. In short, the zoomed picture within Figure 2.3 is as accurate as we like. Now however, convexity together with measure stability would give us a contradiction as we can use convexity to show that in the picture of B_R in the figure, as much of the region above the dotted line as we want cannot be part of the noncontact set. On the other hand, the measure of that region is $\left(\frac{\pi-2}{4}\right)R^2 > 0$, and that leads to a contradiction with measure stability and our assumption that k is sufficiently large.

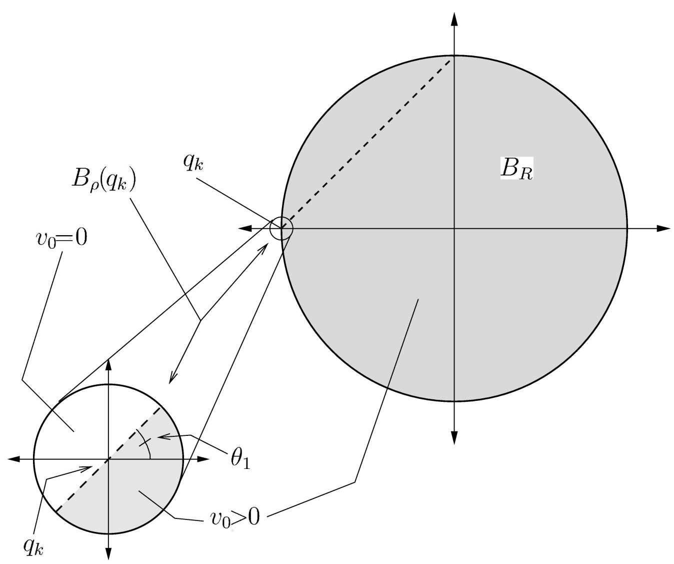

Figure 2.3: Set above dotted line contradicts measure stability

□

2.6 Nonconvexity Continued

In the previous section the discontinuous structure of our $a_k^{ij}(x)$'s seemed to play a major role in the nonconvexity of the associated mean value sets. Here we will show that in taking smooth approximations of such $a_k^{ij}(x)$'s we will still have nonconvex mean value sets, even though the principal coefficients of the operators are no longer discontinuous. In light of these examples we now expect that operators whose mean value sets are all convex are the exception and not the rule.

Let $b^{ij}_{s;k}(x,y)$ be a smooth approximation of $a^{ij}_k(x,y)$ by convolving with a mollifier. Denote the Green's function associated to $b^{ij}_{s;k}(x,y)$ by $G_{s;k}(x,y)$. Then from Theorem 5.4 in [19] we know that $G_{s;k}(x,y)$ converges uniformly to $G_k(x,y)$ on compact sets away from the singularity as $s \to \infty$. Hence, the methods and results in Section 3 can be directly carried over to this setting. In particular we get an analog of Theorem 2.3.3.

Before proving our final nonconvexity statement we will need a lemma to ensure the regions causing $D_{R;k}(x_0,y_0)$ to be nonconvex are not of measure zero. Obviously if this were the case then the measure convergence of the mean value sets would not guarantee nonconvexity. For simplicity we will drop the dependence of R for the notation and simply write $D_k(x_0,y_0)$ and $D_{s;k}(x_0,y_0)$ for the mean value sets associated the operators whose principal coefficients are $a^{ij}_k(x,y)$ and $b^{ij}_{s;k}(x,y)$ respectively.

Lemma 2.6.1. *Let $a^{ij}_k(x,y)$ and $D_k(x_0,y_0)$ be as above. Then for almost every choice of y_0, there exists a constant $K>0$ such that for all $k>K$ we have an open set $E \subset D^c_k(x_0,y_0)$ with $|E|>0$ and for every point $p \in E$ there exists a line segment containing p starting and ending in $int(D_k(x_0,y_0))$.*

Proof. As before we will assume $y_0 = 0$ for simplicity and without loss of generality assume $x_0 = 0$. Again it should be clear that the proof will still hold if $y \neq 0$. Let $q_k := (\tilde{q}_k, 0) \in D_k(0,0)$ such that $\tilde{q}_k \leq p_k$ for all $(p_k,0) \in \partial D_k(0,0)$. Now take a convergent subsequence of the q_k calling the new sequence q_k again. We then have three possible cases, q_k converges either to a point in $int(B^c_R)$, B_R, or ∂B_R.

Case I: q_k converges to a point in $int(B^c_R)$.
However, q_k can not converge to a point in $int(B^c_R)$ due to the lower bound on the density of $D_k(0,0)$ at the point q_k, as shown by Lemma 1.3 in [3].

Case II: q_k converges to a point in B_R.

Then there exists $\delta > 0$ such that $|-R-q_k| > \delta$ for all k sufficiently large. Then we have

$$\frac{|D_k^c(0,0) \cap B_\delta(q_k)|}{|B_\delta(q_k)|} > 0 \quad \text{for all k sufficiently large,}$$

since otherwise we would contradict the definition of q_k. Note that there is no claim that the above positivity be uniform. Then by choosing k large enough so that $|(D_k(0,0)\,\Delta\, B_R)\cap\{x \leq -R+\delta\}| \leq \frac{1}{100}|\{x \leq -R+\delta\} \cap B_R|$ the choice of $E := D_k^c(0,0) \cap B_\delta(q_k)$ would satisfy the desired properties.

Case III: q_k converges to a point in ∂B_R.

