

NUMERICAL SOLUTIONS OF QUEUEING MODELS

by

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CHAPTER I

INTRODUCTION

Queue is a common phenomenon observed almost in every phase of life such as in a gas station, at a checkout counter, at a tool crib, and in hospitals. In most of the industrial systems, men, machines, or materials form a queue for some type of service. The problem of queue is not so obvious in the industrial systems as is the case with the queue encountered in every day life. For example, in production systems where material or semi-finished products wait for processing at each machine causing many delays such that due-dates cannot be met, decisions have to be made regarding the increase in the number of facilities to provide a certain level of service.

In such situations also, queues are formed due to the uncertainty in demand, service time, or in both. Two extreme cases may occur: (1) providing too much service than desired would involve much idle time of the facilities and hence increase costs; (2) not providing enough facilities, thus overloading the existing facilities would cause long queues most of the time. The latter case would cause loss of good will, hence loss of customers. In an industry, this would cause idle employees and resources. Management may like to find a solution which makes a compromise between the two conflicting situations. Many such decisions are based on the intuition and personal judgment.

Queueing theory helps make such decisions in a scientific way by providing vital information about the behavior of the

queueing system. Queueing theory is a body of knowledge which deals with the development of various mathematical models for analyzing and predicting the behavior of stochastic service systems.

Following Lee [15], there are three principal uses of queueing models: (1) In planning the extent and character of new facilities where queueing is likely to be an important factor; (2) In evolving strategies for manpower planning in existing facilities to meet conditions upon service standards and economics of operation; and (3) To compare alternative forms of organization of servicing facilities.

1.1 Basic Structure of Queueing Models

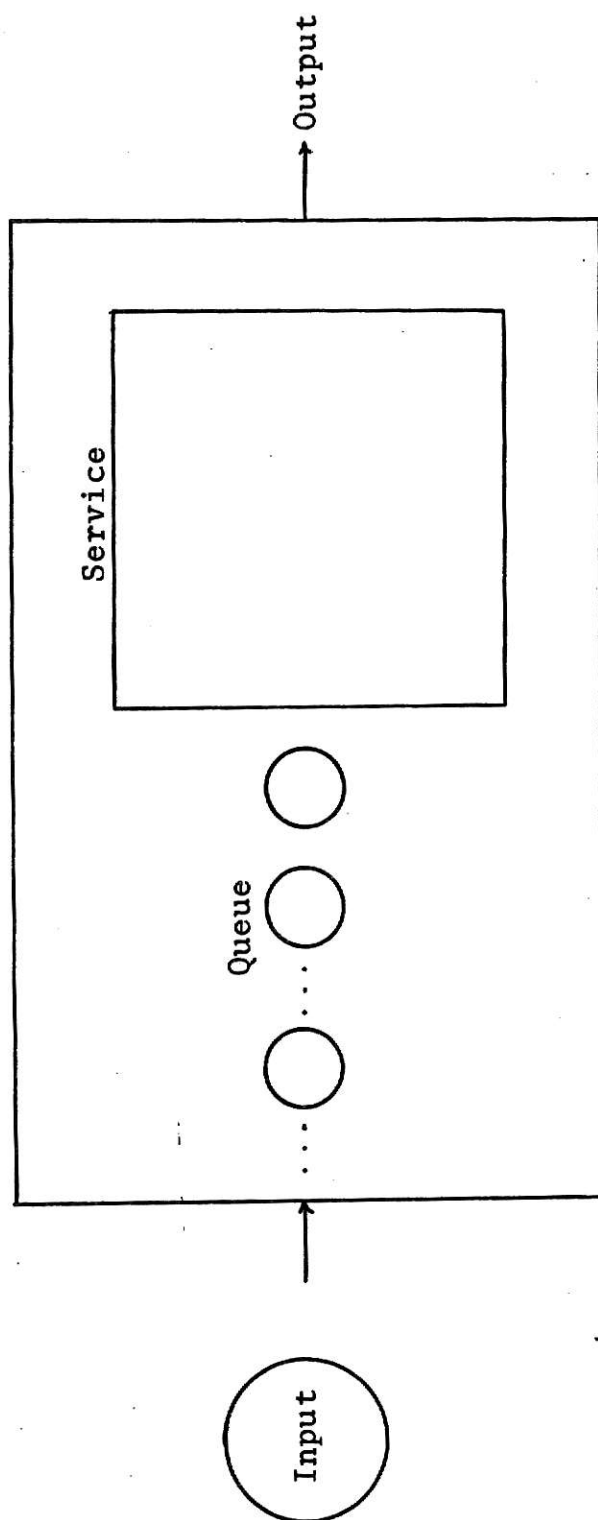
In a queueing system, units arrive for certain types of service, form a queue if the service facility is busy and then depart after the service is completed. In order to analyze such a system, a study of the nature of the input source from which the arrivals are generated, a statistical description of the arrival process, a description of the pattern and behavior of the arriving units, the number and type of servers, the queue discipline, and the service time distribution are required. The basic random variables in such a system are the number of units present in the system at any time t , the inter-arrival time between units, and the waiting time of a unit in the queue or in the system. Thus, a queueing system consists of a set of random variables which depend on time. A queueing system is completely described by specifying: (1) input; (2) queue; (3) queue

discipline; and (4) service mechanism. The basic queueing process is shown in Figure 1.1.

Input. The nature of the source from which arrivals are generated should be specified. If the generation of arrivals from the source is affected by the number of units in the system, then the source is called a finite source; otherwise, it is called infinite. The pattern of arrivals should be specified by whether the units arrive singly or in batches. Arrival behavior should be known, too. There are cases where an arriving unit must wait until the service is completed. Such cases are common for automobiles at toll booths near a turnpike. In some instances, an arrival will not join the queue if, on his arrival, a long queue exists. The arrival may not join the system if the server is busy; that is, balk. In some other instances, the arriving unit may join the queue and after waiting for some time loses patience, and, hence leaves the system; that is, reneges. When the unit remains in the system until service is completed, its behavior is known as patient.

Queue. A queue is formed when all the service stations are busy. The queue size is specified by the maximum number of units that can be allowed to wait. The queue is referred to as limited or unlimited depending on the capacity of the size of the queue.

Queue Discipline. This indicates the order in which the units are selected from the queue for service. Common types of queue disciplines are first-come, first-served; random service;



. Figure 1.1. The Basic Queueing Process.

last-come, first-served; and service according to some other priority rules.

Service Mechanism. Just like arrivals, service pattern should be specified whether the unit is served one at a time or in a batch. The number and type of servers should be specified. Availability of service and the probability distribution of service time should be known.

1.2 Solutions of Queueing Models

There are generally two approaches for the solution of queueing models: (1) analytical approach; and (2) numerical approach.

In the analytical approach, arrival and service time distributions are approximated to one of the well-known probability distributions. Furthermore, the basic elements of the system are defined and then a mathematical model is constructed in terms of the state probabilities which are dependent on time. This model, when solved analytically, provides solutions in the transient state. However, a steady state solution can be obtained by solving the above model independent of time. The time effect in a queueing system is clearly visible at the start of the operation.

When a queueing system has recently started operating, the state of the system, for example, the number of units in the system will be greatly influenced by the initial state and the time that has since elapsed. Due to Hillier [10], the system is

said to be in transient state when such a condition arises. In general, after a sufficient warming up period, the state of the system becomes independent of the initial state. The system is said to be in steady state in such conditions. Steady state will only occur when the mean service rate exceeds the mean arrival rate of units for service, otherwise the system always will remain in the transient state. Generally, steady state solutions are easier to obtain ^{or} than those for transient state.

The study of a transient state solution is quite useful in the following situations: (1) When the effect of rush-hour upon the system is to be determined; (2) When the service mechanism needs occasional checks, rest, or overhaul, and a reserve of servers is not available. In such cases the problem might be one of how many servers may be rested and for how long before the system explodes; (3) When the behavior of the system is to be studied during a short interval of time; and (4) When the effect of sudden changes in system parameters are to be studied [6]. In the above cases, steady state solution is of little use.

Analytical solution for transient state is not available for most of the queueing models. Even if the solutions for simpler models are available, they are in a closed form of Laplace and Z-transforms from which solutions for practical use are difficult to obtain. Hence, numerical approaches should be considered for handling such models. The numerical approach may be typified by two classes: (1) Simulation technique; and (2) Numerical analysis.

When theoretical models are not available, simulation is quite useful. Statistics concerning arrivals and service times are generated from specified distributions. Operating data for any variety of arrival and service time distributions with any queue discipline can be generated.

On the other hand, when the behavior of a queueing process can be represented by a mathematical model, and an analytical solution cannot be obtained, such models can be analyzed numerically providing a good approximation. An advantage of a numerical analysis is that it is relatively easy to estimate the error in the solution. For example, when a numerical integration procedure is used for solving a set of simultaneous linear differential equations for which the initial conditions are known, error due to truncation of higher order terms are easily estimated. The solution obtained by numerical methods are also approximate, but are quite accurate to be applied in most of the practical situations.

1.3 Proposed Research

There have been many applications of queueing theory, such as in telephone traffic problems, scheduling of patients in hospitals, scheduling of personnel, cafeteria design, vehicular traffic, and air line industries. The practical application of queueing theory, however, is far less than the number of theoretical papers published (more than 2000 publications in over a half century). The practical application of the theory has been limited because of the difficulty in obtaining the solutions of

the models. Most of the queueing models have no available analytical solutions. For models having analytical solutions, no transient solutions exist. Transient solutions have been obtained for a number of models in terms of complicated multiple Laplace transforms and generating functions. It is very difficult to find the inverse of these functions, and thus they are difficult to solve explicitly for practical use.

Even when explicit transient solutions for a few simple models are available, they appear in terms of infinite series of rather very complicated functions. For example, the explicit transient solution of a single queueing model which has been developed in [6] is

$$P_n(t) = e^{-(\lambda+\mu)t} \left[\rho^{\frac{n}{2}} I_n(x) + \rho^{\frac{n-1}{2}} I_{n+1}(x) + (1-\rho)\rho^x \sum_{r=2}^{\infty} \rho^{\frac{-(n+r)}{2}} I_{n+2}(x) \right]$$

where

$$x = 2t\sqrt{\lambda\mu}$$

$$\rho = \lambda/\mu$$

and $I_n(x)$ is a Bessel function of order n such that

$$I_n(x) = \frac{x^n}{2^n \Gamma(n+1)} \left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2 \cdot 4(2n+2)(2n+4)} - \dots \right],$$

where

$P_n(t)$ denotes the probability of n units in the system at time t , and $n = 0, 1, 2, \dots$. In applying such a solution, we have to take recourse to a numerical approach. In practice,

numerical solutions are needed to predict the behavior of the system having a certain waiting capacity during a particular time domain.

Because of the lack of success in obtaining explicit solutions to the existing queueing models, which can directly be applied to practical situations in industry, we propose some numerical methods of solution to the various queueing models. This will open avenues for industrial practitioners to apply queueing theory in industry to a great extent. Numerical solutions will provide the state probabilities both in the transient state and in the steady state. From the state probabilities, various measures of effectiveness will be computed. The method used is a numerical integration procedure for solving simultaneous linear differential equations of initial value type.

Several curves for various measures of effectiveness are plotted and their tables also are included. Application of queueing models to job-shop scheduling is discussed also. A job-shop is viewed as a queueing process and the corresponding models are developed. The computation has been done with the use of computer programs in FORTRAN IV executed on the IBM 360/50 computer. The solutions obtained are not of the same accuracy as analytical solutions could have provided, but they are quite useful for most of the practical purposes.

CHAPTER II

NUMERICAL SOLUTIONS OF QUEUEING MODELS

Certain stochastic service systems which possess the Markovian property are attacked analytically. A stochastic system has the Markovian property if its future depends only on the present, regardless of past information. In a Markovian queueing process the distribution of the interarrival time as well as the service time follow the exponential distribution because of its forgetfulness property. This property can be shown as follows:

Let T be a random variable representing the service time which follows the exponential distribution. Then,

$$\begin{aligned} P\{T \leq t\} &= \int_0^t \mu e^{-\mu t} dt \\ &= \mu \left. \frac{e^{-\mu t}}{-\mu} \right|_0^t \\ &= 1 - e^{-\mu t} \end{aligned}$$

or

$$\begin{aligned} P\{T > t\} &= 1 - (1 - e^{-\mu t}) \\ &= e^{-\mu t} \end{aligned}$$

Now, given Δt any non-negative quantity,

$$\begin{aligned} P\{T > t + \Delta t | T > t\} &= \frac{P\{T > t + \Delta t\}}{P\{T > t\}} \\ &= \frac{e^{-\mu(t+\Delta t)}}{e^{-\mu t}} \\ &= e^{-\mu \Delta t} \end{aligned}$$

But,

$$e^{-\mu \Delta t} = P\{T > \Delta t\}$$

Hence

$$P\{T > t + \Delta t | T > t\} = P\{T > \Delta t\}$$

Thus, the probability of service time being greater than $t + \Delta t$ given that it is greater than t is equal to the probability that the service time is greater than Δt . Thus, the past information that the service time is greater than t is completely forgotten. Queueing models of Markovian type involve ordinary difference-differential equations of initial value. The numerical solution is similar to the solution of a system of ordinary linear differential equations of initial value. Differential equations of the initial value type are those in which all the conditions are known at the initial point of the integration interval. There are various methods of solution of ordinary linear differential equations of the initial value type. They can be categorized into two main classes: (1) Single step methods including Euler's, Picard's, Taylor series expansion, and Runge-Kutta's method of various orders; and (2) Multiple step methods including Adams-Bashforth's, Moulton's, Milne's of various orders, and Hamming's. Single step methods are those in which the functional values at a single previous point are required to obtain the functional values ahead. The most well-known and frequently-used method in this class is the fourth order Runge-Kutta method.

In multiple step methods, functional values at more than one previous point are necessary to start the solution. Thus, multiple step methods are not self starting. These methods also are

known as predictor-corrector methods. One of the well-known predictor-corrector methods is Hamming's method. In this thesis, fourth order Runge-Kutta and Hamming's methods are used.

2.1 Fourth Order Runge-Kutta Method

It is a fourth order integration procedure which is stable. The truncation error is of the order of the fifth power of step size. Hence, if the step size is small, the truncation error will be very small. For example, if the step size is 0.1, the truncation error is of the order of 10^{-5} which is quite accurate for most of the queueing problems. Hence, a fourth order integration method is used to solve the difference-differential equations of the queueing models.

Mathematical Development. Let the differential equation be of the following type:

$$\frac{dy}{dt} = f(t, y), \quad t_0 \leq t \leq t_n$$

where t_0 and t_n are the initial and final points of the integration interval, respectively. Let the initial condition be

$$y(t_0) = y_0$$

The sequence of calculations is shown below, step by step.

Step 1. Substitute the initial value of $y(t_0)$, y_0 in the right hand side of the differential equation to obtain $f(t_0, y_0)$. Multiply $f(t_0, y_0)$ by the chosen step size, h , to obtain

$$k_1 = hf(t_0, y_0)$$

Step 2. Calculate k_2 using k_1 and h such that

$$k_2 = hf(t_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$$

Step 3. Calculate k_3 using k_2 and h such that

$$k_3 = hf(t_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$$

Step 4. Calculate k_4 using k_3 and h such that

$$k_4 = hf(t_0 + h, y_0 + k_3)$$

Step 5. Substitute the values of k_1 , k_2 , k_3 , and k_4 computed in Steps 1 to 4 to calculate the value of the first increment in y , Δy such that

$$\Delta y = \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

Step 6. Add the value of y obtained in Step 5 to the previous known value of y to get the next value of y , y_1 such that

$$y_1 = y_0 + \Delta y, \text{ at } t_1 = t_0 + h.$$

Step 7. Repeat Steps 1 to 6 to get the increment in y for the second period in a similar manner, considering the known value of y_1 at t_1 in place of y_0 at t_0 .

In this manner, calculation proceeds from one step to the other in an orderly sequence until the whole integration interval up to t_n is run through.

Thus, in the fourth order Runge-Kutta Method the solution at each step is given by

$$y_{i+1} = y_i + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$k_1 = hf(t_i, y_i)$$

$$k_2 = hf(t_i + h/2, y_i + k_1/2)$$

$$k_3 = hf(t_i + h/2, y_i + k_2/2)$$

$$k_4 = hf(t_i + h, y_i + k_3)$$

Note that h is known as step size. The value of y_i at t_i is already known. Detailed analysis of the method can be found in [23]. Truncation error of the method is of the order of h^5 .

The Runge-Kutta Method can be used to solve a set of simultaneous ordinary differential equations of the initial value type. The method can be demonstrated by considering a pair of simultaneous first-order differential equations of the form

$$\frac{dx}{dt} = f_1(t, x(t), y(t))$$

$$\frac{dy}{dt} = f_2(t, x(t), y(t))$$

with the initial conditions

$$x = x_0, \quad y = y_0, \quad \text{at } t = t_0$$

Using the fourth-order method, the increments in x and y for the first interval are found from the relations

$$\Delta x = \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

and

$$\Delta y = \frac{1}{6} (\ell_1 + 2\ell_2 + 2\ell_3 + \ell_4)$$

where

$$k_1 = hf_1(t_0, x_0, y_0)$$

$$k_2 = hf_1(t_0 + h/2, x_0 + k_1/2, y_0 + \ell_1/2)$$

$$k_3 = hf_1(t_0 + h/2, x_0 + k_2/2, y_0 + \ell_2/2)$$

$$k_4 = hf_1(t_0 + h, x_0 + k_3, y_0 + \ell_3)$$

$$\ell_1 = hf_2(t_0, x_0, y_0)$$

$$\ell_2 = hf_2(t_0 + h/2, x_0 + k_1/2, y_0 + \ell_1/2)$$

$$\ell_3 = hf_2(t_0 + h/2, x_0 + k_2/2, y_0 + \ell_2/2)$$

$$\ell_4 = hf_2(t_0 + h, x_0 + k_3, y_0 + \ell_3)$$

The increments for the succeeding intervals are computed in exactly the same way except that t_0 , x_0 , and y_0 are replaced by t_1 , x_1 , y_1 , etc. In general,

$$x_{i+1} = x_i + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

$$y_{i+1} = y_i + \frac{1}{6} (\ell_1 + 2\ell_2 + 2\ell_3 + \ell_4)$$

The solution is initiated by substituting the known initial values of x and y into the given differential equations to get initial values of the functions f_1 and f_2 . Values of k_1 and ℓ_1 are obtained next by multiplying the initial values of f_1 and f_2 by h as shown above. With values of k_1 and ℓ_1 known, k_2 and ℓ_2 are next calculated; then k_3 and ℓ_3 are calculated, and finally k_4 and ℓ_4 . Then, from the recurrence equation, values of x_{i+1} and y_{i+1} are found at $t = t_i + h$. These new values of x and y are used as the beginning values for the next step to obtain values of x_{i+2} and y_{i+2} at $t = t_i + 2h$ and so on until the desired integration interval has been run through.

The above method, which was demonstrated for two equations, can be generalized to any number of equations by merely using a set of equations for each dependent variable appearing in the system of simultaneous equations. IBM scientific subroutine package (RKGS) has been used for the solution of queueing models.

Since, for the fourth order Runge-Kutta Method, all the terms containing up to h^4 in the Taylor series expansion are retained and all other terms containing h^5 and other higher powers of h have been neglected, the truncation error, e_t , is of the form

$$e_t = kh^5 + O(h^6)$$

where k depends upon $f(t,y)$ and its higher order partial derivatives. If h is sufficiently small so that the error is determined by the first term, then

$$e_t = kh^5$$

Selection of step size depends upon the accuracy that is desired. The step size should be small enough to achieve required accuracy. It should be as large as possible in order to keep rounding errors under control and to avoid an excessive number of derivative evaluations to save computer time. Detailed error analysis can be found in [3].

Sample Problem. For illustration, consider a simple single queueing model having the following specifications:

Arrival Source	: Infinite
Arrival Pattern	: Single

Arrival Behavior : Patient
 Arrival Time Distribution : Exponential
 Queue Size : No Queue Allowed
 Service Discipline : First Come, First Served
 Service Pattern : Single
 Service Mechanism : Single Phase
 Service Time Distribution : Exponential
 Number of Servers : One

The following are the required notations:

$P_0(t)$ probability that there is no unit in the system
at time t

$P_1(t)$ probability that there is one unit in the system
at time t

λ mean arrival rate

μ mean service rate

The state probabilities can be expressed by the difference-differential equations as given in Morse [16]:

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_1(t)$$

and

$$\frac{dP_1(t)}{dt} = \lambda P_0(t) - \mu P_1(t)$$

Use

$$\lambda = \mu = 1$$

for simplifying the computations. Then,

$$\frac{dP_0(t)}{dt} = -P_0(t) + P_1(t) \tag{1}$$

$$\frac{dP_1(t)}{dt} = P_0(t) - P_1(t).$$

Let the initial conditions be

$$P_0(0) = 1$$

and

$$P_1(0) = 0.$$

Fourth Order Runge-Kutta Solution.

Step 1. Multiplying the right hand side of the equations by h , k_1 , and ℓ_1 are determined.

$$k_1 = hf_1(P_0(0), P_1(0))$$

$$= h\{-P_0(0) + P_1(0)\}$$

$$\ell_1 = hf_2(P_0(0), P_1(0))$$

$$= h\{P_0(0) - P_1(0)\}$$

Let the step size h be 0.1, then

$$k_1 = 0.1(-1 + 0) = -0.1$$

$$\ell_1 = 0.1(1 - 0) = 0.1$$

Step 2. Using k_1 and ℓ_1 from Step 1 and step size h , k_2 and ℓ_2 are determined.

$$k_2 = hf_1(P_0(0) + k_1/2, P_1(0) + \ell_1/2)$$

$$= h\{-(1 - 0.1/2) + 0.1/2\} = 0.1(-0.9) = -0.09$$

$$\ell_2 = hf_2(P_0(0) + k_1/2, P_1(0) + \ell_1/2)$$

$$= 0.1(0.95 - 0.1/2) = 0.1 \times 0.90 = 0.09$$

Step 3. Using k_2 and ℓ_2 from Step 2 and step size h , k_3 and ℓ_3 are determined.

$$\begin{aligned}
 k_3 &= hf_1(P_0(0) + k_2/2, P_1(0) + \ell_2/2) \\
 &= 0.1\{-(1 - 0.045) + 0.045\} = 0.1(-1 + 0.09) \\
 &= 0.1(-0.91) = -0.091 \\
 \ell_3 &= hf_2(P_0(0) + k_2/2, P_1(0) + \ell_2/2) \\
 &= 0.1(1 - 0.045 - 0.045) = 0.1(1 - 0.09) = 0.1(0.91) \\
 &= 0.091
 \end{aligned}$$

Step 4. Using k_3 and ℓ_3 from Step 3, k_4 and ℓ_4 are determined.

$$\begin{aligned}
 k_4 &= hf_1(P_0(0) + k_3, P_1(0) + \ell_3) \\
 &= 0.1\{-(1 - 0.091) + 0.091\} = 0.1(-1 + 0.182) \\
 &= 0.1(-0.818) = -0.0818 \\
 \ell_4 &= hf_2(P_0(0) + k_3, P_1(0) + \ell_3) \\
 &= 0.1(1 - 0.091 - 0.091) = 0.1(1 - 0.182) = 0.0818
 \end{aligned}$$

Step 5. Using k_1, k_2, k_3, k_4 , and $\ell_1, \ell_2, \ell_3, \ell_4$ from Steps 1 to 4, the first increment in $P_0(0)$ and $P_1(0)$ is computed.

$$\begin{aligned}
 \Delta P_0(0) &= \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) \\
 &= \frac{1}{6} (-0.1 + 2(-0.09) + 2(-0.091) - 0.0818) \\
 &= \frac{1}{6} (-0.1 - 0.18 - 0.182 - 0.0818) \\
 &= \frac{1}{6} (-0.5638) \\
 &= -0.090633
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 P_1(0) &= \frac{1}{6} (\lambda_1 + 2\lambda_2 + 2\lambda_3 + \lambda_4) \\
 &= \frac{1}{6} \{0.1 + 2(0.09) + 2(0.091) + 0.0818\} \\
 &= \frac{1}{6} (0.1 + 0.18 + 0.182 + 0.0818) \\
 &= \frac{1}{6} (0.5638) \\
 &= 0.090633
 \end{aligned}$$

Step 6. Adding the value of $\Delta P_0(0)$ to $P_0(0)$ and $\Delta P_1(0)$ to $P_1(0)$, $P_0(0.1)$ and $P_1(0.1)$ are obtained.

$$\begin{aligned}
 P_0(0.1) &= P_0(0) + \Delta P_0(0) \\
 &= 1 - 0.090633 \\
 &= 0.909367 \\
 P_1(0.1) &= P_1(0) + \Delta P_1(0) \\
 &= 0 + 0.090633 \\
 &= 0.090633
 \end{aligned}$$

Step 7. Similarly, $P_0(t)$ and $P_1(t)$ can be calculated at some time t , considering the previous values obtained.

Analytical Solution. Having solved the sample problem by Runge-Kutta method, the same sample problem will be solved analytically. From the properties of probability,

$$P_0(t) + P_1(t) = 1$$

Hence,

$$P_1(t) = 1 - P_0(t) \quad (2)$$

Substituting the value of $P_1(t)$ as obtained from Equation (2),

$$\begin{aligned}\frac{dP_0(t)}{dt} &= -P_0(t) + 1 - P_0(t) \\ &= 1 - 2P_0(t)\end{aligned}$$

or

$$\frac{dP_0(t)}{dt} + 2P_0(t) = 1 \quad (3)$$

The solution of the ordinary linear differential equation

$$\frac{dy}{dt} + Py = Q$$

is given by

$$y = \frac{1}{e^{\int P dt}} \int Q e^{\int P dt} dt \quad (4)$$

Comparing Equation (3) with Equation (4),

$$\begin{aligned}y &= P_0(t) \\ P &= 2 \\ Q &= 1 \\ \int P dt &= \int 2 dt \\ &= 2t + c, \quad \text{where } c \text{ is a constant} \\ \int Q e^{\int P dt} dt &= \int 1 e^{2t+c} dt \\ &= \int e^{2t+c} dt \\ &= e^c \int e^{2t} dt \\ &= e^c \left(\frac{e^{2t}}{2} + k \right), \quad \text{where } k \text{ is a constant} \\ &= \frac{e^c}{2} e^{2t} + k e^c\end{aligned}$$

Hence, from Equation (4)

$$y = \frac{1}{e^{2t+c}} \left(\frac{e^c}{2} e^{2t} + k e^c \right)$$

$$\begin{aligned}
 y &= \frac{e^c}{e^{2t} e^c} \left(\frac{e^{2t}}{e} + k \right) \\
 &= \frac{1}{e^{2t}} \left(\frac{e^{2t}}{2} + 2k \right) \\
 &= \frac{1}{2} (1 + 2ke^{-2t})
 \end{aligned}$$

But

$$y = P_0(t)$$

Hence

$$P_0(t) = \frac{1}{2} (1 + 2ke^{-2t}) \quad (5)$$

Applying initial conditions,

$$P_0(0) = 1, \quad \text{at } t = 0$$

or

$$\begin{aligned}
 1 &= \frac{1}{2} (1 + 2ke^0) \\
 &= \frac{1}{2} (1 + 2k)
 \end{aligned}$$

or

$$2 = 1 + 2k$$

or

$$2k = 1$$

or

$$k = \frac{1}{2}$$

Substituting the value of k in Equation (5)

$$\begin{aligned}
 P_0(t) &= \frac{1}{2} \left(1 + 2 \cdot \frac{1}{2} e^{-2t} \right) \\
 &= \frac{1}{2} (1 + e^{-2t}) \quad (6)
 \end{aligned}$$

From Equation (2)

$$P_1(t) = 1 - P_0(t)$$

Hence

$$\begin{aligned} P_1(t) &= 1 - \frac{1}{2} (1 + e^{-2t}) \\ &= \frac{1}{2} (1 - e^{-2t}) \end{aligned} \quad (7)$$

Equations (6) and (7) represent the general solution of $P_0(t)$ and $P_1(t)$.

At $t = 0.1$

$P_0(0.1)$ from Equation (6) is such that

$$\begin{aligned} P_0(0.1) &= \frac{1}{2} (1 + e^{-0.2}) \\ &= \frac{1}{2} (1 + 0.8187) \\ &= \frac{1.8187}{2} \\ &= 0.90935 \end{aligned}$$

$P_0(0.1)$ from the Runge-Kutta Method was 0.909367.

$P_1(0.1)$ from Equation (7) is such that

$$\begin{aligned} P_1(0.1) &= \frac{1}{2} (1 - 0.8187) \\ &= \frac{0.1813}{2} \\ &= 0.09065 \end{aligned}$$

$P_1(0.1)$ from the Runge-Kutta Method was 0.090633.

Thus, we see that numerical solution and analytical solution are quite close to each other. Unfortunately, we do not have the analytical solution for most of the cases. In such cases, we

have to rely on numerical methods. For a comparison between Runge-Kutta Solution and Analytical Solution, see Table I.

2.2 Hamming's Method

In Hamming's Method initial values at four consecutive points must be known. If only the value at the single point is known, the remaining three values can be found by any of the single step methods. The single step method used for starting the solution is generally the fourth order Runge-Kutta Method.

Mathematical Development. Once the starting values are known, the sequence of calculations in Hamming's Method are as shown below, and as described by James [14].

Step 1. Find the predicted value of y_{i+1} such that

$$P(y_{i+1}) = y_{i-3} + \frac{4h}{3} (2y_i' - y_{i-1}' + 2y_{i-2}')$$

where h is the step size. As the values of y_i , y_{i-1} , and y_{i-2} are previously known, their derivatives, y_i' , y_{i-1}' , and y_{i-2}' , are known also.

Step 2. Modify the predicted value of y_{i+1} , $P(y_{i+1})$ to eliminate some error such that

$$M(y_{i+1}) = P(y_{i+1}) - \frac{112}{121} [P(y_i) - C(y_i)]$$

Step 3. Substitute the modified value of y_{i+1} , $M(y_{i+1})$ in the given differential equation to obtain a modified value of $M(y_{i+1})'$ such that

$$M(y_{i+1})' = f[t_{i+1}, M(y_{i+1})]$$

Step 4. Use the modified value of y_{i+1}^1 , $M(y_{i+1}^1)$ in the corrector equation to get the corrected value of y_{i+1} , $C(y_{i+1})$ such that

$$C(y_{i+1}) = \frac{1}{8} \{9y_i - y_{i-2} + 3h[M(y_{i+1}^1) + 2y_i^1 - y_{i-1}^1]\}$$

Step 5. Calculate the final value of y_{i+1} , $F(y_{i+1})$ using $P(y_{i+1})$ and $C(y_{i+1})$ such that

$$F(y_{i+1}) = C(y_{i+1}) + \frac{9}{121} [P(y_{i+1}) - C(y_{i+1})]$$

IBM scientific subroutine package (HPCG) has been used for applying the Hamming's Method for solving queueing models.

Hamming's Method is also a fourth order method. Hence, truncation error is of the order of h^5 . If propagation and round-off errors are assumed to be negligible, estimates of per-step truncation errors in the Hamming Method are given by

$$T_c = \frac{9}{121} [P(y_{i+1}) - C(y_{i+1})]$$

Sample Problem. Let us consider the same problem which has been solved by the Runge-Kutta Method. The difference-differential equations are:

$$\frac{dP_0(t)}{dt} = -P_0(t) + P_1(t)$$

$$\frac{dP_1(t)}{dt} = P_0(t) - P_1(t)$$

The initial conditions are

$$P_0(0) = 1$$

and

$$P_1(0) = 0$$

As there should be four unknown values of each dependent variable to initiate the solution by the Hamming Method and only one value is known, the other three values are first calculated by the fourth order Runge-Kutta Method. For example, let the step size h be 0.05. From the computer output of Runge-Kutta solution, the other three values of the dependent variables are taken. Thus, the four initial known values of the dependent variables are as shown below:

<u>t</u>	<u>$P_0(t)$</u>	<u>$P_1(t)$</u>
0.00	1.00000	0.000000
0.05	0.95242	0.047581
0.10	0.90937	0.090635
0.15	0.87041	0.129590

The procedure is shown below step by step.

Step 1. Substituting the known values of $P_0(t)$ and $P_1(t)$, the corresponding derivatives $P_0^1(t)$ and $P_1^1(t)$ at the four points are determined such that

<u>t</u>	<u>$P_0^1(t)$</u>	<u>$P_1^1(t)$</u>
0.00	-1.000000	1.000000
0.05	-0.904839	0.904839
0.10	-0.818735	0.818735
0.15	-0.740820	0.740820

The predicted value of $P_0(0.2)$, $P(P_0(0.2))$ and predicted value of $P_1(0.2)$, $P(P_1(0.2))$ is calculated from the equation

$$P(y_{i+1}) = y_{i-3} + \frac{4h}{3} (2y_i^1 - y_{i-1}^1 + 2y_{i-2}^1)$$

Hence,

$$\begin{aligned}
 P(P_0(0.2)) &= P_0(0) + \frac{4(0.05)}{3} (2P_0^1(0.15) - P_0^1(0.1) \\
 &\quad + 2P_0(0.05)) \\
 &= 1 + \frac{4(0.05)}{3} \{2(-0.74082) - (-0.81875) \\
 &\quad + 2(-0.904839)\} \\
 &= 1 + \frac{0.2}{3} (-1.139107) \\
 &= 0.92406 \\
 P(P_1(0.2)) &= P_1(0) + \frac{4(0.05)}{3} (2P_1^1(0.15) - P_0^1(0.1) \\
 &\quad + 2P_0(0.05)) \\
 &= 0.0 + \frac{0.2}{3} \{2(0.74082) - (0.818732) \\
 &\quad + 2(0.904839)\} \\
 &= 0.07594
 \end{aligned}$$

Step 2. The predicted values are to be modified by the relation

$$M(y_{i+1}) = P(y_{i+1}) - \frac{112}{121} [P(y_i) - C(y_i)]$$

Hence,

$$\begin{aligned}
 M(P_0(0.02)) &= P(P_0(0.2)) - \frac{112}{121} [P(P_0(0.15)) \\
 &\quad - C(P_0(0.15))]
 \end{aligned}$$

Since the predicted and corrected values of $P_0(0.15)$ are not known as this is the starting point of solution of the Hamming Method, the given value of $P_0(0.15)$ itself is taken as the value

for

$$P(P_0(0.15)) \text{ and } C(P_0(0.15)).$$

Hence,

$$\begin{aligned} M(P_0(0.2)) &= P(P_0(0.2)) - \frac{112}{121} (0.0) \\ &= P(P_0(0.2)) \\ &= 0.92406 \end{aligned}$$

Similarly,

$$\begin{aligned} M(P_1(0.2)) &= P(P_1(0.2)) \\ &= 0.07596 \end{aligned}$$

Step 3. The modified values of $P_0(0.2)$ and $P_1(0.2)$ from Step 2 are substituted in the given differential equation to obtain the modified values of their derivatives, $M(P_0^1(0.2))$ and $M(P_1^1(0.2))$. Hence,

$$\begin{aligned} M(P_0^1(0.2)) &= -M(P_0(0.2)) + M(P_1(0.2)) \\ &= -0.92406 + 0.07594 \\ &= -0.84812 \end{aligned}$$

$$\begin{aligned} M(P_1^1(0.2)) &= M(P_0(0.2)) - M(P_1(0.2)) \\ &= 0.92406 - 0.07594 \\ &= 0.84812 \end{aligned}$$

Step 4. The modified values of the derivatives obtained in Step 3 are used to obtain the corrected value of the dependent variables from the relation

$$C(y_{i+1}) = \frac{1}{8} \{ 9y_i - y_{i-2} + 3h[M(y_{i+1}^1) + 2y_i^1 - y_{i-1}^1] \}$$

Hence,

$$\begin{aligned}
 C(P_0(0.2)) &= \frac{1}{8} \{ 9P_0(0.15) - P_0(0.05) + 3h[M(P_0^1(0.2)) \\
 &\quad + 2P_0^1(0.15) - P_0^1(0.1)] \} \\
 &= \frac{1}{8} \{ 9(0.87041) - 0.95262 + 3(0.05)[-0.84812 \\
 &\quad + 2(-0.74082) - (-0.81874)] \} \\
 &= \frac{1}{8} \{ 7.83369 - 0.95242 + 0.15[-0.84812 \\
 &\quad - 1.48164 + 0.81874] \} \\
 &= \frac{1}{8} \{ 6.88127 + 0.15[-1.51102] \} \\
 &= \frac{1}{8} \{ 6.88127 - 0.22665 \} \\
 &= \frac{1}{8} \{ 6.65462 \} \\
 &= 0.83182
 \end{aligned}$$

Similarly,

$$C(P_1(0.2)) = 0.16818$$

Step 5. Using the predicted and corrected value from Steps 1 and 4, the final values of $P_0(0.2)$ and $P_1(0.2)$ are calculated using the relation

$$F(y_{i+1}) = C(y_{i+1}) + \frac{9}{121} [P(y_{i+1}) - C(y_{i+1})]$$

Hence, the final value of $P_0(0.2)$ is

$$\begin{aligned}
 P_0(0.2) &= C(P_0(0.2)) + \frac{9}{121} [P(P_0(0.2)) - C(P_0(0.2))] \\
 &= 0.83182 + \frac{9}{121} [0.92406 - 0.83182] \\
 &= 0.83668
 \end{aligned}$$

Similarly, the final value of $P_1(0.2)$ is

$$P_1(0.2) = 0.16332$$

Proceeding in this fashion, the probabilities $P_0(t)$ and $P_1(t)$ can be computed at any time t .

For comparison, the values of $P_0(t)$ and $P_1(t)$ during the interval $[0,1]$ at time interval of 0.1 for the sample problem are obtained by using the analytical method, fourth order Runge-Kutta Method, and Hamming's Method. The values of $P_0(t)$ and $P_1(t)$ are displayed in Table I for the three methods. The deviation of the probabilities from the analytical solution is due to the errors caused by the numerical method. Upon examining the various values in the table, it is apparent that the numerical methods provide quite accurate solutions for most of the practical purposes. The solution obtained by Hamming's Method is similar to that of the fourth order Runge-Kutta Method. Regarding computation time, the computer spent 1.8 minutes using IBM scientific subroutine package HPCG, on an IBM 360/50 computer, whereas the fourth order Runge-Kutta Method took only 0.9 minutes of system time using IBM scientific subroutine package RKGS.

Both HPCG and RKGS were used to solve a few more problems; it was found that RKGS took less computer time, and accuracy was comparable to that of HPCG. Hence, RKGS was used to solve the queueing models cited in this thesis. The computer program is discussed and an output of a sample problem is given in Appendix C.

Table 2.1

Comparison of Analytical Solution vs. Numerical Solution of the Sample Problem

Time	Analytical Solution	Numerical Solution							
		Fourth Order Runge-Kutta Method			Hamming Method				
		$P_0(t)$	$P_1(t)$	$P_0(t)$	Error	$P_1(t)$	Error	$P_1(t)$	Error
t									
0.0	1.00000	0.00000	1.00000	0.00000	0.00000	1.00000	0.00000	0.00000	0.00000
0.1	0.90935	0.09065	0.90937	-0.00002	0.09063	0.00002	0.90937	-0.00002	0.09063
0.2	0.83515	0.16485	0.83516	-0.00001	0.16484	0.00001	0.83516	-0.00001	0.16484
0.3	0.77440	0.22560	0.77441	-0.00001	0.22559	0.00001	0.77441	-0.00001	0.22559
0.4	0.72465	0.27535	0.72467	-0.00002	0.27533	0.00002	0.72466	-0.00001	0.27533
0.5	0.68395	0.31605	0.68394	0.00001	0.31606	-0.00001	0.68394	0.00001	0.31606
0.6	0.65060	0.34940	0.65060	0.00000	0.34940	0.00000	0.65060	0.00000	0.34940
0.7	0.62330	0.37670	0.62330	0.00000	0.37670	0.00000	0.62330	0.00000	0.37670
0.8	0.60095	0.39905	0.60095	0.00000	0.39905	0.00000	0.60095	0.00000	0.39905
0.9	0.58265	0.41735	0.58265	0.00000	0.41735	0.00000	0.58265	0.00000	0.41735
1.0	0.56765	0.43235	0.56767	-0.00002	0.43233	0.00002	0.56767	-0.00002	0.43233
2.0	0.50916	0.49084	0.50916	0.00000	0.49084	0.00000	0.50915	0.00001	0.49084
3.0	0.50124	0.40876	0.50124	0.00000	0.49876	0.00000	0.50123	0.00001	0.49875
4.0	0.50017	0.49983	0.50016	0.00001	0.49983	0.00000	0.50016	0.00001	0.49982
5.0	0.50002	0.49998	0.50002	0.00000	0.49997	0.00001	0.50001	0.00001	0.49997

2.3 Dimensionless Time Transient Solution

In solving the above problem considering the dimensionless time units τ , let $\tau = t/T$, $0 \leq \tau \leq 1$ and $0 \leq t \leq T$. Consider the set of difference-differential equations (1) presented above.

For convenience, these two equations are reproduced below:

$$\frac{dP_0(t)}{dt} = -P_0(t) + P_1(t)$$

$$\frac{dP_1(t)}{dt} = P_0(t) - P_1(t)$$

The initial conditions are:

$$P_0(0) = 1$$

and

$$P_1(0) = 0$$

Applying the transformation technique,

$$\text{when } t = 0, \quad \tau = 0,$$

$$\text{when } t = T, \quad \tau = 1$$

$$\text{and } dt = Td\tau$$

The integration limits of $(0, T)$ can be converted to $(0, 1)$ in dimensionless time units.

Thus, the difference-differential equations in dimensionless time units become

$$\frac{dP_0(\tau)}{Td\tau} = -P_0(\tau) + P_1(\tau)$$

and

$$\frac{dP_1(\tau)}{Td\tau} = P_0(\tau) - P_1(\tau)$$

or

$$\frac{dP_0(\tau)}{d\tau} = T [P_0(\tau) + P_1(\tau)]$$

and

$$\frac{dP_1(\tau)}{d\tau} = T [P_0(\tau) - P_1(\tau)]$$

where

$$P_0(0) = 1$$

$$P_1(0) = 0$$

Substituting the value of t in dimensionless time units, which is $T\tau$, in Equations (6) and (7), the analytical solutions are

$$P_0(\tau) = \frac{1}{2} (1 + e^{-2T\tau})$$

and

$$P_1(\tau) = \frac{1}{2} (1 - e^{-2T\tau})$$

Considering $T = 100$, we have

$$P_0(\tau) = \frac{1}{2} (1 + e^{-200\tau})$$

and

$$P_1(\tau) = \frac{1}{2} (1 - e^{-200\tau})$$

For comparison of the values of $P_0(\tau)$ and $P_1(\tau)$ for $0 \leq \tau \leq 1$, computed by using the analytical method, fourth order Runge-Kutta method and Hamming's method, refer to Table 2.2. System time for Hamming's method, using the IBM Scientific subroutine package, HPCG on an IBM 360/50 computer, was 2.16 minutes, and for the fourth order Runge-Kutta method using the IBM scientific subroutine package RKGS, was only 0.96 minutes.

Table 2.2

Comparison of Analytical Solution vs. Numerical Solution of the Sample Problem

Time	Analytical Solution	Numerical Solution							
		Fourth Order Runge-Kutta Method			Hamming Method				
		$P_0(\tau)$	Error	$P_1(\tau)$	Error	$P_0(\tau)$	Error	$P_1(\tau)$	Error
τ									
0.0	1.00000	0.00000	0.00000	0.00000	0.00000	1.00000	0.00000	0.00000	0.00000
0.1	0.50000	0.50000	0.00000	0.50000	0.00000	0.50001	-0.00001	0.49997	0.00003
0.2	0.50000	0.50000	0.00000	0.50000	0.00000	0.49998	0.00002	0.49999	0.00001
0.3	0.50000	0.50000	0.00001	0.49999	0.00001	0.49998	0.00002	0.49998	0.00002
0.4	0.50000	0.50000	0.00001	0.49999	0.00001	0.49997	0.00003	0.49998	0.00002
0.5	0.50000	0.50000	0.00001	0.49999	0.00001	0.50000	0.00000	0.49994	0.00006
0.6	0.50000	0.50000	0.00001	0.49999	0.00001	0.49997	0.00003	0.49996	0.00004
0.7	0.50000	0.50000	0.00001	0.49999	0.00001	0.49997	0.00003	0.49995	0.00005
0.8	0.50000	0.50000	0.00001	0.49999	0.00001	0.49995	0.00005	0.49995	0.00005
0.9	0.50000	0.50000	0.00001	0.49999	0.00001	0.49995	0.00005	0.49995	0.00005
1.0	0.50000	0.50000	0.00001	0.49999	0.00001	0.49993	0.00007	0.49995	0.00005

CHAPTER III

APPLICATION TO MACHINE-SHOP LIKE QUEUEING MODELS

A machine-shop process consists of a group of work centers, each performing a definite set of jobs which require processing in one or more of the work centers, in a pre-determined sequence. Most of the analytical work in the area of machine-shop scheduling has been concerned with the static machine-shop process. The static problem is defined such that given J jobs and M machines, the machine ordering of each job and the associated processing time, find the sequence which optimizes some desired objectives. The number of feasible sequences is very large. It also increases as the number of jobs and machines increases. Some analytical techniques such as combinatorial, mathematical programming are available for solving small size problems. In the static machine-shop models, it is assumed that all jobs arrive simultaneously at times known in advance and also their processing times are known and constant. In most of the real situations, however, jobs arrive at the shop in a random fashion. To illustrate the effect of intermittent job arrivals on the mean flow time, consider two jobs to be processed on a single machine. Jobs 1 and 2 having processing times of 6 and 8 units, arrive at time 0 and 4, respectively. Let us assume that preemption, that is, the processing of a job that has been started is stopped before completion, is not allowed. If job 1 is loaded on the machine at time 0, it will be completed at time 6. Hence, the flow time for job 1 is 6

units. Job 2 arrives at time 4, and thus it waits 2 units of time for the machine since the machine is busy processing job 1. At time 6, job 2 is loaded on the machine and is completed at time 14. The flow time of job 2 is $14 - 4$ or 10 time units. Consequently, the mean flow time of both jobs is $(6 + 10)/2$ or 8 units of time. In the case of simultaneous arrival of both jobs at time 0, the mean flow time of both jobs will be $(6+16)/2$ or 10 units of time. As Reinitz [20] has pointed out, one or more of the following factors cause the random behavior of arrival of jobs and their processing times: (1) unexpected delays in supplying the engineering specifications, tools, and raw materials; (2) variations in worker's performance; (3) delays due to machine breakdowns; (4) changes in customer's specifications and delivery requirements; (5) variations in raw material and product quality; (6) absenteeism; or (7) any unforeseen circumstances not included in the above.

Due to its dynamic nature, the machine-shop is viewed as a queueing system. This chapter is devoted to the study of the basic structure of the machine-shop system. Such a system is classified into flow-shop and job-shop processes. Various single-machine and multiple-machine queueing models are developed.

3.1 Basic Structure

In general, a shop when viewed as a queueing process, consists of four basic elements: (1) Arrival of jobs; (2) Queue; (3) Service or processing; and (4) Output. The basic machine-shop queueing process is shown in Figure 1.1.

An arrival of jobs is said to occur when the jobs have been engineered and are ready for processing in the shop. Jobs may arrive in a machine shop from an outside source or from some other work center.

A queue is said to form from waiting jobs while a machine is busy. Jobs are usually selected from the queue according to some priority rules, referred to as a queue discipline. Various priority rules are as follows:

(1) First-come, first-processed. The job which arrives earliest for processing is processed first. For example, generally, in a flow shop the jobs are processed in the order of their arrival.

(2) Last-come, first-processed. The job which has arrived last is selected for processing first. For example, jobs arriving at a machine center have been piled in such a way that the latest arrival is the most accessible, and thus, the one to be selected for processing first.

(3) Random selection. Jobs are selected at random from the queue. All the jobs in the queue at the moment of selection have equal probability of being selected for processing. For example, when many jobs have arrived for processing in a machine-shop and it is difficult to distinguish the order in which they arrived, any job may be selected at random for processing.

(4) Due-date rule. This is often followed in manufacturing organizations which produce according to customer orders (jobs). A customer, while ordering, specified a delivery date. In order

to fulfill this order, jobs are selected from the queue according to their due dates. The time interval from the job arrival and the required delivery date is determined. These intervals vary from customer to customer. That job waiting in the queue is selected first for which the difference between the promised due date and the present date is a minimum.

(5) Preemptive priority rule. Each job is assigned a priority. The job with the highest priority is selected first from the queue for processing. If a job with a lower priority is being processed, it is interrupted to process a job with a higher priority. If there are many jobs of one class of priority, selection of jobs is generally done on a first-come, first-processed basis or at random. If the processing of the preempted job can be resumed where it has been left off, it is called a preemptive resume priority. If the processing of the preempted job, when it returns to the machine for processing, must be repeated from the beginning, the process is called a preemptive repeat priority. Preemption may be required, for example, when there are emergencies such as in a maintenance shop of a production plant.

(6) Non-preemptive priority rule. When a waiting job is selected according to its priority class, but without interrupting any other job during its process, this is called non-preemptive priority rule. For example, in the case of a job in process, whose processing is very difficult and preemption will mean much extra work and hence will be more costly, non-preemptive priority rule is applicable.

When a job completes its processing on the assigned machines, it departs and is considered as an output. In-process usually is considered when a job as an output departs from a workcenter after its completion and enters another work center as an input.

There are two basic ways in which a machine-shop like queueing system can be considered: (1) flow-shop process and (2) job-shop process.

Flow-shop process. In a flow-shop, machines are placed in a serial order to suit the processing of particular jobs. There is a natural ordering of machines in the shop. The main characteristic of a flow-shop is that the movement of the jobs is unidirectional and noncyclic. There are two classes of flow-shop process: (1) serial and (2) hybrid.

Serial-shop system. In a serial flow-shop process, each job has the same machine ordering. The jobs are processed on machine 1 first, machine 2 second, ..., and machine M last. Serial flow-shop process is shown in Figure 3.1. In such a case, jobs move through the machines until they are completed. In terms of queueing, this type of process is called single-channel, multiphase system. For example, special purpose machines are arranged in a serial order to manufacture crankshafts or cylinder blocks for automobile engines. In an assembly line, where jobs move from a work station to another work station, is another example of flow shop. The work stations represent the machines in series. If more than one operation on a job is to be performed on a machine, the machine is equivalent to a serial flow shop.

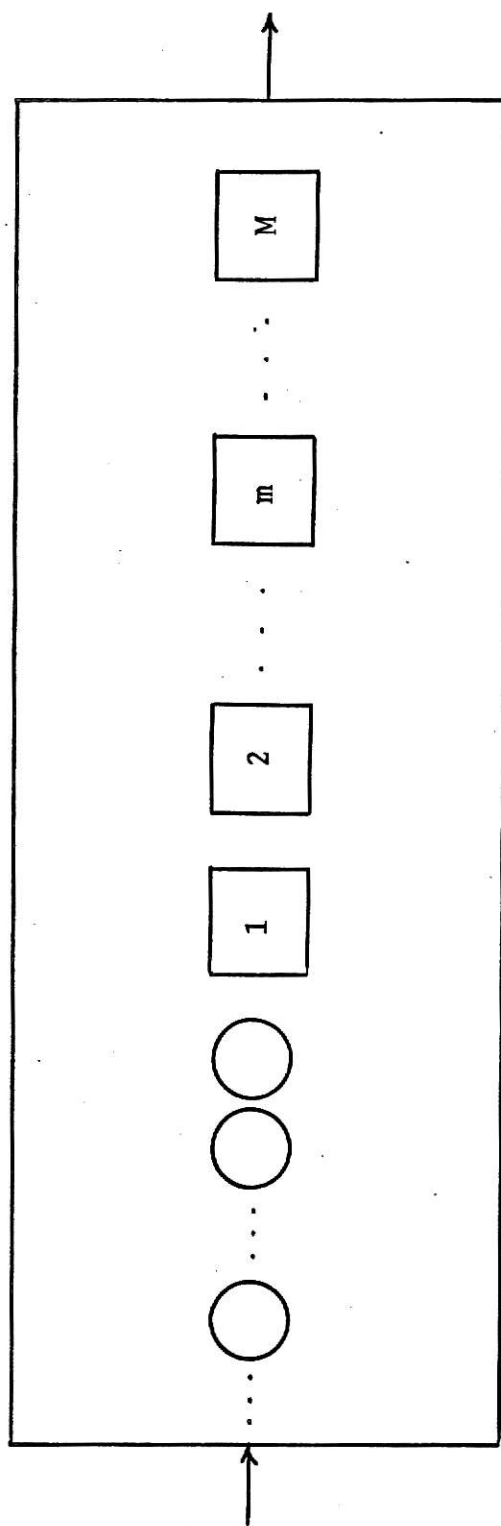


Figure 3.1. Serial-Machine Shop Process.

