INTERIM REPORT OF A VIBRATIONAL
ANALYSIS OF A 3-AXLE, FULLY
ARTICULATED, HIGHWAY VEHICLE

by 583

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NOMENCLATURE

L horizontal distance between front tractor axle and tractor center of gravity (CG)

N horizontal distance between tractor CG and tractor rear axle

P horizontal distance between fifth wheel kingpin and trailer CG

Q horizontal distance between trailer CG and trailer axle

R horizontal distance between tractor CG and fifth wheel kingpin

S vertical distance between tractor CG and fifth wheel kingpin

T vertical distance between trailer CG and fifth wheel kingpin

M_1 sprung mass of the tractor

M_2 sprung mass of the trailer

M_3 mass of the tractor front axle set

M_4 mass of the tractor rear axle set

M_5 mass of the trailer axle set

J_1 mass moment of inertia of the sprung tractor mass about its CG

J_2 mass moment of inertia of the sprung trailer mass about its CG

K_1 combined equivalent spring constant of the tractor front tires

K_2 combined spring constant of the tractor front axle springs

K_3 combined equivalent spring constant of the tractor rear tires

K_4 combined spring constant of the tractor rear axle

K_5 combined equivalent spring constant of the trailer tires

K_6 combined equivalent spring constant of the trailer springs

C_1 combined equivalent damping constant of the tractor tires

C_2 combined damping constant of the tractor front axle dampers

C_3 combined equivalent damping constant of the tractor rear tires

C_4 combined damping constant of the tractor rear axle dampers

C_5 combined equivalent damping constant of the trailer tires

C_6 combined damping constant of the trailer axle dampers

X_1 vertical motion coordinate of the CG of the sprung tractor mass

X_2 vertical motion coordinate of the CG of the sprung trailer mass

X_3 vertical motion coordinate of the tractor front axle

X_4 vertical motion coordinate of the tractor rear axle

X_5 vertical motion coordinate of the trailer axle

X_6 horizontal motion coordinate of the CG of the sprung tractor mass

X_7 horizontal motion coordinate of the CG of the sprung trailer mass

θ_1 rotational coordinate about the CG of the sprung tractor mass

θ_2 rotational coordinate about the CG of the sprung trailer mass

t time

τ_1 time necessary for the truck to travel the distance of the tractor wheelbase $(L + R)$

τ_2 time necessary for the truck to travel a distance measured from the tractor front axle to the trailer axle $(R + P + Q + L)$

$G(t)$ displacement function of the road contour applied to the tractor front axle

$G(t-\tau_1)$ displacement function of the road contour applied to the tractor rear axle

$G(t-\tau_2)$ displacement function of the road contour applied to the trailer axle

$\bar{S}_A$ displacement vector of point A

$\bar{S}_{A/G}$ displacement vector of point A with respect to point G

H mean point of attachment of the tractor front axle spring to the chassis

I mean point of attachment of the tractor rear axle spring to the chassis

J mean point of spring attachment of the trailer axle spring to the chassis

T' kinetic energy of the system due to vibrations

D energy dissipation function due to the damping of the tires and shock absorbers

V potential energy function due to the springing of the tires and suspension springs

λ classical eigenvalue

β eigenvalue solutions to the characteristic equation

$\{q\}$ transient solution vector corresponding to a value of β

c exponential decay rate due to damping

ωd damped natural circular vibrational frequency, radians/second

e 2.7183 ...

$\mu_{\alpha\gamma}$ ratio of the γ'th element of the $\{q\}$ vector corresponding to the α'th β value to the first element of that $\{q\}$ vector

$\{A\}$ complementary solution vector of constant amplitudes of sin ωt

$\{B\}$ complementary solution vector of constant amplitudes of cos ωt

$\angle$ angle

Chapter I

INTRODUCTION

"Over the road" cargo hauling has become a large industry in the United States motivated both by better roads and more efficient trucking facilities. However, most trucks are manufactured to conform first with state trucking laws and second with the operator's desires as regards size and shape.

The state laws govern the weight and geometry of trucks for reasons of traffic safety and road wear. Many times these laws limit the operator to a set of sizes and weights which he considers, at best, arbitrary.

A need exists for a dynamical analysis of a fully articulated (semi-trailer) truck considering the road induced vibrations of the major truck components (wheel-axle assemblies, tractor, and trailer) such that the variable parameters (weight, size, spring constants, damping constants, and truck geometry) of a particular truck can be considered and data regarding the vibrational characteristics of the truck obtained in a short time and with a minimum of expense. Both state and manufacturer-operator interests could thus be better served.

This report is intended to provide a preliminary approach to the problem of a detailed dynamic investigation of the vibrational motions of a three axled, fully articulated, highway vehicle. The derivation of the dynamical and kinematical equations of motion is shown and a theory of their solution discussed. Finally, some future goals of this analysis are presented.

Some dynamical analyses already exist for a fully articulated road vehicle, however, it has been the author's experience to find only the pitch and bounce characteristics of the tractor and trailer considered, completely ignoring the bounce of each axle and the fore-aft motions of the tractor and trailer. See, for instance, Clark (1)* and Huang (2). An accurate determination of the axle bounce is essential for determining force transmission by the tires to the road surface and shock absorber and spring motions. Janeway (3) has researched driver comfort and has found that the fore-aft motions of the tractor are of utmost importance to driver comfort. When the investigations of Huang and Clark were performed, computing facilities of today's calibre and accessibility were not available thus making their simplifications essential.

While the equations presented herein are more detailed than those of Huang and Clark, it is not suggested that these offer any last word on the vibrational analysis of a fully articulated road vehicle. However, a significant updating of the analytical detail is offered, thus pointing the way for the analysis to regain a correspondence with the most modern computing facilities available.

*Numbers in parentheses indicate references at the end of this report.

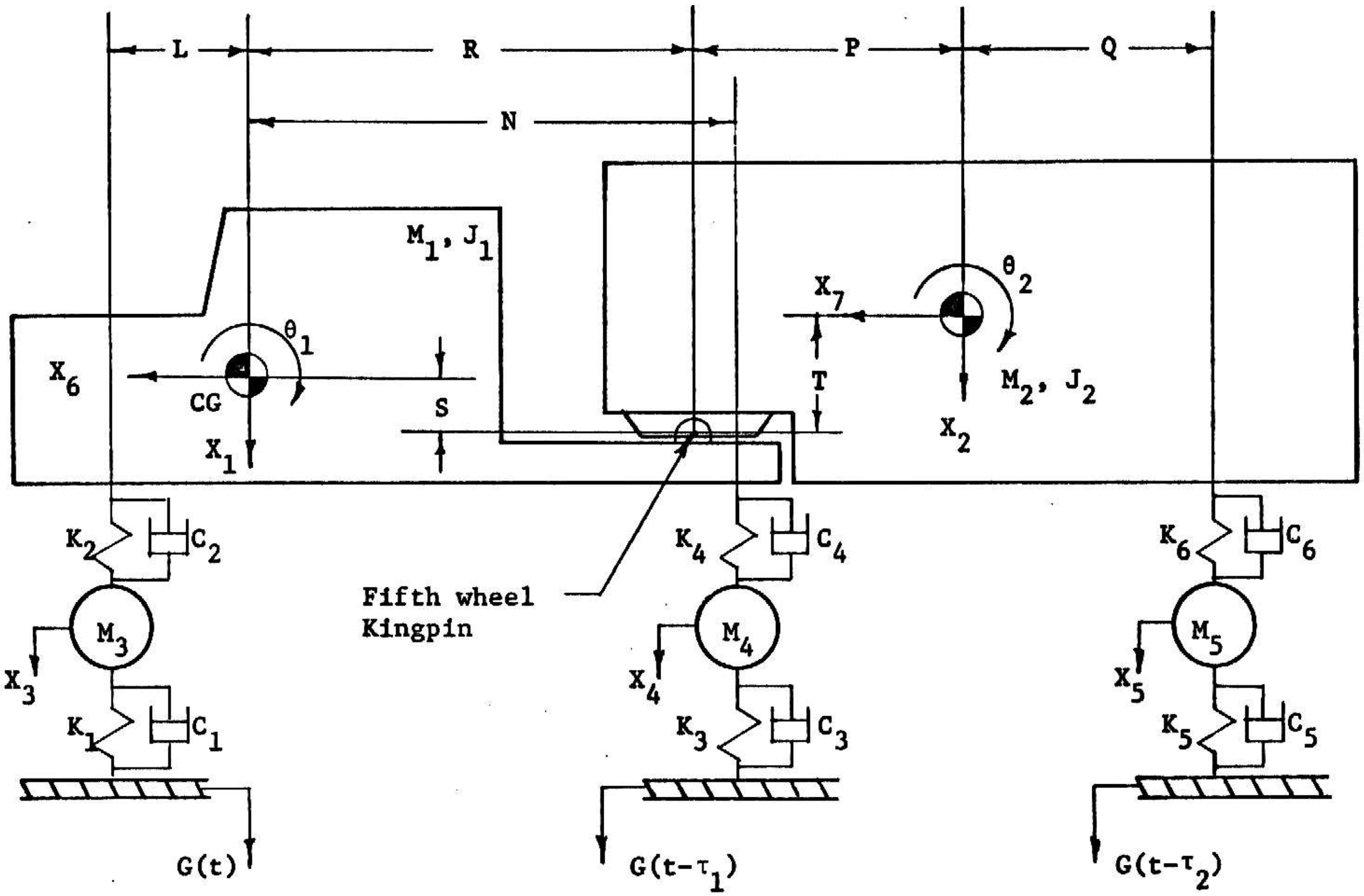
Chapter II

DERIVATION OF EQUATIONS

Assumptions

Even with today's computing facilities certain simplifying assumptions must still be made in order to keep the mathematics tractable. These are enumerated below and are mentioned again as the need arises in the derivation of the equations of motion.

