ANALYSIS OF AN ELONGATED SPLIT-RING DYNAMOMETER

by

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NOMENCLATURE

A	area of cross-section of the ring
α	numerical factor used to obtain the shearing stress; = 1.5 for rectangular cross-section
b	width of the ring
β	dimensionless quantity, $= \frac{h}{r}$
χ	sensitivity ratio
δ_V, δ_H	vertical and horizontal deflections respectively of a quadrant of the ring
$\delta(y)$	increase in fiber length after application of load
Δ_V, Δ_H	vertical and horizontal deflections respectively of the whole ring
Δ_ψ, Δ_γ	angle between end sections of the curved beam before and after application of load respectively
e	distance of the neutral axis from the centroidal axis
E,G	modulus of elasticity and modulus of rigidity respectively
η	dimensionless quantity, $= \frac{\ell}{r}$
$\varepsilon(y)$	strain or unit elongation of the fiber
h	thickness of the ring
H	horizontal component of cutting force
I	area moment of Inertia about the centroidal axis in the plane of the width of the ring
ℓ	length of the straight beam
L	$= \dfrac{\left[\frac{\ell r}{I} + \frac{\ell^2}{2I} - \frac{r}{Ae} + \frac{1}{A} + \frac{\pi r}{2Ae}\right]}{\left[\frac{\ell}{I} + \frac{\pi}{2Ae}\right]}$

λ	function of β, $= 1 - \dfrac{\beta}{\ell n\left(\dfrac{2+\beta}{2-\beta}\right)}$
m	dimensionless parameter; $= \dfrac{e}{r - e}$
M_c, M_A, M_B	bending moments at the fixed end of the cross section
N, V, M	normal force, shear force, and bending moment respectively at any section
ω_1, ω_2	change in angle per unit of angle
ψ	stiffness ratio
P, Q, M_o	normal force, shear force, and bending moment respectively, at the section indicated in figure
r, ϕ	polar co-ordinates to refer points on the centroidal axis of the curved portion of the ring
r_2, r_1	outer and inner radius of the curved portion of the ring
R	radius of curvature after application of load
s	distance along the centroidal axis of the ring
σ_H	stress at the outer fiber of the ring subjected to the horizontal component of the cutting force
σ_{Vo}, σ_{Vi}	stress at the outer and inner fiber respectively of the ring subjected to the vertical component of the cutting force
θ_V, θ_{HR}	angular position of the cross-section in the curved portion of the ring
θ_{Vo}	angular position of the cross-section in the curved portion of the ring for $\sigma_{Vo} = 0$
U	total strain energy
U_1, U_2	strain energy of the straight and curved portions respectively of the ring
V	vertical component of cutting force
V_{max}, H_{max}	maximum vertical and horizontal component of cutting force

y normal distances of the stressed fiber from the centroidal axis

INTRODUCTION

Accurate measurement of the forces generated in cutting metals or refractories is needed for rational design of machine tools and cutting tools. Also, valuable information concerning machinability and tool wear may be obtained from the force data. The cutting forces are usually measured by means of a tool dynamometer.

Machine tool dynamometers typically contain an elastic member such as a metal column or beam, which deforms under applied force. The deformation is transmitted to strain gauges, located at suitable places on the elastic member. The strain gauges are usually connected in a four-arm wheatstone bridge to produce an output signal, which when calibrated, is an accurate indication of the applied force.

A particularly popular type of dynamometer used in practice is a split-ring dynamometer, which utilizes load rings as shown schematically in Figure 1. For analysis purposes, the circular ring is substituted for the octagonal ring as shown in Figure 2a. The active part of the dynamometer is shown schematically in Figure 2b. A split-ring dynamometer with strain gauge locations and four-arm wheatstone bridge wiring diagrams is shown schematically in Figures 3a and 3b. A three-dimensional version of the split-ring dynamometer using the type of load ring discussed, was used by B. King and R. O. Foschi [1]* to obtain the three orthogonal components of a cutting force.

As yet, a single dynamometer has not been designed which can be used for the measurement of desired forces on all machines (lathes, milling machines, drilling machines, grinding machines, etc.), as the forces to be

*Numbers in brackets designate references at the end of report.

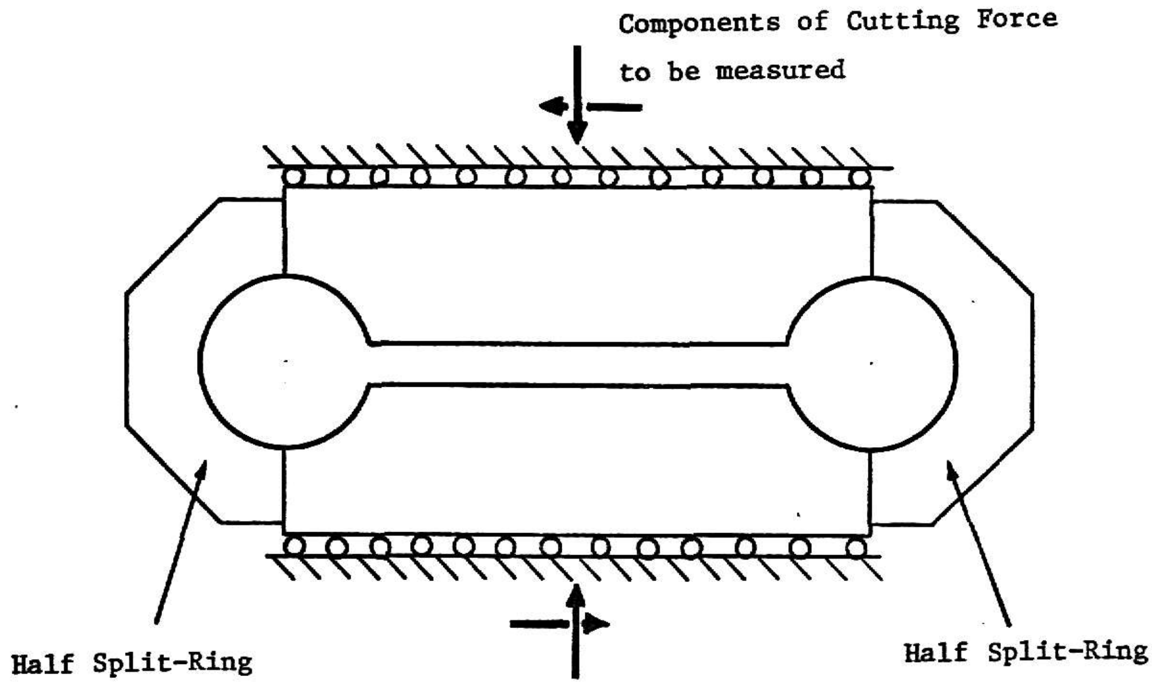

Components of Cutting Force
to be measured
Half Split-Ring
Half Split-Ring

Fig. 1

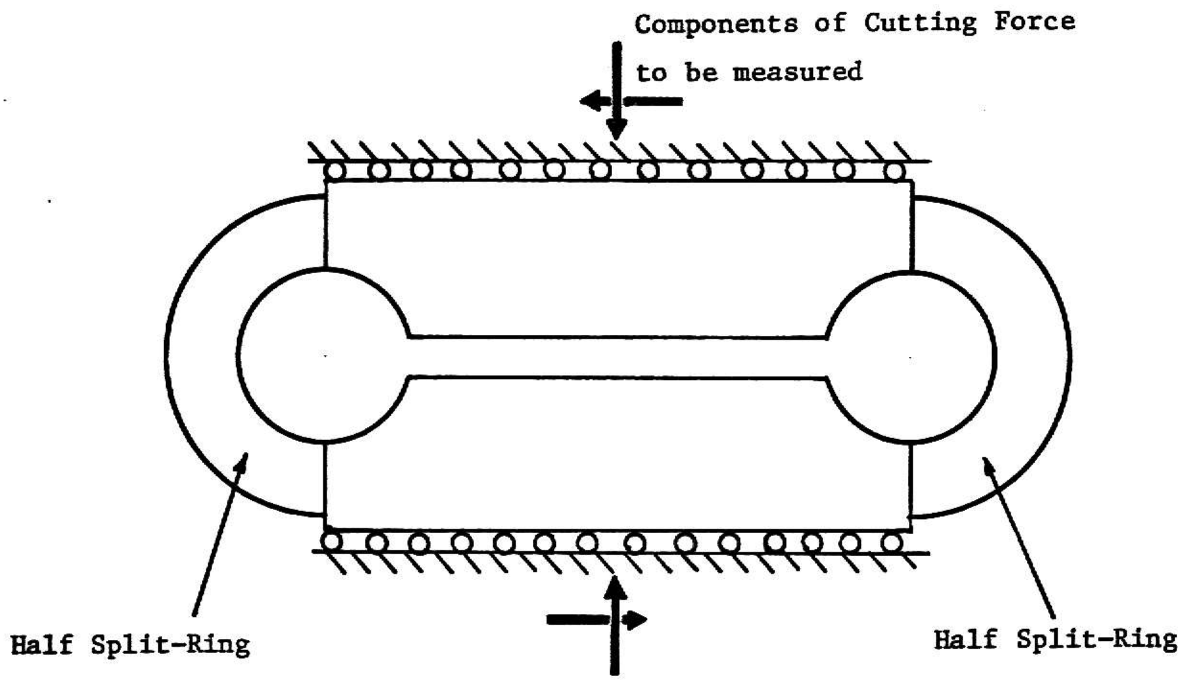

Components of Cutting Force
to be measured
Half Split-Ring
Half Split-Ring

Fig. 2a

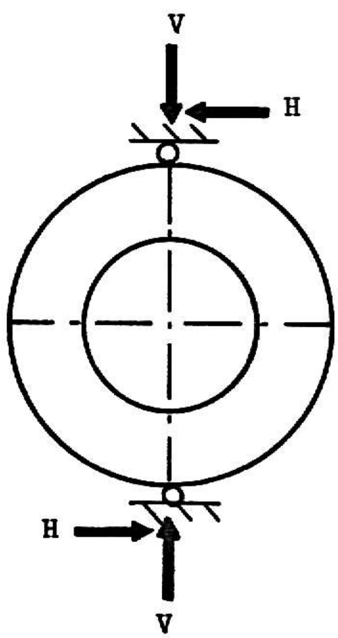

V
H
H
V

Fig. 2b

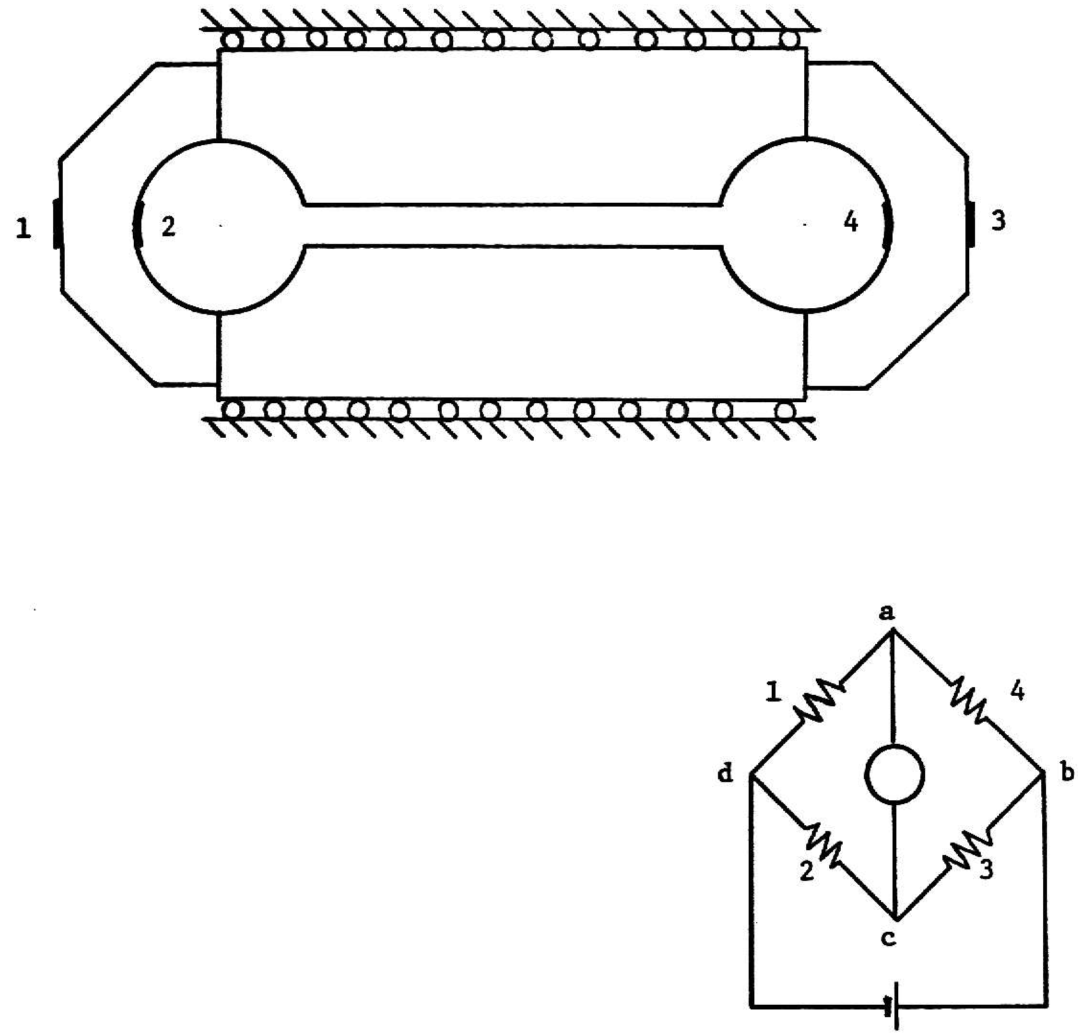


Fig. 3a Wiring Diagrams for Vertical Force Gages

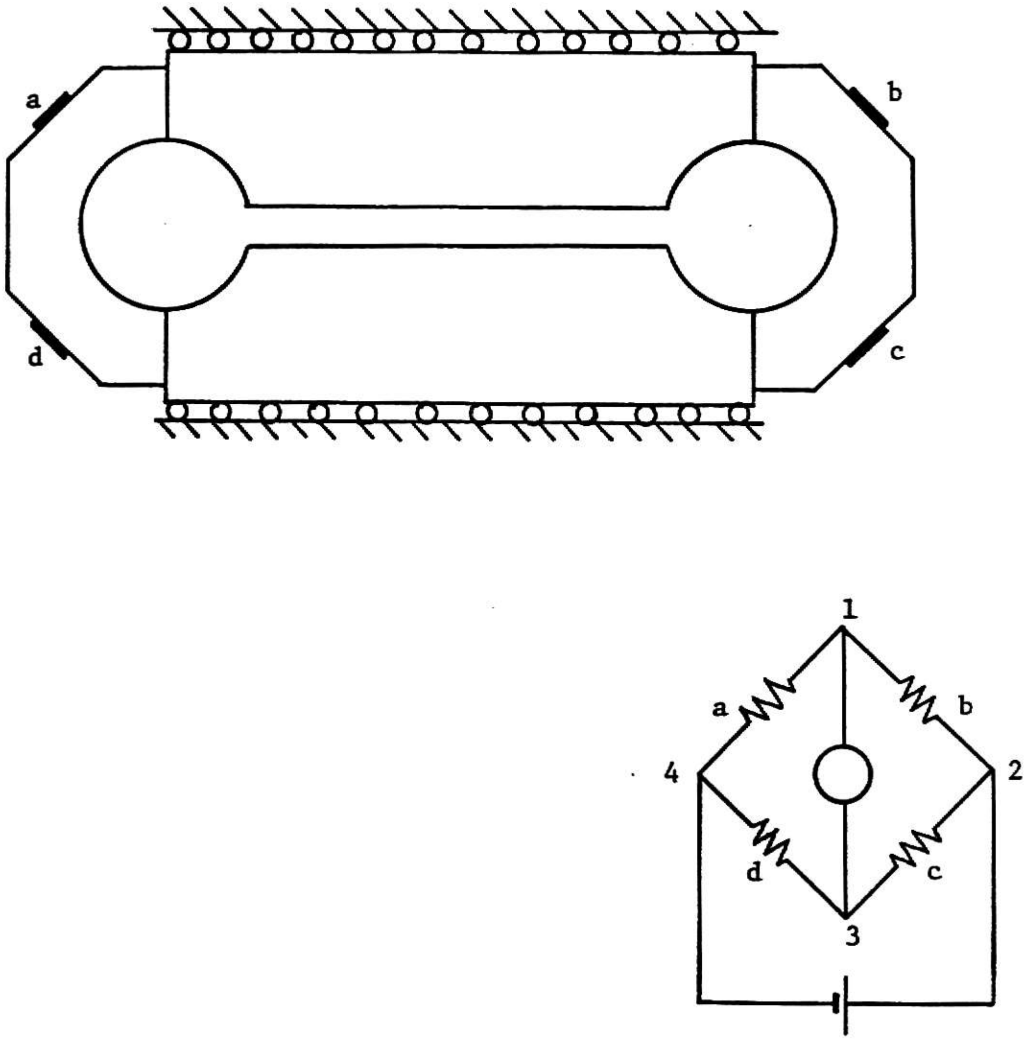


Fig. 3b Wiring Diagrams for Horizontal Force Gages

measured vary in magnitude and direction from one machine to another. Therefore, the type of machine tool and the magnitude of the forces to be measured must be considered in the design of a dynamometer for a specific application. Thus each dynamometer should be individually designed to meet the requirements of the situation.

In some applications, design requirements for tool dynamometers are unusually severe. Maximum stiffness is required to avoid interfering with normal operation of the cutting tool, but this tends to reduce the sensitivity of the dynamometer which makes it difficult to accurately measure small forces. Therefore in designing dynamometers, a compromise must be made between stiffness and sensitivity. This is controlled by the geometrical design of the dynamometer in conjunction with the sensitivity of the strain gauges. Thus there is a need for general design criteria which may be used in determining the most suitable dynamometer design.

The split-ring dynamometer does not satisfy the requirements for some applications since, as will be seen later, the range of the stiffness and sensitivity ratios of vertical and horizontal loading is restricted between approximately 0.2-0.5 and 0.55-1.3 respectively. For some applications a higher stiffness or sensitivity ratio is desired. To improve the split-ring dynamometer, an elongated split-ring dynamometer is considered in this report. The elongated split-ring dynamometer, as shown in Figure 4, differs from the split-ring dynamometer in that it has straight beams included with the split-rings. Within practical limits for the dimensions of the ring, a higher range of stiffness and sensitivity ratio can be obtained. This means that there is more flexibility in the design.

