A resolution of the diagonal for some toric D-M stacks

by

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B.A., Colorado College, 2013

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Abstract

Beilinson's resolution of the diagonal for complex projective spaces gives a locally free resolution of the structure sheaf of the diagonal as a Koszul complex, which gives a Fourier-Mukai transform inducing the identity on objects in the derived category of coherent sheaves of $\mathbb{P}^n$. Since $\mathbb{P}^n$ is a toric variety, we can ask for a resolution of the diagonal of an arbitrary toric variety. While resolutions of the diagonal are known for toric varieties which are unimodular in the sense of Bayer-Popescu-Sturmfels, in order to generalize a resolution of the diagonal for simplicial toric varieties, we must consider smooth Deligne-Mumford toric stacks associated to a given simplicial toric variety. Here, the diagonal can fail to be a closed substack because of non-zero stabilizers at the origin, so that a direct generalization of Beilinson's resolution will not work. For a toric D-M stack associated to a given simplicial toric variety $X'_\Sigma = {}^{X_\Sigma}\!/_{\mu}$ which is a global quotient of the unimodular toric variety X_Σ (in the sense of Bayer-Popescu-Sturmfels) by the finite abelian group μ, let $\tilde{L}$ denote the lattice of principal divisors of X'_Σ. Up to isomorphism, we can choose L, the lattice of principal divisors of X_Σ such that $\tilde{L} \subset L$ is cofinite, L is a unimodular lattice in the sense of Bayer-Popescu-Sturmfels, and ${}^{L}\!/_{\tilde{L}} \cong \mu$. Here, we construct a diagonal object inducing the kernel of a Fourier-Mukai transform by adapting a known resolution of the diagonal for X'_Σ unimodular due to Bayer-Popescu-Sturmfels. We construct a resolution in families $\mathcal{F}^{\bullet}_{\mathcal{H}^{\epsilon}_{L}/L}$ to resolve the diagonal of a smooth toric variety X_Σ corresponding to the parameter ϵ in the effective cone of X_Σ. Furthermore, we strengthen this result to give a resolution of the diagonal for global quotients $X'_\Sigma = {}^{X_\Sigma}\!/_{\mu}$ for X_Σ a smooth toric variety by the finite abelian group μ by showing that the cokernel of an inclusion $coker(M_{\Lambda(L)} \hookrightarrow Im(f))$ from an associated map f is zero, so that the complex $\mathcal{F}^{\bullet}_{H^{\epsilon}_{L}/\tilde{L}}$ gives a resolution of the diagonal $\mathcal{O}_\Delta$, modulo torsion from the irrelevant ideal I_{irr} for the product $X_\Sigma \times X_\Sigma$. This also strengthens the construction of the diagonal object giving the kernel of a Fourier-Mukai transform inducing the identity in

$D^b_{Coh}(X'_\Sigma)$ to the case that X'_Σ a global quotient of a smooth toric variety by a finite abelian group.

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Approved by:

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Nomenclature

$\mathcal{F}^\bullet_{\mathcal{H}_L^\epsilon}$ Cellular resolution of the cellular complex $\mathcal{H}_L^\epsilon$.

$\mathcal{F}^\bullet_{\mathcal{H}_L}$ Cellular resolution of the cellular complex $\mathcal{H}_L$.

$\mathcal{H}_L$ The infinite hyperplane arrangement given by all integral shifts of coordinate hyperplanes in $\mathbb{R}^n$, $\{x_i = r\}$ for $r \in \mathbb{Z}$ and $1 \leq i \leq n$. Here n denotes the number of primitive ray generators in Σ. $\mathcal{H}_L$ is also used to denote the cellular complex given by the intersection of this infinite hyperplane arrangement with $\mathbb{R}L$

$\mathcal{H}_L^\epsilon$ Deformation of the infinite hyperplane arrangement $\mathcal{H}_L$ by the element ϵ in the effective cone of X_Σ. Also used to denote the cellular complex given by the intersection of this infinite hyperplane arrangement with $\mathbb{R}L$.

$\mathbb{R}L$ $L \otimes_{\mathbb{Z}} \mathbb{R}$

$\rho \in \Sigma(1)$ The one-dimensional cone (also called a "ray") ρ in the the fan Σ. At times, we also use $\rho \in \Sigma(1)$ to denote the variable $x_\rho \in S = \mathbb{C}[x_\rho : \rho \in \Sigma(1)]$ in the homogeneous coordinate ring S for X_Σ, which is graded by the class group of X_Σ.

σ A strongly convex rational polyhedral cone in $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$

$Cl(X_\Sigma)$ The class group of the toric variety X_Σ

$D^b_{Coh}(X_\Sigma)$ The bounded derived category of coherent sheaves on the toric variety X_Σ

D_ρ The divisor $x_\rho = 0$ corresponding to the ray generator $\rho \in \Sigma(1)$.

k An algebraically closed field, which can be taken as $k = \mathbb{C}$ throughout.

L	A sublattice of $\mathbb{Z}^n$ for some n. For a complete simplicial toric variety X_Σ with no torus factors, L denotes the lattice of principal divisors
S	The polynomial ring $S = k[\mathbb{A}^{2n}] = k[x_1, \dots, x_n, y_1, \dots, y_n]$.
X_Σ	Toric variety X associated to the fan $\Sigma \subset N_\mathbb{R}$
M	The dual lattice $Hom_\mathbb{Z}(N, \mathbb{Z})$ of characters
N	The lattice of one-parameter subgroups, isomorphic to $\mathbb{Z}^m$ for some $m \in \mathbb{N}$

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I would like to thank my advisor, Gabe Kerr, for his patience, support, and inspiration.

Dedication

This work is dedicated to Mitchell Odell, a lifelong friend; my grandmother Lorraine Sheeley, a teacher; and Sophia Restad, an algebraic geometer and mathematician.

Chapter 1

Introduction

Here, we study bounded derived categories of coherent sheaves $D^b_{Coh}(X)$ on a smooth projective variety X. The derived category $D^b_{Coh}(X)$ forms part of the B-side of Homological Mirror Symmetry (HMS). To motivate $D^b_{Coh}(X)$ as a robust invariant of X, we give the following introduction.

To begin, a variety is taken to be irreducible, reduced, and separated over an algebraically closed field $\overline{k} = k$, with $\emptyset$ not irreducible. Historically, classical algebraic geometry realized a 1-1 order-reversing correspondence between affine varieties and radical ideals given by

$$Y \mapsto I(Y)$$
$$Z(\mathfrak{a}) \leftarrow\!\shortmid \mathfrak{a}$$

Furthermore, the affine coordinate ring $A(Y) := {}^{k[\mathbb{A}^n]}\!/\!_{I(Y)}$ of an affine variety $Y \subseteq \mathbb{A}^n$ determines Y as an isomorphism invariant, in the sense that

$$Y \cong Y' \iff A(Y) \cong A(Y').$$

In order to develop a mature intersection theory, one passes to projective closures $\overline{Y}$ over an algebraically closed field $\overline{k} = k$, so that for instance curves of degree c and d, respectively, intersect in $c \cdot d$ points, or so that any two linear hyperplanes intersect. This requires viewing

$$Y := \overline{Y} \cap U_0 \subseteq \mathbb{P}^n$$

and adding in $\overline{Y} \setminus Y$ which has codimension ≥ 1 since Y is Zariski-dense in $\overline{Y}$. However, the cost here in taking a projective closure is that $S(\overline{Y}) = {}^{k[x_0, \ldots, x_n]}\!\big/_{I(\overline{Y})}$, where $I(\overline{Y})$ is the homogeneous ideal of $\overline{Y}$, is no longer isomorphism invariant:

$$\text{there exist projective varieties } Y, Y' \text{ with } Y \cong Y', S(Y) \ncong S(Y').$$

There are counterexamples from varieties one encounters early in the study of algebraic geometry: the Veronese 2-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^2$ gives $\mathbb{P}^1 \cong \nu_2(\mathbb{P}^1)$, yet

$$S(\nu_2(\mathbb{P}^1)) \cong {}^{k[x_0, x_1, x_2]}\!\big/_{\langle x_0 x_2 - x_1^2 \rangle} \ncong k[x_0, x_1] = S(\mathbb{P}^1).$$

However, the category of finitely generated sheaves of $\mathcal{O}_X$-modules (called "coherent" sheaves) $Coh(X)$ are intrinsic to X in the sense that they do not depend upon the choice of a projective embedding. Gabriel showed that $Coh(X)$ determines X[1] [Corollary 4.4]. Furthermore, a smooth projective variety with ample or anti-ample canonical bundle is determined by $D^b(X)$[1;2].

A smooth projective variety with ample anti-canonical bundle is called *Fano*[3]. While $Coh(X)$ can often be hard to compute, there are many known methods for computing $D^b(X)$ in certain cases[4–6].

Working with $D^b(X)$ now has several advantages: $D^b(X)$ determines X in some cases; $D^b(X)$ is often easier to compute than $Coh(X)$ when we have a (full, strong, complete) exceptional collection; Grothendieck duality holds in $D^b(X)$[7]; the composition of classically derived functors becomes an outright composition of functors in $D^b(X)$, rather than a spectral sequence[8]; $D^b(X)$ has connections to other areas of mathematics, such as Homological Mirror Symmetry, classical algebraic geometry, birational geometry and variation of GIT[9–11].

Derived categories are now also studied in their own right.

Beilinson's resolution of the diagonal[4] was crucial in understanding $D^b(\mathbb{P}^n)$. Since $\mathbb{P}^n$ is a toric variety, we can ask for a generalization of Beilinson's resolution of the diagonal to any toric variety. However, this will be too broad, so we will restrict our attention to simplicial toric varieties. In order to generalize a resolution of the diagonal for any simplicial toric variety, we must reconcile the fact that the grading on the homogeneous coordinate ring comes from $\mathrm{Pic}(X_\Sigma)$, and not just $\mathbb{Z}$ as in the case of projective space.