1) The cab, engine, and tractor chassis are considered to be a rigid body as is the trailer body with its chassis and each axle with its wheels.
2) A "mean" point of spring attachment is assumed to lie on the chassis directly above each axle.
3) Springs and dampers are considered to be described by linear functions of displacement and velocity, respectively. The same assumption is held for the springing and damping characteristics of the tires. This means that the solution will be valid only for small oscillations.
4) All frictional (coulomb) damping is ignored.
5) The springs and dampers are considered weightless.
6) The road input displacement function is assumed to be applied to a point at the center of the tire contact patch.
7) Equal road input displacement functions are considered to be applied to the left and right tracks of the vehicle. This limits the applicability to highway vehicles only and allows a consideration of vibrations only in the plane of Figure 1.



X_6 and X_7 are coordinates which move at the average speed of the truck (as shown on the speedometer) and thus describe only the oscillatory motions of the CG's of the tractor and trailer, respectively. Their origins are defined by the CG positions when all the other coordinates are in their neutral positions.

Figure 1. General Layout of 3-Axle, Fully Articulated, Highway Vehicle Showing Coordinates, Geometry, and Spring and Damper Labels.

8) The wheels must always remain in contact with the road surface (no wheel hop).

9) The cargo remains at rest.

10) Fore-aft slack in the fifth wheel kingpin is permissible as long as a large biasing force (pulling or braking) exists, thus allowing force oscillations, but no fore-aft slapping of the kingpin.

Coordinates as shown in Figure 1 were chosen to describe the vibrational motion of the truck. The G's represent road surface displacement functions and will provide the forcing functions for the steady state vibrational analysis.

Kinematics

Two geometrical equations of constraint enter the problem due to the equal horizontal and vertical motions of the tractor and trailer at the fifth wheel kingpin. Referring to Figure 2 below, the displacement of point A may be found using two equivalent motion equations.

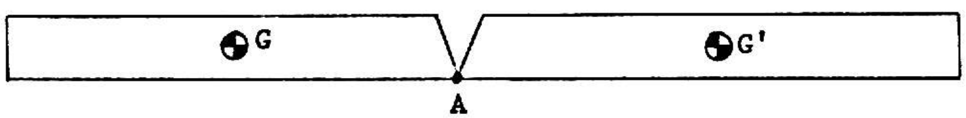


Figure 2. Schematic Representation of Truck Emphasizing Fifth Wheel Kingpin (Hinge) at A.

$$\bar{S}_A = \bar{S}_G + \bar{S}_{A/G} ,$$

$$\bar{S}_A = \bar{S}_{G'} + \bar{S}_{A/G'} .$$

These equations may be combined to yield

$$\bar{S}_G + \bar{S}_{A/G} = \bar{S}_{G'} + \bar{S}_{A/G'} . \tag{II-1}$$

Since this is a vector equation it may be broken down into its vertical and horizontal components.

$$S_{G\text{ vert.}} = X_1 \; ; \quad S_{G\text{ horiz.}} = X_6 .$$

The descriptions of $S_{A/B\text{ vert.}}$ and $S_{A/B\text{ horiz.}}$ are a little more difficult and are derived below.

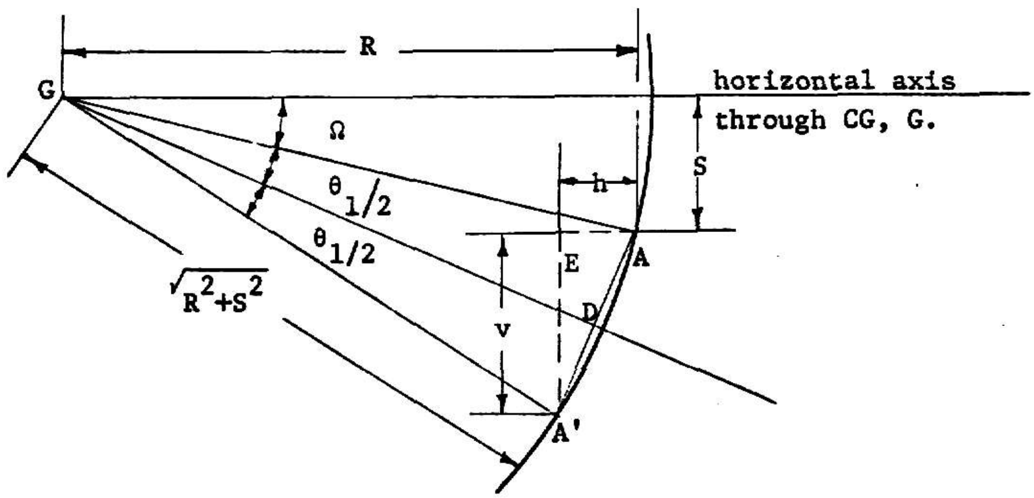


Figure 3. Geometric Representation of the Motion of Hinge Point A Due to a Rotation About Point G.

Where: G = Center of gravity (CG) of tractor,

Ω = Constant angle formed by the horizontal axis through the tractor CG and a line connecting the fifth wheel kingpin A and the tractor CG when all coordinates are at their rest positions,

R = Horizontal distance between the tractor CG and the fifth wheel kingpin A,

S = Vertical distance between the tractor CG and the fifth wheel kingpin A.

Consider the motions of A produced solely by a rotation about point G. AA' is an arc of length $\sqrt{R^2 + S^2}(\theta_1)$ described by point A. This defines the isosceles triangle AA'G which may be divided into two right triangles A'GD and AGD. These may be used in finding the length of the cord AA'.

$$\text{chord } AA' = 2 \sin \left(\frac{\theta_1}{2}\right) \sqrt{R^2 + S^2} .$$

Angle GAE is equal to Ω by the argument of opposite interior angles. Then

$$\angle EAA' = \frac{\pi}{2} - \frac{\theta_1}{2} - \Omega .$$

Also

$$v = \text{chord } AA' \sin (\angle EAA').$$

Therefore

$$v = 2\sqrt{R^2 + S^2}\ \sin\left(\frac{\theta_1}{2}\right) \sin\left(\frac{\pi}{2} - \frac{\theta_1}{2} - \Omega\right).$$

At the same time

$$h = \text{chord AA}'\ \cos\left(\frac{\pi}{2} - \frac{\theta_1}{2} - \Omega\right),$$

$$h = 2\sqrt{R^2 + S^2}\ \sin\left(\frac{\theta_1}{2}\right) \cos\left(\frac{\pi}{2} - \frac{\theta_1}{2} - \Omega\right).$$

These may be greatly simplified by using the trigonometric identities

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y, \tag{II-2a}$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y, \tag{II-2b}$$

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x, \tag{II-2c}$$

$$\sin x \cos x = \frac{1}{2}\sin 2x. \tag{II-2d}$$

Working again with the vertical component v,

$$v = 2\sqrt{R^2 + S^2}\ \sin\left(\frac{\theta_1}{2}\right)\left[\sin\left(\frac{\pi}{2} - \Omega\right)\cos\frac{\theta_1}{2} - \cos\left(\frac{\pi}{2} - \Omega\right)\sin\frac{\theta_1}{2}\right],$$

$$= 2\sqrt{R^2 + S^2}\ \left[-\sin^2\frac{\theta_1}{2}\cos\left(\frac{\pi}{2} - \Omega\right) + \sin\frac{\theta_1}{2}\cos\frac{\theta_1}{2}\sin\left(\frac{\pi}{2} - \Omega\right)\right],$$

$$= 2\sqrt{R^2 + S^2}\,[(-\frac{1}{2} + \frac{1}{2}\cos\theta_1)\sin\Omega + \frac{1}{2}\sin\theta_1\cos\Omega],$$

$$v = \sqrt{R^2 + S^2}\,[(-1 + \cos\theta_1)\sin\Omega + \sin\theta_1\cos\Omega].$$

The following geometric relations may be introduced.

$$\sqrt{R^2 + S^2}\cos\Omega = R,$$

$$\sqrt{R^2 + S^2}\sin\Omega = S.$$

Therefore,

$$v = S_{A/G\ \mathrm{vert.}} = R\sin\theta_1 + S(-1 + \cos\theta_1).$$

Working, now, with the horizontal displacement,

$$h = 2\sqrt{R^2 + S^2}\sin\frac{\theta_1}{2}\,[\cos(\frac{\pi}{2} - \Omega)\cos\frac{\theta_1}{2} + \sin(\frac{\pi}{2} - \Omega)\sin\frac{\theta_1}{2}],$$

$$= 2\sqrt{R^2 + S^2}\,[\sin\Omega\cos\frac{\theta_1}{2}\sin\frac{\theta_1}{2} + \cos\Omega\sin^2\frac{\theta_1}{2}],$$

$$= \sqrt{R^2 + S^2}\,[\sin\Omega\sin\theta_1 + \cos\Omega\,(1 - \cos\theta_1)],$$

$$h = S_{A/G\ \mathrm{horiz.}} = S\sin\theta_1 + R(1 - \cos\theta_1).$$

Therefore,

$$S_{A\ \mathrm{vert.}} = X_1 + R\sin\theta_1 + S(-1 + \cos\theta_1) \qquad \text{(II-3a)}$$

and

$$S_{A\ horiz} = X_6 + S \sin \theta_1 + R(1 - \cos \theta_1). \qquad \text{(II-3b)}$$

The right hand side of equation (II-1) expressed in trailer coordinates proceeds in exactly the same fashion.

$$S_{G'\ horizontal} = X_7,$$

$$S_{G'\ vertical} = X_2.$$

The $S_{A/G'}$ vertical and horizontal components are, again, more difficult and are derived below with the aid of Figure 4.

