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SOME SPECIAL PROBLEMS IN THE GAS DYNAMICS OF VERTICAL FLOW

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by

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P pressure, psfa

P_o stagnation pressure, psfa

P^* pressure at Mach Number unity, psfa

P_B back pressure, psfa

P_e exit pressure, psfa

P_x pressure before the shock, psfa

P_y pressure behind the shock, psfa

P_R pressure ratio: P_2/P_1

Q constant heat flux, Btu/lbm-ft

R gas constant, ft-lbf/lbm-°R

S entropy, Btu/lbm-°R

T temperature, °R

T^* temperature at Mach Number unity, °R

T_o stagnation temperature, °R

T_m mean temperature, °R

T_R temperature ratio: T_2/T_1

T_B temperature where the pressure is the back pressure, °R

T_x temperature before the shock, °R

T_y temperature behind the shock, °R

V gas velocity, fps

V^* gas velocity at Mach Number unity, fps

V_R gas velocity ratio, V_2/V_1

V_x gas velocity before the shock, fps

V_y gas velocity behind the shock, fps

W weight of gas, lb_f

W^* the weight of gas between any section to the section at Mach Number unity, lbf

Z elevation, ft

Z^* the elevation where Mach Number is unity, ft

Z_{max} the maxium elevation, ft

α_1, α_2, α_3, α_4, parameters defined on page 17

β_1, β_2, β_3, β_4,

β_5, β_6, β_7, β_8, parameters defined on page 31

δ_1, δ_2, δ_3, δ_4, parameters defined on page 92

ρ density, lb_m/ft^3

ρ^* density at Mach Number unity, lb_m/ft^3

ρ_x density before the shock, lb_m/ft^3

ρ_y density behind the shock, lb_m/ft^3

ρ_R density ratio: ρ_2/ρ_1

τ_w frictional stress, lb_f/ft^2

INTRODUCTION

Most of the problems encountered in gas dynamics deal with horizontal flow. In this report, vertical flows of a perfect gas are considered, in which the change in elevation is taken as the independent variable. Three cases are considered: (1) reversible adiabatic flow, (2) reversible diabatic flow and (3) irreversible adiabatic flow. Particularly investigated is the weight of gas existing within the pipe between any two levels of elevations, the conditions that produce normal shocks and the locations of these normal shocks.

The four fundamental principles governing the motion of compressible fluids are the law of the conservation of mass, Newton's laws of motion, and the first and the second laws of thermodynamics. The restrictions and hypothesis for the present investigation are as follows:

(1) One dimensional, compressible, fluid flow

(2) Steady flow

(3) Perfect gas

(4) Vertical circular pipe with constant cross-sectional area

(5) Negligible boundary layer effects

(6) Negligible thickness of the normal shock.

Based on the above principles and assumptions, the general analytic expressions are derived and presented in the following pages. Their results are also illustrated with numerical examples.

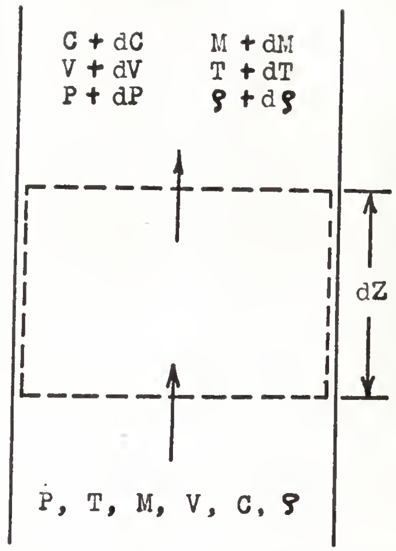

Fig. 1. Control surface for analysis of isentropic, constant-area, vertical flow.

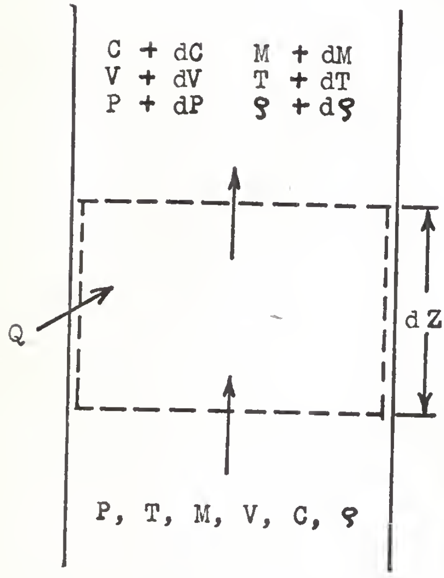

Fig. 2. Control surface for analysis of reversible, diabatic, constant-area, vertical flow.

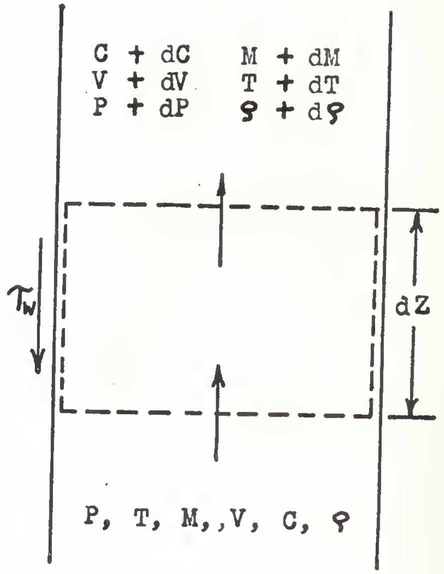

Fig. 3. Control surface for analysis of irreversible, adiabatic, constant-area, verticalfflow.

ISENTROPIC FLOW

Physical Equations

Perfect Gas Equation:

$$\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T} \tag{1}$$

Definition of Mach Number:

$$\frac{dM^2}{M^2} = \frac{dV^2}{V^2} - \frac{dT}{T} \tag{2}$$

Energy Equation:

$$C_p dT + \frac{VdV}{g_c J} + \frac{gdZ}{g_c J} = 0 \tag{3}$$

Equation of Continuity:

$$\frac{d\rho}{\rho} + \frac{1}{2}\frac{dV^2}{V^2} = 0 \tag{4}$$

Momentum Equation:

$$\frac{dP}{\rho} + \frac{gdZ}{g_c} + \frac{VdV}{g_c} = 0 \tag{5}$$

Equation of Sound Speed:

$$\frac{dC}{C} = \frac{1}{2}\frac{dT}{T} \tag{6}$$

Equation of Impulse Function:

$$\frac{dF}{F} = \frac{dP}{P} + \frac{kdM^2}{1 + kM^2} \tag{7}$$

The Change of the Properties

The differential relations of the properties are established as follows.

(a) Pressure:

Combining Eq. (4) and Eq. (5)

$$\frac{gdZ}{g_c} = \frac{dP}{\rho}\left(M^2 - 1 \right) \qquad (8)^1$$

(b) Temperature:

Eqs. (3) and Eq. (5)

$$dT = \frac{dP}{\rho C_p J} \qquad (9)$$

(c) Sound Speed:

Eq. (6)

$$\frac{dC}{C} = \frac{1}{2}\frac{dT}{T} \qquad (6)$$

(d) Gas Speed:

Combining Eqs. (1), (3) and (8) yields

$$\frac{dP}{P}\left[1 + \frac{k-1}{k}(M^2-1)\right] + \frac{1}{2}\left[1 + (k-1)M^2\right]\frac{dV^2}{V^2} = 0 \qquad (10)^2$$

(e) Density:

Eq. (4)

$$\frac{d\rho}{\rho} + \frac{1}{2}\frac{dV^2}{V^2} = 0$$

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[1]Developed in the Appendix, p. 70.

[2]Developed in the Appendix, p. 70.

(f) Mach Number:

Combining Eqs. (9) and (1)

$$\frac{dT}{T} = (k - 1) \frac{d\rho}{\rho} \tag{11}$$

Combining Eqs. (2) and (4)

$$\frac{dM^2}{M^2} = - 2 \frac{d\rho}{\rho} - \frac{dT}{T} \tag{12}$$

Substituting Eqs. (12) into Eqs. (11)

$$\frac{dM^2}{M^2} = - (k + 1) \frac{d\rho}{\rho} \tag{12-a}$$

(g) Stagnation Temperature:

From the definition of stagnation temperature

$$T_o = T (1 + \frac{k-1}{2} M^2)$$

$$\frac{dT_o}{T_o} = \frac{dT}{T} + \frac{(k - 1)M^2}{2 + (k-1)M^2} \frac{dM^2}{M^2}$$

Substituting Eq. (12-a) into the above

$$\frac{dT_o}{T_o} = \frac{2(k - 1)(1 - M^2)}{2 + (k - 1) M^2} \frac{d\rho}{\rho} \tag{13}$$ [1]

(h) Stagnation Pressure:

$$\frac{dP_o}{P_o} = \frac{d\rho}{\rho} \frac{2k(1-M^2)}{2 + (k - 1) M^2} \tag{14}$$ [2]

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[1] Developed in the Appendix, p. 71.

[2] Developed in the Appendix, p. 72.

(i) Impulse Function:

$$\frac{dF}{F} = \frac{d\rho}{\rho} \frac{k(1 - M^2)}{1 + kM^2} \qquad (15)^1$$

From Eqs. (4), (6), (8), (9), (10), (12), (13), (14) and (15), the changes in the fluid properties are summarized for upward and downward flows, and for subsonic and supersonic velocities.

	$dZ > 0$ (Upward Flow)		$dZ < 0$ (Downward Flow)	
Eqs.	$M > 1$ (Supersonic)	$M < 1$ (Subonic)	$M > 1$ (Supersonic)	$M < 1$ (Subonic)
(8)	$dP > 0$	$dP < 0$	$dP < 0$	$dP > 0$
(8)&(9)	$dT > 0$	$dT < 0$	$dT < 0$	$dT > 0$
(6)	$dC > 0$	$dC < 0$	$dC < 0$	$dC > 0$
(10)	$dV < 0$	$dV > 0$	$dV > 0$	$dV < 0$
(4)&(10)	$d\rho > 0$	$d\rho < 0$	$d\rho < 0$	$d\rho > 0$
(12-a), (4)&(10)	$dM < 0$	$dM > 0$	$dM > 0$	$dM < 0$
(13), (4) &(10)	$dT_o < 0$	$dT_o < 0$	$dT_o > 0$	$dT_o > 0$
(14), (4) &(10)	$dP_o < 0$	$dP_o < 0$	$dP_o > 0$	$dP_o > 0$
(15), (4) &(10)	$dF < 0$	$dF < 0$	$dF > 0$	$dF > 0$

The Mach Number always tends toward unity for upward flow

[1]Developed in the Appendix, p. 73.

($\Delta Z > 0$). As the Mach Number approaches unity, dZ approaches zero. Hence, there is a maximum length of the duct for a given initial Mach Number. After the Mach Number reaches unity, a further upward increase in the length of the duct results in a reduction in the flow rate, i.e. the flow is choked.[1] If the flow is downward ($\Delta Z < 0$) the Mach Number decreases for subsonic flow and increases for supersonic flow. There is no choking for this case.

Analytic Equations for the Properties

From the basic physical equations, the following properties are derived as function of Mach Number.

$$\frac{\Delta Z^*}{C^{*2}} g = \frac{1}{k-1}\left[(M)^{\frac{2(1-k)}{1+k}} - 1\right] + \frac{1}{2}\left[(M)^{\frac{4}{k+1}} - 1\right] \quad (16)^2$$

For the definition of Mach Number, (see Eq. 92 Appendix)

$$\frac{T}{T^*} = (M)^{\frac{2(1-k)}{1+k}} \quad (17)$$

With Eq. (17) the other equations are derived:
From the isentropic T- ρ relation and equation of continuity

$$\frac{V}{V^*} = \left(\frac{T^*}{T}\right)^{\frac{1}{k-1}} = \left[M\right]^{\frac{2}{1+k}} \quad (18)$$

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[1]See the section on p. 41.

[2]Developed in the Appendix, p. 75.

$$\frac{\rho}{\rho^*} = \left(\frac{V^*}{V}\right) = \left(\frac{1}{M}\right)^{\frac{2}{1+k}} \tag{19}$$

From the isentropic P-T relation

$$\frac{P}{P^*} = \left(\frac{T}{T^*}\right)^{\frac{k}{k-1}} = \left(\frac{1}{M}\right)^{\frac{2k}{1+k}} \tag{20}$$

From the sonic speed -T relation

$$\frac{C}{C^*} = \left(\frac{T}{T^*}\right)^{\frac{1}{2}} = (M)^{\frac{1-k}{1+k}} \tag{21}$$

And from the definition of stagnation temperature

$$\frac{T_o}{T_o^*} = (M)^{\frac{2(1-k)}{1+k}} \left(\frac{2}{1+k} + \frac{k-1}{k+1} M^2\right)^{\frac{k}{k-1}} \tag{22}$$

Substituting Eq. (20) into the definition of stagnation pressure

$$\frac{P_o}{P_o^*} = \left(\frac{1}{M}\right)^{\frac{2k}{1+k}} \left(\frac{2}{k+1} + \frac{k-1}{k+1} M^2\right)^{\frac{k}{k-1}} \tag{23}$$

Substituting Eq. (20) into the definition of impulse function

$$\frac{F}{F^*} = \left(\frac{1}{M}\right)^{\frac{2k}{1+k}} \left(\frac{1 + kM^2}{1+k}\right) \tag{24}$$

The weight of gas from any section up to the critical level is obtained from the energy equation[1]

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[1]Developed in the Appendix, p. 75.

$$\frac{\Delta Z}{C^{*2}}\, g = \frac{1}{k-1}\left[\left(\frac{\rho^*}{\rho}\right)^{1-k} - 1\right] + \frac{1}{2}\left[\left(\frac{\rho^*}{\rho}\right)^{2} - 1\right]$$

Hence,

$$W^* = \int_0^{z^*} \rho A \frac{g}{g_c}\, dZ$$

$$= \frac{C^*}{g_c} A\rho^* \left[\frac{1+k}{k} - \frac{1}{k}\left(\frac{\rho}{\rho^*}\right)^{k} - \frac{\rho^*}{\rho}\right] \qquad (25)^{1}$$

Numerical Example of Isentropic, Supersonic, Upward Flow of Air

$M_1 = 2$, $T_1 = 500$ R, $P_1 = 100$ psia, $A = 1$ ft^2,

Hence,

$\rho_1 = 0.54$ lbm/ft^3, $V_1 = 2192.14$ fps

$C_1 = 1096.07$ fps, $T_{o1} = 900$ R

$F_1 = 95040$ lbf, $P_{o1} = 783$ psia

Using the energy equation[2] and letting $M_2 = 1.75$

$$\frac{\Delta Z}{C_1^2}\, g = \frac{1}{0.4}\left[1 - 1.14286^{\,1/3}\right] + 2\left[1 - 0.875^{\,4/2.4}\right]$$

$$= 0.2852$$

$\Delta Z = 10{,}640$ ft

$$\frac{T_2}{T_1} = \left(\frac{M_2}{M_1}\right)^{\frac{2(1-k)}{1+k}} = (1.14286)^{1/3} = 1.0455$$

$$\frac{V_2}{V_1} = \left(\frac{T_1}{T_2}\right)^{2.5} = (0.95647)^{2.5} = 0.89471$$

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[1] Developed in the Appendix, p. 75.

[2] Developed in the Appendix, p. 74.

$$\frac{P_2}{P_1} = (1.0455)^{3.5} = 1.1685$$

$$\frac{\rho_2}{\rho_1} = \left(\frac{P_2}{P_1}\right)/\left(\frac{T_2}{T_1}\right) = \frac{1.168514}{1.0455} = 1.1176$$

$$W = F_2 - F_1 = 95{,}040 - 88{,}967 = 6073 \text{ lbf}$$

$$\frac{W}{F_1} = \frac{6073}{95040} = 0.063899$$

$$\frac{C_2}{C_1} = \left(\frac{T_2}{T_1}\right)^{0.5} = 1.0225$$

$$\frac{T_{o2}}{T_{o1}} = \left(\frac{T_2}{T_1}\right)\left(\frac{1+0.2M^2}{1.8}\right) = 1.0455\ (0.89583) = 0.9366$$

$$\frac{P_{o2}}{P_{o1}} = 1.1685\ (0.89583)^{3.5} = 1.1685\ (0.681) = 0.7962$$

$$\frac{F_2}{F_1} = \left(\frac{P_2}{P_1}\right)(0.80114) = 0.9361$$

This process was repeated for other values of M_2, and the results are plotted on Fig. 4. Also plotted on Fig. 5 are the results for isentropic, subsonic, upward flow in which the initial conditions are

$M_1 = 0.5$, $P_1 = 100$ psia, $T_1 = 500$ R

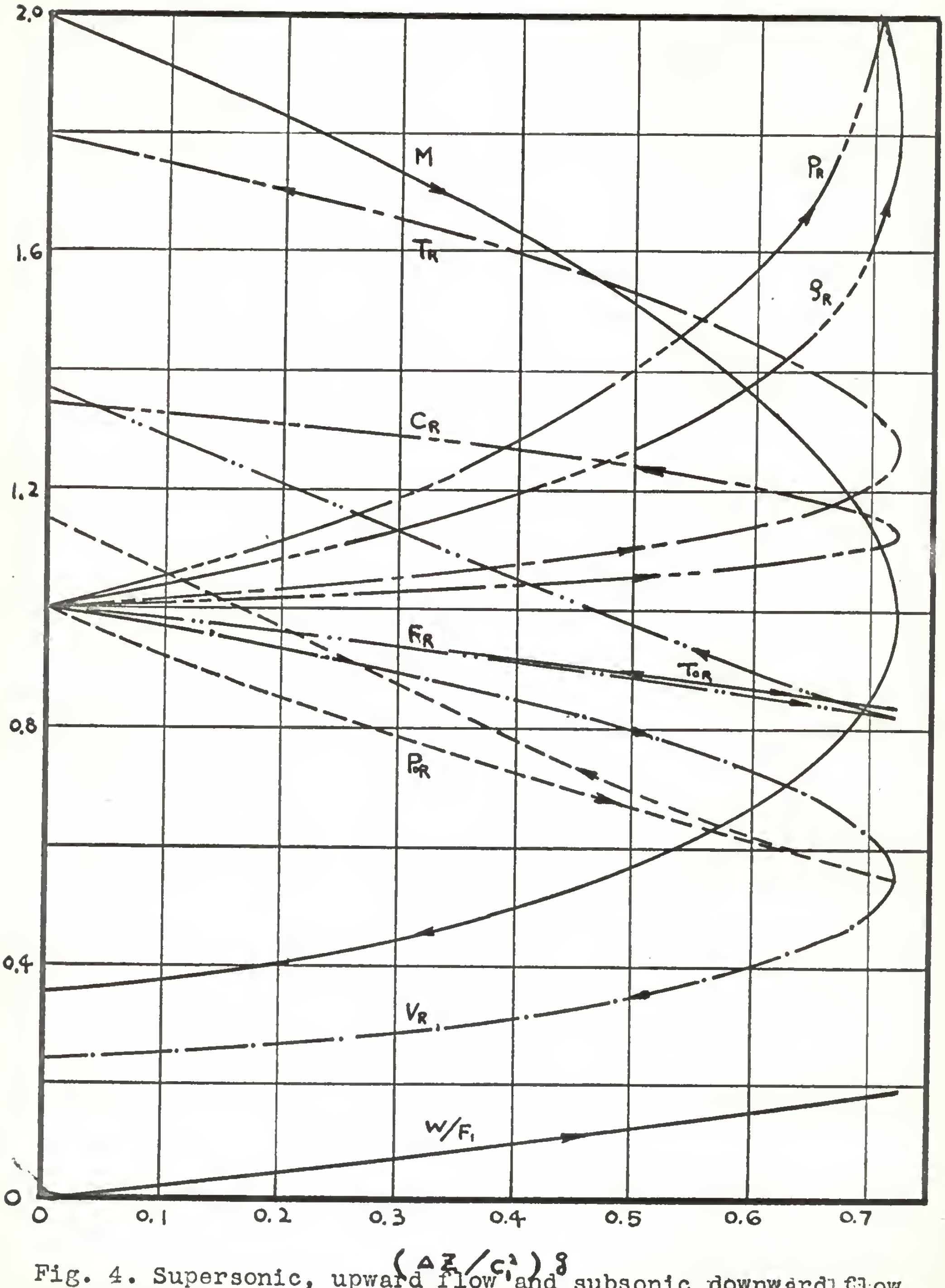

Fig. 4. Supersonic, upward flow and subsonic downward flow in reversible adiabatic process.

NOMENCLATURE

A	flow area, ft^2
a	constant
B	notation for indicating the location where the pressure is the back pressure
b	constant
C	sound speed, fps
C^*	sound speed at Mach Number unity, fps
C'	constant
c	constant
C_p	specific heat at constant pressure, Btu/lbm-R
D	diameter of the pipe, ft
F	impulse function, lbf
f	friction coefficient: $2g_c \tau_w / \rho V^2$
f_m	mean friction coefficient
G	mass velocity: ρV, lbm/ft^2-sec
g	acceleration of gravity, ft/sec^2
g_c	constant of proportionality in Newton's second law, 32.174 lbm-ft/lbf-sec^2
J	conversion factor, 778 ft-lbf/Btu
k	specific heat ratio: C_p/C_v
L	length, ft
M	Mach Number
m	constant
n	constant

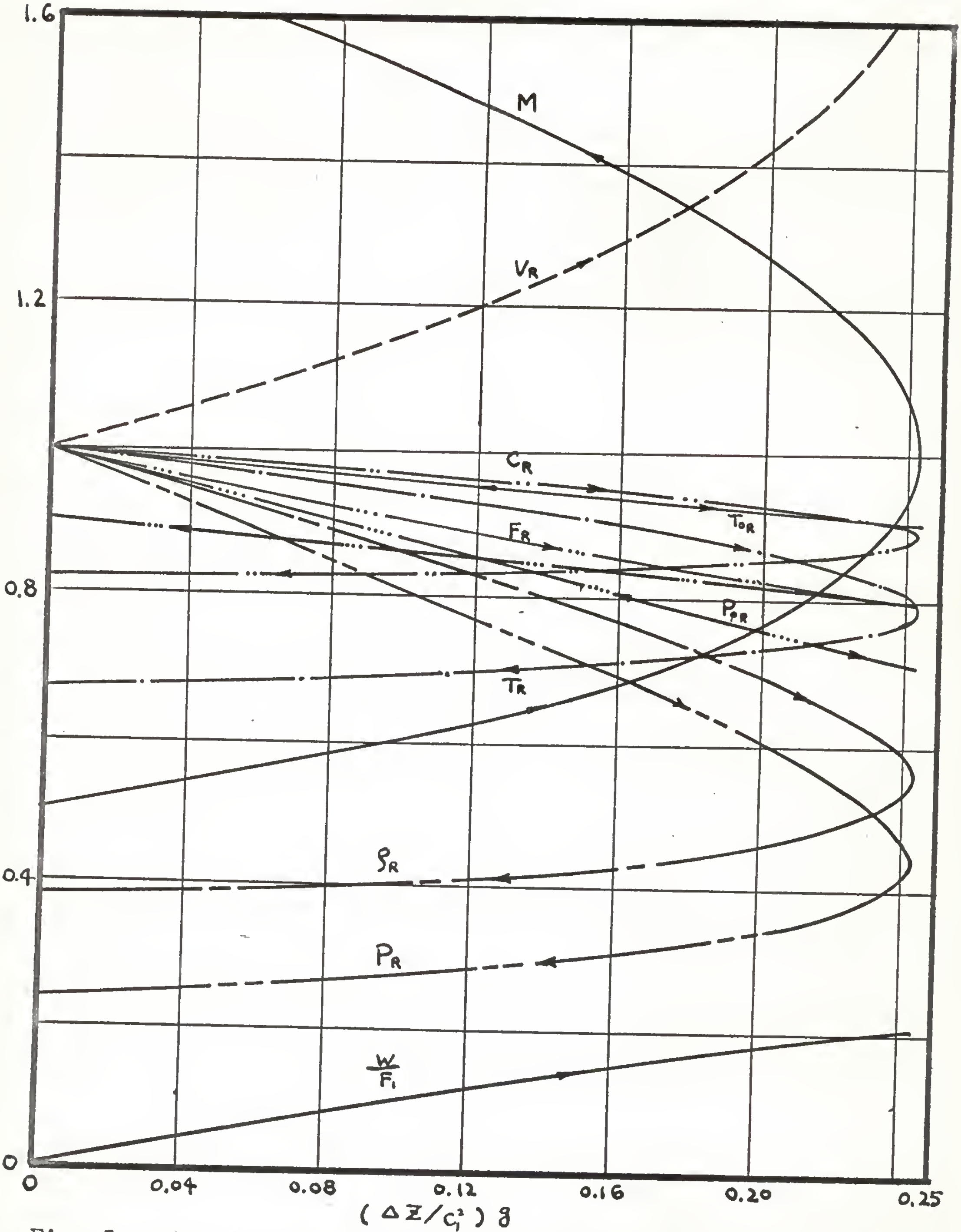

Fig. 5. Subsonic, upward flow and supersonic, downward flow in reversible adiabatic process.