1.1 Results

Bayer-Popescu-Sturmfels[12] give a cellular resolution of the Lawrence ideal J_L corresponding to a unimodular lattice L. Imposing the requirement that L, the lattice of principal divisors of the projective toric variety be unimodular is fairly restrictive: In this case L is given by the image of B in the fundamental exact sequence[3]

$$0 \to M \xrightarrow{B} \mathbb{Z}^{|\Sigma(1)|} \xrightarrow{\pi} Cl(X_\Sigma) \to 0$$

satisfies the property that B has linearly independent columns and every maximal minor of B lies in the set $\{0, +1, -1\}$ (this is the sense of "unimodularity" used in Bayer-Popescu-Sturmfels[12]). This forms a proper subclass of smooth projective toric varieties, as unimodular projective toric varieties can be viewed as subvarieties of products of projective lines. Throughout this dissertation, the word "unimodular" refers to this restricted sense of Bayer-Popescu-Sturmfels.

For a unimodular toric variety X_Σ, the known resolution of J_L gives a finite and minimal resolution of the ideal sheaf $\mathcal{O}_\Delta$ of the diagonal $\Delta \subset X_\Sigma \times X_\Sigma$ by locally free sheaves on $X_\Sigma \times X_\Sigma$. In the well-known example for $X_\Sigma = \mathbb{P}^n_{\mathbb{C}}$, Beilinson's resolution gives the kernel of a Fourier-Mukai transform inducing the identity on $D^b_{Coh}(X_\Sigma)$. Here, we generalize this resolution to simplicial toric varieties X'_Σ which are a global quotient of a smooth toric variety X_Σ by a finite abelian group μ by viewing X'_Σ as a simplicial toric stack. The proof falls into two parts: resolving the diagonal for a smooth toric variety X_Σ, and resolving the diagonal for a simplicial toric stack $\mathcal{X}(\boldsymbol{\Sigma'}) = \left[{X_\Sigma}/{\mu} \right]$ associated to the simplicial toric variety X'_Σ.

Theorem

(a) Let X_Σ be a smooth toric variety. The complex $(\mathcal{F}^\bullet_{\mathcal{H}^\epsilon_L/L}, \partial^\epsilon)$ of $S-$free modules gives a resolution of the diagonal object of $D^b_{Coh}(X_\Sigma \times X_\Sigma)$.

(b) For a simplicial toric stack $\mathcal{X}(\boldsymbol{\Sigma'}) = \left[{X_\Sigma}/{\mu} \right]$ which is a global quotient of the smooth toric variety X_Σ by the finite abelian group μ, the diagonal object in $D^b_{Coh}(X'_\Sigma \times X'_\Sigma)$ is given by

$$\pi(M_{\Lambda(\tilde{L})}) = M_{\Lambda(\tilde{L})} \otimes_{S[\Lambda(\tilde{L})]} S$$

Furthermore, the complex $(\mathcal{F}^\bullet_{\mathcal{H}^\epsilon_L/\tilde{L}}, \partial^\epsilon)$ resolves the diagonal object for $\mathcal{X}'_\Sigma$.

In (b), the cofinite sublattice $\tilde{L} \subset L$ is chosen so that ${L}/{\tilde{L}} \cong \mu$. Now the technique and deformations of the resolution of J_L in Bayer-Popescu-Sturmfels generalize to give a resolution of the diagonal

$$(\mathcal{F}^\bullet_{\mathcal{H}^\epsilon_L/\tilde{L}}, \partial^\epsilon)$$

The proof is given in Section 3.8 and culminates in Section 3.8.2.

Chapter 2

Preliminaries and background

2.1 HAG for DM stacks and orbifolds

For G a topological group, X a topological space with a continuous action of G on X given by

$$m : G \times X \to X, (g, x) \mapsto gx,$$

we denote by $\overline{X} := X/G$ the quotient space and by $q : X \to \overline{X}$ the natural projection. In the category of schemes, a simplicial structure[13] is given on a simplicial scheme $\chi_. = [X/G]$ by $\chi_n = G^n \times X$ and

$$d_i(g_1, \ldots, g_n, x) = \begin{cases} (g_2, \ldots, g_n, g_1^{-1}x) & i = 0 \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_n, x) & 1 \leq i < n, \\ (g_1, \ldots, g_{n-1}, x) & i = n \end{cases}$$

A G-equivariant sheaf has an isomorphism $\Theta : d_0^*\mathcal{F} \simeq d_1^*\mathcal{F}$ satisfying the cocycle condition

$$d_0^*\Theta \circ d_2^*\Theta = d_1^*\Theta, \qquad s_0^*\Theta = id_F.$$

A key point here is: for $G \times X$ a free G space, $q^* : \mathrm{Sh}(\overline{X}) \to \mathrm{Sh}_G(X)$ is an equivalence of categories with inverse functor $q_*^G : \mathrm{Sh}_G(X) \to \mathrm{Sh}(\overline{X})$ where $\mathrm{Sh}_G(X)$ are $G-$equivariant sheaves on X.[14]

This implies that we can define for a free $G-$space $D_G(X) := D(\mathrm{Sh}_G(X)) = D(\mathrm{Sh}(\overline{X}))$, so $D_G(G \times X) := D(\mathrm{Sh}_G(G \times X)) = D(\mathrm{Sh}(\overline{G \times X}))$.

We have a simplicial description of $D_G(X)$ as follows. Denote by $\mathrm{Sh}_{eq}([X/G]_.)$ the full subcategory of $Sh([X/G])$ consisting of simplicial sheaves F for which all structure morphisms are isomorphisms. Then $\mathrm{Sh}_{eq}([X/G])$ is naturally equivalent to $Sh_G(X)$. Now, for P an ∞-acyclic resolution, we have

$$\begin{tikzcd}
D^b_{eq}([X/G]) \arrow[r, "p^*"] \arrow[drr, dashed] & D^b_{eq}([P/G]) \arrow[r, phantom, "\simeq"] & D^b(\overline{P}) \\
 & & D^b_G(X) \arrow[u, phantom, "\cup|"]
\end{tikzcd}$$

is an equivalence from $D^b_{eq}([X/G])$ to $D^b_G(X)$.

In the case of $\mathcal{EG} = [\{*\}/G]$, we use a simplicial structure via $(\mathcal{EG})_n := G^{n+1}$,

$$d_i(g_1, \dots, g_{n+1}) = \begin{cases} (g_2, \dots, g_{n+1}) & i = 0, \\ (g_1, \dots, g_i g_{i+1}, \dots, g_{n+1} & 1 \leq i \leq n. \end{cases}$$

We consider the bisimplicial scheme given by the diagonal object of G acting diagonally on $\mathcal{EG} \times X$ where the rows are given by $G^i \times (\mathcal{EG} \times X)$ with index starting at 0, and maps within the rows d_i^h, s_i^h given by those in $\mathcal{EG} = [G/G]$ and vertical maps $d_0^v : G \times (\mathcal{EG} \times X) \to \mathcal{EG} \times X$ by $d_0^\vee(g, z, x) = (z, x)$ and $d_1^\vee(g, z, x) = (z_1, \ldots, z_n, z_{n+1}g^{-1}, gx)$ for $z = (z_1, \ldots, z_n) \in \mathcal{EG}$, etc. This is to turn a right action of G on $\mathcal{EG}$ into a left action on both $\mathcal{EG}$ and on X. By construction, the diagonal bisimplicial object Δ now has the structure of a quotient stack, given by $\Delta_n = G^n \times G^{n+1} \times X$, with $D_i : \Delta_1 \to \Delta_0$ given by

$$D_0(g_1, g_2, g_3, x) = (g_3, x)$$
$$D_1(g_1, g_2, g_3, x) = (g_2 g_3 g_1^{-1}, g_1 x)$$

and $S_i : \Delta_0 \to \Delta_1$ given by

$$S_0(g, x) = (1, 1, g, x).$$

We have maps D_i from Δ_2 to Δ_1 given by

$$D_0(g_1, g_2, \ldots, g_5, x) = (g_2, g_4, g_5, x)$$
$$D_1(g_1, \ldots, g_5, x) = (g_1 g_2, g_3 g_4, g_5, x)$$
$$D_2(g_1, \ldots, g_5, x) = (g_1, g_3, g_4 g_5 g_2^{-1}, g_2 x)$$

and maps $S_i : \Delta_1 \to \Delta_2$ given by

$$S_0(g_1, g_2, g_3, x) = (1, g_1, 1, g_2, g_3, x)$$
$$S_1(g_1, g_2, g_3, x) = (g_1, 1, g_2, 1, g_3, x)$$

the other maps D_i and S_i on Δ follow similarly.

From $\mathcal{BG} = [\{*\}/G], \mathcal{EG} = [G/G]$, we have that $\mathcal{BG}$ is contractible.

For toric stacks, we must consider how to pull-back, tensor, and pushforward in a cofibrant way. That is, we must consider Deligne-Mumford stacks, rather than derived stacks. To pushforward and pull-back in the derived setting, we can't simply take a derived functor.

2.2 Toric stacks

Following Geraschenko-Satriano[15], a toric stack is a stack quotient of a toric variety X by a subgroup G of its torus T_0. The stack $[X/G]$ has a dense open torus $T = T_0/G$ which acts on $[X/G]$. Toric stacks have an accompanying stacky fan. To start, our examples will take G to be a finite cylic group, so that we have toric Deligne-Mumford stacks[16].