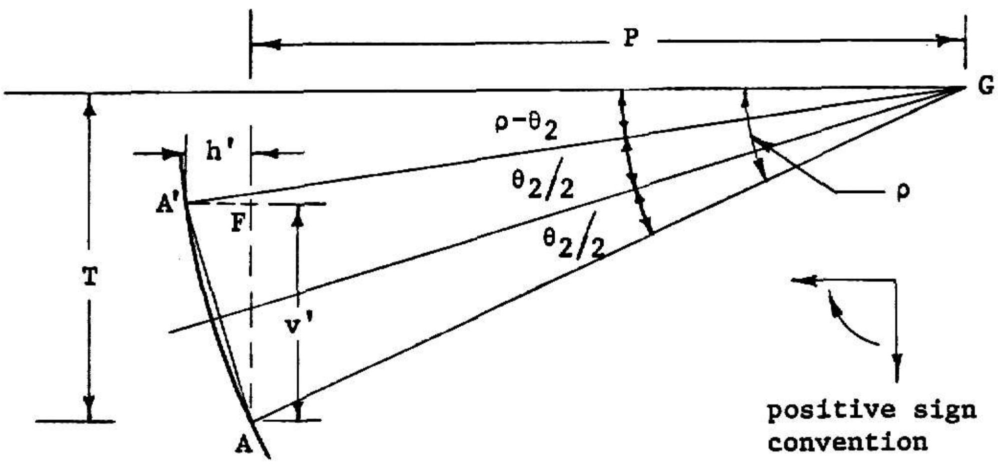


Figure 4. Geometric Representation of Hinge Point A Moving Due to a Rotation About Point G'.

Where: G' = CG of trailer,

ρ = Constant angle formed by the horizontal axis through the trailer CG and a line connecting the fifth wheel kingpin A and the trailer CG when all coordinates are at their rest positions,

P = Horizontal distance between the trailer CG and the fifth wheel kingpin A,

T = Vertical distance between the trailer CG and the fifth wheel kingpin A,

$$\angle G'A'F = \rho - \theta_2,$$

$$\angle AA'F = \frac{\pi}{2} - \frac{\theta_2}{2} - \rho + \theta_2,$$

$$\angle AA'F = \frac{\pi}{2} - \rho + \frac{\theta_2}{2},$$

$$\text{chord } AA' = 2\sqrt{P^2 + T^2} \sin\frac{\theta_2}{2} .$$

According to the positive sign convention indicated, the vertical vector displacement equation becomes

$$- v' = \text{chord } AA' \sin(\angle AA'F),$$

$$- v' = 2\sqrt{P^2 + T^2} \sin\left(\frac{\theta_2}{2}\right) \sin\left(\frac{\pi}{2} - \rho + \frac{\theta_2}{2}\right).$$

The horizontal counterpart is

$$h' = \text{chord } AA' \cos (\measuredangle AA'F),$$

$$h' = 2\sqrt{P^2 + T^2} \sin \frac{\theta_2}{2} \cos \left(\frac{\pi}{2} - \rho + \frac{\theta_2}{2}\right).$$

These equations may be simplified by the application of the trigonometric identities given in equations (II-2a-d).

Then,

$$- v' = 2\sqrt{P^2 + T^2} \left[\cos \rho \cos \frac{\theta_2}{2} \sin \frac{\theta_2}{2} + \sin \rho \sin^2 \frac{\theta_2}{2}\right],$$

$$- v' = \sqrt{P^2 + T^2} \left[\sin \theta_2 \cos \rho + \sin \rho (1 - \cos \theta_2)\right],$$

and

$$h' = 2\sqrt{P^2 + T^2} \left[\sin \frac{\theta_2}{2} \sin \rho \cos \frac{\theta_2}{2} - \cos \rho \sin^2 \frac{\theta_2}{2}\right],$$

$$h' = \sqrt{P^2 + T^2} \left[\sin \rho \sin \theta_2 + (-1 + \cos \theta_2) \cos \rho\right].$$

Again, from the initial geometry of the trailer,

$$P = \sqrt{P^2 + T^2} \cos \rho,$$

$$T = \sqrt{P^2 + T^2} \sin \rho.$$

When these expressions are substituted back into the equations for $-v'$ and h' the following simplification results:

$$-v' = S_{A/G' \text{ vert}} = P \sin \theta_2 + T(1 - \cos \theta_2),$$

$$h' = S_{A/G',\ \text{horiz.}} = T \sin \theta_2 + P(-1 + \cos \theta_2).$$

Therefore,

$$S_{A\ \text{vert.}} = -P \sin \theta_2 + T(-1 + \cos \theta_2) + X_2, \qquad \text{(II-4a)}$$

$$S_{A\ \text{horiz.}} = T \sin \theta_2 + P(-1 + \cos \theta_2) + X_7. \qquad \text{(II-4b)}$$

And by combining equations (II-3,4) there results

$$X_2 = X_1 + R \sin \theta_1 + S(-1 + \cos \theta_1) + P \sin \theta_2 + T(1 - \cos \theta_2), \qquad \text{(II-5a)}$$

$$X_7 = X_6 + S \sin \theta_1 + R(1 - \cos \theta_1) - T \sin \theta_2 + P(1 - \cos \theta_2). \qquad \text{(II-5b)}$$

These are nonlinear, holonomic, scleronomous equations of constraint expressing X_2 as a function of X_1, θ_1, and θ_2 and X_7 as a function of X_6, θ_1, and θ_2. They may be linearized by considering only small deviations of the coordinates in which case $\sin \theta \simeq \theta$ and $\cos \theta \simeq 1$. The linearized equations are as follows:

$$X_2 = X_1 + R\ \theta_1 + P\ \theta_2, \qquad \text{(II-6a)}$$

$$X_7 = X_6 + S\ \theta_1 - T\ \theta_2. \qquad \text{(II-6b)}$$

Further geometric developments are needed in order to find the amount of deflection of springs K_2, K_4, and K_6 and the velocity of deflection of dampers C_2, C_4, and C_6 as X_1, X_3, and X_4, X_5, θ_1, and θ_2 deflect. The following analysis gives the total deflection and velocity of deflection of spring K_2 and damper C_2, respectively.

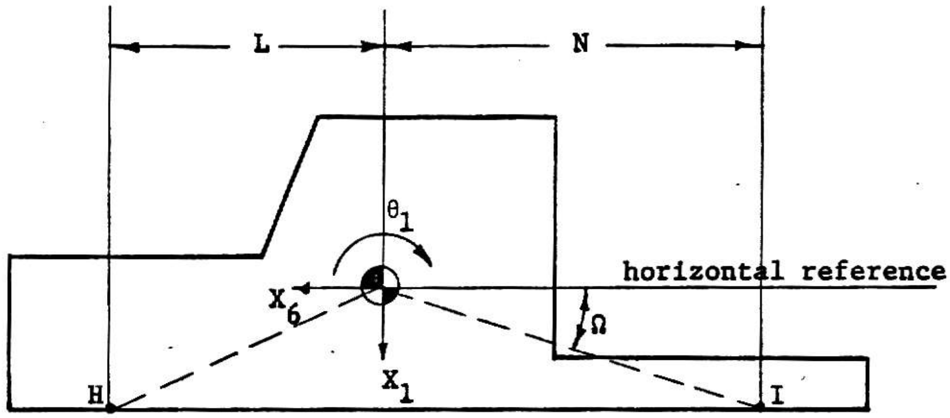


Figure 5. Tractor Diagram Showing "Mean" Points of Spring Attachment H and I.

Points H and I in Figure 5 are directly above the tractor axles and represent "mean" points of spring attachment to the chassis. A single point of spring attachment is contrary to fact since the leaf springs used in trucks have two points of attachment, making the assumed "mean" points valid only for small displacement angles θ_1.

The analysis for the determination of the vertical deflection of point H as a function of X_1 and θ_1 proceeds exactly as the analysis of the vertical motion of the fifth wheel kingpin as a function of X_2 and θ_2 yielding the result for small angles:

$$v_H = -L\,\theta_1 + X_1. \qquad \text{(II-7a)}$$

The vertical motion of point I is analyzed exactly as the vertical motion of point A in terms of the tractor coordinates X_1 and θ_1 (see Figure 3). The result is

$$v_I = N\ \theta_1 + X_1. \tag{II-7b}$$

A point J, shown in Figure 6, may also be defined for the attachment of the trailer springs.

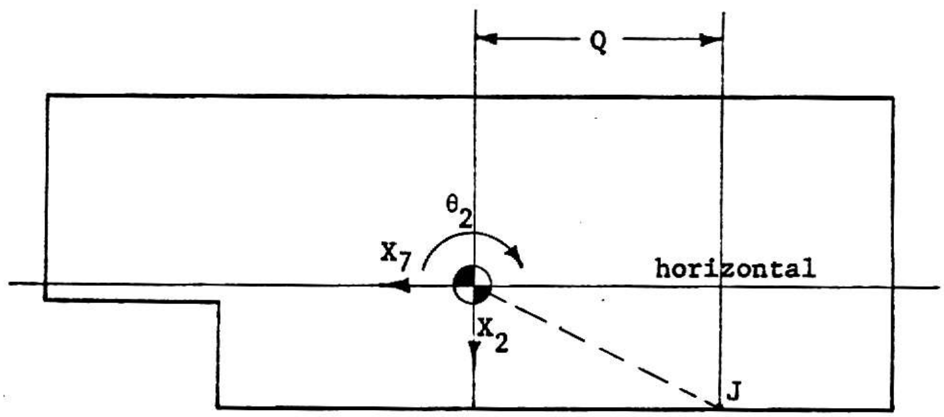


Figure 6. Trailer Diagram Showing the "Mean" point of Spring Attachment J.

The same method of analysis applies to the vertical motion of point J as applied to point I yielding

$$v_J = Q\ \theta_2 + X_2. \tag{II-7c}$$

The horizontal displacements of H, I, and J caused by rotations θ_1 and θ_2 contribute no deflections to the springs and are therefore ignored.

Energy Functions

The Lagrange equations will be used to develop the non-conservative equations of motion. This requires that the kinetic and potential energies and a dissipation function each be expressed in terms of the independent coordinates and their velocities.

The kinetic energy in terms of the velocities of the generalized coordinates is

$$T' = \frac{1}{2}(M_1\dot{X}_1^2 + M_2\dot{X}_2^2 + M_3\dot{X}_3^2 + M_4\dot{X}_4^2 + M_5\dot{X}_5^2 + M_6\dot{X}_6^2 + M_7\dot{X}_7^2 + J_1\dot{\theta}_1^2 + J_2\dot{\theta}_2^2),$$

where

$M_i (i = 1, 2, . . ., 7)$ = Mass associated with coordinate X_i,

$J_i (i = 1, 2)$ = Moment of inertia about a transverse axis (out of paper) through the origin of coordinates (CG) for the tractor and trailer, respectively.