In the analysis of the elongated split-ring, equations for the deflections,

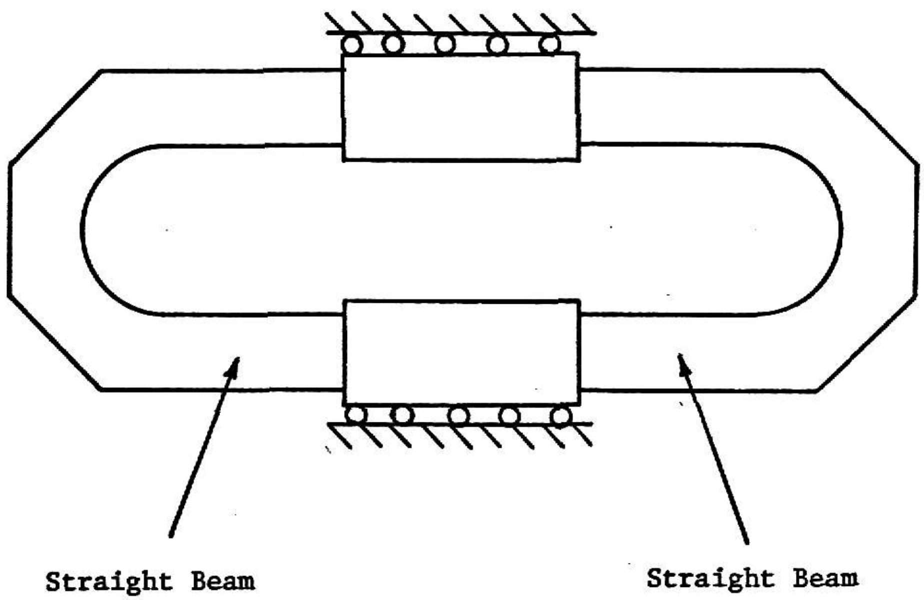


Fig. 4

stresses, stiffness, and sensitivity are set up for each component of force on the ring. The equations derived are based on curved beam theory and Castigliano's theorem for deflection. Graphs are presented for different stiffness and sensitivity ratios between horizontal and vertical loading with respect to two non-dimensional quantities, β and η. A computer program is written to solve the equation set up for the dynamometer design. Finally a brief discussion of the results is presented.

ANALYSIS OF ELONGATED SPLIT-RING DYNAMOMETER

Formulation of the Problem

For the elongated split-ring dynamometer shown in Figure 5, V and H are the vertical and horizontal components of a cutting force to be measured. It is assumed that members E and F are rigid and that relative motion can occur in the vertical or horizontal direction but that no relative angular displacement occurs.

In designing the dynamometer it is proposed to find an expression for stiffness and sensitivity, for each of the loads V and H, in terms of the dimensions of the load ring. The vertical stiffness is defined as the vertical load required for a unit vertical deflection and the vertical sensitivity is defined as the strain output measured due to a unit vertical load.

Load Ring Analysis

Consider one quarter of the load ring as shown in Figure 6a. The ring is assumed to be made of an elastic material and is subjected to forces P, Q and M_o. s represents the distance along the centroidal axis of the ring. The length of the straight portion is ℓ. A polar co-ordinate system together with the length ℓ will be used to denote s for the curved portion of the ring.

To find an expression for the stiffness it is required to find the deflections Δ_V and Δ_H as indicated in Figure 6a. Castigliano's theorem for deflection is used to find Δ_V and Δ_H. To use this theorem it is necessary to determine the total strain energy for the load ring.

At the section located at s, the tensile force N, the shear force V, and the bending moment M are as shown in Figure 6b. Referring to (69) in

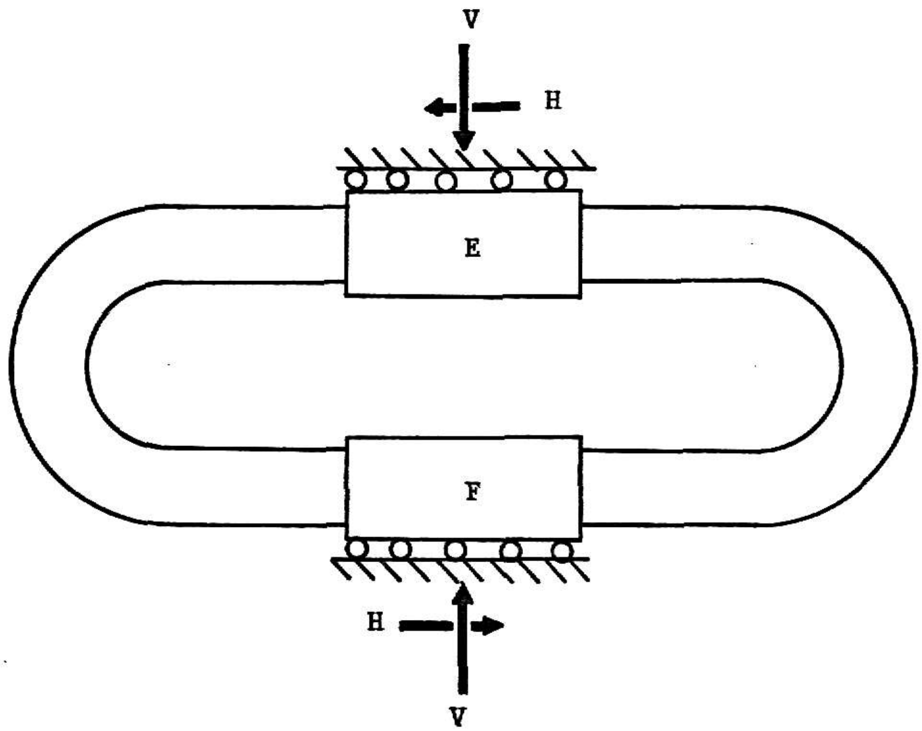


Fig. 5

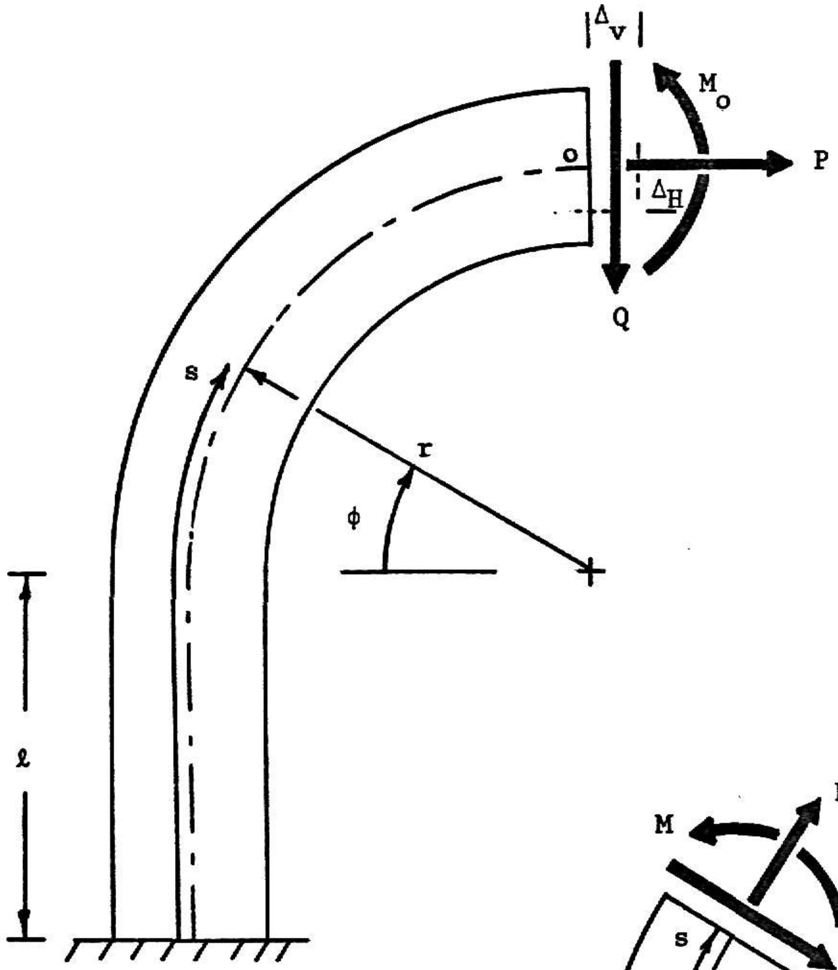


Fig. 6a

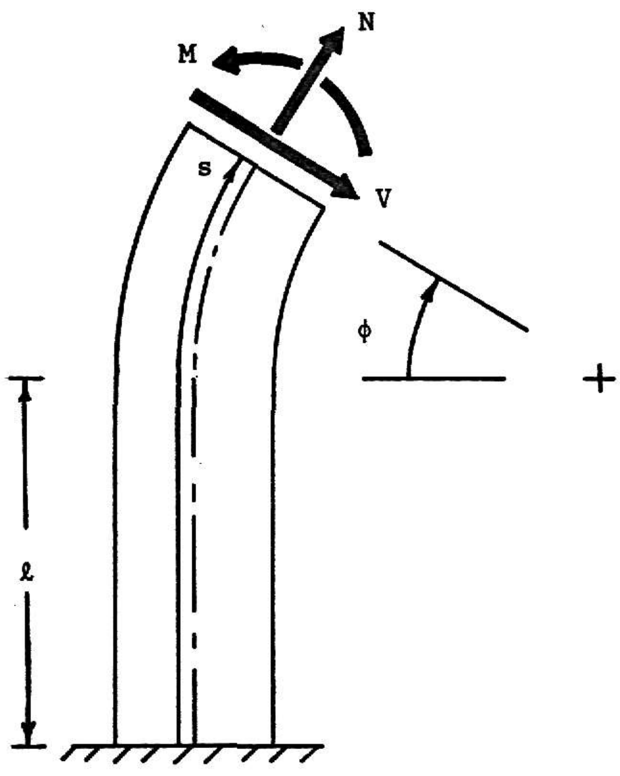


Fig. 6b

Appendix A, for an increment $r\Delta\Psi$ of a curved beam the strain energy has been shown to be

$$dU = \left(\frac{N^2}{2AE} - \frac{MN}{rAE} + \frac{M^2}{2reAE} + \frac{\alpha V^2}{2AG}\right) r\Delta\Psi \tag{1}$$

This form is directly applicable to the curved section of the ring, but must be modified for the straight portion where the radius of curvature becomes infinite.

It is necessary to determine the limit of the quantity [reA] as $r \to \infty$.

Referring to (59) in Appendix A,

$$e = \frac{\int_A \frac{y}{r - y}\, dA}{\int_A \frac{1}{r - y}\, dA} \tag{2}$$

and,
$$r\int_A \frac{y}{r - y}\, dA = \int_A \frac{y^2}{r - y}\, dA \tag{3}$$

Combining (2) and (3), gives,

$$rAe = \frac{rA \int_A \frac{y}{r - y}\, dA}{\int_A \frac{1}{r - y}\, dA} \tag{4a}$$

Substituting (3) into (4a),

$$[rAe] = \frac{A \int_A \frac{y^2}{r - y}\, dA}{\int_A \frac{1}{r - y}\, dA} \tag{4b}$$

Multiplying numerator and denominator by r,

$$[\, rAe] = \frac{A \int_A \frac{y^2}{1 - \frac{y}{r}} dA}{\int_A \frac{1}{1 - \frac{y}{r}} dA} \tag{4c}$$

Taking the limit of each side as $r \to \infty$,

$$\lim_{r \to \infty} [\, rAe] = \frac{A \int_A y^2 dA}{\int_A dA} \tag{5}$$

By the definition of area and moment of inertia of the area,

$$\int_A dA = A \tag{6a}$$

and,
$$\int_A y^2 dA = I \tag{6b}$$

Therefore,

$$\lim_{r \to \infty} [rAe] = I \tag{7}$$

This is in agreement with the straight beam theory. Using (1) and (7) it is possible to write the expression for the total strain energy of the straight portion of the ring as,

$$U_1 = \int_0^{\ell} \left(\frac{N^2}{2AE} + \frac{M^2}{2EI} + \frac{\alpha V^2}{2AG} \right) ds \tag{8a}$$

The total strain energy for the curved portion of the ring is

$$U_2 = \int_0^{\pi/2} \left(\frac{N^2}{2AE} - \frac{MN}{rAE} + \frac{M^2}{2reAE} + \frac{\alpha V^2}{2AG}\right) r d\phi \tag{8b}$$

The total strain energy then is,

$$U = U_1 + U_2 \tag{9}$$

From equilibrium, the forces N, V, and M at any cross-section are given as follows,

For the straight portion

$$M(s) = M_o - P(\ell + r - s) - Qr \qquad o \leq s \leq \ell \tag{10a}$$

$$N(s) = - Q \qquad o \leq s \leq \ell \tag{10b}$$

$$V(s) = P \qquad o \leq s \leq \ell \tag{10c}$$

$$r(s) = \infty \qquad o \leq s \leq \ell \tag{10d}$$

For the curved portion

$$M(\phi) = M_o - P(r - r \operatorname{Sin}\phi) - Qr \operatorname{Cos}\phi \qquad o \leq \phi \leq \frac{\pi}{2} \tag{11a}$$

$$N(\phi) = - Q \operatorname{Cos}\phi + P \operatorname{Sin}\phi \qquad o \leq \phi \leq \frac{\pi}{2} \tag{11b}$$

$$V(\phi) = Q \operatorname{Sin}\phi + P \operatorname{Cos}\phi \qquad o \leq \phi \leq \frac{\pi}{2} \tag{11c}$$

$$r(\phi) = r \qquad o \leq \phi \leq \frac{\pi}{2} . \tag{11d}$$

Using (8a),

$$U_1 = \int_0^{\ell} \left(\frac{N^2(s)}{2AE} + \frac{M^2(s)}{2EI} + \frac{\alpha V^2(s)}{2AG}\right) ds \tag{12}$$

Using (10a), (10b), (10c),

$$M^2(s) = \{M_o - P(\ell + r) - Qr\}^2$$

$$+ 2Ps\,\{M_o - P(\ell + r) - Qr\} + P^2 s^2 \qquad o \le s \le \ell \qquad (13a)$$

$$N^2(s) = Q^2 \qquad o \le s \le \ell \qquad (13b)$$

$$V^2(s) = P^2 \qquad o \le s \le \ell \qquad (13c)$$

Substituting (13a), (13b), (13c) in (12),

$$U_1 = \int_o^{\ell} \Big[\frac{Q^2}{2AE} + \frac{\{M_o - P(\ell + r) - Qr\}^2}{2EI}$$

$$+ \frac{P\{M_o - P(\ell + r) - Qr\}}{EI}\, s \qquad (14a)$$

$$+ \frac{P^2}{2EI} s^2 + \frac{\alpha P^2}{2AG}\Big]\, ds$$

Solving (14a), gives,

$$U_1 = \frac{\ell}{2E}\Big[\frac{Q^2}{A} + \alpha \frac{E}{G}\frac{P^2}{A} + \frac{P^2 \ell^2}{3I}$$

$$+ \frac{\{M_o - r(P + Q)\}\{M_o - P(\ell + r) - Qr\}}{I}\Big] \qquad (14b)$$

using (8b),

$$U_2 = \int_o^{\pi/2} \Big\{ \frac{N^2(\phi)}{2AE} - \frac{M(\phi)\, N(\phi)}{rAE} + \frac{M^2(\phi)}{2reAE} \qquad (15)$$

$$+ \frac{\alpha V^2(\phi)}{2AG}\Big\}\, rd\,\phi$$

Using (11a), (11b), and (11c) gives,

$$\int_0^{\pi/2} M^2(\phi)\,d\phi = \frac{\pi}{2}(M_o - Pr)^2 + 2(M_o - Pr)r[P-Q] + \frac{\pi}{4} r^2P^2 - PQr^2 + \frac{\pi}{4}r^2Q^2 \tag{16a}$$

$$\int_0^{\pi/2} N^2(\phi)\,d\phi = \frac{\pi}{4}Q^2 - PQ + \frac{\pi}{4} P^2 \tag{16b}$$

$$\int_0^{\pi/2} M(\phi)N(\phi)\,d\phi = M_o(P-Q) - \left(\frac{4-\pi}{4}\right)P^2r + \frac{\pi}{4}Q^2r \tag{16c}$$

$$\int_0^{\pi/2} V^2(\phi)\,d\phi = \frac{\pi}{4} Q^2 + PQ + \frac{\pi}{4} P^2 \tag{16d}$$

Substituting (16a),(16b), (16c), and (16d) in (15) gives,

$$U_2' = \frac{r}{2AE}\Big\{\frac{\pi}{4}(P^2 + Q^2)\left(-1 + \frac{r}{e} + \frac{\alpha E}{G}\right) + P^2\left(\frac{\pi r}{2e} - \frac{2r}{e} + 2\right) - PQ\left(1 - \frac{r}{e} - \frac{\alpha E}{G}\right) -2(P-Q)\frac{M_o}{r} + (2P - \pi P - 2Q)\frac{M_o}{e} + \frac{\pi}{2re}M_o^2\Big\} \tag{17}$$

Adding (14b) and (17), and rearranging terms one obtains the expression for the total strain energy for one quarter of the elongated ring, subjected to forces P,Q, and M_o as,

$$U = \frac{Q^2\ell}{2AE} + \frac{\ell}{2EI}\{M_o - r(P + Q)\}\{M_o - r(P + Q) - P\ell\}$$
$$+ \frac{P^2\ell^3}{6EI} + \frac{\alpha P^2\ell}{2AG} + \frac{\pi r(P^2 + Q^2)}{8AE}\left(-1 + \frac{\alpha E}{G} + \frac{r}{e}\right)$$
$$+ \frac{PQr}{2AE}\left(1 + \frac{\alpha E}{G} - \frac{r}{e}\right)$$
$$+ \frac{(M_o - Pr)}{AE}\left\{\left(\frac{r}{e} - 1\right)(P - Q) + \frac{\pi}{4e}(M_o - Pr)\right\} \tag{18}$$

The above expression for strain energy is used for two specific cases, one with the load ring subjected to vertical load only and the other with the load ring subjected to horizontal load only. The two cases are discussed below.

Case 1: Load Ring Subjected to a Vertical Load V

From the condition of symmetry, the elongated load ring subjected to a vertical load V as shown in Figure 7a, can be split into four quadrants as shown in Figure 7b. The distribution of stress in any one quadrant is the same as in the other quadrants. Since the problem is statically indeterminate it is required to find the unknown moment M_o.

Considering one quadrant of the load ring, the forces acting are as shown in Figure 7c.