Definition: For L a lattice (finitely generated free abelian group), we denote by T_L the lattice $D(L^*) = \mathrm{Hom}_{gp}(\mathrm{Hom}_{gp}(L, \mathbb{Z}), \mathbb{G}_m)$ whose lattice of 1-parameter subgroups is naturally isomorphic to L. Here the functor $(-)^*$ denotes $Hom_{gp}(-, \mathbb{Z})$ and the functor $D(-)$ denotes Cartier dual $Hom_{gp}(-, \mathbb{G}_m)$.

Definition: A **stacky fan** is a pair (Σ, β) where Σ is a fan on a lattice L and $\beta : L \to N$ is a homomorphism to a lattice N so that $\mathrm{cok}\beta$ is finite.

For X_Σ the toric variety associated to Σ, the map $\beta^* : N^* \to L^*$ induces a homomorphism of tori $T_\beta : T_L \to T_N$. Since $\mathrm{Cok}(\beta)$ is finite, β^* is injective, so T_β is surjective. Let $G_\beta = \ker(T_\beta)$ and note that T_L is the torus of X_Σ. Note that $G_\beta \subseteq T_L$ is a subgroup.

Using the notation above, if (Σ, β) is a stacky fan, then we define the toric stack $\mathcal{X}_{\Sigma,\beta}$ to be $[X_\Sigma/G_\beta]$ with the torus $T_N = T_L/G_\beta$.

Examples:

$[\mathbb{A}^1/\mu_2]$ is given by the stacky fan (Σ, β) for

$$\begin{array}{ll} \Sigma & \longrightarrow \\ \beta & \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \end{array}$$

Here, Σ consists of the origin and ray generated by $e_1 \in L \cong \mathbb{Z}^1$, together with the map $\beta : \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}$. Here, β induces the map $t \mapsto t^2$ so $G_\beta = \mu_2 \subseteq \mathcal{G}_m$ and $\mathcal{X}_{\Sigma,\beta} = [\mathbb{A}^1/\mu_2]$.

2.3 Combinatorial construction of differential forms on a simplicial toric stack

2.3.1 Pre-Amble

Let $R = k[x_0, \ldots, x_n]$, $I \to R \otimes_k R \to 0$ is exact, and in I/I^2 we have generators $dx_i = \overline{x_i \otimes 1 - 1 \otimes x_i}$, $df = \overline{f \otimes 1 - 1 \otimes f} = \sum \frac{\partial f}{\partial x_i} dx_i$. This description of $\Omega_{\mathbb{A}^n}$ comes from the diagonal embedding $\Delta \subset X \times X$, so that $\ker(\mu) = \mathcal{I}_\Delta$, the ideal sheaf of the diagonal. Then the conormal of Δ is the cotangent, which is dual to tangent.

Grading in B: $|f \otimes g| = |f| + |g|$. So $\Omega_{R/k}$ is a graded module. As R-Mod: $\Omega_{R/k} \cong R^{n+1}$, as a graded R-module: $\Omega_{R/k} \cong R(-1)^{n+1}$. Now, G acts freely on M and $\pi : M \to M/G \cong N$. We can ask: What is TN with respect to TM and TG?

Let $a : G \times M \to M, da : T(G \times M) \to TM$. Then $da_{(e,p)} : T_eG \times \{0\}, a|_{G \times p} : G \to M$, and $da|_{T_eG \times p} : T_eG \to T_pM$. Also, $T_{\pi(p)} \cong T_pM/\mathrm{Im}(da)$. Then

$$0 \to \mathrm{Im}(da) \to T_pM \to T_{\pi(p)}N \to 0$$

is exact. Dualizing gives

$$0 \to T^{\vee}_{\pi(p)}N \to T^*_pM \to (\mathrm{Im}da)^{\vee} \to 0.$$

For $\mathbb{A}^{n+1} \to \mathbb{P}^n$, we can consider $V \to \mathbb{P}(V)$, with trivial line bundle $V^* \otimes \mathcal{O}(-1)$, which gives Ω on V. Now, $\mathbb{G}_m \curvearrowright V$. For the Euler vector field on $\mathbb{A}^{n+1}$, $\nu = \sum x_i \partial_{x_i}$,

$$\Omega_{\mathbb{P}^n} \to \Omega_{R/k} \cong T^*M \overset{\mathrm{ev}}{\to} \mathcal{O}.$$

2.3.2 Finding all one-forms vanishing on Euler vector field ν

We know $\eta \in T^*\mathbb{A}^{n+1}$ can be written $\eta = \sum f_i dx_i$ such that $\eta(\nu) = 0$ with $\nu := \sum_{i=1}^{n+1} x_i \partial_{x_i}$. Then

$$\begin{aligned}
\eta(\nu) :&= \left(\sum_{i=1}^{n+1} f_i dx_i\right)\left(\sum_{j=1}^{n+1} x_j \partial_{x_j}\right) \\
&= \sum_{i,j=1}^{n+1} f_i dx_i \left(x_j \partial_{x_j}\right) \\
&= \sum_{i,j} x_j f_i \delta_i^j && \text{(Kronecker } \delta - \text{function)} \\
&= \sum_{i=1}^{n+1} x_i f_i = 0 && \text{by assumption.}
\end{aligned}$$

Then suppose for the sake of contradiction that there's a unique index i_0 such that $f_{i_0} \neq 0$. Then

$$x_{i_0} f_{i_0} = 0 \implies f_{i_0} = 0$$

in contradiction to $f_{i_0} \neq 0$. Therefore, there exist at least two distinct indices i, j such that $f_i, f_j \neq 0$. For notation, let $e_{ij} = x_i \otimes x_j - x_j \otimes x_i = x_i dx_j - x_j dx_i$. We know $\langle e_{ij} \mid i < j \rangle \subseteq \Omega_{\mathbb{A}^{n+1}}$ as a submodule vanishes on ν. To show that any one-form η vanishing on ν is contained in $\langle e_{ij} \mid i < j \rangle$, from $\eta(\nu) = 0$ we have

$$\sum_{i=1}^{n+1} x_i f_i = 0.$$

Starting with x_1, we can write $f_i = f_i' + x_1 g_i$ for $i > 1$ and $f_i' \in k[x_2, \ldots, x_{n+1}]$. Now

$$\begin{aligned} \eta - \sum_{i=2}^{n+1} g_i e_{1i} &= \eta - \sum_{i=2}^{n+1} g_i \left(x_1 dx_i - x_i dx_1 \right) \\ &= \eta - \sum_{i=2}^{n+1} \left(g_i x_1 dx_i - g_i x_i dx_1 \right) \end{aligned}$$

so

$$\eta - \sum_{i=2}^{n+1} g_i e_{1i} = \left(f_1 + \sum_{i=2}^{n+1} g_i x_i \right) dx_1 + \sum_{i=2}^{n+1} \left(f_i - g_i x_1 \right) dx_i.$$

Pairing $\eta - \sum_{i=2}^{n+1} g_i e_{1i}$ with ν gives $\left(\eta - \sum_{i=2}^{n+1} g_i e_{1i} \right)(\nu) = \left(f_1 + \sum_{i=2}^{n+1} g_i x_i \right) x_1 + F(x_2, \ldots, x_{n+1}) \right)$ for some $F \in k[x_2, \ldots, x_{n+1}]$. Since $\left(\eta - \sum_{i=2}^{n+1} g_i e_{1i} \right)(\nu) = 0$, and since the only term in the RHS above divisible by x_1 is $\left(f_1 + \sum_{i=2}^{n+1} g_i x_i \right) x_1$, it follows that $\left(f_1 + \sum_{i=2}^{n+1} g_i x_i \right) = 0$. Writing the RHS above as

$$\begin{aligned} &= \tilde{f}_1 dx_1 + \sum_{i=2}^{n+1} h_i dx_i \\ &= \sum_{i=2}^{n+1} h_i dx_i \end{aligned}$$

shows that (1) $h_i = f_i' \in k[x_2, \ldots, x_{n+1}]$ and (2) $\tilde{f}_1 = 0$.

Now, set $\eta^{(2)} := \sum_{i=2}^{n+1} h_i dx_i \in k[x_2, \ldots, x_{n+1}] \langle dx_2, \ldots, dx_{n+1} \rangle$, which still vanishes on ν, and restart the above process: writing $h_j = h'_j + x_2 g_j$ for all $j > 2$ with $h'_j \in k[x_3, \ldots, x_{n+1}]$ gives

$$\begin{aligned}
\eta^{(2)} - \sum_{j=2}^{n+1} g_j e_{2j} &= \sum_{j=2}^{n+1} h_j dx_j - \sum_{j=2}^{n+1} g_j \left(x_2 dx_j - x_j dx_2\right) \\
&= \sum_{j=2}^{n+1} h_j dx_j - \sum_{j=2}^{n+1} h_j dx_j - \left(\sum_{j=2}^{n+1} g_j x_2 dx_j - g_j x_j dx_2\right) \\
&= \left(h_2 + \sum_{j=2}^{n+1} g_j x_j\right) dx_2 + \sum_{j=3}^{n+1} \left(h_j - g_j x_2\right) dx_j \\
&= \tilde{g}_2 dx_2 + \sum_{i=3}^{n+1} F_i dx_i
\end{aligned}$$

where $\tilde{g}_2 = 0$ since $\eta^{(2)}$ vanishes on ν (so $\eta^{(2)} - \sum_{j=2}^{n+1} g_j e_{2j}$ vanishes on ν), and $F_i \in k[x_3, \ldots, x_{n+1}]$ for $i > 2$. Continuing in this fashion shows that η minus the maximal linear combinations of $\langle e_{ij} \mid i < j \rangle$ obtained from the above process is zero, so $\eta \in \langle e_{ij} \mid i < j \rangle$.

