PROPERTIES OF ABSTRACT VECTOR SPACES

by

DONALD HUDSON MABRY

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Approved by:

R. L. Yates

Major Professor

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INTRODUCTION

Frequently there are mathematical systems which appear to be quite different and yet if the central theory of each system is examined, common properties may be found. The observer may try to bring such diverse systems under a single heading by extracting all properties common to these systems and listing these as postulates for an otherwise unrestricted system.

It should be mentioned that by a postulate is not meant a self-evident truth or a statement which cannot be proven, but rather an assumed property. The postulates P appearing in a set S are assumptions made about the elements of S. The system S itself consists of elements and operations, assumptions about both and finally consequences or theorems derived from the assumptions. Whenever a system C is found to satisfy the postulates P then the theorems of S can be applied to C. It should be noted that different approaches to a particular system can be made. In one approach a property may be assumed while in another approach this same property may be a theorem derived from other assumed properties.

Often in various parts of mathematics one is confronted with a set in which it is meaningful and interesting to deal with "linear combinations" of the elements of a set. Examples of such linear combinations are found in the calculus and the familiar three-dimensional Euclidean space. In this report a mathematical system which is a useful abstraction of the type mentioned above will be defined and resulting properties examined.

The abstract nature of the material presented enables one to apply the properties of a "vector space" to any set of elements which satisfy the definition. Except for the last section on "inner products", no restriction is made as to the field over which a vector space is defined.

VECTOR SPACES

Definition 1. A vector space consists of the following:

1) a field F of scalars;

2) a set V of vectors;

3) an operation called addition, indicated by +, which is a binary composition in V such that

   a) addition is closed, α, β contained in V implies $\alpha + \beta$ is contained in V,

   b) addition is commutative, $\alpha + \beta = \beta + \alpha$,

   c) addition is associative, $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$,

   d) there exists a unique vector $\mathbb{O}$, such that $\alpha + \mathbb{O} = \mathbb{O} + \alpha = \alpha$, for all α in V,

   e) for each α in V there exists a unique inverse $-\alpha$ such that $\alpha + (-\alpha) = \mathbb{O}$;

4) an operation called scalar multiplication such that for every α, β in V and a, b, in F

   a) scalar multiplication is closed, α in V and a in F implies a α is in V,

   b) $a(\alpha + \beta) = a\alpha + a\beta$,

   c) $(a + b)\alpha = a\alpha + b\alpha$,

   d) $(ab)\alpha = a(b\alpha)$,

e) $1\alpha = \alpha$, where 1 is the identity element in F.

If in any particular discussion, no confusion can arise, the vector space will be denoted by V. However, if the field has not been previously specified, then it will be said that V is over F, where F is the field over which the vector space is defined.

Theorem 1. For 0 in the field F and any α in the vector space V, $0\alpha = \mathbb{0}$.

Proof. From the property of the additive identity of the field F, $(a + 0) = a$ and from (4c) of Definition 1, $(a + 0)\alpha = a\alpha + 0\alpha$. Hence $a\alpha = a\alpha + 0\alpha$ and since the additive identity of the vector space is unique, $0\alpha = \mathbb{0}$.

In the definition of a vector space the vectors -1α and $-\alpha$ are both considered. The distinction between these vectors should be noted. While -1α is a scalar multiple, $-\alpha$ is not. The following theorem gives the exact relation between the two.

Theorem 2. For α and $-\alpha$ in V and -1 in F, $-1\alpha = -\alpha$.

Proof. $\alpha + (-1)\alpha = 1\alpha + (-1)\alpha = (1+(-1))\alpha = 0\alpha = \mathbb{0}$. Hence -1α is the additive inverse of α and thus $-1\alpha = -\alpha$.

Example. Consider the n-tuple space, $V_n(F)$, where F is any field and let V be the set of all n-tuples, $\alpha = (a_1, a_2, \ldots, a_n)$, of scalars where a_i is in F. If $\beta = (b_1, b_2, \ldots, b_n)$ with b_i in F, then the sum is defined by

$$\alpha + \beta = (a_1+b_1, a_2+b_2, \ldots, a_n+b_n).$$

The scalar product is defined by

$$c\alpha = (ca_1, ca_2, \ldots, ca_n)$$

To show that a vector space has been defined, one must show that all of the properties of (3) and (4) of the definition hold.

For (3a), take $\alpha + \beta = (a_1+b_1, a_2+b_2, \ldots, a_n+b_n)$. Since the field is closed under addition $a_i + b_i = c_i$ where c_i is in F. Then $\alpha + \beta = (c_1, c_2, \ldots, c_n)$. Hence the set is closed under addition.

To show that (3b) is satisfied take

$\alpha + \beta = (a_1+b_1, a_2+b_2, \ldots, a_n+b_n)$

$= (b_1+a_1, b_2+a_2, \ldots, b_n+a_n) = \beta + \alpha$, since the scalars in F are commutable.

For (3c) take

$$\begin{aligned}\alpha + (\beta + \gamma) &= (a_1, a_2, \ldots, a_n) + (b_1+c_1, b_2+c_2, \ldots, b_n+c_n)\\ &= (a_1 + (b_1+c_1), a_2 + (b_2+c_2), \ldots, a_n + (b_n+c_n))\\ &= ((a_1+b_1) + c_1, (a_2+b_2) + c_2, \ldots, (a_n+b_n) + c_n)\\ &= (\alpha + \beta) + \gamma,\end{aligned}$$

since the scalars of F are associative.

To verify (3d) take $\alpha + \mathbb{O}$, where $\mathbb{O} = (0, 0, 0, \ldots, 0)$ and 0 is the additive identity of F. Then $\alpha + \mathbb{O}$

$= (a_1, a_2, \ldots, a_n) + (0, 0, \ldots, 0) = (a_1+0, a_2+0, \ldots, a_n+0) = \alpha$.