From statics,

$$M_c = \frac{V}{2}(\ell + r) - M_o \tag{19}$$

Putting $P = -\frac{V}{2}$, $Q = 0$, and $M_o = -M_o$ in equation (18) gives the total strain energy for the present case as

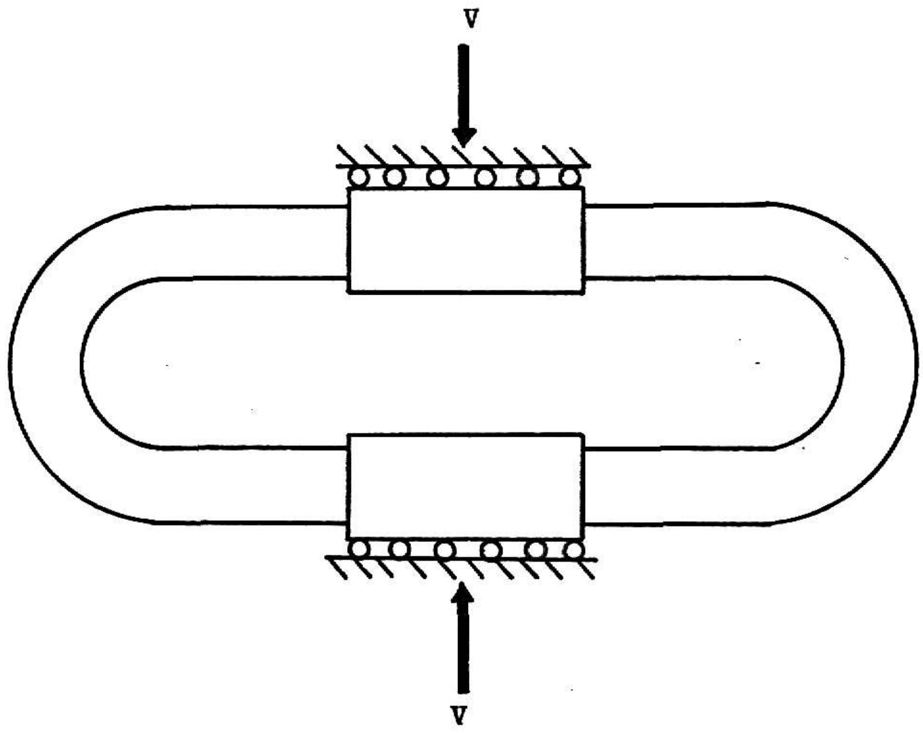


Fig. 7a

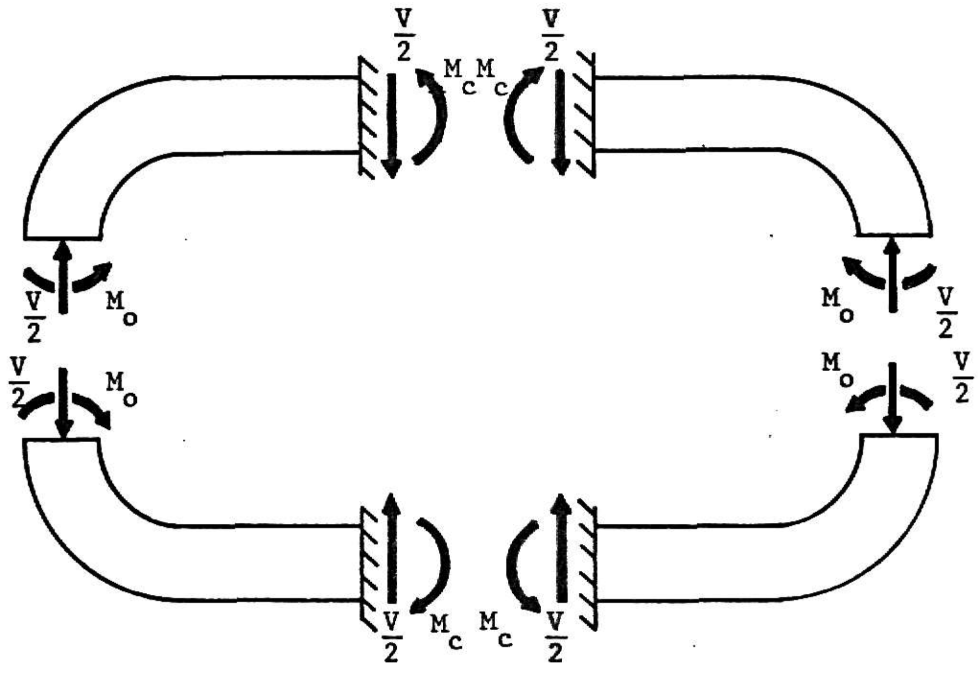


Fig. 7b

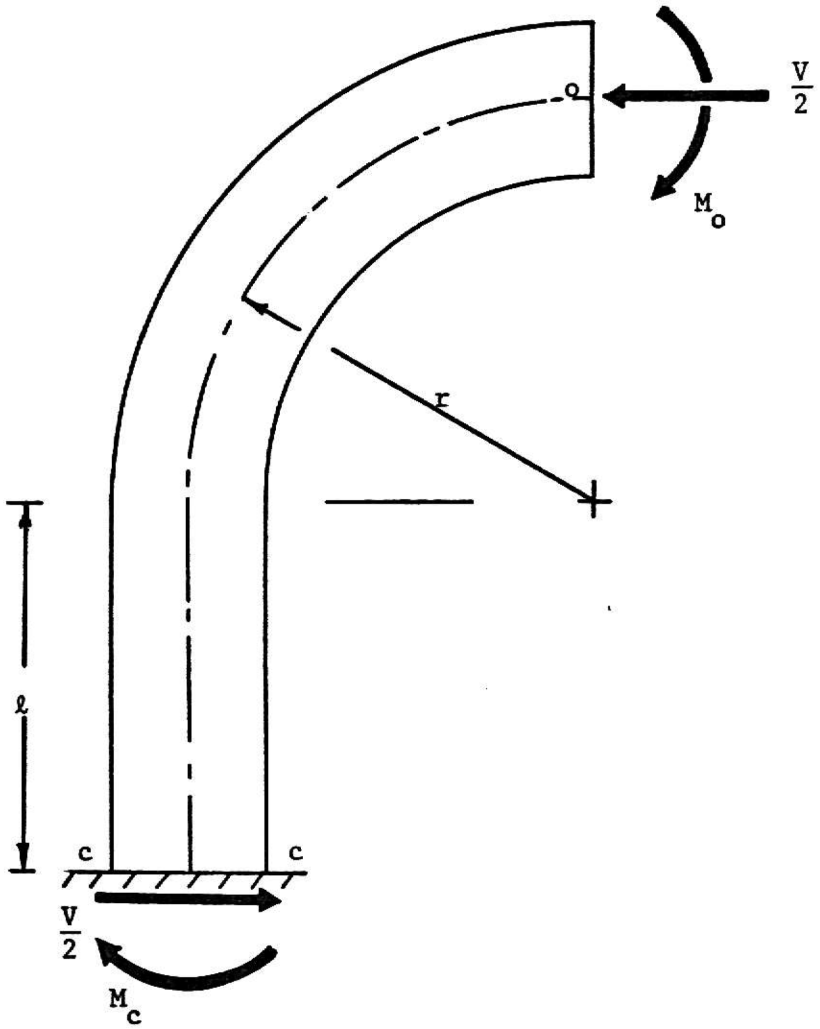


Fig. 7c

$$U = \frac{\ell}{2EI}\left(-M_o + \frac{Vr}{2}\right)\left(-M_o + \frac{Vr}{2} + \frac{V\ell}{2}\right) + \frac{V^2\ell^3}{4(6EI)} \tag{20}$$

$$+ \frac{\alpha V^2\ell}{4(2AG)} + \frac{\pi r\left(\frac{V^2}{4}\right)}{8AE}\left(-1 + \frac{\alpha E}{G} + \frac{r}{e}\right)$$

$$+ \frac{\left(-M_o + \frac{Vr}{2}\right)}{AE}\left\{\left(\frac{r}{e} - 1\right)\left(-\frac{V}{2}\right) + \frac{\pi}{4e}\left(-M_o + \frac{Vr}{2}\right)\right\}$$

The statically indeterminate moment M_o is determined from the condition that the cross-section on which this moment acts does not rotate. That is, $\frac{dU}{dM_o} = 0$.

Differentiating (20) with respect to M_o and equating to zero gives

$$\frac{dU}{dM_o} = \frac{\ell}{2EI}\left\{(-1)\left(-M_o + \frac{Vr}{2} + \frac{V\ell}{2}\right) + \left(-M_o + \frac{Vr}{2}\right)(-1)\right\} \tag{21}$$

$$+ \frac{1}{AE}\left[(-1)\left\{\left(\frac{r}{e} - 1\right)\left(-\frac{V}{2}\right) + \frac{\pi}{4e}\left(-M_o + \frac{Vr}{2}\right)\right\}\right.$$

$$\left. + \left(-M_o + \frac{Vr}{2}\right)\left(-\frac{\pi}{4e}\right)\right] = 0$$

Solving for M_o gives

$$M_o = \frac{\frac{V}{2}\left(\frac{\ell r}{I} + \frac{\ell^2}{2I} - \frac{r}{Ae} + \frac{1}{A} + \frac{\pi r}{2Ae}\right)}{\left(\frac{\ell}{I} + \frac{\pi}{2Ae}\right)} \tag{22}$$

$$\text{Let } \frac{\left(\frac{\ell r}{I} + \frac{\ell^2}{2I} - \frac{r}{Ae} + \frac{1}{A} + \frac{\pi r}{2Ae}\right)}{\left(\frac{\ell}{I} + \frac{\pi}{2Ae}\right)} = L \tag{23}$$

Substituting the value of M_o in (20) gives

$$U = \frac{\ell}{2EI}\left(-\frac{VL}{2}+\frac{Vr}{2}\right)\left(-\frac{VL}{2}+\frac{Vr}{2}+\frac{V\ell}{2}\right)+\frac{V^2\ell^3}{4(6EI)} \tag{24}$$

$$+\frac{\alpha V^2\ell}{4(2AG)}+\frac{\pi r\left(\frac{V^2}{4}\right)}{8AE}\left(-1+\frac{\alpha E}{G}+\frac{r}{e}\right)$$

$$+\frac{\left(-\frac{VL}{2}+\frac{Vr}{2}\right)}{AE}\left\{\left(\frac{r}{e}-1\right)\left(-\frac{V}{2}\right)+\frac{\pi}{4e}\left(-\frac{VL}{2}+\frac{Vr}{2}\right)\right\}$$

According to Castigliano's theorem of deflection, the deflection δ_V, in the direction of the force $\frac{V}{2}$, of the point of application of the force at point o, is equal to the partial derivative, with respect to the force, of the total internal strain energy in the member.

Thus, differentiating (24) with respect to $\frac{V}{2}$ gives,

$$\delta_V = \frac{V\ell^3}{6EI}+\frac{\alpha V\ell}{2AG}+\frac{V\pi r}{8AE}\left(-1+\frac{\alpha E}{G}+\frac{r}{e}\right) \tag{25a}$$

$$+(r-L)\left\{\frac{V\ell^2}{2EI}+\frac{V}{AE}\left(1-\frac{r}{e}\right)\right\}+(r-L)^2\left(\frac{V\ell}{2EI}+\frac{V\pi}{4AEe}\right)$$

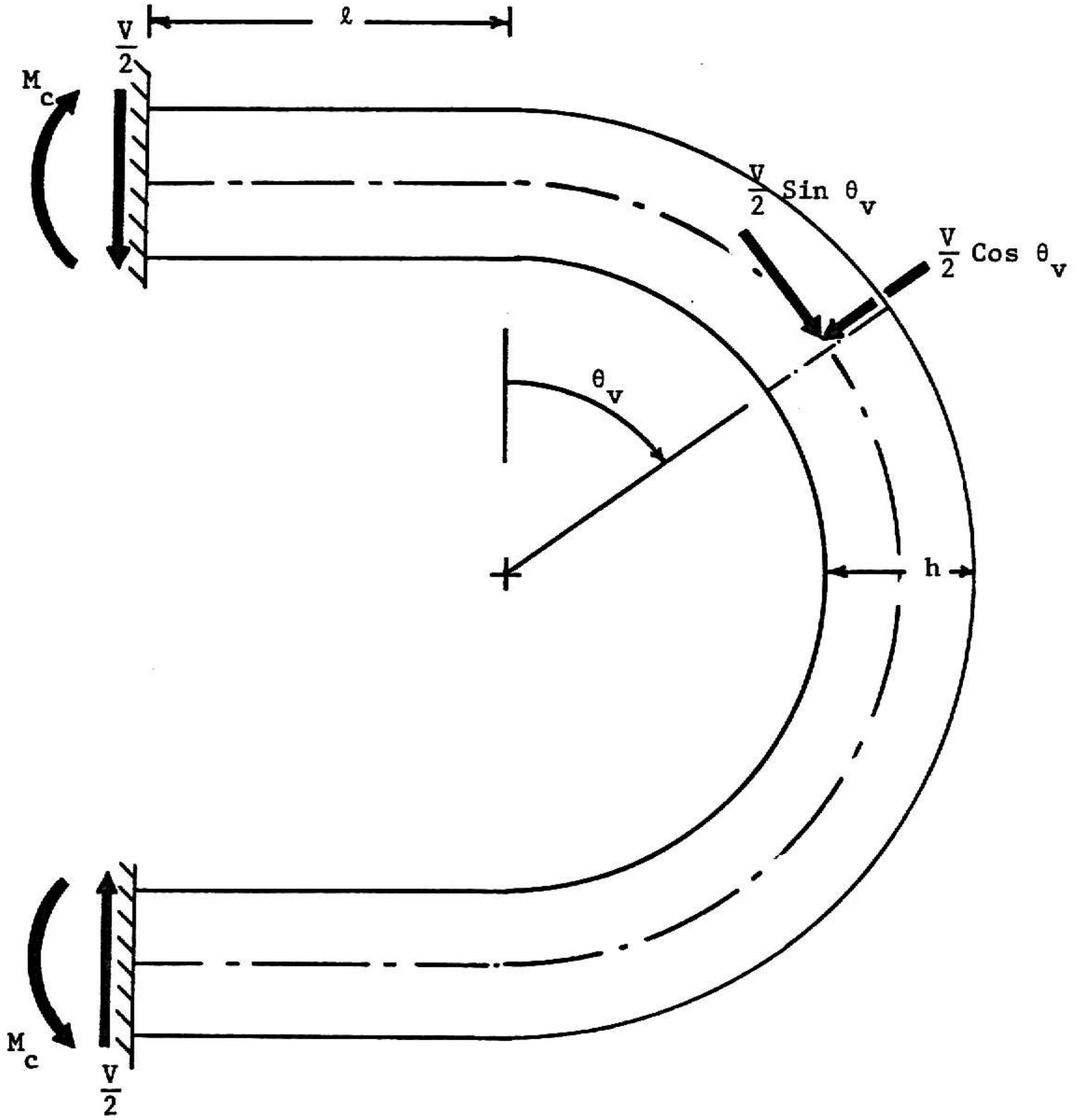
Equation (25a) gives deflection of only one side of the load ring under the load V in Figure 7a. Therefore, the total deflection for the whole ring subjected to a vertical load V is given by

$$\Delta_V = 2\delta_V = \frac{V}{AE}\left[\frac{A\ell^3}{3I}+\frac{\alpha\ell E}{G}+\frac{\pi r}{4}\left(-1+\frac{\alpha E}{G}+\frac{r}{e}\right)\right. \tag{25b}$$

$$\left.+(r-L)\left\{\frac{A\ell^2}{I}+2\left(1-\frac{r}{e}\right)\right\}+(r-L)^2\left(\frac{A\ell}{I}+\frac{\pi}{2e}\right)\right]$$

Determination of stress on the Inside and Outside Fiber of the Load Ring

Consider the half ring with the forces as shown in Figure 8. Since



ℓ
$\frac{V}{2}$
M_c
$\frac{V}{2}$ Sin θ_v
$\frac{V}{2}$ Cos θ_v
θ_v
h
M_c
$\frac{V}{2}$

Fig. 8

strain gauges will be placed only in the curved portion of the ring for the measurement of vertical force, an expression for the stress will be found only for this section of the ring.

From equilibrium, the bending moment M, normal force N, and shear force V at any cross-section in the curved portion of the ring are given as follows,

$$M(\theta_V) = M_c - \frac{V}{2}(\ell + r\, \mathrm{Sin}\theta_V) \qquad 0 \le \theta_V \le \pi \qquad (26a)$$

$$N(\theta_V) = -\frac{V}{2} \mathrm{Sin}\ \theta_V \qquad 0 \le \theta_V \le \pi \qquad (26b)$$

$$V(\theta_V) = -\frac{V}{2} \mathrm{Cos}\ \theta_V \qquad 0 \le \theta_V \le \pi \qquad (26c)$$

Using (19), (26a) becomes

$$M(\theta_V) = \frac{V}{2}\ r(1 - \mathrm{Sin}\theta_V\ ) - M_o \qquad (26d)$$

The shear stress does not enter into the calculation of the stress since its value is zero at the outer and inner fibers.

Applying (62) in Appendix A, and using (26b), (26c), and (26d), the stress at the outer fiber is

$$\sigma_{Vo} = -\frac{V}{2A}\left[\mathrm{Sin}\ \theta_V + \frac{\{r(1 - \mathrm{Sin}^{\theta}V) - L\}\ (\frac{h}{2} + e)}{e(r + \frac{h}{2})}\right] \qquad (27a)$$

and at the inner fiber is

$$\sigma_{Vi} = -\frac{V}{2A}\left[\mathrm{Sin}\ \theta_V - \frac{\{r(1 - \mathrm{Sin}^{\theta}V) - L\}\ (\frac{h}{2} - e)}{e(r - \frac{h}{2})}\right] \qquad (27b)$$

Case 2: Load Ring Subjected to a Horizontal Load H

The elongated load ring subjected to horizontal load H as shown in Figure 9a, can be split into two halfs as shown in Figure 9b. The problem is a

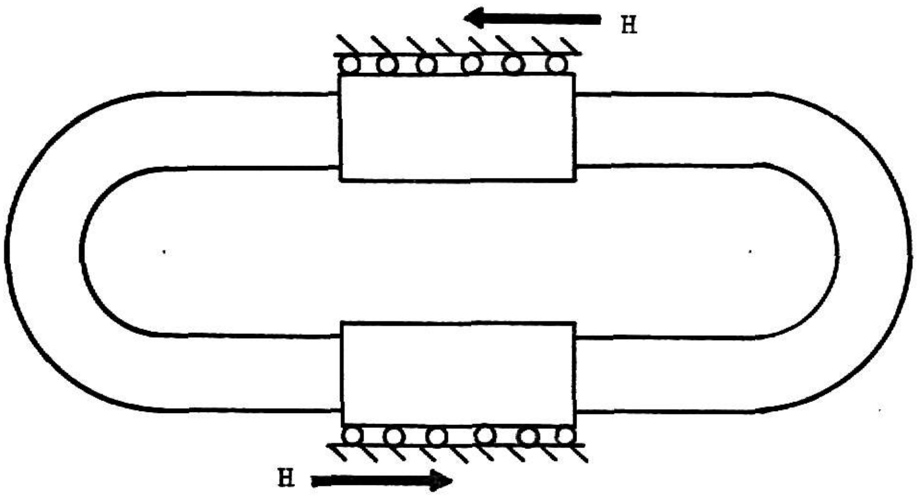


Fig. 9a

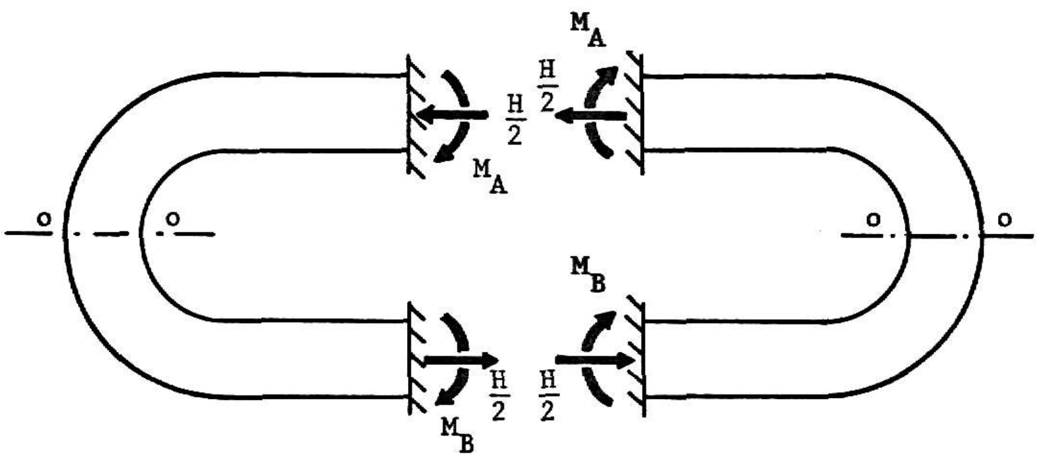


Fig. 9b

statically indeterminate one, and in order to apply (18), it is required to find the moment M_o at section oo.