Hybrid-shop system. In a hybrid flow-shop, there are a number of identical machines in each work center. The number of machines in all the work centers may not be the same. A hybrid flow shop in Figure 3.2 shows that there are a number of work centers arranged in a series to maintain the flow of material in a uniform way. Job 1 is processed in work center 1 first, then goes to work center 2 next, and after being processed in all the previous work centers, is processed in work center M last. Job 2 is processed in work center 1 first and work center 2 last, from where it departs from the system. Job 3 is processed only in work center 2 and, after being processed, leaves the system. Similarly, job j is processed in work center 2 first and then goes directly to the last work center M, from where it departs from the system. Thus, the input to any work center may be from an external source or from previous work centers or from both. It is also possible that any job may depart from the system after being processed by any workcenter. It may be assumed that one and only one operation on a job is completed on a machine in a work center. For example, in a typical overhaul shop for aircraft engines, there may be five work centers, namely: stripping, detailed examination, repair, assembly, and testing, arranged in series. Engines arrive at work center 1 where stripping is done. An arriving engine may join any of the stripping stations of work center 1. After stripping in work center 1, an engine can go to any of the idle stations in work center 2 for detailed examination of defects. In this way, the engine moves from work center to work center,

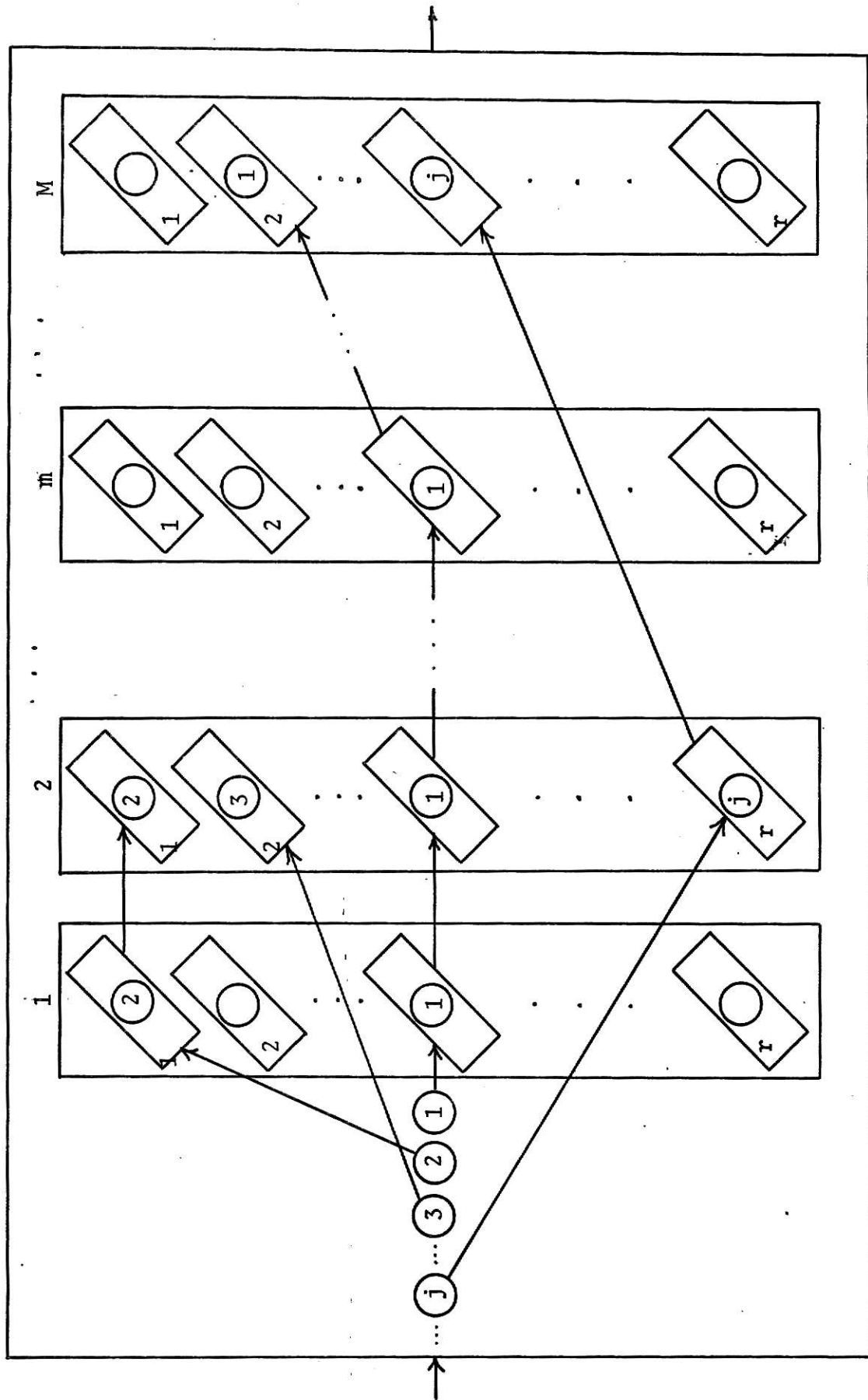


Figure 3.2. Hybrid Flow Shop Process.

until it is out from work center 5 after completion of testing. Other examples are found in production shops where work centers are arranged in series for processing a large number of jobs, each work center performing one type of operation only on a job.

Sometimes an arriving job has the choice to join any of the many independent serial flow-shop systems. These serial flow shops are independent in the sense that once a job has entered in one of the first work centers of a serial flow-shop system, it has to complete all its operations in the same flow shop. Such cases are common in mass production systems where there are several work centers, arranged in parallel. Each work center is independent of the other work centers. Each work center consists of cleaning, washing, primary painting, and finish painting units arranged in serial order. Once any job enters in any of the available work centers, it will complete all its operations in that subsystem.

Job-shop process. In a job-shop process, similar machines are grouped in departments or work centers according to the nature of the work they perform. Machine ordering of any job is independent of the routing of other jobs to be processed. In each work center, all the machines perform the same operation. There are two classes of job-shop processes: (1) parallel and (2) hybrid.

Parallel-shop system. The parallel arrangement can be found in a job-shop with one work center. The work center consists of several similar machines suitable for one operation of a job.

For example, similar grinding machines are placed in one work center. Any job can be processed for the single operation on any of the available machines. A parallel job-shop system is shown in Figure 3.3.

Sometimes, for each machine or a group of machines, standby machines are kept such that if any one machine or more machines break down, operation can be switched to the standby machines. If there are M standby machines for a group of r machines, full production rate can be maintained until there are only M machines broken down. If more than M machines break down, the system production rate is reduced proportionately.

In some cases, out of M machines, any m machine is available for general jobs and $M-m$ machines are reserved for special jobs.

Hybrid-shop system. In a hybrid job-shop process, there are many work centers; each particular work center having similar machines. The machine ordering of any job is independent of any other job. This is the only difference in a hybrid job-shop process from a hybrid flow-shop process. The hybrid job-shop process is shown in Figure 3.4. When a job is completed in a work center, the job goes to other work centers according to a pre-specified machine ordering. Generally, each job has a different machine ordering. From Figure 3.4, it is shown that job 1 is processed in work center 1 first, then goes to work center 2 and, after being processed in work center 2, goes directly to the work center M . After being processed in work center M , the job departs from the system. Job 2 is processed in work center 1 first, then goes to work

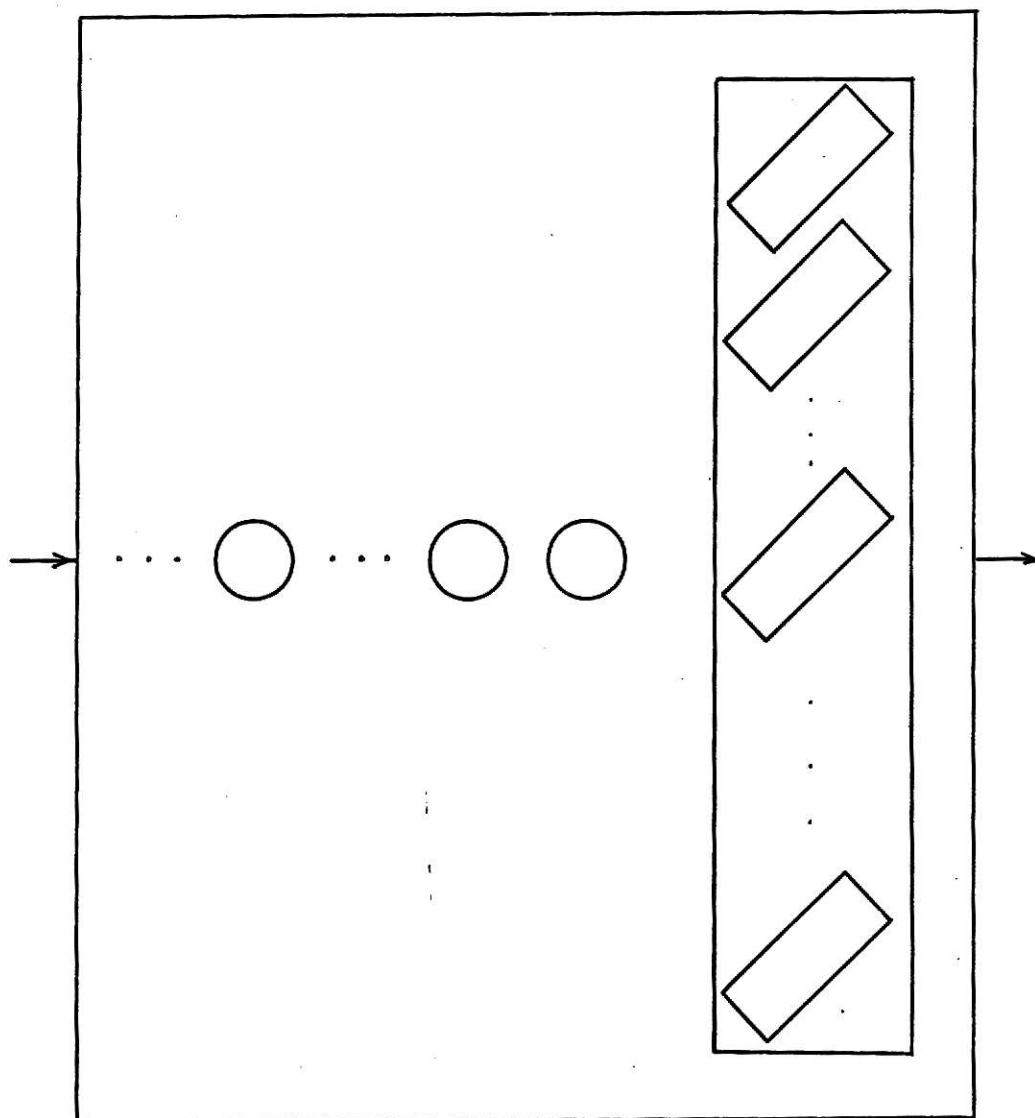


Figure 3.3. Parallel Machine Process.

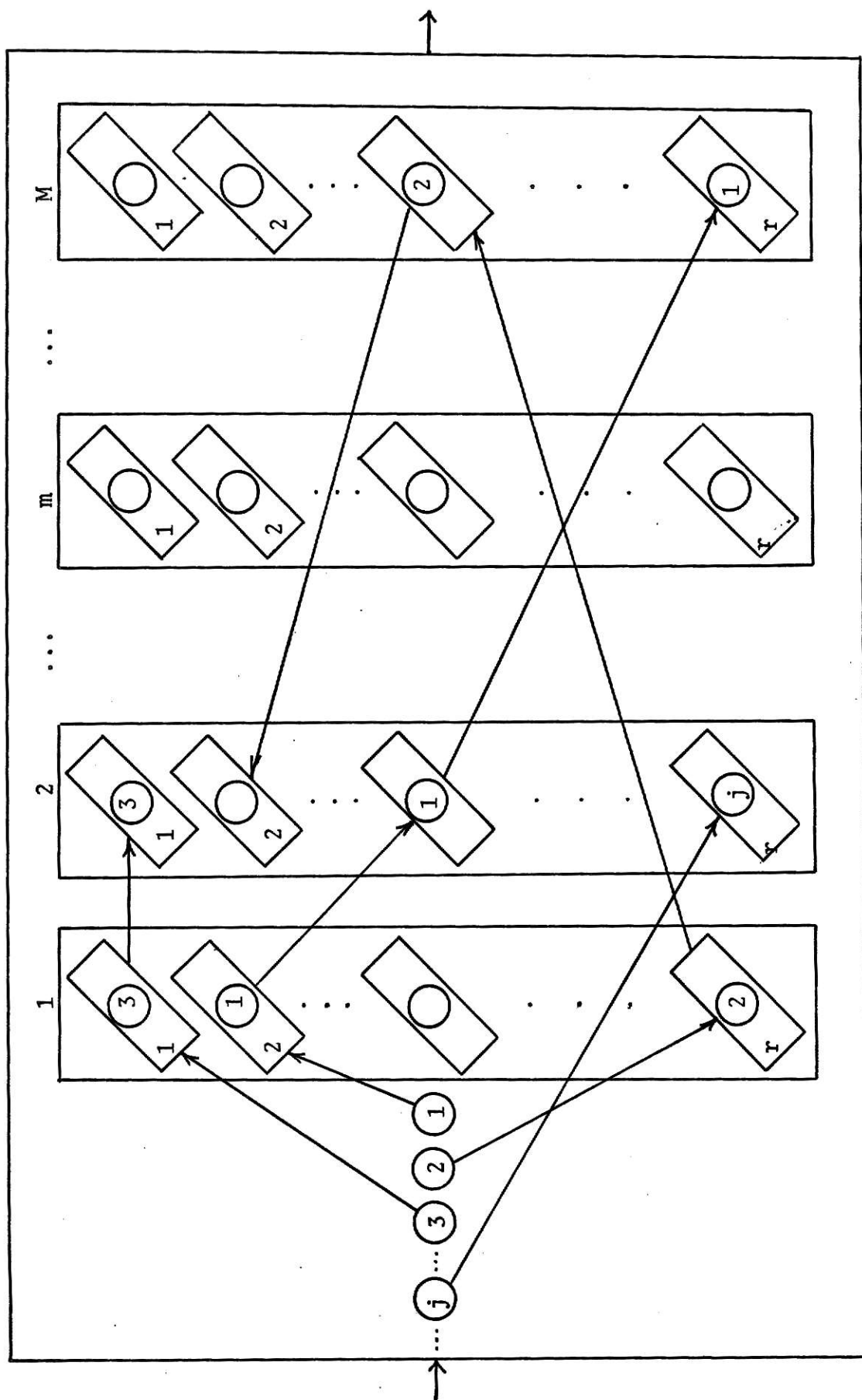


Figure 3.4. Hybrid Job Shop Process.

center M, and comes back to work center 2 and, after being processed there, departs from the system. Job 3 is processed in work center 1 first and work center 2 last. Similarly, job j is processed in work center 2 only and, after being processed, departs from the system. As an example, in a power press production shop where a large range of presses are manufactured, there may be many work centers, namely one work center for turning, one for drilling, one for grinding, one for milling, one for gear cutting, one for polishing, one for painting, and work center 1 for packaging. For a particular component, turning may be required first, then drilling and, last, painting. For some other component, grinding, drilling, and then gear cutting may be required. Thus, all jobs may have a different machine ordering. Such a production shop truly behaves as a hybrid job-shop process.

The mean flow time is a very important measure of performance for machine-shop like queueing processes. Mean flow time is the sum of mean waiting time in the queue and mean processing time. Hence, the solution that minimizes the mean flow time also minimizes the in-process inventory, that is minimizes the mean number of jobs in the system. Other measures of performance are: (1) the mean number of jobs in the system; (2) the mean number of jobs in the queue; (3) the mean waiting time of a job in the queue; (4) machine utilization; (5) mean idle time of the machines; and (6) probability of waiting of jobs.

Next, a machine-shop like queueing model will be developed in which the arrival rate of jobs and processing rate are dependent on the state of the system.

The queue size of the model is assumed to be limited. Then various special cases of the model will be derived. It is assumed that the machine ordering is independent of (1) the state of the system, (2) the processing times, and (3) the knowledge of specific future job arrivals.

Extensive investigations have been carried out as to the arrival and processing time distributions of machine-shop production process by the Management Sciences Research Project at the University of California, Los Angeles. Empirical study has been conducted and the results of chi-square tests of the arrival data have shown that arrivals of jobs can be approximated very closely to the Poisson distribution [20]. The specifications of this model are:

Arrival Source	: Infinite or Finite
Arrival Pattern	: Single
Arrival Behavior	: Patient
Arrival Distribution	: Poisson
Queue Size	: Limited queue
Queue Discipline	: First-come, first-processed, Last-come, first-processed, or Random selection for processing
Processing Pattern	: Single
Processing Mechanism	: Single phase
Processing Time Distribution:	Exponential
Number of Machines	: Single-machine, or Multiple-machine

The following notation is used:

$P_n(t)$ probability that there are exactly n jobs in the

queue system at time t .

λ_n arrival rate of jobs giving n jobs in the system.

μ_n processing rate of jobs on the machine giving n jobs in the system.

N Shop capacity, that is, the number of jobs that can be accommodated in the system.

Thus, the probability of n job arrivals during time t is

$$P_n(t) = \frac{(\lambda_n t)^n}{n!} e^{-\lambda_n t} \quad n = 0, 1, 2, \dots$$

The probability of exactly one job arrival during small time interval Δt is

$$\begin{aligned} P_1(\Delta t) &= (\lambda_n \Delta t) e^{-\lambda_n \Delta t} \\ &= (\lambda_n \Delta t) (1 - \lambda_n \Delta t + o(\Delta t)) \\ &= \lambda_n \Delta t + o(\Delta t) \end{aligned}$$

where $o(\Delta t)$ is any function of Δt which is negligible compared to Δt as Δt becomes sufficiently small.

The probability of no job arrival during the small interval of time Δt is

$$\begin{aligned} P_0(\Delta t) &= e^{-\lambda_n \Delta t} \\ &= 1 - \lambda_n \Delta t + \frac{(\lambda_n \Delta t)^2}{2!} - \frac{(\lambda_n \Delta t)^3}{3!} + \dots \\ &= 1 - \lambda_n \Delta t + o(\Delta t) \end{aligned}$$

The probability of more than one job arrival during the small interval Δt is

$$P_n(\Delta t) = 1 - (1 - \lambda_n \Delta t + 0(\Delta t)) - (\lambda_n \Delta t + 0(\Delta t)),$$

$$n \geq 2$$

$$= 0(\Delta t)$$

Similarly, the probability that a job will be processed completely during time t , when there are n jobs in the system and no arrivals occur in the system at any time, is

$$\begin{aligned} P_{n-1}(t) &= \int_0^t \mu_n e^{-\mu_n t} dt \\ &= \mu_n \left. \frac{e^{-\mu_n t}}{-\mu_n} \right|_0^t \\ &= 1 - e^{-\mu_n t} \end{aligned}$$

The probability that a job will be completed during the small interval of time Δt is

$$\begin{aligned} P_{n-1}(\Delta t) &= 1 - e^{-\lambda_n \Delta t} \\ &= 1 - (1 - \mu_n \Delta t + 0(\Delta t)) \\ &= \mu_n \Delta t + 0(\Delta t) \end{aligned}$$

The probability that a job will not be completed during time Δt when there are n jobs at time t is

$$P_n(t) = 1 - \lambda_n \Delta t + 0(\Delta t)$$

Next, a set of difference-differential equations for $P_n(t)$ will be formulated by using the fundamental properties of probability. In enumerating all possibilities in which there will be n jobs in the system at time $t + \Delta t$, the probability of n jobs in the system at time $t + \Delta t$, may be expressed as follows:

For $n = 0$

$$\begin{aligned}
 P_0(t + \Delta t) &= P_0(t)(1 - \lambda_0 \Delta t + o(\Delta t))(1) + P_1(t) \\
 &\quad \cdot (1 - \lambda_1 \Delta t + o(\Delta t))(\mu_1 \Delta t + o(\Delta t)) \\
 &\quad + o(\Delta t) \\
 &= P_0(t)(1 - \lambda_0 \Delta t) + P_1(t)\mu_1 \Delta t + o(\Delta t),
 \end{aligned}$$

for $1 \leq n < N$, where N is finite

$$\begin{aligned}
 P_n(t + \Delta t) &= P_n(t)(1 - \lambda_n \Delta t + o(\Delta t))(1 - \mu_n \Delta t + o(\Delta t)) \\
 &\quad + P_{n+1}(t)(1 - \lambda_{n+1} \Delta t + o(\Delta t)) \\
 &\quad \cdot (\mu_{n+1} \Delta t + o(\Delta t)) + P_{n-1}(t) \\
 &\quad \cdot (\lambda_{n-1} \Delta t + o(\Delta t))(1 - \mu_{n-1} \Delta t + o(\Delta t)) \\
 &\quad + o(\Delta t) \\
 &= P_n(t)(1 - \lambda_n \Delta t - \mu_n \Delta t) + P_{n+1}(t)\mu_{n+1} \Delta t \\
 &\quad + P_{n-1}(t)\lambda_{n-1} \Delta t + o(\Delta t),
 \end{aligned}$$

and for $n = N$,

$$\begin{aligned}
 P_N(t + \Delta t) &= P_N(t)(1)(1 - \mu_N \Delta t + o(\Delta t)) + P_{N-1}(t) \\
 &\quad \cdot (\lambda_{N-1} \Delta t + o(\Delta t))(1 - \mu_{N-1} \Delta t + o(\Delta t)) \\
 &\quad + o(\Delta t) \\
 &= P_N(t)(1 - \mu_N \Delta t) + P_{N-1}(t)\lambda_{N-1} \Delta t + o(\Delta t)
 \end{aligned}$$

Rearranging the above equations, dividing both sides by Δt , and taking the limit of both sides as $\Delta t \rightarrow 0$, we get

$$\frac{dP_0(t)}{dt} = -\lambda_0 P_0(t) + \mu_1 P_1(t) \tag{1}$$

$$\frac{dP_n(t)}{dt} = \lambda_{n-1} P_{n-1}(t) - (\lambda_n + \mu_n) P_n(t)$$

$$+ \mu_{n+1} P_{n+1}(t), \quad 1 \leq n < N \quad (2)$$

$$\frac{dP_N(t)}{dt} = \lambda_{N-1} P_{N-1}(t) - \mu_N P_N(t) \quad (3)$$

Case A: No Queue. In this case, the number of jobs in the system at any time t is either zero or one. Hence, the stated probability Equations (1) and (3) become

$$\frac{dP_0(t)}{dt} = -\lambda_0 P_0(t) + \mu_1 P_1(t) \quad (4)$$

and

$$\frac{dP_1(t)}{dt} = \lambda_0 P_0(t) - \mu_1 P_1(t) \quad (5)$$

Case B: Unlimited Queue. In this case, Equation (3) is omitted and Equations (1) and (2) are valid. The difference-differential equations are:

$$\frac{dP_0(t)}{dt} = -\lambda_0 P_0(t) + \mu_1 P_1(t) \quad (6)$$

and

$$\begin{aligned} \frac{dP_n(t)}{dt} = & \lambda_{n-1} P_{n-1}(t) + \mu_{n+1} P_{n+1}(t) \\ & - (\lambda_n + \mu_n) P_n(t), \quad n > 0 \end{aligned} \quad (7)$$

In the remaining chapter, various models will be developed.

3.2 Single-Machine Shop Models

In a single-machine shop like queueing models, one work center consisting of one machine only can be formulated as a single queueing model as shown in Figure 1.1. Let us assume that the arrival rates and the processing rates are independent of the

state of the system.

Case A: Infinite Source and No Queue. In this case, no queue, before a machine, is allowed to form. Hence, either the machine is busy when a job is processed or idle when there is no job arrival.

Practical Example. An example is a parking lot with only one working station and there is no waiting space.

Model Analysis. In this case, the arrival and service rates are:

$$\lambda_n = \begin{cases} \lambda & n = 0 \\ 0 & n = 1 \end{cases}$$

$$\mu_n = \begin{cases} 0 & n = 0 \\ \mu & n > 0 \end{cases}$$

Hence, from Equations (4) and (5) we have

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_1(t) \quad (8)$$

and

$$\frac{dP_1(t)}{dt} = \lambda P_0(t) - \mu P_1(t) \quad (9)$$

Case B: Infinite Source and Limited Queue. In this case, the system capacity is limited to N, that is, the maximum number of jobs allowed in the system is N. Hence, the maximum length of the queue is N - 1. A job is turned away if there are N jobs in the system.

Practical Example. A gas station which has one pump unit and there is a limited space for the cars to wait if the pump is

serving a car.

Model Analysis. In this case, the arrival and service rates are:

$$\lambda_n = \begin{cases} \lambda & 0 \leq n < N \\ 0 & n = N \end{cases}$$

$$\mu_n = \begin{cases} \mu & n > 0 \\ 0 & n = 0 \end{cases}$$

Hence, from Equations (1), (2), and (3)

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_1(t) \quad (10)$$

$$\frac{dP_n(t)}{dt} = \lambda P_{n-1}(t) + \mu P_{n+1}(t) - (\lambda + \mu)P_n(t), \quad (11)$$

$$1 \leq n < N$$

$$\frac{dP_N(t)}{dt} = \lambda P_{N-1}(t) - \mu P_N(t) \quad (12)$$

Case C: Infinite Source and Unlimited Queue. In this case, the capacity of the system is unlimited, that is, any number of jobs can queue before a machine for processing.

Practical Example. An example is a special purpose machine, where there is ample space so that a large number of jobs may wait. Another example is a toll booth near a turnpike where any number of cars have to wait for paying the toll before proceeding.

Model Analysis. In this case, the arrival and service rates are:

$$\lambda_n = \lambda \quad n \geq 0$$

$$\mu_n = \begin{cases} \mu & n > 0 \\ 0 & n = 0 \end{cases}$$

Hence, from Equations (6) and (7) we have

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_1(t) \quad (13)$$

and

$$\frac{dP_n(t)}{dt} = \lambda P_{n-1}(t) + \mu P_{n+1}(t) - (\lambda + \mu)P_n(t), \quad (14)$$

$$n > 0$$

Case D: Finite Source and Limited Queue. In this case, the arrival rate is dependent on the state of the system. It is intuitively obvious that the queue is limited.

Practical Example. An example is the repairing of a group of machines, assigned to a single repairman. The machines under breakdown are the jobs and the repairman is the single server. One repairman is assigned to a finite set of M machines. If a machine breaks down while a repairman is idle, it will be served immediately. But if the repairman is busy, the machine waits in the queue. The machine time lost due to waiting for the repairman or the operator is called machine interference. The main aim of a queueing model of this type is to estimate the effect of machine interference on the system's production rate.

Model Analysis. In this case, the number of running machines form the input source. The job arrivals are those which have broken down. Due to the finite number of machines, the breakdown rate, that is, the arrival rate depends on the number of working machines.

The probability that a breakdown will occur is the probability that any one of the machines that are working now has a breakdown. Let there be k machines working at t . Any of the k machines may breakdown, while others will be working. Let the mean breakdown rate be λ per unit time. The probability of any one machine breaking down during small interval Δt is

$$\begin{aligned} &= k(\lambda \Delta t + o(\Delta t))(1 - \lambda \Delta t + o(\Delta t))^{k-1} \\ &= k\lambda \Delta t [1 - (k-1)\lambda \Delta t + o(\Delta t)] + o(\Delta t) \\ &= k\lambda \Delta t + o(\Delta t) \end{aligned}$$

If there are n machines under breakdown out of M , then

$$k = M - n$$

Hence, the probability of a further breakdown when there are n machines already under breakdown, out of M , is $(M - n)\lambda$.

Hence, the arrival and service rates are:

$$\begin{aligned} \lambda_n &= \begin{cases} (M-n)\lambda & 0 \leq n \leq M \\ 0 & n \geq M \end{cases} \\ \mu_n &= \begin{cases} \mu & n > 0 \\ 0 & n = 0 \end{cases} \end{aligned}$$

From Equations (1), (2), and (3), we have

$$\frac{dP_0(t)}{dt} = -M\lambda P_0(t) + \mu P_1(t) \quad (15)$$

$$\begin{aligned} \frac{dP_n(t)}{dt} &= (M - n + 1) P_{n-1}(t) - [(M - n)\lambda + \mu] \\ &\quad \cdot P_n(t) + \mu P_{n+1}(t), \quad 1 \leq n < M \end{aligned} \quad (16)$$

$$\frac{dP_M(t)}{dt} = \lambda P_{M-1}(t) - \mu P_M(t) \quad (17)$$

Having developed the various types of the single machine queueing models, we shall discuss the multiple machine shop models.

3.3 Multiple-Machine Shop Models

The multiple-machine shop queueing models will be developed. They can be classified into two categories: (1) Parallel-Machine shop models and (2) Serial-Machine shop models.

Parallel-Machine Shop Models.

The Parallel-Machine shop process is a class of job-shop process in which there is only one work center consisting of identical machines. Parallel-Machine process is shown in Figure 3.3. When a job arrives for processing, it has three choices in joining the system: (1) If the machines are busy, it joins the single queue. If no queue is allowed before the machine, it is turned away; (2) If a machine is idle, the job can be processed on that machine; and (3) If there are two or more idle machines of the same type, the job can be processed on any one of these machines. Consider the total number of machines is M .

Case A: Infinite Source and No Queue. As no queue is allowed, the system capacity is limited to the number of machines, that is, maximum M jobs may be in the system at any time.

Practical Example. An example is a parking lot where there are M parking spaces and if all the parking spaces are full, an arriving car is lost.

Model Analysis. In this case, the arrival and service rates are:

$$\lambda_n = \begin{cases} \bar{\lambda} & 0 \leq n < M \\ 0 & n \geq M \end{cases}$$

$$\mu_n = \begin{cases} n\mu & 0 \leq n \leq M \\ M\mu & n \geq M \end{cases}$$

Hence, from Equations (1), (2), and (3) we have,

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_1(t) \quad (18)$$

$$\begin{aligned} \frac{dP_n(t)}{dt} = & -(\lambda + n\mu)P_n(t) + \lambda P_{n-1}(t) + (n+1) \\ & \cdot \mu P_{n+1}(t), \quad 1 \leq n < M \end{aligned} \quad (19)$$

$$\frac{dP_M(t)}{dt} = -M\mu P_M(t) + \lambda P_{M-1}(t) \quad (20)$$

Case B: Infinite Source and Limited Queue. In this model, there is a limitation upon the number of jobs which may be waiting before the machines because of limited storage space. Thus, the queue size is limited. If the maximum system capacity is N jobs, the waiting space is only for $N - M$ jobs. If all the machines are busy when a job arrives, it will join the queue feeding all the machines, provided the number of jobs in the system is N . If the number of jobs in the system is already N , the arriving job will not join the queue but will be turned away.

Practical Example. An example is a work center, having identical machines and where the waiting space for the jobs is limited.

Model Analysis. In this case, the arrival and service rates are:

$$\lambda_n = \begin{cases} \lambda & 0 \leq n < N \\ 0 & n \geq N \end{cases}$$

$$\mu_n = \begin{cases} n\mu & 0 \leq n \leq M \\ M\mu & n \geq M \end{cases}$$

From Equations (1), (2), and (3), we get the set of difference-differential equations. Equation (2) is considered when $1 \leq n < M$ and also when $M \leq n < N$

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_1(t) \quad (21)$$

$$\frac{dP_n(t)}{dt} = \lambda P_{n-1}(t) - (\lambda + n\mu)P_n(t) + (n+1)\mu P_{n+1}(t) \quad 1 \leq n < M \quad (22)$$

$$\frac{dP_n(t)}{dt} = \lambda P_{n-1}(t) - (\lambda + M\mu)P_n(t) + M\mu P_{n+1}(t), \quad M \leq n < N \quad (23)$$

$$\frac{dP_N(t)}{dt} = \lambda P_{N-1}(t) - M\mu P_N(t) \quad (24)$$

An application of this model has been made by Taylor and Jackson [28] to the provision of spare aircraft engines. Sometimes for a group of working machines, standby machines are placed in parallel-machine process. Suppose there are M standby machines for a group of r machines. If any one machine breaks down, the operation is switched over to any standby machine. So long as not more than M machines are under break down, full production rates are maintained. Once $M + 1$ or more machines break

down, the production rate of the system is reduced proportionately. Let there be s repairmen to repair the broken machines. Let P_{M+1} be the probability of production emergency as the production rate of the system is reduced as soon as $M + 1$ machines break down. Management might be interested to find out the probability of production emergency and to find out what combination of spare machines and repairmen will reduce the probability of production emergency to a desired level. The M machines break down at random. The time between breakdowns and the service time for the repair follow exponential distribution. As there are s repairmen, s machines can be repaired simultaneously. All machines are assumed similar as to the average number of break downs per unit of working time. The mean break down rate is considered to be independent of the number of machines working.

In this case, the machines under break down form the arriving jobs and s repairmen are the servers.

$$\lambda_n = \begin{cases} \lambda & 0 \leq n < M \\ 0 & n \geq M \end{cases}$$

$$\mu_n = \begin{cases} n\mu & 0 \leq n \leq s \\ s\mu & n \geq s \end{cases}$$

From Equations (21), (22), (23), and (24).

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_1(t) \quad (25)$$

$$\begin{aligned} \frac{dP_n(t)}{dt} = & \lambda P_{n-1}(t) - (\lambda + n\mu)P_n(t) + (n+1) \\ & \cdot \mu P_{n+1}(t), \quad 1 \leq n < s \end{aligned} \quad (26)$$

$$\frac{dP_n(t)}{dt} = \lambda P_{n-1}(t) - (\lambda + s\mu)P_n(t) + s\mu P_{n+1}(t)$$

$$s \leq n \leq M \quad (27)$$

$$\frac{dP_{M+1}(t)}{dt} = \lambda P_M(t) - s\mu P_{M+1}(t) \quad (28)$$

Case C: Infinite Source and Unlimited Queue. Unlimited queue size signifies that the probability of an arriving job being lost is negligibly small. This probability can be determined by management. As soon as a machine becomes idle, the job which has been waiting longest is loaded on the machine instantaneously. If there are no jobs in the system, the machines remain idle until some jobs arrive. If one job arrives when all machines are idle, it joins any one of the machines and the remaining $M - 1$ machines remain idle. If two jobs come one after the other, two machines will be busy and the rest of them will be idle. In this manner, when there are M or more jobs in the system, all the M machines will be busy. If the number of jobs n is greater than M , exactly M jobs will be processed and $n - M$ jobs will be waiting for processing.

Practical Example. An example is a job-shop consisting of one work center having identical machines and where there is ample space for waiting for a large number of jobs.

Model Solution. In this case, arrival and service rates are:

$$\lambda_n = \lambda \quad n \geq 0$$

$$\mu_n = \begin{cases} n\mu & 0 \leq n \leq M \\ M\mu & n \geq M \end{cases}$$

From Equations (6), (7), and (8)

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_1(t) \quad (29)$$

$$\begin{aligned} \frac{dP_n(t)}{dt} = & -(\lambda + n\mu)P_n(t) + \lambda P_{n-1}(t) + (n+1) \\ & \cdot \mu P_{n+1}(t) \quad 1 \leq n < M \end{aligned} \quad (30)$$

and

$$\begin{aligned} \frac{dP_n(t)}{dt} = & -(\lambda + M\mu)P_n(t) + \lambda P_{n-1}(t) + M\mu \\ & \cdot P_{n+1}(t), \quad n \geq M \end{aligned} \quad (31)$$

Case D: Finite Source and Limited Queue. In this case, as the source of the jobs is finite, the arrival rate of jobs depends on the state of the system and, due to the source being finite, the queue size is also limited.

Practical Example. An example is a group of repairmen who are assigned to look after the maintenance of a finite set of M machines. Estimate of the effect of machine interference on the system production rate can be made. Also, the right combination of repairmen or operatives and machines to improve the machine utilization can be determined.

Model Analysis. In this case, the arrival and service rates are:

$$\lambda_n = \begin{cases} (M - n)\lambda & 0 \leq n \leq M \\ 0 & n \geq M \end{cases}$$

$$\mu_n = \begin{cases} n\mu & 0 \leq n \leq s \\ s\mu & n \geq s \end{cases}$$

s is the number of repairmen, n is the number of machines under break down. From Equations (1), (2), and (3)

$$\frac{dP_0(t)}{dt} = -M\lambda P_0(t) + \mu P_1(t) \quad (32)$$

$$\begin{aligned} \frac{dP_n(t)}{dt} = & (M - n + 1)\lambda P_{n-1}(t) - ((M - n)\lambda + n\mu) \\ & \cdot P_n(t) + (n + 1)\mu P_{n+1}(t), \quad 1 \leq n < s \end{aligned} \quad (33)$$

$$\begin{aligned} \frac{dP_n(t)}{dt} = & (M - n + 1)\lambda P_{n-1}(t) - ((M - n)\lambda + s\mu) \\ & \cdot P_n(t) + s\mu P_{n+1}(t), \quad s \leq n < M \end{aligned} \quad (34)$$

and

$$\frac{dP_M(t)}{dt} = \lambda P_{M-1}(t) - s\mu P_M(t) \quad (35)$$

The detailed derivation of the equations can be found in Ashour [1]

Serial-Machine Shop Model

As discussed earlier, each job has the same machine ordering in a serial-machine shop system.

Practical Example. An example is the flow-shop for manufacturing crankshafts or cylinder blocks for automobile engines, where special purpose machines are arranged in a serial order to process the jobs.

Model Analysis. For the sake of simplicity, let us consider the case of two machines arranged in series. The job after completing its operation on machine 1, moves to machine 2 for processing. The specifications of the model are:

Arrival Source	: Infinite
Arrival Pattern	: Single
Arrival Behavior	: Patient
Arrival Distribution	: Poisson
Queue Size	: Unlimited before both phases
Queue Discipline	: First come, first processed, or Last come, first processed, or Random selection
Processing Pattern	: Single
Processing Mechanism	: Two-phase
Processing Time Distribution:	Exponential in both phases
Number of Servers	: One in each phase

The following notation is used:

$P_{0,0}(t)$	Probability that there is no job in phase 1 and phase 2 at time t .
$P_{n_1,n_2}(t)$	Probability that there are n_1 jobs in phase 1 and n_2 jobs in phase 2, at time t .
λ	Arrival rate of jobs in phase 1.
μ_1	Processing rate of jobs on machine 1.
μ_2	Processing rate of jobs on machine 2

Next, a set of difference-differential equations for $P_{n_1,n_2}(t)$ will be formulated by using the fundamental properties of probability. In enumerating all the possibilities in which there will be n_1 jobs in phase 1 and n_2 jobs in phase 2 at time $t + \Delta t$,

the probability of n_1 jobs in phase 1 and n_2 jobs in phase 2 at time $t + \Delta t$, may be expressed as follows: (for $n_1 = 0, n_2 = 0$)

$$\begin{aligned} P_{0,0}(t + \Delta t) &= P_{0,0}(t)(1 - \lambda\Delta t + o(\Delta t))(1) + P_{0,1}(t) \\ &\quad \cdot (1 - \lambda\Delta t + o(\Delta t))(1)(\mu_2\Delta t + o(\Delta t)) \\ &\quad + o(\Delta t) \\ &= P_{0,0}(t)(1 - \lambda\Delta t) + P_{0,1}(t)\mu_2\Delta t + o(\Delta t) \end{aligned}$$

for $n_1 = 0, n_2 > 0$,

$$\begin{aligned} P_{0,n_2}(t + \Delta t) &= P_{0,n_2}(t)(1 - \lambda\Delta t + o(\Delta t))(1) \\ &\quad \cdot (1 - \mu_2\Delta t + o(\Delta t)) + P_{0,n_2+1}(t) \\ &\quad \cdot (1 - \lambda\Delta t + o(\Delta t))(1)(\mu_2\Delta t + o(\Delta t)) \\ &\quad + P_{1,n_2-1}(t)(1 - \lambda\Delta t + o(\Delta t))(\mu_1\Delta t \\ &\quad + o(\Delta t))(1 - \mu_2\Delta t + o(\Delta t)) + o(\Delta t) \\ &= P_{0,n_2}(t)(1 - (\lambda + \mu_2)\Delta t) + P_{0,n_2+1}(t) \\ &\quad \cdot \mu_2\Delta t + P_{1,n_2-1}(t)\mu_1\Delta t + o(\Delta t) \end{aligned}$$

Similarly for $n_1 > 0, n_2 = 0$,

$$\begin{aligned} P_{n_1,0}(t + \Delta t) &= P_{n_1,0}(t)(1 - (\lambda + \mu_1)\Delta t) + P_{n_1,1}\mu_2\Delta t \\ &\quad + P_{n_1-1,0}(t)\lambda\Delta t + o(\Delta t) \end{aligned}$$

Finally for $n_1 > 0, n_2 > 0$,

$$\begin{aligned} P_{n_1,n_2}(t + \Delta t) &= P_{n_1,n_2}(t)(1 - \lambda\Delta t + o(\Delta t))(1 - \mu_1\Delta t) \\ &\quad + o(\Delta t))(1 - \mu_2\Delta t + o(\Delta t)) \end{aligned}$$

$$\begin{aligned}
& + P_{n_1+1, n_2-1}(t)(1 - \lambda \Delta t + 0(\Delta t))(\mu_1 \Delta t) \\
& + 0(\Delta t))(1 - \mu_2 \Delta t + 0(\Delta t)) + P_{n_1, n_2+1} \\
& \cdot (t)(1 - \lambda \Delta t + 0(\Delta t))(1 - \mu_1 \Delta t) \\
& + 0(\Delta t))(\mu_2 \Delta t + 0(\Delta t)) + P_{n_1-1, n_2}(t) \\
& \cdot (\lambda \Delta t + 0(\Delta t))(1 - \mu_1 \Delta t + 0(\Delta t)) \\
& \cdot (1 - \mu_2 \Delta t + 0(\Delta t)) + 0(\Delta t) \\
& = (1 - (\lambda + \mu_1 + \mu_2) \Delta t) P_{n_1, n_2}(t) + \mu_1 \\
& \cdot P_{n_1+1, n_2-1}(t) + \mu_2 P_{n_1, n_2+1}(t) + \lambda \\
& \cdot P_{n_1-1, n_2}(t) + 0(\Delta t)
\end{aligned}$$

Rearranging the above equations, dividing by Δt , and taking the limit of both sides as $\Delta t \rightarrow 0$, we get

$$\frac{dP_{0,0}(t)}{dt} = -\lambda P_{0,0}(t) + \mu_2 P_{0,1}(t) \quad (36)$$

$$\begin{aligned}
\frac{dP_{0,n_2}(t)}{dt} &= \mu_1 P_{1,n_2-1}(t) - (\lambda + \mu_2) P_{0,n_2}(t) \\
&+ \mu_2 P_{0,n_2+1}(t), \quad n_1 = 0, n_2 > 0 \quad (37)
\end{aligned}$$

$$\begin{aligned}
\frac{dP_{n_1,0}(t)}{dt} &= \lambda P_{n_1-1,0}(t) + \mu_2 P_{n_1,1}(t) - (\lambda + \mu) \\
&\cdot P_{n_1,0}(t), \quad n_1 > 0, n_2 > 0 \quad (38)
\end{aligned}$$

$$\begin{aligned}
\frac{dP_{n_1,n_2}(t)}{dt} &= \lambda P_{n_1-1,n_2}(t) - (\lambda + \mu_1 + \mu_2) P_{n_1,n_2}(t) \\
&+ \mu_2 P_{n_1,n_2+1}(t) + \mu_1 P_{n_1+1,n_2-1}(t),
\end{aligned}$$

$$n_1 > 0, n_2 > 0 \quad (39)$$

The steady-state solution is

$$P_{n_1, n_2} = \rho_1^{n_1} \rho_2^{n_2} P_{0,0}$$

where

$$\rho_1 = \lambda/\mu_1$$

and

$$\rho_2 = \lambda/\mu_2$$

Also

$$\sum_{n_2=0}^{\infty} \sum_{n_1=0}^{\infty} P_{n_1, n_2} = 1$$

$$\sum_{n_2=0}^{\infty} \sum_{n_1=0}^{\infty} \rho_1^{n_1} \rho_2^{n_2} P_{0,0} = 1$$

$$P_{0,0} \sum_{n_2=0}^{\infty} \rho_2^{n_2} \sum_{n_1=0}^{\infty} \rho_1^{n_1} = 1$$

$$P_{0,0} \cdot \frac{1}{1-\rho_2} \cdot \frac{1}{1-\rho_1} = 1$$

or

$$P_{0,0} = (1 - \rho_1)(1 - \rho_2)$$

Hence,

$$P_{n_1, n_2} = (1 - \rho_1) \rho_1^{n_1} (1 - \rho_2) \rho_2^{n_2}$$

$$P_{n_1} = \sum_{n_2=0}^{\infty} P_{n_1, n_2}$$

$$\begin{aligned}
&= (1 - \rho_1) \rho_1^{n_1} \sum_{n_2=0}^{\infty} (1 - \rho_2) \rho_2^{n_2} \\
&= (1 - \rho_1) \rho_1^{n_1}
\end{aligned}$$

Similarly,

$$P_{n_2} = \rho_2^{n_2} (1 - \rho_2)$$

Thus, we see that

$$P_{n_1, n_2} = P_{n_1} P_{n_2}$$

It shows that the two phases act independently; each phase has a mean arrival rate of jobs equal to λ which is the mean arrival rate of jobs to the system. The idea can be extended to a flow shop consisting of M phases, each phase consisting of one machine. Thus, each of the M phases could be considered as independent queueing systems and the measures of performance can be computed by finding the measures of performance of individual phases. For the flow shop consisting of M phases,

$$P_{n_1, n_2, \dots, n_M} = \prod_{m=1}^M \rho_m^{n_m} (1 - \rho_m)$$

and

$$\begin{aligned}
P_{n_m} &= \rho_m^{n_m} (1 - \rho_m), \quad m = 1, 2, \dots, M, \\
\rho_m &= \lambda / \mu_m
\end{aligned}$$

3.4 Hybrid-Machine Shop Models

As mentioned earlier, the hybrid-machine shop systems depend on the type of the shop. In the case of the hybrid flow-shop

system, the machine ordering of all jobs is unidirectional; but in the case of the hybrid job-shop system, the machine ordering of any job is independent of the other jobs.

Hybrid Flow-Shop Model

This model was developed and solved in the steady state by Jackson [13].

Practical Example. A typical example is the overhaul shop for aircraft engines, where there are many work centers arranged in series, each consisting of identical machines. Consider, for example, five work centers. Work centers 1, 2, 3, 4, and 5 are for stripping engines, detailed examination, repairs, assembly, and testing.

Model Analysis. Let M be the number of work centers arranged in series, each consisting of r_m identical machines, where $m = 1, 2, \dots, M$. Each work center is referred to as one phase. Each job is processed in phase 1 first, phase 2 second, ..., and phase M last. If a job arrives at a phase and finds all machines busy, it joins a single queue. The machines are loaded according to first-come, first-processed discipline. In-process inventory is allowed in each work center. The arrival of a job at the first work center follows the Poisson distribution with arrival rate λ . The processing time distribution of any job on a machine is exponential with mean $1/\mu_m$ units of time. Let n_m be the number of jobs in phase m . Hence, in this case, the arrival and service rates are:

$$\lambda_{n_m} = \lambda, \quad n_m \geq 0$$

$$\mu_{n_m} = \begin{cases} \bar{n}_m \mu_m & 0 \leq n_m \leq r_m \\ r_m \mu_m & n_m \geq r_m \end{cases}$$

Next, a set of difference-differential equations for the probability of n_1 jobs in the work center 1, n_2 in work center 2, and ..., n_m jobs in work center M, at time t , is formulated using the fundamental properties of probability. So, enumerating all the possibilities during the small time interval Δt , the probability of the above state at time $t + \Delta t$ may be expressed as follows:

$$P_{n_1, \dots, n_m}(t + \Delta t) = [1 - \{\lambda + \sum_{m=1}^M \delta(n_m) \alpha(n_m) \mu_m\} \Delta t$$

$$\cdot P_{n_1, \dots, n_m}(t) + \sum_{m=1}^M \delta(n_m + 1)$$

$$\cdot \alpha(n_m + 1) \mu_m \Delta t P_{n_1, \dots, n_m+1, \dots}$$

$$n_{m+1}-1, n_{m+2}, \dots, n_M(t) + \lambda \Delta t$$

$$\cdot P_{n_1-1, n_2, \dots, n_M}(t) + O(\Delta t)$$

where for m such that $m = 1, 2, \dots, M$

$$\alpha(n_m) = \begin{cases} \bar{n}_m & n_m \leq r_m \\ r_m & n_m \geq r_m \end{cases}$$

$$\delta(n_m) = \begin{cases} 1 & n_m \neq 0 \\ 0 & n_m = 0 \end{cases}$$

$$\delta(n_{m+1}) = 1$$

Rearranging the above equation, dividing both sides by t , and taking the limit of both sides as $t \rightarrow 0$, we get

$$\begin{aligned} \frac{d}{dt} P_{n_1, n_2, \dots, n_M}(t) = & -[\lambda + \sum_{m=1}^M \delta(n_m) \alpha(n_m) \mu_m] P_{n_1, n_2, \dots, n_M}(t) \\ & + \sum_{m=1}^M \delta(n_m + 1) \alpha(n_m + 1) \mu_m \\ & P_{n_1, \dots, n_m + 1, n_m + 1 - 1, n_{m+2}, \dots, n_M}(t) + \lambda \\ & P_{n_1 - 1, n_2, \dots, n_M}(t) \end{aligned}$$

where all the probabilities containing a negative value of n are put equal to zero.

The steady-state solution is

$$P_{n_1, n_2, \dots, n_M} = P_{0, 0, \dots, 0} \prod_{m=1}^M \beta(n_m)$$

where

$$\beta(n_m) = \begin{cases} \frac{1}{n_m!} \left(\frac{\lambda}{\mu_m}\right)^{n_m} & 0 \leq n_m < r_m \\ \frac{1}{r_m!} \left(\frac{\lambda}{\mu_m}\right)^{r_m} \left(\frac{\lambda}{\mu_m r_m}\right)^{n_m - r_m} & n_m \geq r_m \end{cases}$$

Summing over all values of n_m , we have

$$\begin{aligned} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \dots \sum_{n_M=0}^{\infty} [P_{0, 0, \dots, 0} \prod_{m=1}^M \beta(n_m)] \\ = P_{0, 0, \dots, 0} \prod_{m=1}^M \left[\sum_{n_m=0}^{\infty} \beta(n_m) \right] \\ = 1 \end{aligned}$$

or

$$P_{0,0,\dots,0} = 1 / \prod_{m=1}^M \left[\sum_{n_m=0}^{\infty} \beta(n_m) \right]$$

Again, the solution shows that each phase consisting of r_m machines behaves as an independent queueing system. Hence, the hybrid flow-shop system can be analyzed by considering each phase separately as a parallel job-shop model. The measures of performance of the system can be computed from the measures of performance of individual phases. From the solution, it is clear also that arrival rate of jobs in each phase is equal to λ , which is the arrival rate in phase 1. Thus, the job arrivals in each phase follow the Poisson distribution with mean λ .

This type of system is referred to as a multi-channel, multi-phase system. By considering $r_m = 1, m = 1, 2, \dots, M$, we can derive the steady state solution for the serial flow-shop model. In that case,

$$P_{n_1, n_2, \dots, n_M} = P_{0,0,\dots,0} \prod_{m=1}^M (\lambda / \mu_m)^{n_m}$$

where

$$P_{0,0,\dots,0} = \prod_{m=1}^M (1 - \lambda / \mu_m)$$

Hybrid Job-Shop Model

This model has also been developed by Jackson [11]. A hybrid job shop consists also of many work centers; a particular work center has similar machines. Jobs arrive at the shop or from another work center in the system with independent routing according

to the prescribed machine ordering. Note that the hybrid job-shop system differs only from the hybrid flow-shop system in the type of machine ordering of the jobs.

Practical Example. An example is the repair shop of a large production plant. There are many work centers: work center 1 for turning, work center 2 for milling, work center 3 for drilling, work center 4 for fitting and assembly, etc. Machine ordering of any job is independent of the other jobs.

Model Analysis. Let M be the number of work centers each having a group of r_m similar machines. Consider that the arrival of jobs to the system follows the Poisson distribution with mean λ_m , where $m = 1, 2, \dots, M$. The processing time of a job on any machine in a work center is exponentially distributed with mean $1/\mu_m$. The queue discipline in any work center is first-come, first-processed. Once processed in work center m , a job goes to work center k , $k \neq m$, according to the prescribed machine ordering, with probability θ_{mk} ; the probability of the processing being complete is

$$1 - \sum_{k=1}^M \theta_{mk}.$$

Let γ_m be the arrival rate of jobs at work center m from any other work center or from outside the system. Then, for the steady state,

$$\gamma_m = \lambda_m + \sum_{k=1}^M \theta_{km} \gamma_k$$

Let $P_{n_1, n_2, \dots, n_m}(t)$ be the probability of n_1 jobs in work center

1, n_2 in work center 2, ..., and n_M in work center M at time t . Next, a set of difference-differential equations for the probability of n_1 jobs in work center 1, n_2 in work center 2, ..., n_M jobs in work center M at time t is formulated using the fundamental properties of probability. So, enumerating all the possibilities during small time interval Δt , the probability of the above state at time $t + \Delta t$ may be expressed as follows:

$$\begin{aligned}
 P_{n_1, \dots, n_M}(t + \Delta t) = & [1 - \{ \sum_{m=1}^M \lambda_m + \sum_{m=1}^M \alpha_m(n_m) \mu_m \} \Delta t] \\
 & \cdot P_{n_1, \dots, n_M}(t) + [\sum_{m=1}^M \alpha_m(n_m + 1) \\
 & \cdot \mu_m (1 - \sum_{k=1}^M \theta_{km})] \Delta t \\
 & \cdot P_{n_1, \dots, n_m+1, n_{m+1}, \dots, n_M}(t) \\
 & + (\sum_{m=1}^M \lambda_m \delta_m) \Delta t \cdot P_{n_1, \dots, n_m-1, n_{m+1}, \\
 & \dots, n_m}(t) + [[\sum_{j=1}^M \sum_{m=1}^M \alpha_j(n_j + 1) \mu_j \theta_{jm}] \\
 & \cdot \Delta t \cdot P_{n_1, \dots, n_j+1, \dots, n_m-1, \dots, \\
 & n_M}(t) + 0(\Delta t)
 \end{aligned}$$

where

$$\alpha_m(n_m) = \min(n_m; r_m)$$

and

$$\delta_m = \min(n_i; 1)$$

Rearranging the above equations, dividing by Δt , and taking the limit of both sides as $\Delta t \rightarrow 0$, we get

$$\begin{aligned} \frac{d}{dt} P_{n_1, \dots, n_M}(t) = & - \left\{ \sum_{m=1}^M \lambda_m + \sum_{m=1}^M \alpha_m(n_m) \mu_m \right\} \\ & \cdot P_{n_1, \dots, n_M}(t) + \sum_{m=1}^M \alpha_m(n_m + 1) \mu_m \\ & \cdot \left(1 - \sum_{k=1}^M \theta_{km} \right) \cdot P_{n_1, \dots, n_m+1, \dots, n_M} \\ & \cdot (t) + \left(\sum_{m=1}^M \lambda_m \delta_m \right) P_{n_1, \dots, n_m-1, \dots, n_M} \\ & \cdot (t) + \sum_{j=1}^M \sum_{m=1}^M \alpha_j(n_j + 1) \mu_j \theta_{jm} \\ & P_{n_1, \dots, n_j+1, \dots, n_m-1, \dots, n_M}(t) \end{aligned}$$

In steady state case, $\gamma_m / \mu_m r_m < 1$, we have

$$P_{n_1, n_2, \dots, n_M} = P_{n_1}^1 P_{n_2}^2 \dots P_{n_M}^M$$

where

$$P_n^m = \begin{cases} P_0^m (\gamma_m / \mu_m)^n / n! & 0 \leq n \leq r_m, \quad 1 \leq m \leq M \\ P_0^m (\gamma_m / \mu_m)^n / r_m! (r_m)^{n-r_m} & n \geq r_m \end{cases}$$

This result also shows that the system behaves in the steady state as if each work center were an independent queueing system.