We have

$$\lambda \cdot (x_0, \ldots, x_n) = (\lambda x_0, \ldots, \lambda x_n)$$

from $a : \mathbb{C}^* \to \mathbb{C}^{n+1}$,

$$\begin{aligned}
d_a(\partial_\lambda) &= x_0 \partial_{x_0} + x_1 \partial_{x_1} + \cdots + x_n \partial_{x_n} \\
&= \sum_{i=0}^{n} x_i \frac{\partial}{\partial x_i}.
\end{aligned}$$

Now, to show $\psi \in \operatorname{Hom}\left(\mathcal{O}(-1) \boxtimes \Omega_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n}\right)$, where $\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \cong k[x_0, \ldots, x_n] \otimes_k k[y_0, \ldots, y_n]$, we show that relations get mapped to zero. Note that since ψ is a map of modules, multiplication by y_i must go through as follows:

$$\begin{aligned}\psi(1\otimes y_k e_{ij})-\psi(1\otimes y_i e_{kj})+\psi(1\otimes y_j e_{ki}) &= x_i\otimes y_k y_j - x_j\otimes y_k y_i - ...\\ &... -(x_k\otimes y_i y_j - x_j\otimes y_i y_k)+x_k\otimes y_j y_i - x_i\otimes y_j y_k\\ &=0.\end{aligned}$$

We let $\mathcal{F}_1$ have primitive ray generators $e_1=(1,0), e_1+e_2=(1,1), e_2=(0,1)$, and $-e_1-e_2=(-1,-1)$.

The Euler-Jaczewski sequence[17]

$$0\to\Omega_X\to\mathcal{P}_X^\vee\to\mathcal{H}\to 0$$

where $H=H^1(X,\mathcal{O}_X)$ is associated with $\mathrm{Pic}(X)$ for a smooth toric variety X, $\mathcal{H}=H\otimes\mathcal{O}_X$, $\mathcal{P}_X=\bigoplus_{i\in I}\mathcal{O}_X(D_i)$ [note that it's $\mathcal{P}_X^\vee$ in the sequence] and D_i are torus-invariant divisors corresponding to rays in the fan Σ, generalizes the Euler sequence for $\mathbb{P}^n$, and gives an exact sequence computation for Ω_X. Indeed, on $\mathbb{P}^n$, we have the following example.

2.3.3 Example 1

$X=\mathbb{P}^n$, $D_i=\{x_i=0\}$, $\mathcal{O}_X(D_i)=\mathcal{O}(1)$ since $\mathrm{Pic}(\mathbb{P}^n)\cong\mathbb{Z}$. Tensoring with $\mathbb{C}$ gives $H=\mathrm{Pic}_{\mathbb{C}}(\mathbb{P}^n)=\mathbb{C}$. Now, $\mathcal{H}=H\otimes\mathcal{O}=\mathbb{C}\otimes\mathcal{O}$ and we have the short exact sequence

$$0\to\Omega_{\mathbb{P}^n}\to\mathcal{O}(-1)^{n+1}\to\mathcal{O}\to 0$$

where $\Omega_{\mathbb{P}^n}$ can be understood as a bimodule generated by $\langle e_{ij}\mid i<j\rangle\subset R\otimes_k R$.

2.3.4 Example 2

Let $X = \mathcal{F}_1$, $H = \text{Pic}(\mathcal{F}_1) \cong \mathbb{Z}^2$, and $\mathcal{O}^2 = \mathcal{H} = \mathcal{O} \otimes H = R \otimes \text{Pic}(\mathcal{F}_1)$. Pairing primitive ray generators with the vector $(1, -1)$ shows $[D_1] \sim [D_3]$. (D_i here is still $\{x_i = 0\}$). The numbering convention on ray generators here is $(1, 0)$ corresponds to D_1, $(1, 1)$ corresponds to D_2, $(0, 1)$ as corresponds to D_3, and $(-1, -1)$ as corresponds to D_4. We have the short exact sequence

$$0 \to \Omega_{\mathcal{F}_1} \to \bigoplus_{i=1}^{4} \mathcal{O}_{\mathcal{F}_1}(-D_i) \xrightarrow{\alpha} \mathcal{O}^2_{\mathcal{F}_1} \to 0$$

One can view $\Omega_{\mathcal{F}_1} \subset \Omega_{\mathbb{A}^4} \subset R \otimes R$. The homogeneous coordinate ring of $\mathcal{F}_1$ is $R = k[x_1, x_2, x_3, x_4]$ with grading from $\text{Pic}(\mathcal{F}_1)$, which is a bi-grading on $\mathbb{Z}^2$.

The map $\alpha = \sum_{i=1}^4 \alpha_i$ with $\alpha_i(1) = x_i \otimes [D_i]$. To find $\ker(\alpha)$, a sample element is

$$z = (x_3, 0, -x_1, 0) \in \ker(\alpha).$$

The $R-$module associated to $\mathcal{O}_{\mathcal{F}_1}(-D_1) \oplus \cdots \oplus \mathcal{O}_{\mathcal{F}_1}(-D_4)$ is R^4 with grading shifts given by $|x^\alpha| = \alpha - D$. That is, $\mathcal{O}(D)$ as an $R-$module is R with grading $|x^\alpha| = \alpha - D$. Now,

$$\begin{aligned} \alpha(z) &= x_3 x_1 \otimes [D_1] - x_1 x_3 \otimes [D_3] \\ &= (x_3 x_1 - x_1 x_3) \otimes [D_1] = 0. \end{aligned}$$

2.3.5 Finding the kernel

In $\text{Pic}(\mathcal{F}_1)$, we have that $[D_1] \sim [D_3]$ (via dot product with (1,-1)) and explicitly, all relations are given by:

$$0 = \mathrm{div}(\chi^{e_1}) = D_1 + D_2 - D_4,$$

$$0 = \mathrm{div}(\chi^{e_2}) = D_2 + D_3 - D_4$$

since e_1, e_2 form a basis for $M \cong \mathbb{Z}^2$. This gives relations $D_4 \sim D_1 + D_2$ and $D_4 \sim D_2 + D_3$. Together, these imply $[D_1] \sim [D_3]$. An explicit description is given by

$$\mathrm{Pic}(\mathcal{F}_1) \cong {}^{\textstyle \langle D_1, D_2, D_3, D_4 \rangle} \big/ {}_{\textstyle \langle (D_4 \sim D_1 + D_2 \sim D_2 + D_3) \rangle}$$

so that $\mathrm{Pic}(\mathcal{F}_1) \cong \langle [D_1], [D_2] \rangle \cong \mathbb{Z}^2$, but it's useful to keep the explicit relations in mind. To divorce the divisors from the grading, let $\mathrm{Pic}(\mathcal{F}_1) \cong \langle \gamma_1, \gamma_2 \rangle$ where γ_1 and γ_2 are viewed as grading parameters. The sheaf $\mathrm{Pic}(\mathcal{F}_1)$ corresponds to the graded module M with elements of the form $a\gamma_1 + b\gamma_2$.

2.3.6 Kernel of α as submodule of $\mathcal{P}_{\mathcal{F}_1}^{\vee}$

(i) The relation $[D_1] \sim [D_3]$ gives $z_1 = (x_3, 0, -x_1, 0) \in \ker(\alpha)$, since

$$\alpha((x_3, 0, -x_1, 0)) = x_1 x_3 \otimes [D_1] - x_1 x_3 \otimes [D_3] = 0 \in \mathcal{H} \cong \mathcal{O} \otimes \mathrm{Pic}(\mathcal{F}_1).$$

(ii) The relation $[D_4] \sim ([D_1] + [D_2])$ gives

$$z_2 = (x_2 x_4, x_1 x_4, 0, -x_1 x_2) \in \ker(\alpha)$$

since

$$
\begin{aligned}
\alpha((x_2x_4, x_1x_4, 0, -x_1x_2)) &= x_1x_2x_4 \otimes ([D_1] + [D_2]) - x_1x_2x_4 \otimes [D_4] \\
&\sim 0 \otimes ([D_1] + [D_2]) \sim 0 \in \mathcal{H} \cong \mathcal{O} \otimes \mathrm{Pic}(\mathcal{F}_1).
\end{aligned}
$$

(iii) The relation $[D_4] \sim ([D_2] + [D_3])$ gives

$$z_3 = (0, x_3x_4, x_2x_4, -x_2x_3) \in \ker(\alpha)$$

since

$$
\begin{aligned}
\alpha((0, x_3x_4, x_2x_4, -x_2x_3)) &= (x_2x_3x_4) \otimes ([D_2] + [D_3]) - x_2x_3x_4 \otimes [D_4] \\
&\sim 0 \otimes ([D_2] + [D_3])
\end{aligned}
$$

The submodule $\langle z_1, z_2, z_3 \rangle \subset \mathcal{P}^{\vee}_{\mathcal{F}_1}$ is clearly contained in $\ker(\alpha)$. To show the reverse inclusion, note that $\xi = (\xi_1, \xi_2, \xi_3, \xi_4) \in \ker(\alpha)$ satisfies

$$
\begin{aligned}
\alpha(\xi) = \sum_{i=1}^{4} \alpha_i(\xi_i) = 0 \\
&= (x_1\xi_1 + x_3\xi_3) \otimes [D_1] + x_2\xi_2 \otimes [D_2] + x_4\xi_4 \otimes [D_4] \\
&= (x_1\xi_1 + x_3\xi_3) \otimes [D_1] + x_2\xi_2 \otimes [D_2] + x_4\xi_4 \otimes ([D_1] + [D_2]) \\
&= (x_1\xi_1 + x_3\xi_3 + x_4\xi_4) \otimes [D_1] + (x_2\xi_2 + x_4\xi_4) \otimes [D_2] = 0
\end{aligned}
$$

so

$$x_1\xi_1 + x_3\xi_3 + x_4\xi_4 = x_2\xi_2 + x_4\xi_4 = 0$$

and

$$x_2\xi_2 = x_1\xi_1 + x_3\xi_3 = -x_4\xi_4 \tag{2.1}$$

Viewing ξ_1, ξ_3 as parameters of ξ gives

$$\xi = \left(\xi_1, \frac{x_1\xi_1 + x_3\xi_3}{x_2}, \xi_3, \frac{x_1\xi_1 + x_3\xi_3}{-x_4}\right). \tag{2.2}$$

Cases

Note: $\xi_i \in \Bbbk[x_1, \ldots, x_4]$ for $1 \leq i \leq 4$.

