Geometry of mean value sets for general divergence form uniformly elliptic operators

by

Ashok Aryal

B.A., Tribhuvan University, Nepal 2002

M.A., Tribhuvan University, Nepal 2004

M.S., Kansas State University, 2015

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AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the

requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas

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# Abstract

In the Fermi Lectures on the obstacle problem in 1998, Caffarelli gave a proof of the mean value theorem which extends to general divergence form uniformly elliptic operators. In the general setting, the result shows that for any such operator $L$ and at any point $x_0$ in the domain, there exists a nested family of sets $\{D_r(x_0)\}$ where the average over any of those sets is related to the value of the function at $x_0$. Although it is known that the $\{D_r(x_0)\}$ are nested and are comparable to balls in the sense that there exists $c, C$ depending only on $L$ such that $B_{cr}(x_0) \subset D_r(x_0) \subset B_{Cr}(x_0)$ for all $r > 0$ and $x_0$ in the domain, otherwise their geometric and topological properties are largely unknown. In this work we begin the study of these topics and we prove a few results about the geometry of these sets and give a couple of applications of the theorems.

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Major Professor
Dr. Ivan Blank

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# Dedication

To my parents, and

To my wife

# Chapter 1

# Introduction

Based on the great importance of the mean value theorem in understanding harmonic functions, it is clear that analogues for operators other than the Laplacian are automatically of interest. In 1963, Littman, Stampacchia, and Weinberger showed that if $\mu$ is any nonnegative measure on $\Omega$, $L$ is any uniformly elliptic divergence form operator, $u$ is the solution to

$$\begin{aligned} Lu = \mu \quad & \text{in} \;\; \Omega \\ u = 0 \quad & \text{on} \;\; \partial\Omega \,, \end{aligned} \tag{1.1}$$

and $G(x, y)$ is the Green's function for $L$ on $\Omega$, then $u(y)$ enjoys the following mean value property: We have

$$u(y) = \lim_{r \to \infty} \frac{1}{2r} \int_{r \leq G \leq 3r} u(x) a^{ij}(x) D_{x_i} G(x, y) D_{x_j} G(x, y) \, dx \tag{1.2}$$

almost everywhere, and this limit is nondecreasing. (See Equation 8.3 in the paper by Littman, Stampacchia, and Weinberger.[1]) (Note that we will use the Einstein summation convention throughout this work, so if the same index is subscripted and superscripted within an equation, then we assume that it is being summed.) On the other hand, this formula is not as nice as the basic mean value formulas for Laplace's equation for a number of reasons. First, it is an average with weights, and not merely a simple average. Indeed, the weights in question are not even easy to estimate. Second, it is not an average over a ball or something which is even homeomorphic to a ball, but rather an average over a union of level sets of the Green's function which do not include the central point being estimated. Finally, it is also clear that the sets in question are not nested.

The following simpler mean value theorem was stated by Caffarelli [2,3] and proved carefully by Blank and Hao. [4]

**Theorem 1.1** (Mean Value Theorem for Divergence Form Elliptic PDE)**.** *Let $L$ be any divergence form elliptic operator with ellipticity $\lambda$, $\Lambda$. For any $x_0 \in \Omega$, there exists an increasing family of open sets $D_r(x_0)$ which satisfies the following:*

1. $B_{cr}(x_0) \subset D_r(x_0) \subset B_{Cr}(x_0)$, *with $c$, $C$ depending only on $n$, $\lambda$ and $\Lambda$.*

2. *For any $v$ satisfying $Lv \geq 0$ and $r < s$, we have*

$$v(x_0) \leq \frac{1}{|D_r(x_0)|} \int_{D_r(x_0)} v(y)\, dy \leq \frac{1}{|D_s(x_0)|} \int_{D_s(x_0)} v(y)\, dy. \tag{1.3}$$

*Finally, the sets $D_r(x_0)$ are noncontact sets of the following obstacle problem:*
*$u \leq G(\cdot, x_0)$ such that*

$$\begin{aligned} L(u) &= -\chi_{\{u<G\}} r^{-n} \quad && \text{in } \; B_M(x_0) \\ u &= G(\cdot, x_0) && \text{on } \; \partial B_M(x_0) \end{aligned} \tag{1.4}$$

*where $B_M(x_0) \subset \mathbb{R}^n$ and $M > 0$ is sufficiently large.*

Blank and Hao also observed in the course of their work that $\partial D_r(x_0)$ was always a set with dimension strictly less than $n$. (See Remark 3.11 of their paper. [4]) Although the theorem above has already been shown to be useful (see for example the paper by Caffarelli and Roquejoffre [5] as one place where it has already been applied in this form), it is clear that the more that is known about the $D_r(x_0)$ the more useful the theorem is. It is also clear that although the fact that $B_{cr}(x_0) \subset D_r(x_0) \subset B_{Cr}(x_0)$ for all $r > 0$ gives us some information about these sets, there is still much more that is unknown.

The idea of the theorem can perhaps be understood better with some pictures. In Figure 1.1 we see three Euclidean balls $(B_{r_1}(x_1), B_{r_2}(x_2), B_{r_3}(x_3))$, and observe that if $\Delta u = 0$ then it follows that

$$u(x_j) = \frac{1}{|B_{r_j}(x_j)|} \int_{B_{r_j}(x_j)} u(y)\, dy$$

for each $j$, and of course, this works for any harmonic function. Of course, this statement is simply a consequence of the standard MVT for harmonic functions. Similarly, the generalized MVT guarantees that if we are given a fixed operator $L$, then at any point in the domain (and in particular at $x_1, x_2$, and $x_3$ in the figure) and for any $r > 0$ there is a mean value set which is characterized as the noncontact set of the obstacle problem given in Equation (1.4). Now without being given the matrix valued function $A(x)$, there is certainly no way to produce these sets. On the other hand, if we assume that the sets $D_{r_j}(x_j)$ in the figure are three mean value sets for a given operator $L$, then any $L$-harmonic function $u$ (i.e. any function that obeys $Lu = 0$) will satisfy the mean value property:

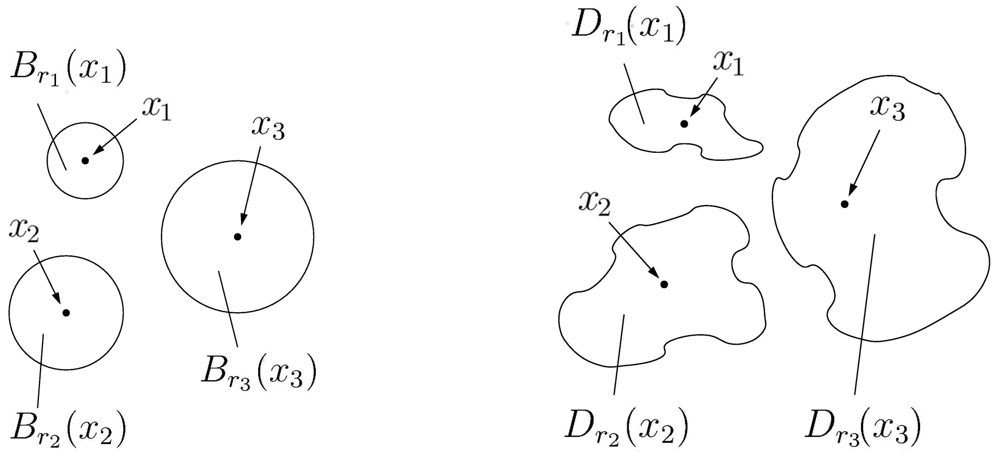
$$u(x_j) = \frac{1}{|D_{r_j}(x_j)|} \int_{D_{r_j}(x_j)} u(y)\, dy$$

for each $j$. Although the $D_{r_j}(x_j)$ that we have drawn are just guesses that might be mean value sets for a given operator, we do already know the following properties:

1. Each $D_r(x)$ contains a ball centered at $x$ with a radius proportional to $r$, and where that proportion depends on nothing other than $\lambda, \Lambda$, and $n$, and similarly each $D_r(x)$ is contained within a ball centered at $x$ with a radius proportional to $r$, and with the same dependencies.

2. If $0 < r < s$, then we always have the inclusion:

$$D_r(x) \subset D_s(x).$$




(a) Euclidean balls (b) MV balls for $L$

Figure 1.1: Mean value balls for $\Delta u = 0$ and for $Lu = 0$

The present work actually originated as an attempt to better understand the solutions of a free boundary problem of Bernoulli type. In the celebrated paper of Alt and Caffarelli in 1981, nonnegative local minimizers of the functional

$$J(u) := \int_D (|\nabla u|^2 + \chi_{\{u>0\}} Q^2) \tag{1.5}$$

are studied.[6] They are shown to exist and satisfy certain Lipschitz regularity estimates, and they obey a linear nondegeneracy statement along their free boundary. From there, Alt and Caffarelli turn to a study of the free boundary. This problem is also found (with $Q \equiv 1$) in the first chapter of the text by Caffarelli and Salsa,[7] and this work originated as an attempt to generalize the work done on that problem. In particular, we were considering the functional

$$J_a(u) := \int_D (a^{ij} D_i u D_j u + \chi_{\{u>0\}}) \tag{1.6}$$

with uniformly elliptic $a^{ij}$, and that will certainly color some aspects of the current work. Unfortunately, after we started our project we learned of very nice and very recent work of dos Prazeres and Teixeira which solved some of the problems[8] that we had intended to publish. Nevertheless, their work had nothing to do with the MVT, and so we can now describe the purposes of the current work: First, we wish to state some theorems related to the geometry of the $D_r(x_0)$. Second, we wish to show two applications in particular which illustrate both the usefulness of the MVT, and the usefulness of our own results which give a more detailed view of properties of the $D_r(x_0)$. Finally, we wish to work toward getting a surface MVT for some generalized divergence form operators by using appropriate test functions constructed by solving the associated Bernoulli problem that we study, but we will see that this program probably cannot be carried out in the same generality as the solid MVT.

The two biggest contributions that we make within this work regarding the properties of the $D_r(x_0)$ appear to be the following:

**Lemma 1.2** (Density Result). *Assume $y_0 \in \partial D_r(x_0)$, and assume that $c$ and $C$ are the constants given in Theorem 2.6. There exists a constant $\tau > 0$ such that for all $h \in (0, 1/2]$, we have*

$$\frac{|B_{chr}(y_0) \cap D_r(x_0)|}{|B_{chr}(y_0)|} \geq \tau \ , \tag{1.7}$$

*and furthermore, $\tau$ is independent of $x_0, y_0$, and $r$.*

This result prevents the $D_r(x_0)$ from having what might be described as an "outward pointing cusp."

**Lemma 1.3** (Continuous Expansion). *Fix $x_0, y_0 \in \Omega$ and assume that there exists $t > 0$ so that $y_0$ is compactly contained within $D_t(x_0)$, and then choose $s \in (0, t)$ so that $y_0 \notin \overline{D_s(x_0)}$ . Then there exists a unique $r \in (s, t)$ such that $y_0 \in \partial D_r(x_0)$.*

This result allows us to state that the boundary of the mean value sets will move in a continuous fashion in the sense that the boundary will never "jump past" any of the points in the domain. (We cannot currently prove any sort of continuity of the boundaries with respect to Hausdorff distance, however.)

We were able to use the mean value theorem above in order to prove positive density of the contact set along the free boundary. Originally, we needed our two lemmas just mentioned in order to prove a nondegeneracy lemma for the Bernoulli problem above. Very recently, Benson, Blank, and LeCrone, have extended many of the results within this work to Riemannian manifolds in the case where $L$ is the Laplace-Beltrami operator.[9] Indeed, all of the results from Chapter 2 can be extended to this case, and when dealing with the obstacle problem on a compact Riemannian manifold $\mathcal{M}$ with boundary, in order to be sure that the $D_r(x_0)$ can be extended until an $r_0$ where $\partial D_{r_0}(x_0)$ collides with $\partial\mathcal{M}$, we need the analogue of Lemma 1.3. (See in particular Corollary 4.9 of the aforementioned paper.[9])

# Chapter 2

# Solid MVT for divergence form elliptic operators

## 2.1 Background and Proof of the MVT for the Laplacian

One of the most fundamental theorems in elliptic PDE is the mean value theorem for Laplace's equation. It can be stated as follows:

**Theorem 2.1** (Mean Value Theorem)**.** *Assume that $u \in C^2(\Omega)$, that $\Delta u \geq (\leq)\ 0$ in $\Omega$, and that $B_r(x_0) \subset \Omega$, then*

$$u(x_0) \leq (\geq)\ \frac{1}{\mathcal{H}^{n-1}(\partial B_r)} \int_{\partial B_r(x_0)} u(y) dS_y =: \fint_{\partial B_r(x_0)} u(y) dS_y \tag{2.1}$$

*and*

$$u(x_0) \leq (\geq)\ \frac{1}{|B_r|} \int_{B_r(x_0)} u(y) dy =: \fint_{B_r(x_0)} u(y) dy \tag{2.2}$$

This theorem is the first thing which is proven in Gilbarg and Trudinger's text,[10] which is viewed by many as the bible of elliptic PDE. When we wish to distinguish between the results which are contained within Equations (2.1) and (2.2) we will refer to the results of Equation (2.1) as a "surface" or "spherical MVT," and we will refer to the results of Equation (2.2) as a "solid MVT" or a "MVT on balls."