(3e) is verified by taking $-\alpha = (-a_1, -a_2, \ldots, -a_n)$ where $-a_i$ is the additive inverse of a_i in F. Then $\alpha + (-\alpha)$

$= (a_1 - a_1, a_2 - a_2, \ldots, a_n - a_n) = \mathbb{O}$.

It is obvious that if 1 is taken as the identity of multiplication in F, that $1\alpha = \alpha$. This shows that property (4e) holds.

To show that (4a) holds, take $c\alpha = (ca_1, ca_2, \ldots, ca_n)$. Since F is closed under multiplication $ca_i = d_i$, with d_i in F. Hence $c\alpha = (d_1, d_2, \ldots, d_n)$ and the set is closed under scalar multiplication.

In showing (4b) take

$$a(\alpha + \beta) = (a(a_1+b_1), a(a_2+b_2), \ldots, a(a_n+b_n))$$
$$= (aa_1+ab_1, aa_2+ab_2, \ldots, aa_n+ab_n)$$
$$= (aa_1, aa_2, \ldots, aa_n) + (ab_1, ab_2, \ldots, ab_n)$$
$$= a\alpha + a\beta .$$

Now take

$$(a + b)\alpha = ((a + b) a_1, (a + b) a_2, \ldots, (a + b) a_n)$$
$$= (aa_1+ba_1, aa_2+ba_2, \ldots, aa_n+ba_n)$$
$$= a\alpha + b\alpha .$$

This shows that (4c) holds.

To show (4d) simply take $(ab)\alpha = (aba_1, aba_2, \ldots, aba_n)$ $= a(ba_1, ba_2, \ldots, ba_n) = a(b\alpha)$.

Since it has been shown that the set of all n-tuples satisfies the definition, it constitutes a vector space.

<u>Definition 2</u>. Let V be a vector space over a field F. A subspace of V is defined as a subset W of V which is also a vector space.

With this definition the following theorem can be stated.

<u>Theorem 3</u>. A nonempty subset W of V is a subspace of V if, and only if, W is closed under addition and scalar multiplication. That is, if α, β are in W and c is in F, then $c\alpha + \beta$ is in W.

Proof: If W is a nonempty subset of V and closed under addition and scalar multiplication then (4a) and (3a) are satisfied. The subset W has at least one vector α such that $-1\alpha + \alpha = -\alpha + \alpha = 0$ is in W, since it is closed under scalar multiplication. Also if α is in W and c is in F, $c\alpha = c\alpha + 0$ is in W, in particular $-\alpha = -1\alpha$ is in W. Again if α, β are in W then (4b), (4c), (4d), and (4e) hold in W since they hold in V and W is over the same field F.

Conversely, if W is a subspace of V, then by definition, W is closed under addition and scalar multiplication.

Theorem 4. If S and T are subspaces of V then those vectors belonging to both S and T form a subspace of V. That is $S \cap T = W$ is a subspace of V, where $S \cap T = W$ represents the intersection of S and T.

Proof: If α, β are contained in W, then $c\alpha + \beta$ is in S and also in T by Theorem 3, and hence in W. If $c\alpha + \beta$ is in W then W is a subspace by Theorem 3.

Definition 3. A vector β is a linear combination of the vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$, if there exist scalars $c_1, c_2, \ldots, c_n$, in F such that

$$\beta = c_1 \alpha_1 + c_2 \alpha_2 + \cdots + c_n \alpha_n$$
$$= \sum_{i=1}^{n} c_i \alpha_i .$$

Taking a fixed set of vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$, over a field F, it can be easily shown from Theorem 3 that the set of all linear combinations, $c_1 \alpha_1 + c_2 \alpha_2 + \cdots + c_n \alpha_n$, constitutes a vector space. The set of all linear combinations of $\alpha_1, \alpha_2, \ldots, \alpha_n$ will be denoted by $[\alpha_1, \alpha_2, \ldots, \alpha_n]$.

The vector space $[\alpha_1, \alpha_2, \ldots, \alpha_n]$ is the smallest subset containing all of the vectors α_1. More generally, starting with an arbitrary set of vectors, the set of all linear combinations of all finite subsets is the smallest subspaces which contains the original set. If this technique is applied to find the smallest subspace containing two subspaces S and T, it is seen that a linear combination of elements of S and T reduces to an element $\alpha + \beta$, with α in S and β in T. This proves Theorem 4.

Theorem 5. If S and T are subspaces of V then the set of all sums, $\alpha + \beta$, with α in S and β in T, is a subspace called the linear sum of S and T and written S + T.

The linear sum clearly contains S and T and is contained in any other subspace P containing S and T. Properties of the linear sum may be stated as follows:

1) $S \leq S + T$, $T \leq S + T$

2) $S \leq P$ and $T \leq P$ imply $S + T \leq P$, where $S \leq P$ means that S is contained in the subspace P.

Definition 4. A set of vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$ is said to be linearly dependent if there exist scalars $c_1, c_2, \ldots, c_n$ in F, not all zero, such that

(5) $\quad c_1 \alpha_1 + c_2 \alpha_2 + \cdots + c_n \alpha_n = \mathbb{O}$.

A set which is not linearly dependent is called linearly independent.

In other words, for $\alpha_1, \alpha_2, \ldots, \alpha_n$ to be linearly independent all of the c_i's must be equal to zero in order for (5) to hold.

Definition 5. If S is the subspace consisting of all linear combinations of $\alpha_1, \alpha_2, \ldots, \alpha_n$, then S is called the space spanned (generated) by the vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$.

No stipulation is made that $\alpha_1, \alpha_2, \ldots, \alpha_n$, be linearly independent in order to span a space. This case will be taken up shortly.

Before this, however, a relation between independence and linear combinations can be stated.

Theorem 6. The nonzero vectors $\alpha_1, \alpha_2, \ldots, \alpha_m$ in a vector space V are linearly dependent if and only if one of the vectors is a linear combination of the others.

Proof: If α_i is a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_m$ then

$\alpha_i = c_1 \alpha_1 + \ldots + c_{i-1} \alpha_{i-1} + c_{i+1} \alpha_{i+1} + \ldots + c_m \alpha_m$.