As in the proof of Theorem 1.1.15 we zoom in at q_k and note that our solution converges to one of the blowup limits in Figure 2.1. Note that blowups as in Figures 2.1c and 2.1f are not possible here as this would lead to a contradiction to how q_k was defined. If the blowup is as in Figure 2.1i we can take E to be defined similarly to that in Case II where δ need only be small enough. The remaining blowups are all dealt with similarly so we will assume the blowup limit to be that of the form in Figure 2.1d. Again as in the proof of Theorem 1.1.15 we can choose k sufficiently large so that the zoomed in picture with in Figure 2.3 is as accurate as we like. Hence the choice of $E := \{(x,y) \in D_k^c(0,0) \cap B_{\rho/2}(q_k)\,|\, x > \tilde{q}_k\}$ would satisfy the desired properties. □

Theorem 2.6.2. *Let $D_k(x_0, y_0)$ and $D_{s;k}(x_0, y_0)$ be as above. If $D_k(x_0, y_0)$ is nonconvex then there exists an $S > 0$ so that $D_{s;k}(x_0, y_0)$ is nonconvex for all $s > S$.*

Proof. Again assuming $y_0 = 0$ for simplicity and without loss of generality assume $x_0 = 0$. Let E be as in the previous lemma. Then from Lemma 2.6.1 we have for any $p \in E$ there exists points q_1, $q_2 \in int(D_k(0,0))$ such that the line containing q_1 and q_2 also contains p. Then let $\epsilon_0 > 0$ be the largest value such that $B_{\epsilon_0}(p) \subset E$. Similarly let ϵ_1, $\epsilon_2 > 0$ be the largest values such that $B_{\epsilon_1}(q_1)$, $B_{\epsilon_2}(q_2) \subset int(D_k(0,0))$ and define $\epsilon := min\{\epsilon_0, \epsilon_1, \epsilon_2\}$. Then every line connecting points from $B_\epsilon(q_1)$ to $B_\epsilon(q_2)$ goes through $B_\epsilon(p)$ and every point in $B_\epsilon(p)$ is contained in a line that crosses $B_\epsilon(q_1)$ and $B_\epsilon(q_2)$.

Using the analog of Theorem 2.3.3 for the sets $D_k(0,0)$ and $D_{s;k}(0,0)$ we can pick s

large enough to ensure that almost every point in $B_{\epsilon/2}(q_1)$ and $B_{\epsilon/2}(q_2)$ belongs to $D_{s;k}(0,0)$ and to ensure that almost every point in $B_{\epsilon/2}(p)$ belongs to $D^c_{s;k}(0,0)$. Hence, $D_{s;k}(0,0)$ is nonconvex. $\square$

Chapter 3

Nondegenerate Motion of Singular Points in Obstacle Problems with Varying Data

3.1 Introduction

In this chapter we aim to prove the compact containment of mean value sets for the operator L in $\mathbb{R}^n$ and for the Laplace-Beltrami operator on Riemannian manifolds. (See Theorem 1.1.16 and 1.1.17.) Interestingly there is *almost* a proof based on the Hopf Lemma that has already been pointed out in [5]. Along these lines, if we assume that $y_0 \in \partial D_r(x_0) \cap \partial D_s(x_0)$, and we assume that $\partial D_r(x_0)$ is regular at y_0, then by invoking Caffarelli's famous free boundary regularity theorem for the obstacle problem (see [9] and/or [10]), then we are guaranteed that there will exist a ball $B_\rho(z_0)$ satisfying:

1. $B_\rho(z_0) \subset D_r(x_0)$, and
2. $\partial B_\rho(z_0) \cap \partial D_r(x_0) \cap \partial D_s(x_0) = y_0$.

Now if we let u_r and u_s be the solutions to the problem in Equation 1.4, then it follows that $v(x) := u_r(x) - u_s(x)$ will satisfy the following:

1. $v \geq 0$ in $B_\rho(z_0)$,

2. $Lv = s^{-n} - r^{-n} < 0$ in $B_\rho(z_0)$, and

3. $v(y_0) = 0$.

In this situation we can apply the Hopf Lemma to guarantee that $\nabla v(y_0) \neq 0$. On the other hand

$$\nabla v(y_0) = \nabla u_r(y_0) - \nabla u_s(y_0) = \nabla G(x_0, y_0) - \nabla G(x_0, y_0) = 0$$

which gives us a contradiction.

Of course the bad news in the "proof" above is that we assumed that y_0 was a regular point of the free boundary. Now in the most typical pictures of free boundaries with singular points, it should be even easier to touch the boundary of $D_r(x_0)$ with a ball, in spite of these examples, Schaeffer gave other examples of contact sets in the obstacle problem with cantor-like structures (see [20]) and the recent work of Figalli and Serra that yields some nice regularity results for the singular set seems to require that the operator be the Laplacian (see [11]).

One can also ask if it is possible to repair the proof above so that it continues to hold even at the singular points, and indeed, that was our first attempt at solving this problem. In joint work by Alvarado, Brigham, Maz'ya, Mitrea, and Ziadé, a sharp form of Hopf's Lemma is shown which does not require touching with a ball; one only needs to touch with a "pseudoball" (see [2, Theorem 4.4]). Furthermore in Caffarelli's original 1977 Acta paper, he shows "almost convexity" conditions which guarantee the existence of a half ball contained in the noncontact set (see [9, Corollary 1 and Corollary 2]). Unfortunately, the union of the half balls described by Caffarelli does not contain a pseudoball of the type described in [2], so it appears that this route will not lead to a proof.

Now of course, one thing that really *is* shown by the argument above is that if there is a situation with $r \neq s$ and where $\partial D_r(x_0) \cap \partial D_s(x_0)$ is nonempty, then it can only happen at singular points. In this respect, and in viewing the flow of $\partial D_t(x_0)$ as we vary t, this situation should be compared to the results of King, Lacey, and Vázquez for the Hele-Shaw problem (see [16]). They show that if there is a corner built into the initial data, and if the angle formed satisfies certain inequalities, then that corner will remain motionless for a while at the beginning of the evolution. It is also worth observing that Serfaty and Serra have shown a normal velocity formula for the free boundary in the obstacle problem at regular points when varying the data, but their result (a *normal* velocity formula) obviously cannot be applied at a point of the free boundary that does not have a normal vector (see [21]).