Determination of M_o

Considering one half of the load ring, the forces acting are as shown in Figure 9c. From statics, taking moments about point A gives

$$M_A + M_B = Hr \tag{28}$$

Again from equilibrium, the bending moment M, normal force N, and shear force V at any cross-section in the three portions of the ring, namely the two straight beams and the curved beam are given as follows,

$$M(s) = M_A \qquad 0 \le s \le \ell \tag{29a}$$

$$N(s) = \frac{H}{2} \qquad 0 \le s \le \ell \tag{29b}$$

$$V(s) = 0 \qquad 0 \le s \le \ell \tag{29c}$$

$$M(\theta_{HR}) = M_A - \frac{Hr}{2}(1 - \cos\theta_{HR}) \qquad 0 \le \theta_{HR} \le \pi \tag{30a}$$

$$N(\theta_{HR}) = \frac{H}{2}\cos\theta_{HR} \qquad 0 \le \theta_{HR} \le \pi \tag{30b}$$

$$V(\theta_{HR}) = -\frac{H}{2}\sin\theta_{HR} \qquad 0 \le \theta_{HR} \le \pi \tag{30c}$$

$$M(s) = M_A - Hr \qquad \ell + \pi r \le s \le 2\ell + \pi r \tag{31a}$$

$$N(s) = \frac{H}{2} \qquad \ell + \pi r \le s \le 2\ell + \pi r \tag{31b}$$

$$V(s) = 0 \qquad \ell + \pi r \le s \le 2\ell + \pi r \tag{31c}$$

The statically indeterminate moment M_o is determined from the condition set forth by the constraints which allows for no change in angle between the planes at A and B.

That is,

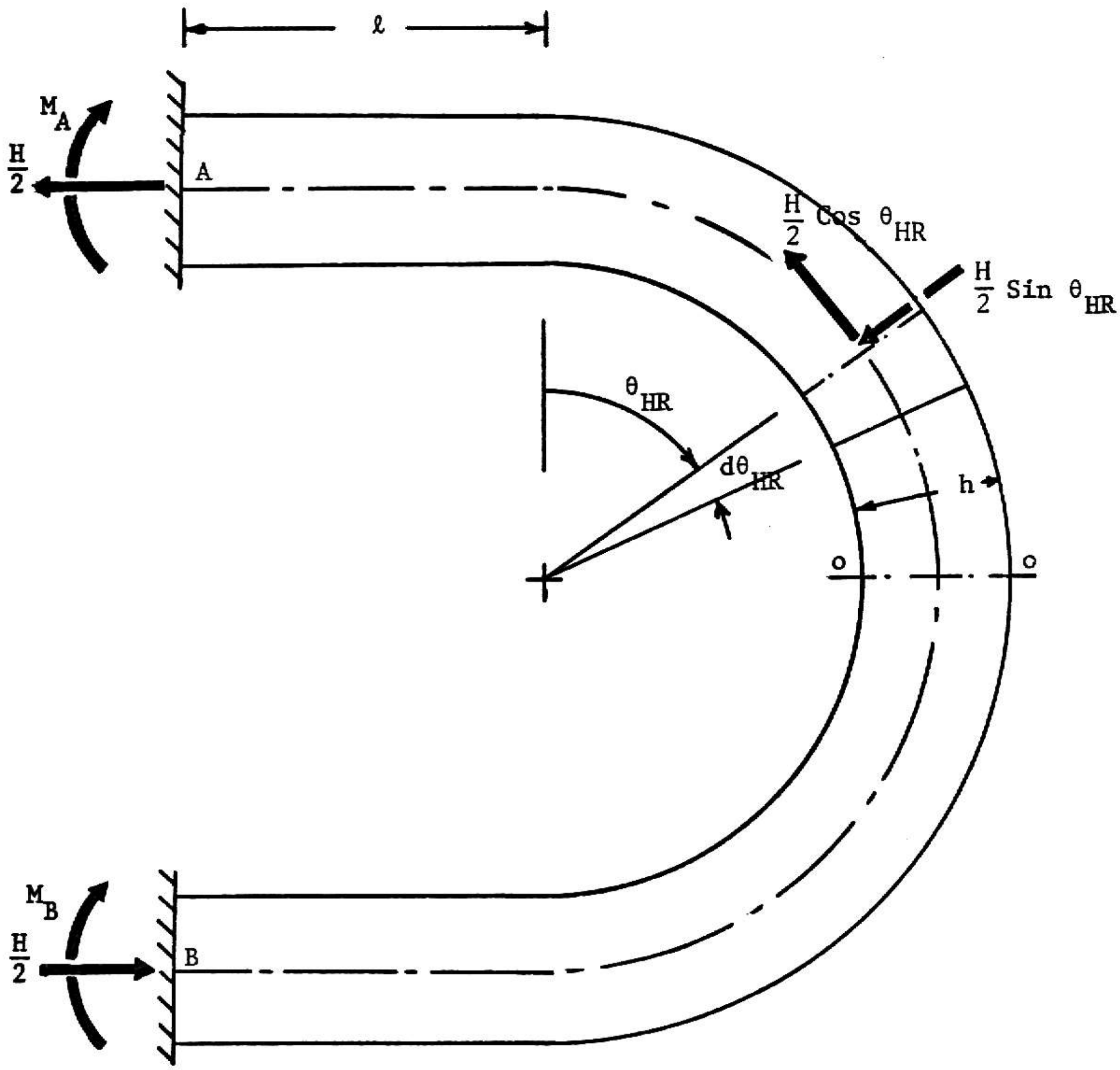

ℓ
M_A
$\frac{H}{2}$
A
$\frac{H}{2}$ Cos θ_{HR}
$\frac{H}{2}$ Sin θ_{HR}
θ_{HR}
$d\theta_{HR}$
h
o
o
M_B
$\frac{H}{2}$
B

Fig. 9c

$$\int_0^{\ell} \frac{M}{EI}\, ds + \int_0^{\pi} \omega \, d\theta_{HR} + \int_{\ell+\pi r}^{2\ell+\pi r} \frac{M}{EI}\, ds = 0 \tag{32}$$

where ω is the change in angle per unit of angle, and

where for an initially straight beam, the rate of change of slope of the elastic curve of deformation is given by,

$$\frac{d^2y}{dx^2} = \frac{M}{EI} \tag{33a}$$

Using (65) in Appendix A, one obtains the equation for ω as

$$\omega = \frac{M}{AeE} - \frac{N}{AE} \tag{33b}$$

Substituting (33a) and (33b) in (32), one obtains

$$\int_0^{\ell} \frac{M(s)}{EI}\, ds + \int_0^{\pi} \left\{\frac{M(\theta_{HR})}{AeE} - \frac{N(\theta_{HR})}{AE}\right\} d\theta_{HR} + \int_{\ell+\pi r}^{2\ell+\pi r} \frac{M(s)}{EI}\, ds = 0 \tag{34a}$$

Using (29a) - (31c) in (34a) gives,

$$\int_0^{\ell} \left(\frac{M_A}{EI}\right) ds + \int_0^{\pi} \left\{\frac{M_A - \frac{Hr}{2}(1 - \cos\theta_{HR})}{AeE} - \frac{\frac{H}{2}\cos\theta_{HR}}{AE}\right\} d\theta_{HR}$$

$$+ \int_{\ell+\pi r}^{2\ell+\pi r} \left(\frac{M_A - Hr}{EI}\right) ds = 0 \tag{34b}$$

Solving the above equation, one obtains,

$$M_A = \frac{H}{2}\, r \tag{35a}$$

Substituting (35a) in (28) gives

$$M_B = \frac{H}{2}\, r \tag{35b}$$

Using (30a), moment M_o at section oo is given by

$$M_o = 0 \tag{36}$$

With $M_o = 0$, the load ring can now be split as shown in Figure 10a. Figure 10b shows one quarter of the ring which corresponds to the right top half of Figure 10a. Putting $P = 0$, $Q = -\frac{H}{2}$, and $M_o = 0$ in equation (18) gives the total strain energy for the present case as

$$U = \frac{(\frac{H^2}{4})\ell}{2AE} + \frac{(\frac{H^2}{4})\ r^2\ell}{2EI} + \frac{\pi r(\frac{H^2}{4})}{8AE}\{-1 + \frac{\alpha E}{G} + \frac{r}{e}\} \tag{37}$$

The deflection in the direction of the force $\frac{H}{2}$, of the point of application of the force at point o is given by,

$$\delta_H = \frac{dU}{d\frac{H}{2}} = \frac{(\frac{H}{2})\ell}{AE} + \frac{(\frac{H}{2})\ r^2\ell}{EI} + \frac{\pi r(\frac{H}{2})}{4AE}\{-1 + \frac{\alpha E}{G} + \frac{r}{e}\} \tag{38a}$$

The total horizontal deflection Δ_H for the whole ring is given by,

$$\Delta_H = 2\delta_H = \frac{H}{AE}\{\ell(1 + \frac{Ar^2}{I}) + \frac{\pi r}{4}(-1 + \frac{\alpha E}{G} + \frac{r}{e})\} \tag{38b}$$

Determination of Stress on the Outside Fiber of the Load Ring

The strain gauges used to measure horizontal force will be placed only on the outside fiber in the curved portion of the ring. Therefore, an expression for stress will be found only for the outside fiber of the curved portion of the ring. The shear stress is zero at the inner and outer fibers of the load ring.

Applying (62) in Appendix A, and using (30a), (30b), and (30c), the stress at the outer fiber is given as

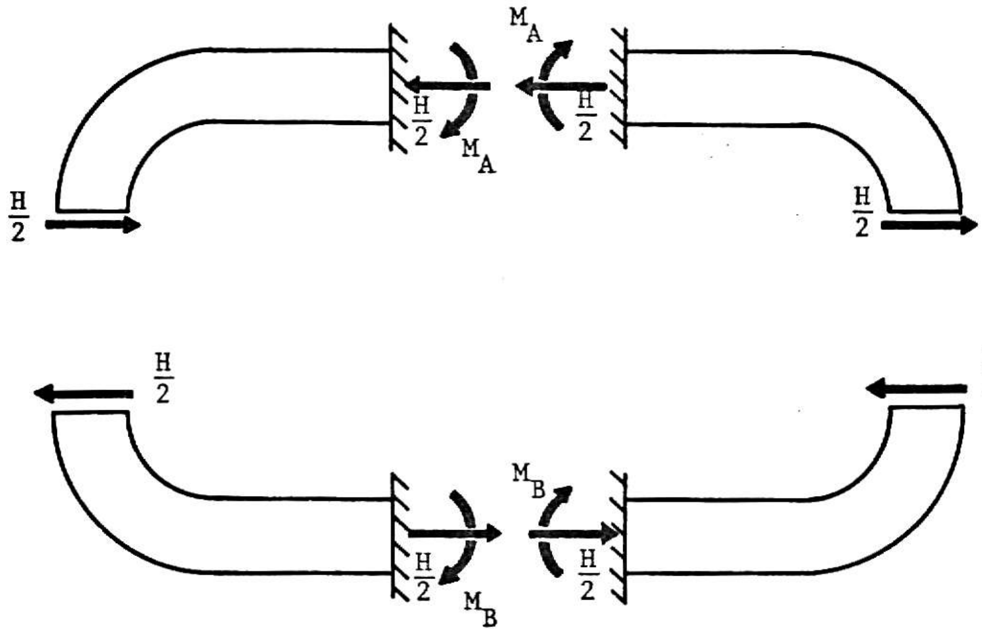

M_A
$\frac{H}{2}$
$\frac{H}{2}$
M_A
$\frac{H}{2}$
$\frac{H}{2}$
$\frac{H}{2}$
$\frac{H}{2}$
M_B
$\frac{H}{2}$
$\frac{H}{2}$
M_B

Fig. 10a

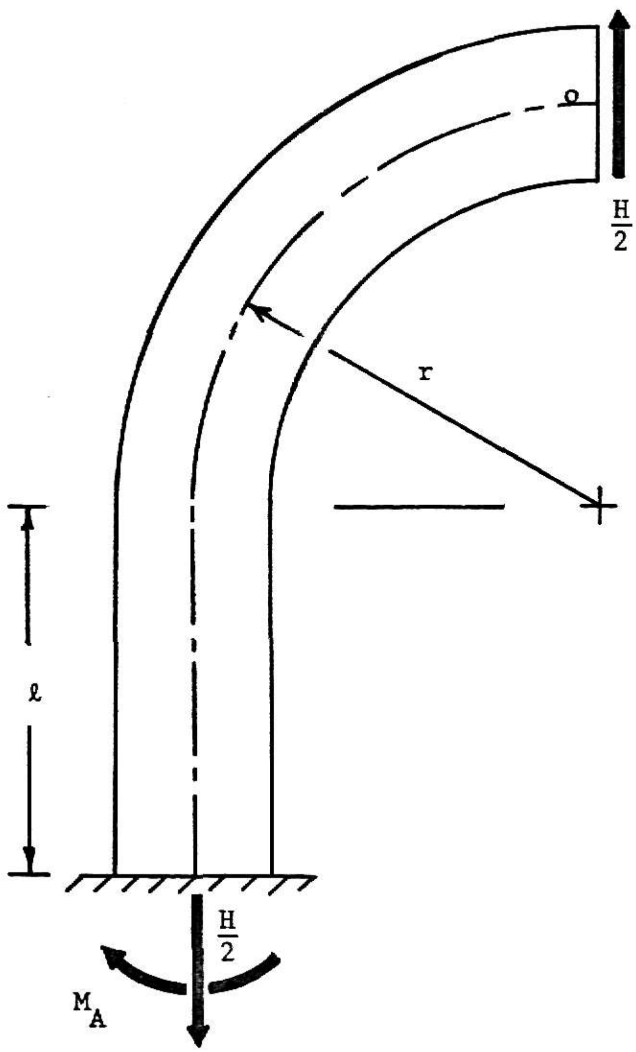


Fig. 10b

$$\sigma_H = \frac{H \, Cos\theta_{HR}}{2A} \{1 - \frac{r(\frac{h}{2} + e)}{e(r + \frac{h}{2})}\} \tag{39}$$

FORMULATION OF DESIGN EQUATIONS

In the analysis of an elongated split-ring dynamometer, expressions for deflection and stress, for the vertical and horizontal loading of the load ring, in terms of its dimensions are obtained. In this section design equations for stiffness and sensitivity ratios will be derived. Before embarking on the derivation of these equations, a particular type of cross-section of the ring, namely rectangular, will be analyzed. The rectangular cross-section of the ring is shown in Figure 11.

Using (60a) in Appendix A gives

$$mA = b\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{y\,dy}{r-y} \qquad (40a)$$

Solving (40a) and using (61a) in Appendix A gives,

$$e = r - \frac{h}{\log_e\left(\frac{r_2}{r_1}\right)} \qquad (40b)$$

Stiffness Ratio

In this analysis, stiffness is defined as the load required to produce a unit deflection at the point of application of the load. Thus, from (25b), one obtains an expression for the vertical stiffness as,

$$\frac{V}{\Delta_V} = \frac{AE}{\left[\frac{A\ell^3}{3I} + \frac{\alpha\ell E}{G} + \frac{\pi r}{4}\left(-1 + \frac{\alpha E}{G} + \frac{r}{e}\right) + (r - L)\left\{\frac{A\ell^2}{I} + 2\left(1 - \frac{r}{e}\right)\right\} + (r - L)^2\left(\frac{A\ell}{I} + \frac{\pi}{2e}\right)\right]} \qquad (41a)$$

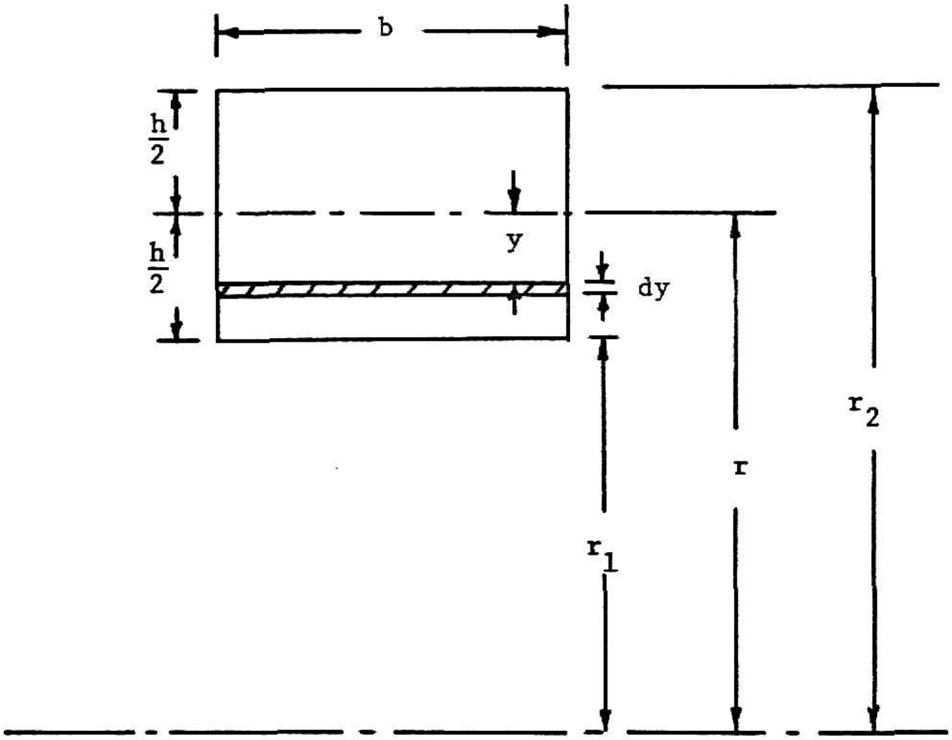


Fig. 11

and using (38b), the horizontal stiffness is given as,

$$\frac{H}{\Delta_H} = \frac{AE}{\{\ell(1 + \frac{Ar^2}{I}) + \frac{\pi r}{4}(-1 + \frac{\alpha E}{G} + \frac{r}{e})\}} \tag{41b}$$