Thus

$c_1 \alpha_1 + c_2 \alpha_2 + \ldots + c_{i-1} \alpha_{i-1} + c_{i+1} \alpha_{i+1} + \ldots + c_m \alpha_m - \alpha_i = \mathbb{O}$. Hence at least one c_i is nonzero, namely -1, thus satisfying the definition of linear dependence.

Conversely, suppose the set of α_i's is linearly dependent. Then $c_1 \alpha_1 + c_2 \alpha_2 + \ldots + c_i \alpha_i + \ldots + c_m \alpha_m = \mathbb{O}$, where $c_i \neq 0$ for some i. Thus

$-c_i \alpha_i = c_1 \alpha_1 + \ldots + c_{i-1} \alpha_{i-1} + c_{i+1} \alpha_{i+1} + \ldots + c_m \alpha_m$,

and

$$\alpha_i = -\frac{c_1}{c_i} \alpha_1 - \frac{c_2}{c_i} \alpha_2 - \ldots - \frac{c_{i-1}}{c_i} \alpha_{i-1}$$

$$-\frac{c_{i+1}}{c_i} \alpha_{i+1} - \ldots - \frac{c_m}{c_i} \alpha_m$$

$$= d_1 \alpha_1 + d_2 \alpha_2 + \ldots + d_m \alpha_m.$$

Hence α_1 is a linear combination of the rest of the vectors. This proves the theorem.

Definition 6. The dimension of a vector space V is the maximum number of linearly independent vectors in V and will be denoted by $d[V]$.

Definition 7. A basis of a vector space V is a set of linearly independent vectors which spans V.

One cannot conclude from the above definition that a vector space V has one and only one basis. For example, consider $V_3(R)$, where R is the real field. The vectors (1, 0, 0), (0, 1, 0), and (0, 0, 1) will generate $V_3(R)$, but this same space can be generated by the vectors (1, 0, 0), (0, 1, 0), and (0, 0, 1). It can easily be shown that each set consists of linearly independent vectors and hence forms a basis for $V_3(R)$.

Special note is made of the vectors (1, 0, 0), (0, 1, 0), and (0, 0, 1). These vectors are called the unit vectors of $V_3(F)$. In general the unit vectors of $V_n(F)$ are

$$u_1 = (1, 0, 0, \ldots, 0)$$
$$u_2 = (0, 1, 0, \ldots, 0)$$
$$\ldots$$
$$u_n = (0, 0, 0, \ldots, 1),$$

where 1 is the identity element of F. The notation u_i will be reserved for the unit vectors. It is easily seen that these vectors form a basis for $V_n(F)$.

Theorem 7. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ form a basis for a vector space V, then every vector in V can be expressed uniquely as a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Proof. Every vector in V is a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_n$ since the α_i's span V. Suppose there are two representations of a vector β. Then $\beta = a_1 \alpha_1 + a_2 \alpha_2 + \ldots + a_n \alpha_n$ $= b_1 \alpha_1 + b_2 \alpha_2 + \ldots + b_n \alpha_n$. Now $(a_1 - b_1) \alpha_1 + (a_2 - b_2) \alpha_2 + \ldots + (a_n - b_n) \alpha_n = \Phi$ and $(a_i - b_i) = 0$ for all i since the α_i's are independent. Thus $a_i = b_i$ and the uniqueness is established.

<u>Theorem 8</u>. The number of vectors in a basis of a vector space V is equal to the dimension of V.

Proof. Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ span the vector space V which has dimension n and let r equal the maximum number of linearly independent vectors in $\alpha_1, \alpha_2, \ldots, \alpha_m$. Now, renumbering if necessary, let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be the linearly independent vectors from the generating set. Obviously then $r \leq n$. Also $\alpha_1, \alpha_2, \ldots, \alpha_r, \alpha_r + j$, j $= 1, 2, \ldots, m - r$, is a linearly dependent set, expressed as $a_1 \alpha_1 + a_2 \alpha_2 + \ldots + a_r \alpha_r + a_{r+j} \alpha_{r+j} = \Phi$, where $a_{r+j} \neq 0$, for $a_{r+j} = 0$ would imply the dependence of $\alpha_1, \alpha_2, \ldots, \alpha_r$. Thus α_{r+j} is a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_r$, so that if the elements of V are represented as linear combinations of $\alpha_1, \alpha_2, \ldots, \alpha_m$, a term involving α_{r+j}, $j > 0$, can be replaced by a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_r$. Thus $\alpha_1, \alpha_2, \ldots, \alpha_r$ is a generating set for V.

To show that $n \leq r$ and hence $r = n$, it must be shown that any set of more than r linear combinations of

$\alpha_1, \alpha_2, \ldots, \alpha_r$ is linearly dependent. To do this let $\beta_1, \beta_2, \ldots, \beta_s$ be a set of vectors in which each β_i is a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_r$ and $s > r$.

Then

$$(6) \qquad {}_i = \sum_{j=1}^{r} a_{ij} \alpha_j, \; i = 1, 2, \ldots, s.$$

The existence of a set of scalars $c_1, c_2, \ldots, c_s$, not all zero such that $\sum_{i=1}^{s} c_i \beta_i = 0$ will show the dependence of the β_i. It is sufficient to choose the c_i's to satisfy the linear system.

$$(7) \qquad \sum_{i=1}^{s} a_{ij} c_i = 0, \; j = 1, 2, \ldots, r,$$

since these expressions will be the coefficients of the α_j's when in $\sum_{i=1}^{s} c_i \beta_i$, each β_i is replaced by its value in (6) and the terms are collected. A non-trivial solution of (7) always exists since the number of unknowns, s, exceeds the number of equations, r. Hence the c_i's exist and the set $\beta_1, \beta_2, \ldots, \beta_s$ is dependent. Thus $n \leq r$ and from above $r \leq n$, hence $r = n$.