Case (i) If $\xi_1 = 0$ and $\xi_3 \neq 0$, then

$$x_3\xi_3 = x_2\xi_2 = -x_4\xi_4.$$

$$\begin{aligned} \xi_2 &= \frac{x_3\xi_3}{x_2} && \text{implies } x_2 \mid \xi_3, \text{ and} \\ \xi_4 &= \frac{x_3\xi_3}{-x_4} && \text{implies } -x_4 \mid \xi_3 \end{aligned}$$

so that $\xi_3 = F \cdot (-x_2x_4)$ for some $F \in \Bbbk[x_1, \ldots, x_4]$. Now, $\xi_2 = \frac{x_3\xi_3}{x_2}$ and $\xi_4 = \frac{x_3\xi_3}{-x_4}$ imply

$$\begin{aligned} \xi &= (0, F \cdot (-x_3x_4), F \cdot (-x_2x_4), F \cdot x_2x_3) \\ &= F \cdot (0, -x_3x_4, -x_2x_4, x_2x_3) \end{aligned}$$

is an $-F$-multiple of $(0, x_3x_4, x_2x_4, -x_2x_3)$. Taking $-F = 1$ is necessary to include $(0, x_3x_4, x_2x_4, -x_2x_3)$ in a generating set.

Case (ii) If $\xi_3 = 0, \xi_1 \neq 0$, then

$$x_1\xi_1 = x_2\xi_2 = -x_4\xi_4$$

so that $\xi_2 = \frac{x_1\xi_1}{x_2}, \xi_4 = \frac{x_1\xi_1}{-x_4}$ imply $x_2, -x_4 \mid \xi_1$. Hence, $\xi_1 = -x_2x_4 \cdot G$ for some

$G \in \Bbbk[x_1, \ldots, x_4]$. So

$$\begin{aligned} \xi &= (-x_2x_4 \cdot G, -x_1x_4 \cdot G, 0, x_1x_2 \cdot G) \\ &= G \cdot (-x_2x_4, -x_1x_4, 0, x_1x_2) \end{aligned}$$

is a G-multiple of $(-x_2x_4, -x_1x_4, 0, x_1x_2)$. Taking $G = 1$ is necessary to include $(-x_2x_4, -x_1x_4, 0, x_1x_2)$ in a generating set.

Case (iii) If $\xi_1 = \xi_3 = 0$, then $\xi = (0, 0, 0, 0)$.

Case (iv) If $x_1\xi_1 + x_3\xi_3 = 0$ with $\xi_1, \xi_3 \neq 0$, then $\xi_2 = \xi_4 = 0$, and

$$x_1\xi_1 = -x_3\xi_3$$

gives $\xi_1 = \frac{-x_3\xi_3}{x_1}$ implies $x_1 \mid \xi_3$ and $\xi_3 = \frac{x_1\xi_1}{-x_3}$ implies $-x_3 \mid \xi_1$. So

$$\begin{aligned} \xi &= (-x_3 \cdot H, 0, x_1 \cdot H, 0) \qquad\qquad \text{for some } H \in \Bbbk[x_1, \ldots, x_4] \\ &= H \cdot (-x_3, 0, x_1, 0). \end{aligned}$$

Taking $H = 1$ is necessary for including $(-x_3, 0, x_1, 0)$ in a generating set.

Case (v) If $\xi_1, \xi_3 \neq 0$ then we can write $\xi_1 = \tilde{\xi}_1 + f \cdot x_3$, where $\tilde{\xi}_1 \in \Bbbk[x_1, x_2, x_4]$. Let $\xi - f \cdot z_1 = (\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3, \tilde{\xi}_4)$. Then, since $z_1 \in \ker(\alpha)$ and α is $\mathcal{O}_{\mathcal{F}_1}$-linear as a homomorphism of $\mathcal{O}_{\mathcal{F}_1}$-modules, $\xi \in \ker(\alpha)$ implies $\xi - f \cdot z_1 \in \ker(\alpha)$. We can therefore apply equation (1) to $\tilde{\xi}$:

$$x_2\tilde{\xi}_2 = x_1\tilde{\xi}_1 + x_3\tilde{\xi}_3 = -x_4\tilde{\xi}_4.$$

Solving for $x_1\tilde{\xi}_1$ gives

$$x_1\tilde{\xi}_1 = x_2\tilde{\xi}_2 - x_3\tilde{\xi}_3 = -x_4\tilde{\xi}_4 - x_3\tilde{\xi}_3$$

From $x_1\tilde{\xi}_1 = x_2\tilde{\xi}_2 - x_3\tilde{\xi}_3$, since $\tilde{\xi}_1 \in \Bbbk[x_1, x_2, x_4]$, we can write $x_1\tilde{\xi}_1 = x_2\tilde{\tilde{\xi}}_2$, since part of $\tilde{\xi}_2$ must cancel with $x_3\tilde{\xi}_3$. Therefore, $x_2 \mid \tilde{\xi}_1$. Similarly, from $x_1\tilde{\xi}_1 = -x_4\tilde{\xi}_4 - x_3\tilde{\xi}_3$, we can write $x_1\tilde{\xi}_1 = -x_4\tilde{\tilde{\xi}}_4$ since part of $\tilde{\xi}_4$ must cancel with $x_3\tilde{\xi}_3$, since $\tilde{\xi}_1 \in \Bbbk[x_1, x_2, x_4]$. Therefore, $x_2x_4 \mid \tilde{\xi}_1$ so that

$$\tilde{\xi}_1 = g \cdot x_2x_4 \qquad \text{some } g \in \Bbbk[x_1, x_2, x_3, x_4], \text{ and}$$

$$\xi - f \cdot z_1 - g \cdot z_2 \text{ has first component zero}$$

which falls into cases (i) or (iii).

This gives

$$\ker(\alpha) = \langle (0, x_3x_4, x_2x_4, x_2x_3), (-x_2x_4, -x_1x_4, 0, x_1x_2), (-x_3, 0, x_1, 0) \rangle \subset \mathcal{P}^{\vee}_{\mathcal{F}_1}.$$

2.3.7 Kernel as submodule of $\Omega_{\mathbb{A}^4}$

We know $\Omega_{\mathbb{A}^4} \cong \langle dx_1, \ldots, dx_4 \rangle$. Also, $\Omega_{\mathbb{A}^n} := \ker(\mu)$, where $R \otimes R \xrightarrow{\mu} R$ is multiplication.

Claim 1: Given $\nu : \mathrm{Pic}(X_\Sigma) \to \mathbb{Q}$ over a field of characteristic zero and $(f_1, f_2, f_3, f_4) \in \ker(\alpha)$ a homogeneous element, then

$$F_\nu(f_1, \ldots, f_4) := \sum_{i=1}^{4} \nu([D_i]) f_i \otimes x_i \in \Omega_{\mathbb{A}^4}.$$

Proof. We prove the general case: X_Σ smooth with $|\Sigma(1)| = n$,

$$\begin{aligned}\mu(F_\nu(f_1, \ldots, f_n)) &= \sum_{i=1}^{n} \nu\left([D_i]\right) f_i x_i \\ &= (1 \otimes \nu)\left(\sum_{i=1}^{n} f_i x_i \otimes [D_i]\right) \\ &= (1 \otimes \nu)(\alpha(f_1, \ldots, f_n)) = 0\end{aligned}$$

so $F_\nu(f_1, \ldots, f_n) \in \ker(\mu) \cong \Omega_{\mathbb{A}^n}$. $\square$

Now, let

$$M := \left\langle F_\nu(f_1, \ldots, f_4) : (f_1, \ldots, f_4) \in \ker(\alpha) \text{ homogeneous}, \nu \in \operatorname{Pic}(X_\Sigma)^\vee_{\mathbb{Q}} \right\rangle.$$

Note: For $\xi = (\xi_1, \xi_2, \xi_3, \xi_4) \in \ker(\alpha) \subset \mathcal{P}^\vee$, the choice of ν given by $\nu([D_1]) = \nu([D_2]) = 1$ enables one to view $\ker(\alpha) \subset \mathcal{P}^\vee_{\mathcal{F}_1}$ by

$$F_\nu(\xi_1, \xi_2, \xi_3, \xi_4) = \xi_1 \otimes x_1 + \xi_2 \otimes x_2 + \xi_3 \otimes x_3 + 2\xi_4 \otimes x_4.$$

Now,

$$\begin{aligned}F_\nu((-x_3, 0, x_1, 0)) &= -x_3 \otimes x_1 + x_1 \otimes x_3 \\ &= e_{13}.\end{aligned}$$

$$
\begin{aligned}
F_\nu(0, x_3x_4, x_2x_4, -x_2x_3) &= x_3x_4 \otimes x_2 + x_2x_4 \otimes x_3 - 2x_2x_3 \otimes x_4 \\
&= x_3(x_4 \otimes x_2 - x_2 \otimes x_4) + x_2(x_4 \otimes x_3 - x_3 \otimes x_4) \\
&= x_3 \cdot e_{42} + x_2 \cdot e_{43}.
\end{aligned}
$$

$$
\begin{aligned}
F_\nu(x_2x_4, x_1x_4, 0, -x_1x_2) &= x_2x_4 \otimes x_1 + x_1x_4 \otimes x_2 - 2x_1x_2 \otimes x_4 \\
&= x_2(x_4 \otimes x_1 - x_1 \otimes x_4) + x_1(x_4 \otimes x_2 - x_2 \otimes x_4) \\
&= x_2 \cdot e_{41} + x_1 \cdot e_{42}.
\end{aligned}
$$