Let the two stiffnesses be related by the stiffness ratio ψ, as given by

$$(\frac{H}{\Delta_H}) = \psi(\frac{V}{\Delta_V}) \tag{42}$$

Normally for a short beam or no beam, ψ will have a value less than 1, since the ring tends to be stiffer in the vertical direction than in the horizontal direction. Putting (41a) and (41b) in (42) one obtains,

$$\frac{AE}{\{\ell(1 + \frac{Ar^2}{I}) + \frac{\pi r}{4}(-1 + \frac{\alpha E}{G} + \frac{r}{e})\}} = \tag{43}$$

$$\psi \frac{AE}{[\frac{A\ell^3}{3I} + \frac{\alpha \ell E}{G} + \frac{\pi r}{4}(-1 + \frac{\alpha E}{G} + \frac{r}{e}) + (r - L)\{\frac{A\ell^2}{I} + 2(1 - \frac{r}{e})\} + (r - L)^2(\frac{A\ell}{I} + \frac{\pi}{2e})]}$$

Now,

$$A = bh \quad \text{and } I = \frac{bh^3}{12} \tag{44}$$

Substituting these into equation (43) yields

$$\begin{aligned}
&144\ \ell^5 e^3 + 30\ \pi\ell^4 h^2 e^2 + (\frac{\alpha E}{G} - \psi - 1)\ 144\ \ell^3 h^2 e^3 + \pi^2 \ell^3 h^4 e \\
&+ 144\ \ell^3 h^2 e^2 r - 1728\ \psi\ \ell^3 e^3 r^2 + (\frac{\alpha E}{G} - \psi - 0.5)\ 12\ \pi\ell^2 h^4 e^2 \\
&+ (1 - \frac{\alpha E}{G})(\psi - 1)\ 36\ \pi\ell^2 h^2 e^3 r + 6\ \pi\ell^2 h^4 er + (1 - 5\psi)\ 36\ \pi\ell^2 h^2 e^2 r^2 \\
&- 12\ \ell h^4 e^3 + (\frac{\alpha E}{G} - \psi)\ 0.25\ \pi^2 \ell h^6 e + \{\frac{\pi^2}{4}(\psi - 1)(1 - \frac{\alpha E}{G}) + 2\}\ 12\ell h^4 e^2 r \\
&+ \{\frac{\pi^2}{4}(1 - 2\psi) - 1\}\ 12\ \ell h^4 er^2 + \{\frac{\pi^3}{16}(\psi - 1)(1 - \frac{\alpha E}{G}) + \pi\}\ rh^6 e \\
&+ \{\frac{\pi^3}{16}(1 - \psi) - \frac{\pi}{2}\}\ r^2 h^6 - 0.5\ \pi h^6 e^2 = 0
\end{aligned} \tag{45}$$

The width of the ring b, cancels out and therefore has no effect on the final result. To solve (45), by plotting a graph, dimensionless quantities will be introduced.

Let $\beta = \frac{h}{r}$ and $\eta = \frac{\ell}{r}$ (46a)

then,

$$\begin{aligned} h &= \beta\, r \\ \ell &= \eta\, r \\ e &= \lambda\, r \end{aligned} \tag{46b}$$

where,

$$\lambda = 1 - \frac{\beta}{\ell n\,(\frac{2+\beta}{2-\beta})}$$

Substituting (46b) in (45) gives,

$$\begin{aligned} &144\,\lambda^3\eta^5 + 30\,\pi\lambda^2\beta^2\eta^4 + (\tfrac{\alpha E}{G} - \psi - 1)\,144\,\lambda^3\beta^2\eta^3 \\ &+ \pi^2\,\lambda\beta^4\eta^3 + 144\,\lambda^2\beta^2\eta^3 - 1728\,\psi\lambda^3\eta^3 \\ &+ (\tfrac{\alpha E}{G} - \psi - 0.5)\,12\,\pi\lambda^2\beta^4\eta^2 + (1 - \tfrac{\alpha E}{G})(\psi - 1)\,36\,\pi\lambda^3\beta^2\eta^2 \\ &+ 6\,\pi\lambda\beta^4\eta^2 + (1 - 5\psi)\,36\,\pi\lambda^2\beta^2\eta^2 - 12\,\lambda^2\beta^4\eta \\ &+ (\tfrac{\alpha E}{G} - \psi)\,0.25\,\pi^2\lambda\beta^6\eta + \{\tfrac{\pi^2}{4}(\psi - 1)(1 - \tfrac{\alpha E}{G}) + 2\}\,12\,\lambda^2\beta^4\eta \\ &+ \{\tfrac{\pi^2}{4}(1 - 2\psi) - 1\}\,12\,\lambda\beta^4\eta + \{\tfrac{\pi^3}{16}(\psi - 1)(1 - \tfrac{\alpha E}{G}) + \pi\}\,\lambda\beta^6 \\ &+ \{\tfrac{\pi^3}{16}(1 - \psi) - \tfrac{\pi}{2}\}\beta^6 - 0.5\,\pi\lambda^2\beta^6 = 0 \end{aligned} \tag{47}$$

Equation (47) helps to design a load ring for a given stiffness ratio. For numerical calculations it will be assumed that the load ring is made of steel and has a rectangular cross-section. For this case

$E = 30 \times 10^6$ lb/in^2, $G = 11.5 \times 10^6$ lb/in^2 and $\alpha = 1.5$.

Sensitivity Ratio

In the practical application of the dynamometer, both the vertical and horizontal components of cutting force will be acting on the ring simultaneously. Therefore, in order to obtain the sensitivity ratio equation, the proper location of the strain gauges should be investigated. The location of the strain gauges should be such that the stress (or strain) measured is due to one component of load only.

For the location of the strain gauges shown in Figures 12a, 12b, it is required to find the angular position of the gauges, that is, θ_V and θ_{HR} for which the stresses σ_{Vo} and σ_{HR} respectively will be zero. The angles θ_V and θ_{HR} will be called the best place for separation.

Determination of Best Place for Separation

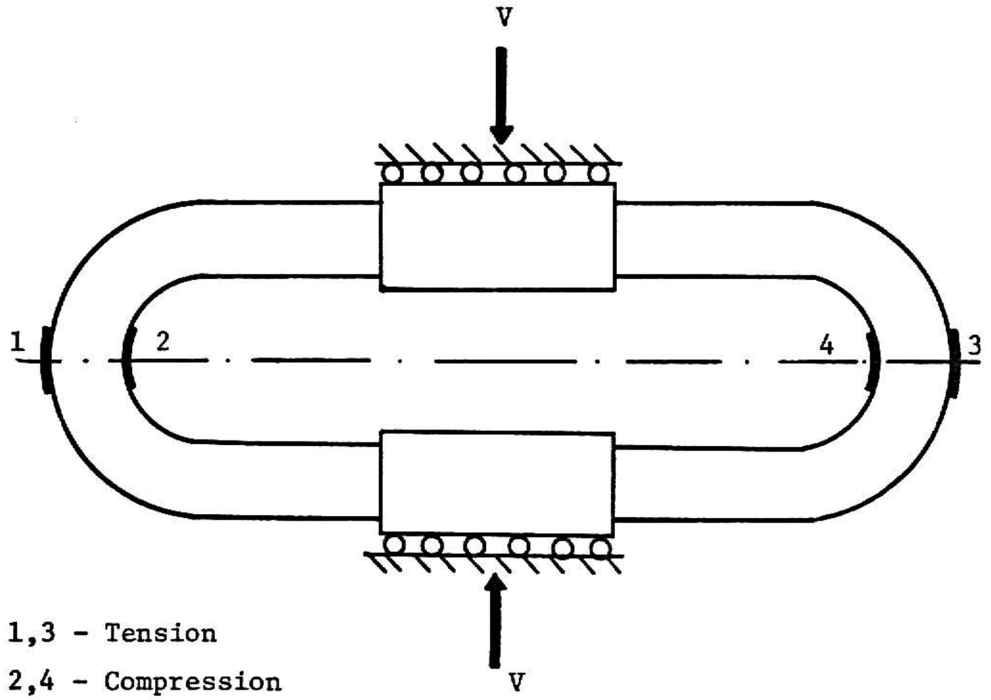
Equating σ_{Vo} given by (27a) to zero, one obtains,

$$-\frac{V}{2A}\left[\operatorname{Sin}\theta_V + \frac{\{r(1 - \operatorname{Sin}\theta_V) - L\}(\frac{h}{2} + e)}{e(r + \frac{h}{2})}\right] = 0 \tag{48a}$$

Solving (48a) for θ_V, and denoting θ_V by θ_{Vo},

$$\theta_{Vo} = \operatorname{Sin}^{-1} \frac{(L - r)(\frac{h}{2} + e)}{[e(r + \frac{h}{2}) - r(\frac{h}{2} + e)]} \tag{48b}$$

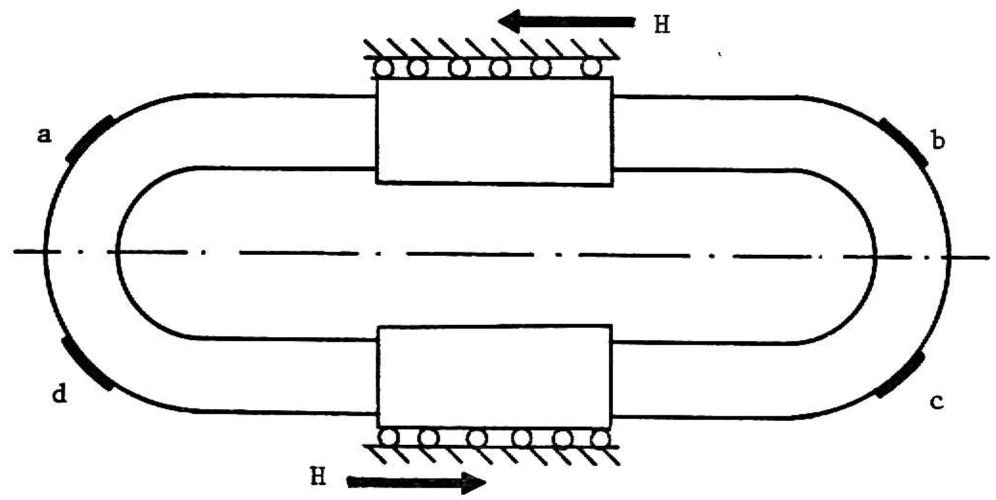
Using dimensionless quantities, as before, in (46a) and (46b), gives



1,3 - Tension

2,4 - Compression

Fig. 12a



a,c - Tension

b,d - Compression

Fig. 12b

$$\theta_{V_o} = \pm \operatorname{Cos}^{-1}\sqrt{1 - \left[\frac{24\lambda^2\eta^2 + 12\lambda\beta\eta^2 + 4\lambda^2\beta^2 - 4\lambda\beta^2 + 2\lambda\beta^3 - 2\beta^3}{24\lambda^2\beta\eta - 24\lambda\beta\eta - \pi\beta^3 + \pi\lambda\beta^3}\right]^2} \qquad (48c)$$

Equating σ_H given by (39) to zero, one obtains

$$\frac{H \operatorname{Cos} \theta_{HR}}{2A}\left\{1 - \frac{r(\frac{h}{2} + e)}{e(r + \frac{h}{2})}\right\} = 0 \qquad (49a)$$

Solving (49a) for θ_{HR} gives

$$\theta_{HR} = \frac{\pi}{2} \qquad (49b)$$

Sensitivity will be defined as the strain output measured for a unit applied load. Since stress is proportional to strain within the elastic limit, it is sufficient to relate the two stresses, one due to vertical component of load and the other due to horizontal component of load, both values corresponding to the location of the strain gauges. Referring to Figures 12a and 12b, the sensitivity ratio equation takes the form,

$$2\left|\frac{\sigma_{Vo} - \sigma_{Vi}}{V}\right| = \chi 4\left|\frac{\sigma_H}{H}\right| \qquad (50a)$$

χ, being the sensitivity ratio.

Taking for σ_H, the stress at the point where the strain gauge b is located, (50a) can be written as,

$$\frac{\sigma_{Vo} - \sigma_{Vi}}{V} = -2\chi\frac{\sigma_H}{H} \qquad (50b)$$

and also θ_{Vo} will take a positive value in (48c).

Putting (27a), (27b), and (39) in (50b), one obtains,

$$\frac{1}{V}\ [-\ \frac{V}{2A}\{\text{Sin}\ \theta_V + \frac{[r(1 - \text{Sin}\ \theta_V) - L](\frac{h}{2} + e)}{e(r + \frac{h}{2})}\}$$

$$+ \frac{V}{2A}\ \{\text{Sin}\ \theta_V - \frac{[r(1 - \text{Sin}\ \theta_V) - L](\frac{h}{2} - e)}{e(r - \frac{h}{2})}\}] \tag{50c}$$

$$= -\ \frac{2\chi}{H}\ [\frac{H\ \text{Cos}\ \theta_{HR}}{2A}\ \{1 - \frac{r(\frac{h}{2} + e)}{e(r + \frac{h}{2})}\}]$$

Simplifying (50c) and substituting for A and I one obtains,

$$24\ \ell^2 e + 48\ \ell re\ (\text{Sin}\ \theta_V - \chi\ \text{Cos}\ \theta_{HR}) + 24\ \chi\ell he\ \text{Cos}\ \theta_{HR}$$

$$+\ \pi\chi h^3\ \text{Cos}\ \theta_{HR} + 2\ \pi h^2 r\ (\text{Sin}\ \theta_V - \chi\ \text{Cos}\ \theta_{HR}) \tag{50d}$$

$$-\ 4\ h^2 r + 4\ h^2 e\ =\ 0$$

Using dimensionless variables given by (46a) and (46b), (50d) takes the form,

$$24\ \lambda\eta^2 + 48\ (\text{Sin}\ \theta_V - \chi\ \text{Cos}\ \theta_{HR})\ \lambda\eta + 24\chi(\text{Cos}\ \theta_{HR})\lambda\beta\eta$$

$$+\ \{2\pi\ (\text{Sin}\ \theta_V - \chi\ \text{Cos}\ \theta_{HR})\ -4\}\ \beta^2 + 4\lambda\beta^2 \tag{50e}$$

$$+\ \pi\chi(\text{Cos}\ \theta_{HR})\ \beta^3\ =\ 0$$

To get an accurate analysis, θ_V and θ_{HR} should assume the values for the best place for separation as given by (48c) and (49b). Thus in equation (50e),

$$\theta_V = \frac{\pi}{2}$$

and $$\theta_{HR} = +\ \text{Cos}^{-1}\sqrt{1-[\frac{24\lambda^2\eta^2 + 12\lambda\beta\eta^2 + 4\lambda^2\beta^2 - 4\lambda\beta^2 + 2\lambda\beta^3 - 2\beta^3}{24\ \lambda^2\beta\eta\ -\ 24\ \lambda\beta\eta\ -\ \pi\beta^3 + \pi\lambda\beta^3}]^2} \tag{51}$$

Many times in practice θ_{HR} is arbitrarily chosen to be $\frac{\pi}{4}$. Therefore two cases will be considered.

Case 1: $\theta_V = \frac{\pi}{2}$; θ_{HR} --- best place for separation.

$$24\ \lambda\eta^2 + 48\ (1-\chi\ \mathrm{Cos}\ \theta_{HR})\ \lambda\eta + 24\ \chi(\mathrm{Cos}\ \theta_{HR})\ \lambda\beta\eta \tag{52a}$$

$$+ \{2\pi\ (1-\chi\ \mathrm{Cos}\ \theta_{HR})\ -4\}\ \beta^2 + 4\lambda\beta^2 + \pi\chi(\mathrm{Cos}\ \theta_{HR})\ \beta^3 = 0$$

where,

$$\mathrm{Cos}\ \theta_{HR} = +\sqrt{1 - [\frac{24\lambda^2\eta^2 + 12\lambda\beta\eta^2 + 4\lambda^2\beta^2 - 4\lambda\beta^2 + 2\lambda\beta^3 - 2\beta^3}{24\lambda^2\beta\eta - 24\lambda\beta\eta - \pi\beta^3 + \pi\lambda\beta^3}]^2} \tag{52b}$$

Case 2: $\theta_V = \frac{\pi}{2}$; $\theta_{HR} = \frac{\pi}{4}$

$$24\lambda\eta^2 + 48(1-\frac{1}{\sqrt{2}}\chi)\lambda\eta + 24(\frac{1}{\sqrt{2}})\chi\lambda\beta\eta + \{2\pi(1-\frac{1}{\sqrt{2}}\chi)\ -4\}\beta^2 + 4\lambda\beta^2 \tag{53}$$

$$+\ \pi(\frac{1}{\sqrt{2}})\chi\beta^3 = 0$$

Equations (52a), (52b), and (53) helps in designing a load ring for a given sensitivity ratio.

All the equations derived thus far, hold true for a split-ring dynamometer by putting ℓ, the length of the straight beam, equal to zero. Equations were derived for a split-ring dynamometer, in a separate analysis and were in agreement with the equations for an elongated split-ring, with the substitution ℓ equal to zero.

NUMERICAL COMPUTATION AND RESULTS

The equations (47), (52a) and (52b), and (53) derived in the theoretical analysis, have to be solved. Equation (53) is second degree in η and is solved by using the quadratic formula and computer program 1 as shown in Appendix B. Since a direct method of solution is not possible for solving (47), (52a) and (52b), these are solved, by an iterative process called the Regula falsi (method of false position) method, using computer programs 2 and 3 shown in Appendix B.

In solving the equations, different values of the stiffness or sensitivity ratios are chosen. Also β is given values between 0.1 to 1.0 in increments of 0.1. For these values, the corresponding values for η are computed.

Graphs in Figure 13 indicate the variation of β with respect to η for different stiffness ratios and those in Figures 14a and 14b, indicate the same for different sensitivity ratios for the two cases. The stiffness and sensitivity ratio graphs are superimposed for each case and are shown in Figures 15a and 15b. A graph is also drawn for the best place for separation with respect to β and is shown in Figure 16.

Figures 17, 18a, and 18b show load rings drawn to scale for a few values taken from the graphs.