The standard proof of the MVT found in most references involves computing the derivative with respect to $r$ of the quantity

$$\int_{\partial B_r(x_0)} u(y) dS_y$$

in order to get the surface formula, and then integrating in $r$ in order to get the formula on the solid ball. This method has two obvious drawbacks: First, it is necessary to assume that $u$ is twice differentiable, and second, the computation is very confusing. In his Fermi lectures,[3] Caffarelli gave a proof of the solid MVT which does not require differentiability and is much more elegant.

In order to state Caffarelli's proof we need a couple of standard definitions.

**Definition 2.2** (Weakly Superharmonic)**.** A function $u \in L^1_{loc}(\Omega)$ is said to be weakly superharmonic if for every $\phi \in C_0^2(\Omega)$ with $\phi \geq 0$ we have:

$$\int_\Omega u(x)\Delta\phi(x)\, dx \leq 0\,.$$

Obviously one can define weakly subharmonic functions similarly.

**Definition 2.3** (Fundamental Solution for the Laplacian)**.** The function defined by:

$$\Gamma(x) = \Gamma(|x|) := \begin{cases} -\left(\dfrac{\ln|x|}{2\pi}\right) & n = 2 \\ \\ \dfrac{1}{n(n-2)\omega_n}|x|^{2-n} & n > 2 \end{cases}$$

is called the fundamental solution for the Laplacian.

The fundamental solution has the following basic properties:

- $\Delta\Gamma(x) = 0$ in $\mathbb{R}^n \setminus \{0\}$, and
- The following formula holds for any harmonic function, $u$, in $\Omega$ :

$$u(x) = \int_{\partial\Omega} \left[ \Gamma(y-x) \frac{\partial}{\partial n} u(y) - u(y) \frac{\partial}{\partial n} \Gamma(y-x) \right] dS_y$$

**Theorem 2.4.** *If $v$ is weakly superharmonic then its average is a decreasing function of $r$. More precisely, if* $0 < s < r$

$$v(x_0) \geq \frac{1}{|B_s|} \int_{B_s(x_0)} v(y)dy \geq \frac{1}{|B_r|} \int_{B_r(x_0)} v(y)dy$$

The key step in the proof that Caffarelli gives is to create a test function with desirable properties and use it in the definition of superharmonic function.

*Proof.* Let $x_0 \in \Omega$. We can assume without loss of generality that $x_0 = 0$.

For $r > 0$ we define $P_r$ to be the polynomial of the form

$$P_r(x) = -\alpha(r)|x|^2 + \beta(r)$$

which is tangent to $\Gamma(x)$ on the sphere $\partial B_r$, and satisfies $P_r(x) \leq \Gamma(x)$ in $\mathbb{R}^n$. Next we define

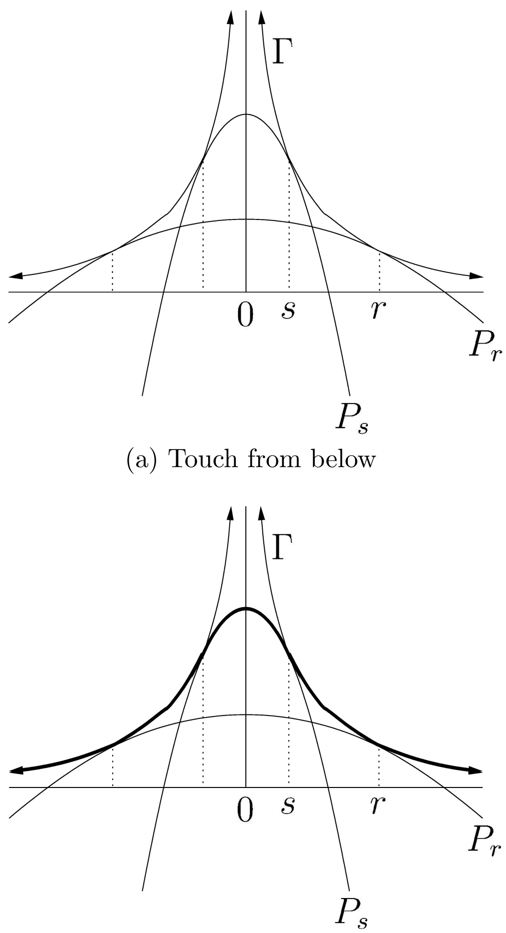
$$\Psi_r(x) := \begin{cases} \Gamma(x) & \text{for } x \in B_r^c \\ \\ P_r(x) & \text{for } x \in B_r \end{cases}$$

and observe that

$$\Delta\Psi_r(x) = \begin{cases} 0 & \text{for } x \in B_r^c \\ \\ -2n\alpha(r) & \text{for } x \in B_r\ . \end{cases}$$

It is also not too hard to show that if $0 \le s \le r$, then $\Psi_s \ge \Psi_r$ in $\mathbb{R}^n$.

A picture of what is going can be seen in the figures below.




(a) Touch from below

(b) The membrane $\Psi_s$

Figure 2.1: Smooth obstacle

Let $\Phi_{s,r} := \Psi_s - \Psi_r$, and observe:

i) $\Phi_{s,r} \ge 0$ in $\mathbb{R}^n$,

ii) $\Phi_{s,r} \equiv 0$ outside $B_r$, and

iii) $\Phi_{s,r} \in C_0^{1,1}(B_r)$

Using the fact that $u$ is weakly superharmonic:

$$\begin{aligned}
0 &\geq \int_\Omega u\Delta\Phi_{s,r} \\
&= \int_{B_r} u\Delta\Psi_s - \int_{B_r} u\Delta\Psi_r \\
&= \int_{B_s} u\Delta\Psi_s - \int_{B_r} u\Delta\Psi_r \\
&= -\int_{B_s} 2n\alpha(s)u + \int_{B_r} 2n\alpha(r)u \ .
\end{aligned}$$

Therefore

$$2n\alpha(r)\int_{B_r} u \leq 2n\alpha(s)\int_{B_s} u \ ,$$

or

$$2n\alpha(r)|B_r| \fint_{B_r} u \leq 2n\alpha(s)|B_s| \fint_{B_s} u \ . \tag{2.3}$$

Now by using the fact that $u \equiv 1$ is both weakly superharmonic and weakly subharmonic, we can observe:

$$\begin{aligned}
0 = \int_\Omega 1\Delta\Phi_{s,r} &= \int_{B_r} \Delta\Psi_s - \int_{B_r} \Delta\Psi_r \\
&= \int_{B_s} \Delta\Psi_s - \int_{B_r} \Delta\Psi_r = -\int_{B_s} 2n\alpha(s) + \int_{B_r} 2n\alpha(r) \\
&= 2n\alpha(r)|B_r| - 2n\alpha(s)|B_s|
\end{aligned}$$

which obviously implies $\alpha(r)|B_r| = \alpha(s)|B_s|$.

Plugging the last equality into Equation (2.3) we obtain:

$$\fint_{B_r} u \leq \fint_{B_s} u \; .$$

Hence, letting s go to zero, we get

$$u(x_0) \geq \fint_{B_r(x_0)} u,$$

which gives us what we need.

■

One may think that the test function we just constructed relies heavily on the symmetry properties of the Laplacian. Another way to look at it, however, is that one is producing a function which is supported on a ball whose Laplacian is a negative constant on a smaller concentric ball, and is a positive constant on the rest of its support. With this, the test function can be viewed as a solution to an obstacle-type free boundary problem. One the other hand, Caffarelli noted in the Fermi lectures[3] that for any divergence form elliptic PDE one could reconstruct a workable analogue of the key test function by solving an appropriate obstacle problem. Blank and Hao filled in the details in their paper.[4]

## 2.2 Generalization of MVT

Let $\Omega$ be an open connected set in $\mathbb{R}^n$, and let $A(x) = (a^{ij}(x))$ be a symmetric uniformly elliptic matrix. That is for each $x \in \Omega$ we have a unique matrix $a^{ij}(x)$ satisfying:

$$a^{ij} \equiv a^{ji} \quad \text{(i.e. symmetry)} \tag{2.4}$$

and there exist $0 < \lambda \le \mu < \infty$ such that

$$0 < \lambda|\xi|^2 \le a^{ij}(x)\xi_i\xi_j \le \mu|\xi|^2 \;\text{ for all }\; \xi \in \mathbb{R}^n \setminus \{0\}, \text{ and } x \in \Omega, \tag{2.5}$$

which is called uniform ellipticity in this setting. Although there are certainly very interesting operators which are not uniformly elliptic, we will content ourselves to assume uniform ellipticity throughout this entire work.

**Remark 2.5** (Analyst's Convention)**.** Notice that with our definition we can have $L = \Delta$, but we won't have $L = -\Delta$.

We consider the divergence form operator $L := \text{div}(A(x)\nabla(u))$. For any $f \in L^2(\Omega)$, we will say that $u$ is a subsolution of $Lu = f$ (or more simply $Lu \ge f$), whenever $u \in W^{1,2}(\Omega)$ and for every $\phi \in W_0^{1,2}(\Omega),\ \phi \ge 0$, we have

$$-\int_\Omega a^{ij} D_i u D_j \phi \ge \int_\Omega f\phi\,. \tag{2.6}$$

Of course, supersolutions are defined in the same way, but with the inequality in Equation (2.6) reversed.

We recall here the main MVT that is the focus of our attention (Stated by Caffarelli,[2] and proved in detail by Blank and Hao[4]):

**Theorem 2.6** (MVT for divergence form elliptic PDE)**.** *Let $L$ be a divergence form elliptic operator as described above. For any $x_0 \in \Omega$, there exists an increasing family of open sets $D_r(x_0)$ which satisfies the following:*

1. *There exists $c$ and $C$ depending only on $n, \lambda,$ and $\mu$, such that for all $r > 0$ such that $B_{Cr}(x_0) \subset \Omega$ we have $B_{cr}(x_0) \subset D_r(x_0) \subset B_{Cr}(x_0)$.*

*2. For any $v$ satisfying $Lv \geq 0$ in $\Omega$ and any $0 < r < s$, we have*

$$v(x_0) \leq \frac{1}{|D_r(x_0)|} \int_{D_r(x_0)} v(y)\, dy \leq \frac{1}{|D_s(x_0)|} \int_{D_s(x_0)} v(y)\, dy. \tag{2.7}$$

*Finally, the sets $D_r(x_0)$ are noncontact sets of the following obstacle problem: $u \leq G(\cdot, x_0)$ such that*

$$\begin{aligned} Lu &= -\chi_{\{u<G\}} r^{-n} \quad && \text{in} \;\; B_M(x_0) \\ u &= G(\cdot, x_0) && \text{on} \;\; \partial B_M(x_0) \end{aligned} \tag{2.8}$$

*where $B_M(x_0) \subset \mathbb{R}^n$ and $M > 0$ is sufficiently large.*

**Remark 2.7** (Dependencies)**.** It is shown in the paper by Blank and Hao[4] that for any $r > 0$, the solution of the obstacle problem above becomes independent of the choice of $M$ as long as it is sufficiently large, and we will always assume that that is the case. (It will be identically equal to the Green's function outside of the compact set $D_r(x_0)$.) We will frequently want to stress the dependence of the solution on $r$, and so, accordingly, we will refer to it as "$u_r$." We will also use "$w_r := G - u_r$" when we wish to look at a function which, at least away from $x_0$ satisfies the usual equations obeyed by the height function for an obstacle problem.

**Remark 2.8** (Technicality)**.** Technically, we cannot use the function $G(x, x_0)$ as boundary values in the sense of having a difference in $W_0^{1,2}$ until we suitably remove the singularity at $x_0$, so in their paper[4] Blank and Hao use a function that they call $G_{sm}$ which agrees with $G$ within a neighborhood of the boundary but which has no singularity in order to bypass this difficulty.

The function $u_r$ is also the minimizer of

$$J_r(u, \Omega) := \int_\Omega (a^{ij} D_i u D_j u - 2r^{-n} u) dx \tag{2.9}$$

among functions less than or equal to $G$ with boundary values equal to $G$. Note that the Green's function $G$ of the general divergence form elliptic operator $L$ is the analogue of the classical obstacle and $u_r$ is that of the membrane, and here the obstacle constrains the membrane from above.