Since the dimension of a given vector space V does not change, an immediate consequence of Theorem 7 is the following corollary.

Corollary 1. All bases of a vector space V include precisely the same number of vectors.

Using the results of Corollary 1 another obvious result of

Theorem 7 is

<u>Corollary 2</u>. Every basis of V, where d $[V]$ = n, contains exactly n vectors.

It has been shown that the basis of a vector space is not unique. The variety of bases which can exist is illustrated by the following theorem.

<u>Theorem 9</u>. Any set of linearly independent vectors, $\alpha_1, \alpha_2, \ldots, \alpha_m$, in a vector space V of dimension n is part of a basis and can be extended to a basis of V.

Proof. Let $\beta_1, \beta_2, \ldots, \beta_n$ be a basis for V. Then the set $\alpha_1, \alpha_2, \ldots, \alpha_m, \beta_1, \beta_2, \ldots, \beta_n$ spans V and is linearly dependent. By Theorem 8, the number of vectors in the basis is equal to the dimension and by definition, the dimension is the maximum number of vectors which are linearly independent. Thus adding one or more vectors results in a linearly dependent set. Since these vectors are dependent, at least one vector, which is a linear combination of the others, can be deleted and still leave a set of spanning vectors. By choosing only those vectors which are linear combinations of those preceding them, none of the α_i's will be eliminated since they are linearly independent. By this process all of the linearly dependent vectors will be eliminated, leaving a set of spanning vectors which are linearly independent and hence forms a basis which will include $\alpha_1, \alpha_2, \ldots, \alpha_m$. This proves the theorem.

<u>Theorem 10</u>. If a vector space V is the linear sum of two

subspaces, S and T, where $S \cap T = \mathbb{O}$, then the union of any basis of S with any basis of T is a basis of V.

Proof: Let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be a basis for S and $\beta_1, \beta_2, \ldots, \beta_s$ be a basis for T. Then these vectors span V since any vector in V can be expressed as a linear combination of them. These vectors are necessarily independent

for if $\sum_{i=1}^{r} a_i \alpha_i + \sum_{j=1}^{s} b_j \beta_j = \mathbb{O}$

then $\gamma = \sum_{i=1}^{r} a_i \alpha_i = - \sum_{j=1}^{s} b_j \beta_j$.

One vector γ is now represented as a linear combination of the α_i's and also the β_i's and hence must lie in both S and T. Since $S \cap T = \mathbb{O}$, $\gamma = \mathbb{O}$ and hence $a_i = b_i = 0$. Therefore the spanning vectors of V are linearly independent and thus are a basis for V. This theorem can be extended to the case of the linear sum of a finite number of subspaces.

By using the result of Theorem 8, the following corollary to Theorem 10 can be shown.

Corollary. If the vector space V is the linear sum of two subspaces, S and T, where $S \cap T = \mathbb{O}$, then $d[V] = d[S] + d[T]$.

Proof. Since Theorem 8 states that the dimension of a vector space is the number of vectors in any basis, the proof of Theorem 10 shows that when $d[S] = r$ and $d[T] = s$ then $d[V] = r + s$.

If the restriction of $S \cap T = \mathbb{O}$ is removed from S and T then a more general statement about a vector space $V = S + T$ can be made.

Theorem 11. Let S and T be any two subspaces of a vector space V. Then

$$d\,[S] + d\,[T] = d\,[S \cap T] + d\,[S + T]\,.$$

Proof. Let $\gamma_1, \gamma_2, \ldots, \gamma_t$ form a basis for $S \cap T$. Then by Theorem 9, $\gamma_1, \ldots, \gamma_t, \alpha_1, \ldots, \alpha_r$ and $\gamma_1, \ldots, \gamma_t, \beta_1, \ldots, \beta_s$ form bases for S and T respectively. The vectors $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s, \gamma_1, \ldots, \gamma_t$ obviously span S + T. To establish independence let

$$\sum_{i=1}^{t} c_i \gamma_i + \sum_{j=1}^{r} a_j \alpha_j + \sum_{k=1}^{s} b_k \beta_k = \mathbb{O}.$$

Then $\sum_{j=1}^{r} a_j \alpha_j = -\sum_{i=1}^{t} c_t \gamma_t - \sum_{k=1}^{s} b_k \beta_k$ is in T.

As a linear combination of $\alpha_1, \ldots, \alpha_r$ this vector is also in S, hence in $S \cap T$, which means $\sum_{j=1}^{r} a_j \alpha_j = \sum_{i=1}^{t} d_i \gamma_i$, whence

$$(8) \qquad \sum_{j=1}^{r} a_j \alpha_j - \sum_{i=1}^{t} d_i \gamma_i = \mathbb{O},$$

for scalars d_i. The independence of the basis vectors of S implies all of the coefficients in (8) are zero. Thus every $a_i = 0$. Hence $\sum_{i=1}^{t} c_i \gamma_i + \sum_{k=1}^{s} b_k \beta_k = \mathbb{O}$. Now the independence of the basis vectors of T implies that all c_i and b_k vanish. The conclusion follows from $d\,[S] + d\,[T] = (t + r) + (t + s) = (t + r + s) + t = d\,[S + T] + [S \cap T]$.

CHANGE OF BASIS

If the set $\alpha_1, \alpha_2, \dots, \alpha_n$ forms a basis for a vector space V, then of course any vector, β, in V can be uniquely expressed in terms of that basis, namely by $\beta = a_1 \alpha_1 + a_2 \alpha_2 + \dots + a_n \alpha_n$. It has been shown that this vector β can be uniquely expressed in terms of other bases of V. If the basis is known, say $\alpha_1, \dots, \alpha_n$, then the vector can be completely described by the scalars $a_1, a_2, \dots, a_n$. This leads to the following definition.

<u>Definition 8</u>. For any vector β in a vector space V described by $a_1 \alpha_1 + a_2 \quad {}_2 + \dots + a_n \alpha_n$ the scalars $[a_1, a_2, \dots, a_n]$ are called the coordinates of β relative to the basis $\alpha_1, \alpha_2, \dots, \alpha_n$.