Balanced Design Load Ring

Given a maximum value for the vertical and horizontal components of the load denoted by V_{max} and H_{max}, let it be desired to have a load ring with an equal deflection, of the point of application of the load, in the vertical and horizontal directions due to loads V_{max} and H_{max} respectively and to have

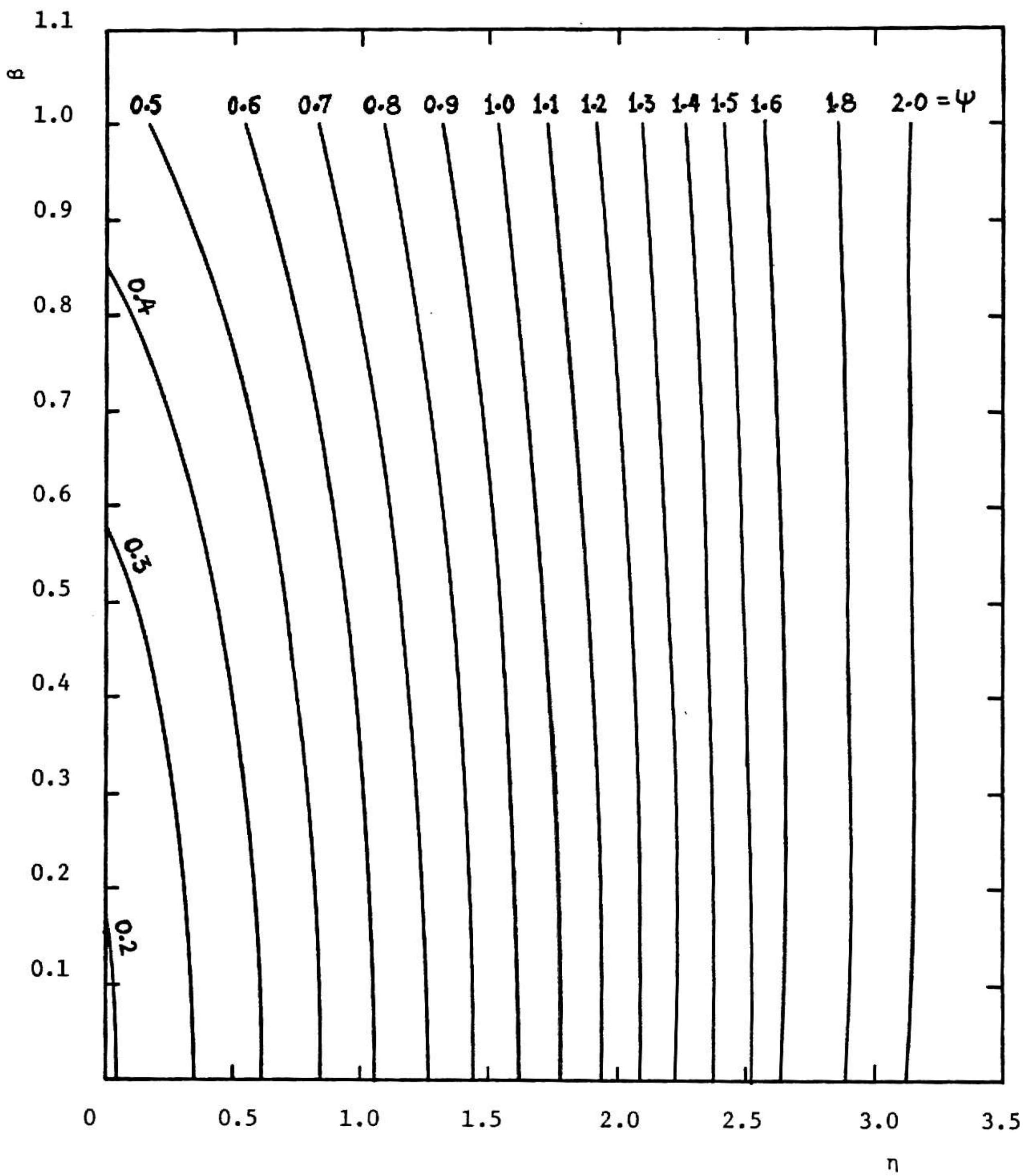


Fig. 13 Dynamometer Stiffness Ψ as a Function of Geometric Parameters η and β.

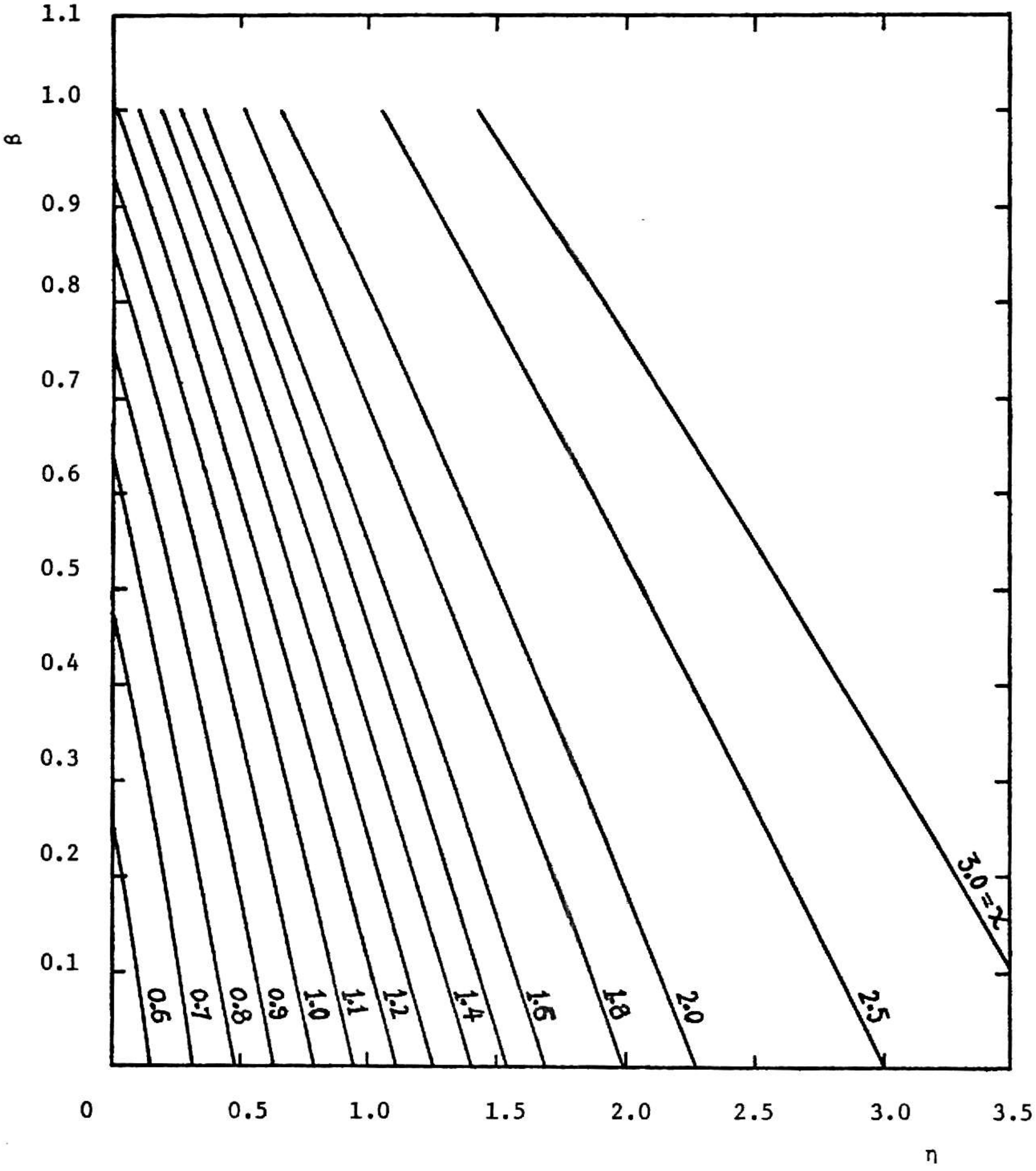


Fig. 14a Dynamometer Sensitivity χ $(\theta_{HR} = \frac{\pi}{4})$ as a Function of Geometric Parameters η and β

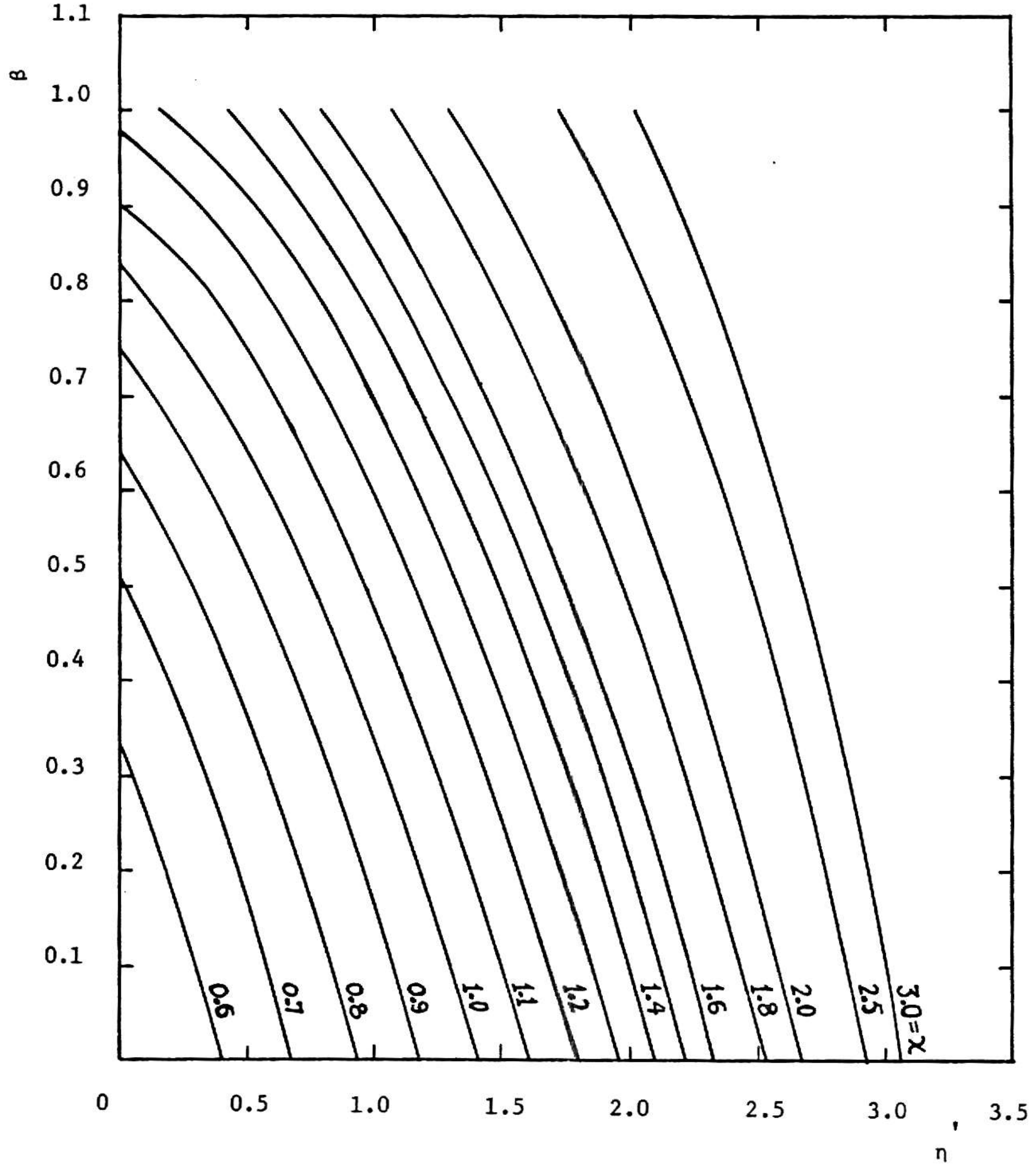


Fig. 14b Dynamometer Sensitivity χ (θ_{HR}= Best Place for Separation) as a Function of Geometric Parameters η and β.

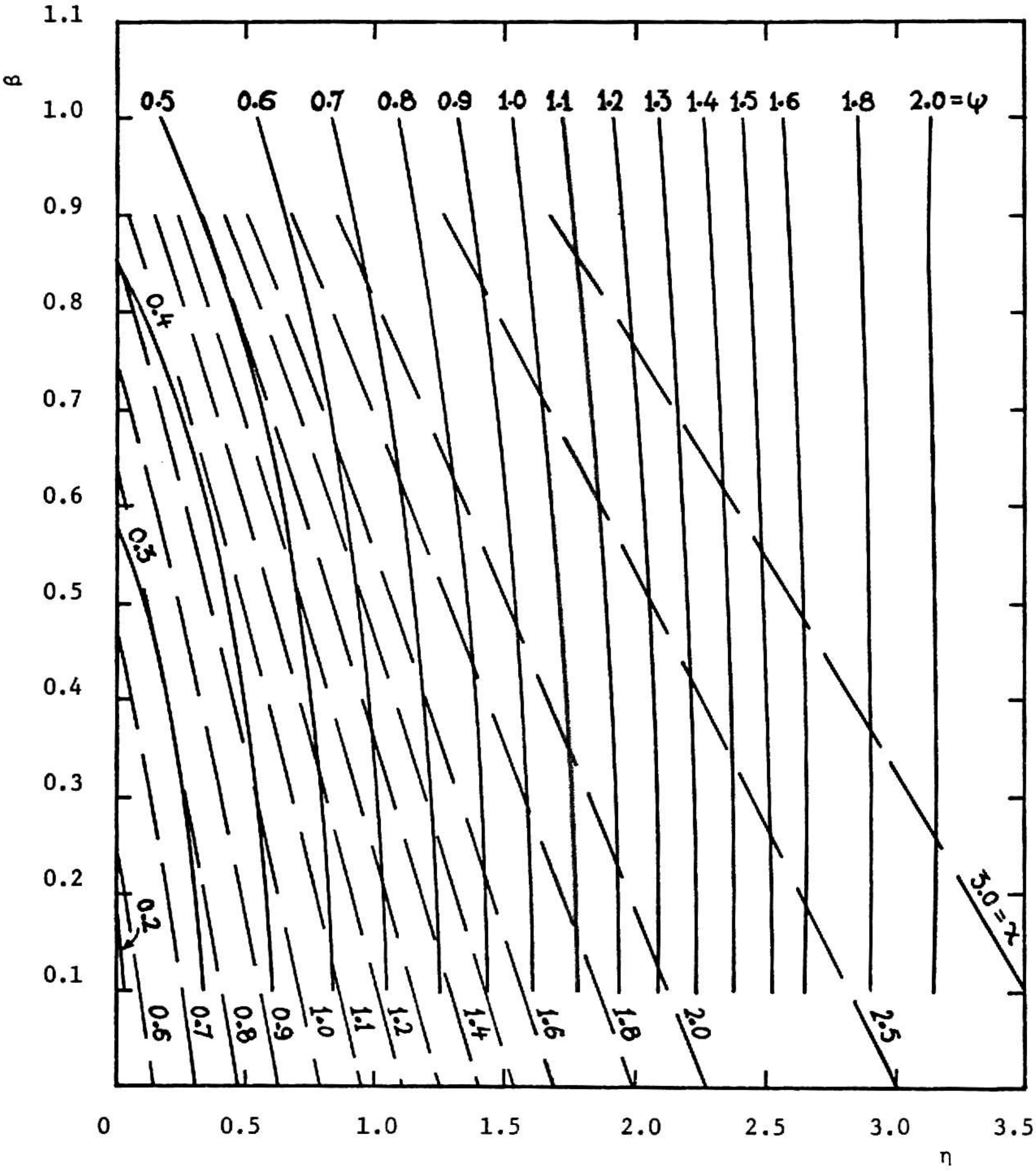


Fig. 15a Dynamometer Stiffness Ψ and Sensitivity χ ($\theta = \frac{\pi}{4}$) as a Function of Geometric Parameters η and β.

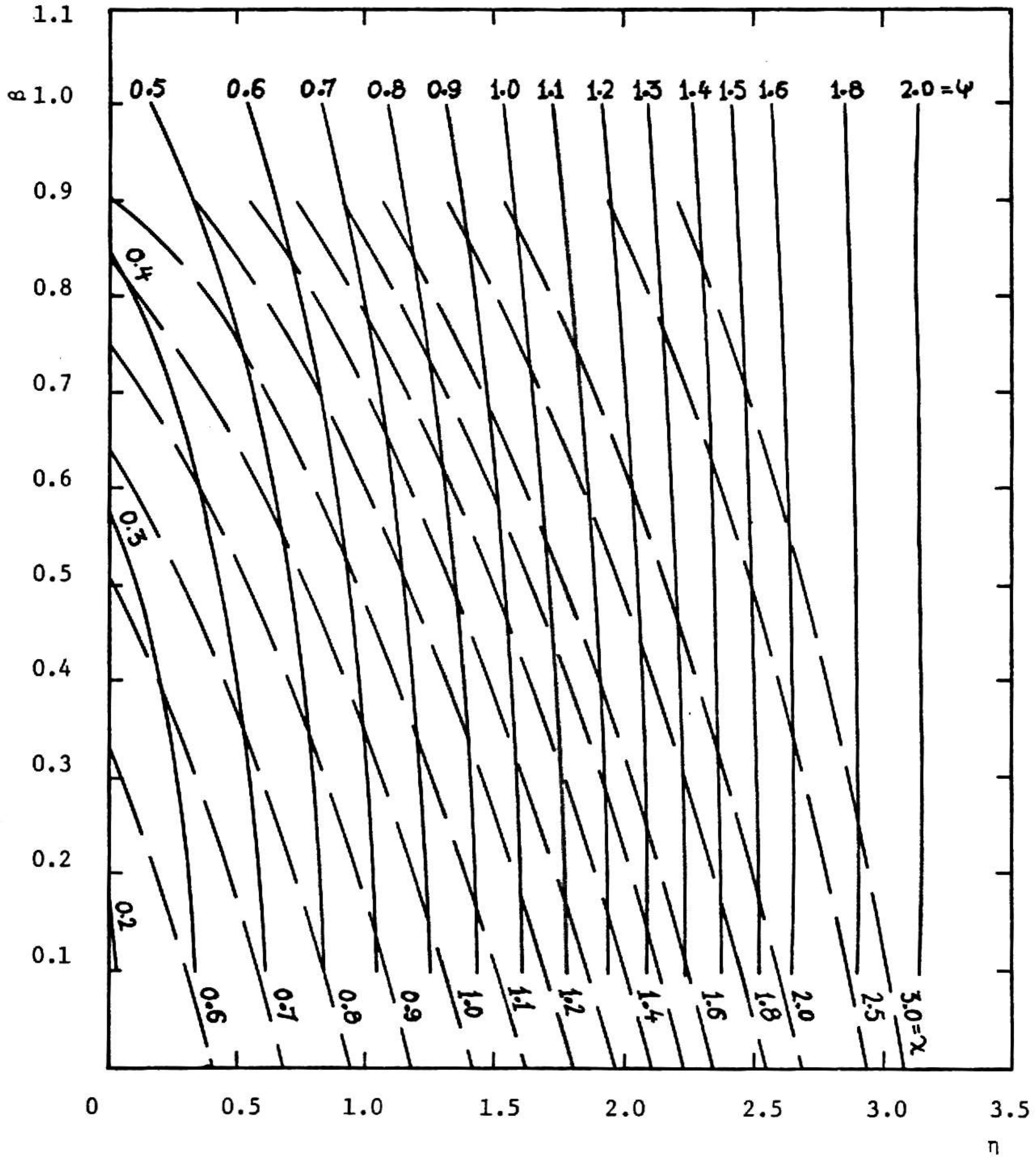


Fig. 15b Dynamometer Stiffness Ψ and Sensitivity χ (θ=Best Place for Separation) as a Function of Geometric Parameters η and β.