Although, as Caffarelli observed, the sets $D_r(x_0)$ are nested and comparable to balls in the sense that:

$$B_{cr}(x_0) \subset D_r(x_0) \subset B_{Cr}(x_0) \ ,$$

we know very little about the topology of the sets. As a first small step in this direction we offer the following lemma:

**Lemma 2.9** (Structure of $D_r(x_0)$)**.** *For any $x_0 \in \Omega$ and for any $r > 0$ such that $B_{Cr}(x_0) \subset \Omega$, the set $D_r(x_0)$ is connected and it contains $x_0$. In particular, the set $D_r(x_0)$ is always a domain.*

*Proof.* Since $x_0 \in B_{cr}(x_0) \subset D_r(x_0)$, it is immediate that $x_0 \in D_r(x_0)$. Although this statement is certainly trivial, we include it because of the observation that the MVT given by Littman, Stampacchia, and Weinberger does not have this property. (See Equation (1.2) above.)

Now for the next part, without loss of generality we can assume $x_0 = 0$. Assume for the sake of obtaining a contradiciton that $D_r(0)$ has a component that we will call $E$ which does not contain 0. Within $E$ we have $LG = 0,\ Lu_0 \leq 0$, and $u_0 < G$. On the other hand, it follows from the paper[4] by Blank and Hao that $E$ is a bounded set, and since $u_0 = G$ on $\partial E$, we contradict the weak maximum principle. ∎

**Lemma 2.10** (Density Result)**.** *Assume $y_0 \in \partial D_r(x_0)$, and assume that $c$ and $C$ are the constants given in Theorem 2.6. There exists a constant $\tau > 0$ such that for all $h \in (0, 1/2]$,* *we have*

$$\frac{|B_{chr}(y_0) \cap D_r(x_0)|}{|B_{chr}(y_0)|} \geq \tau \ , \tag{2.10}$$

*and furthermore,* $\tau$ *is independent of* $x_0, y_0,$ *and* $r$.

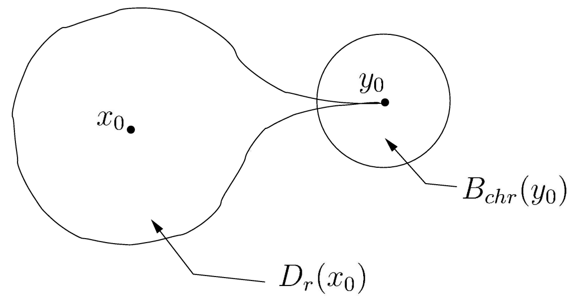



Figure 2.2: Not possible according to the lemma.

*Proof.* Without loss of generality we can rescale so that $r = 1$. Observe that Theorem 2.6 implies that that $x_0$ belongs to the complement of $B_{ch}(y_0)$. Using the characterization of $D_1(x_0)$ as the noncontact set for an obstacle problem (see Equation (2.8)) that there exists a function $u$ satisfying:

$$\begin{aligned} Lu &= -\chi_{\{u<G\}} = -\chi_{D_1(x_0)} && \text{in} \ \ B_M(x_0) \\ u &= G(\cdot, x_0) && \text{on} \ \ \partial B_M(x_0) \end{aligned} \tag{2.11}$$

on a large ball $B_M(x_0)$. Next observe that the height function

$$w(x) := G(x, x_0) - u(x)$$

satisfies

$$Lw = \chi_{\{w>0\}}$$

in any neighborhood not containing $x_0$.

Now we can apply the nondegeneracy theorem for this situation (see Theorem 3.9 of the paper by Blank and Hao[4]) to get the existence of a point $z_0$ at a distance of $ch/2$ to $y_0$ where

$w$ has grown by an amount $\sim h^2 > 0$. Next, by applying optimal regularity (see Theorem 3.2 of paper by Blank and Hao[4]) we can be sure that there is a ball with a radius bounded from below by a constant times $h$ which is centered at $z_0$ which is not in the contact set. ■

**Lemma 2.11** (Convergence of Minimizers)**.** *For any $q > 0$, we let $u_q$ minimize $J_q$ within the set:*

$$K_{M,G} := \{u \in W^{1,2}(B_M) \ : \ u - G \in W_0^{1,2}(B_M), \ \textit{and } u \leq G \ a.e.\} \tag{2.12}$$

*where $J_q$ is as given in Equation (2.9) above. Now fix $r > 0$. Then*

$$u_s \rightharpoonup u_r \ \textit{in } W^{1,2}(B_M) \tag{2.13}$$

*and*

$$\lim_{s \to r} ||u_s - u_r||_{C^\alpha(\overline{B_M})} = 0 \tag{2.14}$$

*for some $\alpha > 0$.*

*Proof.* It is not hard to show that if $s_m$ is a sequence of positive numbers converging to $r$, and if we let $u_m := u_{s_m}$, then the sequence $\{u_m\}$ is uniformly bounded in $W^{1,2}(B_M)$ and uniformly bounded in $C^\alpha(\overline{B_M})$. (See section 4 of the first paper by Blank and Hao[4] for details.) Thus, by using standard functional analysis we can be sure that there is a subsequence of $s_m$ which we will denote by $s_j$ such that we have

$$u_j \rightharpoonup \tilde{u} \text{ in } W^{1,2}(B_M) \ \text{ and } \ \lim_{j \to \infty} ||u_j - \tilde{u}||_{C^\alpha(\overline{B_M})} = 0 \tag{2.15}$$

for some function $\tilde{u} \in W^{1,2}(B_M) \cap C^\alpha(\overline{B_M})$. Since the original sequence $\{s_m\}$ was arbitrary, it remains only to show that $\tilde{u} = u_r$.

First note that for all of the $u_m$ in our sequence, we have:

$$|J_r(u_m) - J_{s_m}(u_m)| \leq \int_{B_M} \left|2(s_m)^{-n} - 2r^{-n}\right| u_m \leq \left|2(s_m)^{-n} - 2r^{-n}\right| \tilde{C} \tag{2.16}$$

where $\tilde{C}$ is the maximum of the $L^1$ norms of the $u_m$. Of course, as we let $m \to \infty$ the right hand side goes to zero. We know

$$\begin{aligned} J_r(u_r) &\leq J_r(\tilde{u}) && \text{because } u_r \text{ minimizes } J_r \\ &\leq \liminf_{j\to\infty} J_r(u_j) && \text{by weak lower semicontinuity} \\ &= \liminf_{j\to\infty} J_{s_j}(u_j) && \text{by using Equation (2.16).} \end{aligned}$$

Now we claim that

$$\liminf_{j\to\infty} J_{s_j}(u_j) \leq J_r(u_r) \tag{2.17}$$

which we can combine with the chain of inequalities in the previous paragraph along with uniqueness of minimizers to show that $\tilde{u} = u_r$. Suppose that this is not the case. Then there exists $s_k \to r$ and an $\epsilon > 0$ such that

$$J_{s_k}(u_k) \geq J_r(u_r) + \epsilon\ . \tag{2.18}$$

On the other hand, for sufficiently large $k$, by using Equation (2.16) again and then Equation (2.18) we have

$$J_{s_k}(u_r) \leq J_r(u_r) + \epsilon/2 \leq J_{s_k}(u_k) - \epsilon/2 < J_{s_k}(u_k)$$

which contradicts the fact that $u_k$ is the minimizer of $J_{s_k}$. ∎

**Remark 2.12** (Statement for the $w_r$)**.** Of course in the language of the height functions

$w_r$, as long as $K$ is compactly contained in the complement of $\{x_0\}$ we have

$$\lim_{r \to s} ||w_r - w_s||_{C^\alpha(K)} = 0 \; . \tag{2.19}$$

**Lemma 2.13** (Continuous Expansion)**.** *Fix $x_0, y_0 \in \Omega$ and assume that there exists $t > 0$ so that $y_0$ is compactly contained within $D_t(x_0)$, and then choose $s \in (0,t)$ so that $y_0 \notin \overline{D_s(x_0)}$ . Then there exists a unique $r \in (s,t)$ such that $y_0 \in \partial D_r(x_0)$.*

*Proof.* We borrow some of the ideas used in the proof of the counter-example within the paper by Blank and Teka.[11] Define the set of real numbers:

$$S := \{ \; t \in \mathbb{R} \; : \; y_0 \notin D_t(x_0) \; \} \; ,$$

and let $r_0$ be the supremum of $S$. Because the $D_r(x_0)$ are an increasing family of sets with respect to $r$, the set $S$ is an interval. We claim that $y_0 \in \partial D_{r_0}(x_0)$. Assuming that $y_0 \notin \partial D_{r_0}(x_0)$, then there exists a $\rho > 0$ so that

$$\mathrm{dist}(y_0, \partial D_{r_0}(x_0)) = \rho \; . \tag{2.20}$$

At this point there are two possible cases: In the first case $B_\rho(y_0) \subset D_{r_0}(x_0)$, and in the second case $B_\rho(y_0) \subset D_{r_0}(x_0)^c$.

Suppose first that $B_\rho(y_0) \subset D_{r_0}(x_0)$. In this case, we have $\Pi := \overline{B_{\rho/2}(y_0)} \subset D_{r_0}(x_0) = \{w_{r_0} > 0\}$, and so if

$$\gamma := \min_{\Pi} w_{r_0},$$

then $\gamma > 0$. By Lemma 2.11 , there exists a sufficiently small $\delta > 0$ such that $|r - r_0| < \delta$ implies

$$||w_r - w_{r_0}||_{L^\infty(\Pi)} \leq \gamma/2 \; . \tag{2.21}$$

Then the triangle inequality implies $w_r \geq \gamma/2 > 0$ in all of $\Pi \subset D_{r_0}(x_0)$ which contradicts the definition of $r_0$.

Next suppose that $B_\rho(y_0) \subset D_{r_0}(x_0)^c = \{w_{r_0} = 0\}$. Within $B_\rho(y_0)$ the function $w_r$ satisfies the obstacle problem:

$$Lw_r = \chi_{\{w_r > 0\}} r^{-n} \tag{2.22}$$

and therefore $w_r$ enjoys the quadratic nondegeneracy property. (See section 3 of the first paper by Blank and Hao[4]) Because of this nondegeneracy, as long as $r > r_0$, (and by using the definition of $r_0$,) we can guarantee that there is a point within $\Pi := B_{\rho/2}(y_0)$ where $w_r$ is greater than a constant $\gamma > 0$. On the other hand, by Lemma 2.11 again, there exists a sufficiently small $\delta > 0$ such that $|r - r_0| < \delta$ implies

$$||w_r - w_{r_0}||_{L^\infty(\Pi)} \leq \gamma/2 \; . \tag{2.23}$$

Thus

$$0 < \gamma \leq ||w_r||_{L^\infty(\Pi)} = ||w_r - w_{r_0}||_{L^\infty(\Pi)} \leq \gamma/2$$

which gives us a contradiction for this case. Hence we must have $y_0 \in \partial D_{r_0}(x_0)$. ∎

# Chapter 3

# A Bernoulli-type free boundary problem and applications of the generalized solid MVT

We turn now to applications of the mean value results to the following problem: Given $a^{ij}(x)$ as above, and boundary data, $\varphi \geq 0$ given on $\partial B_1$, we consider minimizers of the functional:

$$J_a(u) := \int_{B_1} (a^{ij} D_i u D_j u + \chi_{\{u>0\}}) \tag{3.1}$$

which we gave above in Equation (1.6) for a general domain $D$. Now in the case where $a^{ij} \equiv \delta^{ij}$ the functional $J_a(u)$ simplifies to:

$$J(u) := \int_{B_1} (|\nabla u|^2 + \chi_{\{u>0\}}). \tag{3.2}$$

Alt and Caffarelli considered local minimizers of this functional, and indeed this problem was used as a model problem within the text by Caffarelli and Salsa.[7] We will say that $u_0$

is a local minimizer of $J$, if given any subdomain $D_0$ of $B_1$ the value of

$$J(u_0; D_0) := \int_{D_0} (|\nabla u_0|^2 + \chi_{\{u_0>0\}}),$$

is less than or equal to the value of $J(v; D_0)$ for any $v$ which is equal to $u_0$ on $\partial D_0$.