REVERSIBLE DIABATIC FLOW

Physical Equations

Energy Equation:

$$C_p dT + \frac{VdV}{g_c J} + \left(\frac{g}{g_c J} - Q \right) dZ = 0$$

$$\frac{dT}{T} + \frac{k-1}{2} M^2 \frac{dV^2}{V^2} + \left(\frac{g}{g_c} - JQ \right)(k-1) M^2 \left(\frac{dZ}{V^2} g_c \right) = 0 \qquad (26)$$

where Q is the constant heat flux in Btu/lbm-ft.

Momentum Equation:

$$\frac{dP}{\rho} + \frac{g}{g_c} dZ + \frac{VdV}{g_c} = 0$$

$$\frac{dP}{P} \frac{1}{kM^2} + g \frac{dZ}{V^2} + \frac{1}{2} \frac{dV^2}{V^2} = 0 \qquad (27)^{[1]}$$

Perfect Gas Equation:

$$\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T}$$

Definition of Mach Number:

$$\frac{dM^2}{M^2} = \frac{dV^2}{V^2} - \frac{dT}{T}$$

Equation of Continuity:

$$\frac{d\rho}{\rho} + \frac{1}{2} \frac{dV^2}{V^2} = 0$$

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[1]Developed in the Appendix, p. 76.

Analytic Treatment

Combining Eqs. (1), (2), (4), (26) and (27), the differential relation of Mach Number and velocity is obtained. Separating the variables and carrying out the integration, gives

$$V^2 = C' \left[\frac{\frac{g}{g_c} + JQk - JQ + \frac{g}{g_c}k + JQk(k-1)M^2}{M^2} \right]^{-\frac{2(g/g_c + JQk - JQ)}{(g/g_c + JQk - JQ + kg/g_c)}}$$

When $M = 1$, $V = V^*$

Hence,

$$\frac{V}{V^*} = \left[\frac{M^2(\frac{g}{g_c} + JQk^2 - JQ + \frac{g}{g_c}k)}{\frac{g}{g_c} + JQk - JQ + \frac{g}{g_c}k + JQk(k-1)M^2} \right]^{\frac{(g/g_c + JQk - JQ)}{(g/g_c + JQk - JQ + kg/g_c)}}$$

For simplicity, let

$$\alpha_1 = g/g_c + JQk^2 - JQ + kg/g_c$$

$$\alpha_2 = g/g_c + JQk - JQ + kg/g_c$$

$$\alpha_3 = JQk(k-1)$$

$$\alpha_4 = \frac{g/g_c + JQk - JQ}{g/g_c + JQk - JQ + kg/g_c}$$

which results in

$$\frac{V}{V^*} = \left(\frac{\alpha_1 M^2}{\alpha_2 + \alpha_3 M^2} \right)^{\alpha_4} \tag{28}$$[1]

Using the same methods the following are obtained:

$$\frac{\rho}{\rho^*} = \left(\frac{\alpha_2 + \alpha_3 M^2}{\alpha_1 M^2} \right)^{\alpha_4} \tag{29}$$

$$\frac{T}{T^*} = \frac{1}{M^2} \left(\frac{\alpha_1 M^2}{\alpha_2 + \alpha_3 M^2} \right)^{2\alpha_4} \tag{30}$$

$$\frac{P}{P^*} = \frac{1}{M^2} \left(\frac{\alpha_1 M^2}{\alpha_2 + \alpha_3 M^2} \right)^{\alpha_4} \tag{31}$$

$$\frac{T_o}{T_o^*} = \frac{T}{T^*} \left(\frac{1 + \frac{k-1}{2} M^2}{\frac{1+k}{2}} \right)$$

$$= \left(\frac{2M^2}{1+k} + \frac{k-1}{k+1} \right) \left(\frac{\alpha_1 M^2}{\alpha_2 + \alpha_3 M^2} \right)^{2\alpha_4} \tag{32}$$

$$\frac{P_o}{P_o^*} = \left(\frac{2}{k+1} + \frac{k-1}{k+1} M^2 \right)^{\frac{k}{k-1}} \frac{1}{M^2} \left(\frac{\alpha_1 M^2}{\alpha_2 + \alpha_3 M^2} \right)^{\alpha_4} \tag{33}$$

$$\frac{C}{C^*} = \frac{1}{M} \left(\frac{\alpha_1 M^2}{\alpha_2 + \alpha_3 M^2} \right)^{\alpha_4} \tag{34}$$

$$\frac{\Delta Z}{C^{*2}} g_c = \frac{1}{(g/g_c - JQ)(k-1)} \left[\frac{k+1}{2} - \left(\frac{1}{M^2} + \frac{k-1}{2} \right) \left(\frac{\alpha_1 M^2}{\alpha_2 + \alpha_3 M^2} \right)^{2\alpha_4} \right] \tag{35}$$

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[1]Developed in the Appendix, p. 77.

$$\frac{F}{F^*} = \frac{1+kM^2}{M^2(1+k)} \left(\frac{\alpha_1 M^2}{\alpha_2 + \alpha_3 M^2} \right)^{\alpha_4} \tag{36}$$

$$\frac{W^*}{p^*A} = \frac{|F - F^*|}{p^*A} = \left| \left(\frac{1}{M^2} \right) \left(\frac{\alpha_1 M^2}{\alpha_2 + \alpha_3 M^2} \right)^{\alpha_4} - (1+k) \right| \tag{37}$$

$$\frac{S - S^*}{C_p} = \ln \frac{1}{M^{2/k}} \left(\frac{\alpha_1 M^2}{\alpha_2 + \alpha_3 M^2} \right)^{\left(\frac{k+1}{k}\right)\alpha_4} \tag{38}$$ [1]

To determine the Mach Number at which choking occurs[2] (maximum value of entropy), Equation (37) is differentiated.

$$\frac{dS}{C_p} = -\frac{2}{k}\frac{1}{M}\,dM + 2\left(\frac{k+1}{k}\right)\frac{\alpha_4}{M}\left(\frac{\alpha_2}{\alpha_2 + \alpha_3 M^2}\right) dM \tag{38-a}$$

Let $dS/dM = 0$,

$$M = \frac{\alpha_2}{\alpha_3}\left(\alpha_4 k + \alpha_4 - 1\right)^{1/2}$$

Substituting α_2, α_3 and α_4,

$$M^2 = \frac{JQk^2 - JQk}{JQk(k-1)} = 1$$

This is the Mach Number at which the entropy is a maximum. Differentiating Eq. (35) with respect to Mach Number and setting it equal to zero, gives

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[1]Developed in the Appendix, p. 79.

[2]Developed in the Appendix, p. 80.

$$M = \left(\frac{2 - \frac{1}{\alpha_4}}{\frac{\alpha_3}{\alpha_2 \alpha_4} + 1 - k} \right)^{1/2}$$

Substitution of α_2, α_3 and α_4, gives

$$M = 1$$

Hence,

$$\frac{\Delta Z^*}{C^{*2}} g_c = \frac{1}{(g/g_c - JQ)(k-1)} \left(\frac{k+1}{2} - \frac{k+1}{2} \right) = 0$$

These equation show that the length of the duct is at its critical maximum length when the final Mach Number reaches unity; at the same time, the entropy is at its maximum value.

Consequently, there is a limiting length of the vertical duct for each initial Mach Number for the cases in which there are: (1) heat input and upward flow, and (2) heat input and downward flow, which is also the case for horizontal flow.

A further increase in the length of the pipe beyond the critical length for subsonic flow, produces a choked condition in which the initial Mach Number is reduced in value. In the case of supersonic flow, increasing the length of pipe beyond the critical length produces a normal shock within the pipe or in the nozzle which feeds the pipe. This normal-shock condition will be analyzed later.

If heat is rejected by the gas in its vertical flow, the Mach Number is decreased when the flow is subsonic, and the Mach

Number is increased if the flow is supersonic.

It is also interesting to note that the maximum entropy change for heating with downward flow is larger than the maximum entropy change for heating with upward flow, provided that the initial conditions are identical in both cases.

In addition, the Mach Number at the maximum static temperature is considered. Differentiating Eq. (30) and setting it equal to zero, the result is obtained as follows.

$$M^2 = \frac{\alpha_2(2\alpha_4 - 1)}{\alpha_3}$$

and

$$M = \sqrt{\frac{JQ - g/g_c}{JQk}} = \sqrt{\frac{1 - (g/g_c)/(JQ)}{k}} \qquad (39)^1$$

The Mach Number at maximum temperature, for subsonic flow, increases toward $1/\sqrt{k}$ as the heat flow per unit mass of gas (Q) per unit length of pipe increases. But, in the case where the heat input is small, so that Q is equal to $g/(g_c J)$, the highest value of static temperature occurs at the initial state, and decreases thereafter. In the Rayleigh-Line process for horizontal flow, the maximum static temperature occurs when $M = 1/\sqrt{k}$. It is found that the value of the Mach Number at the maximum static temperature is not a constant. It depends on the amount of heat influx or efflux per unit length of pipe.

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[1]Developed in the Appendix, p. 84.

Numerical Example of Reversible Diabatic, Supersonic, Upward Flow of Air

The initial conditions are

$$M_1 = 2,\ P_1 = 100 \text{ psia},\ T_1 = 500 \text{ R}$$

and

$$Q = 100 \text{ Btu/lbm-ft}$$

From Eq. (97) (see Appendix p. 77)

$$\frac{1}{2}(M^2-1)\frac{dV^2}{V^2} + (4.112\ M^2)\ \frac{dZ}{V^2}\ g_c = 0 \qquad (97')$$

From Eq. (98) (see Appendix p. 78)

$$(1+0.2\ M^2)\frac{dV^2}{V^2} - \frac{dM^2}{M^2} - 2.712\ M^2\ \frac{dZ}{V^2}\ g_c = 0 \qquad (98')$$

Combination of Eqs. (97') and (98'), and elimination of $\frac{dZ}{V^2} g_c$, gives

$$(2.756 + 2.1784\ M^2)\ \frac{dV^2}{V^2} = 4.1120\ \frac{dM^2}{M^2}$$

$$\frac{dV^2}{V^2} = \frac{1}{0.6702+0.5297\ M^2}\ \frac{dM^2}{M^2}$$

and

$$\ln\left(\frac{V_2}{V_1} \right)^2 = \ln\left[\left(\frac{M_2}{M_1} \right)^2 \left(\frac{0.6702 + 0.5297\ M^2}{0.6702 + 0.5297\ M^2} \right) \right]^{1.4920}$$

Lot $M_2 = 1.9$

$$\frac{V_2}{V_1} = 0.9810$$

$$\frac{T_2}{T_1} = \left(\frac{M_1}{M_2} \right)^2 \left(\frac{V_2}{V_1} \right)^2 = (1.1080)(0.9810)^2 = 1.0663$$

$$\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2} = 1.0194$$

$$\frac{T_{o2}}{T_{o1}} = 1.0663\left(\frac{1 + 0.2\ M_2^2}{1.8} \right) = 1.0663(0.95667) = 1.02$$

$$\frac{P_2}{P_1} = \left(\frac{\rho_2}{\rho_1} \right)\left(\frac{T_2}{T_1} \right) = 1.0194(1.0663) = 1.0869$$

$$\frac{P_{o2}}{P_{o1}} = \left(\frac{P_2}{P_1} \right)(0.8564) = 0.9306$$

$$\frac{\Delta S}{C_p} = \ln(1.0663) - (0.2857) \ln(1.0868) = 0.04041$$

$$\frac{\Delta Z}{C_1^2} g_c = \left[\left(\frac{T_2}{T_1} \right) + 0.8\left(\frac{V_2}{V_1} \right)^2 - 1.8009 \right] \left(\frac{1}{2.712} \right)$$

$$= 0.01255$$

$$\frac{C_2}{C_1} = 1.0326$$

$$\frac{F_2}{F_1} = \frac{15649.9}{95040}(6.054) = 0.99689$$

$$\frac{W}{F_1} = \frac{95040 - 94744.6}{95040} = \frac{295.4}{95040} = 0.003108$$

This procedure was repeated for other values of M_2, and the results are plotted on Fig. 7. Also plotted on Fig. 8 are the

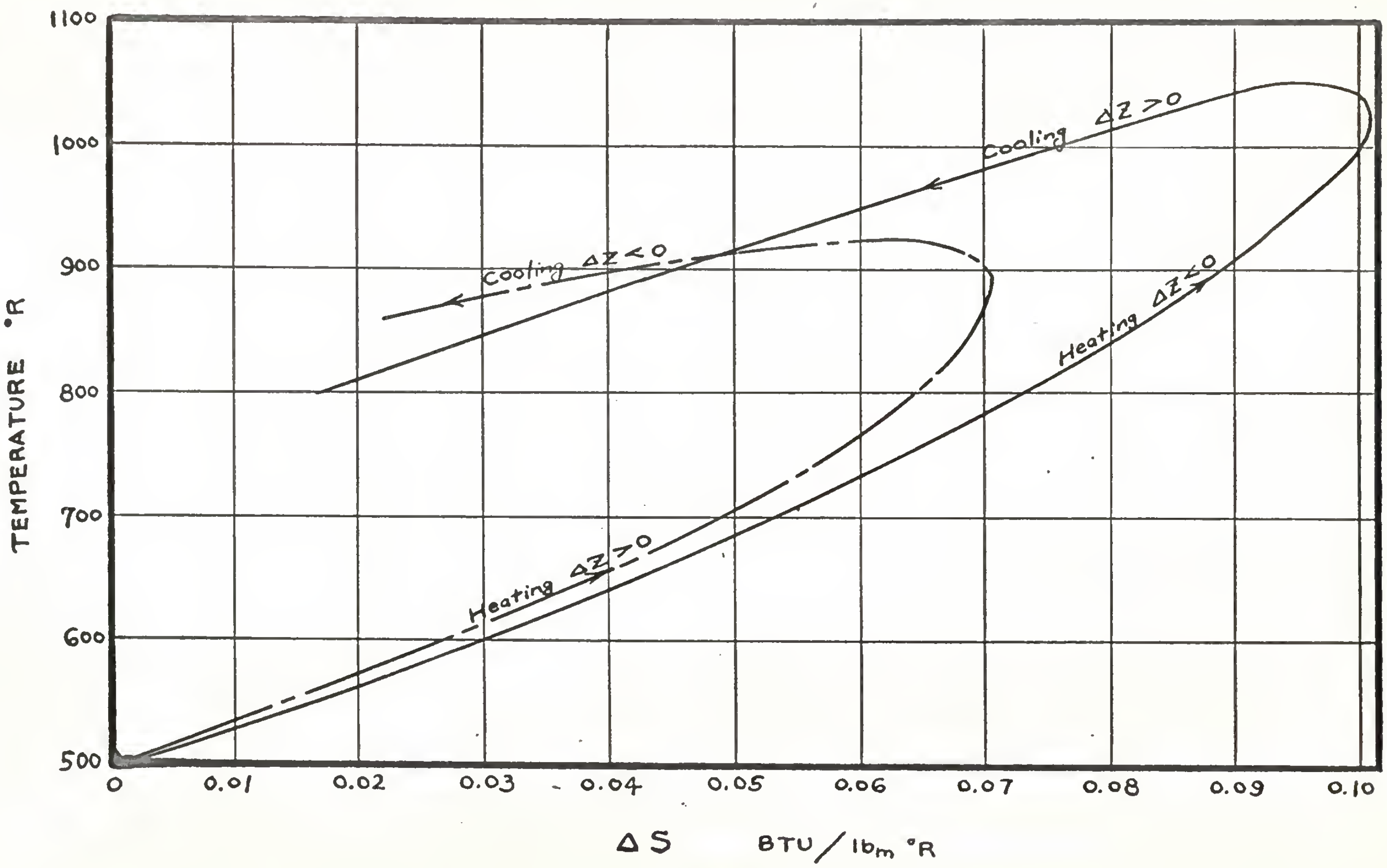

Fig. 6. Rayleigh line for vertical flow at $M_1 = 2$, $P_1 = 100$ psia & $T_1 = 500$°R

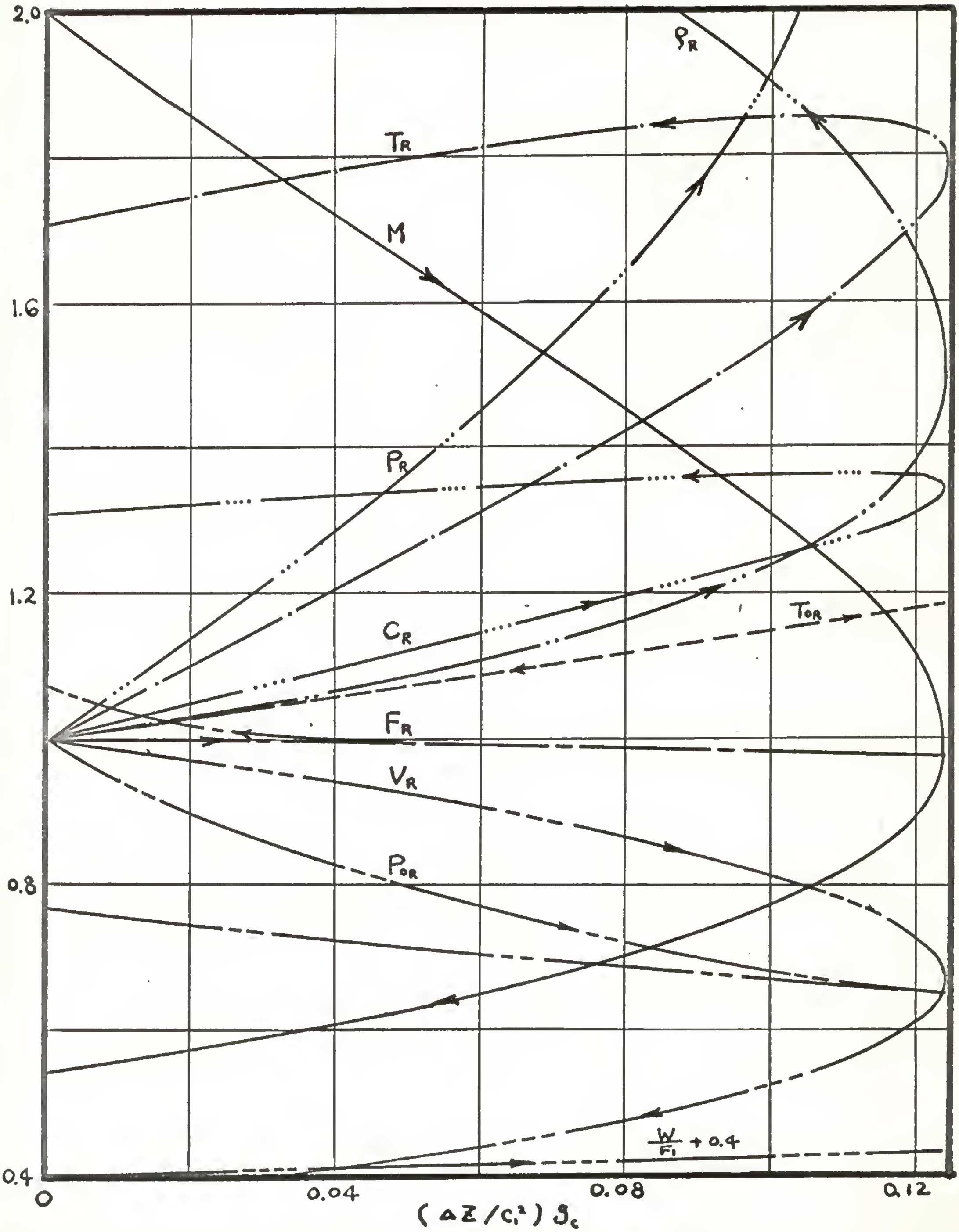

Fig. 7. Reversible heating, supersonic upward flow and reversible cooling, subsonic downward flow.