It is obvious that when a vector is described by its coordinates, these coordinates must be given relative to a definite basis.

A simple example would be the 3-tuple vector (4, 8, 3). The coordinates relative to the unit vectors would of course be $[4, 8, 3]$, while relative to the basis (1, 0, 0), (0, 1, 0), (0, 0, $\frac{1}{2}$) they would be $[4, 8, 6]$.

The problem of expressing a vector, α, in terms of one basis, B, if the coordinates of α are given relative to another basis A, now arises.

However before this problem can be solved another problem must be eliminated. This is the problem of ordering. If $\alpha = [a_1, a_2, \dots, a_n] = \sum_{i=1}^{n} a_i u_i$, then a definite ordering of the basis vectors is determined since it is easily seen which is the "first" vector in the basis, which is the "second" and so on. Here

$$u_i = (\delta_{i1}, \delta_{i2}, \ldots, \delta_{in}), i = 1, 2, \ldots, n,$$

where δ_{ij} is Kronecker's delta.

However, for any other basis, B, such an ordering may not be so natural and therefore it is necessary to impose such an ordering so that the i^{th} coordinate of a vector may be determined. The question is one of how to order a set S of n elements. More than one method can be used but the following will be used here.

Definition 9. An ordering of the set S having n elements is a function f from the subset, $N = \{1, 2, \ldots, n\}$ of the set of positive integers onto the set S such that

1) if p and q are contained in N, $p \neq 1$, then $f(p) \neq f(q)$ are contained in S.

2) if a is contained in S then $f(p) = a$ has a solution p, which is contained in N and is unique. The function f is said to be a one-to-one correspondence.

From this follows directly the definition of an ordered basis.

Definition 10. An ordered basis of a vector space V is a basis together with a fixed ordering of the elements.

Thus a basis B, is an ordered basis if it is clearly understood which vector of B is the i^{th} one.

The actual relation between the coordinated $[b_1, b_2, \ldots, b_n]$

of a vector α relative to a basis B and the coordinates $[a_1, a_2, \ldots, a_n]$ relative to a basis A can be given by

$$[b_1, b_2, \ldots, b_n] = P\ [a_1, a_2, \ldots, a_n]'$$ [1]

where P is an invertable matrix.

Isomorphism

Now let $B = \beta_1, \beta_2, \ldots, \beta_n$ be an ordered basis for a vector space V. Then for every β in V, having coordinates $[b_1, b_2, \ldots, b_n]$, there is a unique n-tuple $(b_1, b_2, \ldots, b_n)$ of scalars such that $\beta = \sum_{i=1}^{n} b_i\ \beta_i$. Since it has been shown that every vector has a unique expression as a linear combination of any set of basis vectors, this n-tuple is unique.

Consider now $\alpha = \sum_{i=1}^{n} a_i\ \beta_i$, then

$$\alpha + \beta = \sum_{i=1}^{n} a_i\ \beta_i + \sum_{i=1}^{n} b_i\ \beta_i = \sum_{i=1}^{n} (a_i + b_i)\ \beta_i$$

so that the i^{th} coordinate of $\alpha + \beta$ in this ordered basis is $a_i + b_i$. Likewise the i^{th} coordinate of $c\beta$ is cb_i.

It is easily noted that for every set of coordinates for a vector space V, there corresponds a set of n-tuples from $V_n(F)$. For the vector $\sum_{i=1}^{n} a_i\ \beta_i$ there corresponds the n-tuple $(a_1, a_2, \ldots, a_n)$.

To describe this type of relation between any two vector spaces the following definition is given.

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[1] Kenneth Hoffman and Ray Kunze, Linear Algebra, p.p. 49-52

<u>Definition 11</u>. A function f on V to V', where both vector spaces are over the same field F, is called an isomorphism between V and V' if

1) f is a one-to-one correspondence,

2) f is a linear function, that is $f(a\alpha + b\beta) = af(\alpha) + bf(\beta)$ for all a, b in F and α, β in V.

The two vector spaces V and V' are said to be isomorphic if such a correspondence exists.

Since it was shown that the n-tuples which correspond to the sums and scalar multiples of vectors in a vector space V are preserved under the correspondence exhibited above and the correspondence is one-to-one, the following theorem is proved.

<u>Theorem 12</u>. A vector space of dimension n over F is isomorphic to $V_n(F)$.

<u>Theorem 13</u>. If V and V' are vector spaces of dimension n over F, each one-to-one correspondence between a basis for V and a basis for V' defines an isomorphism between V and V'. All isomorphisms on V to V' are obtainable in this way.

Proof. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta_1, \beta_2, \ldots, \beta_n$ be bases for V and V' respectively and f be a one-to-one correspondence between them such that $f(\alpha_i) = \beta_i$, $i = 1, 2, \ldots, n$. Now extend f to a one-to-one correspondence of V to V' by defining

$$f\left(\sum_{i=1}^{n} a_i \alpha_i\right) = \sum_{i=1}^{n} a_i f(\alpha_i) = \sum_{i=1}^{n} a_i \beta_i .$$

By definition 11 this is an isomorphism.

Conversely, if f is an isomorphism on V to V' and $\alpha_1, \alpha_2, \ldots, \alpha_n$ is a basis for V, set

$f(\alpha_i) = \beta_i$, $i = 1, 2, \ldots, n$. Since
$a_1 \alpha_1 + \ldots + a_n \alpha_n \neq \mathbb{O}$ and $f(\mathbb{O}) = \mathbb{O}$, then
$f(a_1 \alpha_1 + \ldots + a_n \alpha_n) = a_1 \beta_1 + \ldots + a_n \beta_n \neq \mathbb{O}$.
Hence the β_i's are linearly independent. Now let
$\alpha = a_1 \alpha_1 + \ldots + a_n \alpha_n$ be any bector in V. Then
$f(\alpha) = \beta = f(a_1 \alpha_1 + \ldots + a_n \alpha_n)$
$= a_1 f(\alpha_1) + \ldots + a_n f(\alpha_n)$ is a vector in V' corresponding to α in V. Since β can be any vector in V' and
$\beta = a_1 \beta_1 + \ldots + a_n \beta_n$ and the β_i's are linearly independent, the set $\beta_1, \beta_2, \ldots, \beta_n$ forms a basis for V'.
Since f is linear

$$f\left(\sum_{i=1}^{n} a_i \alpha_i\right) = \sum_{i=1}^{n} a_i f(\alpha_i) = \sum_{i=1}^{n} a_i \beta_i,$$

thus f is an isomorphism of the type above.