3.5 A Special Shop Queueing Model

Consider a machine-shop, consisting of three work centers: A, B, and C, as shown in Figure 3.5. Machines in work center A

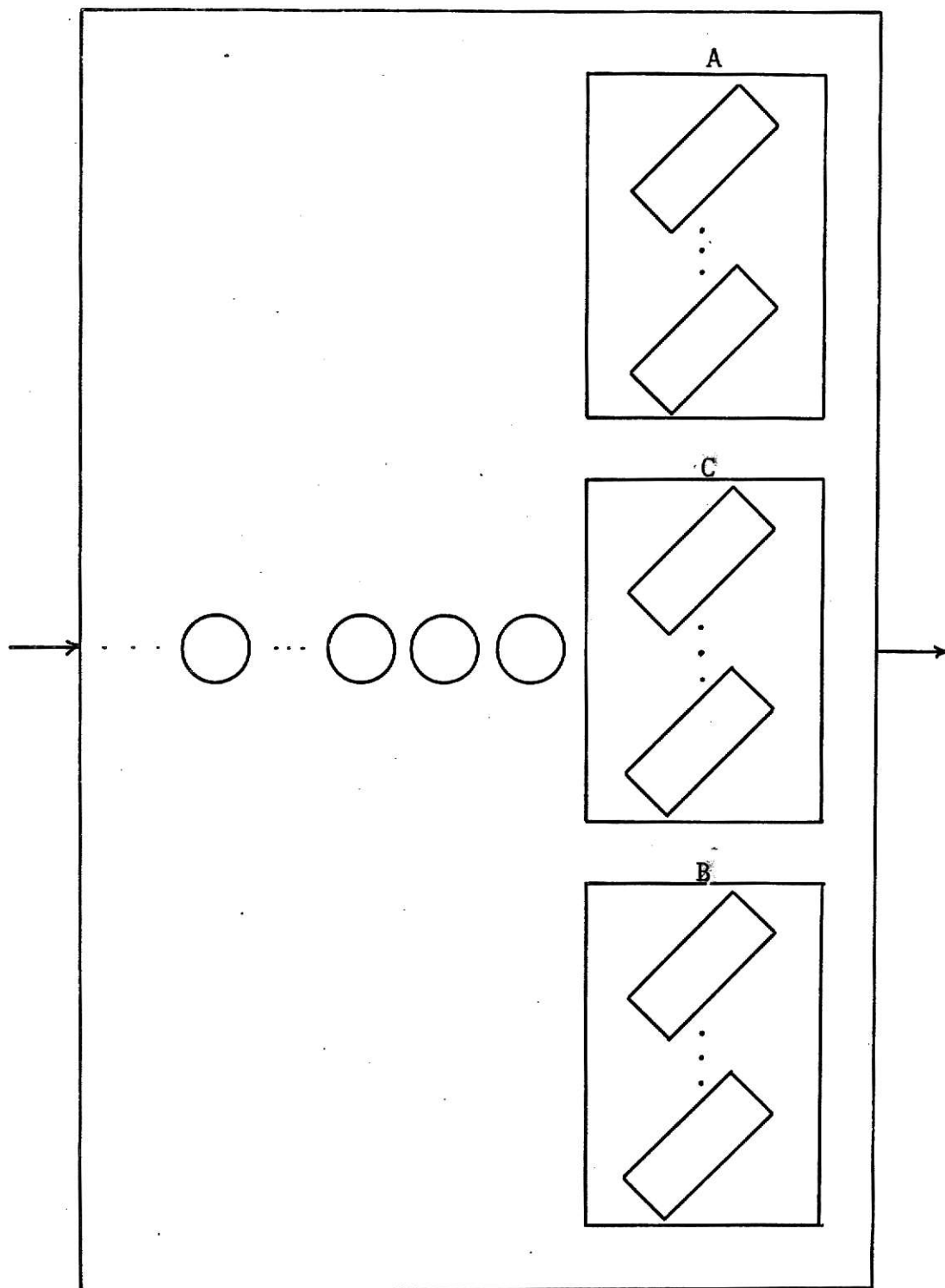


Figure 3.5. Special Machine Shop Process.

are similar and a job can be processed on any one of the machines in this work center. Machines in work center B are different from those of A, hence, a different type of operation can be performed on any of the machines in work center B. Jobs for work center A are different from jobs for work center B. It is assumed that job arrivals and processing times for both work center A and work center B, behave as if the jobs originate from the same process. Machines in work center C can process both the type of jobs arriving at work center A or the type of jobs arriving at work center B. The machines in work center C can perform the operation on any job that requires processing on one of the machines in work center A or B. It is assumed that the source of arrival of jobs is infinite and no queue is allowed.

Let there be M_a , M_b , and M_c machines in work centers A, B, and C, respectively. Also let M denote the total number of machines in all the work centers. Thus, the number of available machines in work center A may be considered as $M_a + M_c$ and similarly in work center B as $M_b + M_c$. But, in place of providing $M_a + M_b + 2M_c$ machines, only $M_a + M_b + M_c$ machines are in the system. So, M_c machines can be used by both the groups A and B. The maximum number of jobs in the system at any time is M as there are only M machines and no queue is allowed.

Practical Example. Consider a job-shop process in which work center consists of M_a drilling machines, work center B consists of M_b turning machines, and work center C consists of M_c general purpose machines which can process both types of jobs—

ones which require only drilling and ones which require only turning operations.

Model Analysis. It is assumed that there are equal numbers of machines in work center A and work center B.

$$\begin{aligned}\text{Let } m &= M_a + M_c \\ &= M_b + M_c\end{aligned}$$

When the number of busy machines is less than m , an arriving job will always find an idle machine. Hence, the probability of transition from state n to $n+1$ during a small interval of time Δt is $\lambda \Delta t$. When $n \geq m$, a job for a particular work center will get an idle machine only when the m machines are accessible to this work center are not all busy. It is assumed that all the machines are equally likely to be busy. Hence, the probability β_n that all m machines accessible to that work center will be busy at any one time, when there are n busy machines among M , is given by the hypergeometric probability function such that

$$\begin{aligned}\beta_n &= \frac{\binom{n}{m} \binom{M-n}{0}}{\binom{M}{m}} \\ &= \frac{\binom{n}{m}}{\binom{M}{m}}\end{aligned}$$

Hence, when $n \geq m$, the probability of transition from state, n to $n + 1$, busy machines is $\lambda(1 - \beta_n)\Delta t$
when $n < m$, $\beta_n = 0$

Thus, the arrival and service rates are:

$$\lambda_n = \begin{cases} \lambda & n < m \\ \lambda(1 - \beta_n) & m \leq n < M \\ 0 & n \geq M \end{cases}$$

$$\mu_n = n\mu \quad n \geq 0$$

From Equations (1), (2), and (3)

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_1(t)$$

$$\frac{dP_n(t)}{dt} = \lambda P_{n-1}(t) - (\lambda + n\mu)P_n(t) + (n+1)\mu P_{n+1}(t)$$

$$1 \leq n < m$$

$$\begin{aligned} \frac{dP_n(t)}{dt} = & \lambda(1 - \beta_{n-1})P_{n-1}(t) - (\lambda(1 - \beta_n) + n\mu)P_n(t) \\ & + (n+1)\mu P_{n+1}(t), \quad m \leq n < M \end{aligned}$$

$$\frac{dP_M(t)}{dt} = \lambda(1 - \beta_{M-1})P_{M-1}(t) - M\mu P_M(t)$$

This type of model was used for studying the problem of limited availability in telephone traffic systems, details of which can be found in Syski [27].

CHAPTER IV

NUMERICAL RESULTS OF QUEUEING MODELS

As has been discussed in Chapter I, the study of queueing models help make scientific decisions for stochastic service systems by predicting their behavior. For example, in a parallel machine-shop process, the decision whether to buy new machines in order to reduce the in-process inventory may be based on assessing the annual profit from additional production obtained due to reduction in mean flow time of jobs and comparing this with the capital cost of the new machines. Mean flow time of jobs can be computed by the existing and proposed arrangements from the study of the queueing models. Sometimes, the mean waiting time of jobs may be small. On the other hand, it may be too large. In such cases, the knowledge of mean waiting time of jobs when there is a non-empty queue will help decide whether to buy new machines. The knowledge of the probability of a lost job will help decide whether to increase the waiting space for jobs or to buy new machines to increase the processing rate by comparing the business lost due to jobs turned away and the cost of new machines. The probability of a lost job may be reduced by increasing the waiting space, but this will increase the mean flow time of jobs. By reducing the system capacity, the mean flow time may be reduced, but the probability of lost jobs will rise. An increase in the number of existing machines will reduce the mean flow time and, hence, will reduce the probability of a lost job. It may be

cheaper, sometimes, to provide more waiting space than to increase the number of machines. Generally, an economic balance is to be maintained between the system capacity and the number of machines.

In practical situations, for decision making, many measures of performance which enable comparisons to be made among various alternatives, are to be computed. For computing the various measures of performance, we have to compute the probabilities by solving the difference-differential equations, describing the state probabilities, using a numerical integration procedure as discussed in Chapter II.

First, the flow time and waiting time distributions are derived.

4.1 Flow-Time and Waiting-Time Distributions for Single-Machine Models

Consider the queue discipline is first-come, first-processed. The flow time is the period a job spends in the shop. In other words, it is the difference between the completion time and release or arrival time. The period the job waits before the processing begins on a machine is termed the waiting time. Thus, flow time is the sum of two independent random variables: the waiting time and the processing time. In queueing terminology, flow time is the waiting time in the system and waiting time is the waiting time in the queue.

Flow-Time Distribution for Transient State. Flow time of an arriving job is a random variable. If there is no job ahead of the arriving job, flow time is the processing time of the job. If there is one job ahead and that job is in process, flow time is the sum of the remaining processing time of the first job and the processing time of the second job. Thus, if an arriving job finds n jobs ahead of it in the system at any time t , it is the $(n + 1)$ th job in the system. The flow time of the $(n + 1)$ th job is then the sum of the remaining processing time of the first job and the sum of all the processing times of the remaining n jobs in the system. Or, simply,

$$f(t) = t_1' + t_2 + t_3 + \dots + t_{n+1}$$

where

$f(t)$ flow time of $(n + 1)$ th job

t_1' remaining processing time of the job being processed at the time of arrival of the $(n + 1)$ th job

t_j processing time of the jobs, $j = 2, 3, \dots, n+1$

Note that $t_j, j = 2, 3, \dots, n + 1$, are identically, independently distributed random variables having the following exponential distribution:

$$\mu e^{-\mu t_j}, \quad j = 2, 3, \dots, n + 1$$

with mean $1/\mu$.

On account of the Markovian property of the exponential distribution, the distribution of the remaining processing time t_1'

is the same as that of t_j , $j = 2, 3, \dots, n + 1$. That is, the remaining processing time of a job in process satisfies the same law as the processing time of the unprocessed jobs.

Thus, the distribution of flow time of $(n + 1)$ th job is the distribution of the sum of the $(n + 1)$ random variables; that is, the distribution of $t_1 + t_2 + t_3 + \dots + t_{n+1}$ where each t_j follows an exponential distribution.

It is known that the distribution of the sum of j random variables, each following an exponential distribution, is a gamma distribution having the probability density function

$$w_j(t) = \frac{\mu(\mu t)^{j-1}}{(j-1)!} e^{-\mu t}$$

If there are n jobs at any time t in the system, flow time probability density function of the $(n + 1)$ th job is

$$w_{n+1}(t) = \frac{\mu(\mu t)^n}{n!} e^{-\mu t}$$

This should be multiplied by the probability that there are n jobs in the system at any time t to give

$$P_n(t) w_{n+1}(t)$$

Since the job considered may occupy any position in the system, the above expression must be summed over the domain, $n = 0, 1, \dots$

This gives the flow time probability density function for the unlimited queue case as

$$f(t) = \sum_{n=0}^{\infty} P_n(t) w_{n+1}(t)$$

The mean flow time is given by

$$\begin{aligned}
\bar{F}(t) &= \int_{t=0}^{\infty} t f(t) dt \\
&= \int_0^{\infty} t \sum_{n=0}^{\infty} P_n(t) w_{n+1}(t) dt \\
&= \sum_{n=0}^{\infty} \int_0^{\infty} t P_n(t) w_{n+1}(t) dt
\end{aligned}$$

If $P_n(t)$ is obtained numerically for each value of t , the numerical value of $P_n(t)$ can be put in the above expression and the mean flow time $\bar{F}(t)$ may be computed as follows:

$$\begin{aligned}
\bar{F}(t) &= \sum_{n=0}^{\infty} P_n(t) \int_0^{\infty} t w_{n+1}(t) dt \\
&= \sum_{n=0}^{\infty} P_n(t) \int_0^{\infty} \frac{\mu(\mu t)^n}{n!} e^{-\mu t} t dt \\
&= \sum_{n=0}^{\infty} P_n(t) \int_0^{\infty} \frac{(\mu t)^{n+1}}{n!} e^{-\mu t} dt \\
&= \sum_{n=0}^{\infty} P_n(t) \int_0^{\infty} \frac{(n+1)\mu}{(n+1)\mu} \frac{(\mu t)^{n+1}}{n!} e^{-\mu t} dt \\
&= \sum_{n=0}^{\infty} P_n(t) \int_0^{\infty} \frac{(n+1)\mu}{\mu} \frac{(\mu t)^{n+1}}{(n+1)!} e^{-\mu t} dt \\
&= \sum_{n=0}^{\infty} P_n(t) \frac{(n+1)}{\mu} \int_0^{\infty} \frac{\mu(\mu t)^{n+1}}{(n+1)!} e^{-\mu t} dt \\
&= \frac{1}{\mu} \sum_{n=0}^{\infty} (n+1) P_n(t) \cdot 1 \\
&= \frac{1}{\mu} \sum_{n=0}^{\infty} n \cdot P_n(t) + \frac{1}{\mu} \sum_{n=0}^{\infty} P_n(t) \\
&= \frac{1}{\mu} \sum_{n=0}^{\infty} n \cdot P_n(t) + \frac{1}{\mu}
\end{aligned}$$

$$\bar{F}(t) = \frac{L_s}{\mu} + \frac{1}{\mu}$$

It will be shown later that

$$\frac{1}{\mu} \sum_{n=0}^{\infty} n P_n(t)$$

is the mean waiting time of jobs in the queue,

where

$$W(t) = L_s / \mu$$

L_s : the mean number of jobs in the system

Thus, the mean flow time is

$$\bar{F}(t) = W(t) + \frac{1}{\mu}$$

where

$W(t)$: mean waiting time

$1/\mu$: mean processing time

Flow time distribution for steady state. In the steady state case,

$$P_n(t) = P_n$$

Hence, the flow time density function

$$\begin{aligned} f(t) &= \sum_{n=0}^{\infty} P_n w_{n+1}(t) \\ &= \sum_{n=0}^{\infty} P_n \frac{\mu (\mu t)^n}{n!} e^{-\mu t} \end{aligned}$$

However,

$$P_n = \rho^n (1 - \rho)$$

Hence,

$$\begin{aligned}
f(t) &= \sum_{n=0}^{\infty} \rho^n (1 - \rho) \frac{\mu(\mu t)^n}{n!} e^{-\mu t} \\
&= \mu(1 - \rho) e^{-\mu t} \sum_{n=0}^{\infty} \rho^n \frac{(\mu t)^n}{n!} \\
&= \mu(1 - \rho) e^{-\mu t} \left[1 + \rho \mu t + \frac{\rho^2 (\mu t)^2}{2!} + \dots \right] \\
&= \mu(1 - \rho) e^{-\mu t} e^{\rho \mu t} \\
&= \mu(1 - \rho) e^{-(1-\rho)\mu t}
\end{aligned}$$

The cumulative distribution function for the flow time is given by the probability of flow time of no more than t ,

$$\begin{aligned}
P(\leq t) &= \int_0^t f(t) dt \\
&= \int_0^t (1 - \rho) e^{-(1-\rho)\mu t} dt \\
&= \mu(1 - \rho) \int_0^t e^{-(1-\rho)\mu t} dt \\
&= \mu(1 - \rho) \left[\frac{e^{-(1-\rho)\mu t}}{-(1-\rho)\mu} \right]_0^t \\
&= \frac{(1-\rho)}{-(1-\rho)\mu} \left[e^{-(1-\rho)\mu t} \right]_0^t \\
&= -[e^{-(1-\rho)\mu t} - 1] \\
&= 1 - e^{-(1-\rho)\mu t}
\end{aligned}$$

From the density function of flow time, it is clearly seen that it is an exponential distribution having a mean of $1/\mu(1 - \rho)$. Hence, mean flow time, which is the waiting time in the system,

is given by

$$P = \frac{1}{\mu(1 - \rho)}$$

Waiting-time distribution for transient state. With a similar analysis, an arriving job will not wait if there is no job in the system; that is, the machine is idle. If there is one job running, the waiting time will be the remaining processing time of the running job. Hence, if there are n jobs in the system ahead of an $(n + 1)$ th arriving job, the waiting time will be the sum of the remaining time of the first job and processing time of $(n - 1)$ jobs. Hence, the waiting time distribution function is the distribution of n exponential random variables, which is a gamma distribution,

$$w_n(t) = \frac{\mu(\mu t)^{n-1}}{(n-1)!} e^{-\mu t}$$

This equation is multiplied by $P_n(t)$, the probability of n jobs in the system and summed over the domain, $n = 1, 2, \dots$. Summation starts from the value of $n = 1$ because if $n = 0$; that is, there is no job in the system, then there will be no waiting.

Hence, waiting time probability density function

$$w(t) = \sum_{n=1}^{\infty} P_n(t) w_n(t)$$

Thus, the mean waiting time is

$$\begin{aligned} W(t) &= \int_0^{\infty} t w(t) dt \\ &= \int_0^{\infty} t \sum_{n=1}^{\infty} P_n(t) w_n(t) dt \end{aligned}$$

$$W(t) = \sum_{n=1}^{\infty} \int_0^{\infty} t P_n(t) w_n(t) dt$$

If the numerical value of $P_n(t)$ is taken, then

$$\begin{aligned} W(t) &= \sum_{n=1}^{\infty} P_n(t) \int_0^{\infty} t w_n(t) dt \\ &= \sum_{n=1}^{\infty} P_n(t) \int_0^{\infty} \frac{\mu (\mu t)^{n-1}}{(n-1)!} e^{-\mu t} t dt \\ &= \sum_{n=1}^{\infty} P_n(t) \int_0^{\infty} \frac{n\mu}{n\mu} \frac{(\mu t)^n}{(n-1)!} e^{-\mu t} dt \\ &= \sum_{n=1}^{\infty} P_n(t) \int_0^{\infty} \frac{n\mu}{\mu} \cdot \frac{\mu (\mu t)^n}{n!} e^{-\mu t} dt \\ &= \sum_{n=1}^{\infty} P_n(t) \frac{n}{\mu} \int_0^{\infty} \frac{\mu (\mu t)^n}{n!} e^{-\mu t} dt \\ &= \frac{1}{\mu} \sum_{n=1}^{\infty} n \cdot P_n(t) \\ &= \frac{L_s}{\mu} \end{aligned}$$

Also,

$$W(t) = \bar{F}(t) - \frac{1}{\mu}$$

Waiting-time distribution for steady state. For the steady state

$$\begin{aligned} P_n(t) &= P_n \\ &= \rho^n (1 - \rho) \end{aligned}$$

Hence, for $t > 0$, the waiting time probability density function is

$$\begin{aligned}
w(t) &= \sum_{n=1}^{\infty} P_n w_n(t) \\
&= \sum_{n=1}^{\infty} \rho^n (1 - \rho) \frac{\mu (\mu t)^{n-1}}{(n-1)!} e^{-\mu t} \\
&= \mu \rho (1 - \rho) e^{-\mu t} \sum_{n=1}^{\infty} \frac{\rho^{n-1} (\mu t)^{n-1}}{(n-1)!} \\
&= \mu \rho (1 - \rho) e^{-\mu t} \left[1 + \rho \mu t + \frac{(\rho \mu t)^2}{2} + \dots \right] \\
&= \mu \rho (1 - \rho) e^{-\mu t} \cdot e^{\rho \mu t} \\
&= \mu \rho (1 - \rho) e^{-(1-\rho)\mu t}
\end{aligned}$$

Thus, the mean waiting time is

$$\begin{aligned}
W &= \int_0^{\infty} t w(t) dt \\
&= \int_0^{\infty} t \mu \rho (1 - \rho) e^{-(1-\rho)\mu t} dt \\
&= \rho \int_0^{\infty} (1 - \rho) \mu t e^{-(1-\rho)\mu t} dt \\
&= \rho \left[\mu (1 - \rho) t \frac{e^{-(1-\rho)\mu t}}{-(1-\rho)\mu} - \int \mu (1 - \rho) \frac{e^{-(1-\rho)\mu t}}{-(1-\rho)\mu} \right]_0^{\infty} \\
&= \rho \left[-t e^{-(1-\rho)\mu t} - \frac{e^{-(1-\rho)\mu t}}{\mu (1-\rho)} \right]_0^{\infty} \\
&= \frac{\rho}{\mu (1-\rho)}
\end{aligned}$$

Also,

$$\begin{aligned}
W &= \bar{F} - \frac{1}{\mu} \\
&= \frac{1}{\mu (1-\rho)} - \frac{1}{\mu}
\end{aligned}$$

$$\begin{aligned}
 W &= \frac{1 - 1 + \rho}{\mu(1 - \rho)} \\
 &= \frac{\rho}{\mu(1 - \rho)}
 \end{aligned}$$

The cumulative distribution, $P(\leq t)$, which gives the probability of waiting no more than time t is obtained as follows:

$$\begin{aligned}
 P(\leq t) &= 1 - P(> t) \\
 &= 1 - \int_t^{\infty} w(t) dt \\
 &= 1 - \int_t^{\infty} \mu \rho (1 - \rho) e^{-(1-\rho)\mu t} dt \\
 &= 1 - \left[\mu \rho (1 - \rho) \frac{e^{-(1-\rho)\mu t}}{-(1-\rho)\mu} \right]_t^{\infty} \\
 &= 1 + \rho \left[e^{-(1-\rho)\mu t} \right]_t^{\infty} \\
 &= 1 - \rho e^{-(1-\rho)\mu t}, \quad t > 0
 \end{aligned}$$

It should be pointed out that the mean flow time and mean waiting time for all queue disciplines, which are independent of the processing time, namely last-come, first-processed and random selection disciplines, are the same as that for first-come, first-processed queue discipline. Refer to Appendix A for details. The flow time probability density function, the waiting time probability density function, and the corresponding mean flow time and mean waiting time, for the limited queue case and no queue case have been derived in Appendix A.

4.2 Flow-Time and Waiting-Time Distributions for Multiple-Machine Models

Consider the queue discipline in first-come, first-processed. First, waiting-time distribution will be derived.

Waiting-time distribution for transient state. As we know, the waiting time is the period during which a job has to wait in queue before the processing on any of the available machines is started. If a job, after arrival, finds j jobs already in the system before it, the job will be loaded on the machine immediately if $j < M$, where M is the number of machines in parallel job-shop system. If $j \geq M$, the job has to wait until $j - M + 1$ jobs are processed completely. The processing of each job is a random variable having exponential distribution with mean $1/M_\mu$. We know that the distribution of the sum of n random variables, each following exponential distribution of mean $1/M_\mu$ is a gamma distribution having the probability density function

$$\frac{M_\mu (M_\mu t)^{n-1}}{(n-1)!} e^{-M_\mu t}$$

Hence, the probability density function for the sum of $j - M + 1$ exponential random variables is

$$\frac{M_\mu (M_\mu t)^{j-M}}{(j-M)!} e^{-M_\mu t}$$

This is to be multiplied by the probability of waiting. When there are M or more jobs in the system, only an arriving job has to wait. Hence, the waiting time probability density function for

unlimited queue case is

$$w(t) = \sum_{j=M}^{\infty} P_j(t) \frac{M_{\mu}(M_{\mu}t)^{j-M}}{(j-M)!} e^{-M_{\mu}t}$$

The mean waiting time is

$$W(t) = \sum_{j=M}^{\infty} P_j(t) \int_0^t \frac{M_{\mu}(M_{\mu}t)^{j-M}}{(j-M)!} e^{-M_{\mu}t} t dt$$

where

$P_j(t)$ is the numerical value of P_j at time t

Also it is known that

$$\begin{aligned} W(t) &= \frac{L_s}{\mu} \\ &= \frac{\sum_{n=0}^{\infty} n P_n(t)}{\mu} \end{aligned}$$

where

L_s is the mean number of jobs in the system

Waiting-time distribution for the steady state. For the steady state case,

$$P_j(t) = P_j$$

Hence, waiting time probability density function

$$\begin{aligned} w(t) &= \sum_{j=M}^{\infty} P_j e^{\frac{M_{\mu}(M_{\mu}t)^{j-M}}{(j-M)!}} e^{-M_{\mu}t} \\ &= \sum_{j=M}^{\infty} \frac{P_0}{M^{j-M} M!} \left(\frac{\lambda}{\mu}\right)^j \frac{M_{\mu}(M_{\mu}t)^{j-M}}{(j-M)!} e^{-M_{\mu}t} \\ &= \sum_{j=M}^{\infty} \frac{P_0}{M^{j-M} M!} \frac{\lambda^j}{\mu^j} \frac{M_{\mu} M^{j-M} (\mu t)^{j-M}}{(j-M)!} e^{-M_{\mu}t} \end{aligned}$$

$$\begin{aligned}
w(t) &= \sum_{j=M}^{\infty} \frac{P_0}{M!} - \lambda^j \frac{M_{\mu} \mu^{-M} t^{j-M}}{(j-M)!} e^{-M_{\mu} t} \\
&= \frac{P_0 M_{\mu}}{M!} \left(\frac{\lambda}{\mu}\right)^M Me^{-M_{\mu} t} \sum_{j=M}^{\infty} \frac{(\lambda t)^{j-M}}{(j-M)!} \\
&= \frac{P_0 M_{\mu}}{M!} \left(\frac{\lambda}{\mu}\right)^M Me^{-M_{\mu} t} \left[1 + \lambda t + \frac{(\lambda t)^2}{2!} + \dots\right] \\
&= \frac{P_0 M_{\mu}}{M!} \left(\frac{\lambda}{\mu}\right)^M e^{-M_{\mu} t} e^{\lambda t} \\
&= \frac{P_0 M_{\mu}}{M!} \left(\frac{\lambda}{\mu}\right)^M e^{-t(M_{\mu} - \lambda)}, \quad 0 < t < \infty
\end{aligned}$$

Mean waiting time is

$$\begin{aligned}
W &= \int_0^{\infty} t w(t) dt \\
&= \int_0^{\infty} \frac{P_0 M_{\mu}}{M!} \left(\frac{\lambda}{\mu}\right)^M e^{-t(M_{\mu} - \lambda)} t dt \\
&= \frac{P_0 M_{\mu}}{M!} \left(\frac{\lambda}{\mu}\right)^M \int_0^{\infty} e^{-t(M_{\mu} - \lambda)} t dt \\
&= \frac{P_0 M_{\mu}}{M!} \left(\frac{\lambda}{\mu}\right)^M \left[t \frac{e^{-t(M_{\mu} - \lambda)}}{-(M_{\mu} - \lambda)} - \int \frac{e^{-t(M_{\mu} - \lambda)}}{-(M_{\mu} - \lambda)} \right]_0^{\infty} \\
&= \frac{P_0 M_{\mu}}{M!} \left(\frac{\lambda}{\mu}\right)^M \left[- t \frac{e^{-t(M_{\mu} - \lambda)}}{M_{\mu} - \lambda} - \frac{e^{-t(M_{\mu} - \lambda)}}{(M_{\mu} - \lambda)^2} \right]_0^{\infty} \\
&= \frac{P_0 M_{\mu}}{M!} \left(\frac{\lambda}{\mu}\right)^M \frac{1}{(M_{\mu} - \lambda)^2} \\
&= \frac{P_0 M_{\mu}}{M!} \left(\frac{\lambda}{\mu}\right)^M \frac{1}{M^2 \mu^2 (1-\rho)^2} \\
&= \frac{P_0 \left(\frac{\lambda}{\mu}\right)^M}{M! M_{\mu} (1-\rho)^2}
\end{aligned}$$

$$W = \frac{P_0 \left(\frac{\lambda}{\mu}\right)^{M\rho}}{\lambda M! (1-\rho)^2}, \quad \rho = \lambda/M\mu$$

Flow-time distribution for transient state. As in the single machine queueing models, the flow time density function is

$$w(t) = \sum_{j=0}^{\infty} P_j(t) w_{j+1}(t)$$

But

w_{j+1} , in this case, is

$$= \frac{M\mu (M\mu t)^{j-M}}{(j-M)!} e^{-M\mu t}$$

Hence,

$$w(t) = \sum_{j=0}^{\infty} P_j(t) \frac{M\mu (M\mu t)^{j-M}}{(j-M)!} e^{-M\mu t}$$

Mean flow time is

$$W(t) = \sum_{j=0}^{\infty} P_j(t) \int_0^{\infty} \frac{M\mu (M\mu t)^{j-M}}{(j-M)!} e^{-M\mu t} t dt$$

where

$P_j(t)$ the numerical value of P_j at time t

Also, the mean number of jobs in the system at time t is

$$L_s = \sum_{n=0}^{\infty} n P_n(t)$$

Hence,

$$\begin{aligned} \bar{F}(t) &= \frac{L_s}{\mu} + \frac{1}{\mu} \\ &= W(t) + \frac{1}{\mu} \end{aligned}$$

$$\bar{F}(t) = \frac{\sum_{n=0}^{\infty} n P_n(t)}{\mu} + \frac{1}{\mu}$$

Flow-time distribution for steady state. In the steady state case,

$$P_j(t) = P_j$$

Hence, the flow time density function,

$$\begin{aligned} f(t) &= \sum_{j=0}^{\infty} P_j w_{j+1}(t) \\ &= \sum_{j=0}^{\infty} P_j \int_0^{\infty} \frac{M\mu (M\mu t)^{j-M}}{(j-M)!} t dt \end{aligned}$$

Mean flow time is computed from the relation

$$\begin{aligned} \bar{F} &= W + \frac{1}{\mu} \\ &= \frac{P_0 \left(\frac{\lambda}{\mu}\right)^M}{\lambda M! (1-\rho)^2} + \frac{1}{\mu} \end{aligned}$$

The mean flow time and mean waiting time for last-come, first-processed and random selection for processing queue disciplines are the same as that for first-come, first-processed. For limited queue and no queue cases, refer to Appendix A.

4.3 Measures of Performance

In order to design a new queueing system or study an existing system, several measures of performance of the system are required for decision making. As the system is stochastic in nature, the measures of performance are expressed in terms of the probabilities and expectations.

The measures of performance and their notations as computed in this thesis are as follows:

PO	probability of no job in the system
PL	probability of a job being turned away
PB	probability of all servers being busy
NS	mean number of jobs in the system
NQ	mean number of jobs in the queue
FT	mean flow time
WT	mean waiting time
N/Q	mean number of jobs in the queue given that there is a non-empty queue
NPROC	mean number of jobs in service
NIDLE	mean number of idle servers
IDLE%	percentage of idle servers
UTIL%	percent of utilization of servers

Some of the relations between these measures of performance are as follows:

1. The probability of all servers being busy is the same as the probability of waiting. From this relation, the probability that all servers are not busy can be computed.

2. The mean flow time and mean waiting time can be computed easily from the known relationship

$$FT = NS/\lambda$$

$$WT = NQ/\lambda$$

3. The mean number of jobs in service is, simply,

$$NPROC = NS - NQ$$

4. The utilization of servers, in percent, is

$$\text{UTIL}\% = \frac{\text{NPROC}}{r} \times 100$$

where r is the number of servers.

5. The mean number of idle servers is given by

$$\text{NIDLE} = r - \text{PROC}$$

6. The idle servers, in percent, is

$$\text{IDLE}\% = 100 - \text{UTIL}\%$$

If the mean number of jobs in the system is reduced, the mean flow time also reduces, and hence the in-process inventory is reduced. The mathematical expressions from which the above measures of performance for the single- and multi-machine shop models, which have been computed, are shown in Appendix B.

4.4 Numerical Results

The various single-machine and multiple-machine queueing models are solved using the IBM subroutine RKGS. Tables for measures of performance for single-machine and multiple-machine models are displayed in Appendix B. From these tables, curves are plotted for the important measures of performance. No queue, limited queue, and unlimited queue sizes have been considered. For each model, three tables are displayed corresponding to three different utilization factors, that is $\rho = 0.75, 1.00$, and 1.25 for the infinite source queueing models and $\rho = 0.50, 0.75$, and 1.00 for the finite source queueing models. For no queue models, curves only for percentage idleness of servers and percentage utilization of servers have been plotted as other measures of

performance can directly be derived from them. Curves for infinite source and limited queue size models are similar to those of infinite source and unlimited queue size models. Hence, measures of performance curves for infinite source and unlimited queue size models only have been plotted. From Tables B.1.1, B.1.2, and B.1.3 and Figures 4.1, 4.2, and 4.3 it is seen that as the values of utilization factor increase from 0.75 to 1.25, percentage idleness of servers decreases. On the other hand, probability of jobs being turned away, that is PL is increased. This, because of the serious limitation on the queue size which is zero, in these cases. As such, when the server is busy processing a job, an arriving job is lost. For the same utilization factor, as the queue size is increased, the percentage idleness of servers and probability of lost jobs decreases, as is evident from Tables B.1.1, B.2.1, and B.3.1. An extreme case is that when the queue size is unlimited, the probability of a lost job is zero, as seen from Table B.3.1. Thus, no queue and unlimited queue cases are two extremes of limited queue cases. When an unlimited queue exists, the probability of a lost job is negligible. For single-machine model, with unlimited queue size, the system capacity is considered to be 50, for which the probability of a lost job, PL is zero, as seen from Table B.3.1. Thus, the large number considered for the infinite capacity of the system is 50 in this case. As seen from Figures 4.5 and 4.6, the mean number of jobs in the system and mean number of jobs in the queue increase with the increase in dimensionless time, that is, the system is in

transient state and after a certain time it is stabilized around the value 2.93 and 2.2, respectively, although the system capacity is 50. In this case, the value of the utilization factor is 0.75. Similarly, the system is stabilized in the case with mean flow time and mean waiting time as seen from Figure 4.7. In this case, the mean number of jobs in the system, mean number of jobs in the queue, mean flow time, and mean waiting time are independent of time and the system capacity. Practically, in this case, the system reached the steady state. Thus, from the tables and figures, we can also find how the system behaves in the transient state and how long the system takes to reach the steady state. The value of P_0 , the probability of no job in the system is given by $1 - \rho$ in the steady state. In the case of this study, P_0 should be equal to $1 - 0.75$, that is 0.25, which agrees with the numerical result obtained as seen in Table B.3.1. When the utilization factor is equal to 1, the mean number of jobs in the system and in the queue is increasing with time and does not reach a steady state. This is evident from Figure 4.9. Similarly, this is apparent in the case with mean flow time and mean waiting time. When the utilization factor is 1.25, mean number of jobs in the system and in the queue has reached the value of 44.259 and 43.259 at $\tau = 1.0$ as seen from Table B.3.3, and does not reach the steady state. As is also evident that although there is unlimited queue allowed in this case, even if we limit the system capacity to a very large number, there will still be some probability of lost jobs as the time passes. Thus, when utilization factor is 1

or more, the system never reaches the steady state and the queue grows without bounds. This fact is not evident in the no queue case, as it is in the limited or unlimited queue cases. Thus, the solutions of a queueing model differ significantly depending on the value of the utilization factor, ρ . When ρ is < 1 , the probability of the length of the queue being equal to the queue size is zero, while, with $\rho \geq 1$ among all the probabilities of lost jobs is the largest among all other probabilities of jobs in the system. When $\rho < 1$, the mean number of jobs in the system is independent of the system capacity and is very small in comparison to system capacity, provided it is large. When the utilization factor is equal to or greater than one, the mean number of jobs increases with time and with increases in the system capacity. From Figures 4.9 and 4.11 it may be inferred that with the increase in utilization factor, the mean number of jobs in the system is very large and behaves as if roughly proportional to the system capacity. Thus, when the utilization factor is equal to or greater than one, the system always remains in the transient state and never reaches the steady state. When $\rho = 1.25$, percentage utilization of servers reaches the value 100. On the other hand, probability of jobs lost also increases, as seen from Tables B.3.3. and Figure 4.10. Similar arguments hold equally well for mean number of jobs in the queue, mean flow time, and mean waiting time. In the finite source and limited queue cases, percentage utilization of servers increases very rapidly to a very high value even when the utilization factor is equal to 0.5.

For multiple-machine models, similar conclusions may be drawn. Special shop model is considered a multi-machine model with infinite source and no queue.

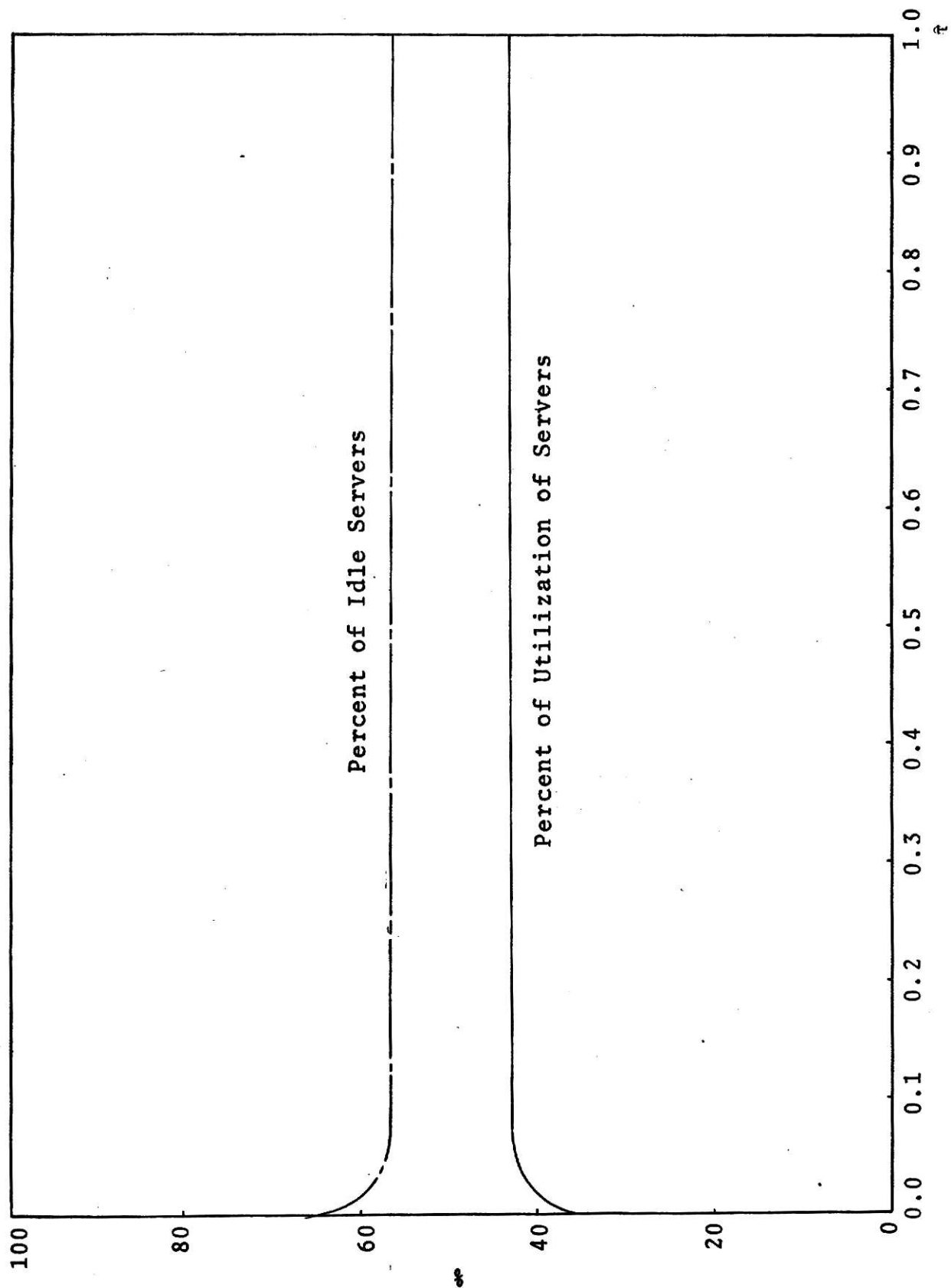


Figure 4.1. Single-Machine Model With Infinite Source, No Queue, and $\rho = 0.75$.

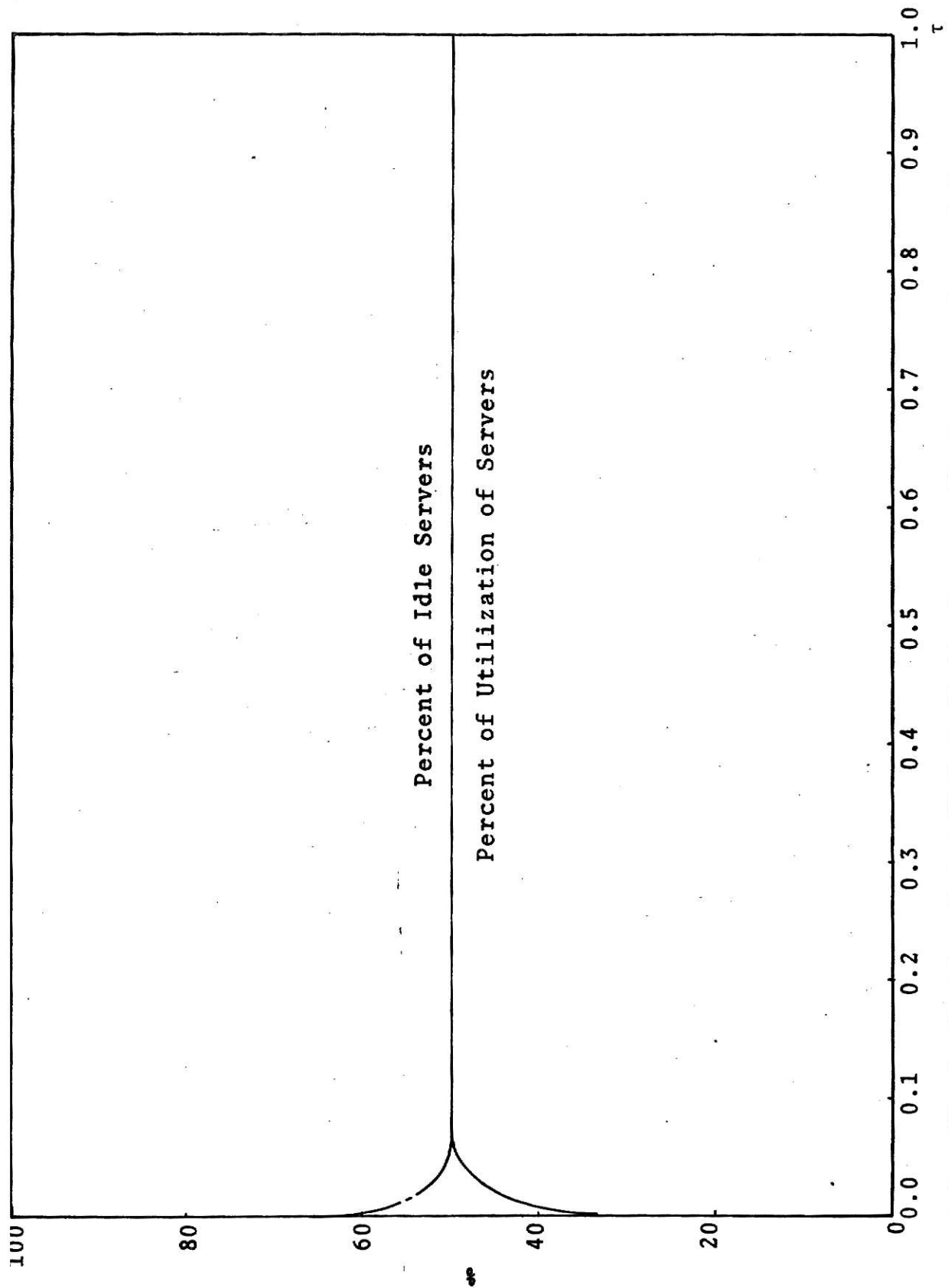


Figure 4.2. Single-Machine Model With Infinite Source, No Queue, and $\rho = 1.0$.

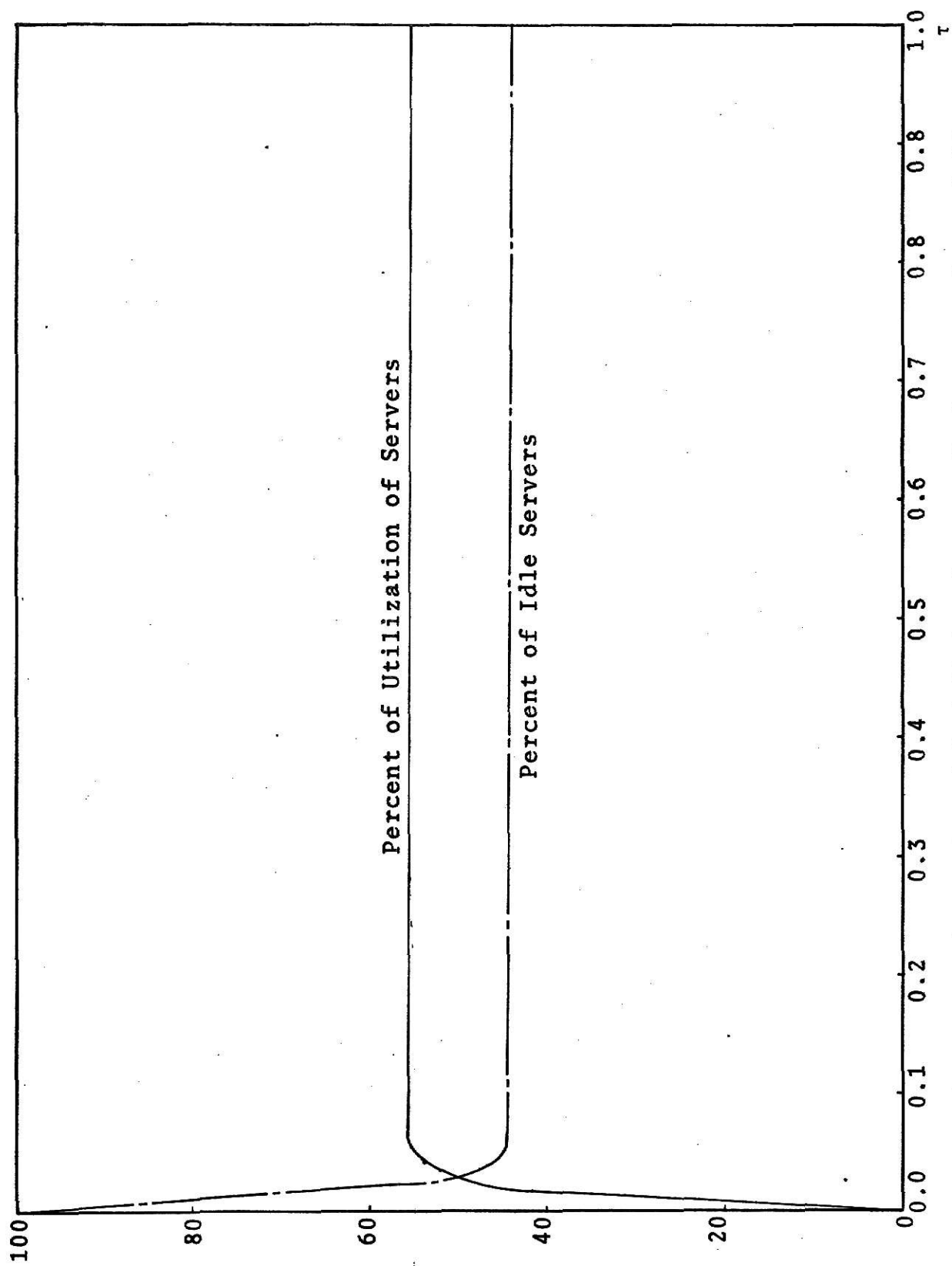


Figure 4.3. Single-Machine Model With Infinite Source, No Queue, and $\rho = 1.25$.

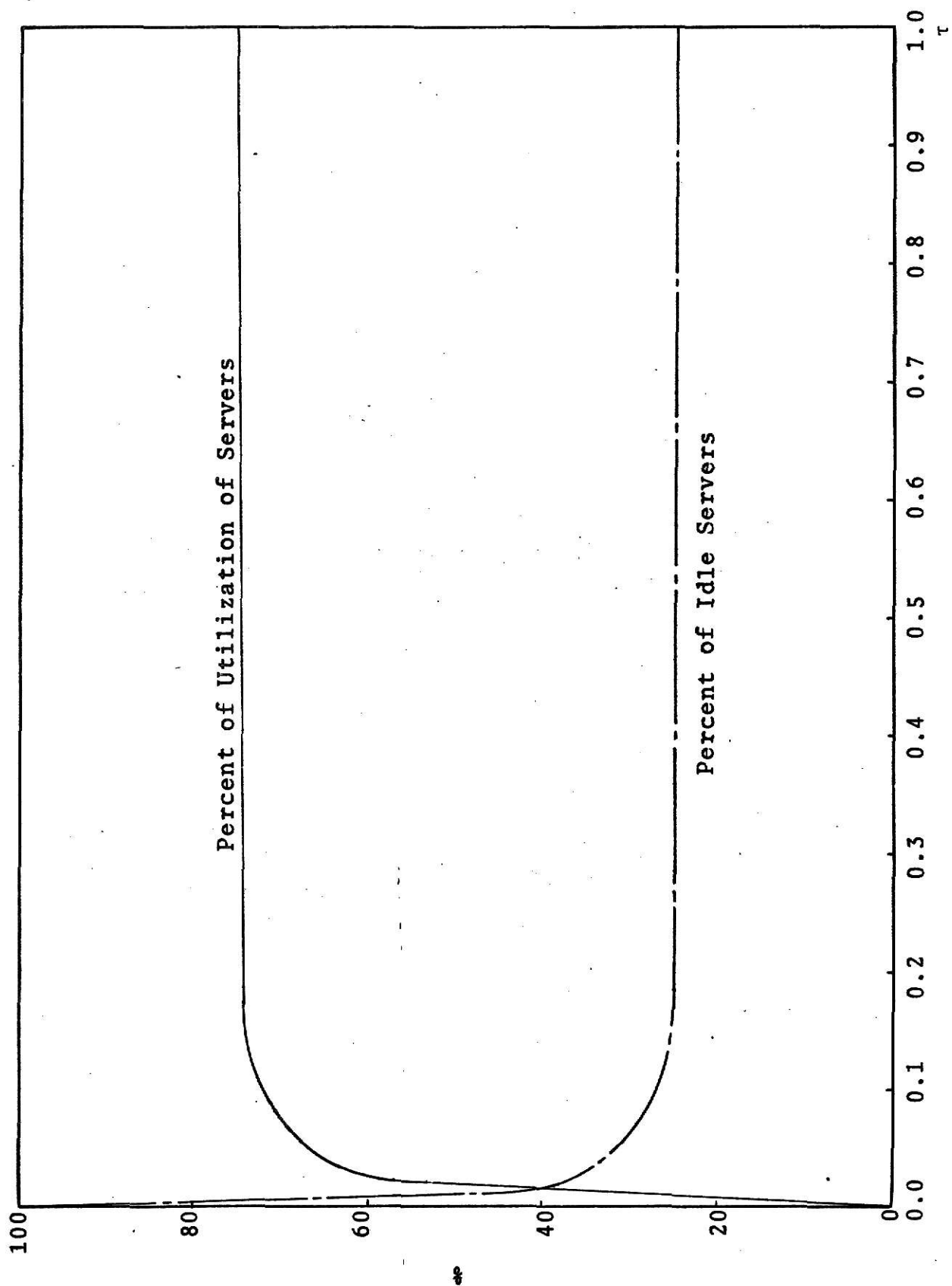


Figure 4.4. Single-Machine Model, Infinite Source, Unlimited Queue, and $\rho = 0.75$.