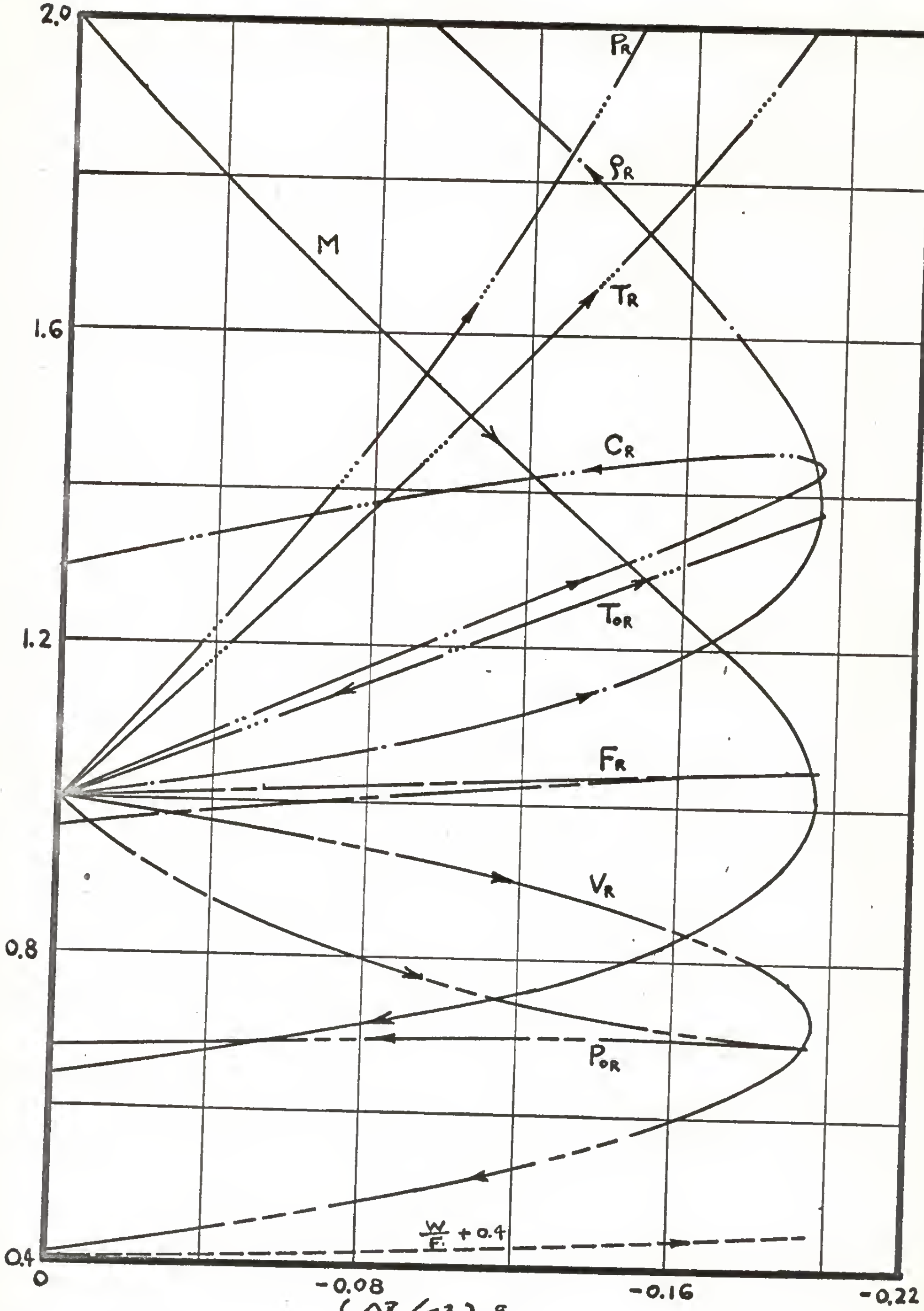

Fig. 8. Supersonic downward flow in reversible heating process & subsonic upward flow in reversible cooling process.

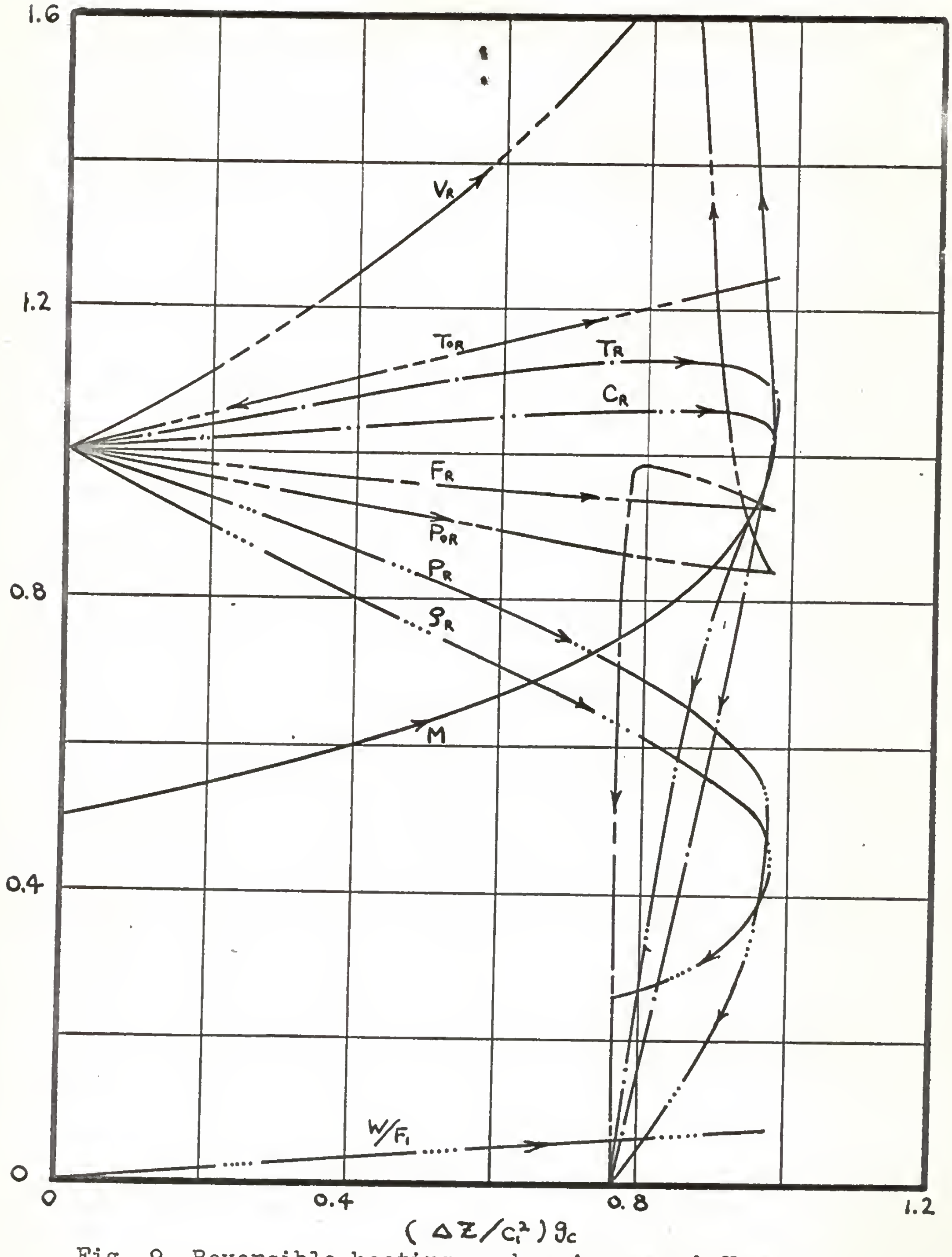

Fig. 9. Reversible heating, subsonic upward flow and reversible cooling, supersonic downward flow.

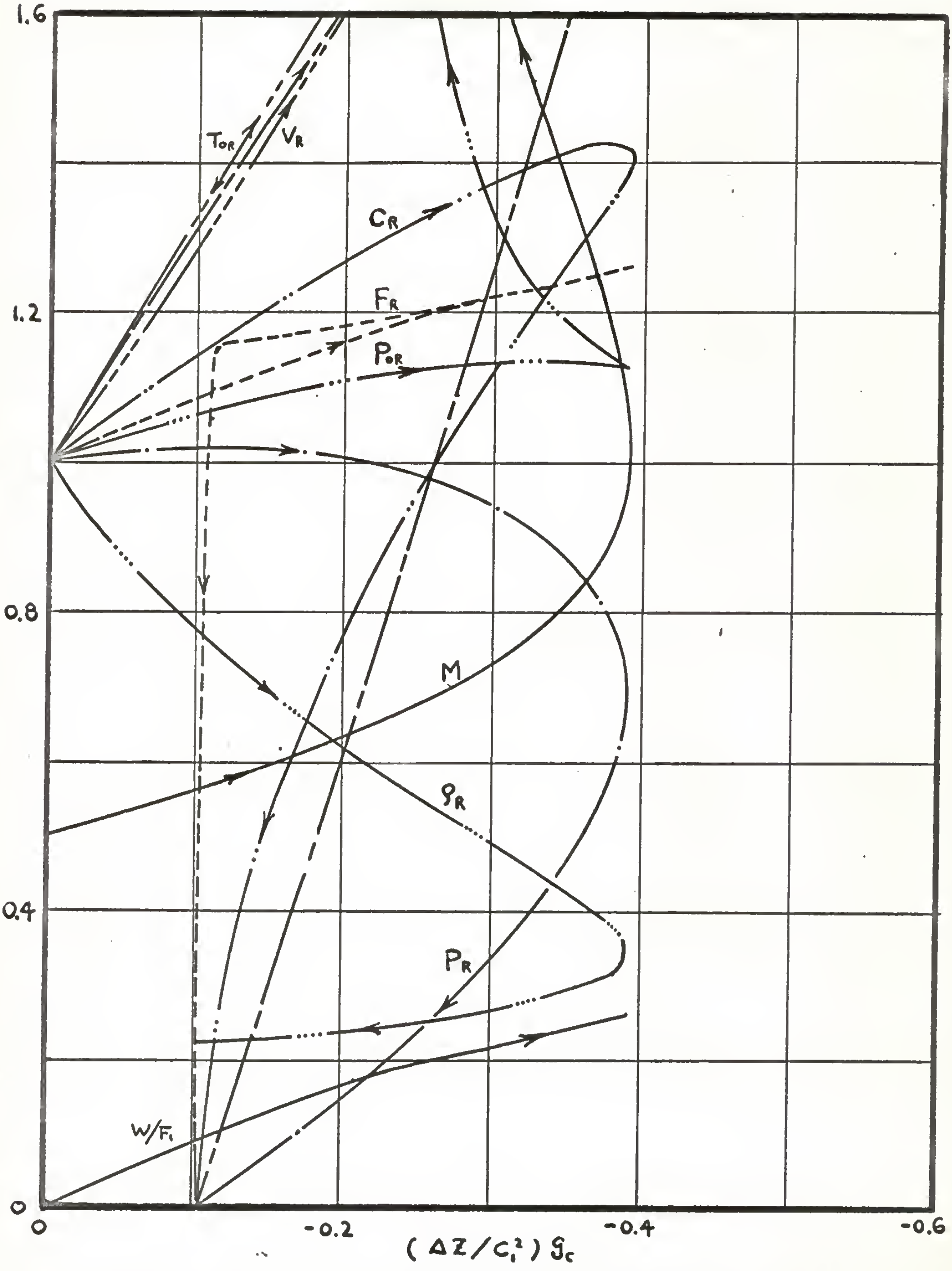

Fig. 10. Reversible heating, subsonic, downward flow and reversible cooling, supersonic upward flow.

results for heating with supersonic, downward flow. Fig. 9 shows the results for heating with subsonic, upward flow, and Fig. 10 shows the results for heating with subsonic, downward flow.

IRREVERSIBLE ADIABATIC FLOW

Physical Equations

Energy Equation:

$$\frac{dT}{T} + \frac{k-1}{2} M^2 \frac{dV^2}{V^2} + \frac{k-1}{k} \frac{1}{RT} \frac{g}{g_c} dZ = 0 \qquad (40)$$

Momentum Equation:

$$\frac{dP}{P} + \frac{g}{g_c} \frac{dZ}{RT} + \frac{kM^2}{2} \frac{dV^2}{V^2} + kM^2 \frac{2f}{D} dZ = 0 \qquad (41)$$ [1]

Perfect Gas Equation:

$$\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T}$$

Definition of Mach Number:

$$\frac{dM^2}{M^2} = \frac{dV^2}{V^2} - \frac{dT}{T}$$

Equation of Continuity:

$$\frac{d\rho}{\rho} = -\frac{1}{2} \frac{dV^2}{V^2}$$

---

[1]Developed in the Appendix, p. 85.

Analytic Treatment

Combining the above equations, eliminating some differential variables, and rearranging gives

$$\left[\frac{g}{g_c}\frac{1}{2RT}\left(\frac{k+1}{k}\right) + k\left(\frac{2f}{D}\right)M^2 + \left(\frac{k-1}{2}\right)k\left(\frac{2f}{D}\right)M^4\right]\frac{dV^2}{V^2}$$

$$= \left[k\left(\frac{2f}{D}\right)M^2 + \frac{g}{g_c}\frac{1}{kRT}\right]\frac{dM^2}{M^2}$$

Integrating from the critical section to a specified section yields

$$\ln\left(\frac{V}{V^*}\right)^2 = \ln\left\{\frac{k(k-1)\left(\frac{2f_m}{D}\right)M^2 + k\left(\frac{2f_m}{D}\right) - \left[k^2\left(\frac{4f_m}{D}\right) - (k^2-1)\frac{g\,2f_m}{g_c RT_m D}\right]^{1/2}}{k(k-1)\left(\frac{2f_m}{D}\right)M^2 + k\left(\frac{2f_m}{D}\right) + \left[k^2\left(\frac{4f_m}{D}\right) - (k^2-1)\frac{g\,2f_m}{g_c RT_m D}\right]^{1/2}}\right\}^{\dfrac{\left(\frac{k^2}{k-1}\right)\left(\frac{2f_m}{D}\right)}{\left[k^2\left(\frac{4f_m^2}{D}\right) - (k^2-1)\frac{g\,f_m^2}{g_c RT_m D}\right]^{1/2}}}$$

$$\left[\frac{M^4}{\left(\frac{k+1}{k}\right)\frac{g}{g_c 2RT_m} + k\left(\frac{2f_m}{D}\right) + k\left(\frac{k-1}{k}\right)\left(\frac{2f_m}{D}\right)M^4}\right]^{\frac{1}{k+1}}$$

$$- \ln\left\{\frac{k^2\left(\frac{2f_m}{D}\right) - \left[k^2\left(\frac{4f_m^2}{D}\right) - (k^2-1)\frac{g}{g_c}\frac{2\,f_m}{RT_m D}\right]^{1/2}}{k^2\left(\frac{2f_m}{D}\right) + \left[k^2(4f_m^2/D) - (k^2-1)\frac{g}{g_c}\frac{2\,f_m}{RT_m D}\right]^{1/2}}\right\}^{\dfrac{\left(\frac{k^2}{k+1}\right)\left(\frac{2f_m}{D}\right)}{\left[k^2\left(\frac{4f_m^2}{D}\right) - (k^2-1)\frac{g}{g_c}\frac{2\,f_m}{RT_m D}\right]^{1/2}}}$$

$$\left[\frac{1}{\frac{k+1}{k}\frac{g}{g_c}\frac{1}{2RT_m}+k\left(\frac{k+1}{2}\right)\frac{2f_m}{D}}\right]^{\frac{1}{k+1}}$$

Where $V = V^*$ at $M = 1$

T_m = Mean temperature between the sections.

$$f_m = \frac{1}{L}\int_o^L f\, dZ$$

The details of derivations are described in the Appendix (p. 86)

For simplicity, let

$$\beta_1 = \left[k^2 4f_m^2/D - (k^2-1)(g2f_m/g_c RT_m D)\right]^{1/2}$$

$$\beta_2 = \frac{k(2f_m/D) - \beta_1}{k(k-1)(2f_m/D)}$$

$$\beta_3 = \frac{k^2(2f_m/D) - \beta_1}{k^2(2f_m/D) + \beta_1}$$

$$\beta_4 = \frac{\left[k(2f_m/D) + \beta_1\right]\left[k^2(2f_m/D) - \beta_1\right]}{\left[k(k-1)(2f_m/D)\right]\left[k^2(2f_m/D) + \beta_1\right]}$$

$$\beta_5 = \frac{\frac{k+1}{k}(g/g_c 2RT_m)}{\frac{k+1}{k}(g/g_c 2RT_m) + k(\frac{k+1}{2})(2f_m/D)}$$

$$\beta_6 = \frac{k2f_m/D}{\frac{k+1}{k}(g/g_c 2RT_m) + k(\frac{k+1}{2})(2f_m/D)}$$

$$\beta_7 = \frac{k(\frac{k-1}{2})(2f_m/D)}{\frac{k+1}{k}(g/g_c 2RT_m) + k(\frac{k+1}{2})(2f_m/D)}$$

$$\beta_8 = \frac{k^2}{k+1}(f_m/D\beta_1)$$

The above equation simplifies to

$$\frac{V}{V^*} = \left[\frac{M^2 + \beta_2}{M^2 + \beta_4}\right]^{\beta_8} \left(\frac{M^4}{\beta_5 + \beta_6 M^2 + \beta_7 M^4}\right)^{\frac{1}{2(k+1)}} \qquad (42)^1$$

Also, the other relations are

$$\frac{\rho}{\rho^*} = \left[\frac{\beta_3 M^2 + \beta_4}{M^2 + \beta_2}\right]^{\beta_8} \left[\frac{\beta_5}{M^4} + \frac{\beta_6}{M^2} + \beta_7\right]^{\frac{1}{2(k+1)}} \qquad (43)$$

$$\frac{T}{T^*} = \frac{1}{M^2}\left(\frac{V}{V^*}\right)^2$$

$$= \frac{1}{M^2} \left[\frac{M^2 + \beta_2}{\beta_3 M^2 + \beta_4}\right]^{2\beta_8} \left[\frac{M^4}{\beta_5 + \beta_6 M^2 + \beta_7 M^4}\right]^{\frac{1}{k+1}} \qquad (44)$$

$$\frac{P}{P^*} = \frac{\rho}{\rho^*}\frac{T}{T^*}$$

$$= \frac{1}{M^2} \left[\frac{M^2 + \beta_2}{\beta_3 M^2 + \beta_4}\right]^{\beta_8} \left[\frac{M^4}{\beta_5 + \beta_6 M^2 + \beta_7 M^4}\right]^{\frac{1}{2(k+1)}} \qquad (45)$$

$$\frac{T_o}{T_o^*} = \left(\frac{T}{T^*}\right)\left(\frac{1 + \left(\frac{k-1}{2}\right) M^2}{\frac{1+k}{2}}\right)$$

$$= \left(\frac{M^2 + \beta_2}{\beta_3 M^2 + \beta_4}\right)^{2\beta_8} \left(\frac{M^4}{\beta_5 + \beta_6 M^2 + \beta_7 M^4}\right)^{\frac{1}{k+1}} \left(\frac{2M^2}{1+k} + \frac{k-1}{k+1}\right) \qquad (46)$$

---

[1]Developed in the Appendix, p. 86.

$$\frac{P_o}{P_o^*} = \left(\frac{P}{P^*}\right)\left(\frac{1+\left(\frac{k-1}{2}\right)M^2}{\frac{1+k}{2}}\right)$$

$$= \frac{1}{M^2}\left(\frac{M^2+\beta_2}{\beta_3M^2+\beta_4}\right)^{\beta_8}\left(\frac{M^4}{\beta_5+\beta_6M^2+\beta_7M^4}\right)^{\frac{1}{2(k+1)}}\left(\frac{2}{k+1}-\frac{k-1}{k+1}M^2\right)^{\frac{k}{k-1}} \quad (47)$$

$$\frac{C}{C^*} = \frac{1}{M}\left(\frac{M^2+\beta_2}{\beta_3M^2+\beta_4}\right)^{\beta_8}\left(\frac{M^4}{\beta_5+\beta_6M^2+\beta_7M^4}\right)^{\frac{1}{2(k+1)}} \quad (48)$$

$$\frac{F}{F^*} = \frac{1+kM^2}{M^2(1+k)}\left(\frac{M^2+\beta_2}{M^2+\beta_4}\right)^{\beta_8}\left(\frac{M^4}{\beta_5+\beta_6M^2+\beta_7M^4}\right)^{\frac{1}{2(k+1)}} \quad (49)$$

$$\frac{W^*}{P^*A} = \frac{|F-F^*|}{P^*A} = \left|\left(\frac{P}{P^*}\right)\left(1+kM^2\right)-\left(1+k\right)\right|$$

$$= \left|\frac{(1+kM^2)}{M^2}\left(\frac{M^2+\beta_2}{\beta_3M^2+\beta_4}\right)^{\beta_8}\left(\frac{M^4}{\beta_5+\beta_6M^2+\beta_7M^4}\right)^{\frac{1}{2(k+1)}}-(1+k)\right| \quad (50)$$

$$\frac{S-S^*}{C_p} = \ln\frac{T/T^*}{(P/P^*)^{(k-1)/k}}$$

$$= \ln\frac{1}{M^{2/k}}\left(\frac{M^2+\beta_2}{\beta_3M^2+\beta_4}\right)^{\frac{\beta_8(k-1)}{k}}\left(\frac{M^4}{\beta_5+\beta_6M^2+\beta_7M^4}\right)^{\frac{1}{2k}} \quad (51)$$

$$\frac{\Delta Z^*}{(C^*)^2} = \frac{1}{k-1}\left[\left(\frac{1}{M^2}+\frac{k-1}{2}\right)\left(\frac{M^2+\beta_2}{\beta_3M^2+\beta_4}\right)^{2\beta_8}\left(\frac{M^4}{\beta_5+\beta_6M^2+\beta_7M^4}\right)^{\frac{1}{k+1}}-\frac{k+1}{2}\right] \quad (52)$$

<u>Numerical Example of Irreversible Adiabatic, Supersonic, Upward Flow of Air</u>

Given $M_1 = 2$, $P_1 = 100$ psia, $T_1 = 500$ R, $D = 1$ ft, $f_m = 0.005$

Determine all the properties at $M_2 = 1.8$

For any two sections Eq. (42) becomes

$$\frac{V_2}{V_1} = \left[\frac{k(k-1)(2f_m/D)M_2^2+(k2f_m/D)-\beta_1}{k(k-1)(2f_m/D)M_2^2+(k2f_m/D)+\beta_1} \cdot \frac{k(k-1)(2f_m/D)M_1^2+k(2f_m/D)+\beta_1}{k(k-1)(2f_m/D)M_1^2+k(2f_m/D)-\beta_1}\right]^{\frac{k^2\ 2f_m}{(k+1)D\beta_1}}$$

$$\cdot\left[\frac{\frac{(k+1)}{k}(g/g_c 2RT_m)+(k2f_m M_1^2/D)+k(\frac{k-1}{2})(2f_m/D)M_1^4}{\frac{(k+1)}{k}(g/g_c 2RT_m)+(k2f_m M_2^2/D)+k(\frac{k-1}{2})(2f_m/D)M_2^4} \cdot \frac{M_2^4}{M_1^4}\right]^{\frac{1}{k+1}}$$

Let $T_m = 523\ ^\circ R$

$$\beta_1 = 0.01(\ 1.96-1.801/523\ )^{1/2} = 0.01(1.9565564)^{1/2}$$

$$= 0.013987696$$

$$\frac{V_2}{V_1} = \left(\ \frac{0.018156304}{0.046131696}\ \frac{0.050387696}{0.022412304}\ \right)^{0.291932}$$

$$\left(\ \frac{10.4976}{16}\ \frac{0.100830749}{0.074784029}\ \right)^{0.20833333}$$

$$= (\ 0.9649143\ )(\ 0.97478068\ )$$

$$= 0.94057982$$

$$\left(\frac{V_2}{V_1}\right)^2 = 0.88469040$$

$$\frac{T_2}{T_1} = \left(\frac{M_1}{M_2}\right)^2 \left(\frac{V_2}{V_1}\right)^2 = \frac{35387616}{3.24} = 1.092210$$

$$T_2 = 546 \ R, \quad T_m = \frac{500 + 546}{2} = 523 \ R$$

which agrees with the assumed value of T_m.