Therefore, as a submodule of $\Omega_{\mathbb{A}^4}$, this choice of ν gives:

$$\Omega_{\mathcal{F}_1} \cong \langle e_{13}, x_3 \cdot e_{42} + x_2 \cdot e_{43}, x_2 \cdot e_{41} + x_1 \cdot e_{42} \rangle \subset \Omega_{\mathbb{A}^4}.$$

2.3.8 Whether $M \hookrightarrow \Omega_{\mathbb{A}^4}$ is an isomorphism on $\Omega_{\mathbb{A}^4_{ss}}$

To compute the unstable locus $\mathcal{Z}$ for $\mathcal{F}_1$, we note that in the fan Σ for $\mathcal{F}_1$, let us denote σ_1 as the cell between rays 1 and 2, σ_2 as the cell between rays 2 and 3, σ_3 as the cell between rays 3 and 4, and σ_4 as the cell between rays 4 and 1. This gives

$$
\begin{aligned}
\sigma_1^\vee &= D_3 + D_4 \leftrightarrow x_3x_4 \\
\sigma_2^\vee &= D_1 + D_4 \leftrightarrow x_1x_4 \\
\sigma_3^\vee &= D_1 + D_2 \leftrightarrow x_1x_2 \\
\sigma_4^\vee &= D_2 + D_3 \leftrightarrow x_2x_3
\end{aligned}
$$

Informally: rays 3 and 4 are opposite σ_1, rays 4 and 1 are opposite σ_2, rays 1 and 2 are opposite σ_3, and rays 2 and 3 are opposite σ_4. This gives the irrelevant ideal

$$B = \langle x_3x_4, x_4x_1, x_1x_2, x_2x_3\rangle .$$

Now

$$\begin{aligned} \mathcal{Z} &= V(B) \\ &= V(\langle x_3x_4, x_1x_4, x_1x_2, x_2x_3\rangle) \\ &= V(x_2, x_4) \cup V(x_1, x_3). \end{aligned}$$

So $\mathbb{A}^4_{ss} = \mathbb{A}^4 \setminus Z = \mathbb{A}^4 \setminus (V(x_2, x_4) \cup V(x_1, x_3)) = \mathbb{A}^4 \setminus (\{x_2 = x_4 = 0\} \cup \{x_1 = x_3 = 0\})$. On $\mathbb{A}^4_{ss}$, we have that $\{$either x_2 or $x_4 \neq 0\}$ and $\{$either x_1 or $x_3 \neq 0\}$.

2.3.9 General case from matroidal setup

Previously, our finite set of points determining α above came from primitive ray generators for the fan Σ associated to $\mathcal{F}_1$. In general, one can ask: given a finite set of points, can one construct a similar map α and find $\ker(\alpha)$? The kernel of this map is matroidal: For $A \subset N$ a $\mathbb{Z}-$linearly spanning set, we have the short exact sequence

$$0 \to L_A \xrightarrow{\delta} \mathbb{Z}^A \xrightarrow{\gamma} N \to 0. \tag{2.3}$$

We'll use the dual sequence

$$0 \to M \xrightarrow{\gamma^\vee} \mathbb{Z}^A \xrightarrow{\delta^\vee} L_A^\vee \to 0. \tag{2.4}$$

Define $R = \mathbb{C}[x_a \mid a \in A]$ which is $L_A^\vee$-graded, a generalization of the Cox homogeneous coordinate ring. Let $|x_a| = \delta^\vee(e_a^\vee)$. Then the generalized α is

$$\alpha : \bigoplus_{a\in A} R(-|x_a|) \to R\otimes L_A^\vee \to 0$$

where $\bigoplus_{a\in A} R(-|x_a|) \ni u_a \mapsto \alpha(u_a) := x_a \otimes |x_a|$.

Characterize $\ker(\alpha)$: a good relation in M can be divided into two positive parts: $M \overset{\gamma^\vee}{\hookrightarrow} \mathbb{Z}^A$ where

$$\begin{aligned} M \ni v \mapsto \gamma^\vee(v) &= \gamma^\vee(v)_+ - \gamma^\vee(v)_- \\ &= \sum_{a\in A(v)_+} c_a e_a - \sum_{a\in A(v)_-} c_a e_a \end{aligned}$$

where $c_a > 0\ \forall a$. Here, $A(v)_+ \cap A(v)_- = \emptyset$, $A(v) = A(v)_+ \cup A(v)_-$. Denote

$$f_{v,a} = \sum_{\tilde{a}\in A(v), \tilde{a}\neq a} e_{\tilde{a}} \in \mathbb{Z}^A.$$

Define

$$z_v = \sum_{a\in A(v)_+} c_a x^{f_{v,a}} u_a - \sum_{a\in A(v)_-} c_a x^{f_{v,a}} u_a$$

Claim: z_1, z_2, z_3 of the previous section are of this type.

Proof. From Σ for $\mathcal{F}_1$ with primitive ray generators $a_1 = (1,0)$ corresponding to D_1 (and x_1 and e_1), $a_2 = (1,1)$ corresponding to D_2 (and x_2 and e_2), $a_3 = (0,1)$ corresponding to D_3 (and x_3 and e_3), and $a_4 = (-1,-1)$ corresponding to D_4 (and x_4 and e_4), we have

$$\{z_v \mid v = (1,0), (0,1), (1,-1)\} = \{(x_2x_4, x_1x_4, 0, -x_1x_2), (0, x_3x_4, x_2x_4, -x_2x_3), (x_3, 0, -x_1, 0)\}$$

so that

$$\langle z_v \mid v \in M \rangle = \ker(\alpha).$$

For $v = (1,0)$, we have $\gamma^\vee((1,0)) = e_1 + e_2 - e_4$ so that $A(v)_+ = a_1, a_2$, $A(v)_- = a_4$ and

$$\begin{aligned} z_v &= 1 \cdot x^{(2,4)} e_1 + 1 \cdot x^{(1,4)} e_2 - 1 \cdot x^{(1,2)} e_4 \\ &= (x_2 x_4, x_1 x_4, 0, -x_1 x_2). \end{aligned}$$

For $v = (0,1)$, we have $\gamma^\vee((0,1)) = 1 \cdot e_2 + 1 \cdot e_3 - 1 \cdot e_4$ so that $A(v)_+ = a_2, a_3$, $A(v)_- = a_4$, and

$$\begin{aligned} z_v &= 1x^{(3,4)} e_2 + 1x^{(2,4)} e_3 - 1x^{(2,3)} e_4 \\ &= (0, x_3 x_4, x_2 x_4, -x_2 x_3). \end{aligned}$$

For $v = (1,-1)$, $\gamma^\vee((1,-1)) = e_1 - e_3$ so that $A(v)_+ = a_1$, $A(v)_- = a_3$, and

$$\begin{aligned} z_v &= 1x^3 e_1 - 1x^1 e_3 \\ &= (x_3, 0, -x_1, 0) \end{aligned}$$

By the previous section, this shows that for $\mathcal{F}_1$,

$$\ker(\alpha) = \langle z_v \mid v \in M \rangle .$$

$\square$

Claim: In general, $\alpha(z_v) = 0$.

Proof. Let $f = \sum_{a \in A(v)} e_a \in \mathbb{Z}^A_{\geq 0}$. Then

$$\begin{aligned}
\alpha(z_v) &= \alpha\left(\sum_{a \in A(v)_+} c_a x^{f_{v,a}} u_a\right) - \alpha\left(\sum_{a \in A(v)_-} c_a x^{f_{v,a}} u_a\right) \\
&= \sum_{a \in A(v)_+} c_a x^{f_{v,a}} \alpha(u_a) - \sum_{a \in A(v)_-} c_a x^{f_{v,a}} \alpha(u_a) \\
&= \sum_{a \in A(v)_+} c_a x^{f_{v,a}} x_a \otimes |x_a| - \sum_{a \in A(v)_-} c_a x^{f_{v,a}} x_a \otimes |x_a| \\
&= x^f \otimes \left(\sum_{a \in A(v)_+} c_a |x_a| - \sum_{a \in A(v)_-} c_a |x_a|\right) \\
&= x^f \otimes \delta^\vee \left(\sum_{a \in A(v)_+} c_a e_a^\vee - \sum_{a \in A(v)_-} c_a e_a^\vee\right) \\
&= x^f \otimes \delta^\vee(\gamma^\vee(v)) = 0
\end{aligned}$$

as the composition of two maps from a short exact sequence. □

2.3.10 Part II: arithmetic rules

Finding a better set of generators for $< z_v : \ v \in M >$**.**

Arithmetic Rule 0:

By linearity of $\gamma^\vee$, $z_{-v} = -z_v$. This implies that if z_v is included in a generating set, we do not also need to include z_{-v}.

Arithmetic Rule 1: If we take disjoint sets $A_-, A_+ \subset A$ and define the cone $C_{A_-,A_+} \subset M_\mathbb{R}$ via

$$C_{A_-,A_+} = \left\{ m \in M_\mathbb{R} : \gamma^\vee(m) = \sum_{a \in A_+} b_a e_a^\vee - \sum_{a \in A_-} c_a e_a^\vee \text{ with } b_a, c_a \geq 0 \right\}$$

(note: $b_a, c_a \geq 0$ gives closed cone). Then over all disjoint pairs C_{A_-,A_+} form a complete convex rational polyhedral fan on $M_\mathbb{R}$. We will use $B = (A_-, A_+)$ to denote a signed subset, with $\overline{B} = A_{-} \cup A_{+}$.