The dissipation function in terms of the velocities of the generalized coordinates is

$$D = \frac{1}{2}\{C_1[\dot{X}_3 - \dot{G}(t)]^2 + C_2[\dot{X}_1 - \dot{X}_3 - L\dot{\theta}_1]^2 + C_3[\dot{X}_4 - \dot{G}(t - \tau_1)]^2$$

$$+ C_4[\dot{X}_1 - \dot{X}_4 + N\dot{\theta}_1]^2 + C_5[\dot{X}_5 - \dot{G}(t - \tau_2)]^2 + C_6[\dot{X}_2 - \dot{X}_5 + Q\dot{\theta}_2]^2\},$$

where

$C_i (i = 1, 2, . . ., 6)$ = damping coefficients,

L, N, Q = dimensions of truck,

$\tau_i (i = 1, 2)$ = time lag between front to rear axle of tractor and front axle of tractor to the trailer axle, respectively.

The potential energy expressed in terms of the generalized coordinates is

$$V = \frac{1}{2}\{K_1[X_3 - G(t)]^2 + K_2[X_1 - X_3 - L\theta_1]^2 + K_3[X_4 - G(t - \tau_1)]^2$$

$$+ K_4[X_1 - X_4 + N\theta_1]^2 + K_5[X_5 - G(t - \tau_2)]^2 + K_6[X_2 - X_5 + Q\theta_2]^2\},$$

where

$K_i (i=1, 2, . . ., 6)$ = spring constants.

The two equations of constraint (II-6a,b) may be substituted into these three functions thus expressing them in terms of a set of independent coordinates. Equations (II-6a,b) are rewritten below.

$$X_2 = X_1 + R\theta_1 + P\theta_2 , \qquad \text{(II-6a)}$$

$$X_7 = X_6 + S\theta_1 - T\theta_2. \qquad \text{(II-6b)}$$

The time derivatives of these equations are

$$\dot{X}_2 = \dot{X}_1 + R\dot{\theta}_1 + P\dot{\theta}_2, \tag{II-8a}$$

$$\dot{X}_7 = \dot{X}_6 + S\dot{\theta}_1 - T\dot{\theta}_2. \tag{II-8b}$$

Taking the last four equations into consideration, the three energy functions become

$$T' = \frac{1}{2}[M_1\dot{X}_1^2 + M_2(\dot{X}_1 + R\dot{\theta}_1 + P\dot{\theta}_2)^2 + M_3\dot{X}_3^2 + M_4\dot{X}_4^2 + M_5\dot{X}_5^2 + M_6\dot{X}_6^2$$

$$+ M_7(\dot{X}_6 + S\dot{\theta}_1 - T\dot{\theta}_2)^2 + J_1\dot{\theta}_1^2 + J_2\dot{\theta}_2^2], \tag{II-9a}$$

$$D = \frac{1}{2}\{C_1[\dot{X}_3 - \dot{G}(t)]^2 + C_2[\dot{X}_1 - \dot{X}_3 - L\dot{\theta}_1]^2 + C_3[\dot{X}_4 - \dot{G}(t - \tau_1)]^2$$

$$+ C_4[\dot{X}_1 - \dot{X}_4 + N\dot{\theta}_1]^2 + C_5[\dot{X}_5 - \dot{G}(t - \tau_2)]^2$$

$$+ C_6[R\dot{\theta}_1 + P\dot{\theta}_2 + \dot{X}_1 - \dot{X}_5 + Q\dot{\theta}_2]^2\}, \tag{II-9b}$$

$$V = \frac{1}{2}\{K_1[X_3 - G(t)]^2 + K_2[X_1 - X_3 - L\theta_1]^2 + K_3[X_4 - G(t - \tau_1)]^2$$

$$+ K_4[X_1 - X_4 + N\theta_1]^2 + K_5[X_5 - G(t - \tau_2)]^2$$

$$+ K_6[R\theta_1 + P\theta_2 + X_1 - X_5 + Q\theta_2]^2\}. \tag{II-9c}$$

Lagrange Equations

The Lagrange Equations of Motion are

$$\frac{d}{dt}\left(\frac{\partial T'}{\partial \dot{y}_i}\right) - \frac{\partial T'}{\partial y_i} + \frac{\partial D}{\partial \dot{y}_i} + \frac{\partial V}{\partial y_i} = 0, \qquad \text{(II-10)}$$

where $y_i (i = 1, 2, \ldots, n)$ = independent coordinates.

When the operations indicated in these equations are performed on the energy functions (II-9a,b,c) seven simultaneous, nonhomogeneous, second order differential equations with constant coefficients result. The equation derived for $y_i = X_6$ is of particular interest.

$$\frac{d}{dt}\left(\frac{\partial T'}{\partial \dot{X}_6}\right) - \frac{\partial T'}{\partial X_6} + \frac{\partial D}{\partial \dot{X}_6} + \frac{\partial V}{\partial X_6} = 0.$$

But,

$$\frac{\partial T'}{\partial X_6} = \frac{\partial D}{\partial \dot{X}_6} = \frac{\partial V}{\partial X_6} = 0.$$

Whenever an independent coordinate, such as X_6, does not appear explicitly in the kinetic energy, potential energy, or energy dissipation functions, it is called an ignorable coordinate and may be substituted out of the differential equations of motion thus reducing the number of degrees of freedom by one.

The Lagrange equation for X_6 is

$$\frac{d}{dt}\left(\frac{\partial T'}{\partial \dot{X}_6}\right) = 0 .$$

Carrying out this operation,

$$\frac{\partial T'}{\partial \dot{X}_6} = M_6\dot{X}_6 + M_7(S\dot{\theta}_1 - T\dot{\theta}_2 + \dot{X}_6),$$

$$\frac{d}{dt}\left(\frac{\partial T'}{\partial \dot{X}_6}\right) = M_6\ddot{X}_6 + M_7(S\ddot{\theta}_1 - T\ddot{\theta}_2 + \ddot{X}_6) = 0.$$

This may be solved for X_6 in terms of θ_1 and θ_2 as follows.

$$\ddot{X}_6 = -\frac{M_7}{M_6 + M_7}(S\ddot{\theta}_1 - T\ddot{\theta}_2). \qquad \text{(II-11a)}$$

This equation may be integrated twice with respect to time in order to get

$$X_6 = -\frac{M_7}{M_6 + M_7}(S\theta_1 - T\theta_2) + \underline{C}_1 t + \underline{C}_2. \qquad \text{(II-11b)}$$

The constants $\underline{C}_1$ and $\underline{C}_2$ may be found by evaluating the velocity and displacement of the X_6 coordinate at $t = 0$.

$$\dot{X}_6(0) = -\frac{M_7}{M_6 + M_7}[S\dot{\theta}_1(0) - T\dot{\theta}_2(0)] + \underline{C}_1,$$

$$X_6(0) = -\frac{M_7}{M_6 + M_7}[S\theta_1(0) - T\theta_2(0)] + \underline{C}_2.$$

At this point, one should recall that X_6 is a coordinate whose origin moves at the average speed of the truck, thus describing only the oscillatory motions of the tractor CG. $\underline{C}_1$ must then be equal to zero, otherwise equation (II-11b) shows that X_6 would grow with time. This restricts the initial velocity to

$$\dot{X}_6(0) = \frac{-M_7}{M_6 + M_7}[S\dot{\theta}_1(0) - T\dot{\theta}_2(0)].$$

The constant $\underline{C}_2$ may also be forced to equal zero if

$$X_6(0) = \frac{-M_7}{M_6 + M_7}[S\theta_1(0) - T\theta_2(0)].$$

This occurs when the origin of the X_6 coordinate is defined to be at the center of gravity of the tractor when all of the independent coordinates are in their neutral positions, in which case

$$X_6 = -\frac{M_7}{M_6 + M_7}(S\theta_1 - T\theta_2). \qquad \text{(II-11c)}$$

Thus, while X_6 was formerly thought to be an independent coordinate (because no kinematical dependence was discovered), it is now seen to be dynamically dependent upon coordinates θ_1 and θ_2.

The remaining six differential equations may be derived quite conventionally except that the above expression for X_6 should be substituted into the remaining differential equations of motion. Since equation (II-11a), above, was shown to be directly integrable, there remain only six unknowns in the differential equations.

X_1 equation:

$$\frac{\partial T'}{\partial \dot{X}_1} = M\dot{X}_1 + M_2(\dot{X}_1 + R\dot{\theta}_1 + P\dot{\theta}_2),$$

$$\frac{d}{dt}\left(\frac{\partial T'}{\partial \dot{X}_1}\right) = M_1\ddot{X}_1 + M_2(\ddot{X}_1 + R\ddot{\theta}_1 + P\ddot{\theta}_2).$$

$$\frac{\partial D}{\partial \dot{X}_1} = C_2(\dot{X}_1 - \dot{X}_3 - L\dot{\theta}_1) + C_4(\dot{X}_1 - \dot{X}_4 + N\dot{\theta}_1)$$

$$+ C_6(R\dot{\theta}_1 + P\dot{\theta}_2 + \dot{X}_1 - \dot{X}_5 + Q\dot{\theta}_2).$$

$$\frac{\partial V}{\partial X_1} = K_2(X_1 - X_3 - L\theta_1) + K_4(X_1 - X_4 + N\theta_1)$$

$$+ K_6(R\theta_1 + P\theta_2 + X_1 - X_5 + Q\theta_2).$$

Assembling these functions and grouping like coordinates yields

$$(M_1 + M_2)\ddot{X}_1 + (C_2 + C_4 + C_6)\dot{X}_1 + (K_2 + K_4 + K_6)X_1 - C_2\dot{X}_3 - K_2X_3$$

$$- C_4\dot{X}_4 - K_4X_4 - C_6\dot{X}_5 - K_6X_5 + M_2R\ddot{\theta}_1 + (C_4N + C_6R - C_2L)\dot{\theta}_1$$

$$+ (-K_2L + K_4N + K_6R)\theta_1 + M_2P\ddot{\theta}_2 + C_6(P + Q)\dot{\theta}_2 + K_6(P + Q)\theta_2 = 0.$$