Thus follows

$$\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2} = 1.06317$$

$$\frac{P_2}{P_1} = \left(\frac{\rho_2}{\rho_1} \frac{T_2}{1} \right) = 1.161205$$

$$\frac{C_2}{C_1} = \left(\frac{T_2}{T_1} \right)^{1/2} = 1.0451$$

$$\frac{\Delta s}{C_p} = \ln\left(\frac{T_2}{T_1} \right) - 0.285714 \ln\left(\frac{P_2}{P_1} \right)$$

$$= 0.0882 - 0.04271 = 0.04548$$

Since

$$\rho_m = \frac{\rho_1 + \rho_2}{2} = 0.5570559$$

and

$$G = \rho V = 1183.7556$$

From momentum equation

$$(P_2 - P_1) + \frac{G}{g_c}(V_2 - V_1) + \left(\frac{G^2 \, 4f}{2 g_c \rho_m} + \rho_m \right) \Delta Z = 0$$

$$2321.352 - 4780.4466 + 779.35512 \, \Delta Z = 0$$

$$\Delta Z = 3.146404 \text{ ft}$$

The weight of gas is

$$W = \rho_m A \Delta Z = 0.5570559(0.7853982)(3.146404)$$

$$= 1.3765854 \text{ lbf}$$

$$\frac{W}{F_1} = \frac{1.3765854}{74644.245} = 0.00001844194$$

and

$$\frac{F_2}{F_1} = \frac{116.1205(144)(1+1.4 \times 3.24)}{95040} = 0.9740046$$

$$\frac{P_{\circ 2}}{P_{\circ 1}} = (1.161205)(0.73433877) = 0.85271785$$

$$\frac{T_{\circ 2}}{T_{\circ 1}} = \left(\frac{T_2}{T_1}\right)(0.915555) = 0.999978$$

This procedure can be used to determine the properties of other values of the Mach Number, and the results are plotted on Fig. 12. Also plotted on Fig. 13 are the results for downward flow.

NORMAL SHOCK FOR UPWARD FLOW

Governing Equations for Normal Shock

$$M_y^2 = \frac{M_x^2 + 2/(k-1)}{\frac{2k}{k-1} M_x^2 - 1} \tag{53}$$

$$T_{\circ x} = T_{\circ y} \tag{54}$$

$$\frac{T_y}{T_x} = \frac{1 + \frac{k-1}{2} M_x^2}{1 + \frac{k-1}{2} M_y^2} \tag{55}$$

$$\frac{P_y}{P_x} = \frac{1 + kM_x^2}{1 + kM_y^2} \tag{56}$$

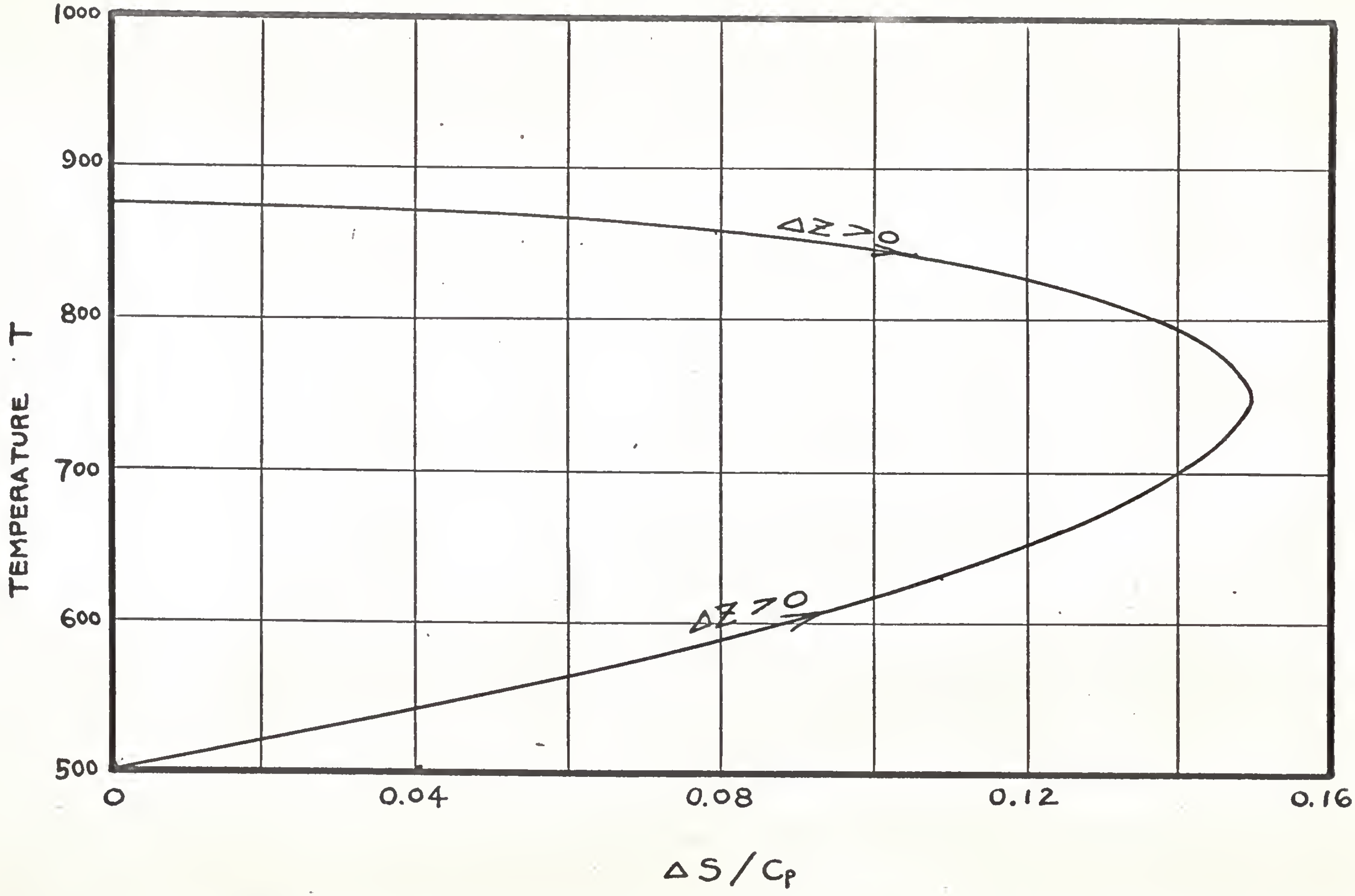

Fig. 11. Fanno line for vertical flow at $M_1 = 2$, $T_1 = 500°R$ & $P_1 = 100$ psia.

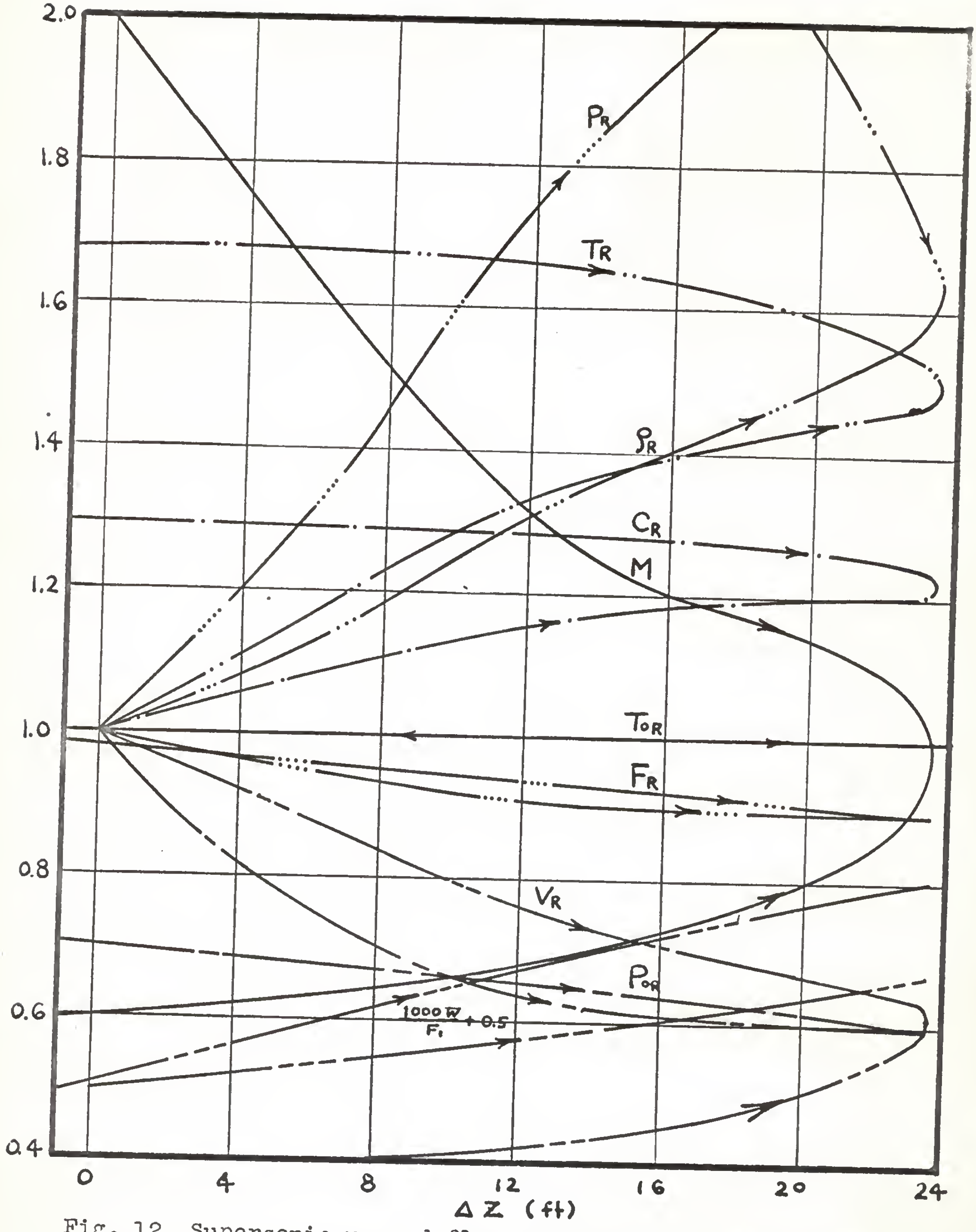

Fig. 12. Supersonic upward flow and subsonic upward flow in irreversible adiabatic process.

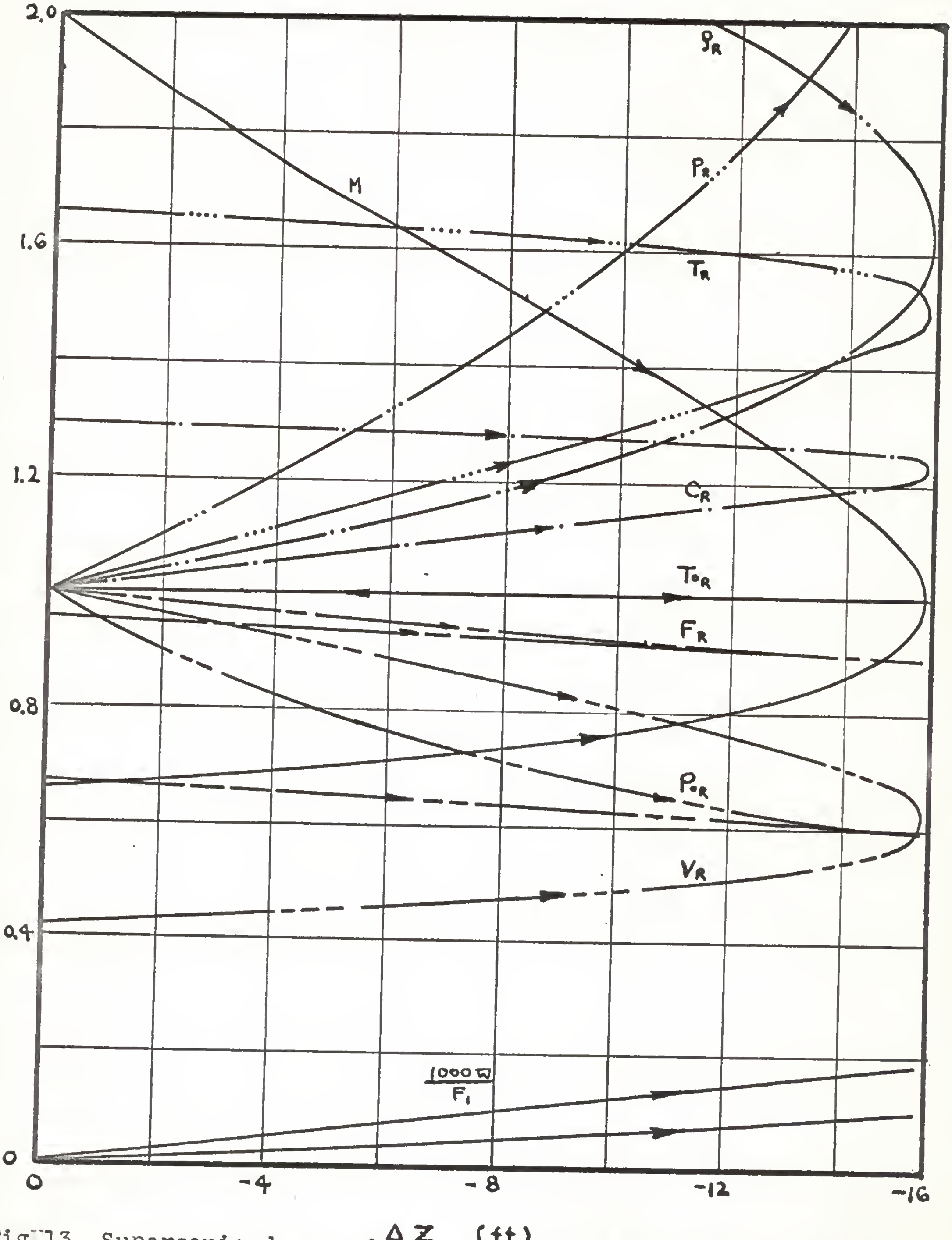

Fig. 13. Supersonic downward flow and subsonic downward flow in irreversible adiabatic process.

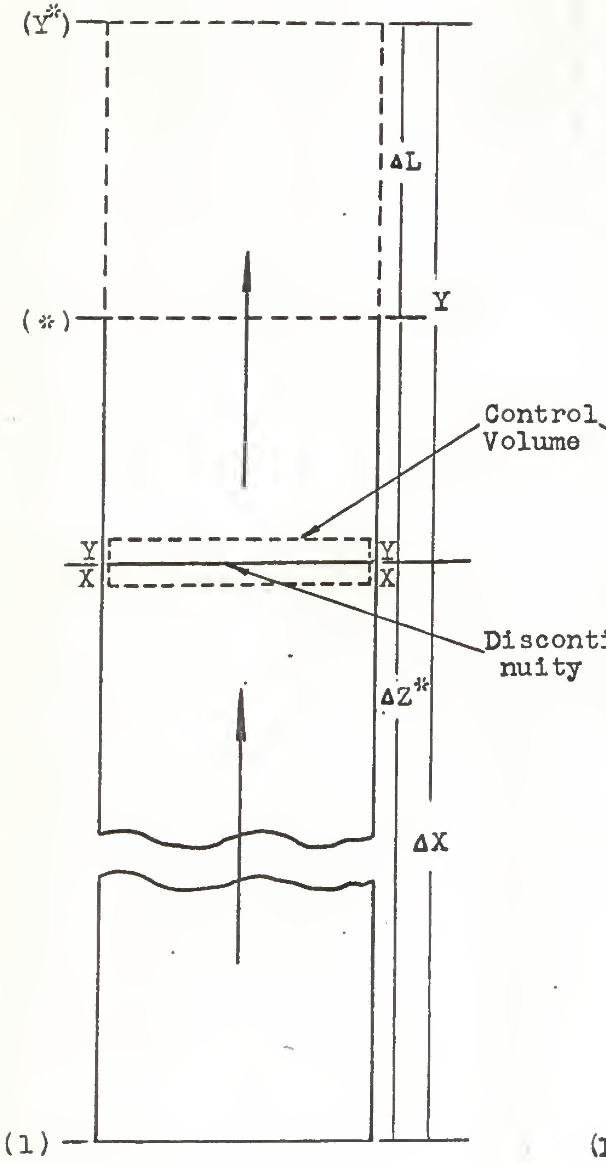

Fig. 14. Normal shock due to variable length of pipe with a given initial condition (1).

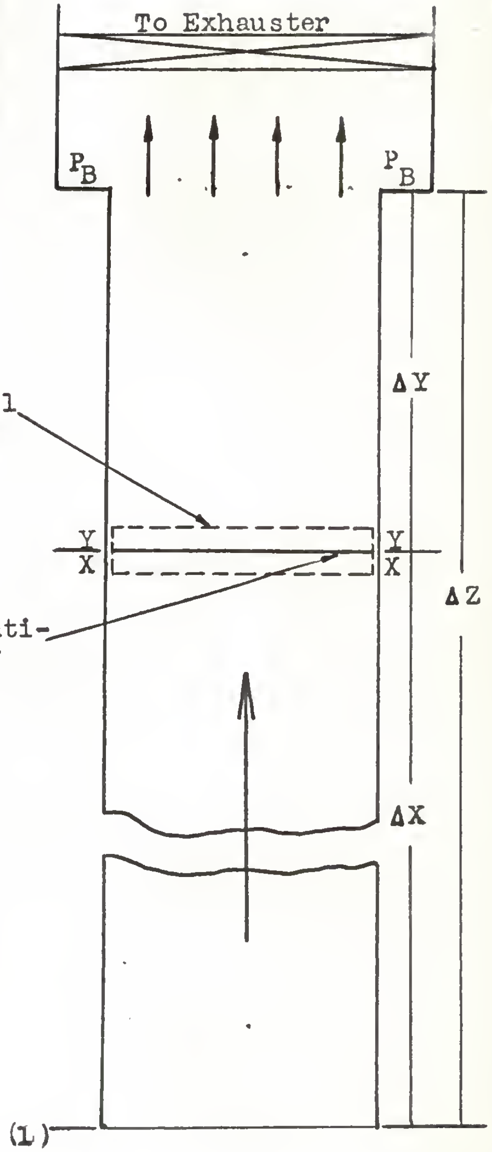

Fig. 15. Normal shock due to variable back pressure and a fixed length of pipe with a given initial condition (1).

Normal Shock Location in Isentropic Upward Flow

1. Variable length of the pipe:

M_1, P_1 and T_1 are given at an initial level. M_1 is always greater than one (supersonic flow). ΔL denotes the increased length over the section where the Mach Number was unity when no shock was present (see Figs. 14 and 15).

For a given initial condition, there exists a corresponding critical condition in the pipe. This section is denoted as *, the critical elevation ΔZ^*, and the temperature T^*.

Suppose a shock occurs within the pipe at the height ΔX:

The energy equation for state (1) to (x) is

$$C_p(T_x - T_1) + \frac{V_x^2 - V_1^2}{2g_c J} + \frac{g}{Jg_c}(Z_x - Z_1) = 0$$

From the bottom equation of page 74 in the Appendix:

$$kRg_cT_1\left\{\frac{1}{k-1}\left[\left(\frac{M_x}{M_1}\right)^{\frac{2(1-k)}{1+k}} - 1\right] + \frac{M_1^2}{2}\left[\left(\frac{M_x}{M_1}\right)^{\frac{4}{k+1}} - 1\right]\right\} + g\Delta X = 0 \qquad (57)$$

The energy equation for state (1) to state (*) is

$$kRg_cT_1\left\{\frac{1}{k-1}\left[\left(\frac{1}{M_1}\right)^{\frac{2(1-k)}{1+k}} - 1\right] + \frac{M_1^2}{2}\left[\left(\frac{1}{M_1}\right)^{\frac{4}{k+1}} - 1\right]\right\} + g\Delta Z^* = 0$$

giving

$$\Delta Z^* = kRT_1 \frac{g_c}{g} \left\{ \frac{1}{k-1} \left[1 - \left(\frac{1}{M_1} \right)^{\frac{2(1-k)}{1+k}} \right] + \frac{M_1^2}{2} \left[1 - \left(\frac{1}{M_1} \right)^{\frac{4}{k+1}} \right] \right\} \tag{58}$$

Since from Eq. (17) on p. 10

$$T^* = T_1 \left(\frac{1}{M_1} \right)^{\frac{2(1-k)}{1+k}} \tag{59}$$

The energy equation for state (x) to (*), provided no shock occurs, is

$$C_p T_{\circ x} = C_p T^* + \frac{V^{*2}}{2 g_c J} + \frac{g}{g_c} \frac{\Delta Z^* - \Delta X}{J}$$

Rearranging, and using the definitions for specific heat and sound speed, gives

$$T_{\circ x} = \frac{k+1}{2} T^* + \frac{g}{g_c} \frac{k-1}{kR} (\Delta Z^* - \Delta X) \tag{60}$$

Similarly, from state (y) to (y^*)

$$T_{\circ y} = \frac{k+1}{2} T_y^* + \frac{g}{g_c} \frac{k-1}{kR} \Delta Y \tag{61}$$

Combining Eqs. (60) and (61) by using Eq. (54), gives

$$T_y^* = T^* - \frac{k-1}{Rk} \frac{2}{1+k} (\Delta Y - \Delta Z^* + \Delta X) \frac{g}{g_c}$$

$$T_y^* = T^* - \frac{k-1}{Rk} \frac{2}{1+k} (\Delta L) \frac{g}{g_c}$$

and with Eq. (59), gives

$$T_y^* = T_1 \left(\frac{1}{M_1} \right)^{\frac{2(1-k)}{1+k}} - \frac{k-1}{Rk}\,\frac{2}{1+k}(\Delta L)\,\frac{g}{g_c} \tag{62}$$

Substitution of Eq. (62) into the energy equation for state (y) to (y*) gives

$$kRg_cT_y^* \left\{ \frac{1}{k-1}\left(1-M_y^{\frac{2(1-k)}{1+k}} \right) + \frac{1}{2}\left(1-M_y^{\frac{4}{1+k}} \right) \right\} + g\,\Delta Y = 0$$

And using Eq. (53), (57), (58) and the relation $\Delta Y = \Delta Z^* + \Delta L - \Delta X$, the resulting equation for locating the shock is

$$kRg_c \left[T_1\left(\frac{1}{M_1} \right)^{\frac{2(1-k)}{1+k}} - \frac{g}{g_c}\,\frac{k-1}{Rk}\,\frac{2}{1+k}\,\Delta L \right]$$

$$\cdot \left\{ \frac{1}{k-1}\left[1-\left(\frac{M_x^2 + \frac{2}{k-1}}{\frac{2k}{k-1}M_x^2 - 1} \right)^{\frac{1-k}{1+k}} \right] + \frac{1}{2}\left[1-\left(\frac{M_x^2 + \frac{2}{k-1}}{\frac{2k}{k-1}M_x^2 - 1} \right)^{\frac{2}{1+k}} \right] \right\}$$

$$+ kRg_cT_1 \left\{ \frac{1}{k-1}\left[\left(\frac{M_1}{M_x}\right)^{\frac{2(k-1)}{1+k}} - M_1^{\frac{2(1-k)}{1+k}} \right] + \frac{M_1^2}{2}\left[\left(\frac{M_x}{M_1}\right)^{\frac{4}{k+1}} - \left(\frac{1}{M_1}\right)^{\frac{4}{k+1}} \right] \right\}$$

$$+ g\,\Delta L = 0 \tag{63}$$

From the above equation M_x can be obtained for a given initial condition, and then from Eq. (57), the location of the shock can be determined.

Numerical example of normal shock location due to variable length of pipe for vertically upward isentropic flow of air. Given $M_1 = 2$, $P_1 = 100$ psia, $T_1 = 500$ R, $A = 1\ ft^2$, find ΔX for a certain ΔL.

For convenience, a value of M_x will be assumed, and the value of ΔL then will be determined.