Proof. Two elements v_1, v_2 of $M_{\mathbb{R}}$ are in the same cone iff $A(v_1)_+, A(v_2)_+ \subseteq A(v)_+$ and $A(v_1)_-, A(v_2)_- \subseteq A(v)_-$. Linearity of $\gamma^\vee : M \to \mathbb{Z}^A$ with rational images of each $m \in M$ gives rationality of C_{A_-,A_+}. Convexity of C_{A_-,A_+} follows from viewing $\gamma^\vee$ as a linear map from $M_{\mathbb{R}} \to \mathbb{R}^{|A|}$, whose orthants are convex. Since C_{A_-,A_+} maps cones to subsets of orthants in $\mathbb{R}^{|A|}$ and the intersection of convex sets is convex, C_{A_-,A_+} is convex. □

Arithmetic Rule 2: Suppose $C_{B_1}, \ldots, C_{B_r}$ are faces of $C_{B_{r+1}}$ and $v_i \in \operatorname{relint}(C_{B_i})$ such that $\sum_{i=1}^r v_i = v_{r+1}$. Then $\sum_{i=1}^r x^{B \setminus B_i} z_{v_i} = z_{v_{r+1}}$.

Proof.

Assumption 1: B is a finite set. [Valid since A is a finite set.]

Reference 1:

$$f_{v,a} = \sum_{\tilde{a} \in A(v), \tilde{a} \neq a} e_{\tilde{a}} \in \mathbb{Z}^{|A|}.$$

Reference 2:

$$z_v = \sum_{a \in A(v)} c_a x^{f_{v,a}} u_a.$$

Suppose $C_{B_1}, \ldots, C_{B_r}$ are faces of $C_{B_{r+1}}$. Then $B_1, \ldots, B_r$ are each given by B minus a finite number of points.

Assumption 2: $v_i \in \mathrm{relint}(C_{B_i})$ for $1 \le i \le r+1$.

Assumption 3: $\sum_{i=1}^{r} v_i = v_{r+1}$.

Construction 1:

$$z_{v_i} = \sum_{a \in A(v_i)} c_{a,i} x^{f_{v_i,a}} u_a$$

Remark: Here, $B = A(v_{r+1})$.

$$\begin{aligned}
\sum_{i=1}^{r} x^{B \setminus B_i} z_{v_i} &= \sum_{i=1}^{r} x^{B \setminus B_i} \sum_{a \in A(v_i)} c_{a,i} x^{f_{v_i,a}} u_a \\
&= \sum_{i=1}^{r} x^{B \setminus B_i} \sum_{a \in B_i} c_{a,i} x^{f_{v_i,a}} u_a \\
&= \sum_{i=1}^{r} x^{B \setminus B_i} \sum_{a \in B_i} c_{a,i} x^{\sum_{\tilde{a} \neq a \in A(v_i)} e_{\tilde{a}}} u_a \\
&= \sum_{i=1}^{r} x^{B \setminus B_i} \sum_{a \in B_i} c_{a,i} x^{\sum_{\tilde{a} \neq a \in B_i} e_{\tilde{a}}} u_a \\
&= \sum_{i=1}^{r} \sum_{a \in B_i} x^{B \setminus B_i} c_{a,i} x^{\sum_{\tilde{a} \neq a \in B_i} e_{\tilde{a}}} u_a \\
&= \sum_{i=1}^{r} \sum_{a \in B_i} x^{B \setminus a} c_{a,i} u_a \\
&= \sum_{a \in B,} \sum_{i \mathrm{s.t.} a \in B_i} x^{B \setminus a} c_{a,i} u_a \\
&= \sum_{a \in B} x^{B \setminus a} (\sum_{a \in B_i} c_{a,i}) u_a \\
&= \sum_{a \in B} x^{B \setminus a} c_{a,r+1} u_a \\
&= z_{v_{r+1}}.
\end{aligned}$$

$\square$

Arithmetic Rule 3: (For tensors: if $K = \mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$): Let $v \in M_K$ and define z_v as before. Then $\exists\ v_1, \ldots, v_r \in M$ and $b_1, \ldots, b_r \in K$ such that

$$\sum_{i=1}^{r} b_i x^{\overline{B} \setminus \overline{B}_i} z_{v_i} = z_v$$

where $v_i \in \text{relint}(C_{B_i})$ and $\overline{B} = A(v)$.

Proof. Over K, primitive ray generators all lie in a one $K-$dimensional face of C_B. Let $\{v_i\}$ be a set of primitive ray generators for the minimal fan containing C_B (which each lie on a ray – meaning, a face of C_B), with corresponding ray $v_i \in C_{B_i}$. Let $v \in \text{relint} C_B$. In the case $K = \mathbb{Q}$, convexity and rationality of C_B imply there exists a subset $\{v_i\}_{i_1}^{i_r} \subset \{v_i\}$ and coefficients $\{b_i\}_{i_1}^{i_r} \in \mathbb{Q}$ such that

$$\sum_{j=1}^{r} b_j v_j = v$$

where j denotes the subscript on i. If $K = \mathbb{R}$, convexity suffices to provide the existence of $\{v_i\}_{i_1}^{i_r}$ and $\{b_i\}_{i_1}^{i_r}$ such that $\sum_{j=1}^{r} b_j v_j = v$. The proof that $\sum_{i=1}^{r} b_i x^{\overline{B} \setminus \overline{B_i}} z_{v_i} = z_v$ goes through as in the proof of Arithmetic Rule 2.

In the case $K = \mathbb{C}$, $v \in M_{\mathbb{C}}$ satisfies $v = \text{Re}(v) + \text{Im}(v)$. We'll consider $M_{\mathbb{C}} \cong M_{\mathbb{R} \oplus \mathbb{R}} \cong M_{\mathbb{R}} \times M_R$ as a real vector space, so that $v \in M_{\mathbb{C}} \cong M_{\mathbb{R}} \times M_{\mathbb{R}}$ with $\text{Re}(v) \in M_{\mathbb{R}} \times 0$ and $\text{Im}(v) \in 0 \times M_{\mathbb{R}}$.

This gives signed subsets $A(\text{Re}(v))_{\text{-}}, A(\text{Re}(v))_{+}$ and signed subsets $A(\text{Im}(v))_{\text{-}}, A(\text{Im}(v))_{+}$. The signed subsets $A(\text{Re}(v))_{\text{-}}, A(\text{Re}(v))_{+}$ yield the real cone $C_{A(\text{Re}(v))_{\text{-}}, A(\text{Re}(v))_{+}} \subseteq M_{\mathbb{R}} \times 0$, and the signed subsets $A(\text{Im}(v))_{\text{-}}, A(\text{Im}(v))_{+}$ yield the real cone $C_{A(\text{Im}(v))_{\text{-}}, A(\text{Im}(v))_{+}} \subseteq 0 \times M_{\mathbb{R}}$. Now, $\overline{B} = A(v) = A(\text{Re}(v))_{\text{-}} \cup A(\text{Re}(v))_{+} \cup A(\text{Im}(v))_{\text{-}} \cup A(\text{Im}(v))_{+}$.

The isomorphism as real vector spaces $M_{\mathbb{R}} \cong M_{\mathbb{R}} \times 0 \cong 0 \times M_{\mathbb{R}}$ lets us consider $\text{Re}(v)$ and $\text{Im}(v)$, separately, as reducing to the case $K = \mathbb{R}$ with $\text{Re}(v), \text{Im}(v) \in M_{\mathbb{R}}$. Now, $\text{Re}(v)$

and $\text{Im}(v)$ both reduce to the $K = \mathbb{R}$ case. This gives the existence of a set of primitive ray generators $\{v_i\}_{i_1}^{i_r}$ in $M_{\mathbb{R}} \times 0$ and coefficients $\{b_i\}_{i_1}^{i_r}$ with $b_i \in \mathbb{R}$ such that

$$\sum_{j=1}^{r} b_j v_j = \text{ Re}(v)$$

and

$$\sum_{i=1}^{r} b_i x^{\overline{B} \setminus \overline{B}_i} z_{v_i} = z_{\text{Re}(v)}.$$

Similarly, $\text{Im}(v) \in M_{\mathbb{R}}$ gives existence of a set of primitive ray generators $\{w_i\}_{i_1}^{i_\ell}$ and real coefficients $\{d_i\}_{i_1}^{i_\ell}$ such that

$$\sum_{j=1}^{\ell} d_j w_j = \text{Im}(v) \qquad \text{and}$$

$$\sum_{i=1}^{\ell} d_i x^{\overline{B} \setminus \overline{B_i}} z_{v_i} = z_{\text{Im}(v)}.$$

Since $\text{Re}(v)$ and $\text{Im}(v)$ are in distinct copies of $M_{\mathbb{R}}$,

$$z_{\text{Re}(v) \oplus \text{Im}(v)} = z_{(\text{Re}(v) \oplus 0) + (0 \oplus \text{Im}(v))}.$$

Therefore, let

$$\{V_j\}_{j=1}^{r+\ell} = \begin{cases} v_{i_j} \times \vec{0} & 1 \leq j \leq r, \\ \vec{0} \times w_{i_j} & r+1 \leq j \leq \ell + r \end{cases}$$

so that $\overline{B_j} = \overline{A(V_j)}$. We also have coefficients

$$g_j = \begin{cases} b_j & 1 \leq j \leq r, \\ d_j & r+1 \leq j \leq r+\ell \end{cases}$$

so that

$$v = \sum_{j=1}^{r+\ell} g_j V_j$$

and

$$\sum_{j=1}^{r+\ell} b_j x^{\overline{B}\setminus\overline{B_j}} z_{V_j} = z_v$$

□

How to Use Arithmetic Rule 3 for Section 9.2, 9.3

Since every nonzero $v \in M$ lies in the relative interior of the cone defined by its nonzero coefficients $C_{A(v)_-,A_+}$, Arithmetic Rule 2 implies that there always exists a set $\{v_i\}_{i=1}^k, \{b_i\}_{i=1}^k$ such that

$$\sum_{a \in B_g} c_a \otimes e_a^\vee = \sum_{i=1}^k \gamma_{\mathbb{C}}^\vee (b_i \otimes_{\mathbb{C}} v_i)$$

and a monomial times $\sum_{i=1}^k b_i z_{v_i}$ can always be used to reduce a positive value of $||w||$ when tensoring over $K = \mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$.