Another attack which could lead to a full proof via the Hopf Lemma would be to expand the work of Figalli and Serra ([11]) to include more than the Laplacian, so that the better regularity allowed us to touch the singular set with an interior ball. Although even the Figalli/Serra results allow for some lower-dimensional "anomalous" points that would need to be handled in order to get a touching ball, so generalization *and* improvement would be needed for that route, and it is quite likely impossible. In any case, an examination of the methods employed in their work reveals arguments that seem to be particular to the Laplacian, and so perturbation arguments seem like a better attack as opposed to trying to do their work from scratch in a more general setting. Even though we have not successfully expanded that work, that perturbation approach is related to our third application of the main idea in this work. The difficulty there is related to the instability of singular free boundaries. Certainly it is a trivial matter to make a singular free boundary that disappears under an appropriate perturbation. For example, $u(x) = x^2$ satisfies $\Delta u = \chi_{\{u>0\}} f$, with $f(x) \equiv 2$, but if you raise the boundary data and/or reduce f anywhere and solve the new problem, then the free boundary will disappear. That observation led us to the question of whether or not we could find a way to make specific perturbations which always led to singular points. In the third application, although we do not get results which are precise enough to allow us to generalize [11], we do successfully find a way to approximate singular free boundaries with other singular free boundaries of solutions to obstacle problems with operators with constant coefficients and which have similar boundary data, and this approximation may be of independent interest.

The main idea that has worked is the following: We use the derivative of the solution to the obstacle problem as a barrier, and using that function we can come to a contradiction where a related function that we can show is nonnegative must also become negative due to standard regularity and nondegeneracy estimates that we have for the obstacle problem if the two distinct free boundaries share a common boundary point. We present three applications of this idea in this chapter.

3.2 Preliminaries and Terminology

We will use the following basic notation and assumptions throughout the chapter:

$\mathcal{M}$	a smooth connected Riemannian n-manifold
g	the metric for our ambient manifold $\mathcal{M}$
$\eta_\delta(S)$	the δ-neighborhood of the set S
Δ_g	the Laplace-Beltrami operator on $\mathcal{M}$.

Frequently be more convenient to work with the height function, and so we define $w_r(x) := G(x_0, x) - u_r(x)$ which obeys either:

$$Lw_r = \chi_{\{w>0\}} r^{-n} - \delta_{x_0} \tag{3.1}$$

or

$$\Delta w_r = \chi_{\{w>0\}} r^{-n} - \delta_{x_0} \tag{3.2}$$

according to which case we are currently studying. (We use δ_{x_0} to denote the usual delta function at x_0.)

Finally, there is a simple lemma in [6] that we will use repeatedly, so we record it here for the reader's convenience:

Lemma 3.2.1 (Theorem 2.7c of [6]). *Suppose that for $i = 1, 2$, the functions $w_i \geq 0$ solve the obstacle problem:*

$$\begin{aligned} \Delta u &= \chi_{\{w>0\}} g \quad &&\textit{in} \;\; B_1 \\ u &= \psi_i &&\textit{on} \;\; \partial B_1 \end{aligned} \tag{3.3}$$

where $0 < \mu_1 \le g \le \mu_2$, *and* $\psi_1 \le \psi_2 \le \psi_1 + \epsilon$, *then*

$$w_1 \le w_2 \le w_1 + \epsilon$$

and in particular

$$||w_1 - w_2||_{L^\infty(B_1)} \le \epsilon\ .$$

3.3 Proof of Compact Containment of Mean Value Sets, Part I

We assume that $L := \partial_i(a^{ij}(x)\partial_j)$, that $||a^{ij}||_{C^{1,1}} < \infty$, and as above we let u_r denote the solution to

$$\begin{aligned} L(u) &= -\chi_{\{u<G\}} r^{-n} && \text{in } \ B_M(x_0) \\ u &= G(\cdot, x_0) && \text{on } \ \partial B_M(x_0) \end{aligned} \tag{3.4}$$

and let $w_r(x) := G(x, x_0) - u_r(x)$.

Lemma 3.3.1. *$L(D_e w_r)$ is a function such that,*

$$|L(D_e w_r)| \le C(\rho) < \infty \ \textit{in} \ \Omega(w_r) \setminus B_\rho(x_0)$$

for any direction e and $\rho > 0$ so that $\overline{B_\rho(x_0)} \subset \Omega(w_r)$.

Proof. Define $E := \Omega(w_r) \setminus B_\rho(x_0)$ and let $\phi \in C_0^\infty(E)$.

$$\begin{aligned} -\int_E a^{ij} D_j(D_e w_r) D_i \phi \, dx &= \int_E D_e(a^{ij} D_i \phi) D_j w_r \, dx \\ &= \int_E D_i \phi (D_e a^{ij}) D_j w_r \, dx + \int_E (D_e D_i \phi) a^{ij} D_j w_r \, dx. \end{aligned}$$

On the other hand, since $D_e \phi \in C_0^\infty(E)$ is a permissible test function, the second integral turns out to be zero:

$$\begin{aligned}\int_E (D_e D_i \phi) a^{ij} D_j w_r \, dx &= \int_E D_i (D_e \phi) a^{ij} D_j w_r \, dx \\ &= -\int_E D_e \phi \, r^{-n} \, dx \\ &= \int_E \phi \, D_e r^{-n} \, dx \\ &= 0 \, .\end{aligned}$$

Hence, we have $L(D_e w_r) = -D_i(D_e \, a^{ij} D_j w_r) \in L^\infty(E)$, with a uniform bound since we have excised a ball around the singularity. $\square$

Proof of Theorem 1.1.16. From [7] we know that $\Omega(w_r) \subset \Omega(w_s)$. Hence, we need only show that there does not exist a point $q \in FB(w_r) \cap FB(w_s)$. In order to show this we will consider the function $v := w_s - w_r$ which satisfies:

1. $v \geq 0$ in B_M
2. $v = w_s \geq 0$ on $\partial \Omega(w_r)$
3. $Lv = s^{-n} - r^{-n} < 0$ in $\Omega(w_r)$
4. $v > 0$ in $\Omega(w_r)$