One might wonder at this point why some ordered basis of V is not selected and each vector described by its corresponding n-tuples of coordinates, since the operation with n-tuples is very convenient. This would defeat the purpose of working with abstract vector spaces for two reasons. First the axiomatic definition of vector spaces indicates the attempt to learn to reason with vectors as abstract algebraic systems. Second, even in those situations in which coordinates are used, the significant results follow from the ability to change the coordinate system, i. e., to change the ordered basis.

INNER PRODUCT SPACES

In this section only vector spaces taken over subsets of the complex fields will be considered. If this restriction is made then the concepts of "angle", "length", and "distance" take on meaning. Taking the field to be the complex field will result in a unitary vector space. The concept of angle will be developed in order to discuss perpendicularity of two vectors. The reader can easily see the application of the theorems to the special and familiar cases of two and three dimensional Euclidean space.

The concept and properties of an inner product will be developed first and then applications of this inner product will be made to vector spaces.

Definition 12. If F is a subfield of the field of complex numbers and V is a vector space over F, then the inner product on V is a function which assigns to an ordered pair of vectors α and β in V, a scalar $(\alpha \mid \beta)$ in F in such a way that

I_1. $(\alpha + \beta \mid \gamma) = (\alpha \mid \gamma) + (\beta \mid \gamma)$

I_2. $(a\alpha \mid \beta) = a(\alpha \mid \beta)$

I_3. $(\alpha \mid \beta) = \overline{(\beta \mid \alpha)}$, where $\overline{(\alpha \mid \beta)}$ is the complex conjugate of $(\alpha \mid \beta)$

I_4. $(\alpha \mid \alpha) > 0$ if $\alpha \neq \mathbb{O}$.

Conditions I_1, I_2 and I_3 imply

I_5. $(\alpha \mid a\beta + \gamma) = \bar{a}(\alpha \mid \beta) + (\alpha \mid \gamma)$.

Of course if F is the real field, the complex conjugates are superfluous. If F is the complex field, then the conjugates are necessary because the obvious contradiction to I_4,

$$(\alpha \mid \alpha) > 0 \text{ and } (i\alpha \mid i\alpha) = -1(\alpha \mid \alpha) > 0,$$

would exist.

Definition 13. If α is any vector in the vector space V, with an inner product defined on V, then the length of α is defined to be the non-negative square root

$$(\alpha \mid \alpha)^{\frac{1}{2}} = \|\alpha\| .$$

If the length of a vector is unity then the vector is normal.

Definition 14. The distance between two vectors α and β in a vector space V is $\|\alpha - \beta\|$.

One important example of the inner product is the inner product of $V_n(F)$, which is called the standard inner product. It is defined on $\alpha = (a_1, a_2, \ldots, a_n)$, $\beta = (b_1, b_2, \ldots, b_n)$ by

$$(\alpha \mid \beta) = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n .$$

This is often called the dot-product.

Attention is now turned to introducing some particular inner product to a vector space. Particular emphasis will be placed on perpendicularity.

Definition 15. An inner product space is a real or complex vector space together with a specified inner product on that space. An inner product space defined over the real field is an Euclidean space while if defined over the complex field it is a unitary space.

Theorem 14. If V is an inner product space, then for any two vectors α and β in V and scalar c in F

L_1. $\|c\,\alpha\| = |c|\ \|\alpha\|$

L_2. $\|\alpha\| > 0$ if $\alpha \neq \mathbb{O}$

L_3. $|(\alpha|\beta)| \leq \|\alpha\|\ \|\beta\|$

L_4. $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$.

Proof: Statements L_1 and L_2 follow directly from Definition 13. The inequality L_3 is obvious when $\alpha = \mathbb{O}$. When $\alpha \neq \mathbb{O}$ put

$$\gamma = \beta - \frac{(\beta|\alpha)}{\|\alpha\|^2}\,\alpha.$$

then

$$0 \leq \|\gamma\|^2 = \left(\beta - \frac{(\beta|\alpha)}{\|\alpha\|^2}\,\alpha \,\middle|\, \beta - \frac{(\beta|\alpha)}{\|\alpha\|^2}\,\alpha\right)$$

$$= (\beta|\beta) - \frac{(\beta|\alpha)}{\|\alpha\|^2}(\alpha|\beta) - \frac{\overline{(\beta|\alpha)}}{\|\alpha\|^2}(\beta|\alpha) + \frac{(\beta|\alpha)^2}{\|\alpha\|^4}(\alpha|\alpha)$$

$$= (\beta|\beta) - \frac{(\beta|\alpha)}{\|\alpha\|^2}\overline{(\beta|\alpha)} - \frac{(\beta|\alpha)\ (\beta|\alpha)}{\|\alpha\|^2} + \frac{(\beta|\alpha)^2\ \|\alpha\|^2}{\|\alpha\|^4}$$

$$= (\beta|\beta) - \frac{(\beta|\alpha)}{\|\alpha\|^2}\left[\overline{(\beta|\alpha)} - (\beta|\alpha) + (\beta|\alpha)\right]$$

$$= (\beta|\beta) - \frac{(\beta|\alpha)}{\|\alpha\|^2}\overline{(\beta|\alpha)} = \|\beta\|^2 - \frac{|(\alpha|\beta)|^2}{\|\alpha\|^2}.$$