(II - 12a)

X_3 equation:

$$\frac{d}{dt}\left(\frac{\partial T'}{\partial \dot{X}_3}\right) = M_3\ddot{X}_3.$$

$$\frac{\partial D}{\partial \dot{X}_3} = C_1[\dot{X}_3 - \dot{G}(t)] - C_2(\dot{X}_1 - \dot{X}_3 - L\dot{\theta}_1).$$

$$\frac{\partial V}{\partial X_3} = K_1[X_3 - G(t)] - K_2(X_1 - X_3 - L\theta_1).$$

Assembling these functions and grouping like coordinates yields

$$- C_2\dot{X}_1 - K_2X_1 + M_3\ddot{X}_3 + (C_1 + C_2)\dot{X}_3 + (K_1 + K_2)X_3 + C_2L\dot{\theta}_1 + K_2L\theta_1$$

$$= C_1\dot{G}(t) + K_1G(t). \qquad \text{(II-12b)}$$

X_4 equation:

$$\frac{d}{dt}\left(\frac{\partial T'}{\partial \dot{X}_4}\right) = M_4\ddot{X}_4.$$

$$\frac{\partial D}{\partial \dot{X}_4} = C_3[\dot{X}_4 - \dot{G}(t - \tau_1)] - C_4(\dot{X}_1 - \dot{X}_4 + N\dot{\theta}_1).$$

$$\frac{\partial V}{\partial X_4} = K_3[X_4 - G(t - \tau_1)] - K_4(X_1 - X_4 + N\theta_1).$$

Assembling these functions and grouping like coordinates yields

$$- C_4\dot{X}_1 - K_4X_1 + M_4\ddot{X}_4 + (C_3 + C_4)\dot{X}_4 + (K_3 + K_4)X_4$$

$$- C_4N\dot{\theta}_1 - K_4N\theta_1 = C_3\dot{G}(t - \tau_1) + K_3G(t - \tau_1). \qquad \text{(II-12c)}$$

$\underline{X_5 \text{ equation:}}$

$$\frac{d}{dt}\left(\frac{\partial T'}{\partial \dot{X}_5}\right) = M_5\ddot{X}_5.$$

$$\frac{\partial D}{\partial \dot{X}_5} = C_5[\dot{X}_5 - \dot{G}(t - \tau_2)] - C_6(R\dot{\theta}_1 + P\dot{\theta}_2 + \dot{X}_1 - \dot{X}_5 + Q\dot{\theta}_2).$$

$$\frac{\partial V}{\partial X_5} = K_5[X_5 - G(t - \tau_2)] - K_6(R\theta_1 + P\theta_2 + X_1 - X_5 + Q\theta_2).$$

Assembling these functions and grouping like coordinates yields

$$- C_6\dot{X}_1 - K_6X_1 + M_5\ddot{X}_5 + (C_5 + C_6)\dot{X}_5 + (K_5 + K_6)X_5 - C_6R\dot{\theta}_1 - K_6R\theta_1$$

$$- C_6 (P + Q)\dot{\theta}_2 - K_6(P + Q)\theta_2 = C_5\dot{G}(t - \tau_2) + K_5G(t - \tau_2). \qquad \text{(II-12d)}$$

$\underline{\theta_1 \text{ equation:}}$

$$\frac{d}{dt}\left(\frac{\partial T'}{\partial \dot{\theta}_1}\right) = M_2(\ddot{X}_1 + R\ddot{\theta}_1 + P\ddot{\theta}_2)R + M_7(\ddot{X}_6 + S\ddot{\theta}_1 - T\ddot{\theta}_2)S + J_1\ddot{\theta}_1 .$$

$\ddot{X}_6$ may be replaced by using equation (II-11a),

$$\ddot{X}_6 = -\frac{M_7}{M_6 + M_7}(S\ddot{\theta}_1 - T\ddot{\theta}_2). \qquad \text{(II-11a)}$$

$$\frac{d}{dt}\left(\frac{\partial T'}{\partial \dot{\theta}_1}\right) = M_2R(\ddot{X}_1 + R\ddot{\theta}_1 + P\ddot{\theta}_2) + M_7S[-\frac{M_7}{M_6 + M_7}(S\ddot{\theta}_1 - T\ddot{\theta}_2) + S\ddot{\theta}_1 - T\ddot{\theta}_2] + J_1\ddot{\theta}$$

$$\frac{\partial D}{\partial \dot{\theta}_1} = - C_2L(\dot{X}_1 - \dot{X}_3 - L\dot{\theta}_1) + C_4N(\dot{X}_1 - \dot{X}_4 + N\dot{\theta}_1) + C_6R(R\dot{\theta}_1 + P\dot{\theta}_2 + \dot{X}_1 - \dot{X}_5 + Q\dot{\theta}_2).$$

$$\frac{\partial V}{\partial \theta_1} = - K_2L(X_1 - X_3 - L\theta_1) + K_4N(X_1 - X_4 + N\theta_1) + K_6R(R\theta_1 + P\theta_2$$

$$+ X_1 - X_5 + Q\theta_2).$$

Assembling these functions and grouping like coordinates yields

$$M_2R\ddot{X}_1 + (-C_2L + C_4N + C_6R)\dot{X}_1 + (-K_2L + K_4N + K_6R)X_1 + C_2L\dot{X}_3 + K_2LX_3$$

$$- C_4N\dot{X}_4 - K_4NX_4 - C_6R\dot{X}_5 - K_6RX_5 + [M_2R^2 + (M_7 - \frac{M_7^2}{M_6 + M_7})S^2 + J_1]\ddot{\theta}_1$$

$$+ (C_2L^2 + C_4N^2 + C_6R^2)\dot{\theta}_1 + (K_2L^2 + K_4N^2 + K_6R^2)\theta_1 + [M_2RP - (M_7 - \frac{M_7^2}{M_6+M_7})ST]\ddot{\theta}_2$$

$$+ C_6R(P + Q)\dot{\theta}_2 + K_6R(P + Q)\theta_2 = 0. \qquad \text{(II-12e)}$$

<u>θ_2 equation</u>

$$\frac{d}{dt}\left(\frac{\partial T'}{\partial \dot{\theta}_2}\right) = M_2P(\ddot{X}_1 + R\ddot{\theta}_1 + P\ddot{\theta}_2) - M_7T(\ddot{X}_6 + S\ddot{\theta}_1 - T\ddot{\theta}_2) + J_2\ddot{\theta}_2.$$

Again, $\ddot{X}_6$ may be replaced by using equation (II-11a).

$$\ddot{X}_6 = - \frac{M_7}{M_6 + M_7}(S\ddot{\theta}_1 - T\ddot{\theta}_2). \qquad \text{(II-11a)}$$

$$\frac{d}{dt}\left(\frac{\partial T'}{\partial \dot{\theta}_2}\right) = M_2P(\ddot{X}_1 + R\ddot{\theta}_1 + P\ddot{\theta}_2) - M_7T[- \frac{M_7}{M_6 + M_7}(S\ddot{\theta}_1 - T\ddot{\theta}_2) + S\ddot{\theta}_1 - T\ddot{\theta}_2]$$

$$+ J_2\ddot{\theta}_2.$$

$$\frac{\partial D}{\partial \dot{\theta}_2} = C_6(P + Q)[R\dot{\theta}_1 + (P + Q)\dot{\theta}_2 + \dot{X}_1 - \dot{X}_5].$$

$$\frac{\partial V}{\partial \theta_2} = K_6(P + Q)[R\theta_1 + (P + Q)\theta_2 + X_1 - X_5].$$

Assembling these functions and grouping like coordinates yields

$$M_2P\ddot{X}_1 + C_6(P + Q)\dot{X}_1 + K_6(P + Q)X_1 + [M_2PR - (M_7 - \frac{M_7^2}{M_6 + M_7})ST]\ddot{\theta}_1 + C_6R(P+Q)\dot{\theta}_1$$

$$+ K_6R(P + Q)\theta_1 + [M_2P^2 + (M_7 - \frac{M_7^2}{M_6 + M_7})T^2 + J_2]\ddot{\theta}_2$$

$$+ C_6(P + Q)^2\dot{\theta}_2 + K_6(P + Q)^2\theta_2 = 0. \qquad \text{(II-12f)}$$

Equations (II-12a-f) may be grouped into the matrix equation as shown below, and written out in full on the following page.

$$(M)\{\ddot{X}\} + (C)\{\dot{X}\} + (K)\{X\} = \{F\}. \qquad \text{(II-13a)}$$

$$\begin{bmatrix} M_1' & 0 & 0 & 0 & M_2R & M_2P \\ 0 & M_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & M_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_5 & 0 & 0 \\ M_2R & 0 & 0 & 0 & M_6' & M_6'' \\ M_2P & 0 & 0 & 0 & M_6'' & M_7' \end{bmatrix} \begin{bmatrix} \ddot{X}_1 \\ \ddot{X}_3 \\ \ddot{X}_4 \\ \ddot{X}_5 \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} \tag{II-13b}$$

$$+ \begin{bmatrix} C_1' & -C_2 & -C_4 & -C_6 & C_2' & C_3' \\ -C_2 & C_4' & 0 & 0 & C_2L & 0 \\ -C_4 & 0 & C_5' & 0 & -C_4N & 0 \\ -C_6 & 0 & 0 & C_6' & -C_6R & -C_3' \\ C_2' & C_2L & -C_4N & -C_6R & C_7' & C_3'R \\ C_3' & 0 & 0 & -C_3' & C_3'R & C_8' \end{bmatrix} \begin{bmatrix} \dot{X}_1 \\ \dot{X}_3 \\ \dot{X}_4 \\ \dot{X}_5 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$+ \begin{bmatrix} K_1' & -K_2 & -K_4 & -K_6 & K_2' & K_3' \\ -K_2 & K_4' & 0 & 0 & K_2L & 0 \\ -K_4 & 0 & K_5' & 0 & -K_4N & 0 \\ -K_6 & 0 & 0 & K_6' & -K_6R & -K_3' \\ K_2' & K_2L & -K_4N & -K_6R & K_7' & K_3'R \\ K_3' & 0 & 0 & -K_3' & K_3'R & K_8' \end{bmatrix} \begin{bmatrix} X_1 \\ X_3 \\ X_4 \\ X_5 \\ \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ C_1 \dot{G}(t) + K_1 G(t) \\ C_3 \dot{G}(t - \tau_1) + K_3 G(t - \tau_1) \\ C_5 \dot{G}(t - \tau_2) + K_5 G(t - \tau_2) \\ 0 \\ 0 \end{bmatrix}.$$