Assume that there exists a point $q \in FB(w_r) \cap FB(w_s)$. Consider the function $D_e w_r$ for some unit vector e to be chosen later. Lemma 3.3.1 ensures that in the set $\Omega(w_r) \setminus B_\rho$, for small ρ, there exists $\epsilon_1 > 0$ such that

$$L(v - \epsilon_1 D_e w_r) < 0 \text{ in } \Omega(w_r) \setminus B_\rho.$$

Also, note that, for ρ small enough, $v > 0$ on ∂B_ρ by [7, Lemma 6.2], nondegeneracy, and optimal regularity. Hence, there exists $\epsilon_2 > 0$ such that

$$v - \epsilon_2 D_e w_r \geq 0 \text{ on } \partial B_\rho \, .$$

Now, in fact, for any ϵ_2 whatsoever, by standard regularity results for the obstacle problem (see [9], [10], or [6]) we automatically have

$$v - \epsilon_2 D_e w_r = v = w_s \geq 0 \text{ on } \partial\Omega(w_r)\,.$$

Then, by the Weak Maximum Principle

$$v - \epsilon D_e w_r \geq 0 \text{ in } \Omega(w_r) \setminus B_\rho$$

for $\epsilon = min\{\epsilon_1, \epsilon_2\}$. However, by optimal regularity and nondegeneracy we know that

$$\sup_{B_\delta(q)} v \leq C_1 \delta^2 \text{ and } \sup_{B_\delta(q)} |\nabla w_r| \geq C_2 \delta$$

for $\delta > 0$ such that $\overline{B_\delta(q)} \subset B_M$. Therefore, for δ small enough, there exists a point $y \in B_\delta(q)$ and a unit vector e so that

$$v(y) - \epsilon D_e w_r(y) \leq C_1 \delta^2 - \epsilon C_2 \delta < 0$$

which gives us a contradiction. □

3.4 Proof of Compact Containment of Mean Value Sets, Part II

Now we turn to the proof of Theorem 1.1.17. Before starting, however, it is worth noting how the previous proof fails in this case. Perhaps the greatest problem is the inability to define a direction e globally. Accordingly, the set $D_r(x_0) \setminus B_\rho(x_0)$ which could be huge (and therefore nowhere close to being contained within a chart of the manifold $\mathcal{M}$) cannot be used for our argument. We must work locally and so instead of working on $D_r(x_0) \setminus B_\rho(x_0)$, we work on $D_r(x_0) \cap B_\delta(q)$ where $q \in \partial D_r(x_0) \cap \partial D_s(x_0)$. On this new set, however, although we have no problem defining directions as long as δ is sufficiently small, we have a new problem of potentially having our test function being negative on parts of the boundary.

The setting we have for this section assumes that we have a point $q \in \partial D_r(x_0) \cap \partial D_s(x_0)$, and a $\delta > 0$ that is small enough so that

1. $B_\delta(q)$ is completely contained within a single chart $(\mathcal{U}, \varphi)$ of $\mathcal{M}$,
2. we let y be points within the original manifold, and x denote points in $\varphi(\mathcal{U})$ so that $x = \varphi(y)$, and
3. we assume that the φ is giving us normal coordinates around q and then the operator Δ_g can be expressed:

$$\begin{aligned} \Delta_g u(y) &= \frac{1}{\sqrt{|det\ g(x)|}} \cdot \frac{\partial}{\partial x_i} \left(g^{ij}(x) \sqrt{|det\ g(x)|}\ \frac{\partial}{\partial x_j} u(x) \right) \\ &=: g^{ij}(x) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} u(x) + b^j(x) \frac{\partial}{\partial x_j} u(x) \\ &=: Lu(x)\ , \end{aligned} \tag{3.5}$$

with $g^{ij}(p) \to \delta^{ij}$, and $b^i(p) \to 0$ as $p \to q$. (We are using δ^{ij} to denote the Kronecker delta.)

So the picture that we have on the manifold is given in Figure 3.1. In terms of a source for

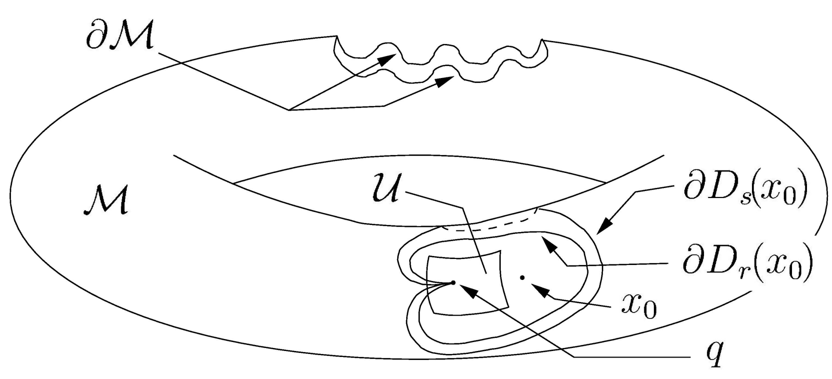

Figure 3.1: The Picture on the Manifold $\mathcal{M}$.

the differential geometry facts and conventions that we needed and used, we found the text [4] by Aubin to be useful for everything above.

Remark 3.4.1. An astute reader might complain that our mean value sets in Figure 3.1 lack the reflection symmetry that would be enjoyed on a piece of a perfect torus, so our picture should be considered to be a "cartoon" in this respect.

Having seen the situation on the manifold above, we observe that in this section we can do all of our work within the chart $\mathcal{U}$ and so we can view our entire problem in the local picture found in $\mathcal{V} := \varphi(\mathcal{U}) \subset \mathbb{R}^n$, and this fact allows us to get away with some obvious abuses of notation. Indeed, we will use $q, D_r(x_0)$, and $D_s(x_0)$ as shorthand for $\varphi(q), \varphi(D_r(x_0) \cap \mathcal{U})$, and $\varphi(D_s(x_0) \cap \mathcal{U})$ respectively. Since it will be convenient to work with a perfect ball in $\mathcal{V}$, we use $B_\epsilon(q)$ to denote the largest ball centered at $\varphi(q)$ which is contained in $\varphi(B_\delta(q))$. So within $\mathcal{V}$ we have a nondivergence form elliptic operator, L, which we can take to be defined on $C^2(\mathcal{V} \cap D_r(x_0))$ and which converges to the Laplacian in the sense described above as we zoom in on q. Lastly, we will obviously view all of our solutions to obstacle problems (so u_r and w_r for example) as being functions defined on $\mathcal{V}$. All of these conventions lead to the local picture shown in Figure 3.2.