Hence $|(\alpha|\beta)|^2 \leq \|\alpha\|^2\ \|\beta\|^2$ and

$$|(\alpha|\beta)| \leq \|\alpha\|\ \|\beta\|.$$

Now using L_3 and denoting the real part of a complex number x, by Re(x), it is found that

$$\|\alpha + \beta\|^2 = \|\alpha\|^2 + (\alpha|\beta) + (\beta|\alpha) + \|\beta\|^2$$
$$= \|\alpha\|^2 + 2\,\mathrm{Re}\left[(\alpha|\beta)\right] + \|\beta\|^2.$$

Since $\mathrm{Re}(x) \leq |\mathrm{Re}(x)| \leq |x|$ and by L_3, $|(\alpha|\beta)| \leq \|\alpha\|\,\|\beta\|$, then $\mathrm{Re}\left[(\alpha|\beta)\right] \leq \|\alpha\|\,\|\beta\|$. Thus

$$\|\alpha + \beta\|^2 \leq \|\alpha\|^2 + 2\,\|\alpha\|\ \|\beta\| + \|\beta\|^2$$
$$= (\,\|\alpha\| + \|\beta\|\,)^2.$$

Thus $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$, which proves L_4.

Remark. The inequality L_3 is known as the Cauchy-Schwartz inequality or just as Schwartz's inequality.

Theorem 15. In a unitary space distance has the following properties.

D_1. $\|\alpha - \beta\|$ 0 if $\alpha \neq \beta$

D_2. $\|\alpha - \beta\| = \|\beta - \alpha\|$

D_3. $\|\alpha - \beta\| + \|\beta - \gamma\| \geq \|\alpha - \gamma\|$.

Proof. Since $\|\alpha - \alpha\| = \|\theta\| = \|0\ \alpha\| = 0\ \|\alpha\| = 0$, by L_1 and $\|\alpha - \beta\| > 0$ if $\alpha \neq \beta$ by L_2, D_1 holds. The equation

$$\|\alpha - \beta\| = \|-1(\beta - \alpha)\| = |-1|\ \|\beta - \alpha\| = \|\beta - \alpha\|$$

proves D_2. Lastly, D_3 follows from L_3 since

$$\|\alpha - \beta\| + \|\beta - \gamma\| \geq \|\alpha - \beta + \beta - \gamma\| = \|\alpha - \gamma\|.$$

Orthogonality

It is convenient when discussing the angle between two vectors in the Euclidean plane, or using notation given earlier, $V_2(R)$, to consider their cosines. If the notation $\angle(\alpha|\beta)$ denotes the angle between nonzero vectors α and β, then applying the law of cosines gives

$$\|\alpha - \beta\|^2 = \|\alpha\|^2 + \|\beta\|^2 - 2\|\alpha\|\ \|\beta\| \cos\angle(\alpha|\beta).$$

Now

$$\|\alpha - \beta\|^2 - \|\alpha\|^2 - \|\beta\|^2$$
$$= ((\alpha-\beta)\,|\,(\alpha-\beta)) - (\alpha|\alpha) - (\beta|\beta)$$
$$= (\alpha - \beta|\alpha) - (\alpha - \beta|\beta) - (\alpha|\alpha) - (\beta|\beta)$$

$$= (\alpha|\alpha) - (\beta|\alpha) - (\alpha|\beta) + (\beta|\beta) - (\alpha|\alpha) - (\beta|\beta)$$
$$= -2\,(\alpha|\beta).$$

Then $\cos \measuredangle (\alpha|\beta) = \dfrac{(\alpha|\beta)}{\|\alpha\|\;\|\beta\|}$.

Also for $V_n(R)$ Schwartz's inequality can be written in the form

$$-1 \leq \frac{(\alpha|\beta)}{\|\alpha\|\;\|\beta\|} \leq 1.$$

The interest in these two illustrations lies in the fact that

$$\frac{(\alpha|\beta)}{\|\alpha\|\;\|\beta\|}$$

corresponds to one and only one cosine of an angle between 0 and π. This suggests a definition of perpendicularity of the two vectors α and β .

Definition 16. Two vectors α and β in a vector space V are said to be orthogonal, $\alpha \perp \beta$, (perpendicular) if their inner product is zero. If S is a set of nonzero vectors in V, S is called an orthogonal set provided any two distinct vectors in S are orthogonal. An orthonormal set has the added restriction that $\|\alpha\| = 1$ for all α in S.

Definition 17. Two vector spaces, V and V', are orthogonal if every vector contained in V is orthogonal to every vector contained in V'.

Orthogonality is a symmetric relation since $(\alpha|\beta) = 0$ implies $(\beta|\alpha) = 0$ by I_3. Next, if $\alpha \perp \beta$, then $a\alpha \perp b\beta$ for all scalars a and b. Moreover, if $\alpha \perp \beta$ and $\alpha \perp \gamma$ then $\alpha \perp (\beta + \gamma)$. The following theorem is an immediate consequence of these facts.

Theorem 16. If in a unitary space every member of the set $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ is orthogonal to every member of the set $\{\beta_1, \beta_2, \ldots, \beta_s\}$, then the space spanned by the α_i's is orthogonal to that space spanned by the β_j's.

Theorem 17. If $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is an orthogonal set of vectors then it is linearly independent.

Proof: If $a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n = \mathbb{O}$ then $(\alpha_i \mid a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n) = 0$ and $a_1(\alpha_i \mid \alpha_1) + a_2(\alpha_i \mid \alpha_2) + \ldots + a_n(\alpha_i \mid \alpha_n) = 0$. Since $(\alpha_i \mid \alpha_j) = 0$ when $i \neq j$, then $a_i(\alpha_i \mid \alpha_i) = 0$. Now since $(\alpha_i \mid \alpha_i) > 0$, $a_i = 0$ and hence the α_i's are linearly independent.