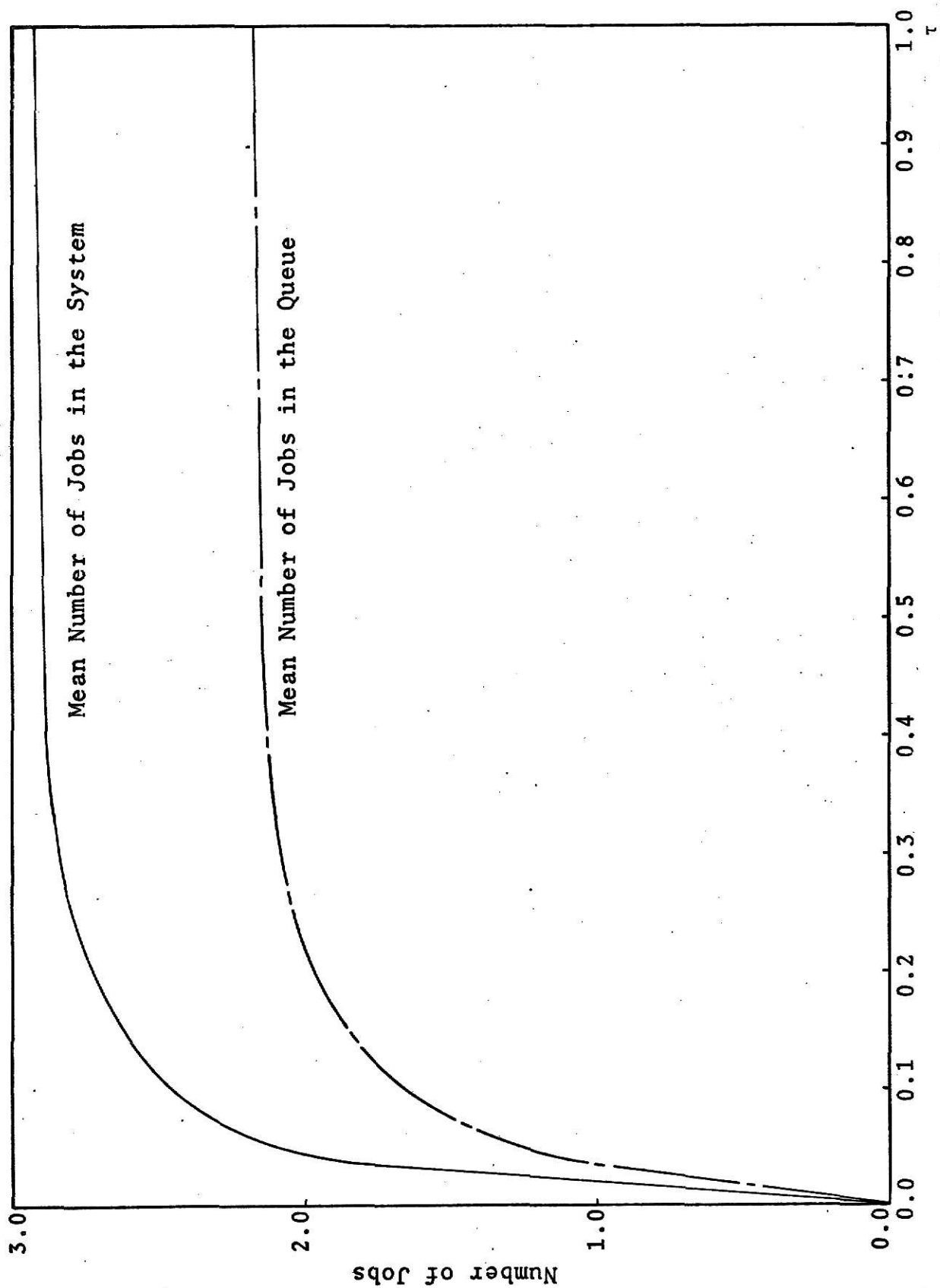


Figure 4.5. Single-Machine Model With Infinite Source, Unlimited Queue, and $\rho = 0.75$.

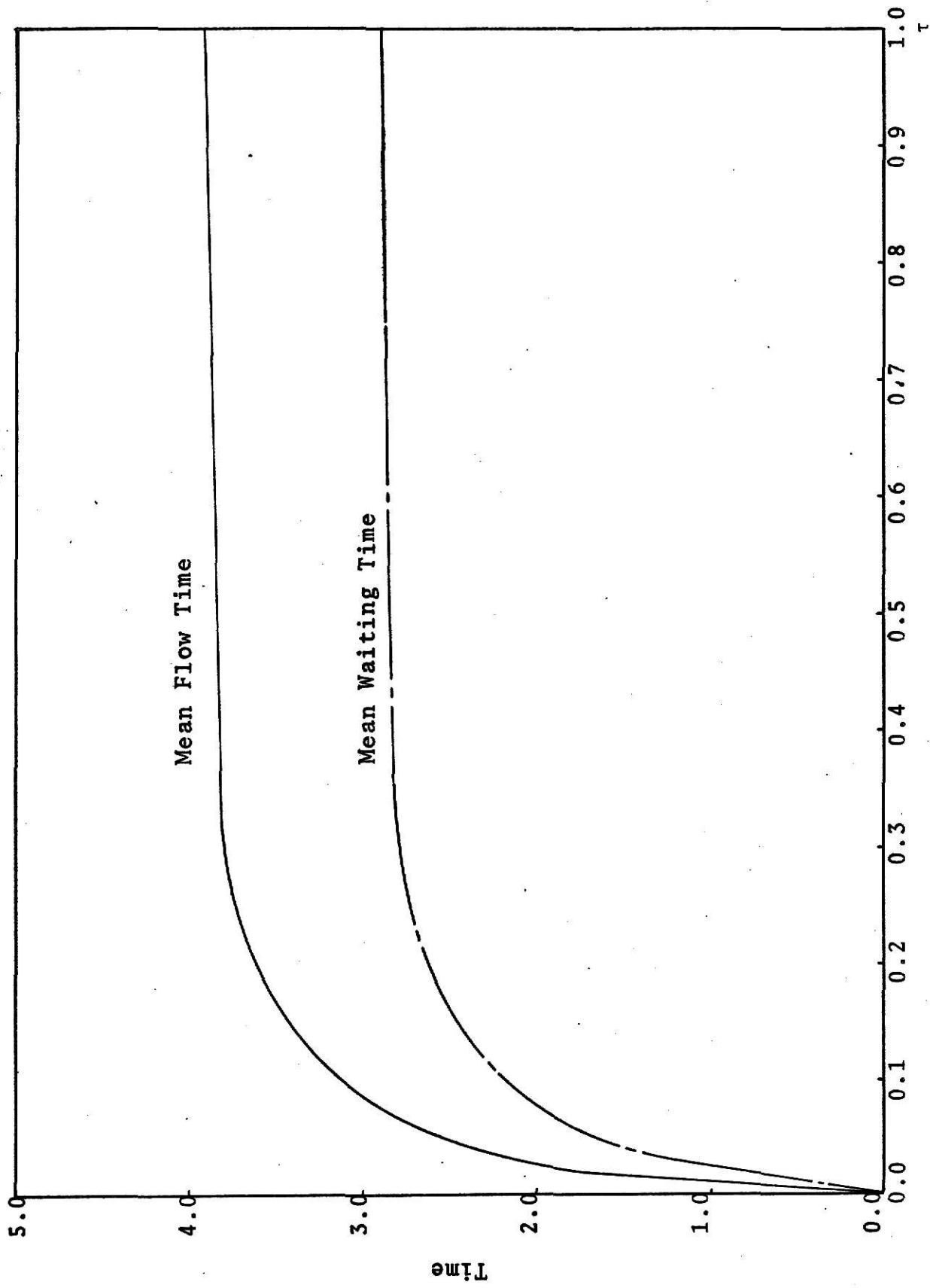


Figure 4.6. Single-Machine Model With Infinite Source, Unlimited Queue, and $\rho = 0.75$.

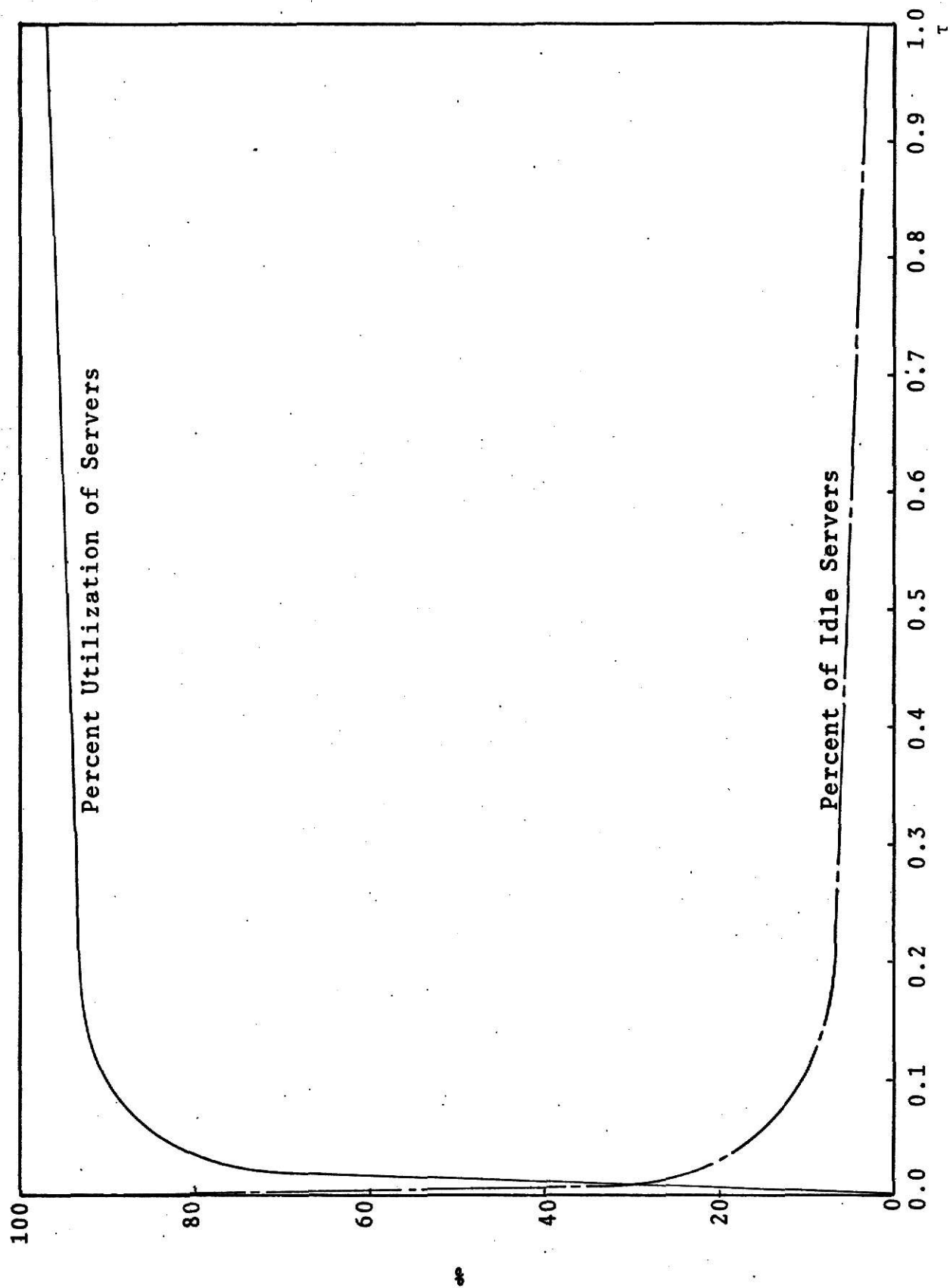


Figure 4.7. Single-Machine Model With Infinite Source, Unlimited Queue, and $\rho = 1.0$.

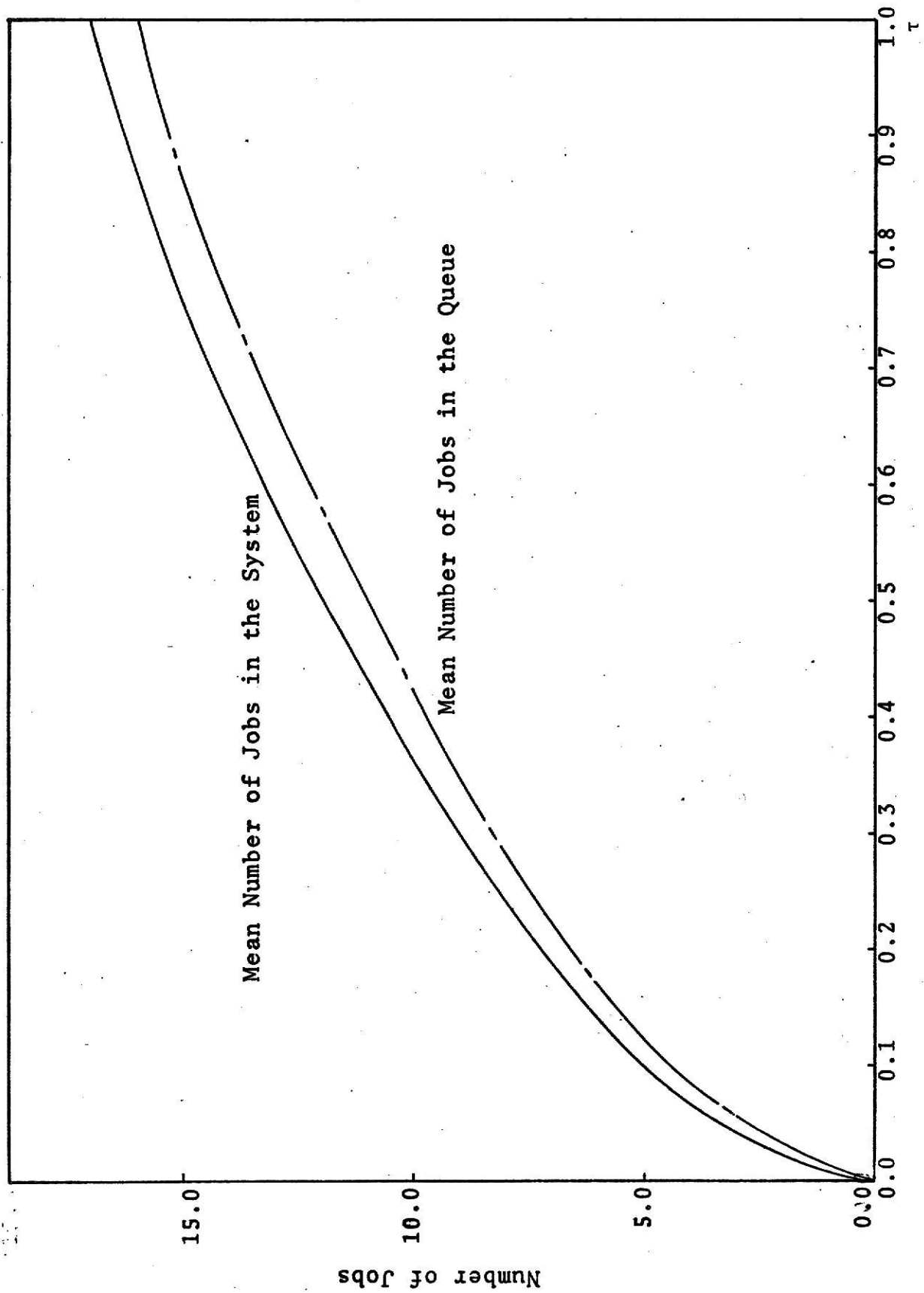


Figure 4.8. Single-Machine Model With Infinite Source, Unlimited Queue, and $\rho = 1.0$.

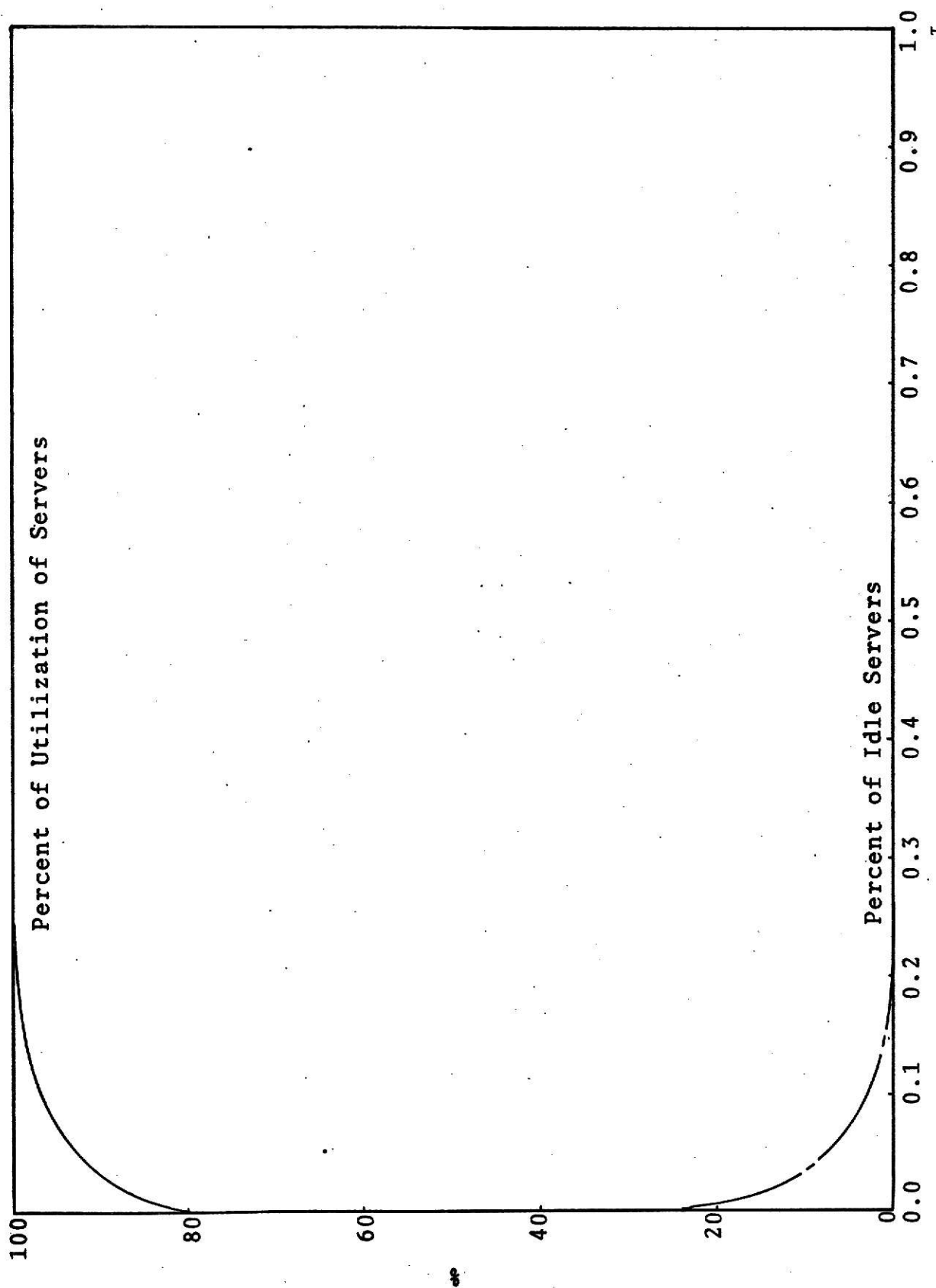


Figure 4.9. Single-Machine Model With Infinite Source, Unlimited Queue, and $\rho = 1.25$.

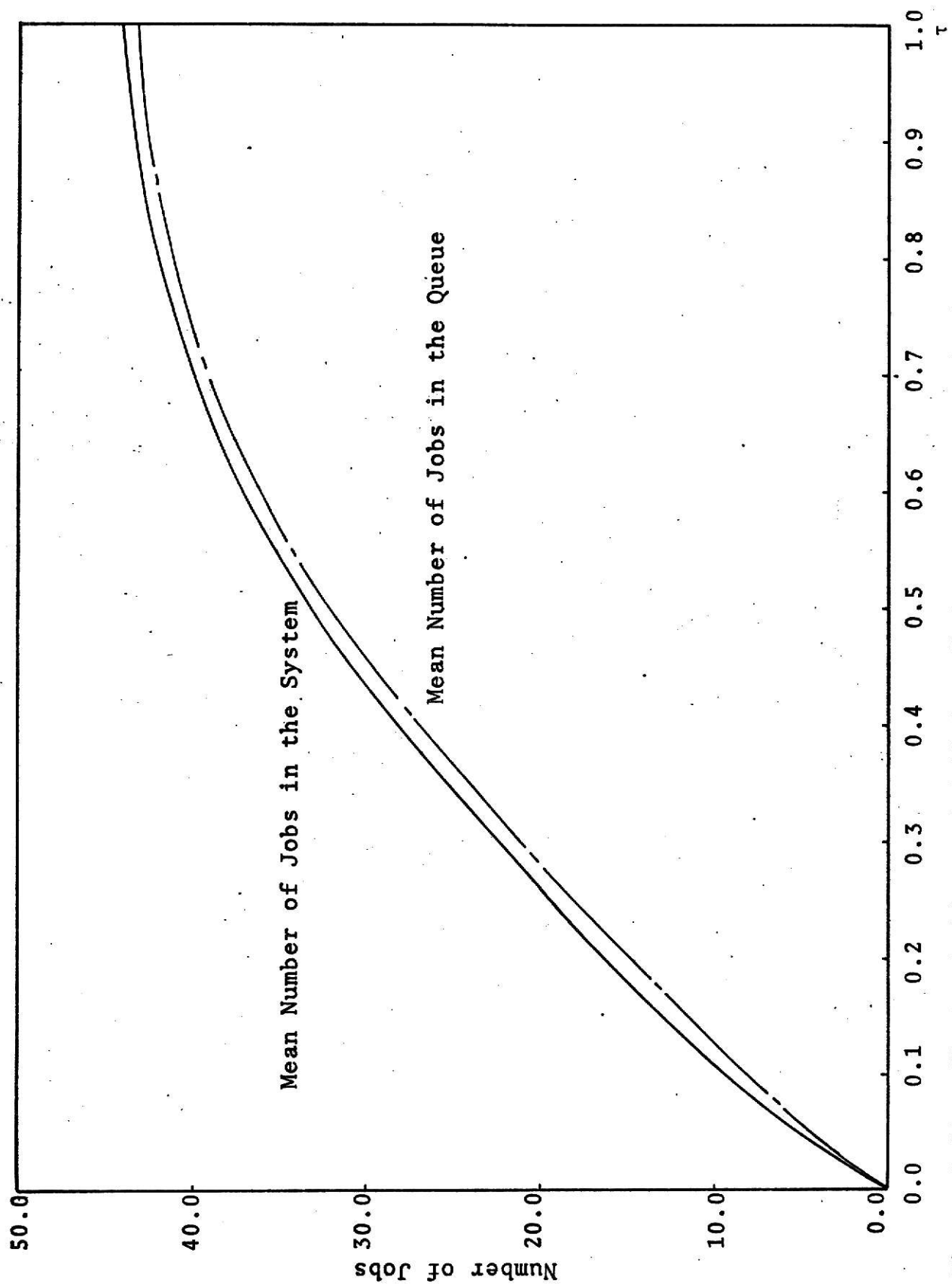


Figure 4.10. Single-Machine Model With Infinite Source, Unlimited Queue, and $\rho = 1.25$.

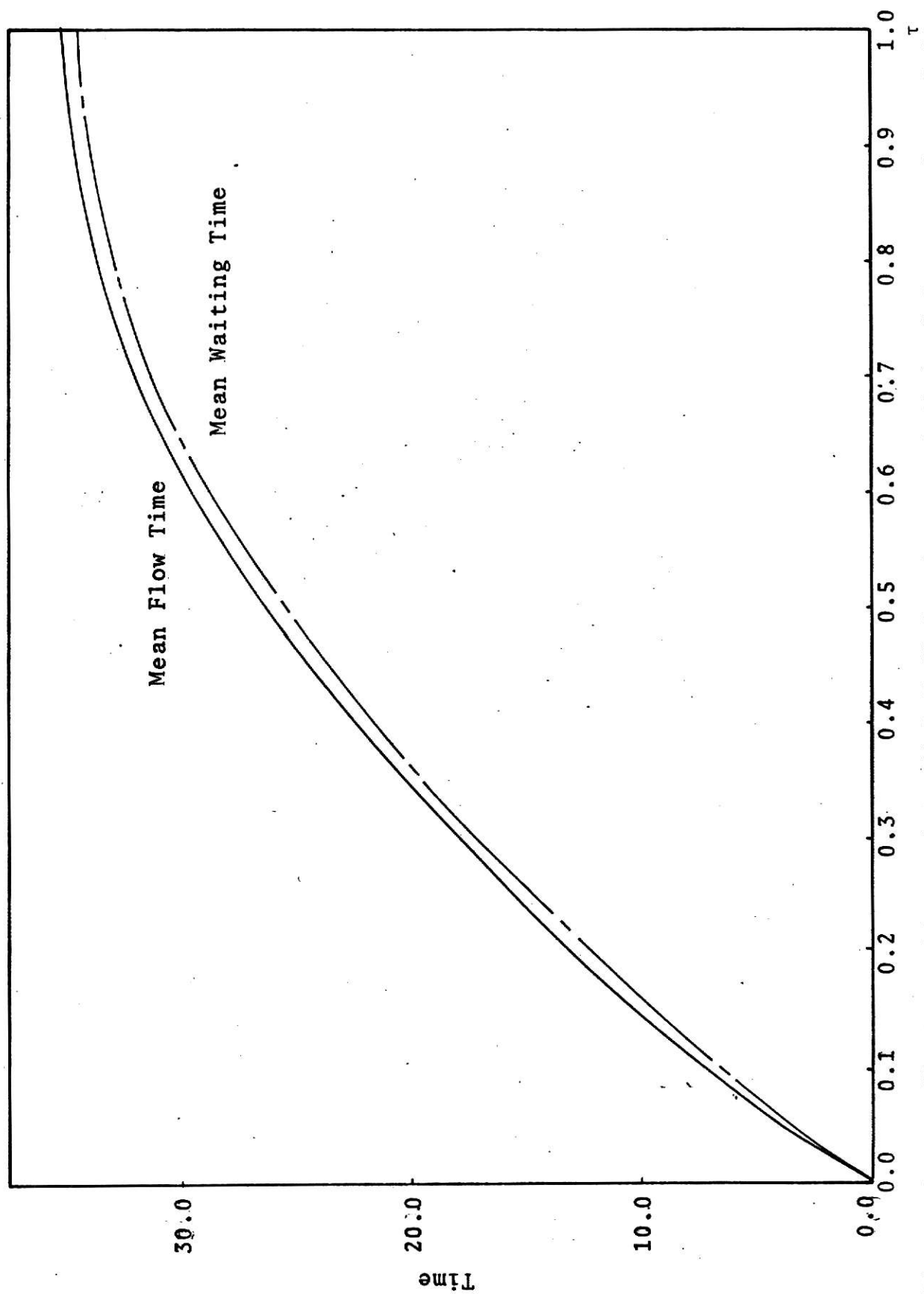


Figure 4.11. Single-Machine Model, With Infinite Source, Unlimited Queue, and $\rho = 1.25$.

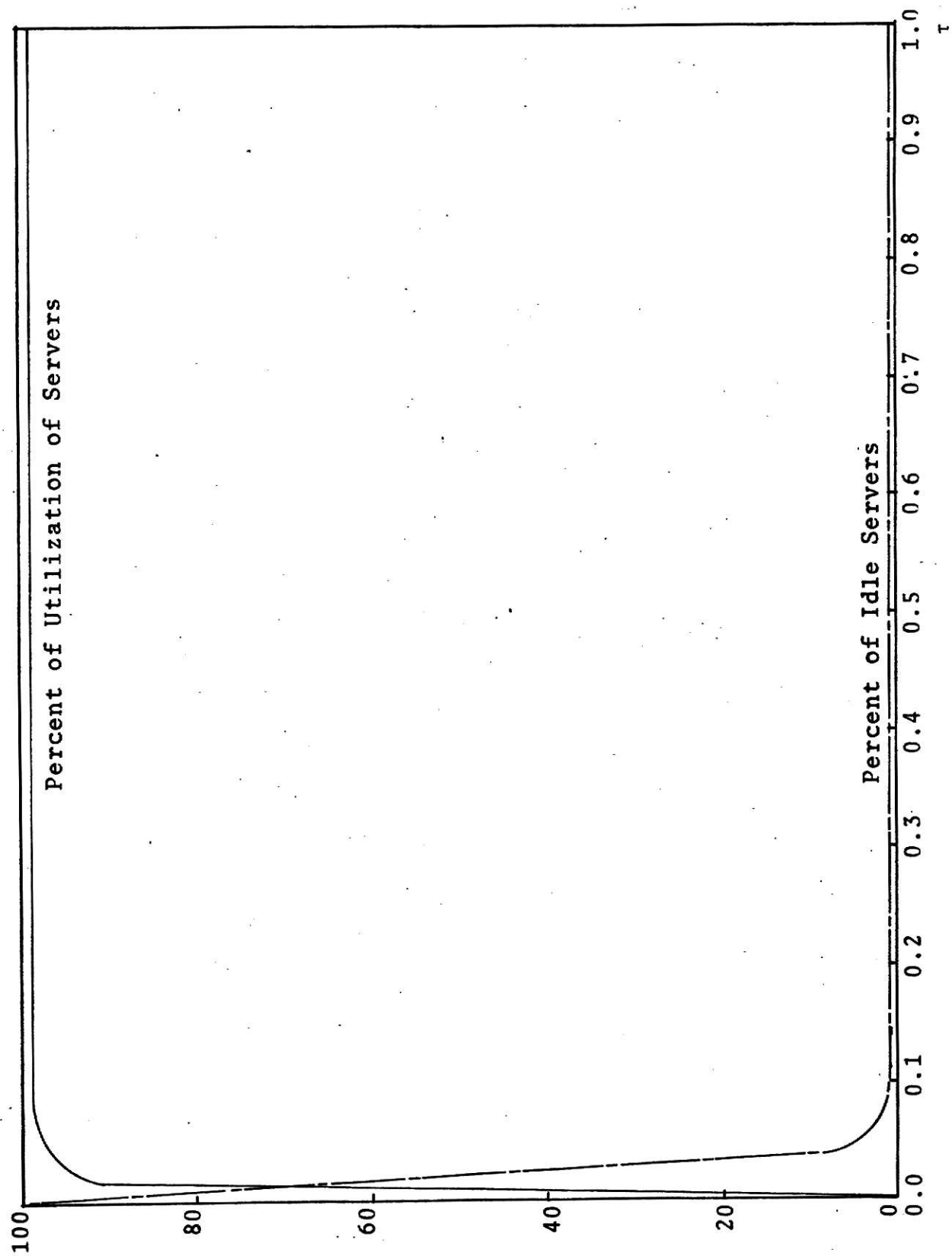


Figure 4.12. Single-Machine Model With Finite Source, Limited Queue, and $\rho = 0.50$.

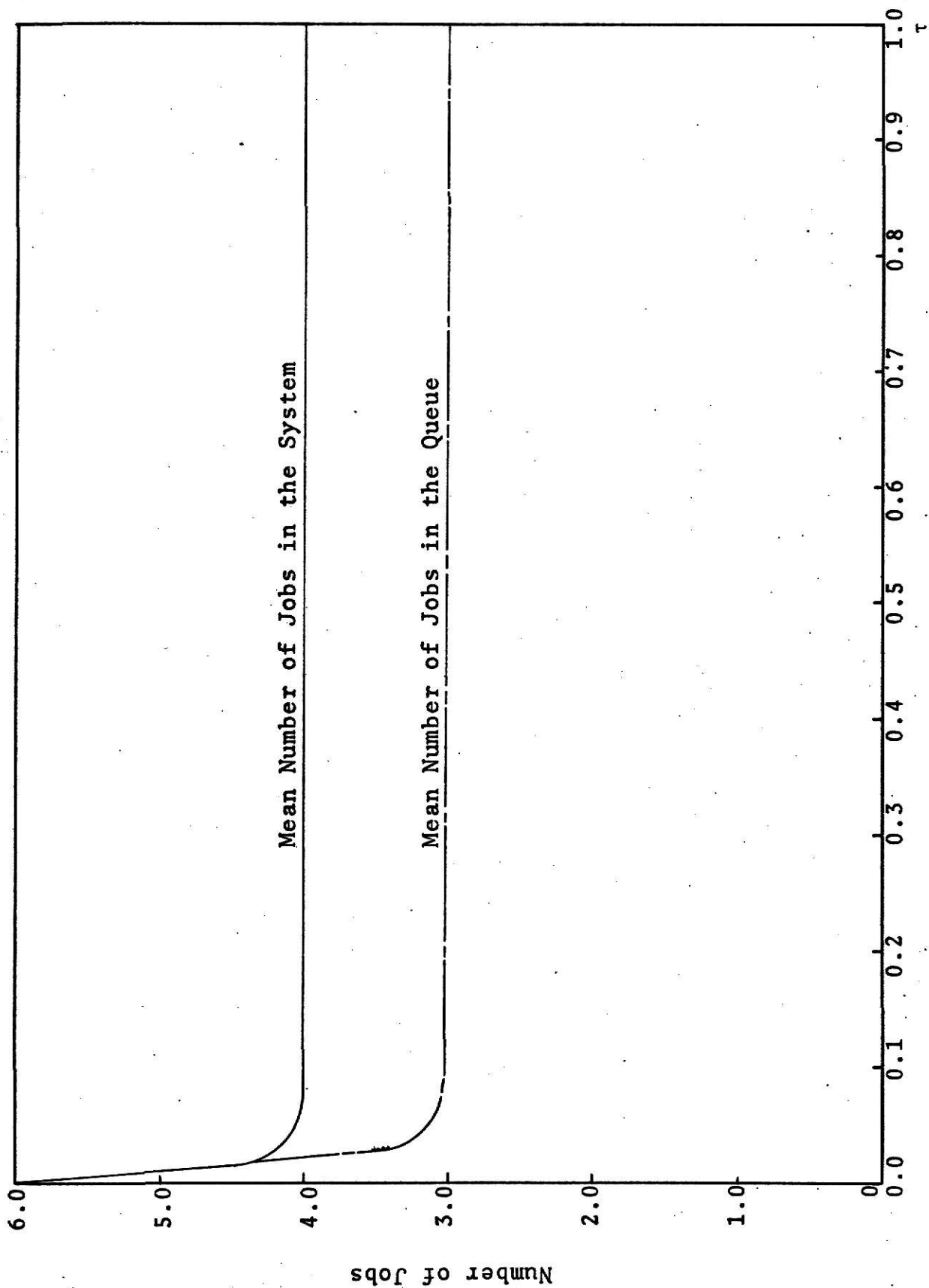


Figure 4.13. Single-Machine Model With Finite Source, Limited Queue, and $\rho = 0.5$.

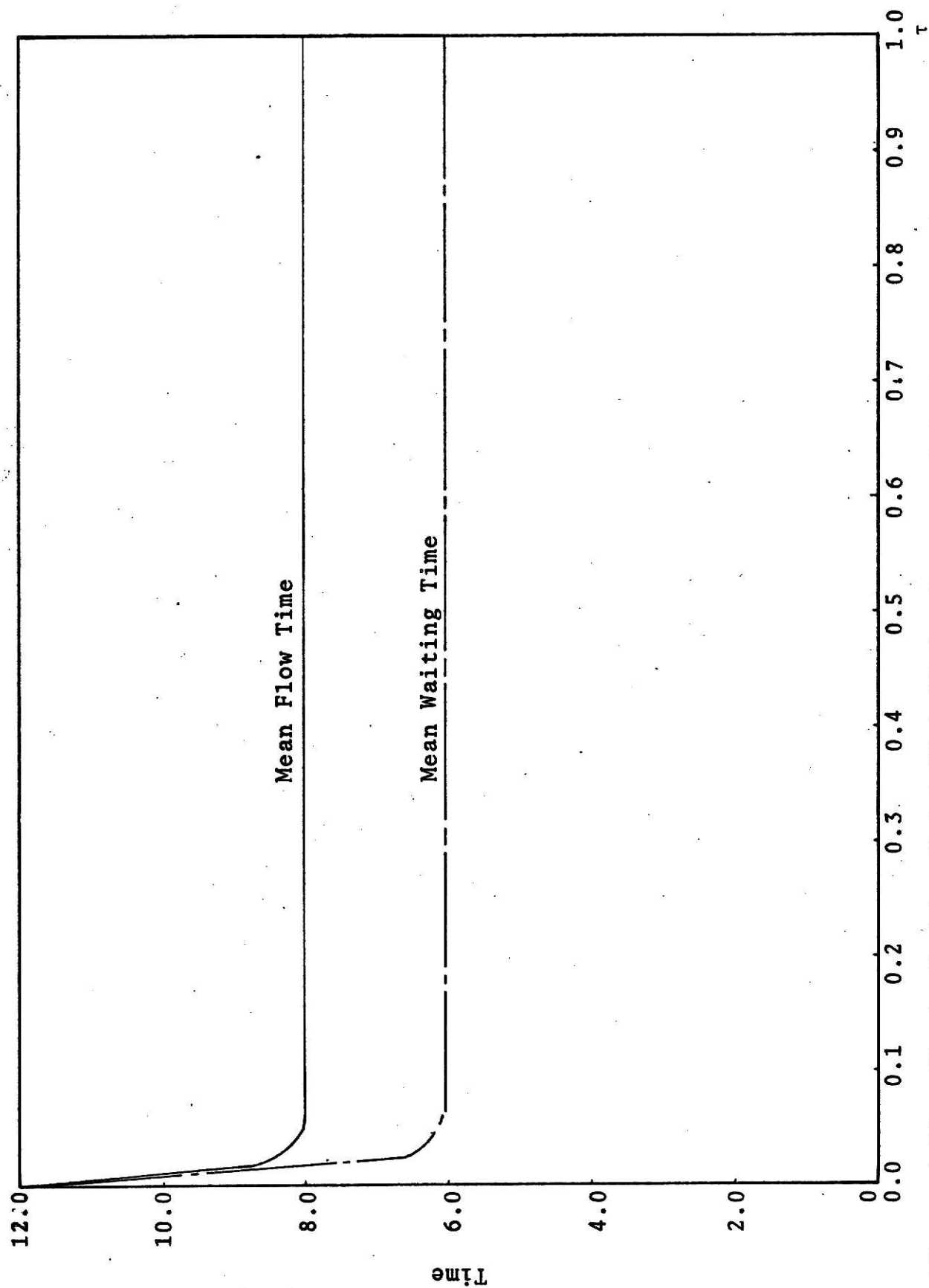


Figure 4.14. Single-Machine Model With Finite Source, Limited Queue, and $\rho=0.5$.

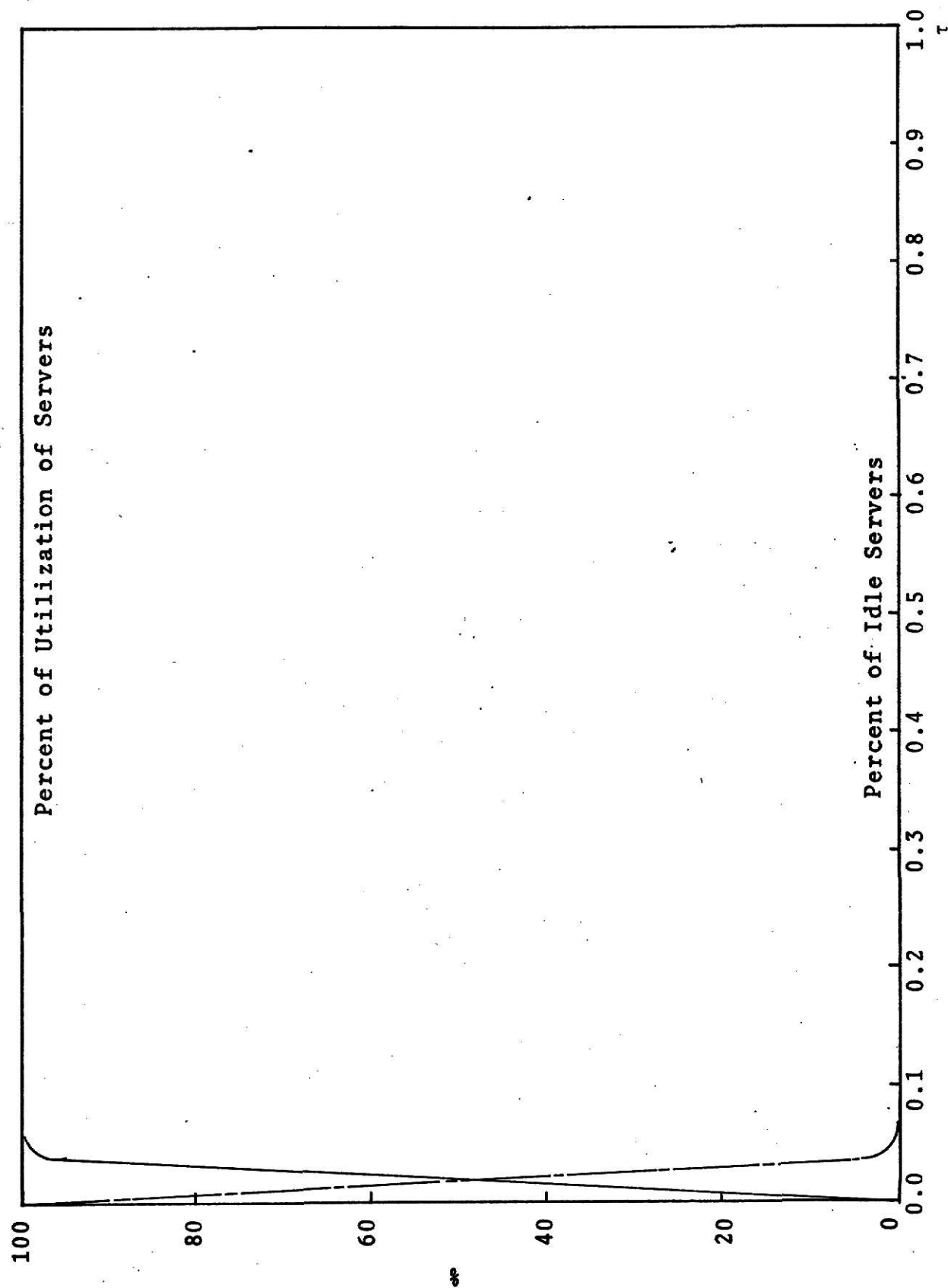


Figure 4.15. Single-Machine Model With Finite Source, Limited Queue, and $\rho = 0.75$.

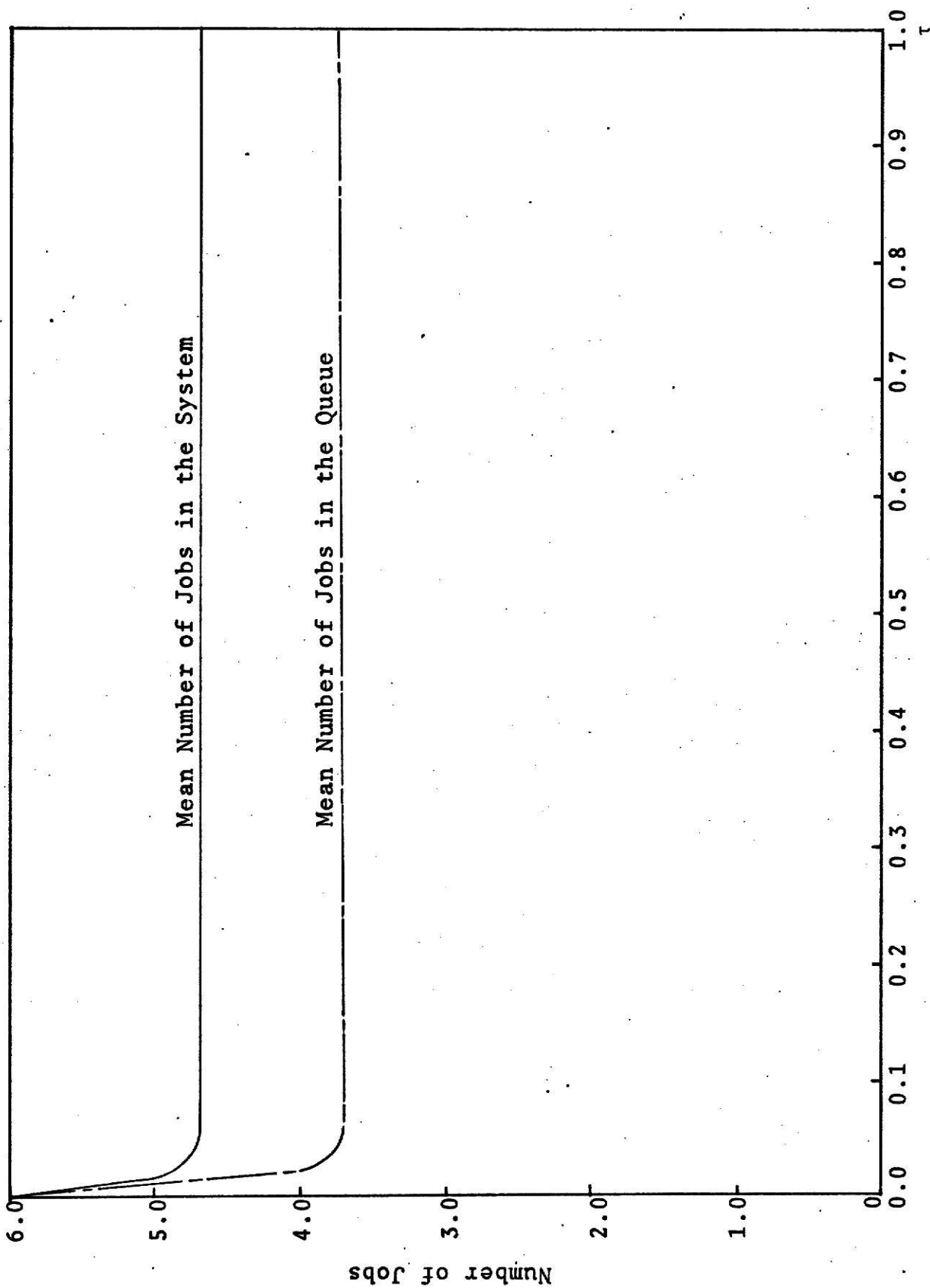


Figure 4.16. Single-Machine Model With Finite Source, Limited Queue, and $\rho = 0.75$.

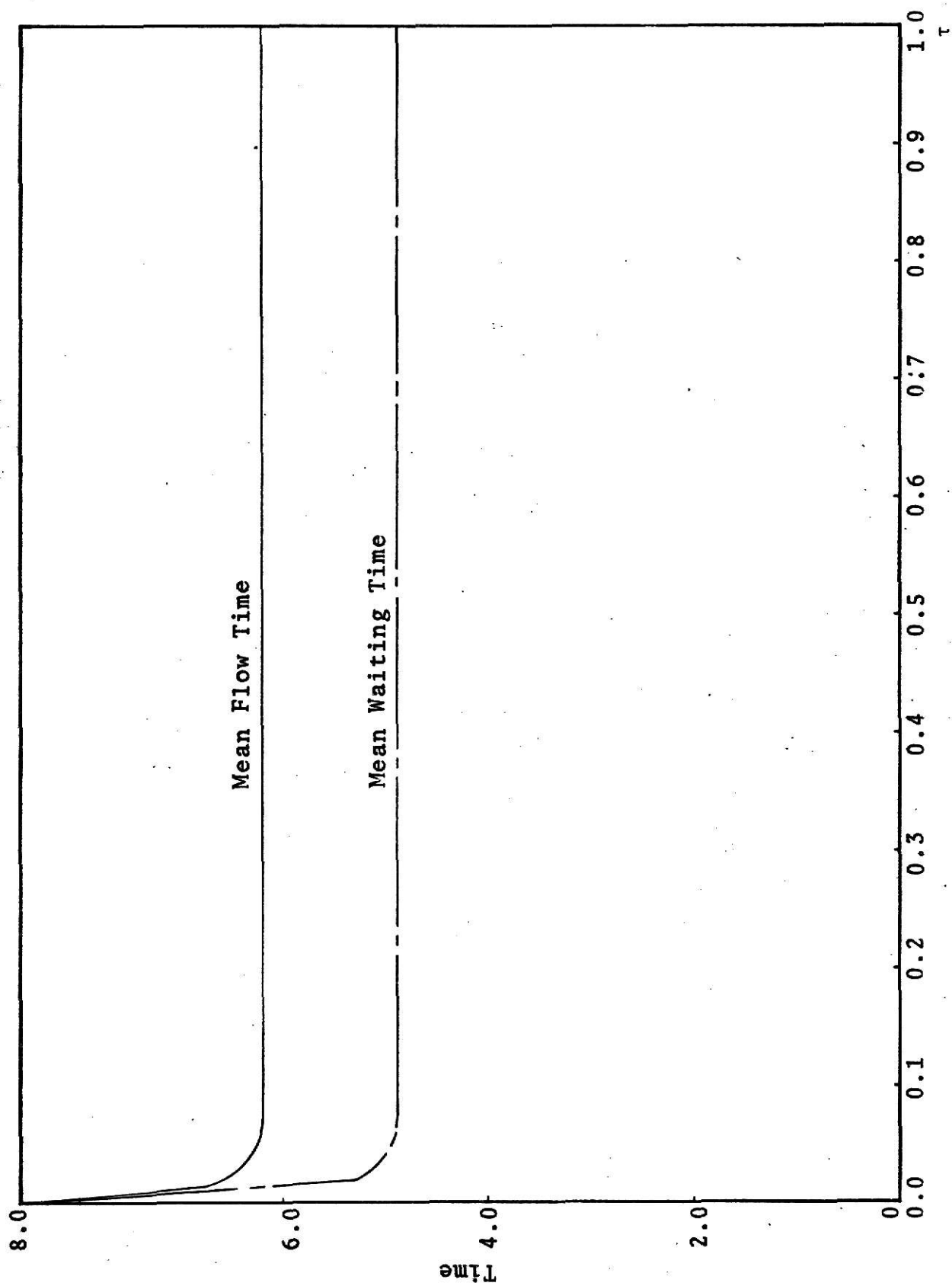


Figure 4.17. Single-Machine Model With Finite Source, Limited Queue, and $\rho = 0.75$.

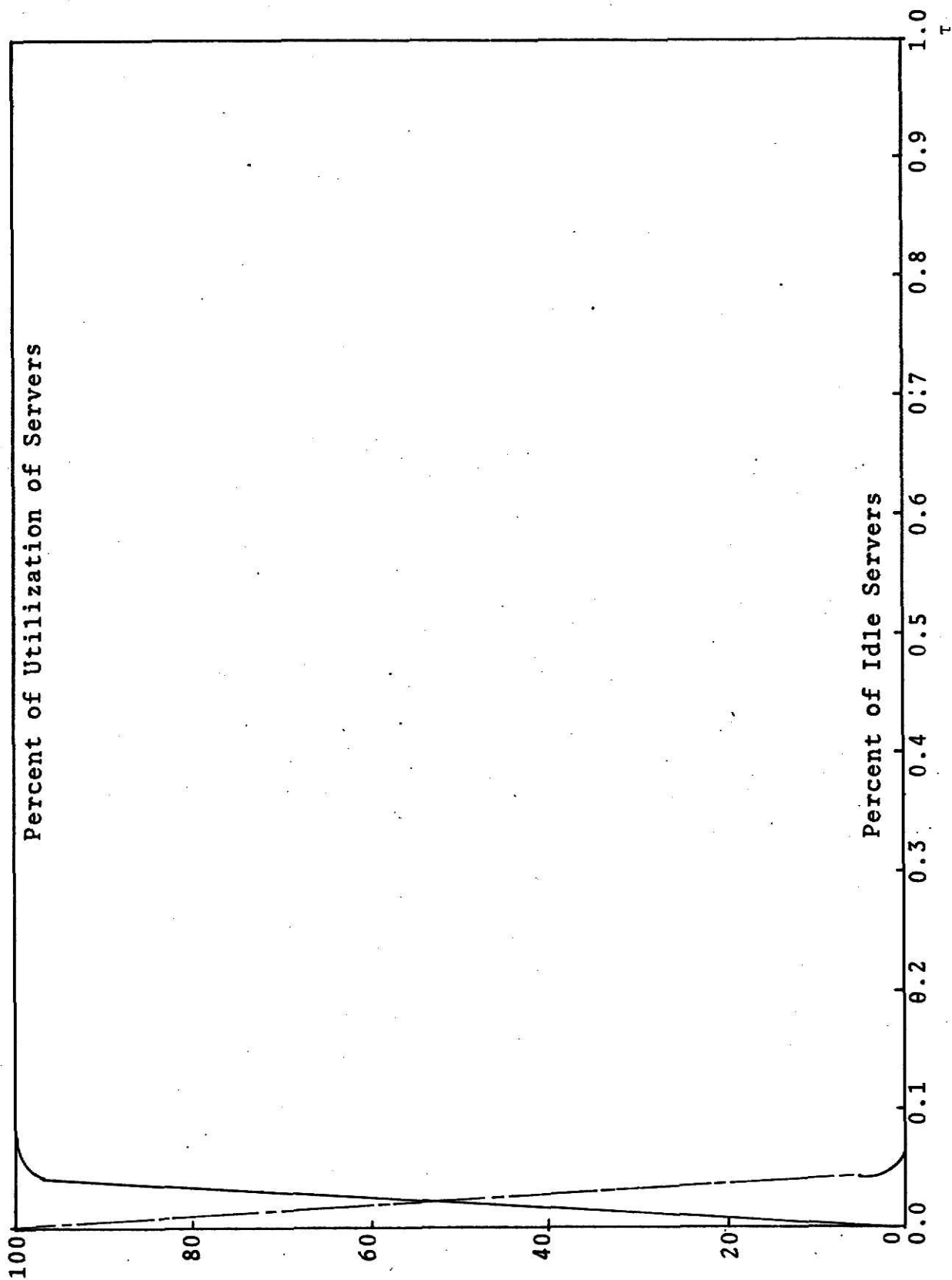


Figure 4.18. Single-Machine Model With Finite Source, Limited Queue, and $\rho = 1.0$.