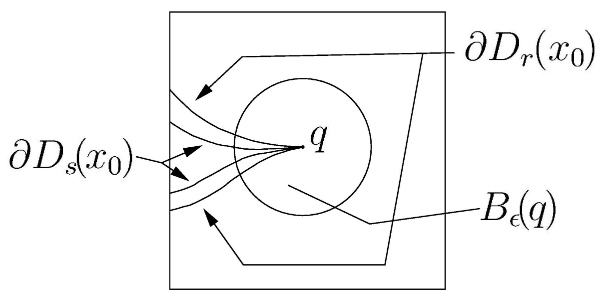

Figure 3.2: The Local Picture in $\mathcal{V}$.

Before jumping into the main proof, we observe the following two lemmas:

Lemma 3.4.2 (Barrier Function Estimates)**.** *By shrinking ϵ if necessary, we have*

$$2n - 1 \leq L(|x - y|^2) \leq 2n + 1 \, , \tag{3.6}$$

for all $x \in B_\epsilon(q)$ and for any fixed $y \in B_\epsilon(q)$.

Proof. This estimate follows immediately by using Equation (3.5) along with the fact that $g^{ij}(x) \to \delta^{ij}$ and $b^i(x) \to 0$ as $x \to q$. $\square$

Lemma 3.4.3 (Boundedness Estimate)**.** *For $x \in D_r(x_0) \cap \mathcal{V}$ and any direction e we have*

$$|L(D_e w_r)| \leq C < \infty \; . \tag{3.7}$$

Proof. We observe that this estimate is very similar to the estimate in the previous section given in Lemma 3.3.1. On the other hand, this time the proof is easier. We know that in $D_r(x_0) \cap \mathcal{V}$ we have

$$Lw_r = g^{ij} D_{ij} w_r + b^i D_i w_r = r^{-n} \; .$$

Differentiating this equation in the e direction, we have:

$$\begin{aligned} 0 &= D_e \left(g^{ij} D_{ij} w_r + b^i D_i w_r \right) \\ &= L(D_e w_r) + (D_e g^{ij}) D_{ij} w_r + (D_e b^i) D_i w_r \; , \end{aligned}$$

so by using regularity known for solutions of the obstacle problem along with the regularity that we have for the coefficients in our operator L, we conclude that

$$|L(D_e w_r)| \leq |D_e g^{ij}| \cdot |D_{ij} w_r| + |D_e b^i| \cdot |D_i w_r| \leq C < \infty.$$

□

Proof of Theorem 1.1.17. We can assume by shrinking ϵ again if necessary, that $x_0 \notin B_\epsilon(q)$ and $B_\epsilon(q)$ has no intersection with $\partial\mathcal{M}$ if $\mathcal{M}$ has boundary. (Lemma 6.2 of [7] guarantees that we can find such an ϵ.) Now we consider the function

$$h := w_s - w_r - \mu D_e w_r \tag{3.8}$$

where $\mu > 0$ will be a very small number and e will be a direction to be chosen later. We are going to arrive at a contradiction by showing that $h \geq 0$ in a ball around q intersected with $D_r(x_0)$ while using the asymptotics of the functions which make up h along with a good choice of the direction e allow us to show that h must be negative arbitrarily close to

q within $D_r(x_0)$.

Now for any positive $\rho < \epsilon$, we consider the set $E_\rho := B_\rho(q) \cap D_r(x_0) = B_\rho(q) \cap \Omega(w_r)$. Within this set we have $w_s - w_r \geq 0$ and $L(w_s - w_r) = s^{-n} - r^{-n} < 0$. Hence, in $E_\rho \setminus \eta_\gamma(\partial E_\rho)$ there exists a κ such that $w_s - w_r \geq \kappa > 0$. Having made this observation, it turns out that we will need a more precise lower bound, and by using the estimate from Lemma 3.4.2 we will succeed. Along these lines we first shrink ϵ (and therefore ρ) if necessary to be sure that that estimate applies, and we assume that $z \in E_\rho \setminus \eta_\gamma(\partial E_\rho)$ and observe that this implies that $B_\gamma(z) \subset E_\rho$. Next we define

$$\Theta(x) := \frac{(s^{-n} - r^{-n})(|x - z|^2 - \gamma^2)}{6n}$$

for use as a barrier function. Indeed, observe that

1. $\Theta = 0 \leq w_s - w_r$ on $\partial B_\gamma(z)$, and

2. recalling that $L(w_s - w_r) = s^{-n} - r^{-n} < 0$ and using the last lemma we get:

$$\begin{aligned} L\Theta &= \frac{s^{-n} - r^{-n}}{6n} \cdot L(|x - z|^2) \\ &\geq (s^{-n} - r^{-n}) \cdot \left(\frac{2n + 1}{6n} \right) \\ &\geq L(w_s - w_r) \quad \text{in} \quad B_\gamma(z). \end{aligned}$$

Thus, by using the weak maximum principle we have

$$w_s(z) - w_r(z) \geq \frac{\gamma^2 (r^{-n} - s^{-n})}{6n} \ \text{ for all } z \text{ within } \ E_\rho \setminus \eta_\gamma(\partial E_\rho) \ .$$

We can now observe the following properties of h :

1. By assuming that μ is sufficiently small, we have

$$Lh = s^{-n} - r^{-n} - \mu L(D_e w_r) \leq -\alpha < 0 \text{ in } E_\epsilon \ . \tag{3.9}$$

2. $h = w_s \geq 0$ on $\partial\Omega(w_r)$.

3. Within $E_\rho \setminus \eta_\gamma(\partial\Omega(w_r))$, by assuming that μ is sufficiently small, we have

$$h \geq \frac{\gamma^2(r^{-n} - s^{-n})}{6n} - \mu D_e w_r \geq \frac{\gamma^2(r^{-n} - s^{-n})}{10n} \ . \tag{3.10}$$

4. Within $E_\rho \cap \eta_\gamma(\partial\Omega(w_r))$, by using the optimal gradient bounds for w_r, we have

$$h \geq -\mu\gamma C \ . \tag{3.11}$$

The picture can be seen in Figure 3.3.