The functional in Equation (3.1) appears in a variety of mathematical models. For example Bernoulli problems, jet flows, and cavity problems can all be formulated as a search for variational minimizers of that functional. In order to study these minimizers we follow the ideas adopted in the text by Caffarelli and Salsa.[7] As in their work we find that it is convenient to produce a minimizer of

$$J_A(u) = \int_\Omega \Big((A(x)\nabla u) \cdot \nabla u + \chi_{\{u_0>0\}}\Big)\, dx, \tag{3.3}$$

within $\mathscr{K}_\psi, := \{u \in W^{1,2}(\Omega) \,:\, u - \psi \in W_0^{1,2}(\Omega)\}$ , by creating it as a limit of minimizers of the following approximating problems. We define:

$$J_{A,\epsilon}(u) := \int_\Omega \left((A(x)\nabla u) \cdot \nabla u + \Phi_\epsilon(u)\right)\, dx, \tag{3.4}$$

and minimize this variational integral within $\mathscr{K}_\psi, := \{u \in W^{1,2}(\Omega) \,:\, u - \psi \in W_0^{1,2}(\Omega)\}$ , where $\Phi_\epsilon$ is smooth monotone function satisfying:

1. $\Phi_\epsilon(s) \equiv 0$ if $s \leq 0$,
2. $\Phi_\epsilon(s) \equiv 1$ if $s \geq \epsilon$, and
3. $0 \leq \Phi_\epsilon'(s) \leq 2/\epsilon$ for all $s$.
4. $\Phi_\epsilon(s) > 0$ for all $s > 0$.

Since $\mathscr{K}_\psi$ is a convex set and since $(A(x)\nabla u)\cdot\nabla u$ is convex in $\nabla u$, the standard existence theorem in the calculus of variations applies. Thus we are guaranteed that there exists a minimizer of $J_\epsilon$ within $\mathscr{K}_\psi$. It will occasionally be convenient to compute energies on subsets of the domain, so if $D \subset \Omega$ then we define:

$$J_{A,\epsilon,D}(u) = \int_D \left( (A(x)\nabla u) \cdot \nabla u + \Phi_\epsilon(u) \right) \, dx \,. \tag{3.5}$$

We would like to highlight some notable properties of the variational integral (3.5) and its minimizers. Since we are studying an elliptic problem, it makes sense to discuss the maximum principle and uniqueness. On the other hand, unlike most elliptic problems, these topics are not quite as straightforward as usual. Indeed, in the interior problem studied by Caffarelli and Salsa,[7] both of these properties fail in certain key ways. Considering the minimizer of

$$\tilde{J}_{0,B_1} := \int_{B_1} \left( |\nabla u|^2 + \chi_{\{u>0\}} \right) \, dx$$

among functions with constant boundary data, it is clear that for a large enough constant the minimizer is simply the constant function. If the constant is small enough, however, then a smaller value of $\tilde{J}_{0,B_1}$ can be achieved with a function that vanishes on a ball which is centered at the origin and is equal to a suitable shifting and scaling of the fundamental solution between that ball and $\partial B_1$. By raising the boundary constant to the right value, one can engineer a "tie" between these two functions thereby contradicting uniqueness.

Based on the discussion above, one might expect nonuniqueness in the exterior problem as well, until one considers that the constant function automatically has infinite energy on our exterior domain. Indeed, we conjecture that the Dirichlet problem for the exterior domain has a unique solution. We will establish some partial results toward proving that conjecture here, and we will establish a weak version of the weak comparison principle, but first, since we will struggle with uniqueness issues, we introduce the following notation and make the following definition:

**Definition 3.1** (Solution Set)**.** Given a functional $\tilde{J}$ that is minimized in the class $\mathscr{K}_\psi$, we define the solution set to be all of the functions $u \in \mathscr{K}_\psi$ which are absolute minimizers of $\tilde{J}$ within $\mathscr{K}_\psi$, and we denote this set by "$\mathcal{A}_{\tilde{J},\psi}$."

With our new notation, the following lemma says that the set $\mathcal{A}_{\tilde{J},\psi}$ is closed under taking a maximum or a minimum.

**Lemma 3.2** (Ordering Lemma)**.** *We assume that $\Omega \subset \mathbb{R}^n$ is open with Lipschitz boundary, that $\psi \in W^{1,2}(\Omega)$, that $A(x)$ satisfies Equations (2.4) and (2.5), and that*

$$\tilde{J}(u) := \int_\Omega \left( (A(x)\nabla u) \cdot \nabla u + G(u) \right) \, dx \tag{3.6}$$

*where $G(u)$ is either equal to $\chi_{\{u>0\}}$ or $\Phi_\epsilon(u)$.*

*Then, if $u_j \in \mathcal{A}_{\tilde{J},\psi}$ then $v_1 := \min\{u_1, u_2\}$, and $v_2 := \max\{u_1, u_2\}$ also belong to $\mathcal{A}_{\tilde{J},\psi}$. Furthermore, if $\mathcal{A}_{\tilde{J},\psi}$ is an infinite set, then it is closed under supremums and infimums as well.*

*Proof.* Let $u_1$ and $u_2$ be any two minimizers of $\tilde{J}(u)$ and assume $m := \tilde{J}(u_1) = \tilde{J}(u_2)$. Let $v_1 := \min\{u_1, u_2\}$, and $v_2 := \max\{u_1, u_2\}$. Then $v_1$, $v_2 \in \mathscr{K}_\psi$. We claim that $v_1$, $v_2 \in \mathcal{A}_{\tilde{J},\psi}$. Since $u_1$ and $u_2$ are minimizers we have, $\tilde{J}(u_1) = \tilde{J}(u_2) \leq \tilde{J}(v_1)$ and $\tilde{J}(u_1) = \tilde{J}(u_2) \leq \tilde{J}(v_2)$. Let $D_1 := \{u_1 > u_2\}$ and $D_2 := \{u_2 \geq u_1\}$, and let $\tilde{J}(u, D)$ be defined to be the functional obtained by restricting the integration in the definition of $\tilde{J}$ to the set $D$. Then

$$\begin{aligned}
\tilde{J}(u_1, D_1) + \tilde{J}(u_1, D_2) &= \tilde{J}(u_1) \\
&\leq \tilde{J}(v_1) \\
&= \tilde{J}(v_1, D_1) + \tilde{J}(v_1, D_2) \\
&= \tilde{J}(u_2, D_1) + \tilde{J}(u_1, D_2)
\end{aligned}$$

which gives $\tilde{J}(u_1, D_1) \leq \tilde{J}(u_2, D_1)$. By a similar computation, we can show that $\tilde{J}(u_2, D_1) \leq$

$\tilde{J}(u_1, D_1)$ which implies

$$\tilde{J}(u_1, D_1) = \tilde{J}(u_2, D_1)\ . \tag{3.7}$$

Now by using Equation (3.7) we have,

$$m = \tilde{J}(u_1) = \tilde{J}(u_1, D_1) + \tilde{J}(u_1, D_2) = \tilde{J}(u_2, D_1) + \tilde{J}(u_1, D_2) = \tilde{J}(v_1)$$

and

$$m = \tilde{J}(u_2) = \tilde{J}(u_2, D_1) + \tilde{J}(u_2, D_2) = \tilde{J}(u_1, D_1) + \tilde{J}(u_2, D_2) = \tilde{J}(v_2)$$

which proves our claims for the finite case.

Before we prove our claims for the infinite case, we introduce the following notation:

$$\begin{aligned} \mathrm{T}(\mathcal{A}_{\tilde{J},\psi}) &:= \sup\{\mathcal{A}_{\tilde{J},\psi}\} \\ \mathrm{B}(\mathcal{A}_{\tilde{J},\psi}) &:= \inf\{\mathcal{A}_{\tilde{J},\psi}\}\ . \end{aligned} \tag{3.8}$$

With this notation we need to show that $\mathrm{T}(\mathcal{A}_{\tilde{J},\psi}) \in \mathcal{A}_{\tilde{J},\psi}$, and $\mathrm{B}(\mathcal{A}_{\tilde{J},\psi}) \in \mathcal{A}_{\tilde{J},\psi}$. We will show the result for the supremum, the infimum case will follow similarly.

We will prove this result in two steps. We start by letting $\{x_k\}$ be a countable dense subset of $\Omega$, and in the first step we will show that there exists an element $U_{x_k}$ of $\mathcal{A}_{\tilde{J},\psi}$ such that

$$U_{x_k}(x_k) = \mathrm{T}(\mathcal{A}_{\tilde{J},\psi})(x_k)\ . \tag{3.9}$$

We recall

$$\mathcal{A}_{\tilde{J},\psi} = \{u \in \mathscr{K}_\psi : \tilde{J}(u) \ \text{ is an absolute minimum }\}.$$

Let $u_n$ be a sequence of elements in $\mathcal{A}_{\tilde{J},\psi}$ such that

$$\lim_{n\to\infty} u_n(x_k) = \mathrm{T}(\mathcal{A}_{\tilde{J},\psi})(x_k).$$

By taking a subsequence and using Arzela-Ascoli, we can get uniform convergence of the $u_n$ to a function $U_k$. We can also use the standard weak compactness theorem for $W^{1,2}$ to be sure that that sequence converges weakly there. Since $\mathscr{K}_\psi$ is a closed under uniform limits, we have $U_k \in \mathscr{K}_\psi$. Then lower semicontinuity of our functional with respect to weak convergence in $W^{1,2}$ guarantees that $U_k \in \mathcal{A}_{\tilde{J},\psi}$.

Now for the second step we define

$$\begin{aligned} \mathrm{T}_1(x) &:= U_1(x) && \text{for all } x \in \Omega \\ \mathrm{T}_{j+1}(x) &:= \max\{U_{j+1}(x),\ \mathrm{T}_j(x)\} && \text{for } j \geq 1 \text{ and for all } x \in \Omega\ . \end{aligned} \tag{3.10}$$

It follows from the finite case that all $T_j \in \mathcal{A}_{\tilde{J},\psi}$. Now we simply take the limit of the $T_j$ and exactly as in the first step we have a function that belongs to $\mathcal{A}_{\tilde{J},\psi}$, but by its construction we know that it agrees with $\mathrm{T}(\mathcal{A}_{\tilde{J},\psi})$ on our countable dense subset of $\Omega$. By uniform continuity of all of the functions in question we know that the limit must equal $\mathrm{T}(\mathcal{A}_{\tilde{J},\psi})$ everywhere in $\Omega$. ■

We have the following weak version of the weak comparison principle:

**Proposition 3.3** (Partial Weak Comparison Principle). *Assume that $\Omega$, $A(x)$, and $\tilde{J}$ have the same definitions and assumptions as in Lemma 3.2. Furthermore, assume that $\psi_\ell \in W^{1,2}(\Omega)$ with $\psi_1 \leq \psi_2$ almost everywhere.*

*Then if $u_\ell \in \mathcal{A}_{\tilde{J},\psi_\ell}$ then we have the following inequalities almost everywhere:*

$$u_1 \leq T(\mathcal{A}_{\tilde{J},\psi_2}) \quad \text{and} \quad u_2 \geq B(\mathcal{A}_{\tilde{J},\psi_1})\ . \tag{3.11}$$

*Proof.* The idea here is identical to the idea for the proof of Lemma 3.2, so we omit it. ■

**Lemma 3.4.** *For any $D \subset B_\delta^c$ the variational integral*

$$J_{\epsilon,D}(u) = \int_D (A(x)\nabla u)\cdot\nabla u + \Phi_\epsilon(u))\,dx$$

*increases as $\epsilon$ decreases for any fixed $u$.*

*Proof.* Note that $\{\Phi_\epsilon(u)\}$ is a monotone decreasing family of functions in $\epsilon$ so, if $\epsilon_1 \leq \epsilon_2$, then we have

$$\begin{aligned} J_{\epsilon_2,D}(u) &= \int_D (A(x)\nabla u)\cdot\nabla u)dx + \int_D \Phi_{\epsilon_2}(u)\,dx \\ &\leq \int_D (A(x)\nabla u)\cdot\nabla u)dx + \int_D \Phi_{\epsilon_1}(u)\,dx \\ &=: J_{\epsilon_1,D}(u)\ . \end{aligned}$$

■

**Proposition 3.5** (Euler-Lagrange Equation)**.** *A minimizer $u$ of $J_\epsilon(u)$ satisfies the Euler Equation*

$$2Lu = f_\epsilon(u) =: \Phi_\epsilon'(u) \tag{3.12}$$

*in $\Omega$.*

*Proof.* Let $\eta \in C_c^\infty(\Omega)$ and let $t \geq 0$. Set

$$\begin{aligned} g(t) &:= J_\epsilon(u+t\eta) \\ &= \int_\Omega \left[(A(x)\nabla(u+t\eta))\cdot\nabla(u+t\eta) + \Phi_\epsilon(u+t\eta)\right]\,dx \\ &= \int_\Omega \left[(A(x)(\nabla u + t\nabla\eta))\cdot(\nabla u + t\nabla\eta) + \Phi_\epsilon(u+t\eta)\right]\,dx \\ &= \int_\Omega \Big[(A(x)\nabla u)\cdot\nabla u + 2t(A(x)\nabla u)\cdot\nabla\eta + t^2((A(x)\nabla\eta)\cdot\nabla\eta) + \Phi_\epsilon(u+t\eta)\Big]\,dx\ . \end{aligned}$$