Let $M_x = 1.3$. Substituting all the known data in Eq. (63)

$$(47007.652 - \frac{\Delta L}{3})\left[\frac{1}{0.4}(1 - 1.618834^{0.4/2.4}) + \frac{1}{2}(1 - 0.6177285^{2/2.4})\right]$$

$$+ 93275(1.5384615^{1/3} - 2^{1/3})$$

$$+ 74620(0.65^{4/2.4} - 0.5^{4/2.4}) + \Delta L = 0$$

$$\Delta L = \frac{-991.597}{1.0146423} = -977.287 \text{ ft}$$

Substituting $M_x = 1.3$ into Eq. (57)

$$37310\left\{\frac{1}{0.4}\left[(\frac{2}{1.3})^{1/3} - 1\right] + 2\left[(\frac{1.3}{2})^{10/6} - 1\right]\right\} + \Delta X = 0$$

$$\Delta X = 37310(0.6384569) = 23820.8269 \text{ ft}$$

The results for different values of ΔL are tabulated as follows and plotted on Fig. 16.

Calculation data for shock location in isentropic flow due to variable length of pipe.

M_x	ΔL	ΔX
1.0	0	26872.03
1.2	- 340.15	25456.15
1.3	- 977.29	23820.83
1.6	- 5304.40	15973.11
1.8	- 10096.63	8683.30
2.0	- 16255.75	0

The results show that decreasing the height of the pipe from its original critical height (ΔZ^*) will cause a shock to be present in the pipe, provided that the back pressure is low enough so that the Mach Number at the end of the decreased pipe is still unity, i.e. the exit pressure is equal to the critical pressure. Increasing the height of the pipe beyond its original critical height will cause a reduction in the rate of mass flow (choking). It will be also found in section 2 below, that when the height of the pipe is over its critical height, the only result is the change of mass flow rate and there is no shock occuring in the pipe whatever the back pressure may be.

2. Normal shock location for fixed length of pipe and variable back pressure.

In the case in which the pressure is the only variable (the length of the duct is fixed), the location of the shock will be

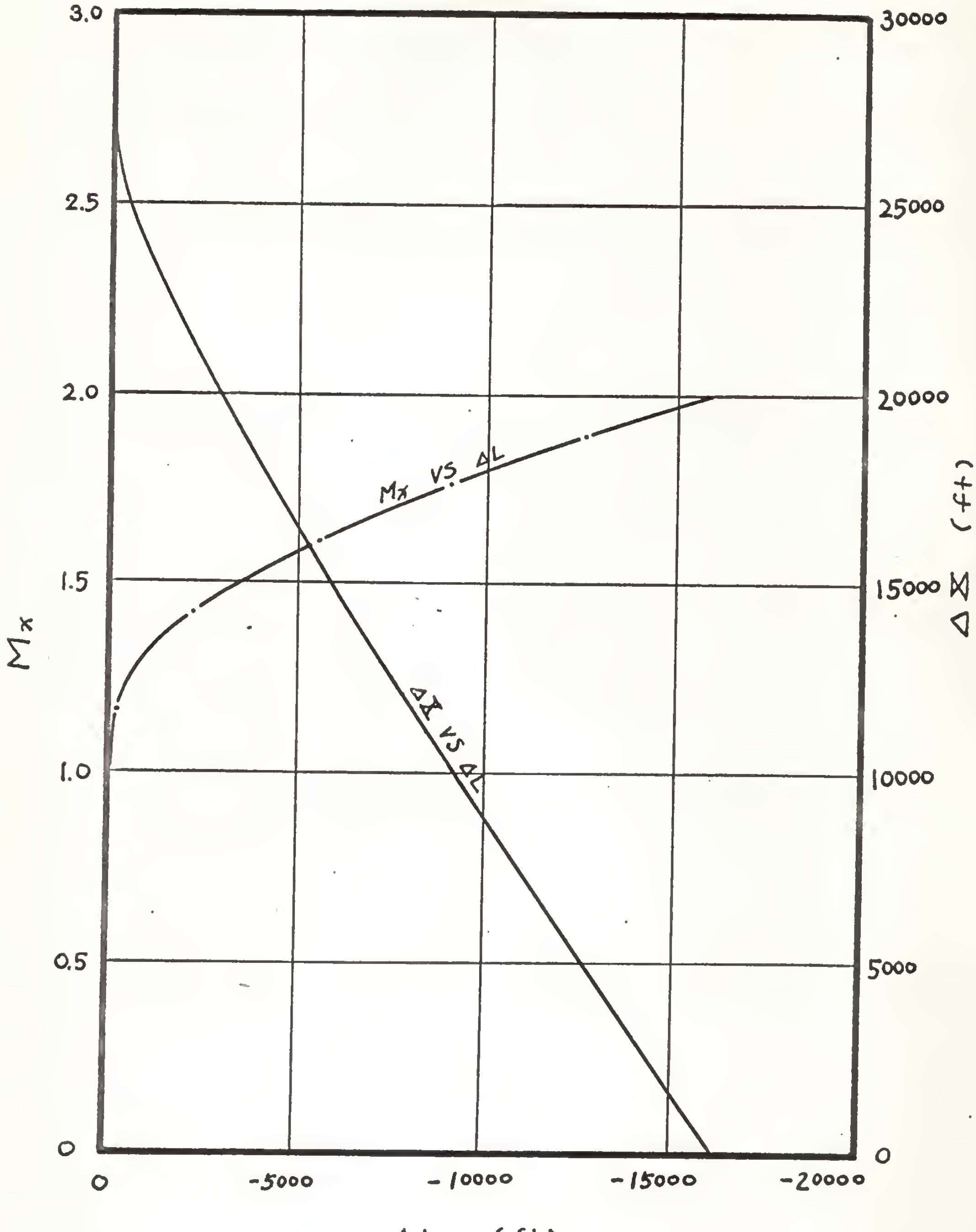

Fig. 16. Normal shock location due to variable length of pipe in reversible adiabatic flow.

found in the following manner.

The energy equation for state (y) to state (B) by using the M-P relation $P_y/P_B = (M_B/M_y)^{2k/(1+k)}$ gives

$$\frac{1}{k-1}\left[\left(\frac{P_y}{P_B}\right)^{\frac{1-k}{k}} - 1\right] + \frac{M_y^2}{2}\left[\left(\frac{P_y}{P_B}\right)^{\frac{2}{k}} - 1\right] + \frac{\Delta Y}{kRT_y}\frac{g}{g_c} = 0 \qquad (64)$$

The P-M isentropic relation and Eq. (56) yields

$$P_y = P_1\left(\frac{M_1}{M_x}\right)^{\frac{2k}{1+k}}\left(\frac{1+kM_x^2}{1+kM_y^2}\right) \qquad (65)$$

Substituting Eqs. (65) and (57) into Eq. (64) with the relation $\Delta Y = \Delta Z - \Delta X$ yields

$$\frac{1}{k-1}\left\{\left[\frac{P_1}{P_B}\left(\frac{M_1}{M_x}\right)^{\frac{2k}{k+1}}\left(\frac{1+kM_x^2}{1+kM_y^2}\right)\right]^{\frac{1-k}{k}} - 1\right\}$$

$$+ \frac{M_y^2}{2}\left\{\left[\frac{P_1}{P_B}\left(\frac{M_1}{M_x}\right)^{\frac{2k}{k+1}}\left(\frac{1+kM_x^2}{1+kM_y^2}\right)\right]^{2/k} - 1\right\}$$

$$+ \frac{\Delta Z}{kRT_1}\frac{g}{g_c}\left(\frac{1+\frac{k-1}{2}M_y^2}{1+\frac{k-1}{2}M_x^2}\right)\left(\frac{M_x}{M_1}\right)^{\frac{2(k-1)}{1+k}}$$

$$= \left(\frac{1+\frac{k-1}{2}M_y^2}{1+\frac{k-1}{2}M_x^2}\right)\left(\frac{M_x}{M_1}\right)^{\frac{2(k-1)}{1+k}}$$

$$\cdot\left\{\frac{1}{k-1}\left[1 - \left(\frac{M_x}{M_1}\right)^{\frac{2(1-k)}{1+k}}\right] + \frac{M_1^2}{2}\left[1 - \left(\frac{M_x}{M_1}\right)^{\frac{4}{k+1}}\right]\right\} \qquad (66)$$

Combining Eqs. (53) and (66) gives the solution of M_x. From Eq. (56) ΔX is determined, which locates the position of the normal shock.

Numerical example of shock location due to variable back pressure.

Given $M_1 = 2$, $P_1 = 100$ psia, $T_1 = 500$ R, $A = 1$ ft^2, find ΔX for a certain P_B.

For a pipe of constant height $\Delta Z = 17742.5$ ft and $M_x = 1.7$, Eq. (66) gives

$$\frac{1}{0.4}\left(\frac{387.41168}{P_B}\right)^{-0.28571428} + 0.2051482\left(\frac{387.41168}{P_B}\right)^{1.4285714}$$

$$= 2.5 + 0.2051482 + 0.21785680 - 0.30889184$$

$$= 2.6141132$$

Hence, two solutions are obtained from the above equation:

$P_B' = 195.0$ psia (Fictitious) & $P_B = 295.0$ psia

Also, using Eq. (57)

$$\frac{\Delta X}{37310} - \frac{1}{0.4}\left[\left(\frac{2}{1.7}\right)^{1/3} - 1\right] + 2\left[\left(\frac{1.7}{2}\right)^{4/2.4} - 1\right] = 0$$

$$\Delta X = 37310(0.33539310) = 12513.52 \text{ ft}$$

The Mach Number at the exit of the pipe, M_B is obtained from the isentropic P-M relation

$$M_B/M_y = \left[P_y/P_B\right]^{(1+k)/2k}$$

where $P_y = P_x\ (1+kM_x^2)/(1+kM_y^2)$

$$= P_1\ (M_1/M_x)^{2k/(1+k)}\ (1+kM_x^2)/(1+kM_y^2)$$

$$= 387.41168 \text{ psia}$$

$$M_B = 0.64055(387.41168/295)^{0.8571428}$$

$$= 0.809089$$

The results are tabulated as follows and plotted on Fig. 17.

Calculation data for the Mach Number before the shock and the location of the shock vs. the back pressure.

M_x	ΔX	P_B	P_B'	M_B
1.550	17555.37	353.0	154.0	0.68729
1.600	15973.11	334.0	163.3	0.72273
1.700	12513.52	295.0	195.0	0.80909
1.762	10181.88	237.5	237.5	0.97867 ($\cong$ 1)

P_B' is the fictitious back pressure.

In this numerical example, the constant height of the pipe ΔZ is equal to 17742.5 ft. A back pressure of 237.5 psia is required to produce a shock at the section where $M_x = 1.762$. For each different value of P_B, there is a corresponding normal shock existing within the pipe. If ΔZ is reduced, the back pressure P_B must be increased in value to keep the normal shock existing at the same location (i.e. $M_x = 1.762$). If ΔZ is increased over the length of 17742.5 ft, no shock would exist at section x ($M_x = 1.762$) for any value of P_B. In general, there is a limiting maximum height of the pipe for a normal shock existing at any specified location. Therefore the "maximum height" (ΔZ_{max}) is introduced. It is the height that for any further addition of pipe, the shock occurrence would be impossible at section x for any back pressure. This result can be checked from the table on p. 45 and the graph on p. 46.

$$\Delta Z_{max}(\text{for } M_x=1.762) = 17742.5 = \Delta Z^* - \Delta L \cong 26872.03-9100$$
$$= 17772.03 \text{ ft}$$

In this numerical example the relation of the "maximum height" to the Mach Number before the shock is found and tabulated in the following.

Calculation data for M_x vs. Z_{max}

M_x	Z_{max}
2.00	10782.8
1.80	16760.0
1.70	19342.4
1.60	21351.8
1.55	22523.5
1.00	26872.0

The "maximum height" for $M_x=1$ is just the critical height (ΔZ^*) of the pipe. When $\Delta Z > 26872.0$, the flow is choked.

<u>Normal Shock Location in Reversible Diabatic Upward Flow</u>

1. Shock location for variable height of the pipe:

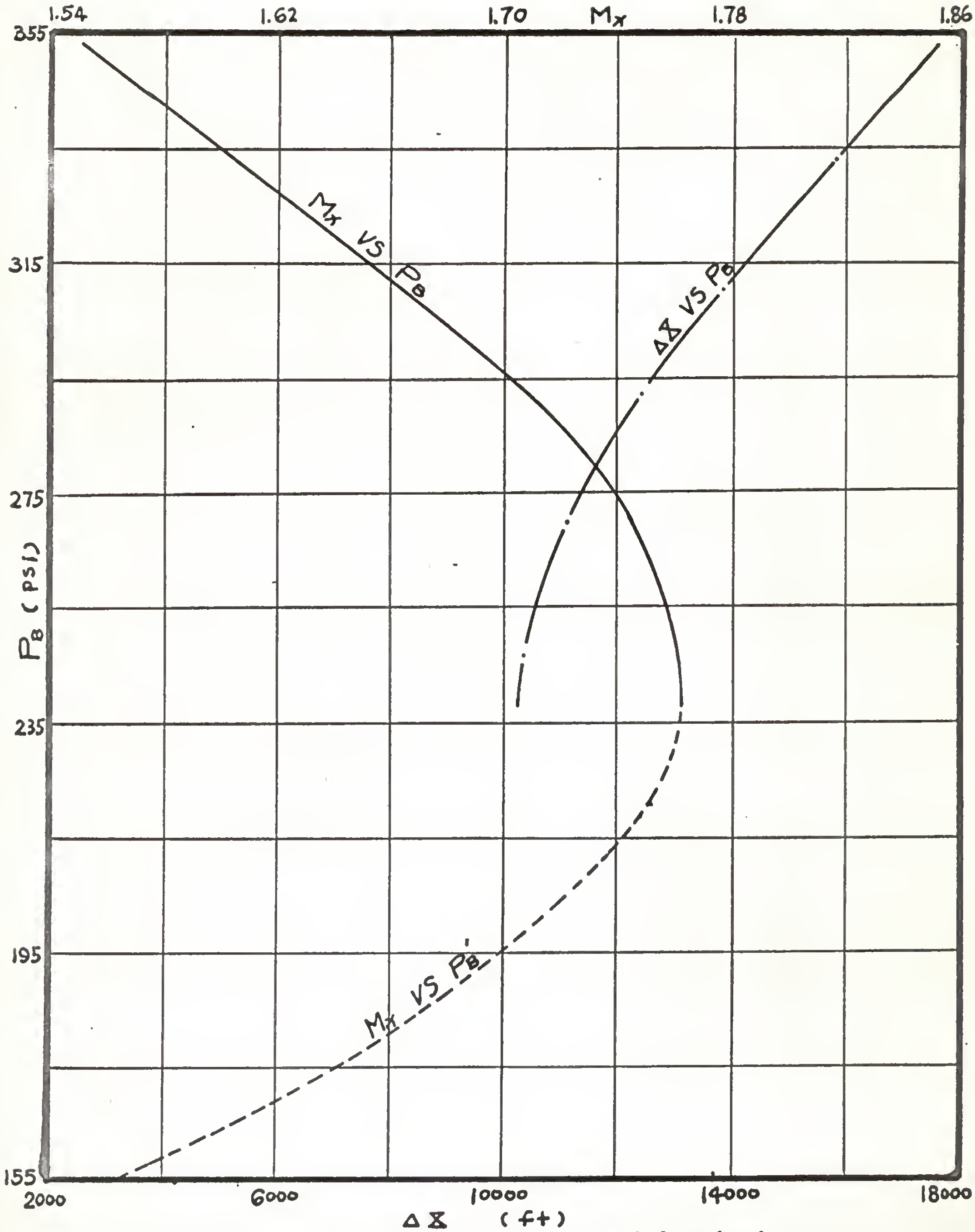

Fig. 17. Normal shock location due to variable back pressure in isentropic upward flow.

The energy equation for state (1) to (x) is

$$1 + \frac{k-1}{2} M_1^2 = \left(\frac{M_1}{M_x} \right)^{2-4\alpha_4} \left(\frac{\alpha_2 + \alpha_3 M_1^2}{\alpha_2 + \alpha_3 M_x^2} \right)^{2\alpha_4}$$

$$+ \frac{k-1}{2} M_1^2 \left(\frac{M_x}{M_1} \right)^{4\alpha_4} \left(\frac{\alpha_2 + \alpha_3 M_1^2}{\alpha_2 + \alpha_3 M_x^2} \right)^{2\alpha_4} + \left(\frac{g}{g_c} - JQ \right)(k-1) \frac{\Delta X}{C_1^2} g_c \tag{67}$$

The energy equation for state (1) to (*) is

$$1 + \frac{k-1}{2} M_1^2 = M_1^{2-4\alpha_4} \left(\frac{\alpha_2 + \alpha_3 M_1^2}{\alpha_1} \right)^{2\alpha_4}$$

$$+ \frac{k-1}{2} M_1^2 \left(\frac{1}{M_1} \right)^{4\alpha_4} \left(\frac{\alpha_2 + \alpha_3 M_1^2}{\alpha_1} \right)^{2\alpha_4} + \left(\frac{g}{g_c} - JQ \right)(k-1) \frac{\Delta Z^*}{C_1^2} g_c \tag{68}$$

The energy equation for state (y) to (y*) is

$$\left(\frac{T_y}{T_y^*} \right) + \frac{k-1}{2} \left(\frac{V_y}{V_y^*} \right)^2 = \frac{k+1}{2} + \left(\frac{g}{g_c} - JQ \right)(k-1) \frac{\Delta Y}{kRT_y^*} \tag{69}$$

And the critical temperature relation is

$$T_y^* = T^* - \frac{2}{(k+1)C_p} \left(\frac{g}{g_c J} - Q \right) \Delta L \tag{70}$$[1]

Combining Eqs. (67), (68), (69) and (70) with the relations

$$\left(\frac{V_2}{V_1}\right)^2 = \left[\left(\frac{M_2}{M_1}\right)^2 \frac{\alpha_2 + \alpha_3 M_1^2}{\alpha_2 + \alpha_3 M_2^2} \right]^{2\alpha_4}$$

---

[1]Developed in the Appendix, p. 89.

$$\frac{T_2}{T_1} = \left(\frac{M_1}{M_2}\right)^{2-4\alpha_4} \left(\frac{\alpha_2 + \alpha_3 M_1^2}{\alpha_2 + \alpha_3 M_2^2} \right)^{2\alpha_4}$$

and

$$\Delta Y = \Delta Z^* + \Delta L - \Delta X$$

yields

$$kR \left[T_1 M_1^2 \left(\frac{\alpha_2 + \alpha_3 M_1^2}{\alpha_1 M_1^2} \right)^{2\alpha_4} - \frac{2}{(k+1)C_p}\left(\frac{g}{g_c J} - Q\right) \Delta L \right]$$

$$\cdot \left[\frac{1}{M_y^2}\left(\frac{\alpha_1 M_y^2}{\alpha_2 + \alpha_3 M_y^2}\right)^{2\alpha_4} + \frac{k-1}{2}\left(\frac{\alpha_1 M_y^2}{\alpha_2 + \alpha_3 M_y^2}\right)^{2\alpha_4} - \frac{k+1}{2} \right]$$

$$= (kRT_1) \left\{ \left(\frac{M_1}{M_x}\right)^{2-4\alpha_4} \left(\frac{\alpha_2 + \alpha_3 M_1^2}{\alpha_2 + \alpha_3 M_x^2}\right)^{2\alpha_4} - M_1^{2-4\alpha_4}\left(\frac{\alpha_2 + \alpha_3 M_1^2}{\alpha_1}\right)^{2\alpha_4} \right.$$

$$\left. + \left(\frac{k-1}{2}\right) M_1^2 \left[\left(\frac{M_x}{M_1}\right)^{4\alpha_4} \left(\frac{\alpha_2 + \alpha_3 M_1^2}{\alpha_2 + \alpha_3 M_x^2}\right)^{2\alpha_4} - \left(\frac{1}{M_1}\right)^{4\alpha_4} \left(\frac{\alpha_2 + \alpha_3 M_1^2}{\alpha_1}\right)^{2\alpha_4} \right] \right\}$$

$$+ \left(\frac{g}{g_c} - JQ \right)(k-1) \Delta L \tag{71}$$

Equation (70) gives the value of M_x. This value of M_x substituted in Eq. (52) gives ΔL.

Numerical example of shock location due to variable length of pipe for a vertically upward flow of air.

Given $M_1 = 2$, $P_1 = 100$ psia, $T_1 = 500$ R, $Q = 100$ Btu/ft.

Find ΔL for a certain M_x.

Eq. (71) becomes

$$(86723.961 + 0.22600000\ \Delta L)$$

$$\cdot \left[\frac{1}{M_y^2} \left(\frac{9.8688 M_y^2}{5.512 + 4.3568 M_y^2} \right)^{1.4920174} + 0.2 \left(\frac{9.8688 M_y^2}{5.512 + 4.3568 M_y^2} \right)^{1.4920174} - 1.2 \right]$$

$$= 37310 \left[\left(\frac{M_x}{2} \right)^{0.9840348} \left(\frac{22.9392}{5.512 + 4.3568 M_x^2} \right)^{1.4920174} + 0.8 \left(\frac{M_x}{2} \right)^{2.9840348} \left(\frac{22.9392}{5.512 + 4.3568 M_x^2} \right)^{1.4920174} \right]$$

$$- 2.712\ \Delta L - 79748.598$$

When $M_x = 1.2$

the above equation simplifies to

$$- 1433.6164 - 0.0037359615\ \Delta L$$

$$= 78515.3766 - 2.712\ \Delta L - 79748.598$$

Thus

$$\Delta L = (200.394)/(2.7082641) = 73.99352 \text{ ft}$$

Eq. (67)

$$1.8 = (0.60491327)(2.7009746) + 0.8(0.21776878)(2.7009746)$$

$$- (2.712\ \Delta X)/37310$$

$$\Delta X = (11357.377)/(2.712) = 4187.823 \text{ ft}$$

The numerical results are tabulated as follows and plotted on Fig. 18.

Calculation data for shock location in reversible diabatic flow due to variable length of pipe.

M_x	ΔL	ΔX
1.0	0	4615.892
1.2	73.99352	4187.823
1.4	199.13912	3248.331
1.6	281.61345	2137.223
1.8	315.13208	1028.284
2.0	315.36435	0

A normal shock will exist somewhere in the pipe if the heat transfer rate is kept constant and height of the pipe is increased provided that the back pressure is low enough.

A special case for Eq. (71) is that in which the heat flux $Q = g/g_cJ$. All the terms that involve ΔL vanish in Eq. (71). In order to determine the location of the shock for this case, the momentum equation, equation of continuity, energy equation, definition of Mach Number and the perfect gas relation are used. By the analogy of the horizontal Fanno Line process to this case, increase of the elevation over the critical height still causes the normal shock to move downward and finally the reductions in flow rate is obtained when the pressure in the throat of the supersonic nozzle, which introduces gas into the lower end of the pipe, is increased.