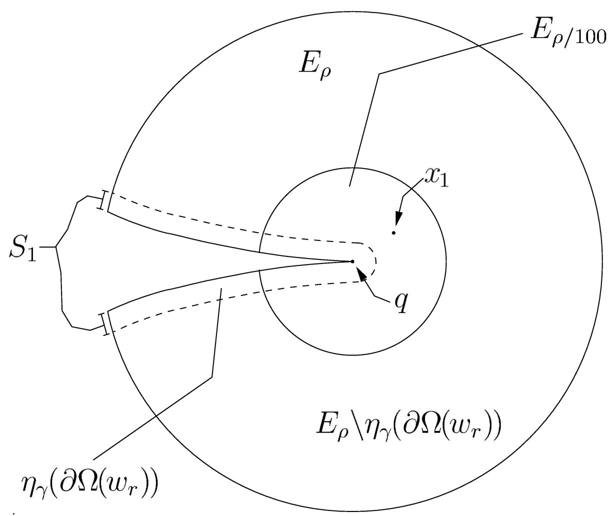

Figure 3.3: The Picture in $B_\rho(q)$

We are now in position to use the ideas within [10, Lemma 11] in order to show that h must be nonnegative everywhere in a small enough ball around q. On the other hand, for the sake of keeping this article more self-contained, and because of slight changes that need to be made (largely because we have an operator which is *close* to the Laplacian, and not exactly the Laplacian) we will present the argument here. In any case we claim that $h \geq 0$ within $E_{\rho/100} = B_{\rho/100}(q) \cap \Omega(w_r)$ provided μ is sufficiently small.

To begin the proof of our claim, we assume that there exists an $x_1 \in E_{\rho/100}$ with $h(x_1) < 0$.

We now define

$$v(x) := h(x) + \delta\left(\frac{r^{-n}}{4n}|x - x_1|^2 - w_r\right) , \tag{3.12}$$

and we observe that $v(x_1) < 0$. We also know that

$$Lv \leq -\alpha + \delta\left(\frac{r^{-n}}{4n}(2n+1) - r^{-n}\right) \leq -\alpha < 0$$

in all of E_ϵ by using Lemma 3.4.2 and Equation (3.9). So, by applying the weak maximum principle, we can be sure that v must attain a negative minimum on ∂E_ρ. On the other hand, all along $\partial\Omega(w_r)$, by using the definition of $h(x)$ we have $v(x) = w_s(x) + C\delta|x - x_1|^2 > 0$. So, we know that $v(x)$ attains its negative mimimum on $\partial B_\rho(q) \cap \partial E_\rho$. For this remaining piece of the boundary, it is convenient to split it into $S_1 := \eta_\gamma(\partial\Omega(w_r)) \cap \partial B_\rho(q)$ and $S_2 := \partial B_\rho(q) \setminus \eta_\gamma(\partial\Omega(w_r))$, and then by employing Equations (3.11) and (3.10) on those sets respectively, we get:

$$v \geq -C_1\mu\gamma + C_2\delta r^{-n}\rho^2 - C_3\delta\gamma^2 \quad \text{on } S_1 \tag{3.13}$$

and

$$v \geq C_4(r^{-n} - s^{-n})\gamma^2 + \delta(C_5 r^{-n}\rho^2 - w_r) \quad \text{on } S_2 . \tag{3.14}$$

On both sets we wish to choose constants so that v is forced to be nonnegative. For S_1 we choose $\gamma \ll \rho$ to force $C_2 r^{-n}\rho^2 > C_3\gamma^2$ and then choose μ as small as we need to give us the desired inequality. For S_2 we choose $\delta \ll \gamma^2$ and then shrink μ again if needed to fix the inequality on S_1. So, at this point we have a contradiction to any negativity of v within $E_{\rho/100}$.

Now, just as in the end of the proof of Theorem 1.1.16, it follows from standard regularity and nondegeneracy estimates for the obstacle problem, that w_s and w_r are bounded by a constant times $|x - q|^2$ within $E_{\rho/100}$, while $D_e w_r(x)$ must grow linearly for some choice of e within the same set. Now by replacing e with $-e$ if necessary, we get $h < 0$ somewhere within $E_{\rho/100}$ and we have the desired contradiction. □

Remark 3.4.4 (Existence of singular points in mean value sets)**.** Currently, it is unknown whether or not mean value sets of the type described in the previous section *ever* possess singular points. So, there is an outside chance that they do not exist. Having made this observation, it is a rather simple matter to show the existence of mean value sets for the Laplace-Beltrami operator on manifolds which have singular points. Indeed, at the moment the topology of one of these sets changes, you will necessarily have singular points. (We can also say that by using the results within [3] the free boundary won't "jump" from a configuration with one topology where the set is smooth to a different topology with smooth boundary; there will always be a moment with a "collision.") For a concrete example, consider harmonic functions on a typical cylinder. Obviously for any such function, one can "unroll" the cylinder and get a periodic harmonic function on $\mathbb{R}^2$. Kuran proved that any connected mean value set for the point $x_0 \in \mathbb{R}^n$ which has positive measure and which contains x_0 must be (up to a set of measure zero) a ball centered at x_0 ([17]). So, the $D_r(x_0)$ which fit within one period should be disks centered at x_0. By increasing the radius of the disk until the diameter is the length of a period, we get a mean value set which when viewed on the original cylinder, will have a "double cusp." Thus, we can be certain that the proof that we gave of the theorem in this section doesn't apply only to the empty set.