Then

$$g'(t) = \int_\Omega [2(A(x)\nabla u) \cdot \nabla\eta + 2t((A(x)\nabla\eta) \cdot \nabla\eta) + \Phi_\epsilon'(u + t\eta)\eta]\ dx\,,$$

and so

$$0 = g'(0) = \int_\Omega [2(A(x)\nabla u) \cdot \nabla\eta + \Phi_\epsilon'(u)\eta]\ dx \tag{3.13}$$

which is

$$-2\text{div}(A(x)\nabla u) + \Phi_\epsilon'(u) = 0$$

by definition. Thus, $2Lu = f_\epsilon(u)$. ∎

Some highlights of what is known about functions $u_0$ which locally minimize $J(u)$ in $B_1$ include the following:

**Theorem 3.6** (Basic Facts for Minimizers of $J$)**.** *Within any $D_0 \subset\subset B_1$ we have:*

1. *$u_0$ is Lipschitz.*

2. *If $z_0 \in D_0 \cap \partial\{u_0 > 0\}$, then there is a constant $C > 0$ depending only on $n$ and $||u_0||_{L^2(B_1)}$ such that*

$$C^{-1}r \leq \sup_{B_r(z_0)} u_0 \leq Cr\,. \tag{3.14}$$

3. *With $z_0 \in D_0 \cap \partial\{u_0 > 0\}$ again, there is a universal $\theta > 0$ such that*

$$\mathcal{L}^n(\{u_0 > 0\} \cap B_r(z_0)) \geq \theta r^n \quad \textit{and} \quad \mathcal{L}^n(\{u_0 = 0\} \cap B_r(z_0)) \geq \theta r^n \tag{3.15}$$

*where we use $\mathcal{L}^n(S)$ to denote the $n$-dimensional Lebesgue measure of $S$.*

4. *Using $\mathcal{H}^\gamma(S)$ to denote the $\gamma$-dimensional Hausdorff measure of $S$, then given $D_0 \subset\subset B_1$ there is a universal $C$ such that*

$$\mathcal{H}^{n-1}(\partial\{u_0 > 0\} \cap D_0) \leq C. \tag{3.16}$$

5. $|\nabla u_0| = 1$ *in a suitable sense on almost all of the free boundary.*

Everything in the theorem above was proven by Alt and Caffarelli.[6,7]

More recently, dos Prazeres and Teixeira studied the local minimizers of the more general functional $J_a$ where the $a^{ij}$ which appear are assumed to be no more than bounded, symmetric, and uniformly elliptic.[8] Now in this case, there is no hope of proving that minimizers are better than the Hölder regularity given by the famous result of De Giorgi and Nash. On the other hand dos Prazeres and Teixeira proved that functions $u_0$ which locally minimize $J_a(u)$ in $B_1$ satisfy the following:

**Theorem 3.7** (Basic Facts for Minimizers of $J_a$)**.** *Within any $D_0 \subset\subset B_1$ we have:*

1. *If $z_0 \in D_0 \cap \partial\{u_0 > 0\}$, then there is a constant $C > 0$ depending only on $n, \lambda, \Lambda$, and $||u_0||_{L^2(B_1)}$ such that*

$$C^{-1} r \leq \sup_{B_r(z_0)} u_0 \leq Cr\ . \tag{3.17}$$

2. *With $g \in D_0 \cap \partial\{u_0 > 0\}$ again, there is a universal $\theta > 0$ such that*

$$\mathcal{L}^n(\{u_0 > 0\} \cap B_r(z_0)) \geq \theta r^n. \tag{3.18}$$

We can find these facts in Theorem 1.1 by dos Prazeres and Teixeira.[8]

Also considered by dos Prazeres and Teixeira were $a^{ij}$ satisfying what they called the "$K$-Lip" property which do allow for Lipschitz estimates of the minimizers, but we never make this assumption. (For those details, we can see the definition 3.3 in the paper by dos Prazeres and Teixeira.[8]) Of course, even without any further hypotheses, one can reasonably view Equation (3.17) as saying that "at the free boundary" the solutions enjoy a Lipschitz-type behavior. On the other hand, for general $a^{ij}$ one can construct a counter-example to the statement: "The one sided gradient exists at the free boundary".

Thus, it seems very difficult to get a successful analogue of the fifth statement in Theorem

3.6 above. It also seems difficult or impossible to prove Equation (3.16) in the general case, although as dos Prazeres and Teixeira observed,[8] the free boundary is necessarily porous, and so if one is willing to weaken $\mathcal{H}^{n-1}$ measure to $\mathcal{H}^{n-\zeta}$ measure for a $\zeta$ which is between 0 and 1, then one can assert the analogue. From a certain point of view, the upshot is that the biggest gap between Theorem 3.6 and Theorem 3.7 that we can hope to close is the fact that Equation (3.18) is only giving half of what Equation (3.15) gave, and that leads to our first application.

## 3.1 Application 1: Positive Density of the Contact Set on the Free Boundary

**Theorem 3.8** (Positive Density of the Contact Set on the Free Boundary)**.** *In the same setting as in Theorem 3.7 and with $x_0 \in D_0 \cap \partial\{u_0 > 0\}$ there exists a $\theta > 0$ depending only on $n, \lambda, \Lambda,$ and $||u_0||_{L^2(B_1)}$ such that*

$$\mathcal{L}^n(\{u_0 = 0\} \cap B_r(x_0)) \geq \theta r^n. \tag{3.19}$$

*Proof.* Let $v$ be a solution of the equation $Lu = 0$ in $B_r(x_0)$ with $v = u_0$ on $\partial B_r(x_0)$. Since $x_0$ is in the free boundary we know that $u_0$ and therefore $v$ is positive on a nontrivial portion of $\partial B_r(x_0)$. Then, the strong maximum principle implies $v > 0$ in $B_r(x_0)$. Since $u_0$ is local minimizer we have,

$$\int_{B_r(x_0)} \big((A(x)\nabla u_0) \cdot \nabla u_0 + \chi_{\{u_0>0\}}\big) \leq \int_{B_r(x_0)} \big((A(x)\nabla v) \cdot \nabla v + \chi_{\{v>0\}}\big)$$

which gives,

$$\int_{B_r(x_0)} \Big((A(x)\nabla u_0)\cdot\nabla u_0 - (A(x)\nabla v)\cdot\nabla v\Big) \le \int_{B_r(x_0)} \chi_{\{v>0\}} - \int_{B_r(x_0)} \chi_{\{u_0>0\}}$$
$$= |B_r(x_0)| - |\{u_0>0\}\cap B_r(x_0)|$$
$$= |\{u_0=0\}\cap B_r(x_0)|$$
$$= |\Omega_0^c \cap B_r(x_0)| \ .$$

On the other hand we claim that,

$$\int_{B_r(x_0)} \Big((A(x)\nabla u_0)\cdot\nabla u_0 - (A(x)\nabla v)\cdot\nabla v\Big) = \int_{B_r(x_0)} \Big(A(x)\nabla(u_0-v)\Big)\cdot\nabla(u_0-v)$$
$$\ge \lambda \int_{B_r(x_0)} |\nabla(u_0-v)|^2$$
$$\ge \frac{C\lambda}{r^2}\int_{B_r(x_0)} |(u_0-v)|^2 \ .$$

Thus, if we grant the claim, then we obviously have

$$|\Omega_0^c \cap B_r(x_0)| \ge \frac{C\lambda}{r^2}\int_{B_r(x_0)} |(u_0-v)|^2 \ . \tag{3.20}$$

Turning to the proof of the claim we see immediately that the last two inequalities in the chain of inequalities above simply use uniform ellipticity and the Poincaré inequality respectively. Thus our claim is proved if we show the first equality. So letting $\varphi := u_0 - v$

and observing that $\varphi \in W_0^{1,2}(B_r(x_0))$ we compute

$$\begin{aligned}
\int_{B_r(x_0)} (A(x)\nabla u_0)\cdot\nabla u_0 - (A(x)\nabla v)\cdot\nabla v - (A(x)\nabla(u_0-v))\cdot\nabla(u_0-v)& \\
&= 2\int_{B_r(x_0)} \Big((A(x)\nabla u_0)\cdot\nabla v - (A(x)\nabla v)\cdot\nabla v\Big) \\
&= 2\int_{B_r(x_0)} (A(x)\nabla v)\cdot\nabla(u_0-v) \\
&= 2\int_{B_r(x_0)} (A(x)\nabla v)\cdot\nabla(\varphi) \\
&= 0
\end{aligned}$$

since $Lv = 0$ in $B_r(x_0)$. Thus, the claim is proved.

Now using the MVT for general divergence form operators we get,

$$\begin{aligned}
v(x_0) &= \frac{1}{|D_r(x_0)|}\int_{D_r(x_0)} v \\
&\geq \frac{1}{|B_{Cr}(x_0)|}\int_{B_{cr}(x_0)} v \\
&= \frac{|B_{cr}(x_0)|}{|B_{Cr}(x_0)|}\cdot\frac{1}{|B_{cr}(x_0)|}\int_{B_{cr}(x_0)} v \\
&\geq \tilde{C}\frac{1}{|B_{cr}(x_0)|}\int_{B_{cr}(x_0)} u_0 \\
&\geq \bar{C}r
\end{aligned}$$

where in the final inequality we have used both the nondegeneracy and the optimal regularity of $u_0$ due to dos Prazeres and Teixeira.[8] Since $v$ is L-harmonic and nonnegative, the Harnack inequality tells us that $v(y) \geq \tilde{C}r$ for all $y \in B_{r/2}(x_0)$. By the Lipschitz continuity of $u_0$ we see that $u_0(y) \leq c_1 hr$ in $B_{hr}(x_0)$. By choosing $h$ to be sufficiently small we get

$$v - u_0 \geq \hat{c}r \quad \text{in} \quad B_{hr}(x_0) \; .$$

Therefore by using Equation (3.20) we get,

$$|\Omega_0^c \cap B_r(x_0)| \geq \frac{c\lambda}{r^2} \int_{B_r(x_0)} |(u_0 - v)|^2 \geq \frac{c\lambda}{r^2} \int_{B_{hr}(x_0)} (\hat{c}r)^2 \geq Cr^n \; .$$

■

By combining this last result with part (2) of Theorem 3.7 we get the following statement simply by definition.

**Corollary 3.9** (Measure Theoretic Boundary)**.** *Every point of the free boundary belongs to the measure theoretic boundary of the zero set and/or of the positivity set.*

Definitions and information about the measure theoretic boundary can be found in a variety of references on geometric measure theory including the book by Evans, and Gariepy[12] and the book by Mattila.[13] We can also observe that a consequence of the results above is the fact that the free boundary, $\partial\Omega$, does not admit cusps either pointing inward or outward.

Turning to the object created by applying $L$ to $u_0$ we will observe in the next lemma that $Lu_0$ is a nonnegative distribution which is supported on the Free Boundary $FB(u_0)$. Of course, then the theorem of Laurent Schwartz tells us that $Lu_0$ is actually a nonnegative measure supported on $FB(u_0)$. As usual, we define $Lu_0$ as the distribution $L_0$ in the following way for suitable test functions $\varphi$ :

$$(Lu_0)(\varphi) = L_0(\varphi) := -\int_{B_r(x_0)} ((A(x)\nabla u_0)) \cdot \nabla\varphi \; . \tag{3.21}$$

**Lemma 3.10** ($u_0$ is a Subsolution)**.** *The function $u_0$ is a subsolution of $Lu = 0$, and more precisely $L_0$ is actually a nonnegative distribution, and therefore a nonnegative measure, and it is supported on $FB(u_0)$.*

*Proof.* Let $\phi \geq 0$ be an appropriate test function. It is clear that $L_0$ is linear and bounded and hence a distribution.

Just as we have defined the distribution $L_0$ using $u_0$, we can define the distribution $L_\epsilon$ by replacing $u_0$ with $u_\epsilon$. Of course, in this case we have:

$$L_\epsilon\phi = -\int_{B_r(x_0)} (A(x)\nabla u_\epsilon) \cdot \nabla\phi = \int_{B_r(x_0)} f_\epsilon(u_\epsilon)\,\phi \geq 0 \tag{3.22}$$

whenever $\phi \geq 0$. Thus, $L_\epsilon$ is a nonnegative distribution.