The following simplifications have been made in the above equations in order to render them systematically presentable.

$$M_1' = M_1 + M_2$$

$$M_6' = M_2R^2 + J_1 + \frac{M_6M_7}{M_6 + M_7}\ S^2$$

$$M_6'' = M_2RP - \frac{M_6M_7}{M_6 + M_7}\ ST$$

$$M_7' = M_2P^2 + J_2 + \frac{M_6M_7}{M_6 + M_7}\ T^2$$

$$C_1' = C_2 + C_4 + C_6$$

$$C_2' = -\ C_2L + C_4N + C_6R$$

$$C_3' = C_6(P + Q)$$

$$C_4' = C_1 + C_2$$

$$C_5' = C_3 + C_4$$

$$C_6' = C_5 + C_6$$

$$C_7' = C_2L^2 + C_4N^2 + C_6R^2$$

$$C_8' = C_6(P + Q)^2$$

$$K_1' = K_2 + K_4 + K_6$$

$$K_2' = -\ K_2L + K_4N + K_6R$$

$$K_3' = K_6(P + Q)$$

$$K_4' = K_1 + K_2$$

$$K_5' = K_3 + K_4$$

$$K_6' = K_5 + K_6$$

$$K_7' = K_2L^2 + K_4N^2 + K_6R^2$$

$$K_8' = K_6(P + Q)^2$$

also

$$M_6 = M_1 + M_3 + M_4 \qquad M_7 = M_2 + M_5$$

Chapter III

THEORY OF SOLUTIONS OF THE DIFFERENTIAL EQUATIONS OF MOTION

The solution to equation (II-13) is, as for a single differential equation, composed of a homogeneous and a complementary part.

Homogeneous Solution

The homogeneous matrix equation

$$(M)\{\ddot{X}\} + (C)\{\dot{X}\} + (K)\{X\} = \{0\} \tag{III-1}$$

has a standard solution of the form

$$\{X\} = \{q\}e^{\beta t} \tag{III-2}$$

where β and the elements of $\{q\}$ (which are all constants) may be complex, but combine to make $\{X\}$ real.

If (III-2) is substituted into (III-1) the following equation results

$$(\beta^2 M + \beta C + K)\{q\} = \{0\}. \tag{III-3a}$$

Nontrivial solutions for the q's are possible only for special values of β (called eigenvalues) which cause the matrix

$$(\beta^2 M + \beta C + K)$$

to be singular; i.e., the determinant

$$|\beta^2 M + \beta C + K| = |\Delta\beta| = 0. \tag{III-4}$$

While these eigenvalues arise for the same reasons as the eigenvalues of the classical eigenvalue problem of

$$(A)\{X\} = \lambda(B)\{X\} \tag{III-5}$$

(where λ = eigenvalue),

equation (III-3) takes the mathematical form

$$(K)\{q\} = \lambda(M)\{q\} + \sqrt{\lambda}\ (C)\{q\} \tag{III-6}$$

due to the inclusion of the damping matrix (C).

Many simple iteration techniques exist for the determination of the λ's in (III-5) which allow easy computer programming with a rapid convergence to its eigenvalues and eigenvectors. However, no similar methods exist for the problem in (III-6) which is considered in this paper. Therefore the more tedious approach of actually evaluating $\Delta\beta$ must be employed. This results in a polynomial in β of degree 2n, where n is the order of the determinant $|\Delta\beta|$ (in this case n = 6). The solution of this polynomial will generally be in the form of twelve complex roots arising in conjugate pairs. The real part of each root expresses a rate of exponential decay while the imaginary part expresses a damped natural frequency of vibration for the system.

If these roots (eigenvalues) are now substituted one at a time back into equation (III-3a):

$$(\beta^2 M + \beta C + K)\{q\} = \{0\} \tag{III-3a}$$

or

$$(\Delta\beta)\{q\} = \{0\} \tag{III-3b}$$

the eigenvectors, or natural vibration mode shapes, can be found. The matrix $(\Delta\beta)$ will be singular since for an eigenvalue β, $|\Delta\beta| = 0$. The rank of $(\Delta\beta)$ will be n-1 if the eigenvalues are distinct (only distinct eigenvalues will be considered here). Then, if a new vector $\{q\}$ is formed by dividing through both sides of (III-3a or b) by q_1 the first algebraic equation of the matrix equation (III-3a,b) will reduce to an identity and a matrix equation of order n-1 in the n-1 unknown ratios $\frac{q_2}{q_1}, \frac{q_3}{q_1}, \ldots, \frac{q_n}{q_1}$ will result.

These n-1 ratios may now be determined uniquely in the n-1 algebraic equations. The first element, q_1, in each eigenvector is actually undetermined and may be chosen arbitrarily, however, it is usual to choose $q_1 = 1$ for the $\frac{q_\alpha}{q_1}$ ratios will then express the q_α solutions directly. Since β is in general complex, these q_α's will also be complex and will express the amplitude and phase angle of the α'th coordinate relative to the amplitude of X_1 (the phase angle of X_1 is chosen to be 0). This may be expressed by vectors on a complex plane as shown in the Figure 7 below.

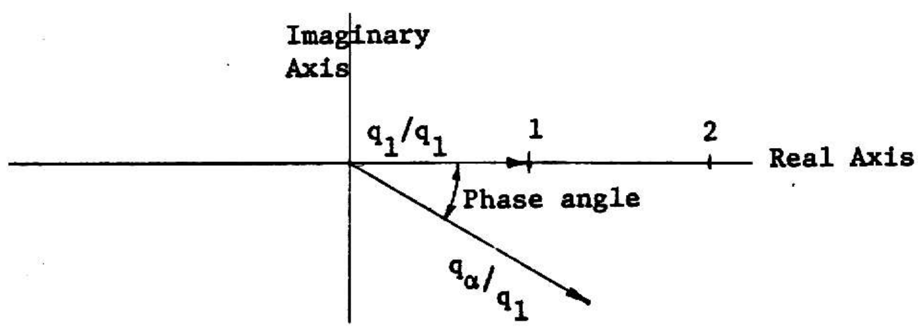


Figure 7. Vector Diagram Showing the Geometric Relationship Between the α'th and the First q Ratio.

It would appear that this theory differs from that of the usual eigenvalue problem (III-5) since 2n eigenvalues and {q} vectors result instead of the usual n. However, on closer examination, it may be seen that when the β eigenvalues occur in complex conjugate pairs, they give rise to a conjugate pair of complex {q} vectors. Each member of the {q} vector pair represents the same eigenvector, which may be demonstrated by the following analysis as developed by the author.

Assume that adjacent pairs of roots of the characteristic equation $|\Delta\beta| = 0$ are conjugate complex, i.e., β_1 and β_2 are conjugate complex roots. Then let $q_{\alpha 1}$ be the α'th element of the {q} vector which results when β_1 is substituted into (III-3b) and $q_{\alpha 2}$ be the α'th element of the {q} vector which results when β_2 is substituted into the same equation. Then when the system is allowed to vibrate freely at the frequency given by the absolute value of the imaginary portion of β_1 and β_2 the α'th coordinate moves as

$$X_\alpha = q_{\alpha 1}\, e^{\beta_1 t} + q_{\alpha 2} e^{\beta_2 t}.$$

Assume, as a worst case, that the arbitrary q_1 is complex. Then

$$q_{11} = a - ib; \quad q_{12} = a + ib; \quad \beta_1 = - c - i\omega d; \quad \beta_2 = - c + i\omega f$$

and

$$X_1 = (a - ib)e^{-(c + i\omega d)t} + (a + ib)e^{-(c - i\omega d)t}.$$

The complex coefficients may be put into exponential form as follows:

$$X_1 = \sqrt{a^2+b^2}\, e^{-(c + i\omega d)t - i\Phi} + \sqrt{a^2+b^2}\, e^{-(c - i\omega d)t + i\Phi}$$

where

$$\Phi = \tan^{-1} \frac{b}{a} .$$

Euler's formula ($e^{i\Phi} = \cos\Phi + i \sin \Phi$) may be used to represent the exponential as trigonometric functions.

$$X_1 = \sqrt{a^2+b^2}\, e^{-ct} [\cos(-\omega dt - \Phi) + i \sin (-\omega dt - \Phi)$$

$$+ \cos (\omega dt + \Phi) + i \sin (\omega dt + \Phi)].$$

The sine terms add out leaving a real expression for X_1.

$$X_1 = 2\sqrt{a^2+b^2}\, e^{-ct} \cos(\omega dt + \Phi).$$

As was mentioned before, since q_1 is arbitrary it is advantageous to choose $q_1 = q_{11} = q_{12} = a = 1$. X_1 then becomes

$$X_1 = 2\, e^{-ct} \cos \omega dt.$$

Similarly, for the α'th coordinate let

$$\beta_1 = - c - i\omega d; \quad \beta_2 = - c + i\omega d; \quad q_{\alpha 1} = g_\alpha - ih_\alpha; \quad q_{\alpha 2} = g_\alpha + ih_\alpha .$$

Then

$$X_\alpha = (g_\alpha - ih_\alpha)\, e^{-(c + i\omega d)t} + (g_\alpha + ih_\alpha)\, e^{-(c - i\omega d)t}.$$

By the preceding argument, this reduces to

$$X_\alpha = 2\sqrt{g_\alpha^2+h_\alpha^2}\, e^{-ct}\, [\cos(\omega dt + \phi_\alpha)]$$

where $\phi_\alpha = \tan^{-1}\dfrac{h_\alpha}{g_\alpha}$ and the signs of g and h are carried into the evaluation of the arctangent.