3.5 Singular Point Approximation

As before in Section 3.3 we consider an operator $L := \partial_i(a^{ij}(x)\partial_j)$, but now, although we are still working with the obstacle problem, we are no longer working with mean-value sets, and currently we will only assume a bound on $||a^{ij}||_{C^0(\overline{B_1})}$. Because our coefficients are always continuous, we can assume without loss of generality that $a^{ij}(0) = \delta^{ij}$ by changing coordinates. In this setting, we let $w \in W^{1,2}(B_1)$ satisfy:

$$\begin{cases} Lu = \chi_{\{u>0\}} & \text{in } B_1 \\ u \geq 0 \\ 0 \in \text{Sing}(u). \end{cases} \tag{3.15}$$

Next, for any $r < 1$, and any $t \in \mathbb{R}$, we let $u_{r;t} \in W^{1,2}(B_r)$ to be the solution to

$$\begin{cases} \Delta u = \chi_{\{u>0\}} & \text{in } B_r \\ u = (w+t)^+ & \text{on } \partial B_r \\ u \geq 0 \end{cases} \tag{3.16}$$

with the goal in this section of getting $u_{r,t}$ to approximate w and to also have a singular free boundary point at 0. One reason why we had this goal, was because we had hoped to generalize the regularity results of Figalli and Serra ([11]) to obstacle problems with more general elliptic operators than simply the Laplacian. Toward this aim, we will work with quadratic rescalings of w and $u_{r;t}$, and with $T := t/r^2$, we make the following list of definitions:

$$\begin{aligned} w_r(x) &:= \frac{w(rx)}{r^2} \\ v_{r;T}(x) &:= \frac{u_{r;t}(rx)}{r^2} \\ a_r^{ij}(x) &:= a^{ij}(rx) \\ L_r &:= D_i(a_r^{ij}(x)D_j). \end{aligned}$$

Then we observe that w_r satisfies

$$\begin{cases} L_r u = \chi_{\{u>0\}} & \text{in } B_1 \\ u \geq 0 \\ 0 \in \mathrm{Sing}(u), \end{cases} \tag{3.17}$$

and $v_{r;T}$ is the solution to

$$\begin{cases} \Delta u = \chi_{\{u>0\}} & \text{in } B_1 \\ u = (w_r + T)^+ & \text{on } \partial B_1 \\ u \geq 0. \end{cases} \tag{3.18}$$

Furthermore, we observe that by decreasing r we can make

$$||a_r^{ij}(\cdot) - \delta^{ij}||_{C^0(\overline{B_1})}$$

as small as we like.

Lemma 3.5.1 (Getting 0 into FB)**.** *Given* $0 < r \leq 1$*, and defining* w_r *and* $v_{r;T}$ *as above, there exists an* $S = S(r)$ *so that* $0 \in FB(v_{r;S})$*.*

Proof. We define the set $I := \left\{T \in \mathbb{R} \,\middle|\, 0 \in \text{int}(\Lambda(v_{r;T}))\right\}$, and observe that I is a bounded nonempty set. Indeed, if T is sufficiently negative, then $v_{r;T} \equiv 0$, and if T is more than $1/2n$, then $v_{r;T}(x) > |x|^2/2n$. Indeed, the following inclusions follow from those observations:

$$(-\infty, -\max_{B_1} w_r) \subset I \subset (-\infty, 1/2n) \tag{3.19}$$

So, we let $S := \sup I$, and we claim that $0 \in FB(v_{r;S})$.

Suppose not. Then either $0 \in \Omega(v_{r;S})$ or $0 \in \text{int}(\Lambda(v_{r;S}))$. In the first case we have a closed ball $\overline{B_\epsilon(0)} \subset \Omega(v_{r;S})$, and so in this case we let

$$\gamma := \min \left\{ v_{r;S}(x) \,\middle|\, x \in \overline{B_\epsilon(0)} \right\} \ .$$

Now we define $\tilde{S} := S - \gamma/2$ and by using Lemma 3.2.1 we have

$$\gamma \leq v_{r;S}(0) \leq v_{r;\tilde{S}}(0) + \frac{\gamma}{2} \ ,$$

and this inequality implies $\sup I \leq \tilde{S} < S$ which is a contradiction.

In the other case, we have a closed ball $\overline{B_\epsilon(0)} \subset \Lambda(v_{r;S})$. Now we let $T_j \downarrow S$ and observe that by the definition of S, we have $0 \in \overline{\Omega(v_{r;T_j})}$ for all T_j. Using the standard nondegeneracy results for the obstacle problem, for every j, we have an x_j in $B_{\epsilon/2}(0)$ with

$$v_{r;T_j}(x_j) \geq C\epsilon^2 \ .$$

This inequality leads to,

$$C\epsilon^2 \leq ||v_{r;S} - v_{r;T_j}||_{L^\infty(B_\epsilon(0))} \leq ||v_{r;S} - v_{r;T_j}||_{L^\infty(B_1(0))} \leq ||v_{r;S} - v_{r;T_j}||_{L^\infty(\partial B_1(0))},$$

where the last inequality is by Lemma 3.2.1 again. However, since

$$||v_{r;S} - v_{r;T_j}||_{L^\infty(\partial B_1(0))} \leq ||S - T_j||_{L^\infty(\partial B_1(0))} \to 0 \ ,$$

we have a contradiction. □

Lemma 3.5.2 (Uniqueness of S)**.** *The $S(r)$ given in Lemma 3.5.1 is the only number S, such that $0 \in FB(v_{r;S})$.*

Proof. Suppose not. Then there exists $S_1 < S_2$ such that $0 \in FB(v_{r;S_1}) \cap FB(v_{r;S_2})$. Now from Lemma 3.2.1 we know that $\Omega(v_{r;S_1}) \subset \Omega(v_{r;S_2})$, and we consider the function

$$V := v_{r;S_2} - v_{r;S_1} \ . \tag{3.20}$$

We observe that $V = S_2 - S_1 > 0$ on $\partial B_1 \cap \Omega(v_{r;S_1})$, and $V \in C^0(\overline{B}_1)$. By using the continuity of V, we have a $\delta \in (3/4, 1)$ so that $V \geq \frac{1}{2}(S_2 - S_1)$ on $\partial B_\delta \cap \Omega(v_{r;S_1})$. We now define the function

$$h := V - \mu D_e v_{r;S_1} \tag{3.21}$$

for a direction e to be chosen later. We observe that h is harmonic, and because $v_{r;S_1} \in C^{1,1}(\overline{B_\delta})$ we can choose μ to be sufficiently small so that $h > 0$ on $\partial B_\delta \cap \Omega(v_{r;S_1})$. Now by the optimal regularity results for the obstacle problem $v_{r;S_1} = D_e v_{r;S_1} = 0$ on all of $\partial\Omega(v_{r;S_1}) \cap B_1$, so we observe that $h \geq 0$ on all of $\partial(B_\delta \cap \Omega(v_{r;S_1}))$. Thus, the maximum principle gives us $h \geq 0$ in all of $B_\delta \cap \Omega(v_{r;S_1})$. Now by proceeding exactly as in the proof of Theorem 1.1.16 where we use the asymptotics of the functions making up h to find a spot where it is negative (and assigning an appropriate direction e) we get a contradiction. □