Now, using the fact that $u_\epsilon \rightharpoonup u_0$ weakly in $W^{1,2}$ along with the symmetry of the matrix $A(x)$ we see

$$\begin{aligned}
L_\epsilon\phi &= -\int_{B_r(x_0)} (A(x)\nabla u_\epsilon) \cdot \nabla\phi \\
&= -\int_{B_r(x_0)} \nabla u_\epsilon \cdot (A(x)\nabla\phi) \\
&\to -\int_{B_r(x_0)} \nabla u_0 \cdot (A(x)\nabla\phi) \\
&= -\int_{B_r(x_0)} (A(x)\nabla u_0) \cdot \nabla\phi) \\
&= L_0\phi
\end{aligned}$$

and this implies that $L_0$ is also a nonnegative distribution.

Now we turn to show that $L_0$ is supported on $FB(u_0)$. First we observe that if $\phi$ is supported in the interior of where $u_0 = 0$, then it is clear that $L_0\phi = 0$. On the other hand, if $\phi$ is supported in a set of the form $\{u_0 > \beta\}$ for some $\beta > 0$, then by using the uniform convergence of the $u_\epsilon$ to $u_0$, by observing the fact that $f_\epsilon(t) \equiv 0$ for all $t > \epsilon$, and by repeating the two inset computations from the last paragraph, we see that $L_0\phi = 0$. Combining these two observations shows us that $L_0$ must be supported on the boundary of $\{u_0 > 0\}$. ■

In the text by Caffarelli and Salsa[7] while doing the problem with $A(x) \equiv I$, they show between Theorem 1.19 and the end of the first chapter that the free boundary condition is a weak version of $|\nabla u| \equiv 1$ on the free boundary. Their weak version establishes that $\Delta u = \mathcal{H}^{n-1}\lfloor FB$. In our case we expect to get a weak version of

$$[A(x)\nabla u_0(x)] \cdot \nu(x) = \nabla u(x) \cdot [A(x)\nu(x)] \equiv 1 \tag{3.23}$$

for $x \in FB(u_0)$. The hope would be $\mu = \mathcal{H}^{n-1}\lfloor FB$, where $\mu$ is the free boundary measure given by $Lu_0$ from Lemma 3.10 .

**Theorem 3.11.** *Let $x_0 \in FB(u_0)$ and let $\mu = Lu_0$. Then $\mu$ is a nonnegative measure and suppported on $F(u_0)$ and for any $r > 0$ we have*

$$\int_{\partial B_r(x_0)} [A(x)\nabla u_0(x)] \cdot \nu(x) dH^{n-1} = \int_{B_r(x_0)} d\mu \sim r^{n-1} \, . \tag{3.24}$$

*Proof.* We have already seen in Lemma 3.10 that $\mu = Lu_0$ is a non-negative measure and supported on $FB(u_0)$. The leading equality in Equation (3.24) is simply the statement of this fact. Next, as $x_0$ is the free boundary point, using the Lipschitz continuity of $u_0$ near free boundary along with the uniform bound on $A(x)$ we know that

$$\int_{\partial B_r(x_0)} [A(x)\nabla u_0(x)] \cdot \nu(x) dH^{n-1} \leq Cr^{n-1} \, , \tag{3.25}$$

so it remains to show the inequality in the opposite direction.

Using the rescaling property it is enough to prove the case for $r = 1$. Let $x_0 \in FB(u_0)$, and let $w$ be an $L$-harmonic function in $B_1(x_0)$ such that $w = u_0$ on $\partial B_1(x_0)$. Then $L(w -$

$u_0) = -Lu_0 = -\mu \le 0$, and by the weak comparison principle $w \ge u_0$. Let $G$ be the Green's function for our operator $L$ on $B_1(x_0)$. Then by using Theorem 6.1[1] in paper by Littman, Stampacchia, and Weinberger we have the following representation formula,

$$(w - u_0)(y) = w(y) - u_0(y) = \int_{B_1(x_0)} G(y,x)d\mu(x).$$

Now since $x_0$ is a free boundary point, we can use nondegeneracy and Lipschitz continuity of $u_0$ in order to guarantee the existence of a point $y \in B_h(x_0)$ (with small $h$), such that $u_0 \sim ch$ within $B_{ch}(y)$ and consequently $u_0 > 0$ in the same ball. Thus $Lu_0 = \mu = 0$ in $B_{ch}(y)$ and because $\mu$ is supported on the free boundary and because $G$ is uniformly bounded away from the singularity, we have:

$$\begin{aligned} w(y) - u_0(y) &= \int_{B_1(x_0)\setminus B_{ch}(y)} G(y,x)d\mu(x) \\ &= \int_{B_1(x_0)\cap\{|x-y|>ch\}} G(y,x)d\mu(x) \\ &\le C\int_{B_1(x_0)} d\mu \ . \end{aligned}$$

On the other hand, for $p > 1$ using the fact that $w \ge u_0$ and that $u_0$ is nondegenerate we observe,

$$⨍_{B_1(x_0)} w^p \ \ge \ ⨍_{B_1(x_0)} u_0^p \ \ge \ C \ .$$

Then by using the Harnack Inequality we get

$$w(y) \ge \left[ ⨍_{B_1(x_0)} w^p \right]^{1/p} \ge C \ ,$$

so that for $h$ small enough we get

$$w(y) - u_0(y) \ge C - ch \ge \tilde{C} \ .$$

Therefore, by chaining together the inequalities above, we get

$$\tilde{C} \leq w(y) - u_0(y) \leq C \int_{B_1(x_0)} d\mu ,$$

and this implies

$$\hat{C} \leq \int_{B_1(x_0)} d\mu . \tag{3.26}$$

By using Equations (3.25) and (3.26) we get the desired conclusion. ■

**Remark 3.12** (Corners but not cusps)**.** Although we have shown that $FB(u_0)$ does not allow cusps, we cannot say whether or not it has corners.

## 3.2 Application 2: A Nondegeneracy Lemma

Although the previous application of the MVT gives us a new result, it does not make use of the new properties that we have shown. On the other hand, by making use of our lemmas in the second section, we can give a new proof of many of the results shown independently by dos Prazeres and Teixeira. Indeed, our method of proof follows the exposition of Caffarelli and Salsa's text almost exactly, and so we will state here only the proof of the key lemma that relies on our statements of the $D_r(x_0)$. This lemma is the analogue of Lemma 1.10 by Caffrelli and Salsa.[7]

**Lemma 3.13** (Nondegeneracy Lemma)**.** *Let $\Omega$ be an open set with $0 \in \partial\Omega$ and $w \geq 0$, $||w||_{C^{0,1}(B_2)} = \bar{\mathcal{K}}$, and $Lw = 0$ in $\Omega \cap B_2$. Suppose $x_0 \in \Omega \cap B_1$ and*

*(i) $w(x_0) = \sigma > 0$, and*

*(ii) in the region $\{w \geq \sigma/3\}$, we have $w(x) \sim dist(x, \partial\Omega)$.*

*Then there exist positive constants $\eta, \beta, \gamma,$ and $\sigma_0$ which all depend on $n, \lambda, \mu,$ and $\bar{\mathcal{K}}$, such that as long as $\sigma \leq \sigma_0$, we have*

$$\beta\sigma \geq \sup_{B_{\eta\sigma}(x_0)} w \geq (1+\gamma)\sigma \ . \tag{3.27}$$

*Proof.* Define $\rho > 0$ by

$$\rho := \sup\{r \in \mathbb{R} \ : \ D_r(x_0) \subset \{w > \sigma/3\}\} \ , \tag{3.28}$$

where $D_r(x_0)$ is the solid mean value set given in Theorem 2.6. Using Lemma 2.13 there exists a $y_0 \in \partial D_\rho(x_0)$ with $w(y_0) = \sigma/3$. By assumptions (i) and (ii) we know that $\rho \sim \sigma$. By the Lipschitz continuity of $w$, for a suitable $h > 0$, we have $w(x) \leq 2\sigma/3$ for all $x \in B_{h\rho}(y_0)$. Now by using Lemma 2.10 we know that $w \leq 2\sigma/3$ in a fixed proportion of $D_\rho(x_0)$. By the basic properties of the mean value sets $D_r(x_0)$, we have:

$$\sigma = w(x_0) = \fint_{D_\rho(x_0)} w(y)dy \ , \tag{3.29}$$

but since there is a fixed proportion of $D_\rho(x_0)$ where $w$ is less than $2\sigma/3$ we must have a point in $D_\rho(x_0)$ which exceeds $\sigma$ by some fixed amount. Since $D_\rho(x_0) \subset B_{C\rho}(x_0)$ with $C$ as given in Theorem 2.6, and since as we observed above we have $\rho \sim \sigma$, we get the right hand side of Equation (3.27). The left hand side of Equation (3.27) follows trivially from Lipschitz continuity so we are done. ∎

**Remark 3.14** (Necessity of Prior Lemmas)**.** Note that both Lemma 1.3 and Lemma 1.2 were needed in the proof.

Iterating this lemma in the same fashion that Caffarelli and Salsa iterate their Lemma 1.10 leads to the key nondegeneracy theorem for solutions to this free boundary problem.

# Chapter 4

# Bernoulli-type free boundary problem for a composite of isotropic medias

Let us recall the Bernoulli type free boundary problem we want to solve: Fix $\delta > 0$ and $x_0 \in \mathbb{R}^n$. We have a nonnegative continuous function $u$ defined on $\Omega$ satisfying:

- $Lu = \mathrm{div}(a_{ij}(x)\nabla u) = 0$ in $\{u > 0\}$ and in $\{u = 0\}$
- $u(x) = \psi(x)$ (prescribed) on $\partial\Omega$.
- $(A(x)\nabla u) \cdot \nabla u = \mu$ (prescribed) in $\partial\{u > 0\}$ the free boundary (FB).

As we discussed earlier, the Bernoulli type free boundary problem above can be formulated as the variational problem (3.1) in the case where $\Omega$ is $B_1$. The goals that we have in studying this problem include in particular, producing an analogue of the surface MVT by creating an appropriate test function by solving this problem on an exterior domain. For the Laplacian, we have carried out the proof in the appendix. Having said this, we cannot really currently hope to deal with all of the bounded, measurable, elliptic $a^{ij}$. Indeed, we have good reason to expect the existence of an example of a minimizer with nonunique blowup limits, and in the second section of this chapter, we will start the process of producing one. Such a function would mean that an analogue of the spherical MVT without weights would probably be unattainable.

On the other hand, in order to produce a nonunique blowup limit, it is likely that we will need the uniqueness of minimizers to our problem, at least with certain types of boundary data. In the first section of this chapter we will prove a few results starting us in this direction. Since the $a(x)$ are just bounded and measurable we can not expect the blow up limit of the $A(x)$ to converge to a unique matrix, so accordingly, we cannot get the corresponding minimizers to converge to unique blow up limits. Indeed, if the $a(x)$ converge to one limit along one sequence of radii and a different limit along a different sequence, then the gradients of the respective blowup limits of the solutions will be different. Consequently it is impossible for the gradient on the free boundary to exist if the free boundary passes through such a point, and so trying to get a function $w$, such that

$$Lw = \mu \mathcal{H}^{n-1} \lfloor \partial\{w > 0\}$$

looks difficult or impossible.

To have a chance at producing a unique blow up limit, in the third section, we will restrict our attention to $a^{ij}(x)$ with the specific form:

$$a^{ij}(x) = a(x)I$$

where the function $a(x)$ satisfies:

$$\lim_{r \downarrow 0} a(r(x - x_0)) \text{ converges in } L^\infty$$

for all $x_0$ in our domain without taking any subsequences. Those $a^{ij}$ still include some interesting cases:

1. $a(x) \in C^0$, and
2. if the domain is chopped up into disjoint pieces and $a(x)$ is a positive constant on each piece, then as long as the pieces are not too irregular, then the limit described above will converge.

These situations can easily occur if we are studying a composite of isotropic materials.

## 4.1 Uniqueness results in specific cases

Our first uniqueness result works for the exterior problem with $A(x) \equiv I$ and constant boundary data.

**Lemma 4.1** (Uniqueness Lemma 1)**.** *If $\psi \equiv C$ on a ball containing $B_\delta$, and $A(x) \equiv I$, and $\Omega := B_\delta^c$, then there is a unique minimizer of the functional $J(u)$ defined in Equation (3.2) among functions in $\mathscr{K}_\psi$. Furthermore, the minimizer is radially symmetric.*

*Proof.* By rotational symmetry of the problem, along with application of Lemma 3.2 we easily deduce that $U := \mathrm{T}(\mathcal{A}_{\tilde{J},\psi})$ and $u := \mathrm{B}(\mathcal{A}_{\tilde{J},\psi})$ are rotationally symmetric. Now if they have the same free boundary, then by uniqueness of solutions to Laplace's equation we are already done. Thus, we can assume that the free boundaries of $U$ and $u$ are concentric spheres centered at the origin, with the radius, $R$, of $FB(U)$ strictly larger than the radius, $r$, of $FB(u)$.