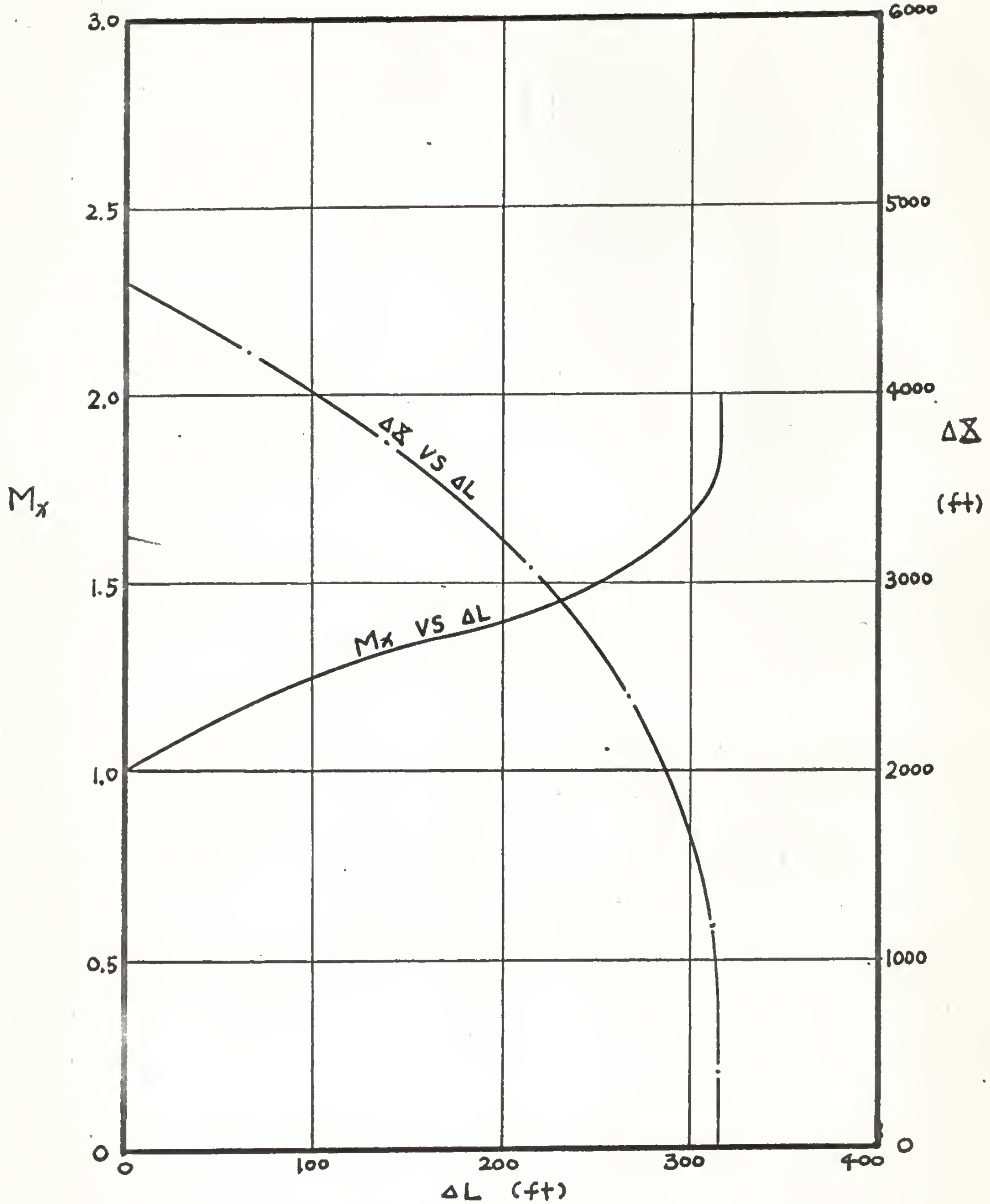

Fig. 18. Normal shock location in reversible heating, upward flow.

2. Shock location due to varying the back pressure with a fixed total height of pipe for vertically upward, reversible dia-batic flow of air:

From the energy equation for state (y) to state (B):

$$1 + \frac{k-1}{2}M_y^2 = \left(\frac{M_y}{M_B}\right)^{2-\alpha_4}\left(\frac{\alpha_2+\alpha_3 M_y^2}{\alpha_2+\alpha_3 M_B^2}\right)^{2\alpha_4}$$

$$+ \frac{k-1}{2}M_y^2\left(\frac{M_B}{M_y}\right)^{4\alpha_4}\left(\frac{\alpha_2+\alpha_3 M_y^2}{\alpha_2+\alpha_3 M_B^2}\right)^{2\alpha_4} + \left(\frac{g}{g_c} - JQ\right)(k-1)\frac{\Delta Z - \Delta X}{kRT_y} \tag{72}$$

From the pressure relation:

$$\frac{P_B}{P_y} = \left(\frac{M_y}{M_B}\right)^{2-2\alpha_4}\left(\frac{\alpha_2+\alpha_3 M_y^2}{\alpha_2+\alpha_3 M_B^2}\right)^{\alpha_4}$$

$$\frac{P_x}{P_1} = \left(\frac{M_1}{M_x}\right)^{2-2\alpha_4}\left(\frac{\alpha_2+\alpha_3 M_1^2}{\alpha_2+\alpha_3 M_x^2}\right)^{\alpha_4}$$

From Eq. (56) and the above two equations:

$$P_B = P_1\left(\frac{M_1}{M_x}\frac{M_y}{M_B}\right)^{2-2\alpha_4}\left(\frac{\alpha_2+\alpha_3 M_1^2}{\alpha_2+\alpha_3 M_x^2}\right)^{\alpha_4} \tag{73}$$

Similarly for the temperature relation:

$$T_y = T_1\left(\frac{M_1}{M_x}\right)^{2-4\alpha_4}\left(\frac{\alpha_2+\alpha_3 M_1^2}{\alpha_2+\alpha_3 M_x^2}\right)^{2\alpha_4}\left(\frac{1+\frac{k-1}{2}M_x^2}{1+\frac{k-1}{2}M_y^2}\right) \tag{74}$$

Five equations, (67), (72), (73), (74) and (53) relate the five unknowns, ΔX, M_x, M_y, M_B and T_y for the given initial and independent of conditions, ΔZ, and back pressure, P_B. Hence the location of the shock can be attained.

<u>Normal Shock Location in Irreversible Adiabatic Upward Flow</u>

1. Shock location for variable height of the pipe:

From the energy equation for state (1) to state (*)

$$\Delta Z^* = \frac{kRT^*}{k-1}\frac{g_c}{g}\left[-\frac{k+1}{2} + \left(\frac{1}{M_1^2} \quad \frac{k-1}{2}\right)\left(\frac{M_1^2+\beta_2}{\beta_3 M_1^2+\beta_4}\right)^{2\beta_8} \left(\frac{M_y^4}{\beta_5+\beta_6 M_y^2+\beta_7 M_y^4}\right)^{\frac{1}{k+1}}\right] \tag{75}$$

From the energy equation for state (1) to state (x)

$$\Delta X = \frac{kRT_1}{k-1}\left(\frac{g_c}{g}\right)\left[\left(1+\frac{k-1}{2}M_1^2\right) - \frac{T_x}{T_1} - \frac{k-1}{2}M_1^2\left(\frac{V_x}{V_1}\right)^2\right] \tag{76}$$

From the energy equation for state (y) to state (y^*)

$$\Delta Y^* = \frac{kRT_y^*}{k-1}\frac{g_c}{g}\left[-\frac{k+1}{2} + \left(\frac{1}{M_y^2}+\frac{k-1}{2}\right)\left(\frac{M_y^2+\beta_2}{\beta_3 M_y^2+\beta_4}\right)^{2\beta_8} \left(\frac{M_y^4}{\beta_5+\beta_6 M_y^2+\beta_7 M_y^4}\right)^{\frac{1}{k+1}}\right] \tag{77}$$

From the critical temperature relation

$$T_y^* = T^* - \frac{k-1}{Rk}\,\frac{2}{k+1}\,\Delta L\,\frac{g}{g_c} \tag{78}$$

From Eq. (42)

$$\left(\frac{V_x}{V_1}\right)^2 = \left[\left(\frac{k(k-1)\frac{2f_m}{D}M_x^2 + k\frac{2f_m}{D} - \beta_1}{k(k-1)\frac{2f_m}{D}M_x^2 + k\frac{2f_m}{D} + \beta_1}\;\frac{k(k-1)\frac{2f_m}{D}M_1^2 + k\frac{2f_m}{D} + \beta_1}{k(k-1)\frac{2f_m}{D}M_1^2 + k\frac{2f_m}{D} - \beta_1}\right)\right]^{\frac{k^2}{k+1}\frac{2f_m}{D}/\beta_1}$$

$$\cdot\left[\left(\frac{M_x}{M_1}\right)^4 \frac{\frac{k+1}{k}\,\frac{g}{g_c}\left(\frac{1}{2RT_m}\right) + k\frac{2f_m}{D}M_1^2 + k\left(\frac{k-1}{2}\right)\frac{2f_m}{D}M_1^4}{\frac{k+1}{k}\,\frac{g}{g_c}\left(\frac{1}{2RT_m}\right) + k\frac{2f_m}{D}M_x^2 + k\left(\frac{k-1}{2}\right)\frac{2f_m}{D}M_x^4}\right]^{\frac{1}{k+1}} \tag{79}$$

$$\frac{T_x}{T_1} = \left(\frac{M_1}{M_x}\right)^2 \left(\frac{V_x}{V_1}\right)^2 \tag{80}$$

And

$$\Delta Y^* = \Delta Z^* + \Delta L - \Delta X \tag{81}$$

Combining Eqs. (75), (76), (77), (78), (79), (80) and (81) and rearranging, gives

$$\frac{kR}{k-1}\,\frac{g_c}{g}\left[T_1 M_1^2\left(\frac{\beta_3 M_1^2 + \beta_4}{M_1^2 + \beta_2}\right)^{2\beta_8} \left(\frac{\beta_5 + \beta_6 M_1^2 + \beta_7 M_1^4}{M_1^4}\right)^{\frac{1}{k+1}} - \frac{k-1}{Rk}\,\frac{2}{1+k}\,\Delta L\,\frac{g}{g_c}\right]$$

$$\cdot\left[\frac{k+1}{2}-\left(\frac{1}{M_y^2}+\frac{k-1}{2}\right)\left(\frac{M_y^2+\beta_2}{\beta_3 M_y^2+\beta_4}\right)^{2\beta_8}\left(\frac{M_y^4}{\beta_5+\beta_6 M_y^2+\beta_7 M_y^4}\right)^{\frac{1}{k+1}}\right]$$

$$=\frac{kR}{k-1}\,\frac{g_c}{g}\left[T_1 M_1^2\left(\frac{\beta_3 M_1^2+\beta_4}{M_1^2+\beta_2}\right)^{2\beta_8}\left(\frac{\beta_5+\beta_6 M_1^2+\beta_7 M_1^4}{M_1^4}\right)^{\frac{1}{k+1}}\right]$$

$$\cdot\left[\frac{k+1}{2}-\left(\frac{1}{M_1^2}+\frac{k-1}{2}\right)\left(\frac{M_1^2+\beta_2}{\beta_3 M_1^2+\beta_4}\right)^{2\beta_8}\left(\frac{M_1^4}{\beta_5+\beta_6 M_1^2+\beta_7 M_y^4}\right)^{\frac{1}{k+1}}\right]$$

$$-\frac{kRT_1}{k-1}\,\frac{g_c}{g}\left\{\left(1+\frac{k-1}{2}M_1^2\right)-\left[\left(\frac{M_1}{M_x}\right)^2+\frac{k-1}{2}M_1^2\right]\right.$$

$$\cdot\left(\frac{k(k-1)\frac{2f_m}{D}M_x^2+k\frac{2f_m}{D}-\beta_1}{k(k-1)\frac{2f_m}{D}M_x^2+k\frac{2f_m}{D}+\beta_1}\;\frac{k(k-1)\frac{2f_m}{D}M_1^2+k\frac{2f_m}{D}+\beta_1}{k(k-1)\frac{2f_m}{D}M_1^2+k\frac{2f_m}{D}-\beta_1}\right)^{\frac{k^2}{(k+1)\beta_1}\frac{f_m}{D}}$$

$$\left.\cdot\left[\left(\frac{\frac{k+1}{k}\frac{g}{g_c}\frac{1}{2RT_m}+k\frac{2f_m}{D}M_1^2+k\left(\frac{k-1}{2}\right)\frac{2f_m}{D}M_1^4}{\frac{k+1}{k}\frac{g}{g_c}\frac{1}{2RT_m}+k\frac{2f_m}{D}M_x^2+k\left(\frac{k-1}{2}\right)\frac{2f_m}{D}M_x^4}\right)\left(\frac{M_x}{M_1}\right)^4\right]^{1/(k+1)}\right\}$$

$$+\Delta L \tag{82}$$

Eq. (82) combined with Eq. (53), determines M_x, thus giving the location of the shock.

Numerical example of normal shock location due to variable length of pipe.

Given $M_1 = 2$, $P_1 = 100$ psia, $T_1 = 500$ R, $f_m = 0.005$, $D = 1$ ft.

Find M_x for a certain ΔL.

Let $M_x = 1.2$, then $M_y^2 = 0.7092511$

From Eq. (56)

$$T_y = 698.69(1.1279938) = 788.11799$$

From Eq. (44)

$$T^* = 749.92$$

From Eq. (78)

$$T_y^* = 749.92 - \frac{\Delta L}{223.86} \tag{83}$$

Assume $T_y^* = 749.943\ ^\circ R$

Thus $T_m = (749.94+788.118)/2 = 769.0305$

$\beta_1 = 0.013991633$

$\beta_2 = 0.001494107$

$\beta_3 = 0.16695725$

$\beta_4 = 0.83453680$

$\beta_5 = 0.001243176$

$\beta_6 = 0.83229730$

$\beta_7 = 0.16645947$

$\beta_8 = 0.29184108$

Substituting into Eq. (77)

$$\Delta Y^* = T_y^*(\ 0.0026260268\ ) \tag{84}$$

Substituting Eq. (83) into Eq. (84) and combining with Eq. (81), gives

$$1.9693100178 - 0.0000117306655 \ \Delta L = 7.157213 + \Delta L$$

Hence

$$\Delta L = -(5.1879030)/(1.00001173066) = -5.1878423 \text{ ft}$$

From Eq. (83)

$$T_y^* = 749.92 - (-5.1878423)/(223.86) = 749.9431 \,^\circ R$$

This result checks with the assumed value, and hence the solution is correct.

All the numerical results are tabulated in the following and plotted on Fig. 19.

Calculation data for shock location in irreversible adiabatic flow due to variable length of pipe.

M_x	ΔL	ΔX
1.0	0.000000	23.432873
1.2	-5.187842	16.275660
1.4	-6.040466	10.225013
1.6	-2.750880	6.582281
1.8	1.661109	3.146404
2.0	6.789773	0.000000

The results show that for a decrease of the height of the pipe from the critical level (no shock and M = 1.0 at the end of the pipe), there are mathematically two possible locations where the shock could appear, when the exit pressure is equal to the critical value, P^*. The mathematical solution shows that (see Fig. 19) for $\Delta Z = 0$ there is a shock within the pipe ($M_x \cong 1.73$); this contradicts the original data of the problem that no shock exists within the pipe. The author recommends a further investigation for this shock phenomenon. Moreover, there is a limiting decrease of the height for a shock existing within the pipe when the exit pressure is kept at P^*. In this example, the limiting lowering length is about -6.6 ft.

2. Shock location for varying the back pressure with a fixed total height of pipe.

The energy equation for state (y) to (B):

$$\Delta Y = \frac{kRT_y}{k-1}\left(\frac{g_c}{g}\right)\left[\left(1+\frac{k-1}{2}M_y^2\right)-\left(\frac{T_B}{T_y}\right)-\frac{k-1}{2}M_y^2\left(\frac{V_B}{V_y}\right)^2\right] \tag{85}$$

Velocity relations:

$$\left(\frac{V_B}{V_y}\right)^2 = \left[\left(\frac{k(k-1)\frac{2f_m}{D}M_B^2+k\frac{2f_m}{D}-\beta_1}{k(k-1)\frac{2f_m}{D}M_B^2+k\frac{2f_m}{D}+\beta_1}\cdot\frac{k(k-1)\frac{2f_m}{D}M_y^2+k\frac{2f_m}{D}+\beta_1}{k(k-1)\frac{2f_m}{D}M_y^2+k\frac{2f_m}{D}-\beta_1}\right)\right]^{\frac{k^2}{(k+1)\beta_1}\frac{2f_m}{D}}$$

$$\cdot\left[\left(\frac{M_B}{M_y}\right)^4\frac{\frac{k+1}{k}\frac{g}{g_c}\frac{1}{2RT_m}+k\frac{2f_m}{D}M_y^2+k\left(\frac{k-1}{2}\right)\frac{2f_m}{D}M_y^4}{\frac{k+1}{k}\frac{g}{g_c}\frac{1}{2RT_m}+k\frac{2f_m}{D}M_B^2+k\left(\frac{k-1}{2}\right)\frac{2f_m}{D}M_B^4}\right]^{\frac{1}{k+1}} \tag{86}$$

$$\left(\frac{V_x}{V_1}\right)^2 = \left(\frac{k(k-1)\frac{2f_m}{D}M_x^2+k\frac{2f_m}{D}-\beta_1}{k(k-1)\frac{2f_m}{D}M_x^2+k\frac{2f_m}{D}+\beta_1}\;\frac{k(k-1)\frac{2f_m}{D}M_1^2+k\frac{2f_m}{D}+\beta_1}{k(k-1)\frac{2f_m}{D}M_1^2+k\frac{2f_m}{D}-\beta_1}\right)^{\frac{k^2}{(k+1)\beta_1}\frac{2f_m}{D}}$$

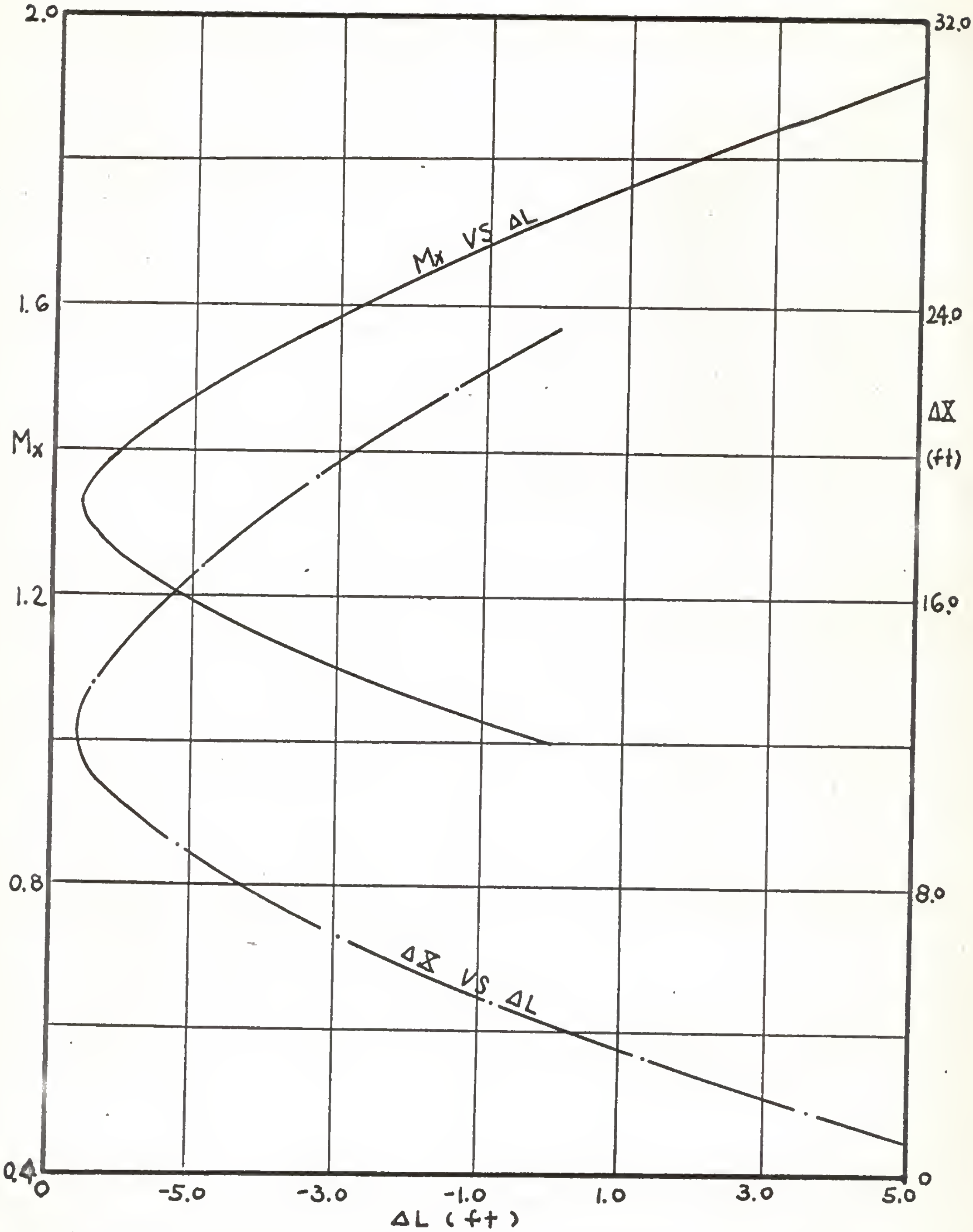

Fig. 19. Normal shock location in irreversible, adiabatic upward flow.

$$\cdot\left[\left(\frac{M_x}{M_1}\right)^4 \frac{\frac{k+1}{k}\frac{g}{g_c}\frac{1}{2RT_m} + k\frac{2f_m}{D}M_1^2 + k\left(\frac{k-1}{2}\right)\frac{2f_m}{D}M_1^4}{\frac{k+1}{k}\frac{g}{g_c}\frac{1}{2RT_m} + k\frac{2f_m}{D}M_x^2 + k\left(\frac{k-1}{2}\right)\frac{2f_m}{D}M_x^4}\right]^{\frac{1}{k+1}} \qquad (87)$$

Temperature relation:

$$\left(\frac{T_B}{T_y}\right) = \left(\frac{M_y}{M_B}\right)^2 \left(\frac{V_B}{V_y}\right)^2 \qquad (88)$$

Pressure relation:

$$\left(\frac{P_B}{P_y}\right) = \left(\frac{T_B}{T_y}\right)\left(\frac{V_y}{V_B}\right) \qquad (89)$$

$$\left(\frac{P_x}{P_1}\right) = \left(\frac{V_1}{V_x}\right)\left(\frac{T_x}{T_1}\right) \qquad (90)$$

And

$$V_y = V_x\left(\frac{P_x}{P_y}\cdot\frac{T_y}{T_x}\right)$$

Twelve equations, (53), (55), (56), (76), (80), (85), (86), (87), (88), (89), (90) and (91) relate the thirteen variables, T_B, P_B, V_B, M_B, T_y, P_y, V_y, M_y, T_x, P_x, V_x, M_x and ΔX. The independently chosen back pressure causes changes in location of the normal shock. Hence, by selecting P_B as the independent variable, ΔX may be found in terms of P_B with the aid of the above equation.

CONCLUSION

The analysis has provided analytical expressions that can be used to evaluate the properties of the flowing gas, the weight of the gas between any two pipe sections, and the location of the normal shock within the pipe for isentropic, reversible diabatic and irreversible adiabatic processes.

All the numerical results are represented graphically so that general orders of magnitude and rates of changes may be found.

Cases in which the Mach Number always tends away from unity are (a) the isentropic, downward flow and (b) heat rejection by the gas for both upward and downward, reversible flows. For the irreversible, adiabatic process the Mach Number always tends toward unity for both upward and downward flows, as in the case of horizontal Fanno-line flow.