Lemma 3.5.3 ($S(r) \to 0$ as $r \to 0$)**.** *For the $S(r)$ given in Lemma 3.5.1 we have*

$$\lim_{r \to 0} S(r) = 0 \, . \tag{3.22}$$

Proof. Suppose not. Then since the $S(r)$ are uniformly bounded, we can find a sequence $r_j \to 0$ such that $S(r_j) \to \tilde{S} \neq 0$. By our assumptions about w_r, and by applying [8, Lemma 3.1 and Lemma 3.2] we know that $w_{r_j} \to w_0 = \frac{1}{2}(x^T M x)$ uniformly where M is a nonnegative matrix and $\Delta w_0 = \text{Trace}(M) = 1$. So, we know that w_0 satisfies

$$\begin{cases} \Delta u = \chi_{\{u>0\}} & \text{in } B_1 \\ u(x) = x^T M x & \text{on } \partial B_1 \\ u \geq 0 \end{cases} \tag{3.23}$$

and additionally, $0 \in \text{Sing}(w_0)$. On the other hand, since $w_{r_j} \to \frac{1}{2}(x^T M x)$ uniformly, we know that the boundary data of $v_{r_j;S_j}$ converges uniformly to $(\frac{1}{2}(x^T M x) + \tilde{S})^+$. So, we have that the limit of the $v_{r_j;S_j}$, which we will call "$v_{0;\tilde{S}}$" satisfies

$$\begin{cases} \Delta u = \chi_{\{u>0\}} & \text{in } B_1 \\ u(x) = (x^T M x + \tilde{S})^+ & \text{on } \partial B_1 \\ u \geq 0 \end{cases} \tag{3.24}$$

and furthermore $0 \in \text{FB}(v_{0;\tilde{S}})$. Now, since $\tilde{S} \neq 0$ we can use the functions w_0 and $v_{0;\tilde{S}}$ along with Lemma 3.5.2 to get a contradiction. □

Remark 3.5.4 ($v_{0;\tilde{S}} = w_0$)**.** It follows from knowing that $\tilde{S} = 0$ along with Equations (3.23) and (3.24) and the uniqueness of the solutions to such problems that $v_{0;\tilde{S}} = w_0$ and in particular $0 \in \mathrm{Sing}(v_{0;\tilde{S}})$. We will use this fact in the next proof.

Finally, to strengthen the statement of our final theorem, we follow [11] and for $m \in \{0, 1, 2, \ldots, n-1\}$ we define the m-th stratum of the singular set to be the subset of the singular set where the dimension of the kernel of the blow up limit is m.

Theorem 3.5.5 (Preserving the Singular Point and Bounding the Stratum)**.** *Given the function w, there exists an $R > 0$ such that $0 \in \mathrm{Sing}(v_{r;S(r)})$ for all $r < R$. Furthermore, by shrinking R if necessary, this singular point is in the same or lower stratum as it is with w. (i.e. If 0 is in the k-th stratum of w, then it will always belong to the strata for $v_{r;S(r)}$ with $m \leq k$ for $r < R$.)*

Proof. Suppose there does not exist an $R > 0$ such that $0 \in \mathrm{Sing}(v_{r;S(r)})$ for all $r < R$. Then there exists $r_j \downarrow 0$ such that $0 \in \mathrm{Reg}(v_{r_j;S(r_j)})$ for all j. Fix $\epsilon > 0$ to be chosen later, and to simplify notation, we will let $v_j := v_{r_j;S(r_j)}$ for the duration of this proof. Using our assumption, for each r_j there exists $n_j \in \partial B_1$ so that with $P_{r_j} := \max\{(x \cdot n_j), 0\}^2$ we have

$$\frac{v_{r_j;S}(\rho x)}{\rho^2} \to P_{r_j}(x) \text{ uniformly as } \rho \to 0.$$

However, there exists a subsequence of r_j, which we denote again by r_j, such that $n_j \to n_0$ and so

$$P_{r_j} \to P_0 = \max\{(x \cdot n_0), 0\}^2$$

uniformly. Hence, given any $\epsilon > 0$ there exists an $\mathcal{R}_1 > 0$ and a $\mathcal{K} > 0$ such that if $r_j < \mathcal{R}_1$ and $\rho < \mathcal{K}$, then

$$\left|\left|\frac{v_j(\rho x)}{\rho^2} - P_0(x)\right|\right|_\infty \leq \epsilon. \tag{3.25}$$

On the other hand $v_j \to w_0 = \frac{1}{2}(x^T M x)$ uniformly in $\overline{B_1}$ by Remark 3.5.4 , so there exists an $\mathcal{R}_2$ such that $r_j \leq \mathcal{R}_2$ implies

$$\left|\left|\frac{v_j(\rho x)}{\rho^2} - x^T M x\right|\right|_\infty \leq \epsilon. \tag{3.26}$$

By applying [22, Lemma 2] we know that there exists a constant $\gamma > 0$ so that

$$\big|\big|x^T Mx - P_0(x)\big|\big|_\infty \geq \gamma \, . \tag{3.27}$$

Now we fix $\epsilon < \gamma/2$ and use the triangle inequality combined with Equations (3.25), (3.26), and (3.27) to get a contradiction.

At this point we have the first part of the theorem. To show the second part we essentially repeat the argument but replace the use of [22, Lemma 2] with the observation that a nonnegative matrix M can be approximated arbitrarily well with matrices with lower dimensional kernels, but it will stay isolated from all of the matrices with higher dimensional kernels. To be more specific, the main difference from the first part of the proof is that in place of P_{r_j} we would have a sequence $x^T M_j \, x$ where the kernels of the M_j have dimension greater than the kernel of the M, and this leads to a contradiction. □

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