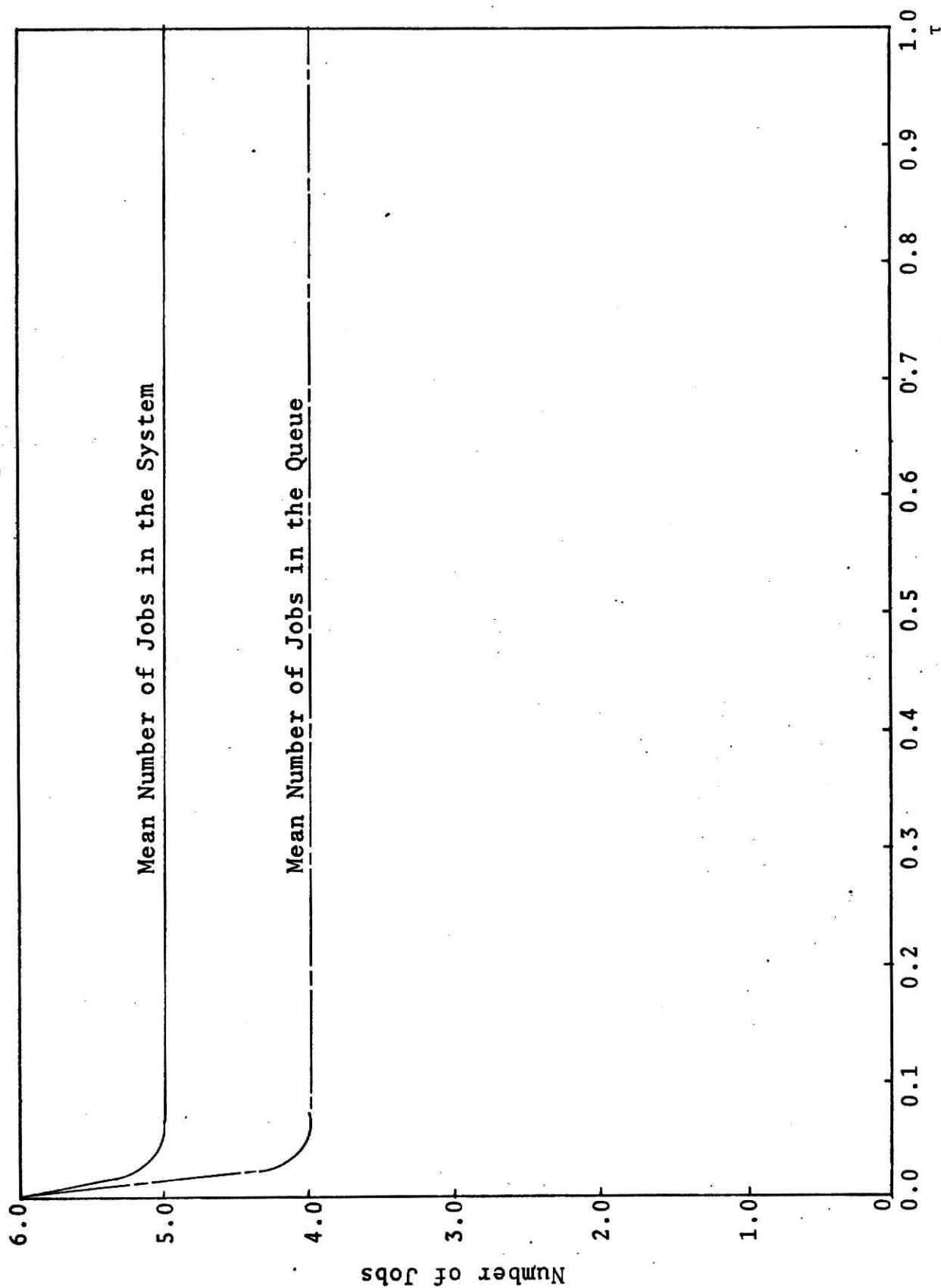


Figure 4.19. Single-Machine Model With Finite Source, Limited Queue, and $\rho = 1.0$.

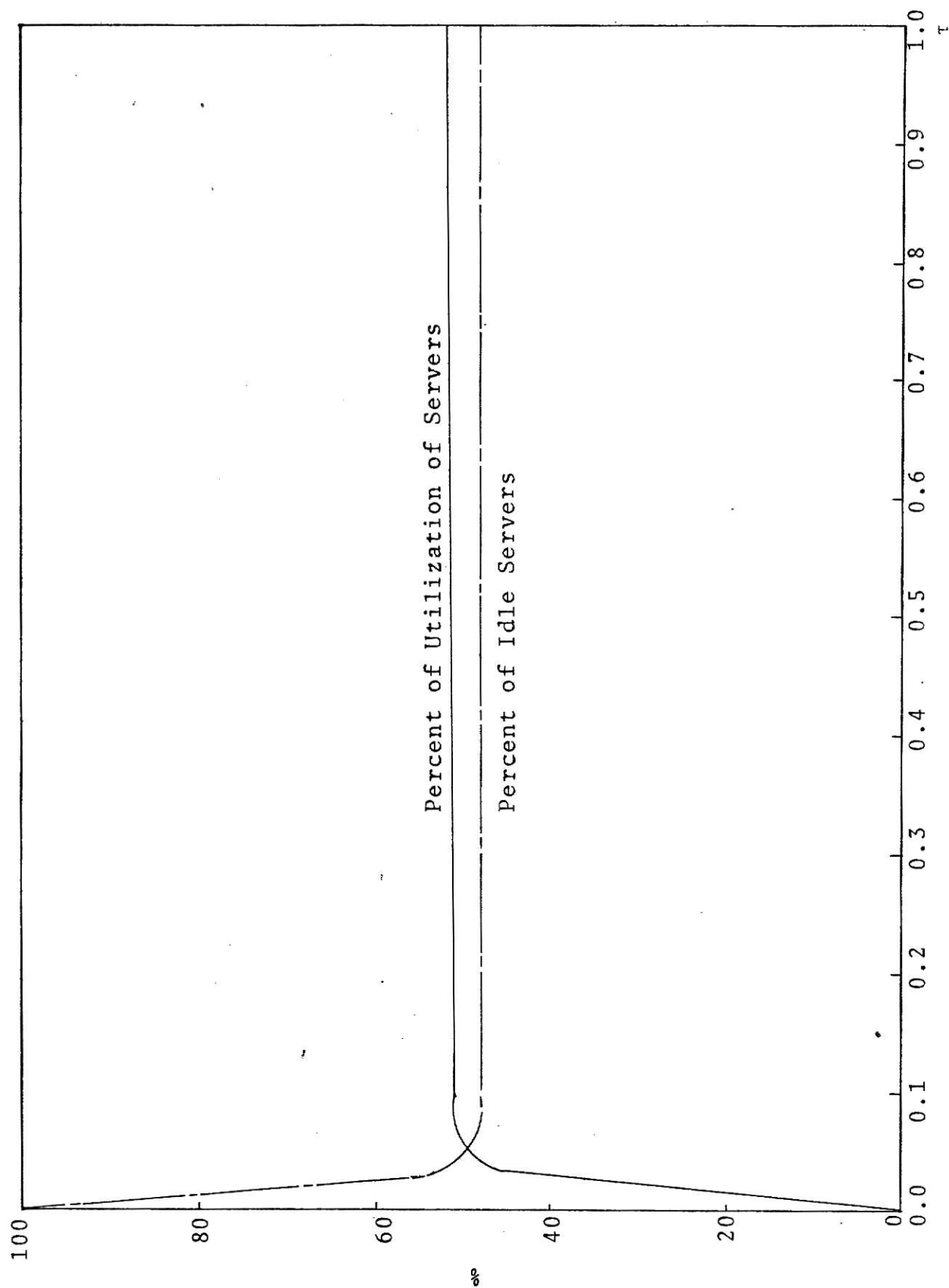


Figure 4.20. Multiple-Machine Model With Infinite Source, No Queue, and $\rho = 0.75$.

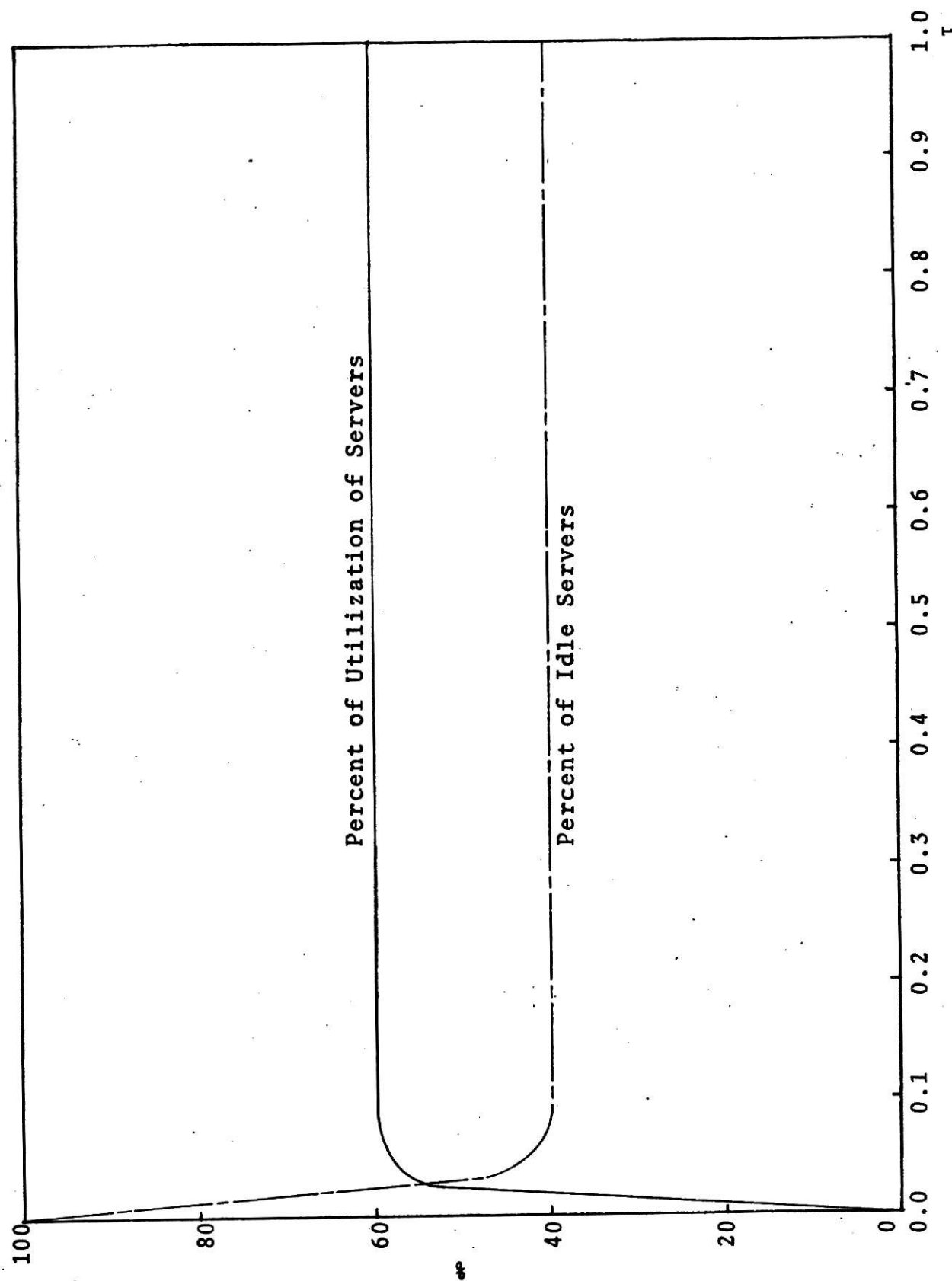


Figure 4.21. Multiple-Machine Model With Infinite Source, No Queue, and $\rho = 1.0$.

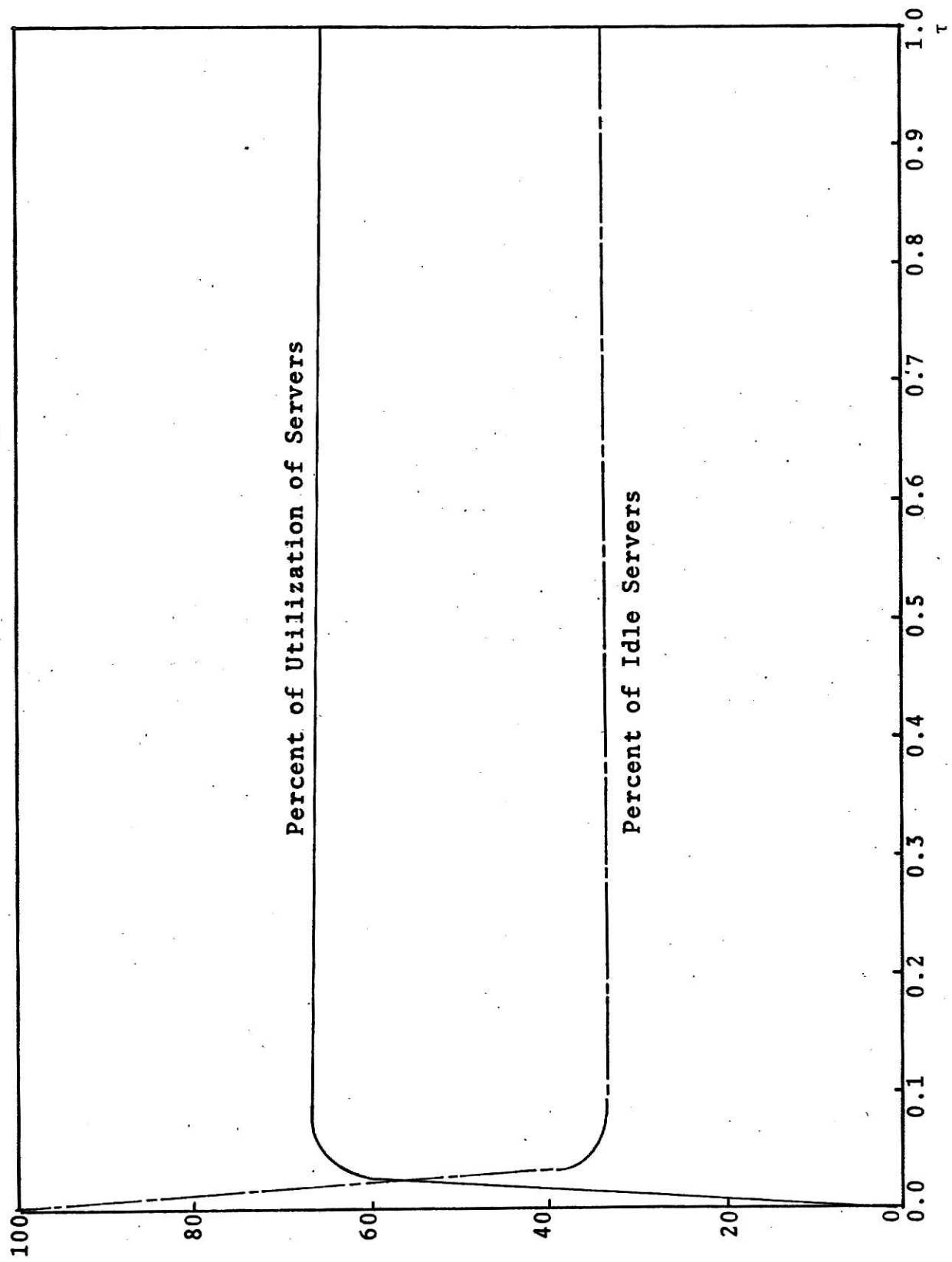


Figure 4.22. Multiple-Machine Model With Infinite Source, No Queue, and $\rho = 1.25$.

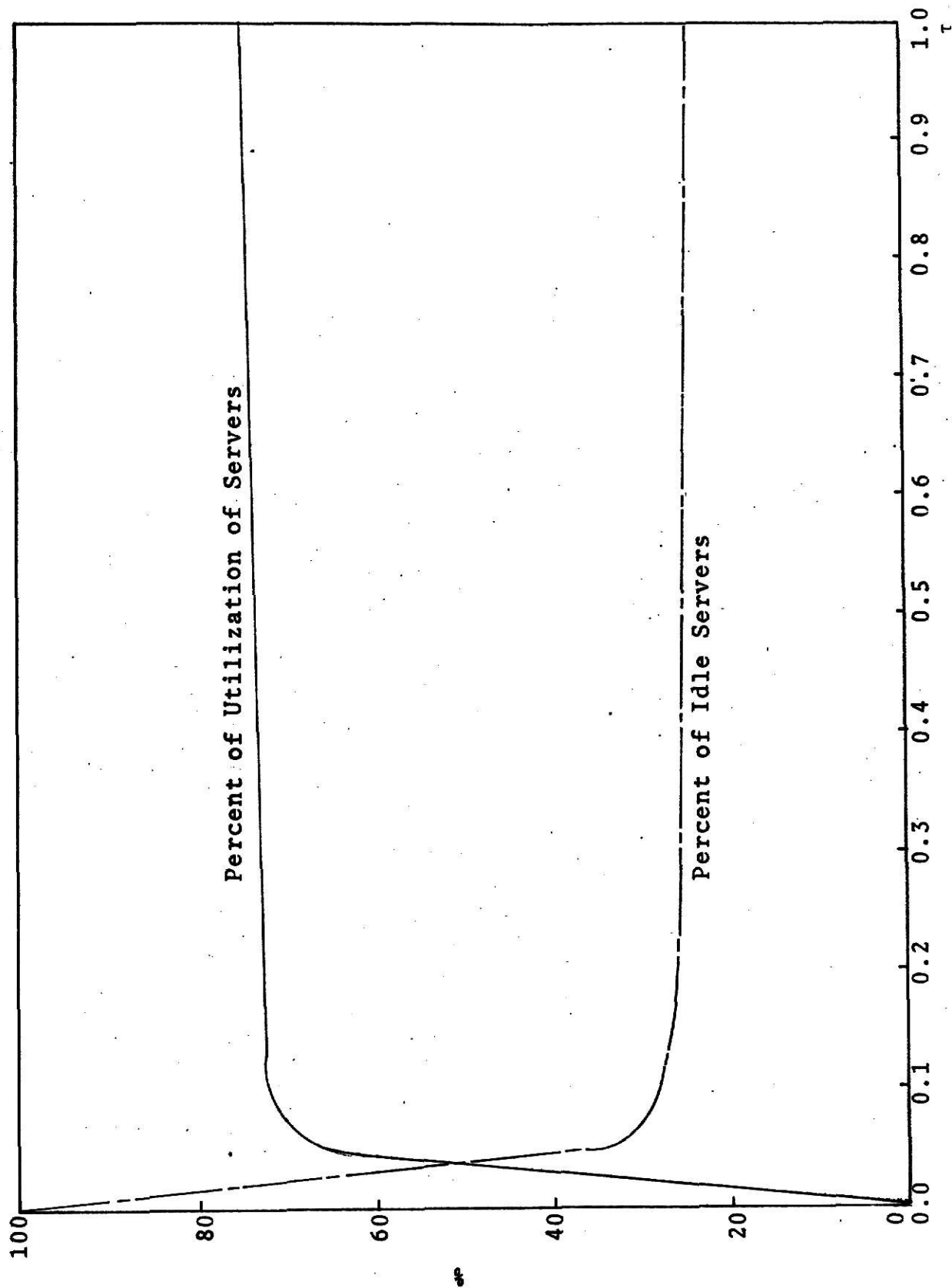


Figure 4.23. Multiple-Machine Model With Infinite Source, Unlimited Queue, and $\rho = 0.75$.

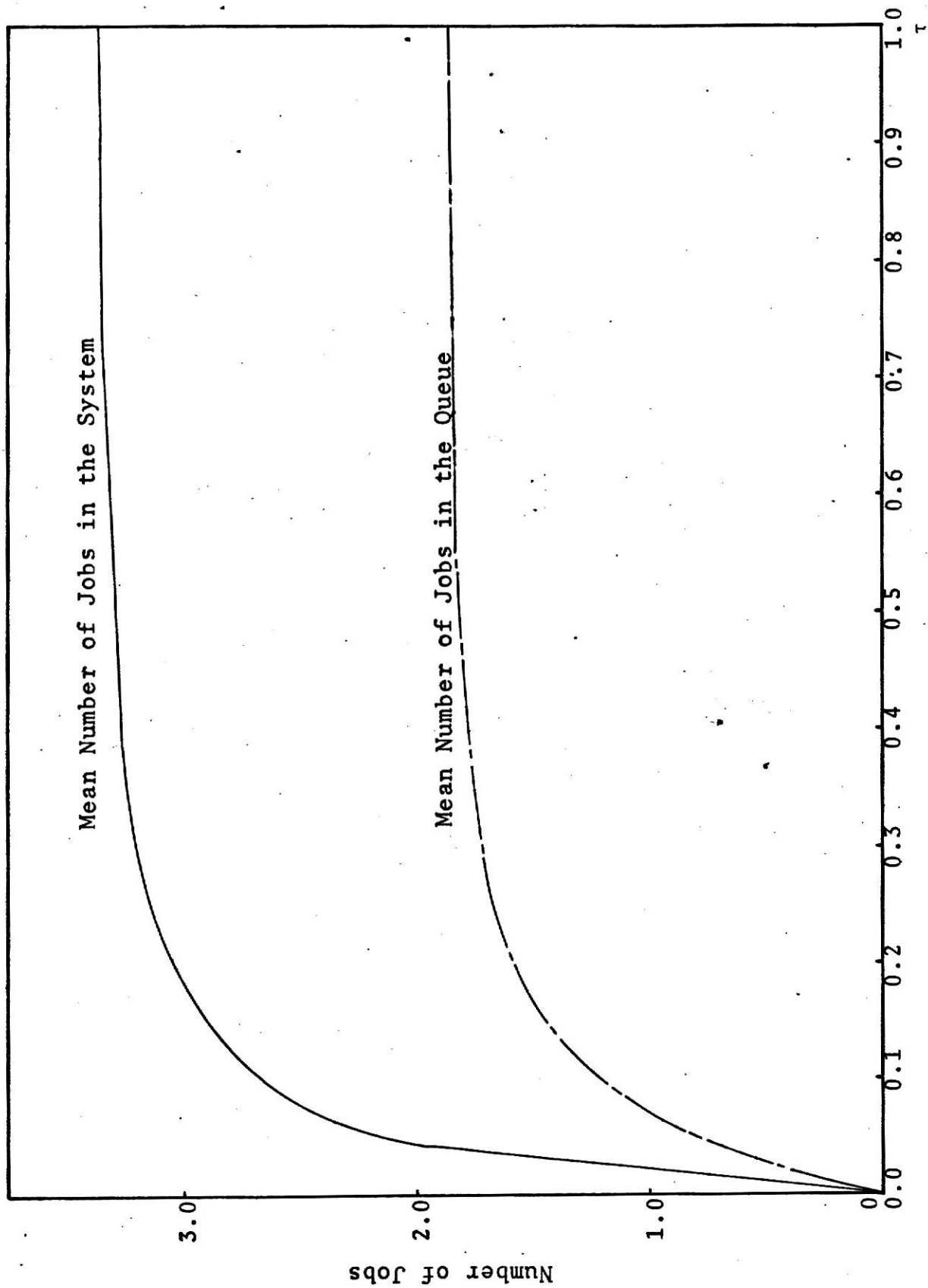


Figure 4.24. Multiple-Machine Model With Infinite Source, Unlimited Queue, and $\rho=0.75$.

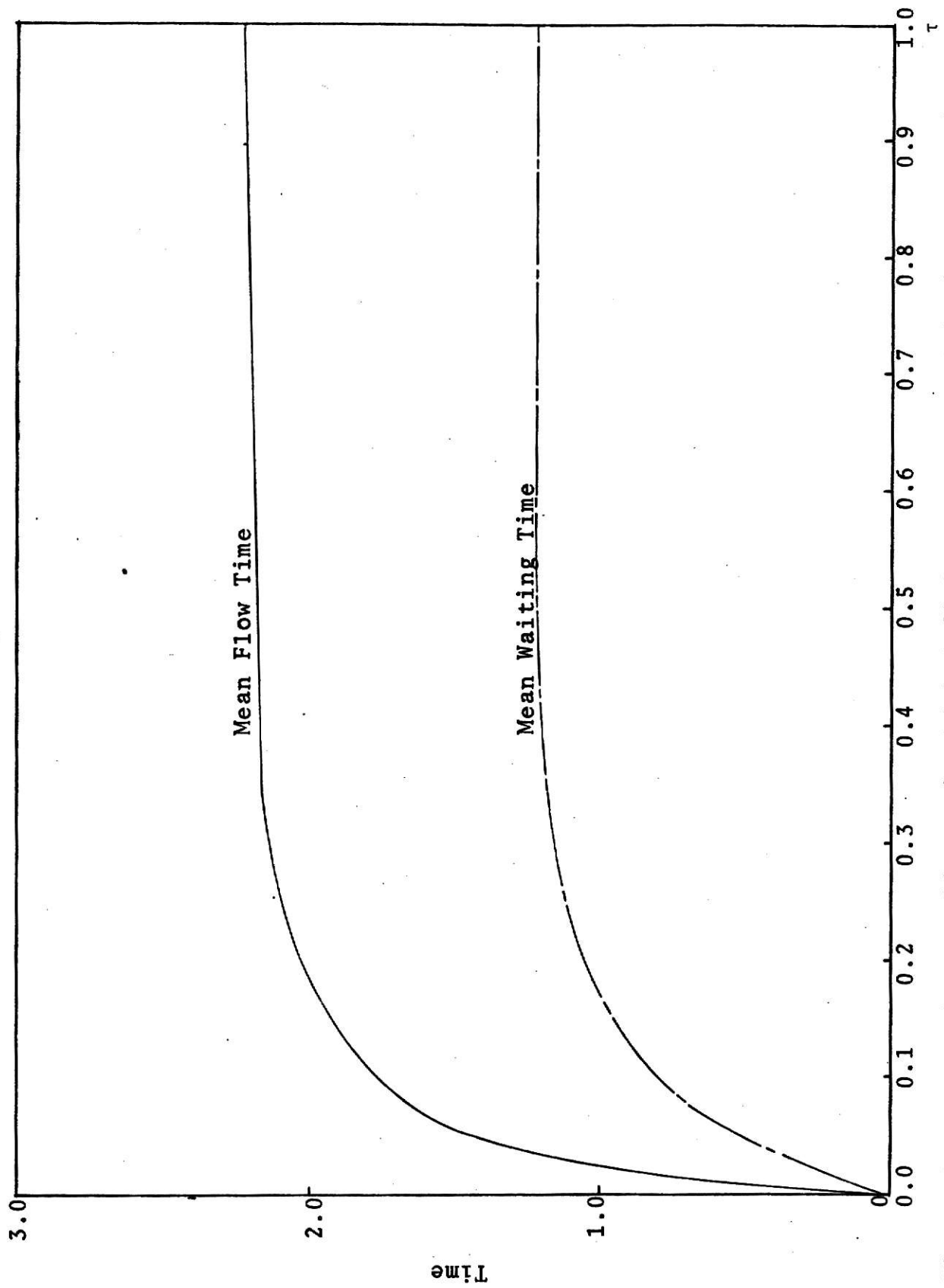


Figure 4.25. Multiple-Machine Model With Infinite Source, Unlimited Queue, and $\rho=0.75$.

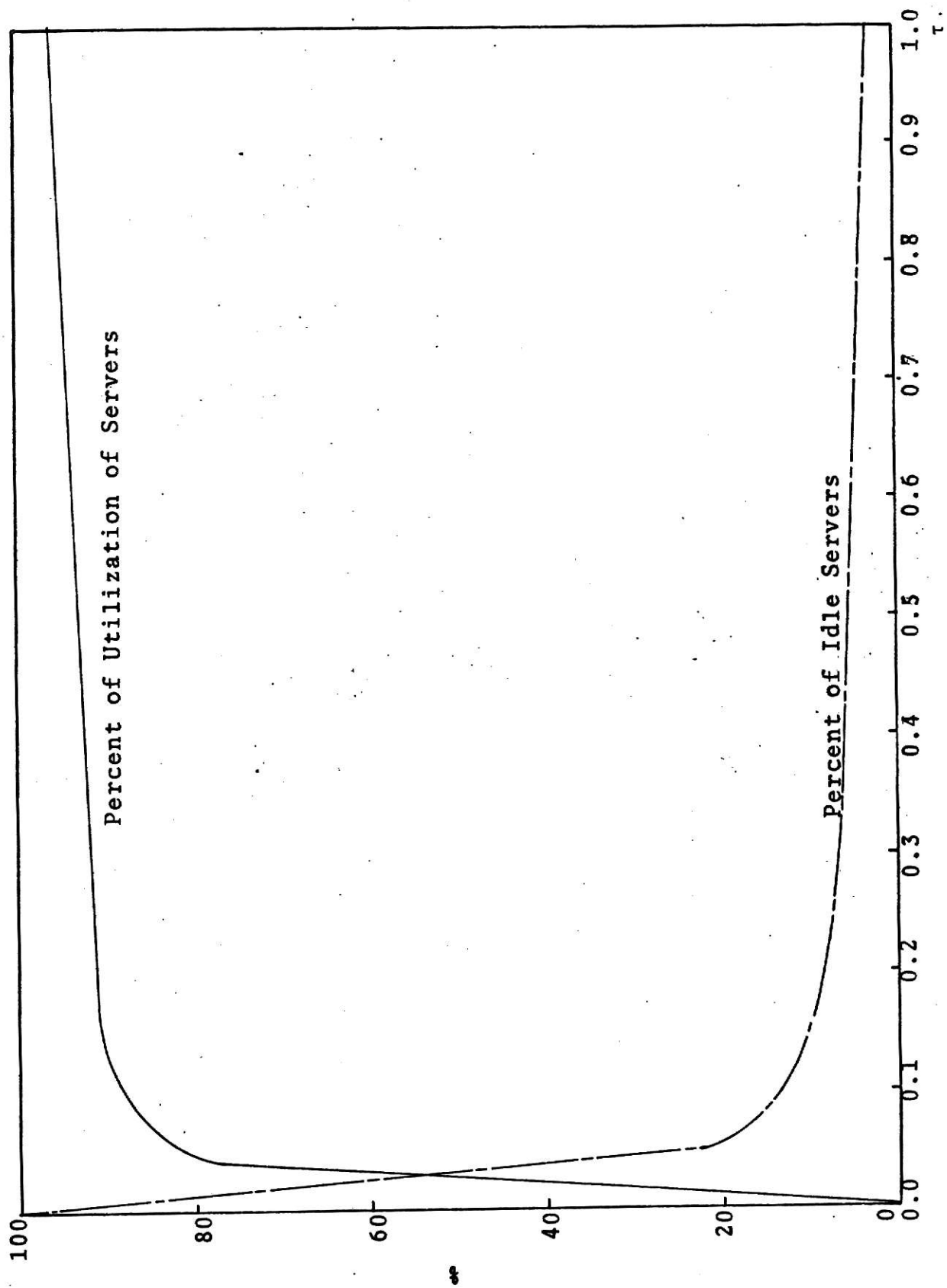


Figure 4.26. Multiple-Machine Model With Infinite Source, Unlimited Queue, and $\rho=1.0$.

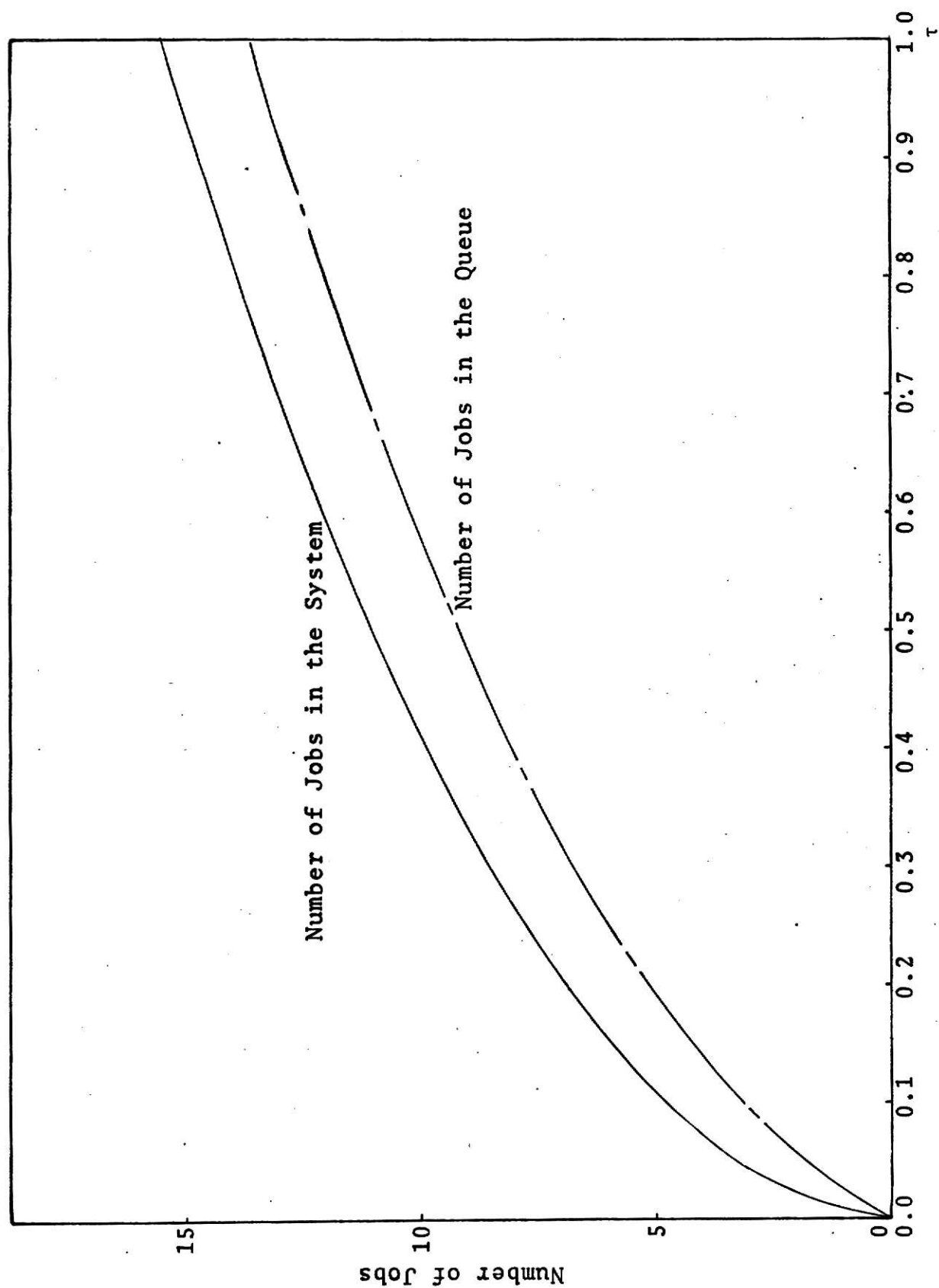


Figure 4.27. Multiple-Machine Model With Infinite Source, Unlimited Queue, and $\rho=1.0$.

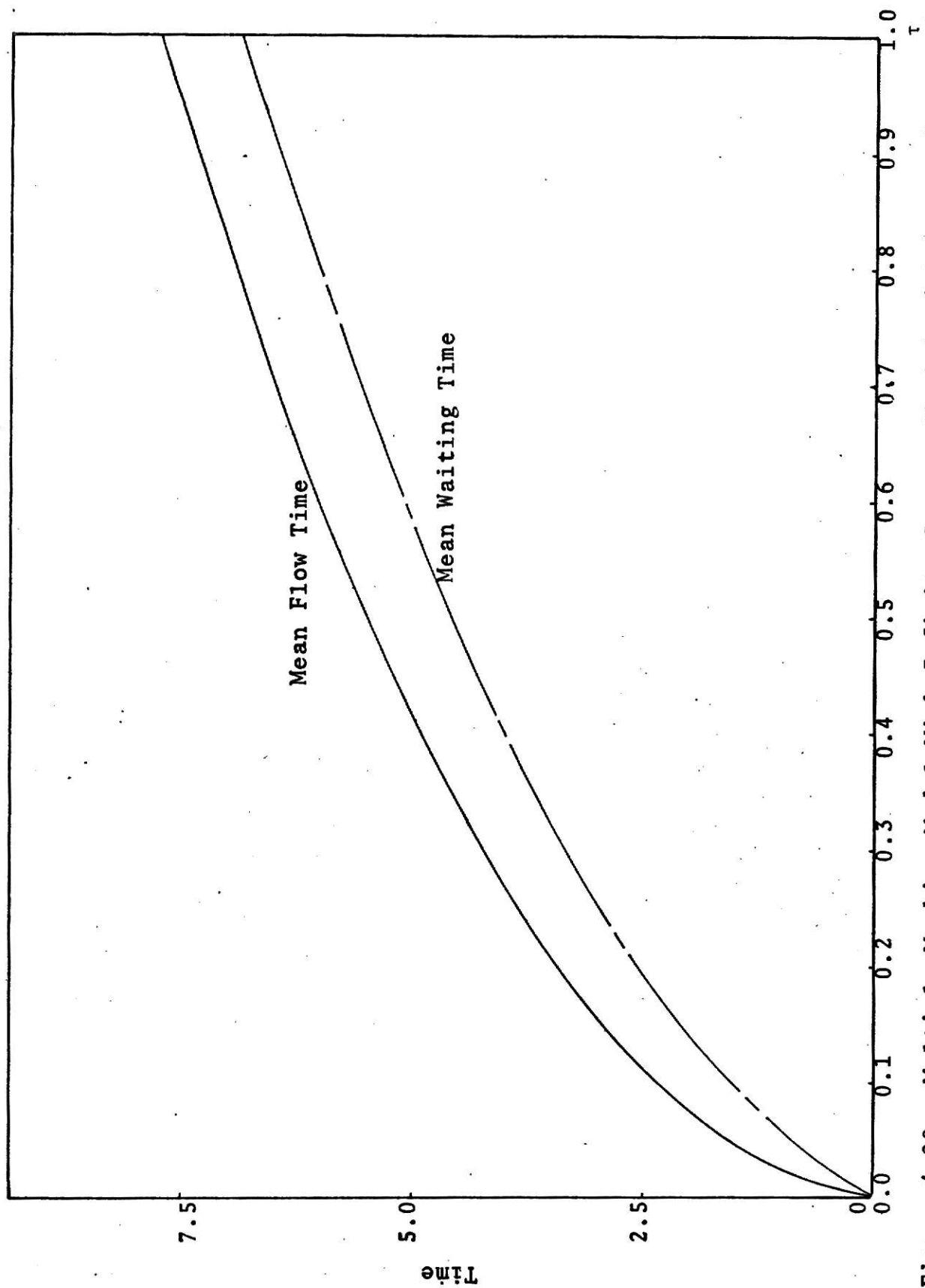


Figure 4.28. Multiple-Machine Model With Infinite Source, Unlimited Queue, and $\rho=1.0$.

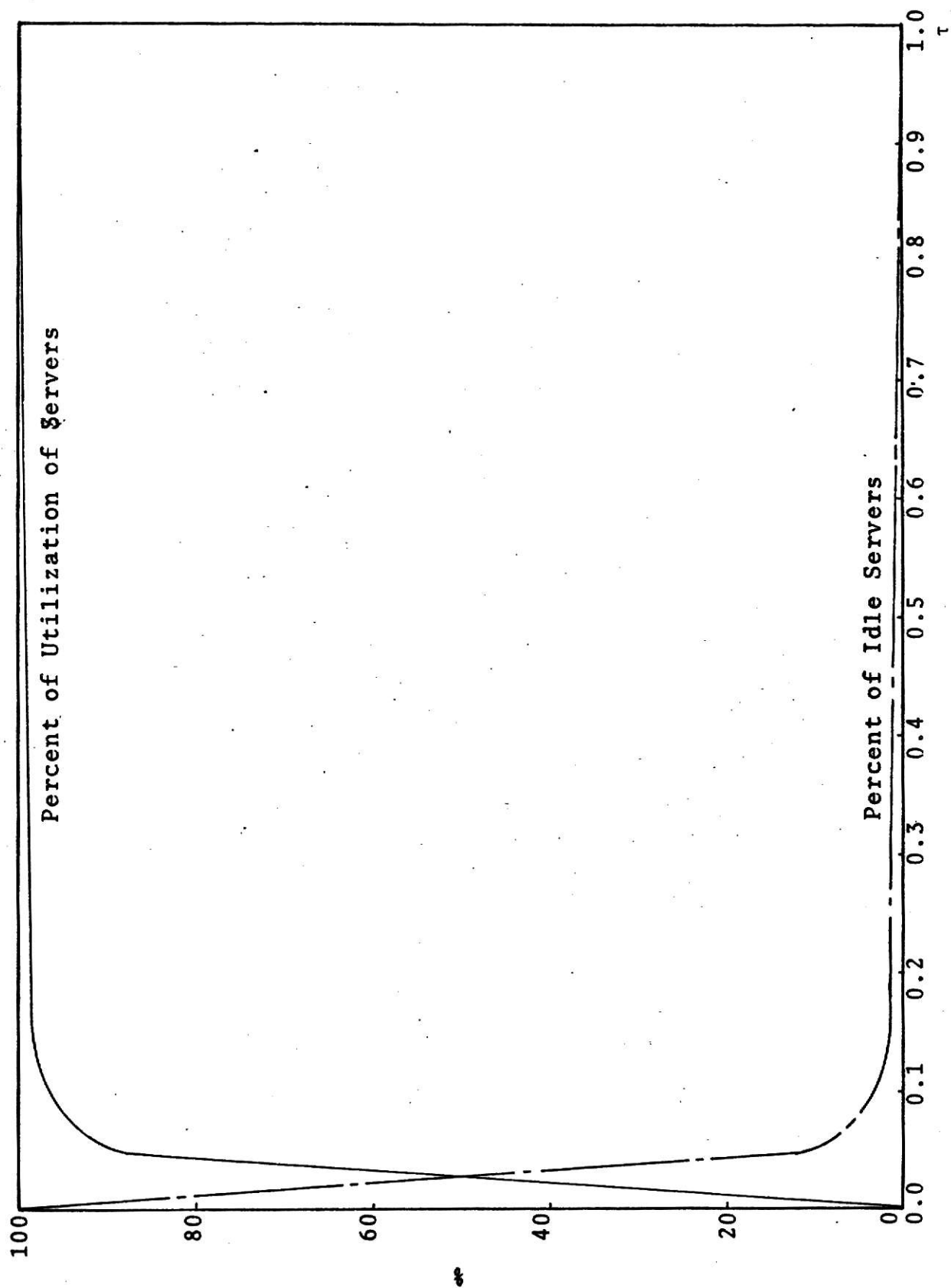


Figure 4.29. Multiple-Machine Model With Infinite Source, Unlimited Queue, and $\rho=1.25$.

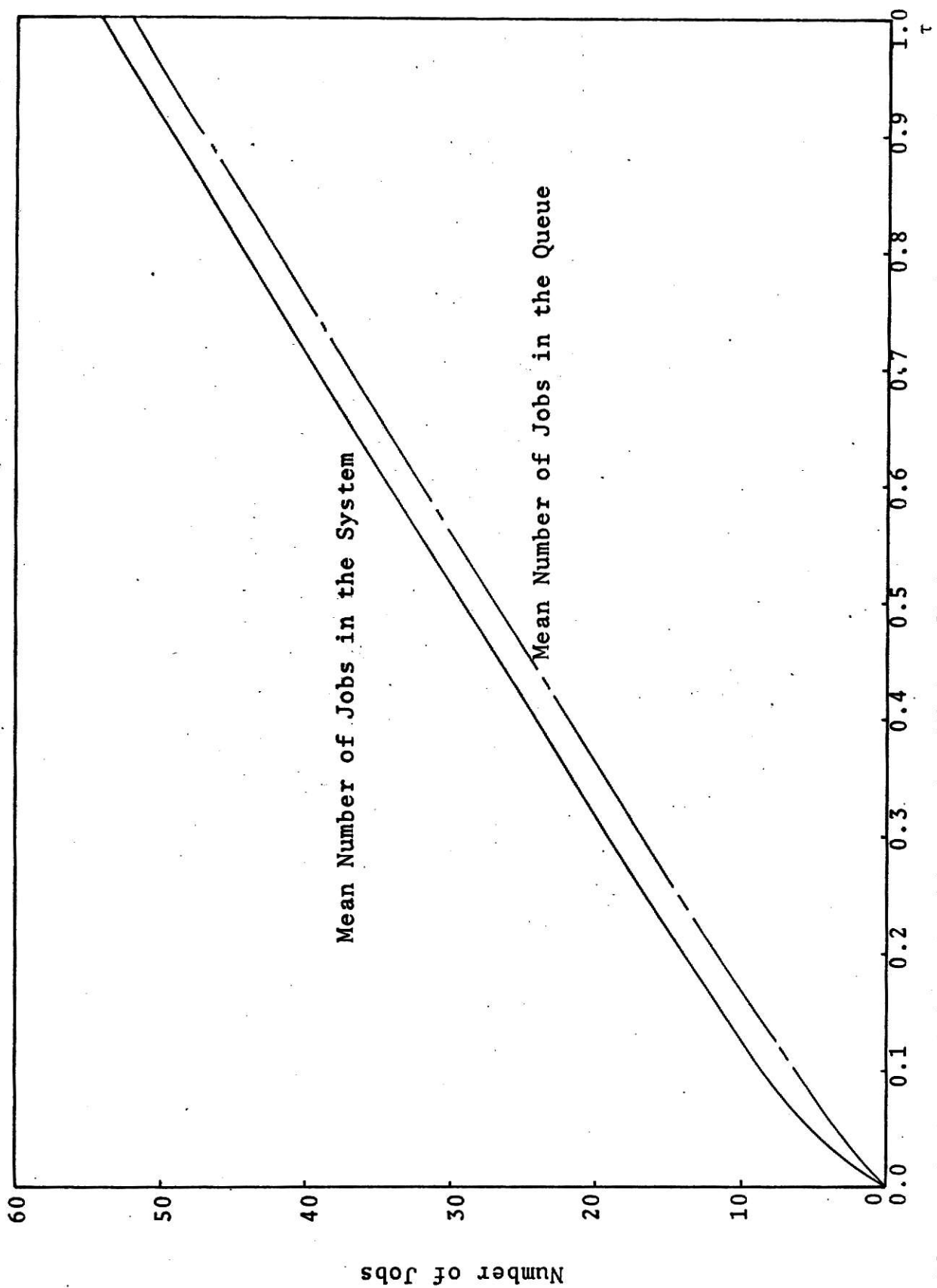


Figure 4.30. Multiple-Machine Model With Infinite Source, Unlimited Queue, and $\rho=1.25$.

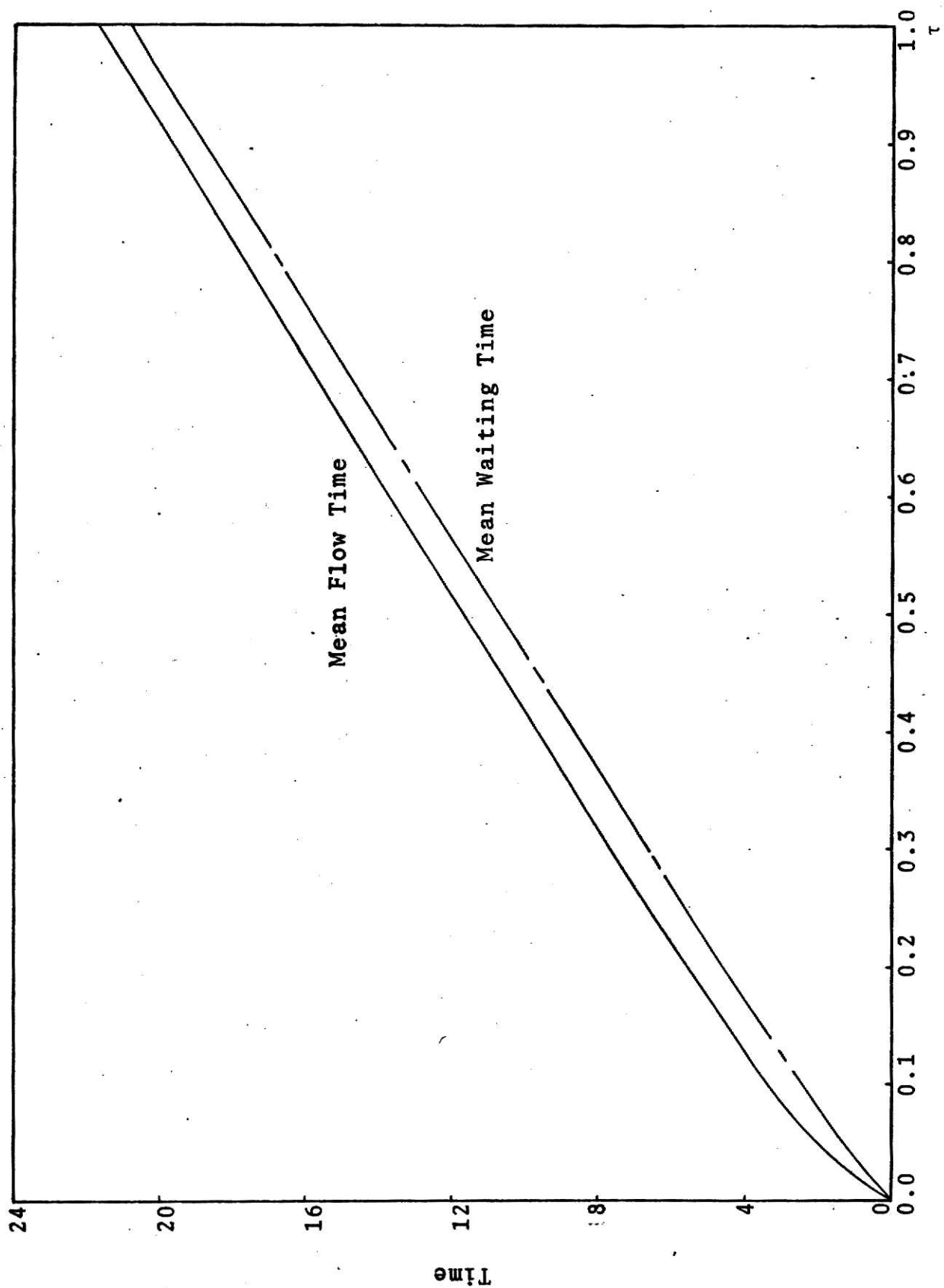


Figure 4.31. Multiple-Machine Model With Infinite Source, Unlimited Queue, and $\rho=1.25$.