When the back pressure is low enough, adding to the height of the pipe over the critical length will not produce a shock within the pipe if the process is isentropic. For reversible diabatic flow, on the contrary, adding to the height will cause a shock to appear in the pipe. For irreversible adiabatic flow, there are mathematically two possible locations of the normal shock when lowering the height of the pipe from the critical level. A further investigation is needed to determine whether both answers are physically possible. Also disclosed is that, there is a maximum height for which a shock could possibly exist at a certain level when the flow is isentropic and upward. The limiting height is always equal to or lower than its original critical height (i.e. ΔZ^*).

The cases concerning a normal shock in all downward flow processes and for all reversible cooling flows whether for upward or downward velocities, have not been investigated.

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APPENDIX

The Derivation of Equations

\# Eq. (8)

Substituting Eq. (4) into Eq. (5)

$$\frac{dP}{\rho} + \frac{g}{g_c}dZ - \frac{V^2}{g_c}\frac{d\rho}{\rho} = 0$$

$$\frac{g}{g_c}dZ = \frac{V^2}{g_c}\frac{d\rho}{\rho} - \frac{dP}{\rho} = \frac{dP}{\rho}\left(\frac{V^2 d\rho}{g_c dP} - 1 \right)$$

$$= \frac{dP}{\rho}\left(M^2 - 1 \right) \qquad (8)$$

\# Eq. (10)

From Eq. (3)

Dividing by C_pT

$$\frac{dT}{T} + \frac{k-1}{2}\frac{dV^2}{kg_cRT} + (k-1)\frac{gdZ}{kg_cRT} = 0$$

Introducing the Mach Number

$$\frac{dT}{T} + \frac{k-1}{2}M^2\frac{dV^2}{V^2} + (k-1)\frac{gdZ}{kg_cRT} = 0$$

Substituting Eqs. (1) and (8)

$$\frac{dP}{P} - \frac{d\rho}{\rho} + \frac{k-1}{2}M^2\frac{dV^2}{V^2} + \frac{k-1}{kRT}\frac{dP}{\rho}\left(M^2 - 1 \right) = 0$$

Substituting Eq. (4), gives

$$\frac{dP}{P} + \frac{k-1}{kRT}\frac{dP}{\rho}(M^2-1) + \frac{1}{2}\frac{dV^2}{V^2} + \frac{k-1}{2}M^2\frac{dV^2}{V^2} = 0$$

$$\frac{dP}{P}\left[1 + \frac{k-1}{k}(M^2 - 1)\right] + \left[1 + (k-1) M^2\right]\frac{1}{2}\frac{dV^2}{V^2} = 0 \qquad (10)$$

Eq. (13)

Substituting Eq. (9) into Eq. (1)

$$\frac{dT}{T} = \frac{dP}{P} - \frac{d\rho}{\rho} = JCpdT\,\frac{\rho}{P} - \frac{d\rho}{\rho}$$

Rearranging

$$\frac{d\rho}{\rho} = \frac{dT}{T}\left(\frac{JCpT\rho}{P} - 1 \right)$$

Thus giving

$$\frac{dT}{T} = (k-1)\,\frac{d\rho}{\rho}$$

For

$$T_o = T\left(1 + \frac{k-1}{2}M^2 \right)$$

$$\frac{dT_o}{T_o} = \frac{dT}{T} + \frac{(k-1)M^2}{2+(k-1)M^2}\frac{dM^2}{M^2}$$

Substituting Eq. (2), gives

$$\frac{dT_o}{T_o} = \frac{dT}{T} + \frac{(k-1)M^2}{2+(k-1)M^2}\left(\frac{dV^2}{V^2} - \frac{dT}{T} \right)$$

$$= \left(\frac{2}{2+(k-1)M^2} \right) \frac{dT}{T} + \frac{(k-1)M^2}{2+(k-1)M^2} \frac{dV^2}{V^2}$$

Substituting Eqs. (12) and (4)

$$\frac{dT_o}{T_o} = \frac{2}{2+(k-1)M^2} (k-1)\frac{d\rho}{\rho} - \frac{2(k-1)M^2}{2+(k-1)M^2} \frac{d\rho}{\rho}$$

Rearranging

$$\frac{dT_o}{T_o} = \frac{2(k-1)(\ 1-M^2\ )}{2+(k-1)\ M^2} \frac{d\rho}{\rho} \qquad (13)$$

Eq. (14)

For stagnation pressure

$$P_o = P\left(1 + \frac{k-1}{2} M^2 \right)^{\frac{k}{k-1}}$$

$$\frac{dP_o}{P_o} = \frac{dP}{P} + \frac{kM^2/2}{1+\frac{k-1}{2}M^2} \frac{dM^2}{M^2}$$

Substituting Eqs. (1) and (2), and rearranging

$$\frac{dP_o}{P_o} = \frac{d\rho}{\rho} + \frac{dT}{T} + \frac{kM^2}{2+(k-1)M^2} \left(\frac{dV^2}{V^2} - \frac{dT}{T} \right)$$

$$= \frac{d\rho}{\rho} + \frac{dT}{T}\left(1 - \frac{kM^2}{2+(k-1)M^2}\right) + \frac{kM^2}{2+(k-1)M^2} \frac{dV^2}{V^2}$$

Eqs. (12) and (4) give

$$\frac{dP_o}{P_o} = \frac{d\rho}{\rho} + (k-1)\left(1 - \frac{kM^2}{2+(k-1)M^2}\right) \frac{d\rho}{\rho} - \frac{2kM^2}{2+(k-1)M^2} \frac{d\rho}{\rho}$$

$$= \frac{d\rho}{\rho}\left[1 + \frac{(2-M^2)(k-1)}{2+(k-1)M^2}\right] - \frac{2kM^2}{2+(k-1)M^2}\frac{d\rho}{\rho}$$

$$= \frac{d\rho}{\rho}\left[\frac{2k(1-M^2)}{2+(k-1)M^2}\right] \tag{14}$$

\# Eq. (15)

Definition of the impulse function

$$F = PA + \rho AV^2/g_c = PA(\ 1+kM^2\ )$$

Differentiating

$$\frac{dF}{F} = \frac{dP}{P} + \frac{kdM^2}{1+kM^2}$$

Substituting Eq. (1)

$$\frac{dF}{F} = \frac{d\rho}{\rho} + \frac{dT}{T} + \frac{kM^2}{1+kM^2}\frac{dM^2}{M^2}$$

Using Eqs. (2) and (4), and rearranging

$$\frac{dF}{F} = \frac{d\rho}{\rho} + \frac{dT}{T} + \frac{kM^2}{1+kM^2}\left(\ \frac{dV^2}{V^2} - \frac{dT}{T}\ \right)$$

$$= \frac{d\rho}{\rho} + \frac{dT}{T}\left(\ 1 - \frac{kM^2}{1+kM^2}\ \right) - \frac{2kM^2}{1+kM^2}\frac{d\rho}{\rho}$$

Using Eq. (12)

$$\frac{dF}{F} = \frac{d\rho}{\rho}\left(\ 1 + \frac{k-1}{1+kM^2} - \frac{2kM^2}{1+kM^2}\ \right) = \frac{d\rho}{\rho}\ \frac{k(1-M^2)}{1+kM^2} \tag{15}$$

The energy equation in terms of Mach Number in isentropic flow

The energy equation for state (1) to (2):

$$C_p\ (T_2 - T_1) + \frac{V_2^2 - V_1^2}{2g_c J} + \frac{g}{Jg_c}(\ Z_2 - Z_1\ ) = 0$$

$$\frac{k}{k-1}\ \frac{R}{J}\ (T_2 - T_1) + \frac{V_2^2 - V_1^2}{2g_c J} + \frac{g}{g_c}\ \Delta Z = 0$$

Dividing by kRT_1:

$$\frac{1}{k-1}(\ \frac{T_2}{T_1} - 1\ ) + \frac{V_2^2 - V_1^2}{2kg_c RT_1} + \frac{g}{kRg_c T_1}\ \Delta Z = 0$$

From the isentropic relations:

$$\frac{T_2}{T_1} = (\frac{\rho_2}{\rho_1})^{k-1} = (\frac{V_1}{V_2})^{k-1} = (\ \frac{M_2}{M_1}\ )^{1-k}\ (\ \frac{T_2}{T_1}\ )^{(1-k)/2}$$

$$\frac{T_2}{T_1} = (\ \frac{M_2}{M_1}\ )^{2(1-k)/(1+k)}$$

and from the Mach Number definition:

$$(\frac{V_2}{V_1})^2 = (\ \frac{M_2}{M_1}\ )^2\ (\ \frac{T_2}{T_1}\ ) = (\ \frac{M_2}{M_1}\ )^{4/(1+k)}$$

The energy equation yields

$$\frac{1}{k-1}\left[(\frac{M_2}{M_1})^{\frac{2(1-k)}{(1+k)}} - 1\right] + \frac{M_1^2}{2}\left[(\frac{V_2}{V_1})^2 - 1\right] + g\frac{\Delta Z}{C_1^2} = 0$$

$$\frac{\Delta Z}{C_1^2} g = \frac{1}{k-1}\left[1 - \left(\frac{M_2}{M_1}\right)^{\frac{2(1-k)}{(1+k)}}\right] + \frac{M_1^2}{2}\left[1 - \left(\frac{M_2}{M_1}\right)^{4/(1+k)}\right]$$

\# Eq. (16)

The energy equation for state (1) to (*):

$$C_p(T^* - T_1) + \frac{V^{*2} - V_1^2}{2g_c J} + \frac{g}{g_c}\frac{\Delta Z^*}{J} = 0$$

Dividing by C_pT^*

$$\left(1 - \frac{T_1}{T^*}\right) + \frac{k-1}{2}\,\frac{V^{*2} - V_1^2}{kg_cRT^*} + \frac{k-1}{k}\,\frac{\Delta Z^*}{g_cRT^*}\,g = 0 \tag{92}$$

From isentropic relations,

$$\frac{T_1}{T^*} = \left(\frac{\rho_1}{\rho^*}\right)^{k-1} = \left(\frac{V^*}{V_1}\right)^{k-1} = M_1^{1-k}\left(\frac{T_1}{T^*}\right)^{\frac{1-k}{2}}$$

$$\frac{T_1}{T^*} = (M_1)^{\frac{2(1-k)}{1+k}} \tag{93}$$

Substituting Eq. (93) into Eq. (92)

$$\frac{\Delta Z^*}{C^{*2}} g = \frac{1}{k-1}\left(M_1^{\frac{2(1-k)}{1+k}} - 1 \right) + \frac{1}{2}\left(M_1^{\frac{4}{k+1}} - 1 \right) \tag{16}$$

\# Eq. (25)

Since

$$M = \left(\frac{\rho^*}{\rho}\right)^{\frac{1+k}{2}} \tag{94}$$

Substitution of Eq. (94) into (16)

$$\frac{\Delta Z^*}{c^{*2}}g = \frac{1}{k-1}\left[\left(\frac{\rho^*}{\rho} \right)^{1-k} - 1 \right] + \frac{1}{2}\left[\left(\frac{\rho^*}{\rho} \right)^{2} - 1 \right]$$

Carrying out the differentiation

$$\frac{dZ}{c^{*2}}g = \left[(\rho^*)^{1-k} \rho^{k-2} - (\rho^*)^2 \rho^{-3} \right] d\rho$$

Hence,

$$W^* = A \int_0^{Z^*} \rho \frac{g}{g_c} dZ = \frac{c^{*2}}{g_c} A \int_{\rho}^{\rho^*} \left[(\rho^*)^{1-k} \rho^{k-1} - (\rho^*)^2 \rho^{-2} \right] d\rho$$

$$W^* = \frac{c^{*2}}{g_c} A \rho^* \left[\frac{1+k}{k} - \frac{1}{k}\left(\frac{\rho}{\rho^*} \right)^k - \frac{\rho^*}{\rho} \right] \tag{25}$$

\# Eq. (27)

Momentum equation:

$$\frac{dP}{\rho} + \frac{g}{g_c} dZ + \frac{VdV}{g_c} = 0$$

Dividing by V^2

$$\frac{dP}{V^2 \rho} + \frac{g}{g_c}\frac{dZ}{V^2} + \frac{1}{2g_c}\frac{dV^2}{V^2} = 0$$

Introducing the Mach Number

$$\frac{dP}{P}\frac{1}{kM^2} + g\frac{dZ}{V^2} + \frac{1}{2}\frac{dV^2}{V^2} = 0 \tag{27}$$

Eq. (28)

Substituting Eq. (1) into Eq. (26)

$$\frac{dP}{P} - \frac{d\rho}{\rho} + \frac{k-1}{2}M^2 \frac{dV^2}{V^2} + \left(\frac{g}{g_c} - JQ \right)(k-1)M^2 \frac{dZ}{V^2}g_c = 0 \qquad (95)$$

Substituting Eq. (4) into Eq. (95)

$$\frac{dP}{P} = -\frac{1}{2}\frac{dV^2}{V^2} - \frac{k-1}{2}M^2 \frac{dV^2}{V^2} - \left(\frac{g}{g_c} - JQ \right) M^2 \frac{dZ}{V^2}g_c \qquad (96)$$

Substituting Eq. (27) into Eq. (96), eliminating dP/P and Rearranging, gives

$$- gkM^2 \frac{dZ}{V^2} - \frac{1}{2}kM^2 \frac{dV^2}{V^2}$$

$$= -\frac{1}{2}\frac{dV^2}{V^2} - \frac{k-1}{2}M^2 \frac{dV^2}{V^2} - \left(\frac{g}{g_c} - JQ \right)(k-1) M^2 \frac{dZ}{V^2}g_c$$

$$\frac{1}{2}\left[kM^2-1 - (k-1) M^2\right] \frac{dV^2}{V^2}$$

$$+\left[\frac{g}{g_c}kM^2 - \left(\frac{g}{g_c} - JQ \right) (k-1) M^2\right] \frac{dZ}{V^2}g_c = 0$$

which gives

$$\frac{1}{2}(M^2-1)\frac{dV^2}{V^2} + \left[\frac{g}{g_c}kM^2 - \left(\frac{g}{g_c} - JQ \right)(k-1)M^2\right] \left(\frac{dZ}{V^2}g_c\right) = 0 \qquad (97)$$

Using Eq. (26)

$$\frac{dT}{T} + \frac{k-1}{2}M^2 \frac{dV^2}{V^2} + \left(\frac{g}{g_c} - JQ \right)(k-1) M^2 \left(\frac{dZ}{V^2}g_c \right) = 0$$

Substitution of Eq. (2) into Eq. (26)

$$\frac{dV^2}{V^2} - \frac{dM^2}{M^2} + \frac{k-1}{2}M^2\,\frac{dV^2}{V^2} + \left(\frac{g}{g_c} - JQ\right)(k-1)\,M^2\left(\frac{dZ}{V^2}g_c\right) = 0 \qquad (98)$$

Combination of Eq. (97) and Eq. (98) for the elimination of $dZ/V^2 g_c$, gives

$$\left[\left(\frac{g}{g_c} - JQ\right)(k-1)\left(\frac{M^2-1}{2}\right) - \left(\frac{g}{g_c} + JQk - JQ\right)\left(1 + \frac{k-1}{2}M^2\right)\right]\frac{dV^2}{V^2}$$

$$+ \left(\frac{g}{g_c} + JQk - JQ\right)\frac{dM^2}{M^2} = 0$$

Rearranged to give

$$\left(\frac{g}{g_c}\,\frac{1}{2} + \frac{1}{2}JQk - \frac{JQ}{2} + \frac{k^2JQ}{2}\,M^2 - \frac{JQk}{2}\,M^2 + \frac{g}{g_c}\,\frac{k}{2}\right)\frac{dV^2}{V^2}$$

$$= \left(\frac{g}{g_c} + JQk - JQ\right)\frac{dM^2}{M^2}$$

$$\frac{dV^2}{V^2} = \frac{2\left(\frac{g}{g_c} + JQk - JQ\right)}{\left(\frac{g}{g_c} + JQk - JQ + \frac{g}{g_c}k\right) + JQk(k-1)\,M^2}\,\frac{dM^2}{M^2}$$

Integration yields

$$\ln V^2 = -\left[\frac{2\left(\frac{g}{g_c} + JQk - JQ\right)}{\frac{g}{g_c} + JQk - JQ + \frac{g}{g_c}\,k}\right]\cdot$$

$$\ln\left[\frac{(g/g_c) + JQk - JQ + kg/g_c + JQk(k-1)M^2}{M^2}\right] + \ln C'$$

Where C' is a constant.

$$V^2 = C'\left[\frac{(\frac{g}{g_c} + JQk-JQ+ \frac{g}{g_c}k) + JQk(k-1)M^2}{M^2}\right]^{-\frac{\left[(g/g_c)+JQk-JQ\right]\;\;2}{(g/g_c)+JQk-JQ+(g/g_c)k}}$$

When $M = 1$, $V = V^*$, and

$$\frac{V}{V^*} = \left[\frac{M^2\,(\frac{g}{g_c} + JQk^2-JQ + \frac{g}{g_c}k)}{(\frac{g}{g_c} + JQk-JQ+ \frac{g}{g_c}k)+ JQk(k-1)M^2}\right]^{\frac{\frac{g}{g_c}+JQk - JQ}{\frac{g}{g_c} + JQk - JQ + \frac{g}{g_c}k}}$$

That is

$$\frac{V}{V^*} = \left(\frac{\alpha_1 M^2}{\alpha_2 + \alpha_3 M^2}\right)^{\alpha_4} \tag{28}$$

Eq. (38)

Since

$$\frac{S-S^*}{C_p} = \ln\left[\frac{T/T^*}{(P/P^*)^{\frac{k-1}{k}}}\right] \tag{99}$$

Substitution of Eqs. (30) and Eq. (31) into Eq. (99)

$$\frac{S-S^*}{C_p} = \ln\left\{\frac{\frac{1}{M^2}\left(\frac{\alpha_1 M^2}{\alpha_2 + \alpha_3 M^2}\right)^{2\alpha_4}}{\left[\frac{1}{M^2}\left(\frac{\alpha_1 M^2}{\alpha_2 + \alpha_3 M^2}\right)^{\alpha_4}\right]^{\frac{k-1}{k}}}\right\}$$

$$= \ln\left[\left(\frac{1}{M^2}\right)^{1-\frac{k-1}{k}}\left(\frac{\alpha_1 M^2}{\alpha_2 + \alpha_3 M^2}\right)^{2\alpha_4-(\frac{k-1}{k})\alpha_4}\right]$$

$$= \ln\left[\frac{1}{M^{2/k}}\left(\frac{\alpha_1 M^2}{\alpha_2+\alpha_3 M^2}\right)^{(\frac{k+1}{k})\alpha_4}\right] \tag{38}$$

Mach Number at the Maximum Value of Entropy, p. 19

Differentiating Eq. (38), gives

$$\frac{dS}{C_p} = \frac{\left(\frac{\alpha_1 M^2}{\alpha_2+\alpha_3 M^2}\right)^{\frac{k+1}{k}\alpha_4}\left(-\frac{2}{k}\right)\left(\frac{1}{M^{(2/k)+1}}\right) + \frac{1}{M^{2/k}}\left(\frac{k+1}{k}\alpha_4\right)\left(\frac{\alpha_1 M^2}{\alpha_2+\alpha_1 M^2}\right)^{\frac{k+1}{k}\alpha_4 - 1}}{\frac{1}{M^{2/k}}\left(\frac{\alpha_1 M^2}{\alpha_2+\alpha_3 M^2}\right)^{(\frac{k+1}{k})\alpha_4}}$$

$$\frac{\left[\frac{(\alpha_2+\alpha_3 M)2\alpha_1 M - \alpha_1 M^2(2\alpha_3 M)}{(\alpha_2+\alpha_3 M)^2}\right]}{} dM$$

$$= -\frac{2}{k}\left(\frac{1}{M}\right)dM + \frac{\frac{k+1}{k}\alpha 4}{\frac{\alpha_1 M^2}{\alpha_2+\alpha_3 M^2}}$$

$$\cdot\left[\frac{(\alpha_2+\alpha_3 M^2)2\alpha_1 M - \alpha_1 M^2(2\alpha_3 M)}{(\alpha_2+\alpha_3 M^2)^2}\right] dM$$

$$= -\frac{2}{k}\left(\frac{1}{M}\right)dM + 2\left(\frac{k+1}{k}\right)\frac{4}{M}\left(1 - \frac{\alpha_3 M^2}{\alpha_2+\alpha_3 M^2}\right) dM$$

$$= -\frac{2}{k}\left(\frac{1}{M}\right)dM + 2\left(\frac{k+1}{k}\right)\frac{4}{M}\left(\frac{\alpha_2}{\alpha_2+\alpha_3 M^2}\right) dM$$

Let $dS/dM = 0$, and

$$\frac{2}{k}\left(\frac{1}{M}\right) = \frac{2\alpha_4}{M}\left(\frac{k+1}{k}\right)\left(\frac{\alpha_2}{\alpha_2 + \alpha_3 M^2}\right)$$

$$\alpha_2 + \alpha_3 M^2 = \alpha_4\,\alpha_2(k+1)$$

$$M^2 = \frac{\alpha_2\,\alpha_4(k+1) - \alpha_2}{\alpha_3} \tag{100}$$

Which is the Mach Number at the maximum entropy. Substitution of the expressions for α_2, α_3 and α_4 into Eq. (100), gives

$$M^2 = \frac{\frac{g}{g_c} + JQk - JQ + \frac{g}{g_c}k}{JQk(k-1)}\left[\frac{\left(\frac{g}{g_c} + JQk - JQ\right)k}{\left(\frac{g}{g_c} + JQk - JQ + \frac{g}{g_c}k\right)} + \frac{\frac{g}{g_c} + JQk - JQ - \frac{g}{g_c} - JQk + JQ - \frac{g}{g_c}k}{\frac{g}{g_c} + JQk - JQ + \frac{g}{g_c}k}\right]$$

$$M^2 = \frac{\frac{g}{g_c}k + JQk^2 - JQk + \frac{g}{g_c} + JQk - JQ - \frac{g}{g_c} - JQk + \frac{J}{Q} - \frac{g}{g_c}k}{JQk(k-1)}$$

$$= \frac{JQk^2 - JQk}{JQk(k-1)} = 1$$

Differentiating Eq. (35) with respect to Mach Number yields

$$\left(\frac{\Delta Z}{C^{*2}}\, g_c\right) = \frac{1}{\left(\frac{g}{g_c} - JQ\right)(k-1)}\left[\frac{k+1}{2} - \left(\frac{1}{M^2} + \frac{k-1}{2}\right)\left(\frac{\alpha_1 M^2}{\alpha_2 + \alpha_3 M^2}\right)^{2\alpha_4}\right]$$

$$\frac{g_c}{c^{*2}}\frac{dZ}{dM} = \frac{-1}{\left(\frac{g}{g_c} - JQ\right)(k-1)}\left[\left(\frac{1}{M^2} + \frac{k-1}{2}\right)2\alpha_4\left(\frac{\alpha_1 M^2}{\alpha_2 + \alpha_3 M^2}\right)^{2\alpha_4 - 1}\right.$$

$$\left.\frac{(\alpha_2 + \alpha_3 M^2)2\alpha_1 M - \alpha_1 M^2 2\alpha_3 M}{(\alpha_2 + \alpha_3 M^2)^2} + \left(\frac{\alpha_1 M^2}{\alpha_2 + \alpha_3 M^2}\right)^{2\alpha_4}\left(-\frac{2}{M^3}\right)\right] = 0$$