The amplitudes and phase angles for all X_α ($\alpha = 2, 3, \ldots, n$) are all that are of importance in expressing the geometric configuration (eigenvector) of the system vibrating in one of its natural modes. That is

$$X_1 = 2\angle 0^\circ \; ; \; X\alpha = 2\sqrt{g_\alpha^2+h_\alpha^2}\,\angle{-\Phi_\alpha}; \quad \Phi_\alpha = \tan^{-1}\frac{h_\alpha}{g_\alpha}, \qquad \text{(III-7)}$$

$$(\alpha = 2, 3, \ldots, n).$$

Furthermore the eigenvectors of a system may always be scaled. It is common to normalize the eigenvector so that $X_1 = 1$. This is easily done in equations (III-7) by dividing out a common factor 2.

The ratios

$$\frac{q_{\alpha\gamma}}{q_{1\gamma}} \quad (\gamma = 1, 2, \ldots, 2n)$$

will be uniquely determined as was pointed out earlier, however, q_{11}, ..., q_{12n} must remain arbitrary. These are the 2n arbitrary constants of integration for the n second order differential equation of motion. Using the simplification

$$\mu_{\alpha\gamma} = \frac{q_{\alpha\gamma}}{q_{1\gamma}}$$

the complete transient solution to the general problem is given by

$$X_1 = q_{11}\, e^{\beta_1 t} + q_{12}\, e^{\beta_2 t} + \ldots + q_{12n}\, e^{\beta_{2n} t}$$

$$X_2 = q_{11}\mu_{21}\, e^{\beta_1 t} + q_{12}\mu_{22}\, e^{\beta_2 t} + \ldots + q_{12n}\mu_{22n}\, e^{\beta_{2n} t}$$

$$\vdots$$

$$X_n = q_{11}\mu_{n1}\, e^{\beta_1 t} + \ldots + q_{12n}\mu_{n\,2n}\, e^{\beta_{2n} t} \; . \qquad \text{(III-8)}$$

The 2n undetermined q constants may be determined if the initial conditions on displacements and velocities for all independent coordinates are known, i.e., $\{X\}$ and $\{\dot{X}\}$ must be known for time $t = 0$. When these are known, the transient solution (III-8) may be differentiated to produce n additional equations needed to complete the set of 2n algebraic equations in the 2n unknowns q_{11}-q_{12n}. At time $t = 0$ all exponential terms equal 1, thus greatly simplifying the evaluation of the q's. Because the μ's directly determine the eigenvectors these equations show that the freely vibrating solution is just a linear combination of the eigenvectors of the system.

The transient solution plays a key role in problems such as determining the motion which occurs immediately following the application or removal of a braking force (while the truck is moving). These problems are important to highway engineers involved in investigating the washboard effect found in downhill sections and stop sign approaches of roads constructed of asphalt pavements. However, since these problems involve a specific application of the analysis, they will not be discussed further. Rather, the next objective of a completely general determination of the truck vibrations is the discussion of the steady state oscillations.

Complementary Solution

The matrix formulation for the complementary (steady state) solution is

$$(M)\{\ddot{X}\} + (C)\{\dot{X}\} + (K)\{X\} = \{F(t)\} \qquad \text{(III-9)}$$

where the matrices (M), (C), and (K) are the same as those used in equations (II-13). A solution for $\{X\}$ is assumed as some function of $F(t)$. For the problem at hand this turns out to be

$$\{X\} = \{A\} \sin\omega t + \{B\} \cos\omega t \qquad \text{(III-10)}$$

because only sinusoidal (cosinusoidal) forcing functions will be involved in $F(t)$. A nonsinusoidal forcing function (if it is periodic and satisfies certain continuity and differentiability conditions) may be expressed as an infinite series of sinusoidal forcing functions. The solution to the nonsinusoidally forced problem, then, merely becomes the superposition of each sinusoidal part of this infinite series. This becomes a Fourier series problem. Of course an infinite number of terms of a Fourier series are involved and may not be practically accommodated, thus an approximation may be effected by using only a few terms of this solution. However, any specified error requirements may be met simply by including additional terms in the answer.

Substituting equation (III-10) back into (III-9) yields

$$-\omega^2(M)\{A\}\sin\omega t - \omega^2(M)\{B\}\cos\omega t + \omega(C)\{A\}\cos\omega t - \omega(C)\{B\}\sin\omega t$$

$$+ (K)\{A\}\sin\omega t + (K)\{B\}\cos\omega t = \{F_1\}\sin\omega t + \{F_2\}\cos\omega t \qquad \text{(III-11)}$$

where the forcing function $F(t)$ has been split into sine and cosine components. The sine and cosine terms of equation (III-11) may be equated as

$$- \omega^2(M)\{A\} - \omega(C)\{B\} + (K)\{A\} = \{F_1\},$$

$$- \omega^2(M)\{B\} + \omega(C)\{A\} + (K)\{B\} = \{F_2\}.$$

These may be further reduced to

$$[- \omega^2(M) + (K)]\{A\} - \omega(C)\{B\} = \{F_1\},$$

$$\omega(C)\{A\} + [- \omega^2(M) + (K)]\{B\} = \{F_2\},$$

and by recombination the following matrix equation results

$$\left[\begin{array}{c|c} -\omega^2(M) + (K) & -\omega(C) \\ \hline \omega(C) & -\omega^2(M) + (K) \end{array}\right] \left[\begin{array}{c} A \\ \hline B \end{array}\right] = \left[\begin{array}{c} F_1 \\ \hline F_2 \end{array}\right] . \tag{III-12a}$$

Or in more compact shorthand

$$(MCK) \ \{{A \atop B}\} = \left\{\begin{matrix} F_1 \\ F_2 \end{matrix}\right\} . \tag{III-12b}$$

This matrix equation represents 2n algebraic equations in the 2n unknowns {A} and {B} (recall that n = 6 for the truck) and for this problem is written out in full on the following page.

$$
\begin{bmatrix}
\underline{M}_1+K'_1 & -K_2 & -K_4 & -K_6 & \underline{M}_2R+K'_2 & \underline{M}_2P+K'_3 & -C'_1 & C''_2 & C''_4 & C''_6 & -C'_2 & -C'_3 \\
-K_2 & \underline{M}_3+K'_4 & 0 & 0 & K_2L & 0 & C''_2 & -C'_4 & 0 & 0 & -C''_2L & 0 \\
-K_4 & 0 & \underline{M}_4+K'_5 & 0 & -K_4N & 0 & C''_4 & 0 & -C'_5 & 0 & C''_4N & 0 \\
-K_6 & 0 & 0 & \underline{M}_5+K'_6 & -K_6R & -K'_3 & C''_6 & 0 & 0 & -C'_6 & C''_6R & C'_3 \\
\underline{M}_2R+K'_2 & K_2L & -K_4N & -K_6R & \underline{M}_6+K'_7 & \underline{M}'_6+K''_3 & -C'_2 & -C''_2L & C''_4N & C''_6R & -C'_7 & -C''_3 \\
\underline{M}_2P+K'_3 & 0 & 0 & -K'_3 & \underline{M}'_6+K''_3 & \underline{M}_7+K'_8 & -C'_3 & 0 & 0 & C'_3 & -C''_3 & -C'_8 \\
C'_1 & -C''_2 & -C''_4 & -C''_6 & C'_2 & C'_3 & \underline{M}_1+K'_1 & -K_2 & -K_4 & -K_6 & \underline{M}_2R+K'_2 & \underline{M}_2P+K'_3 \\
-C''_2 & C'_4 & 0 & 0 & C''_2L & 0 & -K_2 & \underline{M}_3+K'_4 & 0 & 0 & K_2L & 0 \\
-C''_4 & 0 & C'_5 & 0 & -C''_4N & 0 & -K_4 & 0 & \underline{M}_4+K'_5 & 0 & -K_4N & 0 \\
-C''_6 & 0 & 0 & C'_6 & -C''_6R & -C'_3 & -K_6 & 0 & 0 & \underline{M}_5+K'_6 & -K_6R & -K'_3 \\
C'_2 & C''_2L & -C''_4N & -C''_6R & C'_7 & C''_3 & \underline{M}_2R+K'_2 & K_2L & -K_4N & -K_6R & \underline{M}_6+K'_7 & \underline{M}'_6+K''_3 \\
C'_3 & 0 & 0 & -C'_3 & C''_3 & C'_8 & \underline{M}_2P+K'_3 & 0 & 0 & -K'_3 & \underline{M}'_6+K''_3 & \underline{M}_7+K'_8
\end{bmatrix}
\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \end{bmatrix}
=
\begin{bmatrix} 0 \\ F_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ F_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

The following simplifications have been made in the equations of the previous page in order to render them systematically presentable.

$$\underline{M}_1 = -(M_1 + M_2)\omega^2$$
$$\underline{M}_2 = -M_2\omega^2$$
$$\underline{M}_3 = -M_3\omega^2$$
$$\underline{M}_4 = -M_4\omega^2$$
$$\underline{M}_5 = M_5\omega^2$$
$$\underline{M}_6 = -\left(M_2R^2 + J_1 + \frac{M_6M_7}{M_6 + M_7}S^2\right)\omega^2$$
$$\underline{M}_6' = \left(-M_2RP + \frac{M_6M_7}{M_6 + M_7}ST\right)\omega^2$$
$$M_7 = -\left(M_2P^2 + J_2 + \frac{M_6M_7}{M_6 + M_7}T^2\right)\omega^2$$
$$K_1' = K_2 + K_4 + K_6$$
$$K_2' = K_4N + K_6R - K_2L$$
$$K_3' = K_6(P + Q)$$
$$K_4' = K_1 + K_2$$
$$K_5' = K_3 + K_4$$
$$K_6' = K_5 + K_6$$
$$K_7' = K_2L^2 + K_4N^2 + K_6R^2$$
$$K_8' = K_6(P + Q)^2$$
$$K_3'' = K_6R(P + Q)$$

$$C_1' = (C_2 + C_4 + C_6)\omega$$
$$C_2' = (C_4N + C_6R - C_2L)\omega$$
$$C_3' = [C_6(P + Q)]\omega$$
$$C_4' = (C_1 + C_2)\omega$$
$$C_5' = (C_3 + C_4)\omega$$
$$C_6' = (C_5 + C_6)\omega$$
$$C_7' = (C_2L^2 + C_4N^2 + C_6R^2)\omega$$
$$C_8' = [C_6(P + Q)^2]\omega$$
$$C_3'' = [C_6R(P + Q)]\omega$$
$$C_2'' = C_2\omega$$
$$C_4'' = C_4\omega$$
$$C_6'' = C_6\omega$$

Solution of these algebraic equations is nearly automatic with the help of the digital computer, as several IBM Scientific Subroutine programs exist (SIMQ, GELG, DGELG) which are based upon the Gauss reduction scheme.