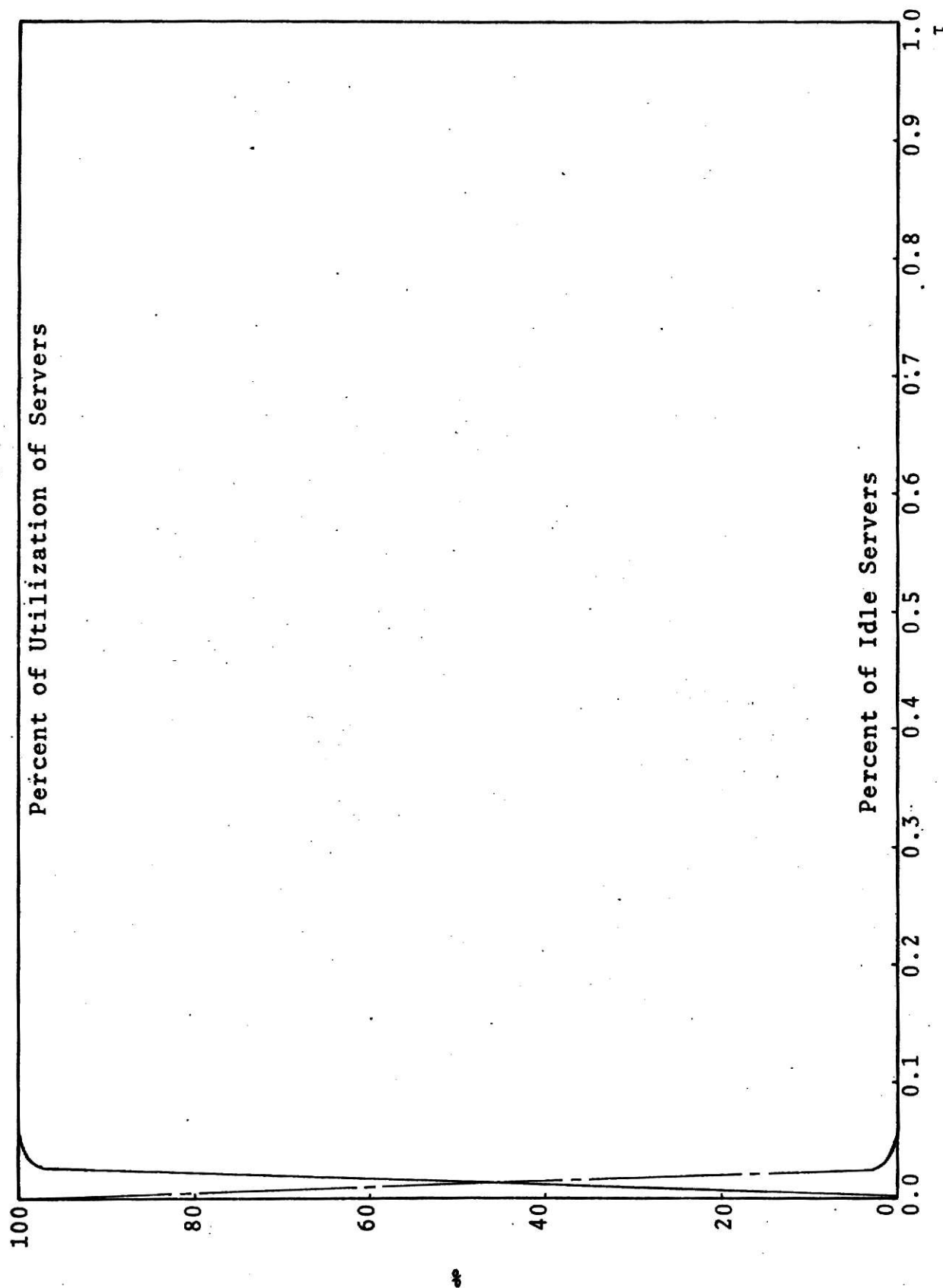


Figure 4.32. Multiple-Machine Model With Finite Source, Limited Queue, and $\rho = 0.5$.

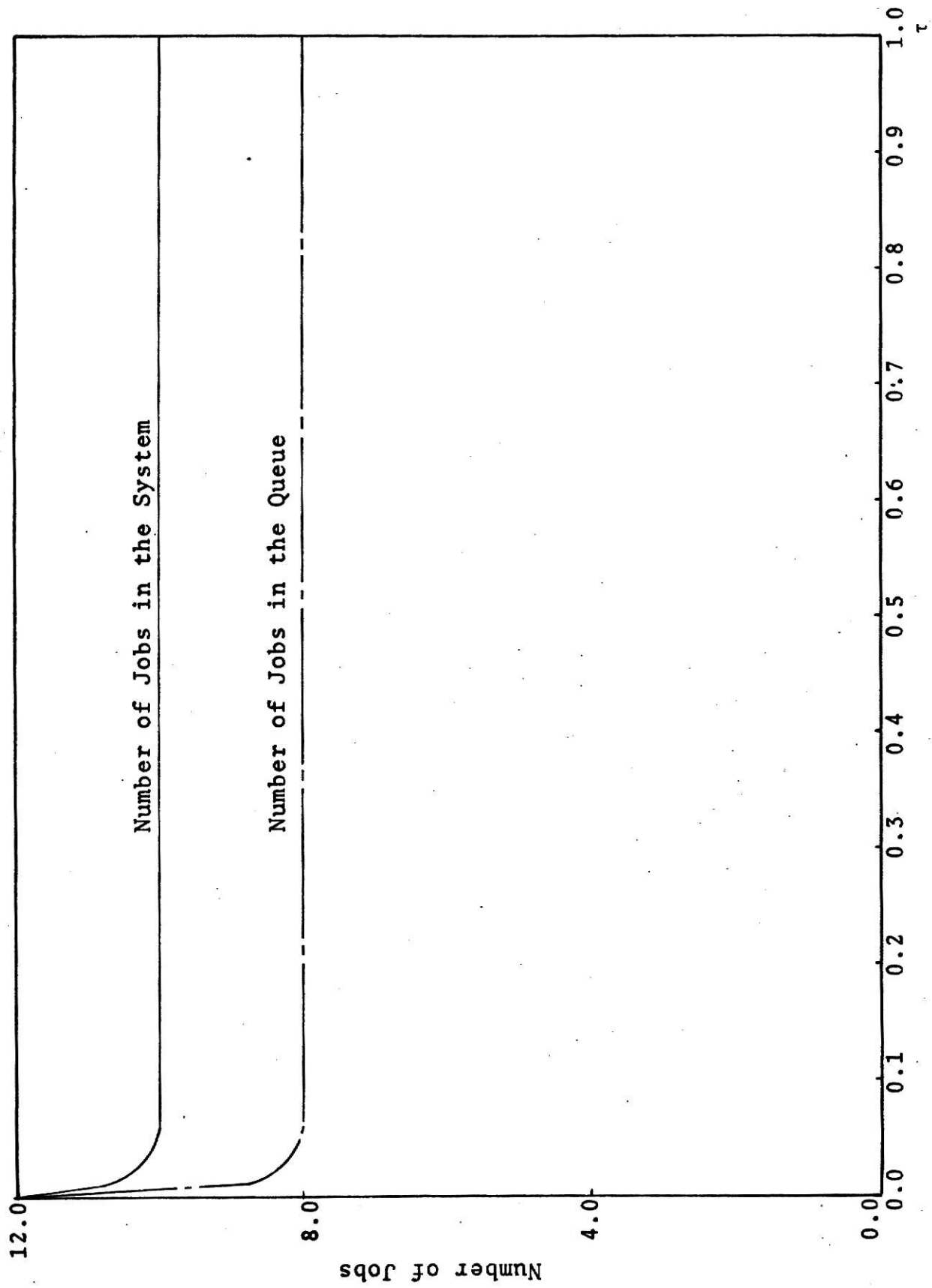


Figure 4.33. Multiple-Machine Model With Finite Source, Limited Queue, and $\rho = 0.5$.

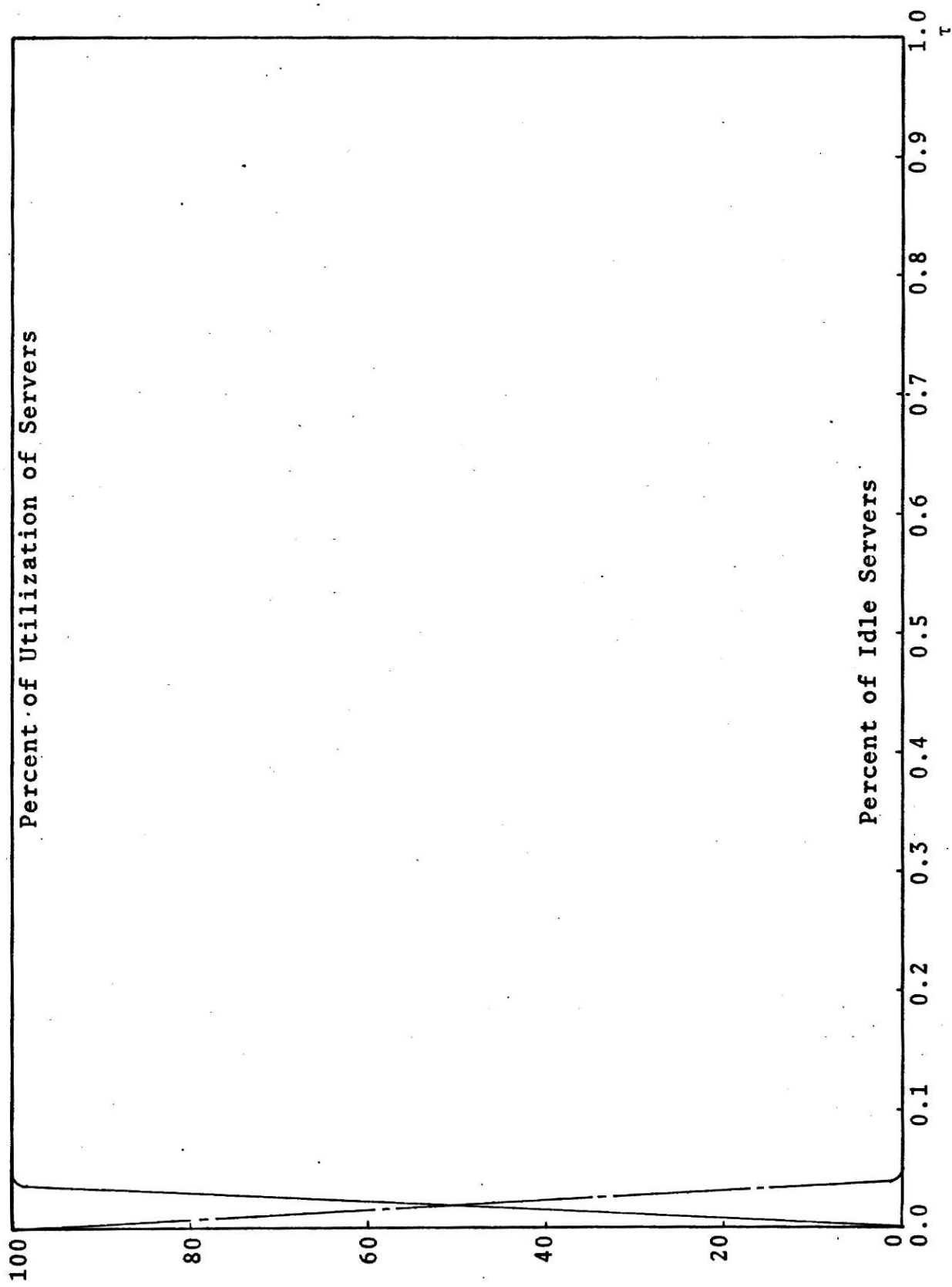


Figure 4.34. Multiple-Machine Model With Finite Source, Limited Queue, and $\rho = 0.75$.

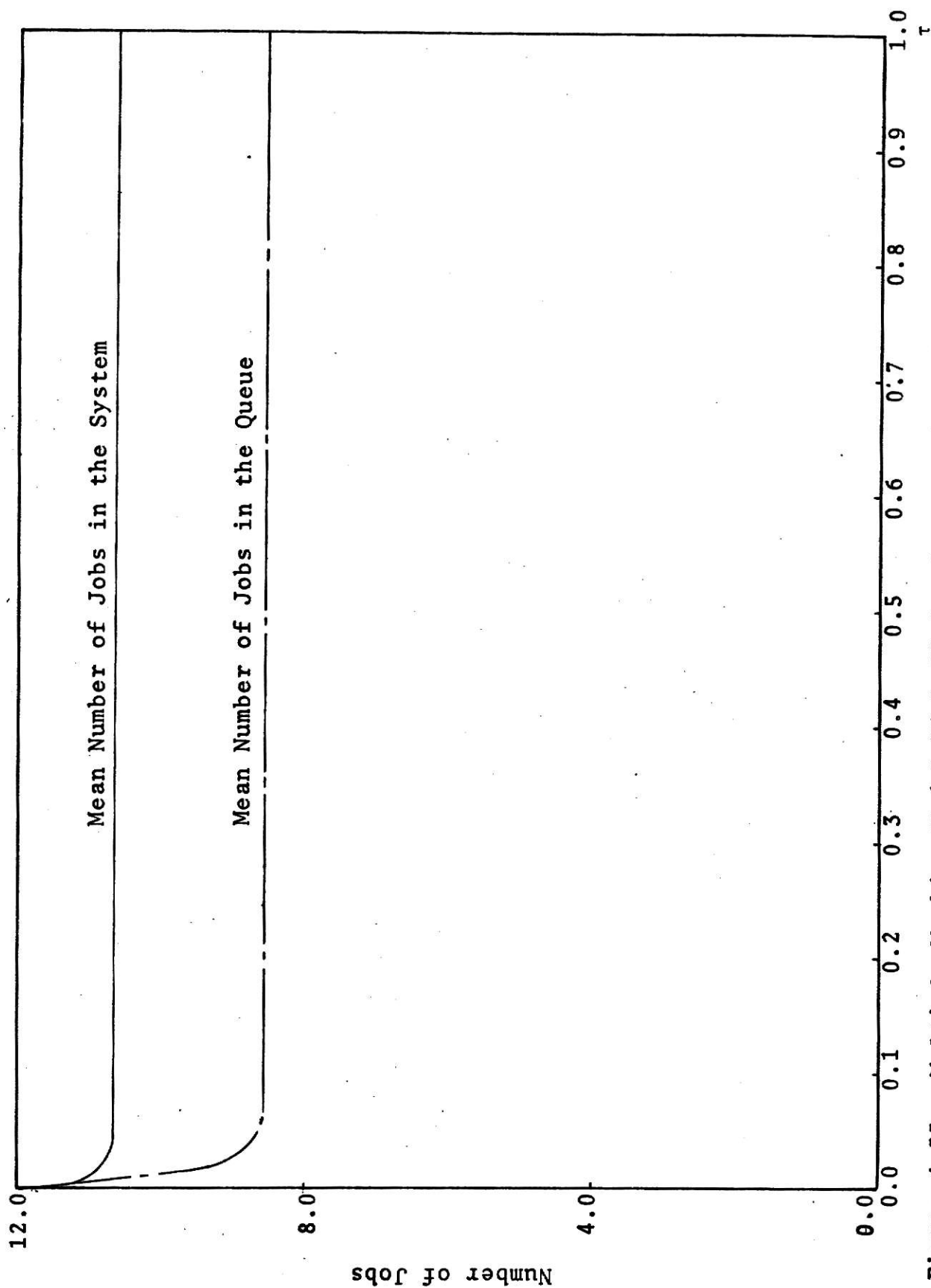


Figure 4.35. Multiple-Machine Model With Finite Source, Limited Queue, and $\rho = 0.75$.

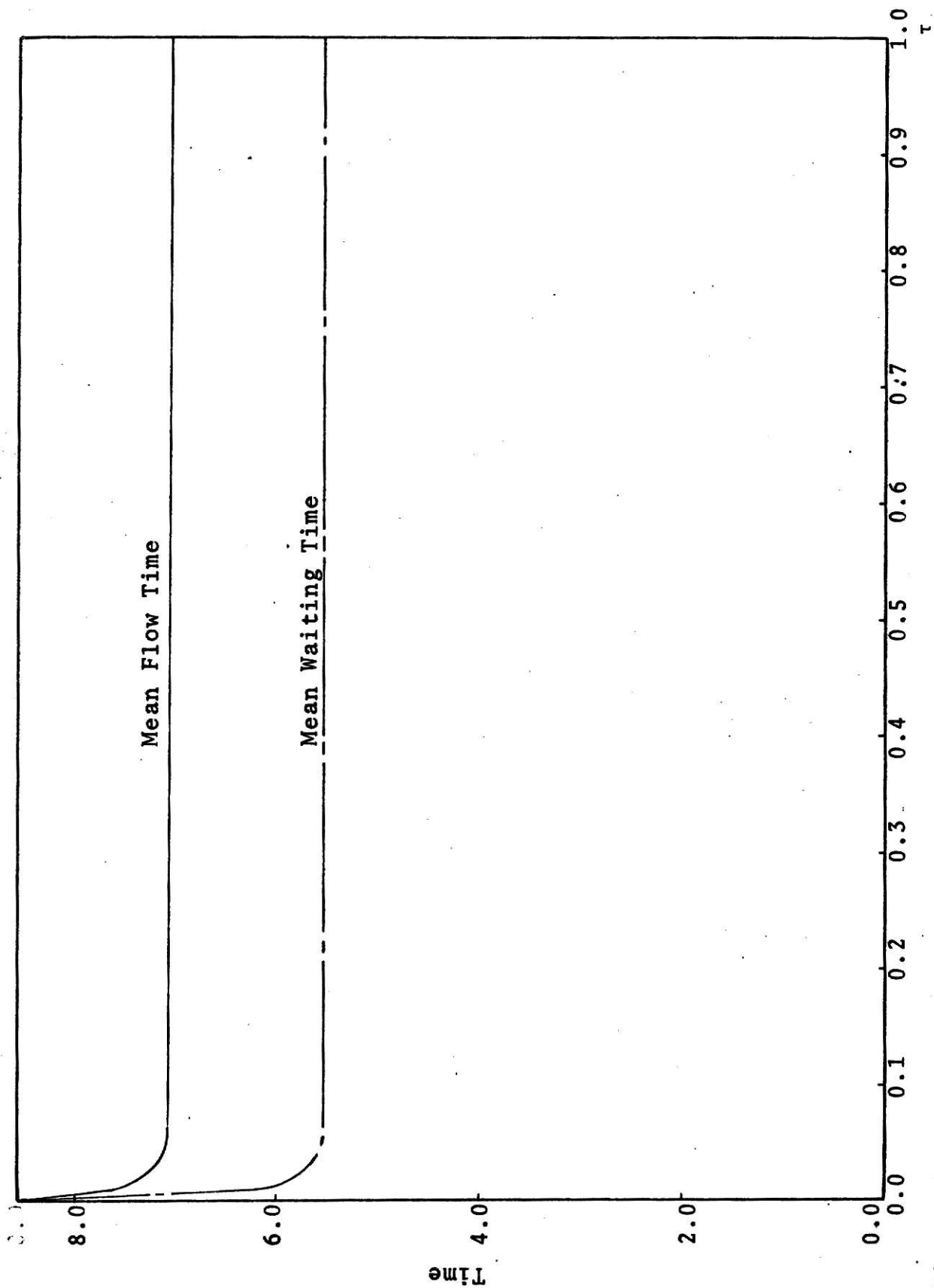


Figure 4.36. Multiple-Machine Model With Finite Source, Limited Queue, and $\rho = 0.75$.

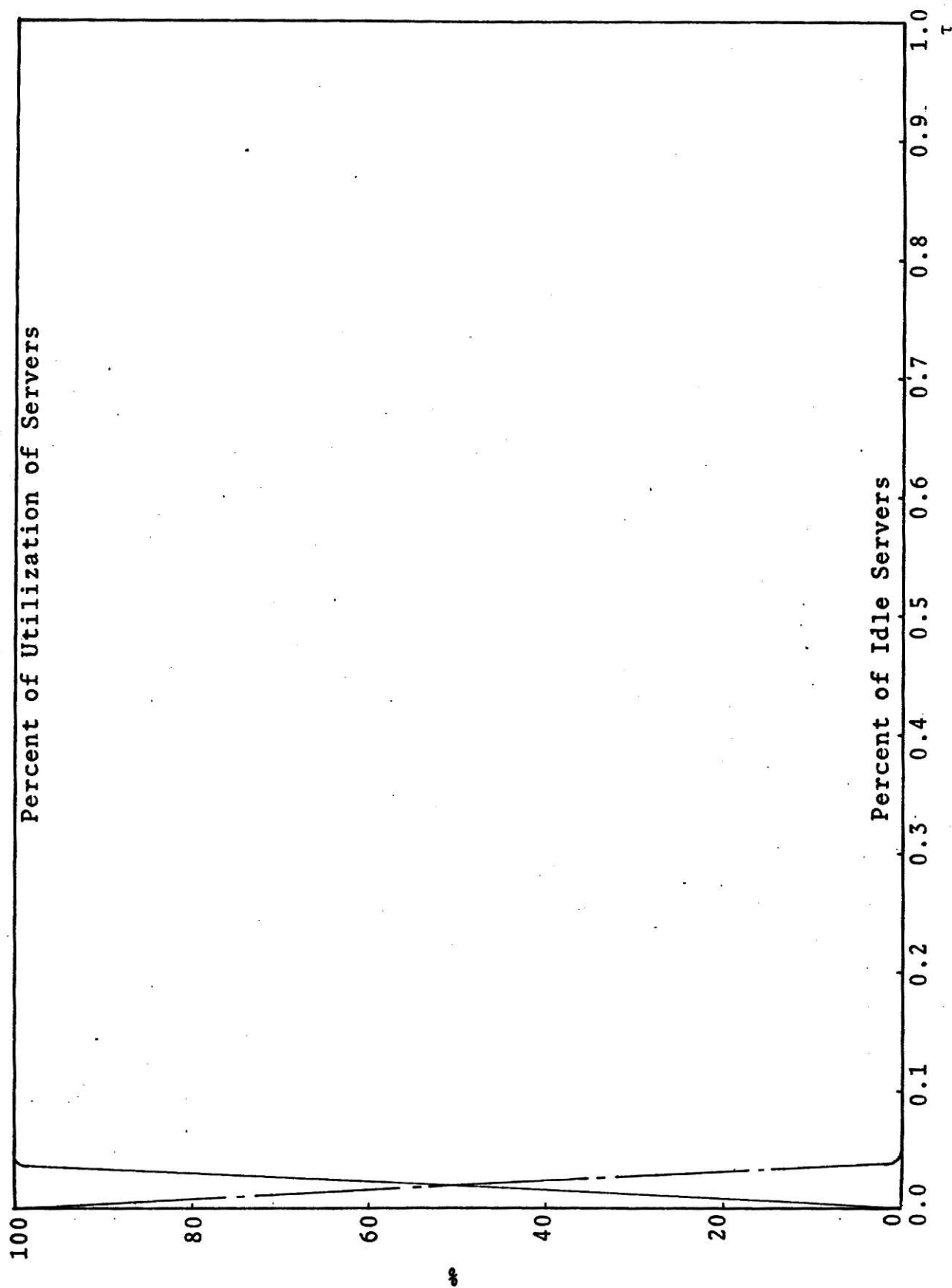


Figure 4.37. Multiple-Machine Model With Finite Source, Limited Queue, and $\rho = 1.0$.

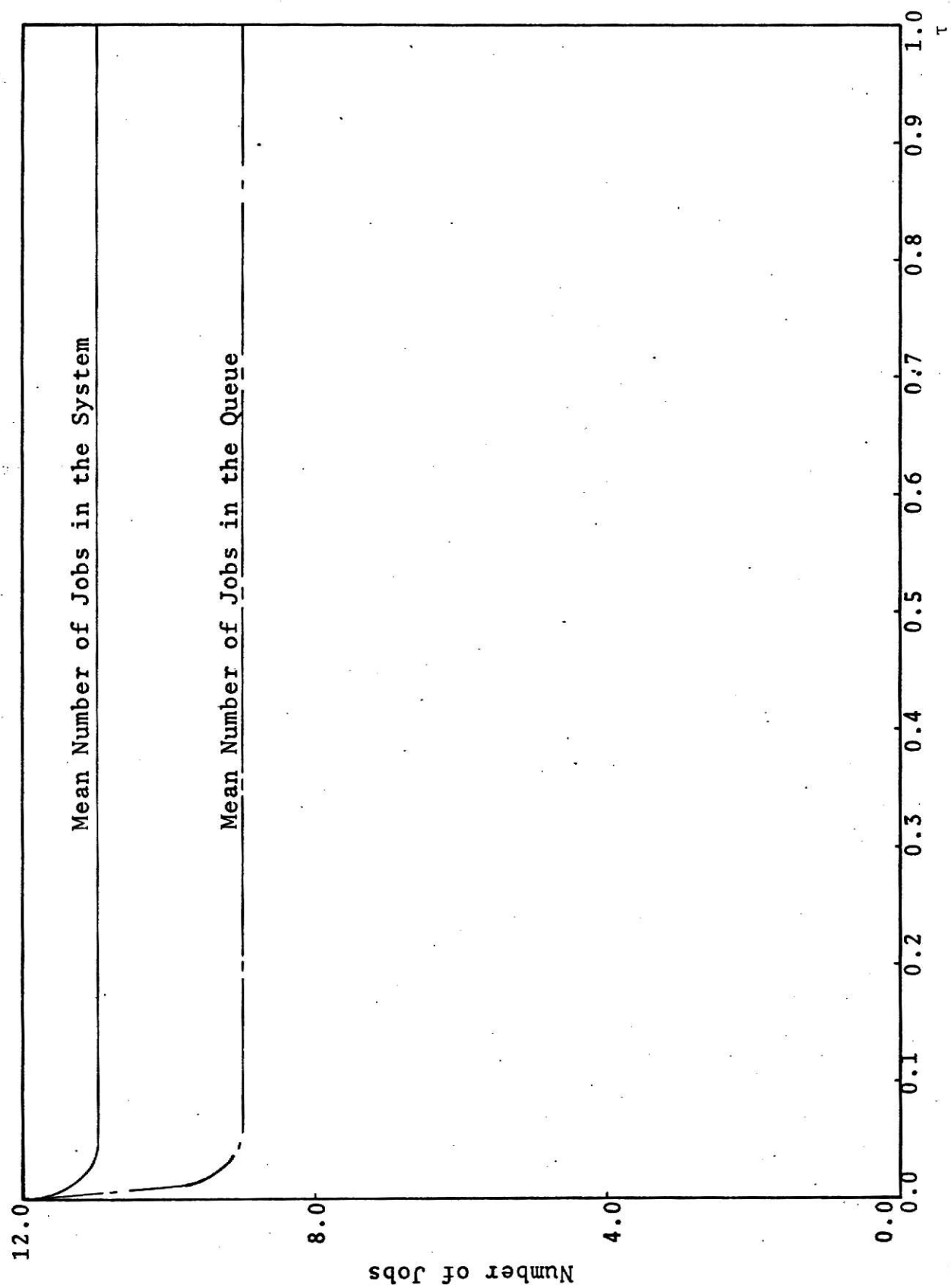


Figure 4.38. Multiple-Machine Model With Finite Source, Limited Queue, and $\rho = 1.0$.

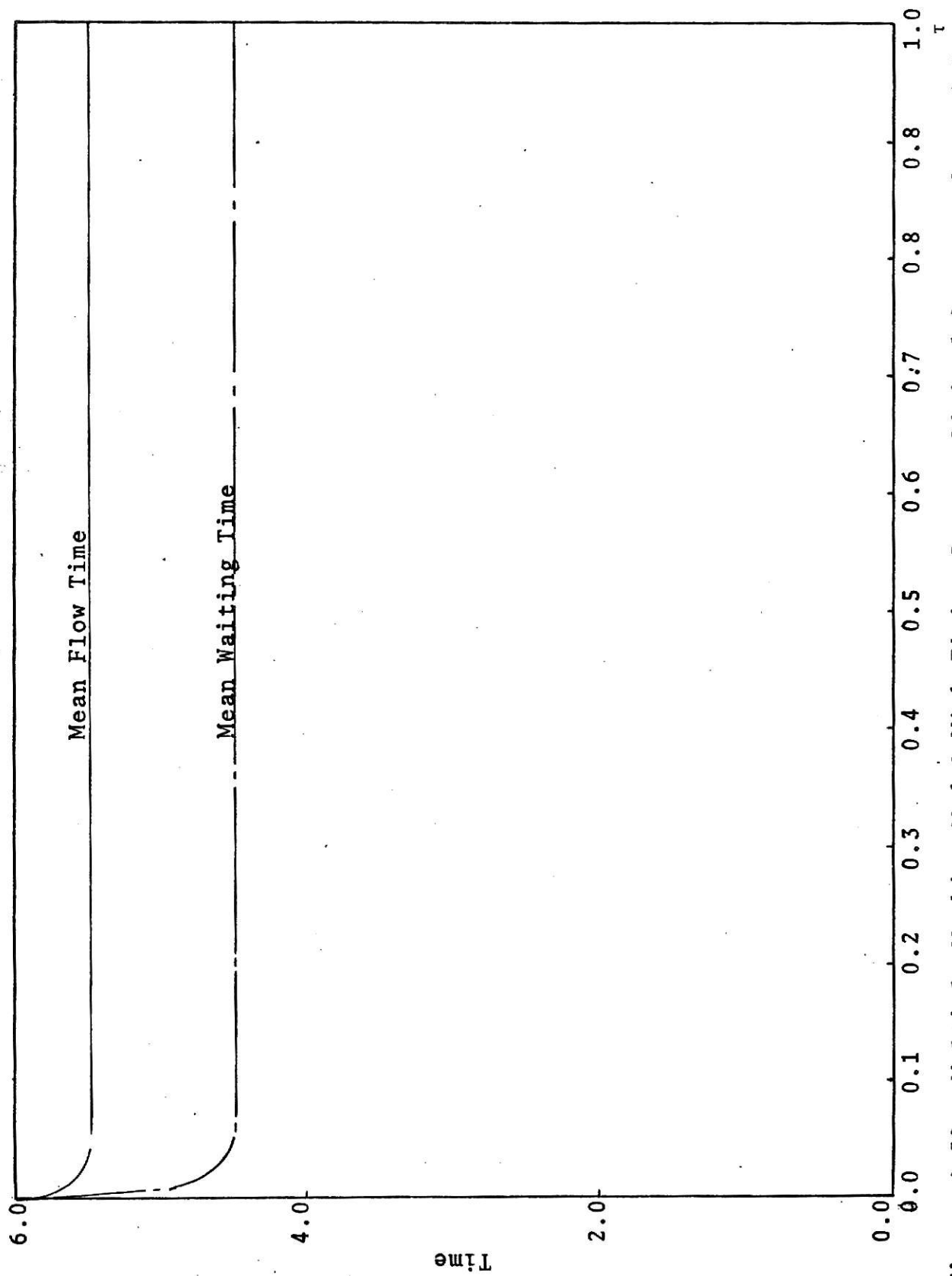


Figure 4.39. Multiple-Machine Model With Finite Source, Limited Queue, and $\rho = 1.0$.

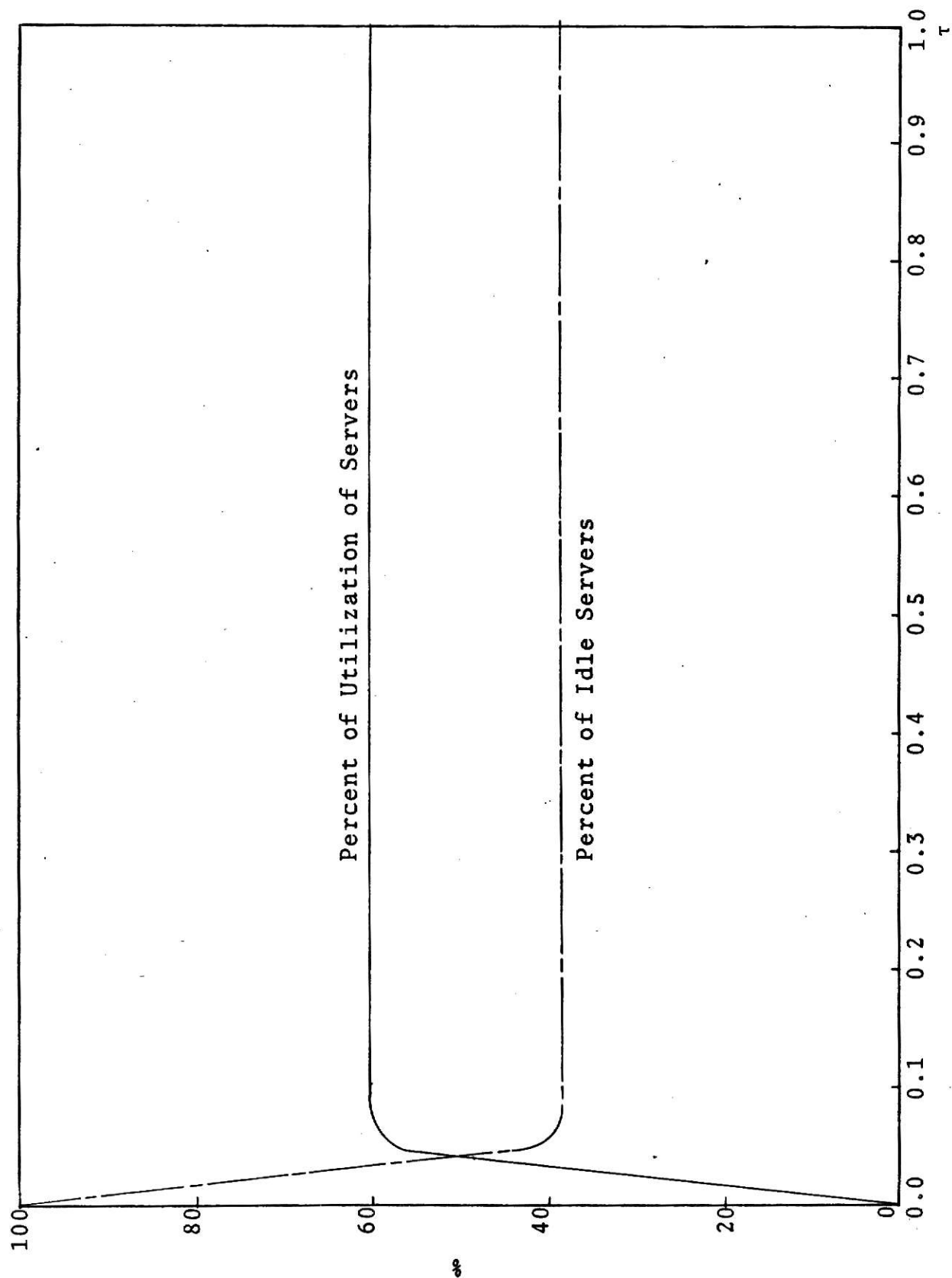


Figure 4.40. Special Machine Shop Model With Infinite Source, No Queue, and $\rho=0.75$.

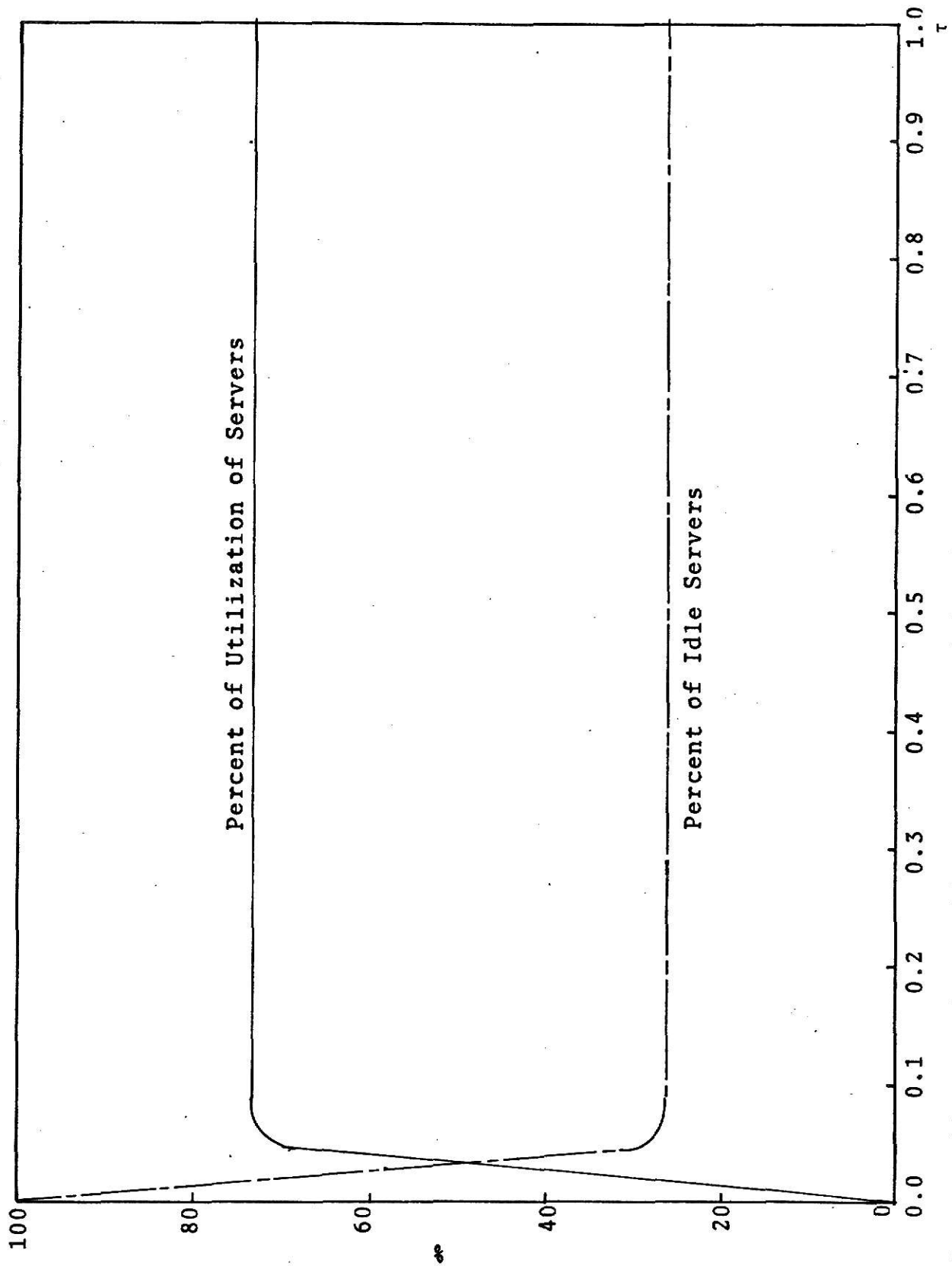


Figure 4.41. Special Machine Shop Model With Infinite Source, No Queue, and $\rho=1.0$.

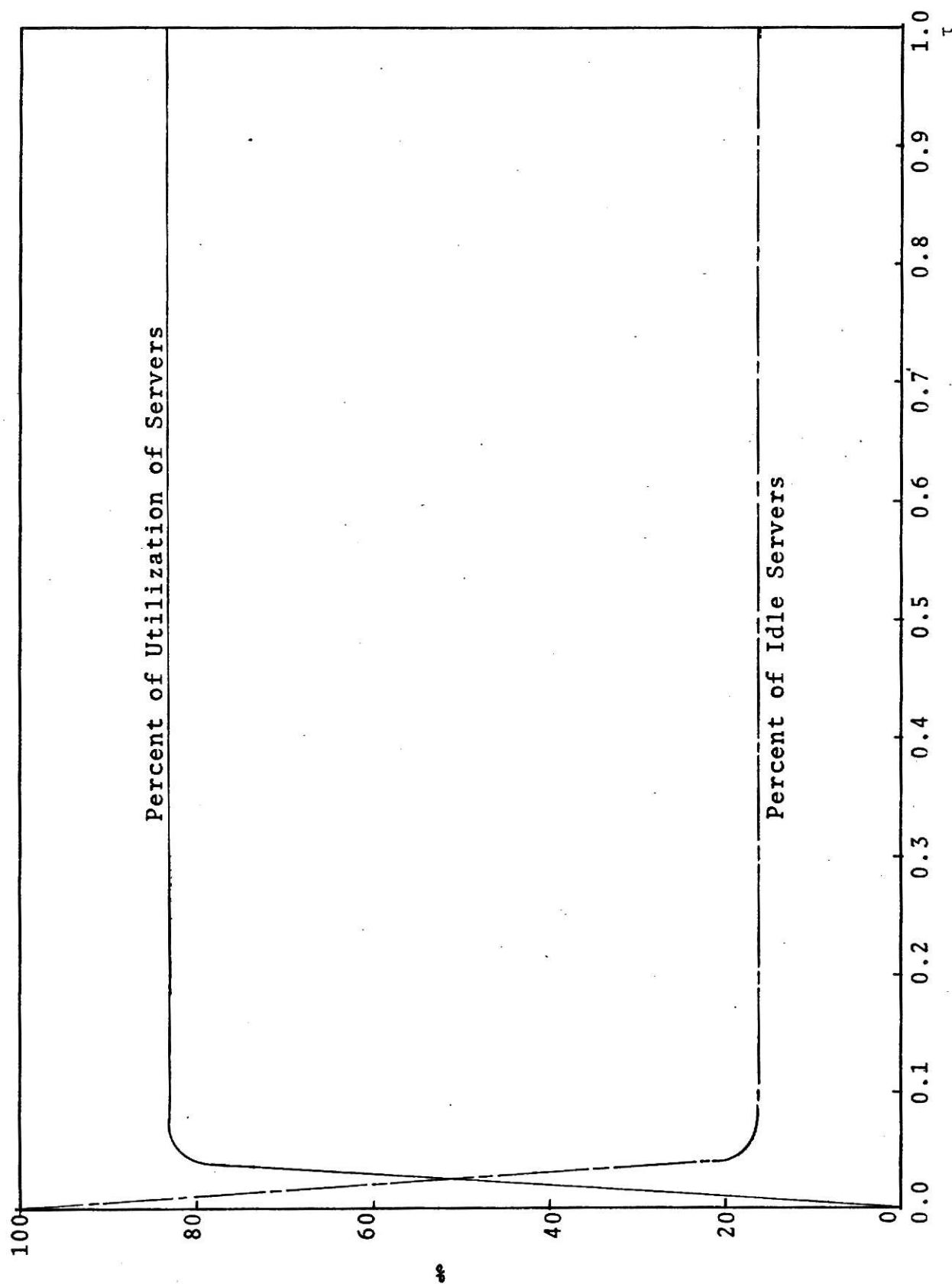


Figure 4.42. Special Machine Shop Model With Infinite Source, No Queue, and $\rho = 1.25$.

CHAPTER V

SUMMARY AND CONCLUSIONS

The practical application of queueing theory is far less than the number of theoretical papers published because of the difficulty in obtaining the explicit transient solutions of the models in a finite time domain. In practical situations, when the effect of rush-hour upon the system is to be determined or when the behavior of the system during a short interval of time is to be predicted, transient solutions are quite useful. In order to apply the steady state solution in practical situations, one has to determine how long the system will remain in that state before reaching the steady state. Analytical solutions for transient state are not available for most of the queueing models. Transient solutions for a number of models are in terms of multiple Laplace transforms and generating functions from which solutions are difficult to obtain for practical use. On the other hand, when the behavior of a queueing process can be represented by a mathematical model, and an analytical explicit solution cannot be obtained, such models can be analyzed numerically providing a good approximation.

In order to motivate the industrial practitioners to apply queueing theory in industry, numerical methods of solution are considered in this thesis. Markovial queueing models which are expressed in terms of difference-differential equations are solved numerically.

First, the basic concepts of queueing theory are discussed. The structure of the basic queueing process is presented.

The methods used for obtaining the numerical solutions are the fourth order Runge Kutta and Hamming for the solution of simultaneous difference-differential equations of initial value type. A sample model is taken and is solved by the IBM scientific sub-routines for the fourth order Runge Kutta (RKGS) and Hamming (HPCG) methods. Solutions by both numerical methods of the sample model are compared with the analytical solution, and it is found that numerical results are quite close to the analytical solution.. The independent variable in the difference-differential equations has been transformed to the dimensionless time units. The RKGS sub-routine takes less computer time than the HPCG. Hence, the RKGS has been used to solve the other models.

Application of queueing theory to machine-shop like systems is considered. Machine-shop process is classified into flow shop and job shop processes. In a machine-shop process, the jobs are the customers and the machines are the servers. Hence, a machine shop process is considered in terms of single-machine models or multiple-machine models. The existing queueing models are developed for various single-machine and multiple-machine systems.

Numerical solutions provide the state probabilities both in the transient and steady state. From the state probabilities, various measures of performance are computed. Nine basic models for the single and multiple-machine systems are investigated. Each model is solved with three different values of utilization factors, that is $\rho = 0.75, 1.00$, and 1.25 for the infinite source queueing models and $\rho = 0.50, 0.75$, and 1.00 for the finite source queueing models.

The measures of performance are very useful in designing a new queueing system, improving an existing system, or comparing the various alternative queueing systems. Hence, several curves for various measures of performance such as mean number of jobs in the system, mean number of jobs in the queue, mean flow time, mean waiting time, percent of idle servers, utilization percentage of servers for the single- and multiple-machine models are plotted and 27 tables for the measures of performance of different models are also included. Computer programming has been written in FORTRAN IV for the IBM 360/50 computer.

Significant results obtained from this are as follows:

1. The behavior of the system differs significantly depending on whether the utilization factor is $\rho < 1$ or $\rho \geq 1$. When ρ is less than one, the system must reach the steady state after a certain time. When ρ is greater or equal to one, the system never reaches the steady state for a large system capacity. The larger the system capacity, the longer the system takes to reach the steady state.
2. For the same utilization factor as the system capacity is increased, the probability of a job being turned away decreases. For no queue case it is maximum, while for unlimited queue case, it is minimum.
3. As the utilization factor increases, percentage of idle servers decreases; on the other hand, the probability that a job is turned away also increases very rapidly.
4. When the system capacity is too small, the difference between transient and steady state is not apparent during a short

period of time.

5. When the source of arriving jobs is also finite, the percentage utilization and probability of a lost job increases very rapidly in comparison with the case of infinite input source.

6. When the utilization factor is less than one, a number of measures of performance such as the mean number of jobs in the system and the mean number of jobs in the queue is much smaller in comparison to the system capacity and is independent of the system capacity provided that it is large.

7. As the utilization factor is increased beyond one, the mean number of jobs in the system or mean number of jobs in the queue and other related measures of performance can be thought of as if they are approximately proportional to the system capacity and rise very rapidly to a very large value.

It is assumed in these models that both the arrival and service patterns are single. In practice, either or both patterns may be in batch. Further research can be done in applying these numerical methods to the solution of Markovian queueing models with bulk arrival or bulk service or both.

SELECTED BIBLIOGRAPHY

1. Ashour, S., "Finite Queueing Theory," Internal Report, Department of Industrial and Management Engineering, University of Iowa, Iowa City, Ia., 1965.
2. Buzacott, J. A., "Prediction of the Efficiency of Production Systems Without Internal Storage," The International Journal of Production Research, Vol. 6, No. 3, 1968, pp. 173-188.
3. Carnahan, B., H. A. Luther and J. O. Wilkers, Applied Numerical Methods, John Wiley & Sons, Inc., New York, 1969, pp. 341-404.
4. Churchman, C. W., R. L. Ackoff and E. L. Arnoff, Introduction to Operations Research, John Wiley & Sons, Inc., New York, 1956, pp. 389-416.
5. Conway, R. W., W. L. Maxwell and L. W. Miller, Theory of Scheduling, Addison-Wesley Publishing Company, Reading, Mass., 1967.
6. Cox, D. R. and W. L. Smith, Queues, Matheun & Co. Ltd., London, 1961, pp. 56-59 and 60-64.
7. Crandall, S. H., Engineering Analysis, McGraw-Hill Book Company, Inc., New York, 1956, pp. 125-194.
8. Elmaghraby, S. E., The Design of Production Systems, Reinhold Publishing Corporation, New York, 1966, pp. 289-327.
9. Ghare, P. M., J. M. Givens and P. E. Torgersen, "A Machine Release Scheme for the Jobshop Work Center," AIIE Transactions, Vol. 1, No. 3, 1969, pp. 208-215.
10. Hillier, F. S. and G. J. Lieberman, Introduction to Operations Research, Holden-Day Inc., San Francisco, 1967, pp. 285-332.
11. Jackson, J. R., "Networks of Waiting Lines," Operations Research, Vol. 5, No. 4, 1957, pp. 518-521.
12. Jackson, J. R., "Jobshop-Like Queueing Systems," Management Science, Vol. 10, No. 1, 1963, pp. 131-142.
13. Jackson, R. R. P., "Queueing Systems With Phase Type Service," Operational Research Quarterly, Vol. 5, No. 4, 1954, pp. 109-120.
14. James, M. L., G. M. Smith and J. C. Walford, Applied Numerical Methods for Digital Computation With Fortran, International Textbook Company, Scranton, Pa., 1967, pp. 313-403.

15. Lee, A. M., Applied Queueing Theory, Macmillan & Co. Ltd., London, 1966, pp. 22-33 and 81-92.
16. Morse, P. M., Queues, Inventories and Maintenance, John Wiley & Sons, Inc., New York, 1958, pp. 59-67 and 116-121.
17. Nelson, R. T., A Simulation Study and Analysis of a Two Station, Waiting Line Network Model, Ph.D. thesis, University of California, Los Angeles, 1965.
18. Prabhu, N. U., Queues and Inventories, John Wiley & Sons, Inc., New York, 1965, pp. 1-22.
19. Ralston, A., A First Course in Numerical Analysis, McGraw-Hill Book Company, New York, 1965, pp. 159-211.
20. Reinitz, R. N. C., An Integrated Job Shop Sequencing Problem, Ph.D. thesis, Case Institute of Technology, Cleveland, Ohio, 1961, pp. 1-15.
21. Riordan, J., Stochastic Service Systems, John Wiley & Sons, Inc., New York, 1962.
22. Saaty, T. L., Elements of Queueing Theory, McGraw-Hill Book Company, Inc., New York, 1961.
23. Scarborough, J. B., Numerical Mathematical Analysis, Sixth Edition, The John Hopkins Press, Baltimore, Md., 1966, pp. 310-370.
24. Sharma, R. P., Bulk Queueing Models and Their Applications, Master's report, Kansas State University, Manhattan, 1969.
25. Shelton, J. R., "Solution Methods for Waiting Line Problems," The Journal of Industrial Engineering, Vol. 11, No. 4, 1960, pp. 293-303.
26. Stanton, R. G., Numerical Methods for Science and Engineering, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1961, pp. 141-154.
27. Syski, R., Introduction to Congestion Theory in Telephone Systems, Oliver and Boyd Ltd., Edinburgh, 1960, pp. 388-392.
28. Tayler, J., and R. R. P. Jackson, "An Application of the Birth and Death Process to the Provision of Spare Machines," Operational Research Quarterly, Vol. 5, No. 4, 1964, pp. 95-108.

APPENDIX A

MEAN FLOW TIME AND MEAN WAITING TIME
FOR VARIOUS QUEUE SIZES AND QUEUE DISCIPLINES

This appendix includes the derivation of mean flow time and mean waiting time for the limited queue and no queue cases and also for the random selection and last-come, first-processed queue disciplines.

Single-Machine Shop Models. The flow time density function for limited queue case is

$$\begin{aligned} f(t) &= \sum_{n=0}^N P_n(t) W_{n+1}(t) \\ &= \sum_{n=0}^N P_n(t) \frac{\mu (\mu t)^n}{n!} e^{-\mu t} \end{aligned}$$

where

N the system capacity

Mean flow time, in this case, is

$$\begin{aligned} F(t) &= \int_{t=0}^{\infty} t f(t) dt \\ &= \int_{t=0}^{\infty} t \sum_{n=0}^N P_n(t) W_{n+1}(t) dt \\ &= \sum_{n=0}^N \int_{t=0}^{\infty} P_n(t) W_{n+1}(t) t dt \end{aligned}$$

where

$P_n(t)$ is obtained numerically,

Hence,

$$\begin{aligned}
\bar{F}(t) &= \sum_{n=0}^N P_n(t) \int_{t=0}^{\infty} W_{n+1}(t) t dt \\
&= \sum_{n=0}^N P_n(t) \int_{t=0}^{\infty} \frac{\mu(\mu t)^n}{n!} e^{-\mu t} t dt \\
&= \sum_{n=0}^N P_n(t) \frac{n+1}{\mu} \\
&= \frac{1}{\mu} \sum_{n=0}^N n P_n(t) + \frac{1}{\mu} \sum_{n=0}^N P_n(t) \\
&= \frac{1}{\mu} \sum_{n=0}^N n P_n(t) + \frac{1}{\mu} \\
&= \frac{L_s}{\mu} + \frac{1}{\mu}
\end{aligned}$$

For the steady state case,

$$\begin{aligned}
\bar{F} &= \frac{1}{\mu} \sum_{n=0}^N n P_n + \frac{1}{\mu} \\
&= \frac{L_s}{\mu} + \frac{1}{\mu}
\end{aligned}$$

Similarly, the waiting time density in the limited queue case is

$$\begin{aligned}
W(t) &= \sum_{n=1}^N P_n(t) W_n(t) \\
&= \sum_{n=1}^N P_n(t) \frac{\mu(\mu t)^{n-1}}{(n-1)!} e^{-\mu t}
\end{aligned}$$

where

$P_n(t)$ the numerical value of P_n at time t

The mean waiting time is

$$W(t) = \int_0^{\infty} t w(t) dt$$

$$\begin{aligned}
W(t) &= \int_0^{\infty} t \sum_{n=1}^N P_n(t) w_n(t) dt \\
&= \sum_{n=1}^N P_n(t) \int_0^{\infty} w_n(t) t dt \\
&= \sum_{n=1}^N P_n(t) \int_0^{\infty} \frac{\mu (\mu t)^{n-1}}{(n-1)!} e^{-\mu t} t dt \\
&= \sum_{n=1}^N P_n(t) \frac{n}{\mu} \\
&= \frac{1}{\mu} \sum_{n=1}^N n P_n(t) \\
&= \frac{L_s}{\mu}
\end{aligned}$$

For the steady state, the mean waiting time is

$$\begin{aligned}
W &= \frac{1}{\mu} \sum_{n=1}^N n P_n \\
&= \frac{L_s}{\mu}
\end{aligned}$$

The flow time density function for no queue case is simply the processing time distribution, that is

$$f(t) = \mu e^{-\mu t}$$

The mean flow time is

$$\bar{F}(t) = 1/\mu$$

Evidently, the mean waiting time is zero.

As the above derivations for flow time distribution and waiting time distribution is independent of the nature of the arrival source, the expression for infinite source and limited queue for flow time and waiting time density functions also hold

equally well for the finite source and limited queue case.