Next we observe that the divergence theorem implies:

$$\int_{FB(U)} \frac{\partial U}{\partial \nu} + \int_{\partial B_\delta^c} \frac{\partial U}{\partial \nu} = \int_{\Omega(U)} \Delta U = 0 \,, \tag{4.1}$$

and

$$\int_{FB(u)} \frac{\partial u}{\partial \nu} + \int_{\partial B_\delta^c} \frac{\partial u}{\partial \nu} = \int_{\Omega(u)} \Delta u = 0 \,. \tag{4.2}$$

(Here we prefer to use $\partial B_\delta^c$ instead of simply $\partial B_\delta$ to stress that the *outward* unit normal $\nu$ is pointing *into* the ball.) Now it is known in this case $a_{ij} = I$ (see the paper by Alt and Caffarelli[6] or the text by Caffarelli and Salsa[7]) that $|\nabla U| \equiv 1$ on $FB(U)$ and $|\nabla u| \equiv 1$ on $FB(u)$, so we easily derive

$$\int_{\partial B_\delta^c} \frac{\partial u}{\partial \nu} = n\omega_n r^{n-1} < n\omega_n R^{n-1} = \int_{\partial B_\delta^c} \frac{\partial U}{\partial \nu} \ , \tag{4.3}$$

where $\omega_n$ is as usual the measure of the unit ball in $\mathbb{R}^n$. On the other hand, since $U > u$ within $\Omega(u)$ we can apply the Hopf Lemma to $U$ and to $U - u$ on $\partial B_\delta^c$ in order to give us:

$$0 < \frac{\partial U}{\partial \nu} < \frac{\partial u}{\partial \nu} \tag{4.4}$$

on all of $\partial B_\delta^c$. When we combine Equations (4.3) and (4.4) we easily derive a contradiction. ∎

Although the lemma above gives us our first uniqueness theorem, we are more interested in boundary data close to $x_n^+$ for the sake of producing counter-examples. We turn to this problem now.

**Lemma 4.2.** *For $A(x) \equiv I$ and given boundary data $\psi = x_n^+$, the supremum $T(\mathcal{A}_{J,\psi})$ of the functional in Equation (3.2) is the minimizer $u = x_n^+$.*

*Proof.* Define $\bar{u} := \mathrm{T}(\mathcal{A}_{J,\psi})$ and suppose that $\bar{u} \neq u = x_n^+$. Then we have $\{u > 0\} \subsetneq \{\bar{u} > 0\}$. Note that the free boundary $\partial\{u > 0\} := \mathrm{FB}(u)$ is a disk which is the intersection of $\{x_n = 0\}$ and a ball, while the free boundary $\partial\{\bar{u} > 0\} := \mathrm{FB}(\bar{u})$ is an $n-1$ dimensional surface with the same boundary. Therefore, since hyperplanes are area minimizing,

$$\mathcal{H}^{n-1}(\mathrm{FB}(u)) < \mathcal{H}^{n-1}(\mathrm{FB}(\bar{u})). \tag{4.5}$$

Next using the divergence theorem and the fact that on the free boundary we have

$$\frac{\partial \bar{u}}{\partial \nu} = -1 \;\text{ and }\; \frac{\partial u}{\partial \nu} = -1$$

we get:

$$\int_{FB(\bar{u})} -1 + \int_{\partial B_1} \frac{\partial \bar{u}}{\partial \nu} = \int_{\Omega(\bar{u})} \Delta \bar{u} = 0\,, \tag{4.6}$$

and

$$\int_{FB(u)} -1 + \int_{\partial B_1} \frac{\partial u}{\partial \nu} = \int_{\Omega(u)} \Delta u = 0\,. \tag{4.7}$$

Equations (4.6) and (4.7) imply:

$$\mathcal{H}^{n-1}(\mathrm{FB}(\bar{u})) = \int_{\partial B_1} \frac{\partial \bar{u}}{\partial \nu} \quad \text{and} \quad \mathcal{H}^{n-1}(\mathrm{FB}(u)) = \int_{\partial B_1} \frac{\partial u}{\partial \nu}\,.$$

Therefore from Equation (4.5) we get,

$$\int_{\partial B_1} \frac{\partial \bar{u}}{\partial \nu} > \int_{\partial B_1} \frac{\partial u}{\partial \nu}\,.$$

On the other hand, by our assumptions we have $\bar{u} > u$ within the positivity set of $u$, so since $\bar{u} = u$ on $\partial B_1$, we must have

$$\frac{\partial u}{\partial \nu} > \frac{\partial \bar{u}}{\partial \nu} \;\text{ on }\; \partial B_1$$

which implies

$$\int_{\partial B_1} \frac{\partial u}{\partial \nu} > \int_{\partial B_1} \frac{\partial \bar{u}}{\partial \nu}\,.$$

This contradiction proves our assertion.

∎

In the following Lemma we will show that the minimizer $\bar{u} := \mathrm{T}(\mathcal{A}_{J,\psi})$ of the functional in Equation (3.2) over $\mathscr{K}_\psi$ is unique when $\psi = x_n^+$. In the proof, we will adopt an idea from an unpublished work of Zheng Hao.[14]

**Lemma 4.3** (Uniqueness Lemma 2)**.** *If $\psi \equiv x_n^+$ on $\partial B_1$, and $A(x) \equiv I$, then $x_n^+$ is the unique minimizer of the functional $J(u)$ defined in Equation (3.2) among the functions in $\mathscr{K}_\psi := \{u \in W^{1,2}(\Omega) \,:\, u - \psi \in W_0^{1,2}(\Omega)\}$.*

*Proof.* Define $u := x_n^+$ on $B_1$, and suppose that we have another minimizer $\bar{u} = x_n^+$ on $\partial B_1$ with the same boundary data. To prove our lemma by contradiction, we will assume that $u \not\equiv \bar{u}$. By the previous lemma, we know that $u \leq \bar{u}$ everywhere.

In the following we define,

$$\Omega(w) := \{w > 0\} \ ,$$

and note that by our assumption of nonuniqueness we have

$$\Omega(u) \subsetneq \Omega(\bar{u}) \ .$$

By the strong maximum principle, we know that within $\Omega(u)$, we have $u < \bar{u}$ and $\Delta u = \Delta \bar{u} = 0$. So $\Delta(\bar{u} - u) = 0$ in $\Omega(u)$ and the minimum of $\bar{u} - u$ on $\overline{\Omega(u)}$ occurs and is zero everywhere on $\partial B_1 \cap \{x_n > 0\}$. Thus, by the Hopf lemma $\nabla u(e_n) \neq \nabla \bar{u}(e_n)$.

On the other hand, we can define the following extensions:

$$U := u = x_n^+ \ \text{ in } \ B_{1.1}, \ \text{ and}$$

$$\bar{U} = \begin{cases} \bar{u} & \text{in } \ B_{1.1} \\ u = x_n^+ & \text{in } \ B_{1.1} \setminus B_1 \end{cases}$$

and note the following facts:

1. When considered on all of $B_{1.1}$, we have $J(u) = J(\bar{u})$ and therefore $\bar{u}$ is a minimizer of $J$ on all of $B_{1.1}$. As an immediate consequence, $\bar{u}$ is harmonic everywhere within $\Omega(\bar{u})$.

2. Because $\nabla u(e_n) \neq \nabla \bar{u}(e_n)$ when considered from within $B_1$, and because $u \equiv \bar{u}$ within $B_{1.1} \setminus B_1$, the gradient of $\bar{u}$ cannot exist and be continuous at $e_n$.

By the nondifferentiability of our minimizer $\bar{u}$ at the point $e_n \in \Omega(\bar{u})$ when considered as a function on $B_{1.1}$, we have our contradiction. ∎

Unfortunately, my proof of uniqueness above seems to rely heavily on the assumption that $A(x) \equiv I$. I have the following conjecture:

**Conjecture 4.4.** *If $A(x)$ is sufficiently close to the identity matrix, then there will be a unique minimizer whenever the boundary data $\psi = (x_n - \beta)^+$, and $\beta$ is sufficiently small. Furthermore, the free boundary will depend continuously on $\beta$.*

## 4.2 Counter-examples for arbitrary $a(x)$

As in the paper by Blank and Teka,[11] we define the function $f_k(x)$ by letting $f_k(x) := \gamma_k(|x|)$ where $\gamma_k(r)$ is defined by

$$\gamma_k(r) := \begin{cases} 2 & \text{for } r \geq \omega_k \\ \\ \dfrac{5 + \cos(\pi \log |\log r|)}{2} & \text{for } r < \omega_k \end{cases} \tag{4.8}$$

and $\omega_k := \exp(-\exp(2k+1))$ . (Note $\omega_k \downarrow 0$, as $k \to \infty$.) Now we observe the following properties:

1.

$$2 \leq f_k \leq 3 \text{ in } B_1,$$

2.

$$\text{for any } q < \infty, \quad \lim_{k\to\infty} ||f_k - 2||_{L^q(B_1)} = 0 \ , \quad \text{and}$$

3.

$$\lim_{r\downarrow 0} r\gamma_k'(r) = 0 \ .$$

We refer to the paper by Blank and Teka for the definition of the space of functions of Vanishing Mean Oscillation, and note that within their work, by using a theorem due to Bramanti, they show that the function $f_k$ just given belongs to VMO$(B_1)$.[11] We also follow Blank and Teka by defining $A(x) := f_k(x)I$, and $a^{ij,k}(x) := f_k(x)\delta^{ij}$. Now the main idea in constructing the counter-example is to observe that $||f_k - 2||_{L^1(B_1)}$ is arbitrarily small, and so we expect that there should be a solution to

$$Lw = \mathcal{H}^{n-1}\lfloor\partial\{w > 0\}$$

which is arbitrarily close to the function

$$w(x) := \frac{1}{2}(x_n)^+ \ .$$

The bad news is that because we have altered the operator slightly, if we take $\frac{1}{2}(x_n)^+$ as the boundary data, then it would be a fantastic stroke of luck if the origin was still in the free boundary and not merely very close to it. Thus, it is necessary to prove that we can "wiggle" the boundary data slightly to force the free boundary back onto the origin.

Let us suppose the boundary data is wiggled by an amount $\beta$ and denote $p_\beta := \frac{1}{2}(x_n - \beta)^+$ . We observe that it solves the equation $2Lw = \chi_{\{w>0\}}$.

Now for $-1/10 \leq \beta \leq 1/10$ and $k \in \mathbb{N}$ let $w_{\beta,k}$ be the minimizer of

$$J_a(u) := \int_{B_1} (A_k(x)\nabla u) \cdot \nabla u + \chi_{\{u>0\}})$$

over the set $\mathscr{K}_{p_\beta}, := \{w_{\beta,k} \in W^{1,2}(B_1) \,:\, w_{\beta,k} - p_\beta \in W_0^{1,2}(B_1)\}$ , where $A_k := (a^{ij,k})$. Then the minimizer solves the following PDE:

$$\begin{cases} w \geq 0 \\ -D_i(a^{ij,k}(x)D_j w) = \chi_{\{w>0\}} \text{ in } B_1 \\ w = p_\beta \text{ on } \partial B_1 \end{cases}$$

The issue we have here is that without proving our conjecture in the previous section, we can't say that the minimizer is unique. On the other hand, if we assume that Conjecture 4.4 holds, then we have the existence of a $\beta_0$ very close to zero, such that the minimizer of $J_a$ with boundary data $p_{\beta_0}$ has a free boundary point at the origin.

By rescaling space along the right radii, and therefore considering the rescaled matrix:

$$\tilde{A}_\epsilon(x) := A(\epsilon x)$$

we can make $||\tilde{A}_\epsilon(x) - 2I||_{L^1(B_1)}$ as small as we like. Along this set of radii, our solution must converge to a rotation of

$$w(x) := \frac{1}{2}(x_n)^+ \ .$$

On the other hand, along a different set of radii going to zero we can make $||\tilde{A}_\epsilon(x) - 3I||_{L^1(B_1)}$ as small as we like, and along this rescaling, the blowup limit must converge to:

$$w(x) := \frac{1}{3}(x_n)^+ \ .$$

## 4.3 Results for nicer media

Since arbitrary $a^{ij}$ leads to nonuniqueness of blow up limits, we study the simpler case of minimizers of the functional

$$J(u) = \int_{B_\delta^c} \left( a(x)|\nabla u|^2 + \frac{\mu^2}{a(x)} \chi_{\{u>0\}} \right) dx \tag{4.9}$$

in an appropriate space with appropriate boundary data.