Corollary 1. If β is a vector which is a linear combination of an orthogonal set of vectors $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$, then β is the linear combination.

$$\beta = \sum_{k=1}^{m} \frac{(\beta \mid \alpha)}{\|\alpha_k\|^2}\alpha_k.$$

Proof. If $\beta = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_m\alpha_m$ then

$$(\beta \mid \alpha_k) = a_1(\alpha_1 \mid \alpha_k) + a_2(\alpha_2 \mid \alpha_k) + \ldots + a_m(\alpha_m \mid \alpha_k).$$

Since $\|\alpha_k\| = (\alpha_k \mid \alpha_k)^{\frac{1}{2}}$ and $(\alpha_i \mid \alpha_j) = 0$ if $i \neq j$, then $a_k = \dfrac{(\beta \mid \alpha_k)}{\|\alpha_k\|^2}$.

Hence

$$\beta = \sum_{k=1}^{m} \frac{(\beta \mid \alpha_k)}{\|\alpha_k\|^2}\alpha_k.$$

Corollary 2. If $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ is an orthogonal set in an inner product space V, then $m \leq d[V]$.

This corollary is an obvious consequence of the above theorem and the definition of the dimension of a vector space.

Theorem 18. Every inner product space has an orthonormal basis.

Proof. Let V be an inner product space and $\beta_1, \beta_2, \ldots, \beta_m$ be a basis for V. To obtain an orthogonal basis a construction called the Gram-Schmidt orthogonalization process is used.

First let $\alpha_1 = \beta_1$. Then set

$$\alpha_2 = \beta_2 - \frac{(\beta_2 | \alpha_1)}{\|\alpha_1\|^2} \alpha_1.$$

Since β_1, β_2 are linearly independent, $\alpha_2 \neq 0$ and since

$$(\alpha_2 | \alpha_1) = \left(\beta_2 - \frac{(\beta_2 | \alpha_1)}{\|\alpha_1\|^2} \alpha_1 \middle| \beta_1\right)$$

$$= (\beta_2 | \beta_1) - \frac{(\beta_2 | \beta_1)}{\|\beta_1\|^2} (\beta_1 | \beta_1)$$

$$= 0,$$

$\alpha_2 \perp \alpha_1$.

Next let

$$\alpha_3 = \beta_3 - \frac{(\beta_3 | \alpha_1)}{\|\alpha_1\|^2} \alpha_1 - \frac{(\beta_3 | \alpha_2)}{\|\alpha_2\|^2} \alpha_2.$$

Then $\alpha_3 \neq 0$, for if it were, β_3 is a linear combination of β_2 and β_1; furthermore $(\alpha_3 | \alpha_1) = (\alpha_3 | \alpha_2) = 0$. Now suppose nonzero orthogonal vectors $\alpha_1, \alpha_2, \ldots, \alpha_k$ have been constructed in such a way that α_j is β_j minus some linear combination of $\beta_1, \beta_2, \ldots, \beta_{j-1}$ for $1 \leq j \leq k$. Let

$$\alpha_{k+1} = \beta_{k+1} - \sum_{j=1}^{k} \frac{(\beta_{k+1} | \alpha_j)}{\|\alpha_j\|^2} \alpha_j.$$

Then $(\alpha_{k+1} | \alpha_i) = (\beta_{k+1} | \alpha_i) - \sum_{j=1}^{k} \frac{(\beta_{k+1} | \alpha_j)}{\|\alpha_j\|^2} (\alpha_j | \alpha_i)$.

Since $(\alpha_j | \alpha_i) = 0$ when $i \neq j$, by induction,

$$(\alpha_{k+1} | \alpha_i) = (\beta_{k+1} | \alpha_i) - (\beta_{k+1} | \alpha_i)$$

$$= 0, \quad \text{for } 1 \leq i \leq k.$$

Thus α_{k+1} is orthogonal to each of the vectors $\alpha_1, \alpha_2, \ldots, \alpha_k$. Suppose $\alpha_{k+1} = \mathbb{O}$. Then β_{k+1} is a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_k$ and hence of $\beta_1, \beta_2, \ldots, \beta_k$. Thus $\alpha_{k+1} \neq \mathbb{O}$. Ultimately an orthogonal set of n vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$, is obtained. By Theorem 17, this set is independent and hence a basis. To obtain an orthonormal basis, replace α_i by $\frac{\alpha_i}{\|\alpha_i\|}$.

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PROPERTIES OF ABSTRACT VECTOR SPACES

by

DONALD HUDSON MABRY

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Often in mathematics there are systems which appear to be quite different and yet have properties which are common to all. In order to bring such systems under a single heading, one must extract all properties common to all the systems and list these as postulates for an otherwise unrestricted system.

In this report such a system has been defined. The postulates appearing in the definition of the vector space are not to be considered as self-evident truths or statements which cannot be proved, but rather as assumed properties. The vector space V, consists of elements and operations, assumptions about both, and finally consequences or theorems derived from the assumptions. Whenever a system satisfies the postulates given in the definition, then the theorems about elements of V can be applied to this new system.

Linear combinations of elements are found frequently in mathematics and are particularly useful in studying properties of vector spaces. Such linear combinations are defined in this report and from this results the discussions of linear dependence and linear independence of vectors. This leads to the concept of a basis for a vector space and the resulting properties of bases. Closely connected to the concept of a basis is the concept of dimension of a vector space. These properties all have application to subspaces of a vector space.

It is often desirable to change from one basis of a vector space to another. This involves describing a vector in terms of its coordinates relative to a given basis. A description of how to describe these coordinates is given, but since the actual calculation involves the use of matrices, a complete discussion is not given. A direct consequence of a change of basis is the isomorphism between two vector spaces.

The last topic discussed is that of inner product spaces. In this section, abstract concepts of length, distance and angle are defined and discussed. The concept of angle is used only to introduce perpendicularity, or orthogonality, of two vectors. A special case of length is noted. This is the case of the normal vector, that is one which has length one. Applying both of the properties of normality and orthogonality, it is found that it is possible to construct an orthonormal basis for any inner product space. The method of construction concludes the report.