Random Selection Discipline. In this discipline, jobs are selected at random from those waiting without regard to when the unit entered the system. As such, all the jobs waiting have equal probability of being selected for processing. As in deriving the equations for the state probabilities, the queue discipline is not considered; the probability distribution for the number of jobs in the system is the same as in first-come, first-processed discipline. Thus, for steady state with unlimited queue,

$$P_n = \rho^n (1-\rho), \quad \rho = \lambda/\mu, \quad n = 0, 1, 2, \dots$$

where

P_n probability of n jobs in the system

Next, the analytical expression for mean waiting time, W , is derived [16]. Let

$q_n(t)$ probability that a specified job which arrived when n other jobs were already in the queue, is still in the queue a time t after arrival.

Also, let us suppose that the job arrived after a small interval of time Δt instead of arriving at time 0. Then, the interval of time to the end will be $t - \Delta t$. Considering all the possibilities during the small interval Δt , we get

$$\begin{aligned} q_n(t) = & \frac{\mu n}{n+1} \Delta t q_{n-1}(t-\Delta t) + \lambda \Delta t q_{n+1}(t-\Delta t) \\ & + (1-\mu \Delta t - \lambda \Delta t) q_n(t-\Delta t) \end{aligned}$$

or

$$q_n(t) - q_n(t-\Delta t) = \frac{\mu n}{n+1} \Delta t q_{n-1}(t-\Delta t) + \lambda \Delta t q_{n+1}(t-\Delta t) - (\lambda + \mu) \Delta t q_n(t-\Delta t)$$

or

$$\lim_{\Delta t \rightarrow 0} \frac{q_n(t) - q_n(t-\Delta t)}{\Delta t} = \lim_{\Delta t \rightarrow 0} \left[\frac{\mu n}{n+1} q_{n-1}(t-\Delta t) + \lambda q_{n+1}(t-\Delta t) - (\lambda + \mu) q_n(t-\Delta t) \right]$$

$$\frac{dq_n(t)}{dt} = \frac{\mu n}{n+1} q_{n-1}(t) + \lambda q_{n+1}(t) - (\lambda + \mu) q_n(t)$$

$$\frac{dq_n(t)}{\mu dt} = \frac{n}{n+1} q_{n-1}(t) + \frac{\lambda}{\mu} q_{n+1}(t) - \left(\frac{\lambda + \mu}{\mu}\right) q_n(t)$$

$$\frac{dq_n(t)}{\mu dt} = \frac{n}{n+1} q_{n-1}(t) + \rho q_{n+1}(t) - (1 + \rho) q_n(t) \quad (1)$$

with the initial conditions,

$$q_n(0) = 1$$

$$q_n(\infty) = 0$$

Let

$G_q(t)$ probability that any job would have to wait for at least time t

Thus,

$$G_q(t) = \sum_{n=0}^{\infty} P_{n+1}(t) q_n(t)$$

For the steady state

$$\begin{aligned} P_{n+1}(t) &= P_{n+1} \\ &= (1-\rho) \rho^{n+1} \end{aligned}$$

Hence,

$$G_q(t) = \sum_{n=0}^{\infty} (1-\rho)\rho^{n+1} q_n(t)$$

The mean waiting time is obtained by integrating $G_q(t)$ over the whole range of t from 0 to ∞ .

Hence, the mean waiting time,

$$\begin{aligned} W &= \int_{t=0}^{\infty} G_q(t) dt \\ &= \int_0^{\infty} (1-\rho) \sum_{n=0}^{\infty} \rho^{n+1} q_n(t) dt \\ &= (1-\rho) \sum_{n=0}^{\infty} \rho^{n+1} \int_0^{\infty} q_n(t) dt \\ &= (1-\rho) \sum_{n=0}^{\infty} \rho^{n+1} W_{nq} \end{aligned} \quad (2)$$

Where

$$W_{nq} = \int_0^{\infty} q_n(t) dt$$

Multiplying Equation (1) by $(n+1)\rho^{n+1}$, integrating both sides over t from 0 to ∞ , and summing over n from 0 to ∞ , we have

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(n+1)\rho^{n+1}}{\mu} \int_0^{\infty} dq_n(t) &= \sum_{n=0}^{\infty} \frac{n}{n+1} n+1 \rho^{n+1} \int_0^{\infty} q_{n-1}(t) dt \\ &+ \sum_{n=0}^{\infty} \rho^{n+2} (n+1) \int_0^{\infty} q_{n+1}(t) dt \\ &- \sum_{n=0}^{\infty} (1+\rho) \rho^{n+1} (n+1) \int_0^{\infty} q_n(t) dt \\ &= \sum_{n=0}^{\infty} n \rho^{n+1} \int_0^{\infty} q_{n-1}(t) dt \end{aligned}$$

$$\begin{aligned}
& + \sum_{n=0}^{\infty} (n+1) \rho^{n+2} \int_0^{\infty} q_{n+1}(t) dt \\
& - \sum_{n=0}^{\infty} (1+\rho) \rho^{n+1} (n+1) \int_0^{\infty} q_n(t) dt \\
& = \sum_{n=0}^{\infty} n \rho^{n+1} W_{nq} + \sum_{n=0}^{\infty} (n+1) \rho^{n+1} \\
& \quad \cdot W_{nq} - \sum_{n=0}^{\infty} (1+\rho) (n+1) W_{nq} \rho^{n+1} \\
& = \sum_{n=0}^{\infty} \rho^{n+1} W_{nq} (n + (n+1) \rho - (1+\rho) \\
& \quad \cdot (n+1)) \\
& = \sum_{n=0}^{\infty} \rho^{n+1} W_{nq} (n + (n+1) \rho - (n+1) \\
& \quad \cdot \rho (n+1)) \\
& = \sum_{n=0}^{\infty} - \rho^{n+1} W_{nq} \\
& = - \sum_{n=0}^{\infty} \rho^{n+1} W_{nq}
\end{aligned}$$

On the other hand, in expanding the left hand side of the above equation, we get

$$\begin{aligned}
\frac{1}{\mu} \sum_{n=0}^{\infty} (n+1) \rho^{n+1} \left[q_n(t) \right]_0^{\infty} &= \frac{1}{\mu} \sum_{n=0}^{\infty} (n+1) \rho^{n+1} (-1) \\
&= - \frac{1}{\mu} \sum_{n=0}^{\infty} (n+1) \rho^{n+1}
\end{aligned}$$

or

$$\begin{aligned}
-\sum_{n=0}^{\infty} \rho^{n+1} W_{nq} &= -\frac{1}{\mu} \sum_{n=0}^{\infty} (n+1) \rho^{n+1} \\
\sum_{n=0}^{\infty} \rho^{n+1} W_{nq} &= \frac{1}{\mu} \sum_{n=0}^{\infty} (n+1) \rho^{n+1} \quad (3)
\end{aligned}$$

But, from Equations (2) and (3), the mean waiting time is

$$\begin{aligned}
W &= (1-\rho) \sum_{n=0}^{\infty} \rho^{n+1} W_{nq} \\
&= (1-\rho) \frac{1}{\mu} \sum_{n=0}^{\infty} (n+1) \rho^{n+1} \\
&= \frac{1-\rho}{\mu} (\rho + 2\rho^2 + 3\rho^3 + \dots) \\
&= \frac{1-\rho}{\mu} \frac{\rho}{(1-\rho)^2} \\
&= \frac{\rho}{\mu(1-\rho)}
\end{aligned}$$

The above expression is the same as in the first-come, first-processed discipline. The units as individuals are affected; their waiting time distributions differ in comparison to the first-come, first-processed discipline, but the mean waiting time or mean flow time does not differ. The variance of waiting time and flow time differ in comparison to that of first-come, first-processed. The relation has been derived by Riordan [21], and is as follows:

$$E(W'^2) \text{ in random selection} = \frac{1}{(1-\rho/2)} E(W''^2)$$

where

$E(W'^2)$ The second moment of waiting time in the random selection discipline

$E(W''^2)$ the second moment of waiting time in the first-come, first-processed discipline.

Last-come, first-processed discipline. In this discipline, the job which has arrived last in the waiting line is selected for processing. At many times, jobs arriving in a machine shop have been stacked in such a way that the job which has arrived last is the most accessible. In such cases, it is more convenient to select such jobs for processing.

The probability distribution of the number of jobs in the system is the same as in the first-come, first-processed discipline. In the steady state, the probability of n jobs in the system is given, in the unlimited queue case, by

$$\begin{aligned} P_n &= (1-\rho)\rho^n \\ &= \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n, \quad n = 0, 1, \dots \end{aligned}$$

In the last-come, first-processed queue discipline, the waiting time of an arriving job j will not be affected by the number of jobs n already in the system because the job j will be processed before the other jobs in the system. If other jobs arrive during the time between the arrival of job j and completion of another job i already in the process, the arriving jobs will receive service before the job j . Thus, the distribution of waiting time of an arriving job is the distribution of the period starting when there are n jobs in the system and ending when the number in the system again falls back to n . This period is called the busy period. Thus, the mean flow time in the last-come, first-processed

case is the same as the mean length of busy period. Let us suppose that the machine is idle at some time. If a job arrives at this time, it will be loaded on the machine for processing immediately. Thus, this job will not have to wait. During the processing time of the job, let more jobs arrive and be processed on the machine according to the selection discipline after the first job is completed. This process will continue until all jobs are completed, after which the machine will be idle.

The busy period is referred to, also, when the machine is busy processing the jobs in that length of period. Since a machine becomes idle in some periods, the busy period and idle period of a machine alternate one after the other [6].

When the jobs arrive according to poisson distribution, there is a $\lambda \Delta t$ chance of an idle period being terminated by the arrival of a job during any short interval of time Δt . Therefore, the distribution of length of idle periods is exponential with mean $1/\lambda$.

Now, whatever the distribution of job processing time is, the steady state probability that the machine is idle is $1-\rho$, where ρ is the utilization factor = λ/μ . If a long time period T is considered, the machine will be idle for a time $T(1-\rho)$. But mean length of an idle period is $1/\lambda$. Therefore, during a period T , there are $\frac{T(1-\rho)}{1/\lambda}$ idle periods, that is $T\lambda(1-\rho)$ idle periods. There must be $T\lambda(1-\rho)$ busy periods also during time T occupying a total time $T\lambda$. Hence, the mean duration of a busy period is

$$\frac{\text{Total busy time}}{\text{Number of busy periods}}$$

$$\begin{aligned}
 &= \frac{T\rho}{T\lambda(1-\rho)} \\
 &= \frac{\rho}{\lambda(1-\rho)} \\
 &= \frac{\lambda/\mu}{\lambda(1-\rho)} \\
 &= \frac{1}{\mu(1-\rho)}
 \end{aligned}$$

The above expression for the mean length of busy period is the same as that of mean flow time in the first-come, first-processed case.

Similarly, the mean waiting time, W in the last-come, first-processed discipline is the same as in the case of first-come, first-processed discipline, that is,

$$W = \frac{\rho}{\mu(1-\rho)}$$

Although the mean waiting time, W and mean flow time, F are the same as those in the first-come, first-processed case, the variance of waiting time or flow time is less in first-come, first-processed than in the last-come, first-processed discipline. This is explained by the following relation:

$$E(W'^2) = \frac{1}{1-\rho} E(W''^2)$$

where

$E(W'^2)$ second moment of waiting time in the last-come, first-processed discipline

$E(W''^2)$ second moment of waiting time in the first-come, first-processed discipline

The second moment of waiting time thus differs by a factor

which depends on utilization. When ρ is very high, the relation shows that the first-come, first-processed discipline is preferable, as when ρ tends to 1, the factor $1/1-\rho$ becomes very large indicating that the variance in waiting time in the last-come, first-processed discipline will be higher in comparison to the first-come, first-processed discipline.

Thus, it is seen that out of the three selection disciplines discussed, the first-come, first-processed case has the least variance of waiting time, while the last-come, first-processed has the greatest variance of waiting time.

Multi-machine shop models. For the limited queue case, with system capacity, N , the mean flow time is

$$\begin{aligned}\bar{F}(t) &= \frac{1}{\mu} \sum_{n=0}^N nP_n(t) + \frac{1}{\mu} \\ &= L_s/\mu + 1/\mu\end{aligned}$$

For the steady state case

$$\begin{aligned}\bar{F} &= \frac{1}{\mu} \sum_{n=0}^N nP_n + \frac{1}{\mu} \\ &= L_s/\mu + 1/\mu\end{aligned}$$

The mean waiting time is

$$\begin{aligned}W(t) &= \frac{1}{\mu} \sum_{n=0}^N nP_n(t) \\ &= L_s/\mu\end{aligned}$$

For steady state,

$$\begin{aligned}
 W &= \frac{1}{\mu} \sum_{n=0}^N n P_n \\
 &= L_s / \mu
 \end{aligned}$$

For no queue case, the mean flow time is the mean processing time of a job, that is

$$F(t) = 1/\mu$$

Evidently, the mean waiting time is zero.

For finite source and limited queue case, the mean flow time and mean waiting time is the same as that of infinite source and limited queue.

APPENDIX B

ANALYTICAL EXPRESSIONS AND TABLES FOR MEASURES OF PERFORMANCE

This appendix includes the mathematical expressions from which the measures of performance for the single- and multiple-machine models have been computed. Tables for these measures of performance are presented also. The following notation is used:

N system capacity

M number of machines

r number of repairmen

λ arrival rate

μ processing rate per server

$P_n(\tau)$ probability of n jobs in the system at dimensionless time units

τ dimensionless time = t/T

where T is the final value of the interval for which the measures are computed

Single Machine Models

Measure of Performance	Notation	Model A		Model B		Model C		Model D	
		infinite source	no queue	infinite source	limited queue	infinite source	unlimited queue	finite source	limited queue
Probability of no job in the system	P0	$P_0(\tau)$		$P_0(\tau)$		$P_0(\tau)$		$P_0(\tau)$	
Probability of a job being turned away	PL	$P_1(\tau)$		$P_N(\tau)$		0		0	
Probability of all servers being busy	PB	$P_1(\tau)$		$1-P_0(\tau)$		$1-P_0(\tau)$		$1-P_0(\tau)$	
Mean number of jobs in the system	NS	$P_1(\tau)$		$\sum_{n=0}^N n P_n(\tau)$		$\sum_{n=0}^N n P_n(\tau)$		NQ+NPROC	
Mean number of jobs in the queue	NQ	0		$NS-1+P_0(\tau)$		$NS-1+P_0(\tau)$		$M-NPROC(1+\mu/\lambda)$	
Mean flow time	FT	$1/\mu$		NS/λ		NS/λ		NS/λ	
Mean waiting time	WT	0		NQ/λ		NQ/λ		NQ/λ	
Mean number of jobs given that there is a non-empty queue	N/Q	0		$NQ/(1-P_0(\tau)-P_1(\tau))$		$NQ/(1-P_0(\tau)-P_1(\tau))$		$NQ/(1-P_0(\tau)-P_1(\tau))$	
Mean number of jobs in service	NPROC	$P_1(\tau)$		$1-P_0(\tau)$		$1-P_0(\tau)$		$1-P_0(\tau)$	
Mean number of idle servers	NIDLE	$P_0(\tau)$		$P_0(\tau)$		$P_0(\tau)$		$P_0(\tau)$	
Percentage of idle servers	IDLE%	$P_0(\tau) \times 100$		$P_0(\tau) \times 100$		$P_0(\tau) \times 100$		$P_0(\tau) \times 100$	
Percentage of utilization of servers	UTIL%	$100-IDLE\%$		$100-IDLE\%$		$100-IDLE\%$		$100-IDLE\%$	

Multiple-Machine Models

Measures of Performance	Notation	Case A Infinite Source no queue	Case B Infinite Source limited queue	Case C Infinite Source unlimited queue	Case D Finite Source limited queue	Special Shop Model
Probability of no job in the system	PO	$P_0(\tau)$	$P_0(\tau)$	$P_0(\tau)$	$P_0(\tau)$	$P_0(\tau)$
Probability of a job being turned away	PL	$P_M(\tau)$	$P_N(\tau)$	0	0	$P_M(\tau)$
Probability of all ser- vers being busy	PB	$P_M(\tau)$	$\sum_{n=M}^N P_n(\tau)$	$\sum_{n=M}^{\infty} P_n(\tau)$	$\sum_{n=r}^M P_n(\tau)$	$P_M(\tau)$
Mean number of jobs in the system	NS	$\sum_{n=0}^M n P_n(\tau)$	$\sum_{n=0}^N n P_n(\tau)$	$\sum_{n=0}^{\infty} n P_n(\tau)$	NQ+PROC	$\sum_{n=0}^M n P_n(\tau)$
Mean number of jobs in the queue	NQ	0	$\sum_{n=M}^N (n-M) P_n(\tau)$	$\sum_{n=M}^{\infty} (n-M) P_n(\tau)$	$M - N P R O C (1 + \frac{\mu}{\lambda})$	0
Mean flow time	FT	$1/\mu$	NS/ λ	NS/ λ	NS/ λ	$1/\mu$
Mean waiting time	WT	0	NQ/ λ	NQ/ λ	NQ/ λ	0
Mean number of jobs given that there is a non-empty queue	N/Q	0	$\sum_{n=0}^M NQ / (1 - \sum_{n=0}^M P_n(\tau))$	$\sum_{n=0}^M NQ / (1 - \sum_{n=0}^M P_n(\tau))$	$\sum_{n=0}^r NQ / (1 - \sum_{n=0}^r P_n(\tau))$	0
Mean number of jobs in service	NPROC	NS	NS-NQ	NS-NQ	$\sum_{n=0}^{r-1} n P_n(\tau) + r \sum_{n=r}^M P_n(\tau)$	

Multiple-Machine Models continued

Measures of Performance	Notation	Case				Special
		A	B	C	D	
		Infinite Source no queue	Infinite Source limited queue	Infinite Source unlimited queue	Finite Source limited queue	Shop Model
Mean number of idle servers	NIDLE	M-NPROC	M-NPROC	M-NPROC	r-NPROC	M-NPROC
Percentage of idle servers	IDLE%	$(\text{NIDLE}/M) \times 100$	$(\text{NIDLE}/M) \times 100$	$(\text{NIDLE}/M) \times 100$	$(\text{NIDLE}/r) \times 100$	$(\text{NIDLE}/M) \times 100$
Percent of utilization of servers	UTIL%	100-IDLE%	100-IDLE%	100-IDLE%	100-IDLE%	100-IDLE%

Next, the tables for the measures of performance for the various single- and multiple-machine models are presented. The list of figures as shown in Chapter IV corresponds to the tables in this appendix as shown below:

<u>Table</u>	<u>Figure</u>	<u>Table</u>	<u>Figure</u>
B.1.1	4.1	B.5.3	4.22
B.1.2	4.2	B.7.1	4.23
B.1.3	4.3		4.24
B.3.1	4.4	B.7.2	4.25
	4.5		4.26
	4.6		4.27
B.3.2	4.7	B.7.3	4.28
	4.8		4.29
B.3.3	4.9		4.30
	4.10	B.8.1	4.31
	4.11		4.32
B.4.1	4.12	B.8.2	4.33
	4.13		4.34
	4.14		4.35
B.4.2	4.15	B.8.3	4.36
	4.16		4.37
	4.17		4.38
B.4.3	4.18	B.9.1	4.39
	4.19		4.40
B.5.1	4.20	B.9.2	4.41
B.5.2	4.21	B.9.3	4.42

TABLE B.1.1

MEASURES OF PERFORMANCE FOR SINGLE-MACHINE MODEL
INFINITE SOURCE AND NO QUEUE

CAPACITY = 1, LAMDA = 0.75, MUE = 1.00, INTERVAL = (0, 100.0)

TAU	P0	PL	PB	NS	NQ	FT	WT	N/Q	NPROC	NIDLE	IDLE%	UTIL%
0.0	1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.000	100.000	0.0
0.05	0.571	0.428	0.428	0.428	0.0	1.000	0.0	0.0	0.428	0.571	57.150	42.850
0.10	0.571	0.429	0.429	0.429	0.0	1.000	0.0	0.0	0.429	0.571	57.143	42.857
0.15	0.571	0.429	0.429	0.429	0.0	1.000	0.0	0.0	0.429	0.571	57.142	42.857
0.20	0.571	0.429	0.429	0.429	0.0	1.000	0.0	0.0	0.429	0.571	57.142	42.857
0.25	0.571	0.429	0.429	0.429	0.0	1.000	0.0	0.0	0.429	0.571	57.142	42.857
0.30	0.571	0.429	0.429	0.429	0.0	1.000	0.0	0.0	0.429	0.571	57.142	42.857
0.35	0.571	0.429	0.429	0.429	0.0	1.000	0.0	0.0	0.429	0.571	57.142	42.857
0.40	0.571	0.429	0.429	0.429	0.0	1.000	0.0	0.0	0.429	0.571	57.142	42.857
0.45	0.571	0.429	0.429	0.429	0.0	1.000	0.0	0.0	0.429	0.571	57.142	42.857
0.50	0.571	0.429	0.429	0.429	0.0	1.000	0.0	0.0	0.429	0.571	57.142	42.857
0.55	0.571	0.429	0.429	0.429	0.0	1.000	0.0	0.0	0.429	0.571	57.142	42.856
0.60	0.571	0.429	0.429	0.429	0.0	1.000	0.0	0.0	0.429	0.571	57.142	42.856
0.65	0.571	0.429	0.429	0.429	0.0	1.000	0.0	0.0	0.429	0.571	57.142	42.856
0.70	0.571	0.429	0.429	0.429	0.0	1.000	0.0	0.0	0.429	0.571	57.142	42.856
0.75	0.571	0.429	0.429	0.429	0.0	1.000	0.0	0.0	0.429	0.571	57.142	42.856
0.80	0.571	0.429	0.429	0.429	0.0	1.000	0.0	0.0	0.429	0.571	57.142	42.856
0.85	0.571	0.429	0.429	0.429	0.0	1.000	0.0	0.0	0.429	0.571	57.142	42.856
0.90	0.571	0.429	0.429	0.429	0.0	1.000	0.0	0.0	0.429	0.571	57.142	42.856
0.95	0.571	0.429	0.429	0.429	0.0	1.000	0.0	0.0	0.429	0.571	57.142	42.856
1.00	0.571	0.429	0.429	0.429	0.0	1.000	0.0	0.0	0.429	0.571	57.141	42.856

TABLE B.1.2

MEASURES OF PERFORMANCE FOR SINGLE-MACHINE MODEL
INFINITE SOURCE AND NO QUEUE

CAPACITY = 1, LAMDA = 1.00, MUE = 1.00, INTERVAL =(0, 100.0)

TAU	P0	PL	PB	NS	NQ	FT	WT	N/Q	NPROC	NIDLE	IDLE%	UTIL%
0.0	1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.000	100.000	0.0
0.05	0.500	0.500	0.500	0.500	0.0	1.000	0.0	0.0	0.500	0.500	50.002	49.997
0.10	0.500	0.500	0.500	0.500	0.0	1.000	0.0	0.0	0.500	0.500	50.000	50.000
0.15	0.500	0.500	0.500	0.500	0.0	1.000	0.0	0.0	0.500	0.500	50.000	50.000
0.20	0.500	0.500	0.500	0.500	0.0	1.000	0.0	0.0	0.500	0.500	50.000	50.000
0.25	0.500	0.500	0.500	0.500	0.0	1.000	0.0	0.0	0.500	0.500	50.000	50.000
0.30	0.500	0.500	0.500	0.500	0.0	1.000	0.0	0.0	0.500	0.500	49.999	49.999
0.35	0.500	0.500	0.500	0.500	0.0	1.000	0.0	0.0	0.500	0.500	49.999	49.999
0.40	0.500	0.500	0.500	0.500	0.0	1.000	0.0	0.0	0.500	0.500	49.999	49.999
0.45	0.500	0.500	0.500	0.500	0.0	1.000	0.0	0.0	0.500	0.500	49.999	49.999
0.50	0.500	0.500	0.500	0.500	0.0	1.000	0.0	0.0	0.500	0.500	49.999	49.999
0.55	0.500	0.500	0.500	0.500	0.0	1.000	0.0	0.0	0.500	0.500	49.999	49.999
0.60	0.500	0.500	0.500	0.500	0.0	1.000	0.0	0.0	0.500	0.500	49.999	49.999
0.65	0.500	0.500	0.500	0.500	0.0	1.000	0.0	0.0	0.500	0.500	49.999	49.999
0.70	0.500	0.500	0.500	0.500	0.0	1.000	0.0	0.0	0.500	0.500	49.999	49.999
0.75	0.500	0.500	0.500	0.500	0.0	1.000	0.0	0.0	0.500	0.500	49.999	49.999
0.80	0.500	0.500	0.500	0.500	0.0	1.000	0.0	0.0	0.500	0.500	49.999	49.999
0.85	0.500	0.500	0.500	0.500	0.0	1.000	0.0	0.0	0.500	0.500	49.999	49.999
0.90	0.500	0.500	0.500	0.500	0.0	1.000	0.0	0.0	0.500	0.500	49.999	49.999
0.95	0.500	0.500	0.500	0.500	0.0	1.000	0.0	0.0	0.500	0.500	49.999	49.999
1.00	0.500	0.500	0.500	0.500	0.0	1.000	0.0	0.0	0.500	0.500	49.999	49.999

TABLE B.1.3

MEASURES OF PERFORMANCE FOR SINGLE-MACHINE MODEL
INFINITE SOURCE AND NO QUEUE

CAPACITY = 1, LAMDA = 1.25, MUE = 1.00, INTERVAL = (0, 100.0)

TAU	P0	PL	PB	NS	NQ	FT	WT	N/Q	NPROC	NIDLE	IDLE%	UTIL%
0.0	1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.000	100.000	0.0
0.05	0.444	0.556	0.556	0.556	0.0	1.000	0.0	0.0	0.556	0.444	44.445	55.554
0.10	0.444	0.556	0.556	0.556	0.0	1.000	0.0	0.0	0.556	0.444	44.444	55.555
0.15	0.444	0.556	0.556	0.556	0.0	1.000	0.0	0.0	0.556	0.444	44.444	55.555
0.20	0.444	0.556	0.556	0.556	0.0	1.000	0.0	0.0	0.556	0.444	44.444	55.555
0.25	0.444	0.556	0.556	0.556	0.0	1.000	0.0	0.0	0.556	0.444	44.444	55.555
0.30	0.444	0.556	0.556	0.556	0.0	1.000	0.0	0.0	0.556	0.444	44.444	55.555
0.35	0.444	0.556	0.556	0.556	0.0	1.000	0.0	0.0	0.556	0.444	44.444	55.555
0.40	0.444	0.556	0.556	0.556	0.0	1.000	0.0	0.0	0.556	0.444	44.444	55.555
0.45	0.444	0.556	0.556	0.556	0.0	1.000	0.0	0.0	0.556	0.444	44.444	55.555
0.50	0.444	0.556	0.556	0.556	0.0	1.000	0.0	0.0	0.556	0.444	44.444	55.555
0.55	0.444	0.556	0.556	0.556	0.0	1.000	0.0	0.0	0.556	0.444	44.444	55.555
0.60	0.444	0.556	0.556	0.556	0.0	1.000	0.0	0.0	0.556	0.444	44.444	55.555
0.65	0.444	0.556	0.556	0.556	0.0	1.000	0.0	0.0	0.556	0.444	44.444	55.555
0.70	0.444	0.556	0.556	0.556	0.0	1.000	0.0	0.0	0.556	0.444	44.444	55.555
0.75	0.444	0.556	0.556	0.556	0.0	1.000	0.0	0.0	0.556	0.444	44.444	55.555
0.80	0.444	0.556	0.556	0.556	0.0	1.000	0.0	0.0	0.556	0.444	44.444	55.555
0.85	0.444	0.556	0.556	0.556	0.0	1.000	0.0	0.0	0.556	0.444	44.444	55.555
0.90	0.444	0.556	0.556	0.556	0.0	1.000	0.0	0.0	0.556	0.444	44.444	55.555
0.95	0.444	0.556	0.556	0.556	0.0	1.000	0.0	0.0	0.556	0.444	44.444	55.555
1.00	0.444	0.556	0.556	0.556	0.0	1.000	0.0	0.0	0.556	0.444	44.444	55.555

TABLE B.2.1

MEASURES OF PERFORMANCE FOR SINGLE-MACHINE MODEL
INFINITE SOURCE AND LIMITED QUEUE

CAPACITY = 4, LAMDA = 0.75, MUE = 1.00, INTERVAL = (0, 100.0)

TAU	P0	PL	PB	NS	NQ	FT	WT	N/Q	NPROC	NIDLE	IDLE%	UTIL%
0.0	1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.000	100.000	0.0
0.05	0.380	0.070	0.620	1.222	0.603	1.630	0.804	1.706	0.620	0.380	38.027	61.973
0.10	0.337	0.098	0.663	1.406	0.742	1.874	0.990	1.795	0.663	0.337	33.687	66.313
0.15	0.329	0.103	0.671	1.437	0.767	1.917	1.022	1.808	0.671	0.329	32.937	67.063
0.20	0.328	0.104	0.672	1.443	0.771	1.924	1.028	1.810	0.672	0.328	32.805	67.195
0.25	0.328	0.104	0.672	1.444	0.772	1.925	1.029	1.811	0.672	0.328	32.783	67.217
0.30	0.328	0.104	0.672	1.444	0.772	1.926	1.029	1.811	0.672	0.328	32.778	67.222
0.35	0.328	0.104	0.672	1.444	0.772	1.926	1.029	1.811	0.672	0.328	32.778	67.222
0.40	0.328	0.104	0.672	1.444	0.772	1.926	1.029	1.811	0.672	0.328	32.777	67.223
0.45	0.328	0.104	0.672	1.444	0.772	1.926	1.029	1.811	0.672	0.328	32.777	67.223
0.50	0.328	0.104	0.672	1.444	0.772	1.926	1.029	1.811	0.672	0.328	32.777	67.223
0.55	0.328	0.104	0.672	1.444	0.772	1.926	1.029	1.811	0.672	0.328	32.777	67.223
0.60	0.328	0.104	0.672	1.444	0.772	1.926	1.029	1.810	0.672	0.328	32.777	67.223
0.65	0.328	0.104	0.672	1.444	0.772	1.926	1.029	1.810	0.672	0.328	32.777	67.223
0.70	0.328	0.104	0.672	1.444	0.772	1.926	1.029	1.810	0.672	0.328	32.777	67.223
0.75	0.328	0.104	0.672	1.444	0.772	1.926	1.029	1.810	0.672	0.328	32.776	67.224
0.80	0.328	0.104	0.672	1.444	0.772	1.926	1.029	1.810	0.672	0.328	32.776	67.224
0.85	0.328	0.104	0.672	1.444	0.772	1.926	1.029	1.810	0.672	0.328	32.776	67.224
0.90	0.328	0.104	0.672	1.444	0.772	1.926	1.029	1.810	0.672	0.328	32.776	67.224
0.95	0.328	0.104	0.672	1.444	0.772	1.926	1.029	1.810	0.672	0.328	32.776	67.224
1.00	0.328	0.104	0.672	1.444	0.772	1.926	1.029	1.810	0.672	0.328	32.776	67.224

TABLE B2.2

MEASURES OF PERFORMANCE FOR SINGLE-MACHINE MODEL
INFINITE SOURCE AND LIMITED QUEUE

CAPACITY = 4, LAMDA = 1.00, MUE = 1.00, INTERVAL = (0, 100.0)

TAU	P0	PL	PB	NS	NQ	FT	WT	N/Q	NPROC	NIDLE	IDLE%	UTIL%
0.0	1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.000	100.000	0.0
0.05	0.254	0.147	0.746	1.719	0.973	1.719	0.973	1.897	0.746	0.254	25.385	74.615
0.10	0.208	0.192	0.792	1.958	1.166	1.958	1.166	1.986	0.792	0.208	20.793	79.207
0.15	0.201	0.199	0.799	1.994	1.195	1.994	1.195	1.998	0.799	0.201	20.117	79.883
0.20	0.200	0.200	0.800	1.999	1.199	1.999	1.199	2.000	0.800	0.200	20.017	79.983
0.25	0.200	0.200	0.800	2.000	1.200	2.000	1.200	2.000	0.800	0.200	20.002	79.998
0.30	0.200	0.200	0.800	2.000	1.200	2.000	1.200	2.000	0.800	0.200	20.000	80.000
0.35	0.200	0.200	0.800	2.000	1.200	2.000	1.200	2.000	0.800	0.200	19.999	80.001
0.40	0.200	0.200	0.800	2.000	1.200	2.000	1.200	2.000	0.800	0.200	19.999	80.001
0.45	0.200	0.200	0.800	2.000	1.200	2.000	1.200	2.000	0.800	0.200	19.999	80.001
0.50	0.200	0.200	0.800	2.000	1.200	2.000	1.200	2.000	0.800	0.200	19.999	80.001
0.55	0.200	0.200	0.800	2.000	1.200	2.000	1.200	2.000	0.800	0.200	19.999	80.001
0.60	0.200	0.200	0.800	2.000	1.200	2.000	1.200	2.000	0.800	0.200	19.999	80.001
0.65	0.200	0.200	0.800	2.000	1.200	2.000	1.200	2.000	0.800	0.200	19.999	80.001
0.70	0.200	0.200	0.800	2.000	1.200	2.000	1.200	2.000	0.800	0.200	19.999	80.001
0.75	0.200	0.200	0.800	2.000	1.200	2.000	1.200	2.000	0.800	0.200	19.999	80.001
0.80	0.200	0.200	0.800	2.000	1.200	2.000	1.200	2.000	0.800	0.200	19.999	80.001
0.85	0.200	0.200	0.800	2.000	1.200	2.000	1.200	2.000	0.800	0.200	19.999	80.002
0.90	0.200	0.200	0.800	2.000	1.200	2.000	1.200	2.000	0.800	0.200	19.998	80.002
0.95	0.200	0.200	0.800	2.000	1.200	2.000	1.200	2.000	0.800	0.200	19.998	80.002
1.00	0.200	0.200	0.800	2.000	1.200	2.000	1.200	2.000	0.800	0.200	19.998	80.002

TABLE B-2.3

MEASURES OF PERFORMANCE FOR SINGLE-MACHINE MODEL
INFINITE SOURCE AND LIMITED QUEUE

CAPACITY = 4, LAMDA = 1.25, MUE = 1.00, INTERVAL = (0, 100.0)

TAU	PO	PL	PB	NS	NQ	FT	WT	N/Q	NPROC	NIDLE	IDLE%	UTIL%
0.0	1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.000	100.000	0.0
0.05	0.165	0.237	0.835	2.168	1.333	1.734	1.066	2.058	0.835	0.165	16.516	83.484
0.10	0.127	0.291	0.873	2.407	1.534	1.926	1.227	2.139	0.873	0.127	12.661	87.339
0.15	0.122	0.297	0.878	2.434	1.556	1.947	1.245	2.146	0.878	0.122	12.237	87.763
0.20	0.122	0.297	0.878	2.437	1.558	1.949	1.247	2.147	0.878	0.122	12.190	87.810
0.25	0.122	0.297	0.878	2.437	1.559	1.949	1.247	2.147	0.878	0.122	12.185	87.815
0.30	0.122	0.297	0.878	2.437	1.559	1.949	1.247	2.147	0.878	0.122	12.184	87.816
0.35	0.122	0.297	0.878	2.437	1.559	1.949	1.247	2.147	0.878	0.122	12.184	87.816
0.40	0.122	0.297	0.878	2.437	1.559	1.949	1.247	2.147	0.878	0.122	12.184	87.816
0.45	0.122	0.297	0.878	2.437	1.559	1.949	1.247	2.147	0.878	0.122	12.184	87.816
0.50	0.122	0.297	0.878	2.437	1.559	1.949	1.247	2.147	0.878	0.122	12.184	87.816
0.55	0.122	0.297	0.878	2.437	1.559	1.949	1.247	2.147	0.878	0.122	12.184	87.816
0.60	0.122	0.297	0.878	2.437	1.559	1.949	1.247	2.147	0.878	0.122	12.184	87.816
0.65	0.122	0.297	0.878	2.437	1.559	1.949	1.247	2.147	0.878	0.122	12.184	87.816
0.70	0.122	0.297	0.878	2.437	1.559	1.949	1.247	2.147	0.878	0.122	12.184	87.816
0.75	0.122	0.297	0.878	2.437	1.559	1.949	1.247	2.147	0.878	0.122	12.184	87.816
0.80	0.122	0.297	0.878	2.437	1.559	1.949	1.247	2.147	0.878	0.122	12.184	87.817
0.85	0.122	0.297	0.878	2.437	1.559	1.949	1.247	2.147	0.878	0.122	12.184	87.816
0.90	0.122	0.297	0.878	2.437	1.559	1.949	1.247	2.147	0.878	0.122	12.183	87.817
0.95	0.122	0.297	0.878	2.437	1.559	1.949	1.247	2.147	0.878	0.122	12.184	87.817
1.00	0.122	0.297	0.878	2.437	1.559	1.949	1.247	2.147	0.878	0.122	12.183	87.817

TABLE B.3.1

MEASURES OF PERFORMANCE FOR SINGLE-MACHINE MODEL
INFINITE SOURCE AND UNLIMITED QUEUE

CAPACITY = 50, LAMDA = 0.75, MUE = 1.00, INTERVAL = (0, 250.0)

TAU	PC	PL	PB	NS	NC	FT	WT	N/Q	NPROC	NIDLE	IDLE%	UTIL%
0.0	1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.000	100.000	0.0
0.05	0.305	0.0	0.695	1.938	1.243	2.583	1.657	2.643	0.695	0.305	30.542	69.458
0.10	0.274	0.0	0.726	2.384	1.659	3.179	2.212	3.176	0.726	0.274	27.446	72.554
0.15	0.263	0.0	0.737	2.603	1.866	3.470	2.487	3.453	0.737	0.263	26.303	73.697
0.20	0.258	0.0	0.742	2.721	1.979	3.629	2.639	3.603	0.742	0.258	25.775	74.225
0.25	0.255	0.0	0.745	2.781	2.036	3.708	2.715	3.674	0.745	0.255	25.492	74.508
0.30	0.254	0.0	0.746	2.838	2.092	3.785	2.789	3.752	0.746	0.254	25.351	74.649
0.35	0.252	0.0	0.748	2.860	2.112	3.813	2.816	3.779	0.748	0.252	25.215	74.785
0.40	0.251	0.0	0.749	2.875	2.126	3.833	2.835	3.797	0.749	0.251	25.145	74.855
0.45	0.251	0.0	0.749	2.885	2.136	3.846	2.848	3.809	0.749	0.251	25.101	74.899
0.50	0.251	0.0	0.749	2.922	2.173	3.896	2.898	3.863	0.749	0.251	25.144	74.856
0.55	0.250	0.0	0.750	2.896	2.146	3.861	2.862	3.822	0.750	0.250	25.049	74.951
0.60	0.251	0.0	0.749	2.928	2.179	3.904	2.905	3.873	0.749	0.251	25.071	74.929
0.65	0.250	0.0	0.750	2.901	2.152	3.869	2.869	3.828	0.750	0.250	25.026	74.974
0.70	0.250	0.0	0.750	2.926	2.177	3.902	2.902	3.869	0.750	0.250	25.036	74.964
0.75	0.250	0.0	0.750	2.923	2.173	3.898	2.898	3.865	0.750	0.250	25.012	74.988
0.80	0.250	0.0	0.750	2.924	2.174	3.899	2.899	3.866	0.750	0.250	25.007	74.993
0.85	0.250	0.0	0.750	2.925	2.175	3.900	2.900	3.867	0.750	0.250	25.004	74.996
0.90	0.251	0.0	0.749	2.973	2.224	3.964	2.965	3.946	0.749	0.251	25.065	74.935
0.95	0.250	0.0	0.750	2.926	2.176	3.901	2.901	3.867	0.750	0.250	25.000	75.000
1.00	0.250	0.0	0.750	2.923	2.174	3.898	2.898	3.860	0.750	0.250	25.034	74.966

TABLE B.3.2

MEASURES OF PERFORMANCE FOR SINGLE-MACHINE MODEL
INFINITE SOURCE AND UNLIMITED QUEUE

CAPACITY = 50, LAMDA = 1.00, MUE = 1.00, INTERVAL =(0, 250.0)

TAU	P0	PL	PB	NS	NQ	FT	WT	N/Q	NPROC	NIDLE	IDLE%	UTIL%
0.0	1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.000	100.000	0.0
0.05	0.159	0.0	0.841	3.482	2.641	3.482	2.641	3.832	0.841	0.159	15.897	84.103
0.10	0.113	0.0	0.887	5.109	4.221	5.109	4.221	5.432	0.887	0.113	11.256	88.744
0.15	0.092	0.0	0.908	6.343	5.435	6.343	5.435	6.650	0.908	0.092	9.197	90.803
0.20	0.080	0.0	0.920	7.370	6.450	7.370	6.450	7.664	0.920	0.080	7.977	92.023
0.25	0.071	0.0	0.929	8.314	7.385	8.314	7.385	8.607	0.929	0.071	7.130	92.870
0.30	0.065	0.0	0.935	9.114	8.179	9.114	8.179	9.398	0.935	0.065	6.509	93.491
0.35	0.060	0.0	0.940	9.890	8.951	9.890	8.951	10.173	0.940	0.060	6.026	93.974
0.40	0.056	0.0	0.944	10.623	9.679	10.623	9.679	10.905	0.944	0.056	5.641	94.359
0.45	0.053	0.0	0.947	11.280	10.334	11.280	10.334	11.560	0.947	0.053	5.316	94.684
0.50	0.050	0.0	0.950	11.914	10.964	11.914	10.964	12.191	0.950	0.050	5.043	94.957
0.55	0.048	0.0	0.952	12.584	11.633	12.584	11.633	12.868	0.952	0.048	4.809	95.191
0.60	0.046	0.001	0.954	13.291	12.337	13.291	12.337	13.585	0.954	0.046	4.606	95.394
0.65	0.044	0.002	0.956	13.837	12.882	13.837	12.882	14.127	0.956	0.044	4.432	95.568
0.70	0.043	0.002	0.957	14.357	13.399	14.357	13.399	14.646	0.957	0.043	4.263	95.737
0.75	0.041	0.003	0.959	14.851	13.892	14.851	13.892	15.136	0.959	0.041	4.123	95.877
0.80	0.040	0.003	0.960	15.322	14.362	15.322	14.362	15.605	0.960	0.040	3.989	96.012
0.85	0.039	0.004	0.961	15.771	14.809	15.771	14.809	16.050	0.961	0.039	3.869	96.131
0.90	0.038	0.004	0.962	16.198	15.236	16.198	15.236	16.473	0.962	0.038	3.760	96.240
0.95	0.037	0.005	0.963	16.606	15.642	16.606	15.642	16.875	0.963	0.037	3.664	96.336
1.00	0.036	0.005	0.964	16.994	16.030	16.994	16.030	17.260	0.964	0.036	3.568	96.432

TABLE B.3.3

MEASURES OF PERFORMANCE FOR SINGLE-MACHINE MODEL
INFINITE SOURCE AND UNLIMITED QUEUE

CAPACITY = 50, LAMDA = 1.25, MUE = 1.00, INTERVAL =(0, 250.0)

TAU	P0	PL	PB	NS	NQ	FT	WT	N/Q	NPROC	NIDLE	IDLE%	UTIL%
0.0	1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.000	100.000	0.0
0.05	0.072	0.0	0.928	5.516	4.588	4.413	3.670	5.438	0.928	0.072	7.197	92.804
0.10	0.034	0.0	0.966	9.217	8.251	7.374	6.601	8.911	0.966	0.034	3.367	96.633
0.15	0.019	0.0	0.981	12.607	11.627	10.086	9.301	12.149	0.981	0.019	1.944	98.056
0.20	0.012	0.0	0.988	15.896	14.909	12.717	11.927	15.327	0.988	0.012	1.232	98.768
0.25	0.008	0.002	0.992	19.290	18.298	15.432	14.638	18.639	0.992	0.008	0.824	99.176
0.30	0.006	0.006	0.994	22.446	21.452	17.957	17.162	21.728	0.994	0.006	0.572	99.429
0.35	0.004	0.014	0.996	25.480	24.484	20.384	19.588	24.707	0.996	0.004	0.406	99.594
0.40	0.003	0.026	0.997	28.336	27.339	22.669	21.871	27.519	0.997	0.003	0.294	99.707
0.45	0.002	0.042	0.998	30.963	29.965	24.770	23.972	30.110	0.998	0.002	0.216	99.785
0.50	0.002	0.059	0.998	33.325	32.327	26.660	25.861	32.443	0.998	0.002	0.161	99.840
0.55	0.001	0.077	0.999	35.408	34.409	28.327	27.528	34.503	0.999	0.001	0.121	99.879
0.60	0.0	0.094	1.000	37.214	36.214	29.771	28.971	36.255	1.000	0.0	0.0	100.000
0.65	0.0	0.110	1.000	38.757	37.757	31.005	30.205	37.757	1.000	0.0	0.0	100.000
0.70	0.0	0.124	1.000	40.056	39.056	32.045	31.245	39.056	1.000	0.0	0.0	100.000
0.75	0.0	0.137	1.000	41.144	40.144	32.915	32.115	40.144	1.000	0.0	0.0	100.000
0.80	0.0	0.148	1.000	42.040	41.040	33.632	32.832	41.040	1.000	0.0	0.0	100.000
0.85	0.0	0.157	1.000	42.783	41.783	34.226	33.426	41.783	1.000	0.0	0.0	100.000
0.90	0.0	0.165	1.000	43.383	42.383	34.706	33.906	42.383	1.000	0.0	0.0	100.000
0.95	0.0	0.171	1.000	43.869	42.869	35.095	34.295	42.869	1.000	0.0	0.0	100.000
1.00	0.0	0.177	1.000	44.259	43.259	35.407	34.607	43.259	1.000	0.0	0.0	100.000

TABLE B.4.1

MEASURES OF PERFORMANCE FOR SINGLE-MACHINE MODEL
FINITE SOURCE AND LIMITED QUEUE

CAPACITY = 6, LAMDA = 0.50, MUE = 1.00, INTERVAL =(0, 100.0)

TAU	P0	PL	PB	NS	NQ	FT	WT	N/Q	NPROC	NIDLE	IDLE%	UTIL%
0.0	1.000	0.0	0.0	6.000	6.000	12.000	12.000	0.0	0.0	1.000	100.000	0.0
0.05	0.018	0.100	0.982	4.037	3.055	8.073	6.110	3.284	0.982	0.018	1.835	98.165
0.10	0.012	0.133	0.988	4.025	3.037	8.050	6.075	3.197	0.988	0.012	1.249	98.751
0.15	0.012	0.136	0.988	4.024	3.036	8.048	6.073	3.191	0.988	0.012	1.211	98.789
0.20	0.012	0.136	0.988	4.024	3.036	8.048	6.073	3.191	0.988	0.012	1.208	98.792
0.25	0.012	0.136	0.988	4.024	3.036	8.048	6.073	3.190	0.988	0.012	1.208	98.792
0.30	0.012	0.136	0.988	4.024	3.036	8.048	6.073	3.190	0.988	0.012	1.208	98.792
0.35	0.012	0.136	0.988	4.024	3.036	8.048	6.073	3.190	0.988	0.012	1.208	98.792
0.40	0.012	0.136	0.988	4.024	3.036	8.048	6.073	3.190	0.988	0.012	1.208	98.792
0.45	0.012	0.136	0.988	4.024	3.036	8.048	6.073	3.190	0.988	0.012	1.208	98.792
0.50	0.012	0.136	0.988	4.024	3.036	8.048	6.073	3.190	0.988	0.012	1.208	98.792
0.55	0.012	0.136	0.988	4.024	3.036	8.048	6.073	3.190	0.988	0.012	1.208	98.792
0.60	0.012	0.136	0.988	4.024	3.036	8.048	6.072	3.190	0.988	0.012	1.208	98.792
0.65	0.012	0.136	0.988	4.024	3.036	8.048	6.072	3.190	0.988	0.012	1.208	98.792
0.70	0.012	0.136	0.988	4.024	3.036	8.048	6.073	3.190	0.988	0.012	1.208	98.792
0.75	0.012	0.136	0.988	4.024	3.036	8.048	6.073	3.190	0.988	0.012	1.208	98.792
0.80	0.012	0.136	0.988	4.024	3.036	8.048	6.072	3.190	0.988	0.012	1.208	98.792
0.85	0.012	0.136	0.988	4.024	3.036	8.048	6.072	3.190	0.988	0.012	1.208	98.792
0.90	0.012	0.136	0.988	4.024	3.036	8.048	6.072	3.190	0.988	0.012	1.208	98.792
0.95	0.012	0.136	0.988	4.024	3.036	8.048	6.072	3.190	0.988	0.012	1.208	98.792
1.00	0.012	0.136	0.988	4.024	3.036	8.048	6.072	3.190	0.988	0.012	1.207	98.793

TABLE B.4.2

MEASURES OF PERFORMANCE FOR SINGLE-MACHINE MODEL
FINITE SOURCE AND LIMITED QUEUE

CAPACITY = 6, LAMDA = 0.75, MUE = 1.00, INTERVAL = (0, 100.0)

TAU	P0	PL	PB	NS	NQ	FT	WT	N/Q	NPROC	NIDLE	IDLE%	UTIL%
0.0	1.000	0.0	0.0	6.000	6.000	8.000	8.000	0.0	0.0	1.000	100.000	0.0
0.05	0.003	0.238	0.997	4.670	3.673	6.227	4.897	3.727	0.997	0.003	0.275	99.725
0.10	0.002	0.263	0.998	4.669	3.672	6.226	4.895	3.714	0.998	0.002	0.207	99.793
0.15	0.002	0.264	0.998	4.669	3.671	6.226	4.895	3.714	0.998	0.002	0.206	99.794
0.20	0.002	0.264	0.998	4.669	3.671	6.226	4.895	3.713	0.998	0.002	0.206	99.794
0.25	0.002	0.264	0.998	4.669	3.671	6.226	4.895	3.713	0.998	0.002	0.206	99.794
0.30	0.002	0.264	0.998	4.669	3.671	6.226	4.895	3.713	0.998	0.002	0.206	99.794
0.35	0.002	0.264	0.998	4.669	3.671	6.226	4.895	3.713	0.998	0.002	0.206	99.794
0.40	0.002	0.264	0.998	4.669	3.671	6.226	4.895	3.713	0.998	0.002	0.206	99.794
0.45	0.002	0.264	0.998	4.669	3.671	6.226	4.895	3.713	0.998	0.002	0.206	99.794
0.50	0.002	0.264	0.998	4.669	3.671	6.226	4.895	3.713	0.998	0.002	0.206	99.794
0.55	0.002	0.264	0.998	4.669	3.671	6.226	4.895	3.713	0.998	0.002	0.206	99.794
0.60	0.002	0.264	0.998	4.669	3.671	6.226	4.895	3.713	0.998	0.002	0.206	99.794
0.65	0.002	0.264	0.998	4.669	3.671	6.226	4.895	3.713	0.998	0.002	0.206	99.794
0.70	0.002	0.264	0.998	4.669	3.671	6.226	4.895	3.713	0.998	0.002	0.206	99.794
0.75	0.002	0.264	0.998	4.669	3.671	6.226	4.895	3.713	0.998	0.002	0.206	99.794
0.80	0.002	0.264	0.998	4.669	3.671	6.226	4.895	3.713	0.998	0.002	0.206	99.794
0.85	0.002	0.264	0.998	4.669	3.671	6.226	4.895	3.713	0.998	0.002	0.206	99.794
0.90	0.002	0.264	0.998	4.669	3.671	6.226	4.895	3.713	0.998	0.002	0.206	99.794
0.95	0.002	0.264	0.998	4.669	3.671	6.226	4.895	3.713	0.998	0.002	0.206	99.794
1.00	0.002	0.264	0.998	4.669	3.671	6.226	4.895	3.713	0.998	0.002	0.206	99.794

TABLE B.4.3

MEASURES OF PERFORMANCE FOR SINGLE-MACHINE MODEL
FINITE SOURCE AND LIMITED QUEUE

CAPACITY = 6, LAMDA = 1.00, MUE = 1.00, INTERVAL =(0, 100.0)

TAU	P0	PL	PB	NS	NQ	FT	WT	N/Q	NPRCC	NIDLE	IDLE%	JTIL%
0.0	1.000	0.0	0.0	6.000	6.000	6.000	6.000	0.0	0.0	1.000	100.000	0.0
0.05	0.0	0.356	1.000	5.000	4.000	5.000	4.000	4.014	1.000	0.0	0.0	100.000
0.10	0.0	0.368	1.000	5.000	4.000	5.000	4.000	4.012	1.000	0.0	0.0	100.000
0.15	0.0	0.368	1.000	5.000	4.000	5.000	4.000	4.012	1.000	0.0	0.0	100.000
0.20	0.0	0.368	1.000	5.000	4.000	5.000	4.000	4.012	1.000	0.0	0.0	100.000
0.25	0.0	0.368	1.000	5.000	4.000	5.000	4.000	4.012	1.000	0.0	0.0	100.000
0.30	0.0	0.368	1.000	5.000	4.000	5.000	4.000	4.012	1.000	0.0	0.0	100.000
0.35	0.0	0.368	1.000	5.000	4.000	5.000	4.000	4.012	1.000	0.0	0.0	100.000
0.40	0.0	0.368	1.000	5.000	4.000	5.000	4.000	4.012	1.000	0.0	0.0	100.000
0.45	0.0	0.368	1.000	5.000	4.000	5.000	4.000	4.012	1.000	0.0	0.0	100.000
0.50	0.0	0.368	1.000	5.000	4.000	5.000	4.000	4.012	1.000	0.0	0.0	100.000
0.55	0.0	0.368	1.000	5.000	4.000	5.000	4.000	4.012	1.000	0.0	0.0	100.000
0.60	0.0	0.368	1.000	5.000	4.000	5.000	4.000	4.012	1.000	0.0	0.0	100.000
0.65	0.0	0.368	1.000	5.000	4.000	5.000	4.000	4.012	1.000	0.0	0.0	100.000
0.70	0.0	0.368	1.000	5.000	4.000	5.000	4.000	4.012	1.000	0.0	0.0	100.000
0.75	0.0	0.368	1.000	5.000	4.000	5.000	4.000	4.012	1.000	0.0	0.0	100.000
0.80	0.0	0.368	1.000	5.000	4.000	5.000	4.000	4.012	1.000	0.0	0.0	100.000
0.85	0.0	0.368	1.000	5.000	4.000	5.000	4.000	4.012	1.000	0.0	0.0	100.000
0.90	0.0	0.368	1.000	5.000	4.000	5.000	4.000	4.012	1.000	0.0	0.0	100.000
0.95	0.0	0.368	1.000	5.000	4.000	5.000	4.000	4.012	1.000	0.0	0.0	100.000
1.00	0.0	0.368	1.000	5.000	4.000	5.000	4.000	4.012	1.000	0.0	0.0	100.000