Rearranging and simplifying, gives

$$2\alpha_4\left(\frac{1}{M^2} + \frac{k-1}{2}\right)\left(\frac{\alpha_1 M^2}{\alpha_2 + \alpha_3 M^2}\right)^{2\alpha_4 - 1}\left[\frac{(\alpha_2 + \alpha_3 M^2)2\alpha_1 M - \alpha_1 M^2 2\alpha_3 M}{(\alpha_2 + \alpha_3 M^2)^2}\right]$$

$$= \frac{2}{M^2}\left(\frac{\alpha_1 M^2}{\alpha_2 + \alpha_3 M^2}\right)^{2\alpha_4}$$

$$2\alpha_4\left(\frac{1}{M^2} + \frac{k-1}{2}\right)\left(\frac{\alpha_2 + \alpha_3 M^2}{\alpha_1 M^2}\right)\frac{(\alpha_2 + \alpha_3 M^2)2\alpha_1 M - \alpha_1 M^2 2\alpha_3 M}{(\alpha_2 + \alpha_3 M^2)^2} = \frac{2}{M^3}$$

$$\left(\frac{1}{M^2} + \frac{k-1}{2}\right)\frac{(\alpha_2 + \alpha_3 M^2)2M^2 - M^4 2\alpha_3}{(\alpha_2 + \alpha_3 M^2)} = \frac{1}{\alpha_4}$$

$$\left(\frac{2 + kM^2 - M^2}{2M^2}\right)\left(\frac{2\alpha_2 M^2 + 2\alpha_3 M^4 - 2\alpha_3 M^4}{\alpha_2 + \alpha_3 M^2}\right) = \frac{1}{\alpha_4}$$

$$\frac{(2 + kM^2 - M^2)\alpha_2 M^2}{M^2(\alpha_2 + \alpha_3 M^2)} = \frac{1}{\alpha_4}$$

The resulting relation is

$$M^2 = \frac{2 - \frac{1}{\alpha_4}}{\frac{\alpha_3}{\alpha_2\,\alpha_4} + 1 - k} \tag{101}$$

Since

$$\frac{1}{\alpha_4} = \frac{\frac{g}{g_c} + JQk - JQ + \frac{g}{g_c}k}{\frac{g}{g_c} + JQk - JQ}$$

$$\frac{\alpha_3}{\alpha_2\,\alpha_4} = \frac{JQk(k-1)}{\left(\frac{g}{g_c} + JQk - JQ + \frac{g}{g_c}k\right)\frac{g/g_c + JQk - JQ}{g/g_c + JQk - JQ + kg/g_c}}$$

$$= \frac{JQk(\ k-1\ )}{\left(\frac{g}{g_c} + JQk - JQ\right)}$$

Eq. (101) yields

$$M^2 = \frac{2 - \frac{g/g_c + JQk - JQ + (g/g_c)k}{g/g_c + JQk - JQ}}{\frac{JQk(k-1)}{g/g_c + JQk - JQ} + 1 - k}$$

$$= \frac{\frac{(g/g_c) + JQk - JQ - (g/g_c)k}{(g/g_c) + JQk - JQ}}{\frac{JQk^2 - JQk + (g/g_c) + JQk - JQ - (g/g_c)k - JQk^2 + JQk}{(g/g_c) + JQk - JQ}}$$

$$= \frac{(g/g_c) + JQk - JQ - kg/g_c}{(g/g_c) + JQk - JQ - kg/g_c} = 1$$

Furthermore, when $M = 1$

$$\frac{\Delta Z^{*}}{C^{*2}}g_c = \frac{1}{(\frac{g}{g_c} - JQ)(k-1)}\left[\frac{k+1}{2} - (\frac{k+1}{2})(\frac{\alpha_1}{\alpha_2 - \alpha_3})\right]^{2\alpha_4} \quad (102)$$

Since

$$\frac{\alpha_1}{\alpha_2 + \alpha_3} = \frac{(g/g_c + JQk^2 - JQ + kg/g_c)}{g/g_c + JQk - JQ + kg/g_c + JQk^2 - JQk} = 1$$

Eq. (102) yields

$$\frac{\Delta Z^{*}}{C^{*2}}g_c = \frac{1}{(\frac{g}{g_c} - 1)(k-1)}(\frac{k+1}{2} - \frac{k+1}{2}) = 0$$

The Mach Number at the maximum static temperature for a reversible diabatic, vertical flow.

From Eq. (30), by logarithmic differentiation

$$\frac{dT}{T} = -\frac{dM^2}{M^2} + 2\alpha_4(\frac{\alpha_1 2MdM}{\alpha_1 M^2} - \frac{\alpha_3 2MdM}{\alpha_2 - \alpha_3 M^2})$$

$$= -\frac{dM^2}{M^2} + 2\alpha_4(\frac{2dM}{M} - \frac{2\alpha_3 MdM}{\alpha_2 - \alpha_3 M^2})$$

$$= 2(\frac{2\alpha_4 - 1}{M} - \frac{2\alpha_3\alpha_4}{\alpha_2 + \alpha_3 M^2})\,dM$$

Let $dT = 0$

$$(2\alpha_4 - 1)(\alpha_2 + \alpha_3 M^2) = 2\alpha_3\alpha_4 M^2$$

$$M^2 = \alpha_2(2\alpha_4 - 1)/\alpha_3$$

That is

$$M^2 = \frac{(g/g_c) + JQk - JQ - (g/g_c)k}{JQk(k-1)} \quad \frac{(g/g_c) + JQk - JQ - (g/g_c)k}{(g/g_c) + JQk - JQ - (g/g_c)k}$$

$$= \frac{(g/g_c)(1-k) + JQ(\ k-1\ )}{JQk(k-1)}$$

$$M = \left(\frac{JQ - g/g_c}{JQk} \right)^{1/2}$$

Eq. (41)

Momentum equation:

$$AdP + \rho \frac{g}{g_c} AdZ + \frac{\rho AV\ dV}{g_c} + \tau_w \pi DdZ = 0 \qquad (103)$$

Dividing by A and introducing $\tau_w = \rho V^2 f/2g_c$, Eq. (103) yields

$$dP + \rho \frac{g}{g_c} dZ + \frac{\rho VdV}{g_c} + \frac{\rho V^2}{2g_c} \frac{4f}{D} dZ = 0$$

Dividing by ρV^2, yields

$$\frac{dP}{\rho V^2} + \frac{g}{g_c} \frac{dZ}{V^2} + \frac{dV^2}{2g_c V^2} + \frac{1}{2g_c} \frac{4f}{D} dZ = 0$$

Upon noting that $\rho V^2/g_c = kPM^2$

$$\frac{dP}{P} + kM^2 g \frac{dZ}{V^2} + \frac{kM^2}{2} \frac{dV^2}{V^2} + kM^2 \frac{2f}{D} dZ = 0$$

$$\frac{dP}{P} + \frac{g}{g_c} \frac{dZ}{RT} + \frac{kM^2}{2} \frac{dV^2}{V^2} + kM^2 \frac{2f}{D} dZ = 0 \qquad (41)$$

\# Eq. (42)

Combining Eq. (1) and Eq. (4) yields

$$\frac{dP}{P} = -\frac{1}{2}\frac{dV^2}{V^2} + \frac{dT}{T} \qquad (104)$$

Substitution of Eq. (104) into Eq. (41)

$$\frac{dT}{T} - \frac{1}{2}\frac{dV^2}{V^2} + \frac{kM^2}{2}\frac{dV^2}{V^2} + \frac{g}{g_c}\frac{dZ}{RT} + kM^2\frac{2f}{D}\,dZ = 0 \qquad (105)$$

Combination of Eq. (105) and Eq. (40) for eliminating dT/T gives

$$\left(\frac{M^2-1}{2}\right)\frac{dV^2}{V^2} + \left(\frac{g}{g_c}\frac{1}{RT} + kM^2\frac{2f}{D} - \frac{k-1}{k}\frac{g}{g_c}\frac{1}{RT}\right)dZ = 0$$

$$\left(\frac{M^2-1}{2}\right)\frac{dV^2}{V^2} + \left(kM^2\frac{2f}{D} + \frac{1}{k}\frac{g}{g_c}\frac{1}{RT}\right)dZ = 0 \qquad (106)$$

Also, substitution of Eq. (2) into Eq. (40) yields

$$\frac{dV^2}{V^2} - \frac{dM^2}{M^2} + \frac{k-1}{2}M^2\frac{dV^2}{V^2} + \frac{k-1}{k}\frac{g}{g_c}\frac{dZ}{RT} = 0$$

$$\left(1 + \frac{k-1}{2}M^2\right)\frac{dV^2}{V^2} - \frac{dM^2}{M^2} + \left(1 - \frac{1}{k}\right)\frac{g}{g_c}\frac{dZ}{RT} = 0 \qquad (107)$$

Combination of Eq. (106) and Eq. (107) for eliminating dZ, gives

$$\left[\left(\frac{M^2-1}{2}\right)\left(\frac{k-1}{k}\frac{g}{g_c}\frac{1}{RT}\right) - \left(1 + \frac{k-1}{2}M^2\right)\left(kM^2\frac{2f}{D} + \frac{1}{k}\frac{g}{g_c}\frac{1}{RT}\right)\right]\frac{dV^2}{V^2}$$

$$+ \left(kM^2 \frac{2f}{D} + \frac{1}{k}\frac{g}{g_c}\frac{1}{RT}\right)\frac{dM^2}{M^2} = 0$$

Rearranging

$$\left[\frac{k-1}{k}\frac{g}{g_c}\frac{1}{RT}\left(\frac{M^2}{2}\right) - \left(\frac{k-1}{k}\right)\frac{g}{g_c}\frac{1}{RT}\left(\frac{1}{2}\right) - \left(1 + \frac{k-1}{2}M^2\right)\left(kM^2\frac{2f}{D}\right)\right.$$

$$\left. - \frac{1}{k}\frac{g}{g_c}\frac{1}{RT} - \frac{kM^2}{2}\left(\frac{1}{k}\frac{g}{g_c}\frac{1}{RT}\right) + \frac{M^2}{2}\frac{1}{k}\frac{g}{g_c}\frac{1}{RT}\right]\frac{dV^2}{V^2}$$

$$+ \left(kM^2 \frac{2f}{D} + \frac{1}{k}\frac{g}{g_c}\frac{1}{RT} \right)\frac{dM^2}{M^2} = 0$$

$$\left[\frac{g}{g_c}\frac{1}{RT}\frac{1}{2}\left(\frac{k+1}{k}\right) + k\frac{2f}{D}M^2 + \left(\frac{k-1}{2}\right)k\frac{2f}{D}M^4\right]\frac{dV^2}{V^2}$$

$$= \left(kM^2 \frac{2f}{D} + \frac{1}{k}\frac{g}{g_c}\frac{1}{RT} \right)\frac{dM^2}{M^2}$$

Separating the variables, yields

$$\frac{dV^2}{V^2} = \frac{\left(k\frac{2f}{D}M^2 + \frac{1}{k}\frac{g}{g_c}\frac{1}{RT} \right) dM^2}{\left[\left(\frac{k+1}{k}\right)\frac{g}{g_c}\frac{1}{2RT} + k\frac{2f}{D}M^2 + k\left(\frac{k-1}{2}\right)\frac{2f}{D}M^4\right]M^2} \qquad (108)$$

Since

$$\int\frac{(mx+n)\,dx}{(a+bx+cx^2)x} = \int\frac{m\,dx}{a+bx+cx^2} + \int\frac{n\,dx}{(a+bx+cx^2)x}$$

$$= \int\frac{m\,dx}{a+bx+cx^2} + \frac{n}{2a}\ln\frac{x^2}{a+bx+cx^2} - \frac{nb}{2a}\int\frac{dx}{a+bx+cx^2}$$

$$= (m - \frac{nb}{2a}) \int \frac{dx}{a+bx+cx^2} + \frac{n}{2a} \ln \frac{x^2}{a+bx+cx^2}$$

$$= (m - \frac{nb}{2a}) \frac{1}{(b^2-4ac)^{1/2}} \ln \frac{2cx+b-(b^2-4ac)^{1/2}}{2cx+b+(b^2-4ac)^{1/2}}$$

$$+ \frac{n}{2a} \ln \frac{x^2}{a+bx+cx^2}$$

Integration of Eq. (108) yields

$$\ln V^2 \Big|_{V^*}^{V} = \ln \left[\frac{2cx+b-(b^2-4ac)^{1/2}}{2cx+b+(b^2-4ac)^{1/2}}\right]^{\frac{m-nb/2a}{(b^2-4ac)^{1/2}}} + \ln\left(\frac{x^2}{a+bx+cx^2}\right)^{\frac{n}{2a}} \Bigg|_{1}^{M}$$

And

$$\frac{V}{V^*} = \left\{ \frac{k(k-1)\frac{2f_m}{D}M^2 + \frac{k2f_m}{D} - \left[k^2 \frac{4f_m^2}{D} - (k^2-1)\frac{g}{g_c}\frac{1}{RT_m}\frac{2f_m}{D}\right]^{1/2}}{k(k-1)\frac{2f_m}{D}M^2 + \frac{k2f_m}{D} + \left[k^2 \frac{4f_m^2}{D} - (k^2-1)\frac{g}{g_c}\frac{1}{RT_m}\frac{2f_m}{D}\right]^{1/2}} \cdot \right.$$

$$\frac{\frac{k^2}{k+1} \cdot \frac{2f_m}{D}}{\left[k^2 \frac{4f_m^2}{D^2} - (k^2-1)\frac{g}{g_c}\frac{1}{RT_m}\frac{2f_m}{D}\right]^{1/2}}$$

$$\left. \cdot \frac{k^2 \frac{2f_m}{D} + k^2 \frac{4f_m^2}{D^2} + \left[(k^2-1)\frac{g}{g_c}\frac{1}{RT_m}\frac{2f_m}{D}\right]^{1/2}}{k^2 \frac{2f_m^2}{D} - k^2 \frac{4f_m^2}{D^2} - \left[(k^2-1)\frac{g}{g_c}\frac{1}{RT_m}\frac{2f_m}{D}\right]^{1/2}} \right\}$$

$$\cdot \left\{ \frac{M^4 \left[\frac{k+1}{k} \frac{g}{g_c} \frac{1}{2RT_m} + k(\frac{k+1}{2} \frac{2f_m}{D}) \right]}{(\frac{k+1}{k})\frac{g}{g_c} \frac{1}{2RT_m} + k\frac{2f_m}{D}M^2 + k(\frac{k-1}{2})\frac{2f_m}{D} M^4} \right\}^{\frac{1}{2(k+1)}}$$

Using the notations β_1, β_2, β_3, β_4, β_5, β_6 and β_7, gives

$$\frac{V}{V^*} = \left(\frac{M^2 + \beta_2}{\beta_3 M^2 + \beta_4}\right)^{\beta_8} \left(\frac{M^4}{\beta_5 + \beta_6 M^2 + \beta_7 M^4}\right)^{\frac{1}{2(k+1)}} \quad (42)$$

Eq. (70)

From the energy equation for state (x) to (*)

$$T_{ox} = \frac{k+1}{2} T^* + \frac{1}{C_p} \left(\frac{g}{g_c J} - Q \right)(\Delta Z^* - \Delta X) \quad (109)$$

From the energy equation for state (y) to (y^*)

$$T_{oy} = \frac{k+1}{2} T_y^* + \frac{1}{C_p} \left(\frac{g}{g_c J} - Q \right)(\Delta Y) \quad (110)$$

Combination of Eq. (109) and Eq. (110) with the normal shock property $T_{ox} = T_{oy}$, gives

$$T_y^* = T^* - \frac{2}{(k+1)C_p} \left(\frac{g}{g_c J} - Q \right)\Delta L \quad (70)$$

An Analysis for a Vertical Flow with Irreversible Diabatic Flow.

Energy equation:

$$C_p dT + \frac{VdV}{g_c J} + \left(\frac{g}{g_c J} - Q \right) dZ = 0$$

$$\frac{dT}{T} + \frac{k-1}{2} M^2 \frac{dV^2}{V^2} + \left(\frac{g}{g_c} - JQ \right)(k-1) M^2 \frac{dZ}{V^2} g_c = 0 \qquad (26)$$

Momentum equation:

$$\frac{dP}{P} + \frac{g}{g_c} \frac{dZ}{RT} + \frac{kM^2}{2} \frac{dV^2}{V^2} + kM^2 \frac{2f}{D} dZ = 0 \qquad (41)$$

Combination of the equation of continuity and the equation of state gives

$$\frac{dP}{P} = - \frac{1}{2} \frac{dV^2}{V^2} + \frac{dT}{T} \qquad (104)$$

Substitution of Eq. (104) into Eq. (41)

$$\frac{dT}{T} - \frac{1}{2} \frac{dV^2}{V^2} + \frac{g}{g_c} \frac{dZ}{RT} + \frac{kM^2}{2} \frac{dV^2}{V^2} + kM^2 \frac{2f}{D} dZ = 0 \qquad (111)$$

Combining Eq. (111) and Eq. (26)

$$\left(\frac{kM^2}{2} - \frac{1}{2} - \frac{k-1}{2} M^2\right) \frac{dV^2}{V^2} + \left[\frac{g}{g_c} \frac{1}{RT} + kM^2 \frac{2f}{D} - \left(\frac{g}{g_c} - JQ\right)(k-1) \frac{M^2}{V^2} \right] dZ$$

$$= 0$$

Rearranging

$$\frac{1}{2}(M^2-1) \frac{dV^2}{V^2} + \left[\frac{g}{g_c} \frac{1}{RT} + kM^2 \frac{2f}{D} - \left(\frac{g}{g_c} - JQ\right)(k-1) \frac{M^2}{V^2} g_c \right] dZ = 0$$

$$(112)$$

Substitution of the definition of Mach Number into Eq. (26) gives

$$\frac{dV^2}{V^2} - \frac{dM^2}{M^2} + \frac{k-1}{2} M^2 \frac{dV^2}{V^2} + \left(\frac{g}{g_c} - JQ\right)(k-1)\left(\frac{M^2}{V^2} g_c\right) = 0$$

$$\left(1+ \frac{k-1}{2}M^2\right)\frac{dV^2}{V^2} - \frac{dM^2}{M^2} + \left(\frac{g}{g_c} - JQ\right)(k-1)M^2 \frac{dZ}{V^2} g_c = 0 \qquad (113)$$

Combination of Eq. (112) and Eq. (113) for eliminating dZ gives

$$\left[\frac{1}{2}\left(M^2-1 \right)\left(\frac{g}{g_c} - JQ\right)(k-1) \frac{1}{kRT}\right] \frac{dV^2}{V^2}$$

$$- \left(1 + \frac{k-1}{2}M^2\right) \left[\frac{g}{g_c} \frac{1}{RT} +kM^2 \frac{2f}{D} -\left(\frac{g}{g_c} -JQ\right)(k-1) \frac{1}{kRT}\right] \frac{dV^2}{V^2}$$

$$+\left[\frac{g}{g_c} \frac{1}{RT} +kM^2 \frac{2f}{D} -\left(\frac{g}{g_c} -JQ \right) \frac{1}{kRT}\right] \frac{dM^2}{M^2} = 0$$

Rearrangement yields

$$\left\{\frac{1}{2}\left(\frac{g}{g_c} -JQ\right)(k-1)\frac{1}{kRT} - \frac{g}{g_c} \frac{1}{RT}\right.$$

$$\left. +\left[\frac{1}{2}\left(\frac{g}{g_c} -JQ\right)(k-1)\frac{1}{RT} -k\frac{2f}{D} - \frac{k-1}{2} \frac{g}{g_c} \frac{1}{RT}\right] M^2 - \left[k(k-1)\frac{f}{D} M^4\right]\right\}\frac{dV^2}{V^2}$$

$$= - \left[\frac{g}{g_c} \frac{1}{RT} -\left(\frac{g}{g_c} -JQ\right)(k-1)\frac{1}{kRT} + \frac{2fk}{D} M^2\right] \frac{dM^2}{M^2}$$

And

$$\frac{dV^2}{V^2} = \frac{\left[\left(\frac{g}{g_c} - JQ\right)(k-1)\frac{1}{kRT} - \frac{g}{g_c}\frac{1}{RT}\right] - k\,\frac{2f}{D}\,M^2}{\left\{\begin{array}{l} \frac{1}{2}\left(\frac{g}{g_c} - JQ\right)(k-1)\frac{1}{kRT} - \frac{g}{g_c}\frac{1}{RT} \\ + \left[\frac{1}{2}\left(\frac{g}{g_c} - JQ\right)(k-1)\frac{1}{RT} - k\,\frac{2f}{D} - \frac{k-1}{2}\,\frac{g}{g_c}\,\frac{1}{RT}\right]M^2 \\ \qquad - \left[k(k-1)\frac{f}{D}\right]M^4 \end{array}\right\}}\;\frac{dM^2}{M^2}$$

Let

$$\delta_1 = \left(\frac{g}{g_c} - JQ\right)(k-1)\frac{1}{kRT_m} - \frac{g}{g_c}\frac{1}{RT_m}$$

$$\delta_2 = \frac{2kf_m}{D}$$

$$\delta_3 = \frac{1}{2}\left(\frac{g}{g_c} - JQ\right)(k-1)\frac{1}{RT_m}\,\frac{g}{g_c}\,\frac{1}{RT_m}$$

$$\delta_4 = \frac{1}{2}\left(\frac{g}{g_c} - JQ\right)(k-1)\frac{1}{RT_m} - \frac{k2f_m}{D} - \frac{k-1}{2}\,\frac{g}{g_c}\,\frac{1}{RT_m}$$

$$\delta_5 = k(k-1)\,\frac{f_m}{D}$$

And integrations

$$\left(\frac{V}{V^*}\right)^2 = \left[\frac{-2\delta_5 M^2 + \delta_4 - (\delta_4^2 + 4\delta_3\delta_5)^{\frac{1}{2}}}{-2\delta_5 M + \delta_4 + (\delta_4^2 + 4\delta_3\delta_5)^{1/2}} \cdot \frac{\delta_4 - 2\delta_5 + (\delta_4^2 - 4\delta_3\delta_5)^{\frac{1}{2}}}{\delta_4 - 2\delta_5 - (\delta_4^2 + 4\delta_3\delta_5)^{1/2}}\right]^{\dfrac{-\delta_2 - (\delta_1\delta_4/2\delta_3)}{(\delta_4^2 + 4\delta_3\delta_5)^{1/2}}}$$

$$\cdot\left[\frac{M^4(\delta_3+\delta_4+\delta_5)}{(\delta_3+\delta_4 M^2+\delta_5 M^4)}\right]^{\frac{\delta_1}{2\delta_3}} \qquad (114)$$

Using the same method, the equations for other properties for irreversible, diabatic, vertical flow can be developed.

SOME SPECIAL PROBLEMS IN THE GAS DYNAMICS OF VERTICAL FLOW

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ABSTRACT

A one-dimensional analysis relating to the variations of the gas properties along a vertical pipe with constant cross-sectional area, is presented.

Three cases are considered: isentropic process, reversible diabatic process, and irreversible adiabatic process.

Particularly investigated is the weight of the gas between any two sections, the locations of the normal shock and the conditions for producing the normal shock within the pipe.

Numerical examples are illustrated and their results are given.