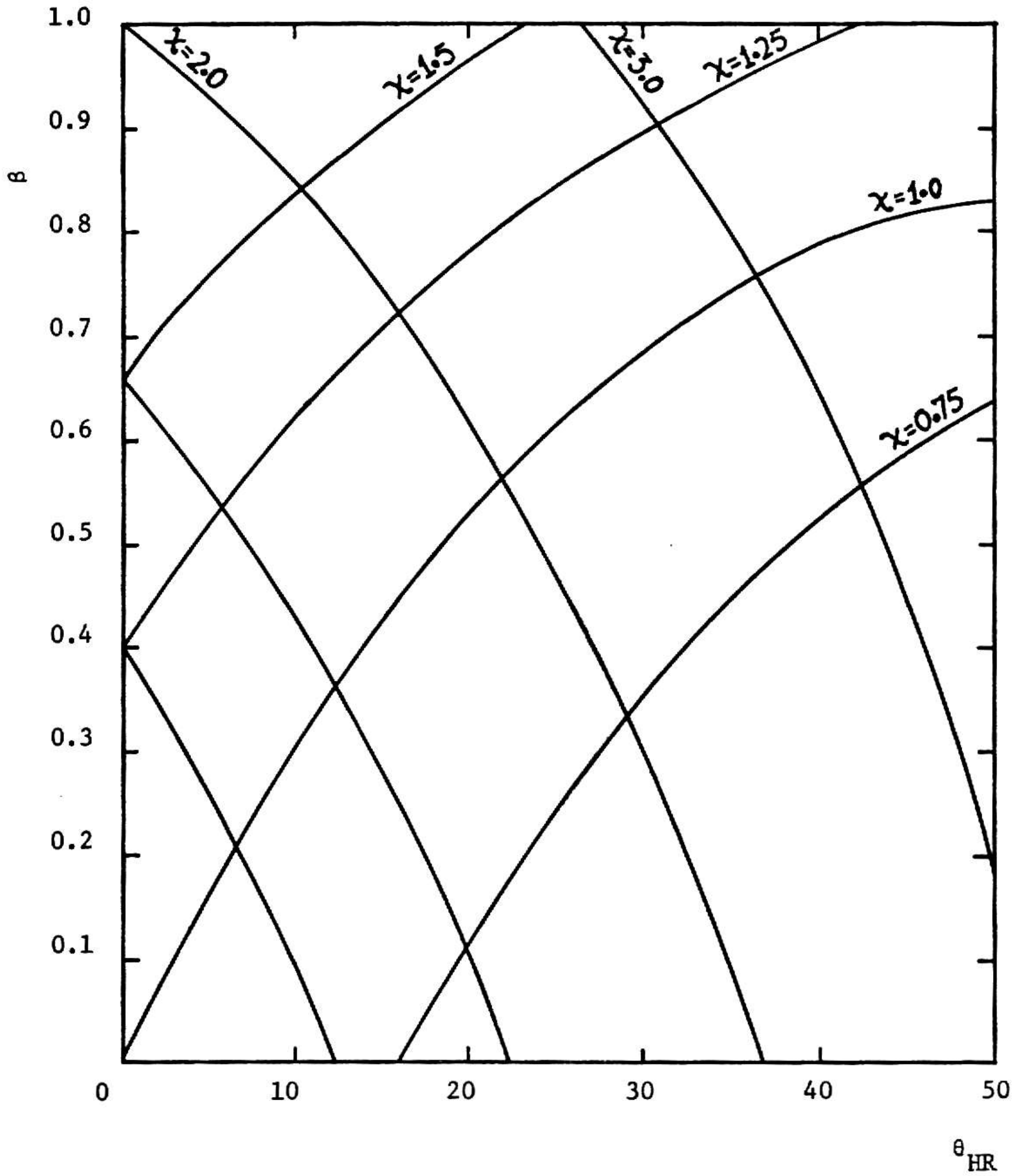


Fig. 16 Best Place for Separation θ_{HR} as a Function of Geometric Parameters η and β

$\Psi = 0.2$

$\chi = 0.55$

$\ell = 0$

$h = 0.24$ in.

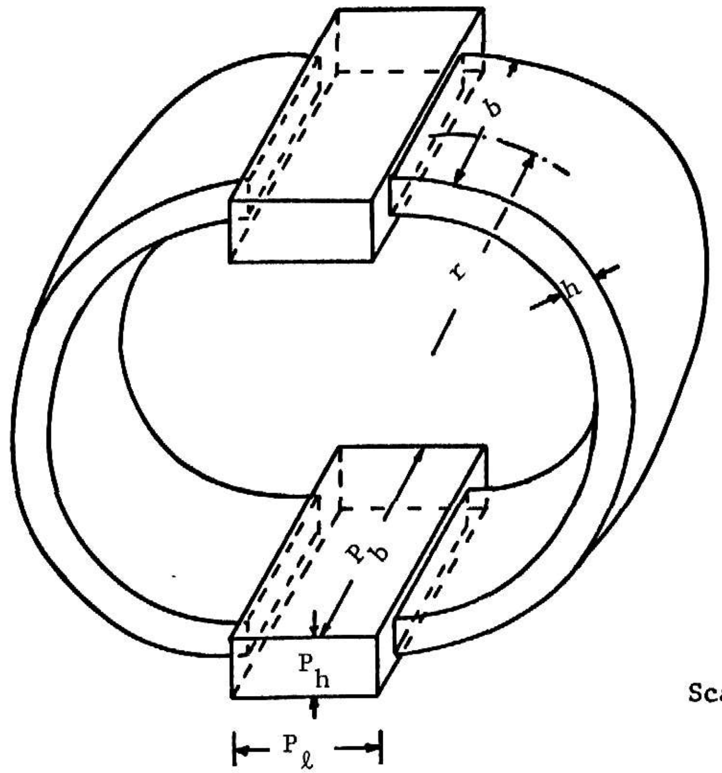
$r = 1.5$

$b = 1.0$

$P_\ell = 1.0$

$P_h = 0.4$

$P_b = 1.5$



Scale: 3/4 Full Size

Fig. 17 Scale Drawing of a Plain Split-Ring Dynamometer

$\Psi = 0.4$

$\chi = 0.75$

$\ell = 0.879$ in.

$h = 0.3$

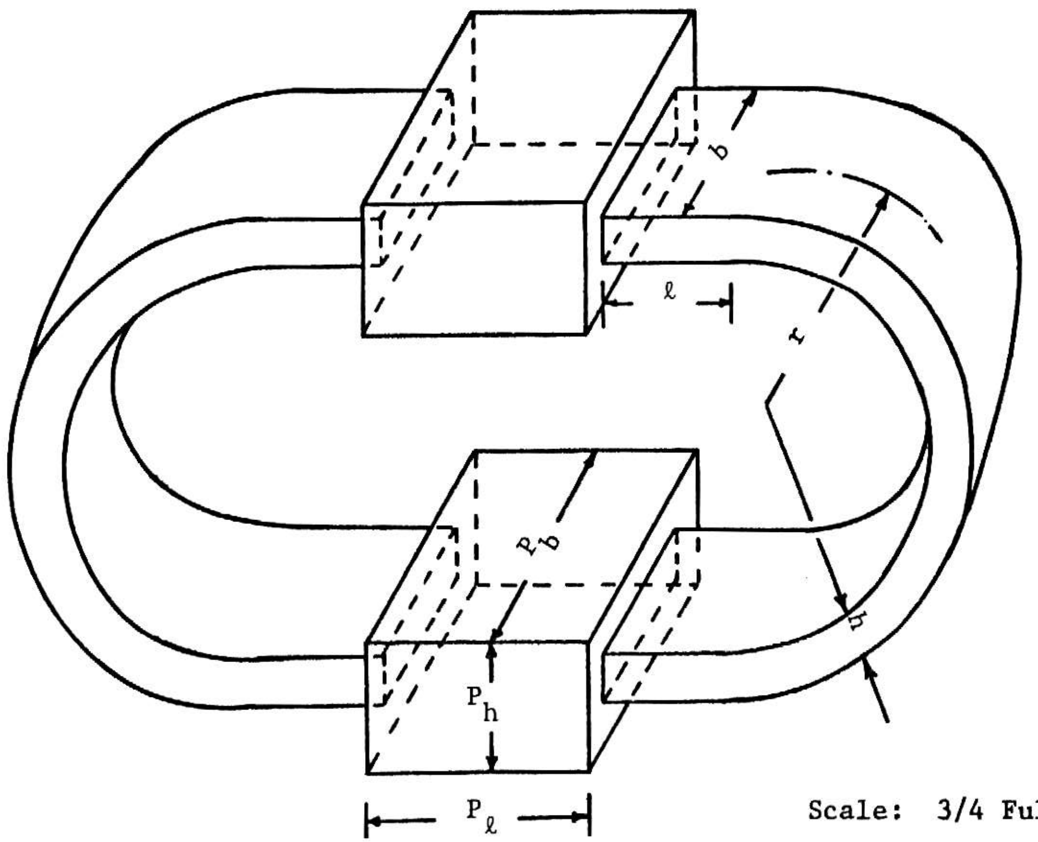
$r = 1.5$

$b = 1.0$

$P_\ell = 1.5$

$P_h = 0.9$

$P_b = 1.5$



Scale: 3/4 Full Size

Fig. 18a Scale Drawing of an Elongated Split-Ring Dynamometer

$\Psi = 1.0$

$\chi = 1.32$

$\ell = 2.685$ in.

$h = 0.3$

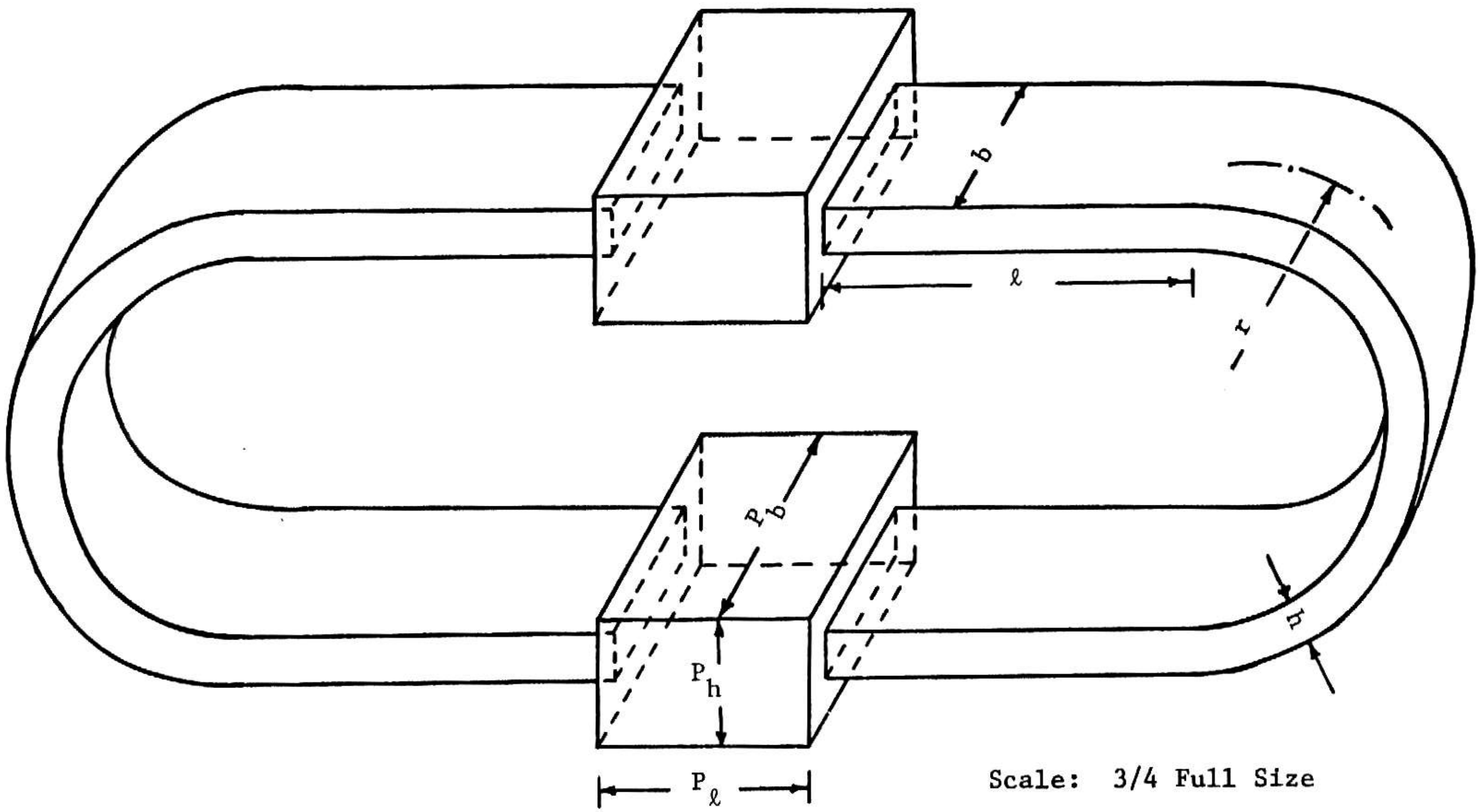
$r = 1.5$

$b = 1.0$

$P_\ell = 1.5$

$P_h = 0.9$

$P_b = 1.5$



Scale: 3/4 Full Size

Fig. 18b Scale Drawing of an Elongated Split-Ring Dynamometer the same value of strain readings for the loads V_{max} and H_{max}. This might be considered as a "balanced design" load ring since the maximum deflections are equal and the relative accuracies of measurement of the two components are equal. For this case, then, the design criteria may be expressed as follows,

$$\left(\frac{\Delta_V}{V}\right) V_{max} = \left(\frac{\Delta_H}{H}\right) H_{max}$$

and,

$$\left(\frac{\sigma_V}{V}\right) V_{max} = \left(\frac{\sigma_H}{H}\right) H_{max}$$

From these one obtains,

$$\chi = \frac{H_{max}}{V_{max}} \text{ and } \Psi = \frac{H_{max}}{V_{max}} \tag{54a}$$

For simplicity, it has been assumed that $\sigma_{Vo} = \sigma_{Vi} = \sigma_V$, since they differ in value only by a small amount.

For a "balanced design" it is desired that

$$\Psi = \chi \tag{54b}$$

Unfortunately it is found from numerical calculations that it is not feasible to satisfy this requirement in practice and hence the "balanced design" cannot be achieved with the present type of dynamometer. It is however possible to closely approximate "balanced design" in many situations. It is significant to note that it is possible to more closely approximate "balanced design" with an elongated split-ring dynamometer than with an ordinary split-ring dynamometer.

DISCUSSION AND CONCLUSIONS

The analysis in this report concerns the design of a dynamometer subjected to only two components of the cutting force. The design equations and graphs help in designing a load ring for a desired combination of the stiffness and sensitivity ratios. However, it was found that the two ratios Ψ and χ cannot have the same value, for the type of load ring considered. Thus it is not possible to obtain "balanced design" in practical situations. Nevertheless, a dynamometer can be designed which closely approximates "balanced design."

In designing a dynamometer, one of the ratios, either stiffness or sensitivity whichever is most important in a specific application can be assigned the desired value and the corresponding value of the other ratio can be determined from the graph. The smaller the value of β, the closer the two ratios become. It should be borne in mind however that since β is directly proportional to h, the thickness of the load ring, it cannot be too small, for this will reduce the load carrying capacity of the load ring. It may be preferable to let both Ψ and χ deviate from the desired values. Then the best results are obtained if Ψ is decreased and χ increased from the desired values.

There are advantages in including straight beam, as in elongated split-ring dynamometer. The main advantage is that greater stiffness and sensitivity ratios can be obtained than those obtainable with a plain split-ring dynamometer. For example, referring to Figure 13, (limiting the value of β to 1.0) the range of stiffness ratio for the split-ring will be between 0.2-0.5 approximately; whereas for the elongated split-ring (with the length of the arm within twice r) the range of stiffness ratio is increased to 0.2-1.2 approximately. Thus there is more flexibility using the elongated split-ring dynamometer.

The other advantage is depicted in Table 1. From the values in this table, it can be concluded that as the length of the straight beam is increased, the two ratios come closer to each other and thereby the design approaches "balanced design."

Table 1. Variation of χ/Ψ Ratio With Respect to ℓ.

		$\ell = 0$	ℓ increasing →			
$\beta = 0.2$	Ψ	0.21	0.4	0.6	0.8	1.0
	χ	0.56	0.75	0.93	1.12	1.32
	Ψ/χ	2.67	1.875	1.55	1.4	1.32
$\beta = 0.8$	Ψ	0.38	0.4	0.6	0.8	1.0
	χ	0.97	0.99	1.28	1.58	1.92
	Ψ/χ	2.55	2.48	2.14	1.975	1.92

REFERENCES

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APPENDICES

APPENDIX A

Curved Beam Theory

Consider the free body diagram of a section of a curved portion of a beam ABCD subjected to a shear force V, bending moment M, and normal force N, as indicated in Figure 19. Let s represent the unstrained position of the cross-section measured along the locus of centroids. It is assumed that plane sections remain plane and normal to the locus of centroids.

Let the unstressed radius of curvature be r,

the radius of curvature after application of loads be R,

the normal distances of the stressed fibers from the centroidal axis be y

and the increase in fiber length after application of loads be $\delta(y)$

The sign convention shown in Figure 20 will be followed.