If $w$ is such a function, then it should satisfy:

1. $w$ is Lipschitz on the free boundary $\partial\{w > 0\}$.
2. $w$ grows linearly away from the free boundary.
3. $Lw = 0$ in $\{w > 0\}$ and $Lw = 0$ in $\{w = 0\}$.
4. $|\nabla w| = \frac{\mu}{a(x)}$ on $\partial\{w > 0\}$.
5. $Lw = \mu \mathcal{H}^{n-1} \lfloor \partial\{w > 0\}$ in a neighborhood of the free boundary.

Indeed, the first three items are immediate from the work of dos Prazeres and Teixeira.[8] The fourth and fifth items will certainly hold when $a(x)$ is continuous by doing a blow up argument, but they should hold more generally as well.

We will like to establish a lemma which will give us an ordering on the minimizers with respect to the boundary data and will give an estimate about the free boundary location. Indeed, as should be clear from the proof of the surface MVT found in the appendix, we will want to solve the exterior problem outside of an arbitrarily small ball, where our data on the ball matches the data of the fundamental solution. In particular, since that data goes to infinity as the radius of the ball goes to zero, it is not a trivial task to show that the free boundary does not behave erratically.

**Lemma 4.5.** *Let $u_j$, $j = 1, 2$ be the minimizers of*

$$J(u; B_\delta^c) := \int_{B_\delta^c} a(x)|\nabla u|^2 + \frac{\mu_j^2}{a(x)}\chi_{\{u>0\}} \tag{4.10}$$

*subject to $u = h_j$ on $\partial B_\delta^c$. If $h_2 \geq h_1$ and $\mu_1 > \mu_2$, then $u_2 \geq u_1$.*

*Proof.* Suppose there exists a non empty set $D := \{u_1 > u_2\}$. By the applicatioin of WMP the case $D \subset \{u_2 > 0\}$ cannot happen. So we assume $D$ also contains some parts of zero set of $u_2$.

Let $m := \min\{u_1, u_2\}$, and $M = \max\{u_1, u_2\}$ then $m$ competes with $u_1$ and $M$ competets with $u_2$. Then we get,

$$\int_D a(x)|\nabla u_1|^2 + \frac{\mu_1^2}{a(x)}\chi_{\{u_1>0\}} \leq \int_D a(x)|\nabla u_2|^2 + \frac{\mu_1^2}{a(x)}\chi_{\{u_2>0\}}$$

which gives

$$\int_D \frac{\mu_1^2}{a(x)}(1 - \chi_{\{u_2>0\}}) \leq \int_D a(x)(|\nabla u_2|^2 - |\nabla u_1|^2), \tag{4.11}$$

and

$$\int_D a(x)|\nabla u_2|^2 + \frac{\mu_2^2}{a(x)}\chi_{\{u_2>0\}} \leq \int_D a(x)|\nabla u_1|^2 + \frac{\mu_2^2}{a(x)}\chi_{\{u_1>0\}}$$

which gives,

$$\int_D \frac{\mu_2^2}{a(x)}(1 - \chi_{\{u_2>0\}}) \geq \int_D a(x)(|\nabla u_2|^2 - |\nabla u_1|^2). \tag{4.12}$$

Thus, from Equations (4.11) and (4.12) we get $\mu_2^2 \geq \mu_1^2$ which is a contradiction.

∎

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# Proof of the spherical MVT

In this appendix we show how the same idea that Caffarelli used to prove the solid mean value inequalities can be used to prove the surface mean value inequalities. The exact inequality that we are trying to prove is found in Equation (2.1).

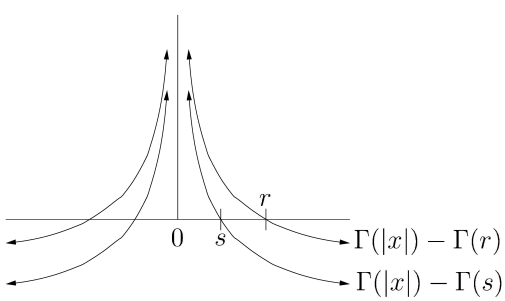
*Proof.* The test function, $\tilde{\Phi}_{r,s}$, that we eventually want will satisfy the following:

- $\tilde{\Phi}_{r,s}$ is a nonnegative radial function.
- $\tilde{\Phi}_{r,s} \equiv 0$ outside $B_r(0)$
- $\Delta\tilde{\Phi}_{r,s} = 0$ in $\mathbb{R}^n \setminus \{\partial B_r \cup \partial B_s\}$
- $\Delta\tilde{\Phi}_{r,s}$ "picks up a distribution" on the spheres $\partial B_r$ and $\partial B_s$.

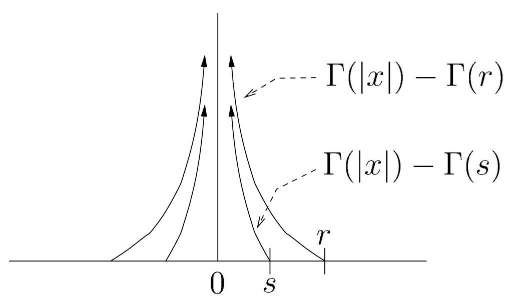
As before we create $\tilde{\Phi}_{r,s}$ as a difference $\tilde{\Psi}_r - \tilde{\Psi}_s$. We define

$$\tilde{\Psi}_s(x) := \max\left\{\Gamma(|x|) - \Gamma(s),\ 0\right\}.$$

A picture of what is going can be seen in Figures A.1 and A.2 below.



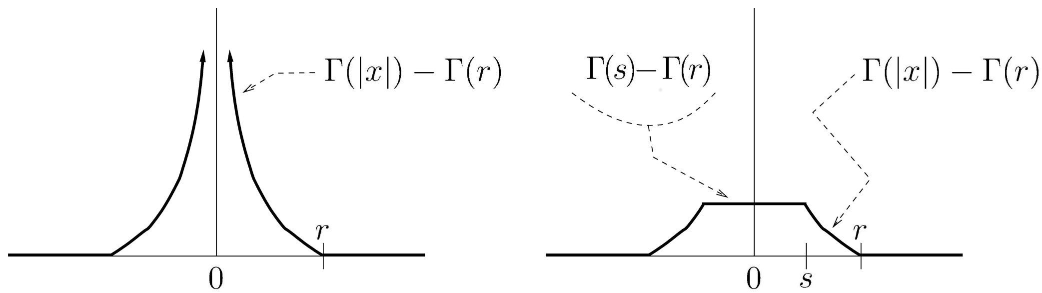

(a) Γ shifted




(b) Γ shifted: The positive part

Figure A.1: Shifting the fundamental solution

The bad news is that $\tilde{\Phi}_{r,s}$ is not in $C^{1,1}$ and so is not an admissible test function. (In fact, it is only Lipschitz.) Therefore, unlike in the solid MVT case, now we need to approximate




(a) $\tilde{\Psi}_r$ (b) $\tilde{\Phi}_{(s,r)} := \tilde{\Psi}_r - \tilde{\Psi}_s$

Figure A.2: Defining our test function $\tilde{\Phi}_{(s,r)}$

$\tilde{\Phi}_{r,s}$ by functions which are $C^{1,1}$ and which converge in a suitable way. To this end, we let

$$\tilde{\Psi}_r^{\epsilon}(x) := \begin{cases} 0 & \text{for} \quad r + \epsilon \le |x| \\ \dfrac{|\Gamma'(r)|}{2\epsilon}\left(|x| - r - \epsilon\right)^2 & \text{for} \quad r \le |x| \le r + \epsilon \\ \Gamma(|x|) - \Gamma(r) + \dfrac{|\Gamma'(r)|\epsilon}{2} & \text{for} \quad |x| \le r \end{cases}$$

Note the following properties of $\tilde{\Psi}_r^{\epsilon}$ :

- $\tilde{\Psi}_r^{\epsilon}$ is radial.
- $\Delta \tilde{\Psi}_r^{\epsilon} \equiv 0$ except at zero and within the annular region: $B_{r+\epsilon} \setminus B_r$.
- $\tilde{\Psi}_r^{\epsilon} \in C^{1,1}$ away from zero.
- $\tilde{\Psi}_r^{\epsilon}$ converges uniformly to $\tilde{\Psi}_r$ as $\epsilon \downarrow 0$.

Finally, observe that if $0 < s < r$, then

$$\tilde{\Phi}_{r,s}^{\epsilon} := \tilde{\Psi}_r^{\epsilon} - \tilde{\Psi}_s^{\epsilon}$$

is an admissible test function which converges uniformly to $\tilde{\Phi}_{r,s}$. It remains to plug our test function into the definition of weakly superharmonic and take a limit as $\epsilon \downarrow 0$.

$$
\begin{aligned}
0 &\geq \int_{\Omega} u\left(\Delta \tilde{\Phi}^{\epsilon}_{r,s}\right) \\
&= \int_{\Omega} u\left(\Delta\left[\tilde{\Psi}^{\epsilon}_{r} - \tilde{\Psi}^{\epsilon}_{s}\right]\right) \\
&= \int_{B_{r+\epsilon}\setminus B_r} u\left(\Delta \tilde{\Psi}^{\epsilon}_{r}\right) - \int_{B_{s+\epsilon}\setminus B_s} u\left(\Delta \tilde{\Psi}^{\epsilon}_{s}\right)
\end{aligned}
$$

So we have:

$$
\int_{B_{r+\epsilon}\setminus B_r} u\left(\Delta \tilde{\Psi}^{\epsilon}_{r}\right) \leq \int_{B_{s+\epsilon}\setminus B_s} u\left(\Delta \tilde{\Psi}^{\epsilon}_{s}\right) \tag{A.1}
$$

Computing the Laplacian in spherical coordinates:

$$
\begin{aligned}
&\int_{B_{r+\epsilon}\setminus B_r} u\left(\Delta \tilde{\Psi}^{\epsilon}_{r}\right) \\
&\quad = \int_{r<|x|<r+\epsilon} u(x)\left(\frac{\partial^2}{\partial |x|^2} + \frac{1}{|x|}\frac{\partial}{\partial |x|} + \frac{1}{|x|^2}\Delta_\theta\right)\tilde{\Psi}^{\epsilon}_{r} \\
&\quad = \int_{r<|x|<r+\epsilon} u(x)\frac{|\Gamma'|}{\epsilon}\left(1 + \frac{1}{|x|}(|x| - r - \epsilon)\right) \\
&\quad = \int_{r<|x|<r+\epsilon} u(x)\frac{|\Gamma'|}{\epsilon}\left(1 + o(\epsilon)\right) \\
&\quad = \omega_n[(r+\epsilon)^n - r^n] ⨍_{r<|x|<r+\epsilon} u(x)\frac{|\Gamma'|}{\epsilon}\left(1 + o(\epsilon)\right) \\
&\quad = \gamma(n)[nr^{n-1}\epsilon + O(\epsilon^2)]\frac{(r^{1-n} + o(\epsilon))}{\epsilon} \cdot \\
&\qquad\qquad ⨍_{r<|x|<r+\epsilon} u(x)\left(1 + o(\epsilon)\right)
\end{aligned}
$$

In the previous computation, the fact that

$$
|B_{r+\epsilon} \setminus B_r| = \epsilon \cdot \mathcal{H}^{n-1}(\partial B_r) + O(\epsilon^2)
$$

is essential and relies on the smoothness of $B_r$. Another essential component of the proof was that $|\Gamma'|$ was constant on $\partial B_r$ and so we were able to pull it outside of the integral while keeping the error small. Having the derivative constant on the free boundary is the main requirement we will have for our free boundary problem; The point is that we need to be able to pull the derivative outside of the integral while still having control of the error.

From the previous computation after some algebraic cancellation we see that:

$$\int_{B_{r+\epsilon}\setminus B_r} u\left(\Delta\tilde{\Psi}_r^{\epsilon}\right)$$
$$= \tilde{C}(n)[1+o(\epsilon)] \fint_{r<|x|<r+\epsilon} u(x)\,(1+o(\epsilon))$$

and if we let $\epsilon \downarrow 0$, then the right hand side converges to

$$\tilde{C}(n) \fint_{\partial B_r} u(x)\ .$$

By taking the limit as $\epsilon \downarrow 0$ in Equation (A.1) and then dividing by the constant $\tilde{C}(n)$, we get:

$$\fint_{\partial B_r} u(x) \leq \fint_{\partial B_s} u(x)\ .$$

Now we can send $s \to 0$ to get the desired result. ■

Note that the key test function is the difference of two solutions to the Bernoulli problem on an exterior domain where the functional being minimized locally can be found in Equation (3.2). Of course if you wanted to produce the test function for a more general elliptic operator, then you would need to replace the functional in Equation (3.2) with the functional found in Equation (3.1).