Thus only a program which calculates the coefficient matrix (MCK), reads $\{{F_1 \atop F_2}\}$, calls one of the above mentioned subroutines, and writes out the answer $\{{A \atop B}\}$ is necessary for the solution to equation (III-12).

Chapter IV

PRESENT STATE OF RESEARCH AND FUTURE GOALS

Present State

An elaborate digital computer program consisting of a main supervisory program and twenty-five subprograms has been written in the Fortran IV language for execution on the IBM 360/50 computer located at the Kansas State University Computing Center. This program solves all of the equations derived in Chapter II by the methods of Chapter III and incorporates the following features.

1) A preface output page states the parameters (masses, spring constants, damping constants, and vehicle geometry) which are considered in the vibrational analysis leading to the following output.

2) Results of a standard undamped eigenvalue analysis yielding resonant vibrational frequencies which should lie near the damped resonant frequencies are presented. The undamped eigenvectors have also been included so that the effect of damping on the mode shape may be evaluated.

3) The damped eigenvalues are next presented. These are generally complex numbers arising in conjugate pairs. The real part of one of these numbers gives the damping rate for the particular mode arising for that eigenvalue and the imaginary part represents the damped natural frequency (radians/second) for the same mode.

4) The eigenvalues are then used to find each damped eigenvector of the system.

5) Bode Plots are calculated for each generalized coordinate of the vehicle. The Bode Plots are graphs of the amplitude ratio (magnification factor) of the output which results when a purely sinusoidal input is applied as a road input displacement function, to the amplitude which would result if the forcing frequency were zero radians per second (static case). Since the forcing frequency may vary due either to a change in the speed of the vehicle or to a change in the frequency of the road input sinusoid (or to a combination of the two), a Bode Surface may be defined for this problem. However, the program produces only section plots of this surface, one for each coordinate for each speed specified in the data input to the program. Actual graphs are produced on the IBM 1403 Printer by this program. The range of frequencies for these graphs is 0-- 10+ highest damped natural frequency (radians/second).

6) The program next accepts information in one or more of four different forms defining actual road input displacement functions. These include concave upward or downward parabolas, a list of coefficients defining a polynomial of up to degree 49, or a tabulated list of road displacements from some level datum, measured at equal intervals along the road. The parabolas are considered periodic over a short length (15-50 feet) and

approximate a condition existing on some concrete highways wherein the surface raises or sinks between the tar strips. The polynomial form, and especially the list of tabulated displacements, may be considered periodic over a much longer distance, say one mile. This enables the output to be studied for a set of random input displacements, which is the general case of road input displacements. The reason for considering periodicity is that the program then converts the given road displacement function information into a Fourier series which is applied term by term as input to the differential equations of motion. A finite number of the resulting outputs are then superimposed to yield the output function. The first five terms of both the input Fourier series and output series are printed, however the output series must be printed in matrix form to accommodate all nine generalized coordinates.

7) Each Bode Plot is scanned to find points of relative maximum magnification factors. The circular frequencies of these relative maxima are then compared with the frequencies of the first fifty terms of the input Fourier series. Any Fourier frequencies which lie within five radians per second of a frequency of a relative maximum magnification factor are printed out along with the circular frequency of the relative maximum point of the Bode Plot. An analyst may then decide whether or not a Fourier series input term, which was outside the range of the five term Fourier series included in the

solution, may produce an importatant contribution to the output and should thus be included in the input. This was found to be an important feature of the program since, due to the application of the forcing function at three points of the vehicle, (displaced by appropriate time lags to allow the vehicle to travel over a specific bump on the road before encountering the rear axle of the tractor and the trailer axle) force restrictions are imposed which, at an eigenvalue frequency, do not allow the mode shape predicted by the eigenvalue to be developed. This leaves the relative maxima of the Bode Plots unknown, until these plots are created, and shows that a simple comparison of the eigenvalue frequencies with the Fourier frequencies is invalid for this problem.

The program was written with a high degree of generality in mind and with minor modifications may find application in the following areas.

Highway engineers may evaluate the loads imparted to the pavement quantitatively for any given truck, speed, and road simply by knowing the tire spring constants, damping constants, and the tire deflection and deflection velocity at any given instant. Thus a supplementary program could be created either for (or by) the highway engineer which could keep a running account of these pavement loads imparted at each axle as a function of time. These data could be graphed, much as the Bode Plots, or printed out in tabular form.

The truck designer could use the program to arrive at a compromise of suspension and loading parameters which satisfies the state and federal regulations, gives an acceptable cargo ride, provides an acceptable level of suspension wear, and (finally) considers driver comfort. Since this program is an analysis rather than a synthesis it must be immediately admitted that this compromise would not represent a maximum based upon any maximization criterion, but would be a "best" set of parameters out of all the parameter sets tried.

Transfer and cargo hauling companies might employ the program to perform a statistical analysis of many different "typical" road contours and an analysis of a statistical contour. The output information could be used to evaluate a mean location for the instant center of rotation of the trailer. The fragile transfer items could then be placed at least nearest this spot whenever a loading choice occurs.

An automobile pulling a single axle trailer is also described by the mathematical model derived in Chapter II, thus small trailer manufacturers (i.e., boat trailers, utility trailers) might utilize this program to reduce vibrational difficulties sometimes encountered in pulling these trailers.

Tire manufacturers could evaluate the deflection amplitudes and velocities in order to predict tire life and (if unable to provide a better tire design) might at least be able to recommend a tire type for a truck whose vibrational characteristics are known. The same comment applies to designers of the actual structural suspension members.

The foregoing list of applications is in no way intended to be exhaustive and might be easily extended by any member of the engineering personnel of the automotive industry.

Future Goals

Very little nonlinear work has been accomplished for vehicle vibrations. It is unlikely that any analytically promising method will be developed in the near future for the solution of the immense number of nonlinearities which may arise in vehicle vibrations and in view of these dim prospects it would be desirable to at least be able to gauge the inaccuracies present in a linear analysis.

A stepwise integration program needs to be developed which will consider the nonlinearities of the springs and dampers as well as the coulomb damping, the nonlinearities shown to arise in the kinematical relations (II-5a, 5b) developed in Chapter II, the smoothing of the bumps due to the finite tire dimensions, and the piece-wise linear wheel hop. These could then be entered into such a program and an output curve obtained for a given input function.

A Runge-Kutta integration process could be used yielding solutions for various step sizes in order to compare the change in the solution curves with a change in the step size. When a decrease in the step size is found to produce no significant change in the output curves, it may be assumed that a reasonably accurate solution has been obtained. This solution may then be compared with the linear solution to evaluate the error due to linearization. This may be done by many different methods, two of which are maximum deviation error or mean square error. It would not be economically feasible to use the nonlinear solution for all vibrational investigations of the vehicle because such stepwise integration programs have proven in the past to be very expensive to execute, thus giving incentive to use the more economical, if less accurate, linear solution whenever reasonable fidelity exists between the linear and nonlinear solution for a certain vehicle.

Finally, an experimental verification should be employed. An ideal experiment would be one employing electromagnetic shakers to excite an actual vehicle with a known input displacement function. Verification could also be effected by using a vehicle equipped with accelerometers at the centers of gravity of the tractor and trailer (and at other positions on these components in order to sense rotational motions) and at each axle. The accelerations could then be integrated twice as the vehicle is rolled over a road of known contour. Experimental verification, in any event, may be quite expensive and may be disappointing due to a lack of available funds.

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VITA

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INTERIM REPORT OF A VIBRATIONAL ANALYSIS OF A 3-AXLE, FULLY ARTICULATED, HIGHWAY VEHICLE

by

GERALD RICK POTTS

B.S., Wichita State University, 1966

AN ABSTRACT OF A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mechanical Engineering

KANSAS STATE UNIVERSITY
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1969

Name: Gerald Rick Potts   Date of Degree: June, 1969

Institution: Kansas State University

Location: Manhattan, Kansas

Title of Study: INTERIM REPORT OF A VIBRATIONAL ANALYSIS OF A 3-AXLE, FULLY ARTICULATED, HIGHWAY VEHICLE

Pages in Study: 50

Major Field: Mechanical Engineering

Scope and Method of Study: The differential equations of vibratory motion of a 3-axled, fully articulated highway vehicle (tractor -- semi-trailer combination) are derived using the energy method and Lagrange's Equations. A method of obtaining a complete solution to these equations (transient and steady state) is discussed and a computer program which utilizes this method is described.

Findings and Conclusions: It is concluded that, since a truck possesses many nonlinearities (e.g., damping, springing, kinematical relations, and wheelhop), a linear analysis may or may not give acceptable results. It is, therefore, proposed that a non-linear computer simulation be performed to evaluate the error present in the linear analysis. An experimental investigation using a full sized truck equipped with accelerometers and rolling over a known surface is suggested as a basic approach to an overall verification of the theories which would be developed by such an analysis.

MAJOR PROFESSOR'S APPROVAL Hugh S. Walker