TABLE B.5.1

MEASURES OF PERFORMANCE FOR MULTIPLE-MACHINE MODEL
INFINITE SOURCE AND NO QUEUE

CAPACITY = 2, NUMBER OF SERVERS = 2, LAMDA = 1.50, MUE = 1.00, INTERVAL = (0, 100.0)

TAU	PO	PL	PB	NS	NQ	FT	WT	N/Q	NPROC	NIDLE	IDLE%	UTIL%
0.0	1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	2.000	100.000	0.0
0.05	0.276	0.310	0.310	1.034	0.0	1.000	0.0	0.0	1.034	0.966	48.295	51.705
0.10	0.276	0.310	0.310	1.034	0.0	1.000	0.0	0.0	1.034	0.966	48.276	51.724
0.15	0.276	0.310	0.310	1.034	0.0	1.000	0.0	0.0	1.034	0.966	48.276	51.724
0.20	0.276	0.311	0.311	1.035	0.0	1.000	0.0	0.0	1.035	0.965	48.261	51.739
0.25	0.276	0.310	0.310	1.034	0.0	1.000	0.0	0.0	1.034	0.966	48.276	51.724
0.30	0.276	0.310	0.310	1.034	0.0	1.000	0.0	0.0	1.034	0.966	48.276	51.724
0.35	0.276	0.311	0.311	1.035	0.0	1.000	0.0	0.0	1.035	0.965	48.271	51.729
0.40	0.276	0.310	0.310	1.034	0.0	1.000	0.0	0.0	1.034	0.966	48.276	51.724
0.45	0.276	0.310	0.310	1.034	0.0	1.000	0.0	0.0	1.034	0.966	48.275	51.725
0.50	0.276	0.310	0.310	1.034	0.0	1.000	0.0	0.0	1.034	0.966	48.276	51.724
0.55	0.276	0.310	0.310	1.034	0.0	1.000	0.0	0.0	1.034	0.966	48.275	51.725
0.60	0.276	0.310	0.310	1.034	0.0	1.000	0.0	0.0	1.034	0.966	48.276	51.724
0.65	0.276	0.310	0.310	1.035	0.0	1.000	0.0	0.0	1.035	0.965	48.274	51.726
0.70	0.276	0.310	0.310	1.034	0.0	1.000	0.0	0.0	1.034	0.966	48.276	51.724
0.75	0.276	0.310	0.310	1.034	0.0	1.000	0.0	0.0	1.034	0.966	48.276	51.724
0.80	0.277	0.312	0.312	1.035	0.0	1.000	0.0	0.0	1.035	0.965	48.245	51.755
0.85	0.276	0.310	0.310	1.034	0.0	1.000	0.0	0.0	1.034	0.966	48.276	51.724
0.90	0.276	0.310	0.310	1.034	0.0	1.000	0.0	0.0	1.034	0.966	48.276	51.724
0.95	0.276	0.310	0.310	1.034	0.0	1.000	0.0	0.0	1.034	0.966	48.275	51.725
1.00	0.276	0.310	0.310	1.034	0.0	1.000	0.0	0.0	1.034	0.966	48.276	51.724

TABLE B.5.2

MEASURES OF PERFORMANCE FOR MULTIPLE-MACHINE MODEL
INFINITE SOURCE AND NO QUEUE

CAPACITY = 2, NUMBER OF SERVERS = 2, LAMDA = 2.00, MUE = 1.00, INTERVAL =(0, 100.0)

TAU	PO	PL	PB	NS	NQ	FT	WT	N/Q	NPROC	NIDLE	IDLE%	UTIL%
0.0	1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	2.000	100.000	0.0
0.05	0.200	0.400	0.400	1.200	0.0	1.000	0.0	0.0	1.200	0.800	40.007	59.993
0.10	0.200	0.400	0.400	1.200	0.0	1.000	0.0	0.0	1.200	0.800	40.000	60.000
0.15	0.200	0.400	0.400	1.200	0.0	1.000	0.0	0.0	1.200	0.800	39.995	60.005
0.20	0.200	0.400	0.400	1.200	0.0	1.000	0.0	0.0	1.200	0.800	40.000	60.000
0.25	0.200	0.400	0.400	1.200	0.0	1.000	0.0	0.0	1.200	0.800	39.999	60.001
0.30	0.200	0.401	0.401	1.200	0.0	1.000	0.0	0.0	1.200	0.800	39.986	60.014
0.35	0.200	0.400	0.400	1.200	0.0	1.000	0.0	0.0	1.200	0.800	40.000	60.000
0.40	0.200	0.400	0.400	1.200	0.0	1.000	0.0	0.0	1.200	0.800	39.998	60.002
0.45	0.200	0.400	0.400	1.200	0.0	1.000	0.0	0.0	1.200	0.800	40.000	60.000
0.50	0.200	0.400	0.400	1.200	0.0	1.000	0.0	0.0	1.200	0.800	40.000	60.000
0.55	0.200	0.400	0.400	1.200	0.0	1.000	0.0	0.0	1.200	0.800	39.995	60.005
0.60	0.200	0.400	0.400	1.200	0.0	1.000	0.0	0.0	1.200	0.800	40.001	59.999
0.65	0.200	0.400	0.400	1.200	0.0	1.000	0.0	0.0	1.200	0.800	40.000	60.000
0.70	0.200	0.401	0.401	1.200	0.0	1.000	0.0	0.0	1.200	0.800	39.986	60.014
0.75	0.200	0.400	0.400	1.200	0.0	1.000	0.0	0.0	1.200	0.800	40.001	59.999
0.80	0.200	0.400	0.400	1.200	0.0	1.000	0.0	0.0	1.200	0.800	39.998	60.002
0.85	0.200	0.400	0.400	1.200	0.0	1.000	0.0	0.0	1.200	0.800	40.000	60.000
0.90	0.200	0.400	0.400	1.200	0.0	1.000	0.0	0.0	1.200	0.800	40.000	60.000
0.95	0.200	0.400	0.400	1.200	0.0	1.000	0.0	0.0	1.200	0.800	39.995	60.005
1.00	0.200	0.400	0.400	1.200	0.0	1.000	0.0	0.0	1.200	0.800	40.001	59.999

TABLE B.5.3

MEASURES OF PERFORMANCE FOR MULTIPLE-MACHINE MODEL
INFINITE SOURCE AND NO QUEUE

CAPACITY = 2, NUMBER OF SERVERS = 2, LAMDA = 2.50, MUE = 1.00, INTERVAL = (0, 100.0)

TAU	PO	PL	PB	NS	NQ	FT	WT	N/Q	NPROC	NIDLE	IDLE%	UTIL%
0.0	1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	2.000	100.000	0.0
0.05	0.151	0.472	0.472	1.321	0.0	1.000	0.0	0.0	1.321	0.679	33.958	66.042
0.10	0.164	0.501	0.501	1.336	0.0	1.000	0.0	0.0	1.336	0.664	33.180	66.820
0.15	0.151	0.472	0.472	1.321	0.0	1.000	0.0	0.0	1.321	0.679	33.954	66.046
0.20	0.151	0.472	0.472	1.321	0.0	1.000	0.0	0.0	1.321	0.679	33.947	66.053
0.25	0.151	0.473	0.473	1.321	0.0	1.000	0.0	0.0	1.321	0.679	33.932	66.068
0.30	0.152	0.474	0.474	1.322	0.0	1.000	0.0	0.0	1.322	0.678	33.903	66.097
0.35	0.153	0.476	0.476	1.323	0.0	1.000	0.0	0.0	1.323	0.677	33.847	66.153
0.40	0.151	0.472	0.472	1.321	0.0	1.000	0.0	0.0	1.321	0.679	33.962	66.038
0.45	0.151	0.472	0.472	1.321	0.0	1.000	0.0	0.0	1.321	0.679	33.957	66.043
0.50	0.151	0.472	0.472	1.321	0.0	1.000	0.0	0.0	1.321	0.679	33.952	66.048
0.55	0.151	0.472	0.472	1.321	0.0	1.000	0.0	0.0	1.321	0.679	33.942	66.058
0.60	0.152	0.473	0.473	1.322	0.0	1.000	0.0	0.0	1.322	0.678	33.923	66.077
0.65	0.152	0.475	0.475	1.322	0.0	1.000	0.0	0.0	1.322	0.678	33.885	66.115
0.70	0.154	0.477	0.477	1.324	0.0	1.000	0.0	0.0	1.324	0.676	33.811	66.189
0.75	0.152	0.473	0.473	1.321	0.0	1.000	0.0	0.0	1.321	0.679	33.930	66.070
0.80	0.151	0.472	0.472	1.321	0.0	1.000	0.0	0.0	1.321	0.679	33.956	66.044
0.85	0.151	0.472	0.472	1.321	0.0	1.000	0.0	0.0	1.321	0.679	33.949	66.051
0.90	0.151	0.473	0.473	1.321	0.0	1.000	0.0	0.0	1.321	0.679	33.936	66.064
0.95	0.152	0.474	0.474	1.322	0.0	1.000	0.0	0.0	1.322	0.678	33.910	66.090
1.00	0.153	0.476	0.476	1.323	0.0	1.000	0.0	0.0	1.323	0.677	33.860	66.140

TABLE B+.6.1

MEASURES OF PERFORMANCE FOR MULTIPLE-MACHINE MODEL
INFINITE SOURCE AND LIMITED QUEUE

CAPACITY = 12, NUMBER OF SERVERS = 2, LAMDA = 1.50, MUE = 1.00, INTERVAL =(0, 100.0)

TAU	P0	PL	PB	NS	NQ	FT	WT	N/Q	NPROC	NIDLE	IDLE%	UTIL%
0.0	1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	2.000	100.000	0.0
0.05	0.190	0.0	0.536	2.153	0.807	1.435	0.538	2.362	1.346	0.654	32.675	67.325
0.10	0.163	0.004	0.594	2.653	1.222	1.769	0.815	2.938	1.431	0.569	28.443	71.557
0.15	0.154	0.007	0.615	2.873	1.412	1.915	0.942	3.186	1.461	0.539	26.974	73.026
0.20	0.151	0.008	0.624	2.978	1.504	1.985	1.003	3.299	1.474	0.526	26.313	73.687
0.25	0.149	0.009	0.629	3.029	1.549	2.019	1.033	3.353	1.480	0.520	25.996	74.004
0.30	0.148	0.009	0.631	3.054	1.571	2.036	1.047	3.379	1.483	0.517	25.842	74.158
0.35	0.147	0.009	0.632	3.066	1.581	2.044	1.054	3.391	1.485	0.515	25.767	74.233
0.40	0.147	0.009	0.632	3.072	1.587	2.048	1.058	3.397	1.485	0.515	25.731	74.269
0.45	0.147	0.009	0.633	3.075	1.589	2.050	1.059	3.400	1.486	0.514	25.712	74.288
0.50	0.147	0.009	0.633	3.076	1.590	2.051	1.060	3.401	1.486	0.514	25.703	74.297
0.55	0.147	0.009	0.633	3.077	1.591	2.051	1.061	3.402	1.486	0.514	25.699	74.301
0.60	0.147	0.009	0.633	3.077	1.591	2.052	1.061	3.402	1.486	0.514	25.697	74.303
0.65	0.147	0.009	0.633	3.077	1.591	2.052	1.061	3.402	1.486	0.514	25.695	74.305
0.70	0.147	0.009	0.633	3.077	1.591	2.052	1.061	3.404	1.486	0.514	25.688	74.312
0.75	0.147	0.009	0.633	3.078	1.591	2.052	1.061	3.402	1.486	0.514	25.694	74.306
0.80	0.147	0.009	0.633	3.078	1.591	2.052	1.061	3.402	1.486	0.514	25.694	74.306
0.85	0.147	0.009	0.633	3.078	1.591	2.052	1.061	3.402	1.486	0.514	25.694	74.306
0.90	0.147	0.009	0.633	3.078	1.591	2.052	1.061	3.402	1.486	0.514	25.693	74.307
0.95	0.147	0.009	0.633	3.078	1.591	2.052	1.061	3.402	1.486	0.514	25.693	74.307
1.00	0.147	0.009	0.633	3.078	1.591	2.052	1.061	3.402	1.486	0.514	25.691	74.309

TABLE B.6.2

MEASURES OF PERFORMANCE FOR MULTIPLE-MACHINE MODEL
INFINITE SOURCE AND LIMITED QUEUE

CAPACITY = 12, NUMBER OF SERVERS = 2, LAMDA = 2.00, MUE = 1.00, INTERVAL =(0, 100.0)

TAU	P0	PL	PB	NS	NQ	FT	WT	N/Q	NPROC	NIDLE	IDLE%	UTIL%
0.0	1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	2.000	100.000	0.0
0.05	0.095	0.007	0.726	3.440	1.809	1.720	0.904	3.223	1.631	0.369	18.449	81.551
0.10	0.065	0.035	0.808	4.746	3.003	2.373	1.502	4.374	1.743	0.257	12.836	87.164
0.15	0.053	0.055	0.843	5.442	3.652	2.721	1.826	4.932	1.789	0.211	10.527	89.473
0.20	0.047	0.067	0.860	5.813	4.000	2.907	2.000	5.207	1.813	0.187	9.350	90.650
0.25	0.044	0.073	0.869	6.012	4.186	3.006	2.093	5.346	1.825	0.175	8.726	91.274
0.30	0.042	0.076	0.874	6.118	4.285	3.059	2.143	5.419	1.832	0.168	8.394	91.606
0.35	0.041	0.078	0.877	6.174	4.338	3.087	2.169	5.457	1.836	0.164	8.217	91.783
0.40	0.041	0.079	0.878	6.204	4.367	3.102	2.183	5.477	1.838	0.162	8.124	91.876
0.45	0.040	0.079	0.879	6.220	4.382	3.110	2.191	5.488	1.839	0.161	8.075	91.925
0.50	0.040	0.080	0.879	6.229	4.390	3.114	2.195	5.493	1.839	0.161	8.050	91.950
0.55	0.040	0.080	0.880	6.233	4.394	3.117	2.197	5.496	1.839	0.161	8.036	91.964
0.60	0.040	0.080	0.880	6.236	4.396	3.118	2.198	5.498	1.839	0.161	8.030	91.970
0.65	0.040	0.080	0.880	6.237	4.397	3.118	2.199	5.499	1.839	0.161	8.028	91.972
0.70	0.040	0.080	0.880	6.237	4.398	3.119	2.199	5.499	1.839	0.161	8.028	91.972
0.75	0.040	0.080	0.880	6.238	4.398	3.119	2.199	5.500	1.839	0.161	8.028	91.972
0.80	0.040	0.080	0.880	6.238	4.398	3.119	2.199	5.500	1.839	0.161	8.029	91.971
0.85	0.040	0.080	0.880	6.238	4.398	3.119	2.199	5.500	1.839	0.161	8.031	91.969
0.90	0.040	0.080	0.880	6.238	4.398	3.119	2.199	5.500	1.839	0.161	8.031	91.969
0.95	0.040	0.080	0.880	6.238	4.398	3.119	2.199	5.500	1.839	0.161	8.032	91.968
1.00	0.040	0.080	0.880	6.238	4.398	3.119	2.199	5.500	1.839	0.161	8.034	91.966

TABLE B.6.3

MEASURES OF PERFORMANCE FOR MULTIPLE-MACHINE MODEL
INFINITE SOURCE AND LIMITED QUEUE

CAPACITY = 12, NUMBER OF SERVERS = 2, LAMDA = 2.50, MUE = 1.00, INTERVAL =(0, 100.0)

TAU	P0	PL	PB	NS	NQ	FT	WT	N/Q	NPROC	NIDLE	IDLE%	UTIL%
0.0	1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	2.000	100.000	0.0
0.05	0.044	0.032	0.855	4.979	3.168	1.992	1.267	4.264	1.812	0.188	9.423	90.577
0.10	0.021	0.123	0.929	7.128	5.220	2.851	2.088	5.992	1.908	0.092	4.579	95.421
0.15	0.013	0.174	0.955	8.090	6.147	3.236	2.459	6.701	1.942	0.058	2.893	97.107
0.20	0.010	0.196	0.966	8.505	6.549	3.402	2.620	6.991	1.956	0.044	2.187	97.813
0.25	0.008	0.206	0.971	8.684	6.722	3.474	2.689	7.113	1.962	0.038	1.886	98.114
0.30	0.008	0.210	0.973	8.761	6.796	3.504	2.719	7.164	1.965	0.035	1.757	98.243
0.35	0.008	0.212	0.974	8.794	6.828	3.518	2.731	7.185	1.966	0.034	1.704	98.296
0.40	0.007	0.213	0.974	8.808	6.842	3.523	2.737	7.196	1.966	0.034	1.679	98.321
0.45	0.007	0.213	0.974	8.814	6.848	3.526	2.739	7.200	1.967	0.033	1.670	98.330
0.50	0.007	0.213	0.974	8.817	6.850	3.527	2.740	7.202	1.967	0.033	1.667	98.333
0.55	0.007	0.213	0.974	8.818	6.851	3.527	2.741	7.202	1.967	0.033	1.666	98.334
0.60	0.007	0.213	0.974	8.818	6.852	3.527	2.741	7.203	1.967	0.033	1.666	98.334
0.65	0.007	0.213	0.974	8.819	6.852	3.527	2.741	7.203	1.967	0.033	1.667	98.333
0.70	0.007	0.213	0.974	8.819	6.852	3.527	2.741	7.203	1.967	0.033	1.668	98.332
0.75	0.007	0.213	0.974	8.818	6.852	3.527	2.741	7.203	1.967	0.033	1.670	98.330
0.80	0.007	0.213	0.974	8.818	6.852	3.527	2.741	7.203	1.967	0.033	1.670	98.330
0.85	0.007	0.213	0.974	8.818	6.852	3.527	2.741	7.203	1.967	0.033	1.672	98.328
0.90	0.007	0.213	0.974	8.818	6.852	3.527	2.741	7.203	1.967	0.033	1.673	98.327
0.95	0.007	0.213	0.974	8.818	6.852	3.527	2.741	7.203	1.967	0.033	1.674	98.326
1.00	0.007	0.213	0.974	8.818	6.851	3.527	2.741	7.203	1.966	0.034	1.675	98.325

TABLE B.7.1

MEASURES OF PERFORMANCE FOR MULTIPLE-MACHINE MODEL
INFINITE SOURCE AND UNLIMITED QUEUE

CAPACITY =100, NUMBER OF SERVERS = 2, LAMDA = 1.50, MUE = 1.00, INTERVAL =(0, 100.0)

TAU	P0	PL	PB	NS	NQ	FT	WT	N/Q	NPROC	NIDLE	IDLE%	UTIL%
0.0	1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	2.000	100.000	0.0
0.05	0.190	0.0	0.536	2.152	0.805	1.434	0.537	2.358	1.346	0.654	32.676	67.324
0.10	0.163	0.0	0.595	2.655	1.223	1.770	0.816	2.942	1.431	0.569	28.435	71.565
0.15	0.154	0.0	0.616	2.897	1.436	1.931	0.957	3.236	1.461	0.539	26.933	73.067
0.20	0.150	0.0	0.626	3.043	1.567	2.029	1.045	3.421	1.476	0.524	26.216	73.784
0.25	0.148	0.0	0.631	3.140	1.656	2.093	1.104	3.552	1.484	0.516	25.805	74.195
0.30	0.146	0.0	0.635	3.210	1.721	2.140	1.147	3.658	1.489	0.511	25.535	74.465
0.35	0.145	0.0	0.637	3.245	1.753	2.163	1.169	3.695	1.492	0.508	25.388	74.612
0.40	0.144	0.0	0.639	3.270	1.775	2.180	1.184	3.725	1.494	0.506	25.279	74.721
0.45	0.144	0.0	0.640	3.287	1.791	2.191	1.194	3.746	1.496	0.504	25.203	74.797
0.50	0.144	0.0	0.641	3.316	1.819	2.211	1.213	3.797	1.497	0.503	25.149	74.851
0.55	0.143	0.0	0.641	3.326	1.828	2.217	1.219	3.808	1.498	0.502	25.110	74.890
0.60	0.143	0.0	0.642	3.332	1.834	2.221	1.222	3.818	1.498	0.502	25.079	74.921
0.65	0.143	0.0	0.642	3.337	1.838	2.225	1.226	3.823	1.499	0.501	25.061	74.939
0.70	0.143	0.0	0.642	3.341	1.842	2.227	1.228	3.827	1.499	0.501	25.046	74.954
0.75	0.143	0.0	0.642	3.343	1.844	2.229	1.229	3.830	1.499	0.501	25.034	74.966
0.80	0.143	0.0	0.643	3.345	1.846	2.230	1.231	3.833	1.499	0.501	25.025	74.975
0.85	0.143	0.0	0.643	3.347	1.847	2.231	1.232	3.834	1.500	0.500	25.018	74.982
0.90	0.143	0.0	0.643	3.348	1.848	2.232	1.232	3.836	1.500	0.500	25.013	74.987
0.95	0.143	0.0	0.643	3.349	1.849	2.233	1.233	3.837	1.500	0.500	25.009	74.991
1.00	0.143	0.0	0.643	3.350	1.850	2.233	1.233	3.837	1.500	0.500	25.005	74.995

TABLE B.7.2

MEASURES OF PERFORMANCE FOR MULTIPLE-MACHINE MODEL
INFINITE SOURCE AND UNLIMITED QUEUE

CAPACITY =100, NUMBER OF SERVERS = 2, LAMDA = 2.00, MUE = 1.00, INTERVAL =(0, 100.0)

TAU	P0	PL	PB	NS	NQ	FT	WT	N/Q	NPROC	NIDLE	IDLE%	UTIL%
0.0	1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	2.000	100.000	0.0
0.05	0.095	0.0	0.726	3.438	1.807	1.719	0.903	3.220	1.631	0.369	18.440	81.560
0.10	0.065	0.0	0.809	4.928	3.184	2.464	1.592	4.637	1.744	0.256	12.817	87.183
0.15	0.053	0.0	0.844	6.059	4.268	3.030	2.134	5.739	1.792	0.208	10.406	89.594
0.20	0.045	0.0	0.866	7.001	5.180	3.500	2.590	6.663	1.820	0.180	8.984	91.016
0.25	0.040	0.0	0.880	7.834	5.994	3.917	2.997	7.483	1.839	0.161	8.029	91.971
0.30	0.037	0.0	0.890	8.595	6.741	4.297	3.371	8.238	1.854	0.146	7.322	92.678
0.35	0.034	0.0	0.899	9.272	7.408	4.636	3.704	8.906	1.865	0.135	6.773	93.227
0.40	0.032	0.0	0.905	9.905	8.032	4.953	4.016	9.532	1.873	0.127	6.332	93.668
0.45	0.030	0.0	0.911	10.503	8.622	5.251	4.311	10.125	1.881	0.119	5.967	94.033
0.50	0.028	0.0	0.915	11.071	9.184	5.535	4.592	10.690	1.887	0.113	5.659	94.341
0.55	0.027	0.0	0.919	11.576	9.684	5.788	4.842	11.186	1.892	0.108	5.394	94.606
0.60	0.026	0.0	0.923	12.100	10.203	6.050	5.101	11.708	1.897	0.103	5.163	94.837
0.65	0.025	0.0	0.926	12.562	10.661	6.281	5.330	12.164	1.901	0.099	4.960	95.040
0.70	0.024	0.0	0.928	13.052	11.147	6.526	5.574	12.654	1.904	0.096	4.778	95.222
0.75	0.023	0.0	0.931	13.482	11.574	6.741	5.787	13.079	1.908	0.092	4.616	95.384
0.80	0.022	0.0	0.933	13.946	12.036	6.973	6.018	13.544	1.911	0.089	4.468	95.532
0.85	0.022	0.0	0.935	14.352	12.439	7.176	6.219	13.946	1.913	0.087	4.335	95.665
0.90	0.021	0.0	0.937	14.745	12.830	7.373	6.415	14.335	1.916	0.084	4.212	95.788
0.95	0.021	0.0	0.939	15.180	13.262	7.590	6.631	14.772	1.918	0.082	4.099	95.901
1.00	0.020	0.0	0.940	15.556	13.635	7.778	6.818	15.145	1.920	0.080	3.995	96.005

TABLE B.7.3

MEASURES OF PERFORMANCE FOR MULTIPLE-MACHINE MODEL
INFINITE SOURCE AND UNLIMITED QUEUE

CAPACITY =100, NUMBER OF SERVERS = 2, LAMDA = 2.50, MUE = 1.00, INTERVAL =(0, 100.0)

TAU	PO	PL	PB	NS	NQ	FT	WT	N/Q	NPROC	NIDLE	IDLE%	UTIL%
0.0	1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	2.000	100.000	0.0
0.05	0.044	0.0	0.855	5.043	3.232	2.017	1.293	4.350	1.812	0.188	9.420	90.580
0.10	0.021	0.0	0.930	8.168	6.259	3.267	2.504	7.179	1.909	0.091	4.554	95.446
0.15	0.012	0.0	0.958	10.979	9.033	4.391	3.613	9.796	1.945	0.055	2.738	97.262
0.20	0.008	0.0	0.972	13.677	11.713	5.471	4.685	12.351	1.964	0.036	1.808	98.192
0.25	0.006	0.0	0.981	16.283	14.308	6.513	5.723	14.844	1.975	0.025	1.259	98.741
0.30	0.004	0.0	0.986	18.854	16.873	7.542	6.749	17.325	1.982	0.018	0.909	99.091
0.35	0.003	0.0	0.990	21.408	19.422	8.563	7.769	19.804	1.987	0.013	0.672	99.328
0.40	0.002	0.0	0.992	23.898	21.908	9.559	8.763	22.232	1.990	0.010	0.507	99.493
0.45	0.002	0.0	0.994	26.437	24.445	10.575	9.778	24.720	1.992	0.008	0.388	99.612
0.50	0.001	0.0	0.995	28.914	26.920	11.566	10.768	27.154	1.994	0.006	0.300	99.700
0.55	0.001	0.0	0.996	31.387	29.391	12.555	11.757	29.591	1.995	0.005	0.234	99.766
0.60	0.0	0.0	0.998	33.859	31.861	13.544	12.745	32.005	1.998	0.002	0.102	99.898
0.65	0.0	0.0	0.998	36.330	34.332	14.532	13.733	34.455	1.998	0.002	0.081	99.919
0.70	0.0	0.0	0.999	38.803	36.804	15.521	14.722	36.909	1.999	0.001	0.065	99.935
0.75	0.0	0.0	0.999	41.277	39.278	16.511	15.711	39.368	1.999	0.001	0.052	99.948
0.80	0.0	0.0	1.000	43.756	41.756	17.503	16.703	41.799	2.000	0.0	0.0	100.000
0.85	0.0	0.001	1.000	46.352	44.352	18.541	17.741	44.352	2.000	0.0	0.0	100.000
0.90	0.0	0.002	1.000	49.328	47.328	19.731	18.931	47.328	2.000	0.0	0.0	100.000
0.95	0.0	0.003	1.000	51.797	49.797	20.719	19.919	49.797	2.000	0.0	0.0	100.000
1.00	0.0	0.005	1.000	54.246	52.246	21.698	20.898	52.246	2.000	0.0	0.0	100.000

TABLE B.8.1

MEASURES OF PERFORMANCE FOR MULTIPLE-MACHINE MODEL
FINITE SOURCE AND LIMITED QUEUE

CAPACITY = 12, NUMBER OF SERVERS = 2, LAMDA = 1.00, MUE = 1.00, INTERVAL = (0, 100.0)

TAU	PO	PL	PB	NS	NQ	FT	WT	N/Q	NPRCC	NIDLE	IDLE%	JTIL%
0.0	1.000	0.0	0.0	12.000	12.000	12.000	12.000	0.0	0.0	2.000	100.000	0.0
0.05	0.0	0.058	1.000	10.001	8.002	10.001	8.002	8.006	1.999	0.001	0.049	99.951
0.10	0.0	0.127	1.000	10.001	8.001	10.001	8.001	8.004	1.999	0.001	0.034	99.966
0.15	0.0	0.135	0.999	10.002	8.005	10.002	8.005	8.014	1.998	0.002	0.117	99.883
0.20	0.0	0.135	0.999	10.002	8.005	10.002	8.005	8.014	1.998	0.002	0.116	99.883
0.25	0.0	0.135	0.999	10.002	8.005	10.002	8.005	8.014	1.998	0.002	0.117	99.883
0.30	0.0	0.135	0.999	10.002	8.005	10.002	8.005	8.014	1.998	0.002	0.118	99.882
0.35	0.0	0.135	0.999	10.002	8.005	10.002	8.005	8.014	1.998	0.002	0.119	99.881
0.40	0.0	0.135	0.999	10.002	8.005	10.002	8.005	8.014	1.998	0.002	0.120	99.880
0.45	0.0	0.135	0.999	10.002	8.005	10.002	8.005	8.015	1.998	0.002	0.121	99.879
0.50	0.0	0.135	0.999	10.002	8.005	10.002	8.005	8.015	1.998	0.002	0.122	99.878
0.55	0.0	0.135	0.999	10.002	8.005	10.002	8.005	8.015	1.998	0.002	0.123	99.877
0.60	0.0	0.135	0.999	10.002	8.005	10.002	8.005	8.015	1.998	0.002	0.124	99.876
0.65	0.0	0.135	0.999	10.003	8.005	10.003	8.005	8.015	1.997	0.003	0.125	99.875
0.70	0.0	0.135	0.999	10.003	8.005	10.003	8.005	8.015	1.997	0.003	0.126	99.874
0.75	0.0	0.135	0.999	10.003	8.005	10.003	8.005	8.015	1.997	0.003	0.127	99.873
0.80	0.0	0.135	0.999	10.003	8.005	10.003	8.005	8.015	1.997	0.003	0.128	99.872
0.85	0.0	0.135	0.999	10.003	8.005	10.003	8.005	8.016	1.997	0.003	0.129	99.871
0.90	0.0	0.135	0.999	10.003	8.005	10.003	8.005	8.016	1.997	0.003	0.130	99.870
0.95	0.0	0.135	0.999	10.003	8.005	10.003	8.005	8.016	1.997	0.003	0.131	99.869
1.00	0.0	0.135	0.999	10.003	8.005	10.003	8.005	8.016	1.997	0.003	0.132	99.868

TABLE B.8.2

MEASURES OF PERFORMANCE FOR MULTIPLE-MACHINE MODEL
FINITE SOURCE AND LIMITED QUEUE

CAPACITY = 12, NUMBER OF SERVERS = 2, LAMDA = 1.50, MUE = 1.00, INTERVAL =(0, 100.0)

TAU	P0	PL	PB	NS	NQ	FT	WT	N/Q	NPROC	NIDLE	IDLE%	UTIL%
0.0	1.000	0.0	0.0	12.000	12.000	8.000	8.000	0.0	0.0	2.000	100.000	0.0
0.05	0.0	0.205	0.999	10.668	8.671	7.112	5.781	8.681	1.998	0.002	0.123	99.877
0.10	0.0	0.262	0.999	10.667	8.668	7.112	5.779	8.673	1.999	0.001	0.052	99.947
0.15	0.0	0.264	0.999	10.667	8.668	7.112	5.779	8.673	1.999	0.001	0.053	99.947
0.20	0.0	0.264	0.999	10.667	8.668	7.112	5.779	8.673	1.999	0.001	0.054	99.946
0.25	0.0	0.264	0.999	10.667	8.668	7.112	5.779	8.673	1.999	0.001	0.055	99.945
0.30	0.0	0.264	0.999	10.667	8.668	7.112	5.779	8.673	1.999	0.001	0.055	99.945
0.35	0.0	0.264	0.999	10.667	8.668	7.112	5.779	8.673	1.999	0.001	0.055	99.945
0.40	0.0	0.264	0.999	10.667	8.668	7.112	5.779	8.673	1.999	0.001	0.055	99.945
0.45	0.0	0.264	0.999	10.667	8.668	7.112	5.779	8.673	1.999	0.001	0.055	99.945
0.50	0.0	0.264	0.999	10.667	8.668	7.112	5.779	8.673	1.999	0.001	0.055	99.945
0.55	0.0	0.264	0.999	10.667	8.668	7.112	5.779	8.673	1.999	0.001	0.055	99.945
0.60	0.0	0.264	0.999	10.667	8.668	7.112	5.779	8.673	1.999	0.001	0.055	99.945
0.65	0.0	0.264	0.999	10.667	8.668	7.112	5.779	8.673	1.999	0.001	0.055	99.945
0.70	0.0	0.264	0.999	10.667	8.668	7.112	5.779	8.673	1.999	0.001	0.055	99.945
0.75	0.0	0.264	0.999	10.667	8.668	7.112	5.779	8.673	1.999	0.001	0.055	99.945
0.80	0.0	0.264	0.999	10.667	8.668	7.112	5.779	8.673	1.999	0.001	0.055	99.945
0.85	0.0	0.264	0.999	10.667	8.668	7.112	5.779	8.673	1.999	0.001	0.055	99.945
0.90	0.0	0.264	0.999	10.667	8.668	7.112	5.779	8.673	1.999	0.001	0.055	99.945
0.95	0.0	0.264	0.999	10.667	8.668	7.112	5.779	8.673	1.999	0.001	0.055	99.945
1.00	0.0	0.264	0.999	10.667	8.668	7.112	5.779	8.673	1.999	0.001	0.055	99.945

TABLE B.8.3

MEASURES OF PERFORMANCE FOR MULTIPLE-MACHINE MODEL
FINITE SOURCE AND LIMITED QUEUE

CAPACITY = 12, NUMBER OF SERVERS = 2, LAMDA = 2.00, MUE = 1.00, INTERVAL =(0, 100.0)

TAU	P0	PL	PB	NS	NQ	FT	WT	N/Q	NPROC	NIDLE	IDLE%	UTIL%
0.0	1.000	0.0	0.0	12.000	12.000	6.000	6.000	0.0	0.0	2.000	100.000	0.0
0.05	0.0	0.342	0.999	11.001	9.003	5.500	4.501	9.011	1.998	0.002	0.089	99.911
0.10	0.0	0.368	0.999	11.001	9.002	5.500	4.501	9.008	1.999	0.001	0.055	99.935
0.15	0.0	0.368	0.999	11.001	9.002	5.500	4.501	9.008	1.999	0.001	0.055	99.934
0.20	0.0	0.368	0.999	11.001	9.002	5.500	4.501	9.008	1.999	0.001	0.056	99.934
0.25	0.0	0.368	0.999	11.001	9.002	5.500	4.501	9.008	1.999	0.001	0.055	99.934
0.30	0.0	0.368	0.999	11.001	9.002	5.500	4.501	9.008	1.999	0.001	0.056	99.934
0.35	0.0	0.368	0.999	11.001	9.002	5.500	4.501	9.008	1.999	0.001	0.055	99.934
0.40	0.0	0.368	0.999	11.001	9.002	5.500	4.501	9.008	1.999	0.001	0.056	99.934
0.45	0.0	0.368	0.999	11.001	9.002	5.500	4.501	9.008	1.999	0.001	0.055	99.934
0.50	0.0	0.368	0.999	11.001	9.002	5.500	4.501	9.008	1.999	0.001	0.056	99.934
0.55	0.0	0.368	0.999	11.001	9.002	5.500	4.501	9.008	1.999	0.001	0.055	99.934
0.60	0.0	0.368	0.999	11.001	9.002	5.500	4.501	9.008	1.999	0.001	0.056	99.934
0.65	0.0	0.368	0.999	11.001	9.002	5.500	4.501	9.008	1.999	0.001	0.055	99.934
0.70	0.0	0.368	0.999	11.001	9.002	5.500	4.501	9.008	1.999	0.001	0.056	99.934
0.75	0.0	0.368	0.999	11.001	9.002	5.500	4.501	9.008	1.999	0.001	0.055	99.934
0.80	0.0	0.368	0.999	11.001	9.002	5.500	4.501	9.008	1.999	0.001	0.056	99.934
0.85	0.0	0.368	0.999	11.001	9.002	5.500	4.501	9.008	1.999	0.001	0.055	99.934
0.90	0.0	0.368	0.999	11.001	9.002	5.500	4.501	9.008	1.999	0.001	0.056	99.934
0.95	0.0	0.368	0.999	11.001	9.002	5.500	4.501	9.008	1.999	0.001	0.055	99.934
1.00	0.0	0.368	0.999	11.001	9.002	5.500	4.501	9.008	1.999	0.001	0.056	99.934

TABLE B.9.1

MEASURES OF PERFORMANCE FOR SPECIAL-SHOP MODEL
INFINITE SOURCE AND NO QUEUE

CAPACITY = 3, NUMBER OF SERVERS = 2, LAMDA = 1.50, MUE = 1.00, INTERVAL =(0, 100.0)

TAU	P0	PL	PB	NS	NQ	FT	WT	N/Q	NPROC	NIDLE	IDLE%	UTIL%
0.0	1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	2.000	100.000	0.0
0.05	0.266	0.087	0.087	1.177	0.0	1.000	0.0	0.0	1.177	0.823	41.133	58.867
0.10	0.251	0.094	0.094	1.217	0.0	1.000	0.0	0.0	1.217	0.783	39.128	60.872
0.15	0.250	0.094	0.094	1.219	0.0	1.000	0.0	0.0	1.219	0.781	39.065	60.935
0.20	0.250	0.094	0.094	1.219	0.0	1.000	0.0	0.0	1.219	0.781	39.063	60.937
0.25	0.250	0.094	0.094	1.219	0.0	1.000	0.0	0.0	1.219	0.781	39.064	60.936
0.30	0.250	0.094	0.094	1.219	0.0	1.000	0.0	0.0	1.219	0.781	39.064	60.936
0.35	0.250	0.094	0.094	1.219	0.0	1.000	0.0	0.0	1.219	0.781	39.064	60.936
0.40	0.250	0.094	0.094	1.219	0.0	1.000	0.0	0.0	1.219	0.781	39.064	60.936
0.45	0.250	0.094	0.094	1.219	0.0	1.000	0.0	0.0	1.219	0.781	39.064	60.936
0.50	0.250	0.094	0.094	1.219	0.0	1.000	0.0	0.0	1.219	0.781	39.064	60.936
0.55	0.250	0.094	0.094	1.219	0.0	1.000	0.0	0.0	1.219	0.781	39.064	60.936
0.60	0.250	0.094	0.094	1.219	0.0	1.000	0.0	0.0	1.219	0.781	39.065	60.935
0.65	0.250	0.094	0.094	1.219	0.0	1.000	0.0	0.0	1.219	0.781	39.065	60.935
0.70	0.250	0.094	0.094	1.219	0.0	1.000	0.0	0.0	1.219	0.781	39.065	60.935
0.75	0.250	0.094	0.094	1.219	0.0	1.000	0.0	0.0	1.219	0.781	39.065	60.935
0.80	0.250	0.094	0.094	1.219	0.0	1.000	0.0	0.0	1.219	0.781	39.065	60.935
0.85	0.250	0.094	0.094	1.219	0.0	1.000	0.0	0.0	1.219	0.781	39.065	60.935
0.90	0.250	0.094	0.094	1.219	0.0	1.000	0.0	0.0	1.219	0.781	39.065	60.935
0.95	0.250	0.094	0.094	1.219	0.0	1.000	0.0	0.0	1.219	0.781	39.066	60.934
1.00	0.250	0.094	0.094	1.219	0.0	1.000	0.0	0.0	1.219	0.781	39.066	60.934

TABLE B.9.2

MEASURES OF PERFORMANCE FOR SPECIAL-SHOP MODEL
INFINITE SOURCE AND NO QUEUE

CAPACITY = 3, NUMBER OF SERVERS = 2, LAMDA = 2.00, MUE = 1.00, INTERVAL = (0, 100.0)

TAU	PO	PL	PB	NS	NQ	FT	WT	N/Q	NPROC	NIDLE	IDLE%	UTIL%
0.0	1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	2.000	100.000	0.0
0.05	0.180	0.144	0.144	1.441	0.0	1.000	0.0	0.0	1.441	0.559	27.938	72.063
0.10	0.170	0.151	0.151	1.471	0.0	1.000	0.0	0.0	1.471	0.529	26.444	73.556
0.15	0.170	0.151	0.151	1.472	0.0	1.000	0.0	0.0	1.472	0.528	26.416	73.584
0.20	0.170	0.151	0.151	1.472	0.0	1.000	0.0	0.0	1.472	0.528	26.416	73.584
0.25	0.170	0.151	0.151	1.472	0.0	1.000	0.0	0.0	1.472	0.528	26.416	73.584
0.30	0.170	0.151	0.151	1.472	0.0	1.000	0.0	0.0	1.472	0.528	26.416	73.584
0.35	0.170	0.151	0.151	1.472	0.0	1.000	0.0	0.0	1.472	0.528	26.416	73.584
0.40	0.170	0.151	0.151	1.472	0.0	1.000	0.0	0.0	1.472	0.528	26.417	73.583
0.45	0.170	0.151	0.151	1.472	0.0	1.000	0.0	0.0	1.472	0.528	26.417	73.583
0.50	0.170	0.151	0.151	1.472	0.0	1.000	0.0	0.0	1.472	0.528	26.417	73.583
0.55	0.170	0.151	0.151	1.472	0.0	1.000	0.0	0.0	1.472	0.528	26.417	73.583
0.60	0.170	0.151	0.151	1.472	0.0	1.000	0.0	0.0	1.472	0.528	26.417	73.583
0.65	0.170	0.151	0.151	1.472	0.0	1.000	0.0	0.0	1.472	0.528	26.417	73.583
0.70	0.170	0.151	0.151	1.472	0.0	1.000	0.0	0.0	1.472	0.528	26.417	73.583
0.75	0.170	0.151	0.151	1.472	0.0	1.000	0.0	0.0	1.472	0.528	26.417	73.583
0.80	0.170	0.151	0.151	1.472	0.0	1.000	0.0	0.0	1.472	0.528	26.417	73.583
0.85	0.170	0.151	0.151	1.472	0.0	1.000	0.0	0.0	1.472	0.528	26.417	73.583
0.90	0.170	0.151	0.151	1.472	0.0	1.000	0.0	0.0	1.472	0.528	26.417	73.583
0.95	0.170	0.151	0.151	1.472	0.0	1.000	0.0	0.0	1.472	0.528	26.418	73.582
1.00	0.170	0.151	0.151	1.472	0.0	1.000	0.0	0.0	1.472	0.528	26.418	73.582

TABLE B.9.3

MEASURES OF PERFORMANCE FOR SPECIAL-SHOP MODEL
INFINITE SOURCE AND NO QUEUE

CAPACITY = 3, NUMBER OF SERVERS = 2, LAMDA = 2.50, MUE = 1.00, INTERVAL = (0, 100.0)

TAU	P0	PL	PB	NS	NQ	FT	WT	N/Q	NPROC	NIDLE	IDLE%	UTIL%
0.0	1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	2.000	100.000	0.0
0.05	0.125	0.202	0.202	1.649	0.0	1.000	0.0	0.0	1.649	0.351	17.529	82.471
0.10	0.120	0.208	0.208	1.669	0.0	1.000	0.0	0.0	1.669	0.331	16.540	83.460
0.15	0.120	0.208	0.208	1.669	0.0	1.000	0.0	0.0	1.669	0.331	16.530	83.470
0.20	0.120	0.208	0.208	1.669	0.0	1.000	0.0	0.0	1.669	0.331	16.529	83.471
0.25	0.120	0.208	0.208	1.669	0.0	1.000	0.0	0.0	1.669	0.331	16.530	83.470
0.30	0.120	0.208	0.208	1.669	0.0	1.000	0.0	0.0	1.669	0.331	16.530	83.470
0.35	0.120	0.208	0.208	1.669	0.0	1.000	0.0	0.0	1.669	0.331	16.530	83.470
0.40	0.120	0.208	0.208	1.669	0.0	1.000	0.0	0.0	1.669	0.331	16.530	83.470
0.45	0.120	0.208	0.208	1.669	0.0	1.000	0.0	0.0	1.669	0.331	16.531	83.469
0.50	0.120	0.208	0.208	1.669	0.0	1.000	0.0	0.0	1.669	0.331	16.531	83.469
0.55	0.120	0.208	0.208	1.669	0.0	1.000	0.0	0.0	1.669	0.331	16.531	83.469
0.60	0.120	0.208	0.208	1.669	0.0	1.000	0.0	0.0	1.669	0.331	16.531	83.469
0.65	0.120	0.208	0.208	1.669	0.0	1.000	0.0	0.0	1.669	0.331	16.531	83.469
0.70	0.120	0.208	0.208	1.669	0.0	1.000	0.0	0.0	1.669	0.331	16.532	83.468
0.75	0.120	0.208	0.208	1.669	0.0	1.000	0.0	0.0	1.669	0.331	16.532	83.468
0.80	0.120	0.208	0.208	1.669	0.0	1.000	0.0	0.0	1.669	0.331	16.532	83.468
0.85	0.120	0.208	0.208	1.669	0.0	1.000	0.0	0.0	1.669	0.331	16.532	83.468
0.90	0.120	0.208	0.208	1.669	0.0	1.000	0.0	0.0	1.669	0.331	16.532	83.468
0.95	0.120	0.208	0.208	1.669	0.0	1.000	0.0	0.0	1.669	0.331	16.533	83.467
1.00	0.120	0.208	0.208	1.669	0.0	1.000	0.0	0.0	1.669	0.331	16.533	83.467

APPENDIX C

COMPUTER PROGRAM

This appendix includes the sample computer program for the numerical solutions of queueing models. The computer program has been written in FORTRAN IV language for the IBM 360/50 computer. For the numerical integration of difference-differential equations, IBM scientific subroutine package RKGS (fourth order Runge-Kutta subroutine) has been used. To use the RKGS, to external subroutines, FCT and OUP for the functional equations and output, respectively, are required in addition to the main program. FCT is the name of the external subroutine for the functional equations. Its parameters are X, Y, and DERY. This subroutine computes the right hand sides of DERY of the system at given values of X and Y. OUP is the name of the output subroutine. Its parameters must be X, Y, DERY, IHLF, NDIM, and PRMT. The parameters FCT and OUP require an external statement in the main program. The parameters of RKGS to be specified in the main program are as follows:

PRMT(1) lower bound of the integration interval
PRMT(2) upper bound of the integration interval
PRMT(3) step size for the independent variable
PRMT(4) upper bound for the truncation error
Y input vector of initial conditions (destroyed at subsequent steps). At subsequent steps Y is replaced by the computed vector of dependent variables, computed at intermediate points X.

DERY input vector of error weights (destroyed at subsequent steps). The sum of the components of the weights should be equal to 1. At subsequent steps DERY is the vector of derivatives of Y at a point X.

NDIM number of equations in the system

AUX auxiliary storage array with eight rows and NDIM columns.

The parameter IHLF gives the output value of the number of bisections of the initial increment. If IHLF is greater than 10, RKGS returns with error message IHLF = 11 into main program. If PRMT(3) = 0, error message may be IHLF = 12. If sign of PRMT(3) is not equal to the sign of PRMT(2) - PRMT(1), error message may be IHLF = 13.

Next, a sample computer program for the infinite source and limited queue case of the multiple-machine model is shown by which the data for Tables B.6.1, B.6.2, and B.6.3 have been computed.

```

EXTERNAL  OUTP,FCT
DIMENSION PRMT(5),Y(105),DERY(105),AUX(8,105)
COMMON N,NS,XL,GM,T,MK
MK=0.0
READ(1,100)N,NS,XL,GM,T
100 FORMAT(2I4,3F10.5)
N1=N+1
Y(1)=1.0
DO 105 I=2,N1
105 Y(I)=0.0
DO 110 I=1,N1
110 DERY(I)=1./N1
PRMT(1)=0.0
PRMT(2)=1.
PRMT(3)=0.01
PRMT(4)=0.0001
NDIM=N1
6 FORMAT(1H1/////62X,'TABLE  4.6.1')
WRITE(3,6)
7 FORMAT(1H0,43X,'MEASURES OF PERFORMANCE FOR MULTIPLE-MACHINE MODEL
1'/1H ,50X,'INFINITE SOURCE AND LIMITED QUEUE')
WRITE(3,7)
40 FORMAT(1H0,25X,'CAPACITY ='I3,', NUMBER OF SERVERS ='I3,', LAMDA =
1'F5.2,', MUE ='F5.2,', INTERVAL =(0,'F6.1,')')
WRITE(3,40)N,NS,XL,GM,T
8 FORMAT(1H0,13X' TAU',5X,'PO',7X,'PL',7X,'PB',7X,'NS',7X,'NQ',7X,'F
1T',7X,'WT',7X,'N/Q',5X,'NPROC',4X,'NIDLE',4X,'IDLE%',4X,'UTIL%'/)
WRITE(3,8)
CALL RKGS(PRMT,Y,DERY,NDIM,IHLF,FCT,OUTP,AUX)
STOP
END

```

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FCT

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```
SUBROUTINE FCT(X,Y,DERY)
  DIMENSION Y(1),DERY(1)
  COMMON N,NS,XL,GM,T,MK
  N1=N+1
  NF=NS+1
  DERY(1)=T*(GM*Y(2)-XL*Y(1))
  DO 10 J=2,NS
    D=J-1.
  10 DERY(J)=T*(XL*Y(J-1)-(XL+D*GM)*Y(J)+(D+1.)*GM*Y(J+1))
    DO 20 J=NF,N
  20 DERY(J)=T*(XL*Y(J-1)+NS*GM*Y(J+1)-(XL+NS*GM)*Y(J))
  DERY(N+1)=T*(XL*Y(N)-NS*GM*Y(N+1))
  RETURN
END
```

```
SUBROUTINE OUTP(X,Y,DERY,IHLF,NDIM,PRMT)
DIMENSION Y(1),DERY(1),PRMT(1)
COMMON N,NS,XL,GM,T,MK
N1=N+1
NF=NS+1
DO 80 J=1,N1
IF (Y(J).LT.0.001)Y(J)=0.000
80 CONTINUE
V=20.*X
NV=V
IF(NV-V+0.002)11,10,10
11 NV=NV+1
IF(NV-V-0.002)10,10,12
10 SUMJ=0.
MK=MK+1
IF(MK.GT.21)GO TO 12
DO 13 J=NF,N1
13 SUMJ=SUMJ+(J-NF)*Y(J)
CLQ=SUMJ
SUMK=0.0
DO 21 K=1,NS
21 SUMK=SUMK+(K-1.)*Y(K)
SUML=0.0
DO 30 L=1,NS
30 SUML=SUML+Y(L)
PB=1.-SUML
CLS=SUMK+CLQ+NS*PB
B=CLS-CLQ
FT=CLS/XL
WT=CLQ/XL
Q=PB-Y(NF)
IF(Q)15,20,15
15 CLN=CLQ/Q
GO TO 25
20 CLN=0.0
25 CLD=NS-B
ALPHA=(CLD/NS)*100.0
BETA=100.0-ALPHA
50 FORMAT(/13XF5.2,12F9.3)
WRITE(3,50)X,Y(1),Y(N1),PB,CLS,CLQ,FT,WT,CLN,B,CLD,ALPHA,BETA
12 RETURN
END
```

NUMERICAL SOLUTIONS OF QUEUEING MODELS

by

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AN ABSTRACT OF A MASTER'S THESIS

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1970

ABSTRACT

Because of the lack of success in obtaining explicit transient solutions to the existing models in a finite time domain, numerical methods of solution are considered for the existing queueing models. This will motivate the industrial practitioners to increase their application of queueing theory in industry. In practice, the transient solutions are very useful. Examples of several situations are: (1) when behavior of the system during a short interval of time is to be predicted, (2) when the effect of rush-hour upon the system is to be studied, and (3) when the time period after which the system will reach steady state is to be determined. Numerical solutions to several shop-like queueing models are obtained by fourth order Runge-Kutta and Hamming methods. The numerical solutions are compared to the analytical solution of a sample model and it is found that numerical solution provides a good approximation.

The machine-shop is viewed as a queueing process and various single-machine and multiple-machine models are developed in the light of existing queueing models. Only Markovian queueing models are considered.

Numerical solutions provide the state probabilities both in the transient and steady state. Measures of performance are computed for nine single- and multiple-machine models, each with three different values of utilization factors. Several measures of performance are tabulated for all models and the corresponding curves are plotted.

It is observed that for the same utilization factor, the probability of a job being lost: (1) is negligible in cases where unlimited queue exists, (2) increases as the system capacity is reduced, and (3) becomes a maximum when no queue is allowed. It is also observed that when the utilization factor is less than one, the measures of performance are independent of the system capacity and are very small in comparison to system capacity. Some other significant results are observed.