Then,

the length of an unstressed fiber along the centroidal axis is

$$\Delta s = r\Delta\Psi$$

The extension of any fiber is

$$\delta(y) = (R-y)\Delta\gamma - (r-y)\Delta\Psi \tag{55}$$

The unit elongation of the fiber or strain is

$$\varepsilon(y) = \frac{\delta(y)}{(r-y)\Delta\Psi} \tag{56a}$$

The stress at y is

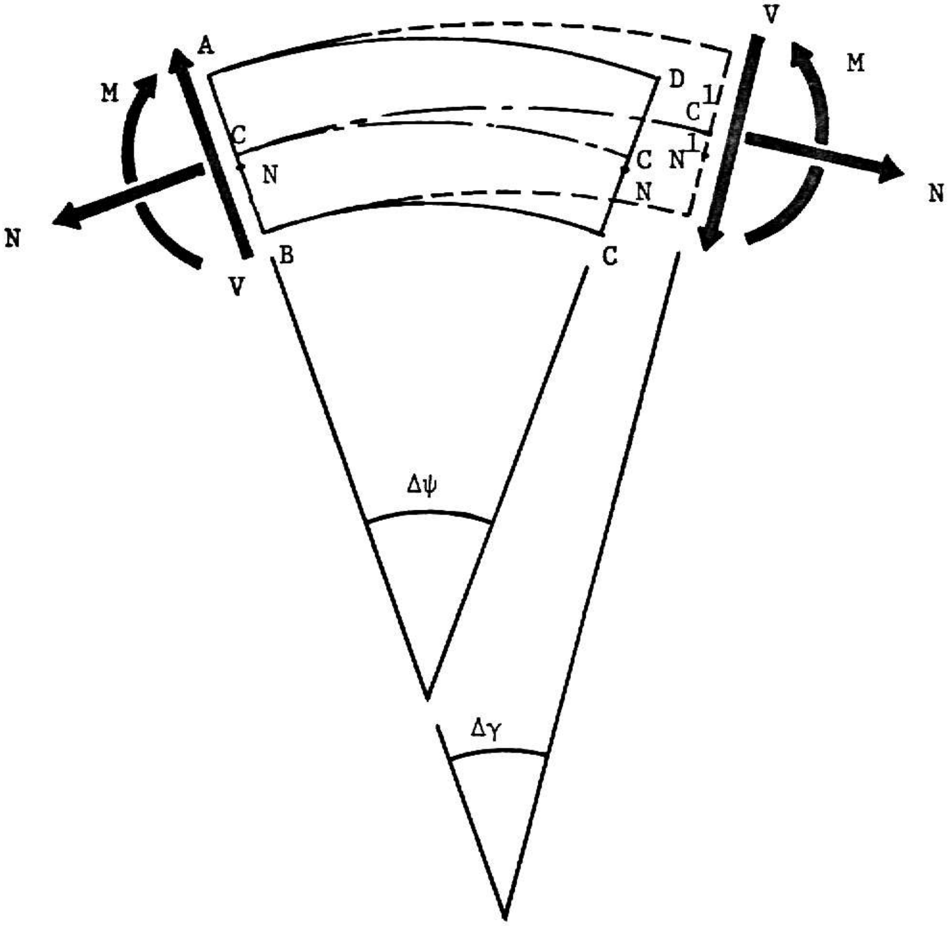

Fig. 19

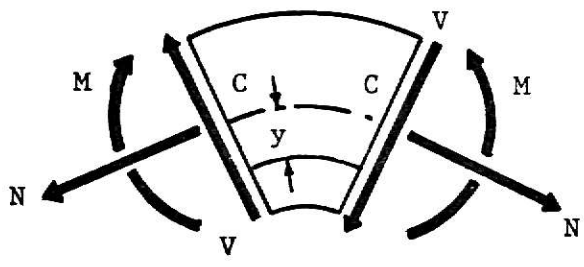

Fig. 20

$$\sigma(y) = \frac{E\ \delta(y)}{(r - y)\Delta\Psi} \tag{56b}$$

Two equations of statics are used

$$\int_A \sigma dA = N \tag{57a}$$

and $$\int_A \sigma y dA = M \tag{57b}$$

Substituting (55) in (56b) gives,

$$\sigma(y) = \frac{E\ (R - y)}{(r - y)} \frac{\Delta\gamma}{\Delta\Psi} - E \tag{58a}$$

Let e represent the distance of the neutral axis NN from the centroidal axis CC.

Then the stress at e is

$$\sigma(e) = \frac{N}{A} \tag{58b}$$

Combining (58a) and (58b),

$$\frac{N}{A} = \frac{E\ (R - e)}{(r - e)} \frac{\Delta\gamma}{\Delta\Psi} - E \tag{58c}$$

Subtracting (58c) from (58a),

$$\sigma(y) - \frac{N}{A} = \frac{E(R - r)(y - e)}{(r - e)(r - y)} \frac{\Delta\gamma}{\Delta\Psi} \tag{58d}$$

Substituting (58d) in (57a) gives

$$\int_A \frac{y}{r - y}\, dA - e \int_A \frac{1}{r - y}\, dA = 0 \tag{59}$$

Let the modified area, mA, of the cross-section be defined by

$$mA = \int_A \frac{y}{r - y}\, dA \tag{60a}$$

where m is a pure number to be determined for each particular shape of the cross-section by performing the indicated integration.

Then,

$$\int_A \frac{dA}{r-y} = (m + 1)\, \frac{A}{r} \tag{60b}$$

And, using $\int_A y dA = 0$, from the definition of centroid,

$$\int_A \frac{y^2}{r - y}\, dA = mrA \tag{60c}$$

Substituting (60a) and (60b) into (59) gives,

$$e = r\, (\frac{m}{m + 1}) \tag{61a}$$

Solving for m, gives

$$m = \frac{e}{r - e} \tag{61b}$$

Substituting (58b) in (57b), the stress at a distance y from the centroidal axis and normal to the cross-section is obtained as,

$$\sigma(y) = \frac{N}{A} + \frac{M(y - e)}{Ae(r - y)} \tag{62}$$

Determination of Change of Curvature

Substituting (58d) in (57b) gives

$$\frac{E(R - r)\ \Delta\gamma}{\Delta\Psi} = \frac{M}{mA} \tag{63a}$$

Combining (58a) and (62),

$$\sigma(0) = \frac{E(R\Delta\gamma - r\Delta\Psi)}{r\Delta\Psi} = \frac{N}{A} - \frac{M}{Ar} \qquad (63b)$$

Simplifying (63b) with the help of (63a) gives

$$\frac{Nr}{A} - \frac{M}{A} + rE - \frac{Er\ \Delta\gamma}{\Delta\Psi} = \frac{M}{mA} \qquad (63c)$$

Solving (63c) gives

$$\left(\frac{\Delta\Psi - \Delta\gamma}{\Delta s}\right) = \frac{M(m+1)}{mr^2AE} - \frac{N}{rAE} \qquad (64)$$

Taking the limit in (64) and using (61a), the expression for the change in curvature of the center line of the beam is given by

$$\left(\frac{d\Psi}{ds} - \frac{d\gamma}{ds}\right) = \frac{M}{erAE} - \frac{N}{rAE} \qquad (65)$$

Determination of Total Energy

Let the strain energy of the differential element ABCD shown in Figure 19, be dU. For an individual fiber the strain energy due to bending and tensile load, using (56b), is given by

$$\delta U_1 = \frac{1}{2}\ \frac{EdA\ \delta^2(y)}{(r-y)\Delta\Psi} \qquad (66a)$$

The total strain energy of the element becomes

$$dU_1 = \frac{E}{2} \int_A \frac{\delta^2(y)}{(r-y)\Delta\Psi}\, dA \qquad (66b)$$

Substituting (63b) in (55) and simplifying,

$$\delta\ (y) = \left\{\frac{(r-y)\ N}{AE} + \frac{(y-e)\ N}{eAE}\right\}\ \Delta\Psi \qquad (67a)$$

Squaring (67a),

$$\frac{E\delta^2(y)}{2(r - y)\Delta\Psi} = \frac{\Delta\Psi}{2A^2E}\{(r - y)\,N^2 + 2\frac{(y - e)\,MN}{e} + \frac{(y - e)^2M^2}{e^2(r - y)}\} \tag{67b}$$

Solving by putting (67b) in (66b) gives

$$dU_1 = \{\frac{N^2}{2AE} - \frac{MN}{rAE} + \frac{M^2}{2reAE}\}r\,\Delta\Psi \tag{68a}$$

To include the effect of shear, the relative radial displacement of the cross-section is $\frac{\alpha V}{AG}\, r\Delta\Psi$, where α is a numerical factor with which the average shearing stress must be multiplied in order to obtain the shearing stress at the centroid of the cross-section.

The energy due to shear then, is given by

$$dU_2 = \frac{\alpha V^2}{2AG}\, r\Delta\Psi \tag{68b}$$

Hence the total strain energy of the element due to the combined action of forces, M, N, and V is given by

$$dU = dU_1 + dU_2 = (\frac{N^2}{2AE} - \frac{MN}{rAE} + \frac{M^2}{2reAE} + \frac{\alpha V^2}{2AG})\, r\Delta\Psi \tag{69}$$

APPENDIX B

This appendix contains a listing of computer programs 1, 2, and 3 for solving equations (53), (52a) and (52b), and (47) respectively.

ILLEGIBLE DOCUMENT

THE FOLLOWING DOCUMENT(S) IS OF POOR LEGIBILITY IN THE ORIGINAL

THIS IS THE BEST COPY AVAILABLE

Program 1

```
      REAL LAM
   10 FORMAT (2F10.5,I10)
   20 FORMAT (F10.5)
   30 FORMAT (' CHI=',F10.5)
   40 FORMAT (4X,'BET',19X,'ETA1',18X,'ETA2',16X,'RESIDUAL1',13X,
     1'RESIDUAL2')
   50 FORMAT (F10.5,4(12X,F10.5))
   60 FORMAT (///)
      PI=3.1415927
      READ(5,10) BETO,DELBET,NBET
      DO 100 J=1,17
      READ(5,20) CHI
      WRITE(6,30) CHI
      WRITE(6,40)
      DO 300 I=1,NBET
      BET=BETO+(I-1)*DELBET
      CALL LAMDA (BET,LAM)
      A=24.*LAM
      B=48.*(1.-(CHI/2.**0.5))*LAM+(24./(2.**0.5))*LAM*BET*CHI
      C=(2.*PI*(1.-(CHI/2.**0.5))-4.)*BET**2+4.*LAM*BET**2+(PI/2.**0.5)
     1*BET**3*CHI
      DSCRM=B**2-4.*A*C
C     TEST THE DISCRIMINANT
      IF(DSCRM.LT.0.0) GO TO 400
      ROOT=DSCRM**0.5
      ETA1=(-B+ROOT)/(2.*A)
      ETA2=(-B-ROOT)/(2.*A)
      RES1=24.*LAM*ETA1**2+(48.*(1.-(CHI/2.**0.5))*LAM+(24./(2.**0.5))
     1*LAM*BET*CHI)*ETA1+(2.*PI*(1.-(CHI/2.**0.5))-4.)*BET**2
     2+4.*LAM*BET**2+PI*(CHI/2.**0.5)*BET**3
      RES2=24.*LAM*ETA2**2+(48.*(1.-(CHI/2.**0.5))*LAM+(24./(2.**0.5))
     1*LAM*BET*CHI)*ETA2+(2.*PI*(1.-(CHI/2.**0.5))-4.)*BET**2
     2+4.*LAM*BET**2+PI*(CHI/2.**0.5)*BET**3
      WRITE(6,50) BET,ETA1,ETA2,RES1,RES2
  300 CONTINUE
      WRITE(6,60)
  100 CONTINUE
  400 STOP
      END
      SUBROUTINE LAMDA(BET,LAM)
      REAL LAM
      LAM=1.-BET/ALOG((2.+BET)/(2.-BET))
      RETURN
      END
```

Program 2

```
      REAL LAM
   10 FORMAT (2F10.5,I10)
   15 FORMAT (2F10.5)
   20 FORMAT (F10.5)
   25 FORMAT (' CHI=',F10.5)
   30 FORMAT (7X,'BET',12X,'ETA',12X,'RES',12X,'ETL',12X,'ETR',13X,'RL',
     113X,'RR',12X,'THETA')
   35 FORMAT (F10.5,6(3X,F13.8),3X,F10.5)
   45 FORMAT (///)
      PI=3.1415927
      READ(5,10) BETO,DELBET,NBET
      READ(5,15) ETAO,DELETA
      DO 700 L=1,3
      READ(5,20) CHI
      WRITE(6,25) CHI
      WRITE(6,30)
      DO 300 I=1,NBET
      BET=BETO+(I-1)*DELBET
      J=0
      K2=1
      CALL LAMDA (BET,LAM)
      CALL COST (BET,LAM,ETAO,ROOT,COSTH,THETA)
      CALL RESDAL (BET,LAM,ETAO,CHI,COSTH,RES)
      ETL=ETAO
      RL=RES
      K1=1
      IF(RES.LT.0.0) K1=2
  110 J=J+1
      ETA=ETAO+J*DELETA
  115 CALL LAMDA (BET,LAM)
      CALL COST (BET,LAM,ETA,ROOT,COSTH,THETA)
      IF(ROOT.LT.0.0) GO TO 300
      CALL RESDAL (BET,LAM,ETA,CHI,COSTH,RES)
      IF(K2.EQ.2) GO TO 130
      IF(RES/RL.LT.0.0) K2=2
      IF(K2.EQ.2) GO TO 120
      RL=RES
      ETL=ETA
      GO TO 110
  120 RR=RES
      ETR=ETA
      GO TO 150
  130 IF(((K1.EQ.1).AND.(RES.LT.0.0)).OR.((K1.EQ.2).AND.(RES.GT.0.0)))
     1 GO TO 140
      RL=RES
      ETL=ETA
      GO TO 150
  140 RR=RES
      ETR=ETA
  150 CONTINUE
      IF(ETR.LT.ETL) STOP
      ETA=(ETR+ETL)/2.0
      IF(((ETR-ETL)/ETR).LT.0.000001) GO TO 200
      GO TO 115
  200 CONTINUE
      WRITE(6,35) BET,ETA,RES,ETL,ETR,RL,RR,THETA
  300 CONTINUE
      WRITE(6,45)
  700 CONTINUE
      STOP
```

Program 2 (cont.)

```
      END
      SUBROUTINE LAMDA(BET,LAM)
      REAL LAM
      LAM=1.-BET/ALOG((2.+BET)/(2.-BET))
      RETURN
      END
      SUBROUTINE COST (BET,LAM,ETA,ROOT,COSTH,THETA)
      REAL LAM
      REAL NUM
      PI=3.1415927
      NUM=24.*LAM**2*ETA**2+12.*LAM*BET*ETA**2+4.*LAM**2*BET**2-4.*LAM
     1*BET**2+2.*LAM*BET**3-2.*BET**3
      DEN=24.*LAM**2*BET*ETA-24.*LAM*BET*ETA-PI*BET**3+PI*LAM*BET**3
      EXP=(NUM/DEN)**2
      ROOT=1.-EXP
      IF(ROOT.LT.0.0) GO TO 600
      COSTH=ROOT**0.5
      THETA=(180./PI)*ARCOS(COSTH)
  600 RETURN
      END
      SUBROUTINE RESDAL (BET,LAM,ETA,CHI,COSTH,RES)
      REAL LAM
      PI=3.1415927
      RES=24.*LAM*ETA**2+48.*(1.-CHI*COSTH)*LAM*ETA+24.*CHI*COSTH*LAM*
     1BET*ETA+(2.*PI*(1.-CHI*COSTH)-4.)*BET**2+4.*LAM*BET**2+PI*CHI*
     2COSTH*BET**3
      RETURN
      END
```

Program 3

```
      REAL LAM
   10 FORMAT (2F10.5,I10)
   15 FORMAT (2F10.8)
   18 FORMAT (2F10.5)
   19 FORMAT (' ALP=',F10.5,5X,' EOG=',F10.5)
   20 FORMAT (F10.5)
   25 FORMAT (' PSI=',F10.5)
   30 FORMAT (7X,'BET',12X,'ETA',12X,'RES',12X,'ETL',12X,'ETR',13X,'RL',
     113X,'RR')
   35 FORMAT (F10.5,6(3X,F13.8))
   45 FORMAT (///)
      PI=3.1415927
      READ(5,10) BETO,DELBET,NBET
      READ(5,15) ETAO,DELETA
      DO 250 K=1
      READ(5,18) ALP,EOG
      WRITE(6,19) ALP,EOG
      DO 700 L=1
      READ(5,20) PSI
      WRITE(6,25) PSI
      WRITE(6,30)
      DO 300 I=1,NBET
      BET=BETO+(I-1)*DELBET
      J=0
      K2=1
      CALL LAMDA (BET,LAM)
      CALL RESDAL (BET,LAM,ETAO,PSI,ALP,EOG,RES)
      ETL=ETAO
      RL=RES
      K1=1
      IF(RES.LT.0.0) K1=2
  110 J=J+1
      ETA=ETAO+J*DELETA
  115 CALL LAMDA (BET,LAM)
      CALL RESDAL (BET,LAM,ETA,PSI,ALP,EOG,RES)
      IF(J.EQ.100) GO TO 300
      IF(K2.EQ.2) GO TO 130
      IF(RES/RL.LT.0.0) K2=2
      IF(K2.EQ.2) GO TO 120
      RL=RES
      ETL=ETA
      GO TO 110
  120 RR=RES
      ETR=ETA
      GO TO 150
  130 IF(((K1.EQ.1).AND.(RES.LT.0.0)).OR.((K1.EQ.2).AND.(RES.GT.0.0)))
     1 GO TO 140
      RL=RES
      ETL=ETA
      GO TO 150
  140 RR=RES
      ETR=ETA
  150 CONTINUE
      IF(ETR.LT.ETL) STOP
      ETA=(ETR+ETL)/2.0
      IF(((ETR-ETL)/ETR).LT.0.000001) GO TO 200
      GO TO 115
  200 CONTINUE
      WRITE(6,35) BET,ETA,RES,ETL,ETR,RL,RR
  300 CONTINUE
```

Program 3 (cont.)

```
      WRITE(6,45)
  700 CONTINUE
      STOP
      END
      SUBROUTINE LAMDA (BET,LAM)
      REAL LAM
      LAM=1.-BET/ALOG((2.+BET)/(2.-BET))
      RETURN
      END
      SUBROUTINE RESDAL (BET,LAM,ETA,PSI,ALP,EOG,RES)
      REAL LAM
      PI=3.1415927
      A=144.*LAM**3*ETA**5+30.*PI*LAM**2*BET**2*ETA**4+(ALP*EOG-PSI-1.)*
     1144.*LAM**3*BET**2*ETA**3+PI**2*LAM*BET**4*ETA**3+144.*LAM**2*BET
     2**2*ETA**3-1728.*PSI*LAM**3*ETA**3+(ALP*EOG-PSI-0.5)*12.*PI*LAM**2
     3*BET**4*ETA**2+(1.-ALP*EOG)*(PSI-1.)*36.*PI*LAM**3*BET**2*ETA**2
     4+6.*PI*LAM*BET**4*ETA**2+(1.-5.*PSI)*36.*PI*LAM**2*BET**2*ETA**2
     5-12.*LAM**3*BET**4*ETA
      B=(ALP*EOG-PSI)*0.25*PI**2*LAM*BET**6*ETA+((PI**2/4.)*(PSI-1.)*
     1(1.-ALP*EOG)+2.)*12.*LAM**2*BET**4*ETA+((PI**2/4.)*(1.-2.*PSI)-1.)
     2*12.*LAM*BET**4*ETA+((PI**3/16.)*(PSI-1.)*(1.-ALP*EOG)+PI)*LAM*BET
     3**6+((PI**3/16.)*(1.-PSI)-(PI/2.))*BET**6-0.5*PI*LAM**2*BET**6
      RES=A+B
      RETURN
      END
```

ANALYSIS OF AN ELONGATED SPLIT-RING
DYNAMOMETER

by

SRIKISHEN N. BINDIGANAVALE

B. E., Osmania University, India, 1970

AN ABSTRACT OF A MASTER'S REPORT

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1973

The present investigation is an analysis of an elongated split-ring dynamometer subjected to two components of the cutting force. The variation of dynamometer stiffness and sensitivity as functions of the geometric parameters are studied. The equations derived for deflections, stresses, stiffness and sensitivity are based on curved beam theory and Castigliano's theorem for deflection.

The design equations and graphs are useful in designing a load ring for a desired combination of stiffness and sensitivity ratios. It was found, however, that the two ratios cannot have the same value for the type of load ring considered. There is a marked improvement with an elongated split-ring dynamometer over a plain split-ring dynamometer, in that a greater number of stiffness and sensitivity ratios can be obtained.