Claim: $\ker(\alpha) = \langle z_v : v \in M \rangle$.

Proof. To fix notation, let $A = \{a_1, \ldots, a_m\}$. The above computation shows $z_v \in \ker(\alpha)$ for arbitrary $v \in M$. Next, fix the basis $R = \mathbb{C}[x_a \mid a \in A] = \bigoplus_{(i_1,\ldots,i_m)\in\mathbb{N}^m} \mathbb{C}\left\langle x_{a_1}^{i_1}\cdots x_{a_m}^{i_m}\right\rangle$ for R. Then $R \otimes L_A^\vee$ can be viewed as

$$\begin{aligned} R_\otimes L_A^\vee &\cong \bigoplus_{(i_1,\ldots,i_m)\in\mathbb{N}^m} c_{i_1,\ldots,i_m} x_{a_1}^{i_1}\cdots x_{a_m}^{i_m} \otimes L_A^\vee && \text{for } c_{i_1,\ldots,i_m} \in \mathbb{C} \\ &\cong \bigoplus_{(i_1,\ldots,i_m)\in\mathbb{N}^m} \mathbb{C} \otimes L_A^\vee && \text{by keeping track of the ordering of the basis of } R. \end{aligned}$$

Suppose $w = \sum_{a\in A} w_a u_a \in \ker(\alpha)$. Let $\pi_g : R \to \mathbb{C}\cdot x^g$ be natural projection, and let $B_g := \{a \in A : x_a w_a \notin \ker \pi_g\}$. Then $a \in B_g$ implies $x_a w_a$ contains a monomial $c_a x^g$ with $c_a \neq 0$. Therefore, $x_a \mid x^g$. Since the $\{x_a\}$ are mutually prime,

$$\prod_{a\in B_g} x_a \mid x^g.$$

Now $a \in B_g$ implies that when $x_a w_a$ is not in $\ker B_g$, we have that $c_a x^{g-e_a}$ appears in w_a with $c_a \neq 0$, so that $w_a = \tilde{F} + c_a x^{g-e_a}$ for some $\tilde{F} \in R(-|x_a|)$. Now

$$\begin{aligned} 0 = \pi_g(\alpha(w)) &= \sum_{a\in B_g} c_a \otimes |x_a| \\ &= \sum_{a\in B_g} c_a \otimes \delta^\vee(e_a) \\ &= \delta_{\mathbb{C}}^\vee\Big(\sum_{a\in B_g} c_a e_a\Big) \implies \sum_{a\in B_g} c_a e_a \in \ker(\delta_{\mathbb{C}}^\vee) = \mathrm{Im}(\gamma_{\mathbb{C}}^\vee) \end{aligned}$$

Since we've tensored over $\mathbb{C}$, we can't necessarily say that there exists one single $v \in M$ such that $\sum_{a\in B_g} c_a \otimes e_a^\vee = \gamma^\vee(v)$. However, we can say two things:

2.3.11 In nice cases

In the event that there does exist $v \in M$ such that $\gamma^\vee(v) = \sum_{a \in B_g} c_a e_a$, then we can say: for a given $g \in \mathbb{N}^m$,

$$\begin{aligned}
||\tilde{w}|| := ||w - x^{g - \sum_{a \in B_g} e_a} z_v|| &= ||w - x^{g - \sum_{a \in B_g} e_a} \sum_{a \in A(v)} c_a x^{\sum_{\tilde{a} \neq a \in A(v)} e_{\tilde{a}}} u_a|| \\
&= ||\bigoplus_{a \in A} \left(w_a - x^{g - \sum_{a \in B_g} e_a} \overline{c_a} \cdot x^{\sum_{\tilde{a} \neq a \in A(v)}} u_{\tilde{a}} \right) u_a|| \\
&\qquad \text{where the } \overline{c}_a \text{ are possibly } 0 \\
&= ||\bigoplus_{a \in A} \left((\tilde{F} + \overline{c_a} x^{g - e_a}) - x^{g - \sum_{a \in B_g} e_a} \overline{c_a} x^{\sum_{\tilde{a} \neq a \in B_g} u_{\tilde{a}}} \right) u_a|| \\
&\qquad \text{where } \tilde{F} \in \ker(\pi_g), \text{ since } B_g = A(v) \text{ by construction of v} \\
&= ||\bigoplus_{a \in A} \left(\tilde{F} + (\overline{c}_a - \overline{c}_a) x^{g - e_a} \right) u_a||
\end{aligned}$$

This shows that if we can select g such that $|B_g| > 0$, then $||w - x^{g - \sum_{a \in B_g} e_a} z_v|| < ||w||$.

2.3.12 Provided arithmetic rule 3, this reduces norm on $\tilde{w}$ in all cases

We can always say there exists $\{v_i\}_{i=1}^k, \{b_i\}_{i=1}^k$ such that

$$\sum_{a \in B_g} c_a \otimes e_a^\vee = \sum_{i=1}^k \gamma_{\mathbb{C}}^\vee (b_i \otimes_{\mathbb{C}} v_i)$$

We have that for a given $g \in \mathbb{N}^m$ with $B_g \neq \emptyset$,

$$
\begin{aligned}
||\tilde{w}|| &= ||w - x^{g-\sum_{a\in B_g} e_a} \sum_{i=1}^{k} b_i z_{v_i}|| \\
&= ||\bigoplus_{a\in A} \left(w_a - x^{g-\sum_{a\in B_g} e_a} \sum_{i=1}^{k} b_i \sum_{a\in A(v_i)} d_a x^{\sum_{\tilde{a}\neq a\in A(v_i)} e_{\tilde{a}}} \right) u_a|| \\
&\qquad \text{where } \gamma^\vee(v_i) = \sum_{a\in A(v_i)} d_a e_a \\
&= ||\bigoplus_{a\in A} \left(\tilde{F} + \overline{c_a} x^{g-e_a} - x^{g-\sum_{a\in B_g} e_a} \sum_{i=1}^{k} b_i \sum_{a\in A(v_i)} d_a x^{\sum_{\tilde{a}\neq a\in A(v_i)} e_{\tilde{a}}} \right) u_a|| \\
&= ||\bigoplus_{a\in A} \tilde{F} + (\overline{c_a} - \overline{c_a}) x^{g-e_a}|| && \text{by assumption}
\end{aligned}
$$

which gives $||\bigoplus_{a\in A}(\tilde{F} + (\overline{c_a} - \overline{c_a})x^{g-e_a}||$ if $\exists\, g$ with $B_g \neq \emptyset$.

$\square$

2.3.13 Group action in the general case

The action in the generalized setting is given by tensoring the following sequence with $\mathbb{C}^*$:

$$0 \to L_A \xrightarrow{\delta} \mathbb{Z}^A \xrightarrow{\gamma} N \to 0.$$

For $v \in M$, we have $\gamma^\vee(v) = \sum_{a\in A(v)} c_a e_a$, and we can differentiate to find δ.

$$
\begin{tikzcd}
L_A \arrow[r, "\delta"] \arrow[d, phantom, "\cong" rotate=-90] & \mathbb{Z}^A \\
\mathbb{Z}^k \arrow[r, "B"] & \mathbb{Z}^A
\end{tikzcd}
$$

where $B = \begin{bmatrix} \tilde{\nu}_1 \\ \tilde{\nu}_2 \\ \vdots \\ \tilde{\nu}_{|A|} \end{bmatrix}$ and $B(v) = \begin{bmatrix} \nu_1(v) \\ \nu_2(v) \\ \vdots \\ \nu_{|A|}(v) \end{bmatrix}$. We can write $\nu_i = \sum_{j=1}^{|A|} b_i^j e_j$ with $b_i^j \in \mathbb{Z}$. Now, $(B \otimes \mathbb{C}^*)(t_1, \ldots, t_k) = \left(\prod t_j^{b_1^j}, \prod t_j^{b_2^j}, \ldots, \prod t_j^{b_{|A|}^j} \right)$ gives a subtorus which acts. A derivative of this action at a particular point will give δ. To define the action, we have

$$(t_1, \ldots, t_k) \cdot (x_1, \ldots, x_{|A|}) = \left(\prod t_j^{b_1^j} x_1, \ldots, \prod t_j^{b_{|A|}^j} x_{|A|} \right)$$

where $(t_1, \ldots, t_k) \in L_A$. The $i-$th Euler field (where i comes from an element u in L_A) is

$$\begin{aligned} X_i &= \frac{\partial}{\partial t_i} \left(\prod t_j^{b_1^j} x_1, \ldots, \prod t_j^{b_{|A|}^i} x_{|A|} \right) \Bigg|_{t=1} \\ &= b_1^i x_1, b_2^i x_2, \ldots, b_{|A|}^i x_{|A|} \qquad \text{as coordinates on the basis vectors } \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{|A|}}, \text{ so} \\ X_i &= \sum_{k=1}^{|A|} b_k^i x_k \frac{\partial}{\partial x_k} \\ &= \sum_{k=1}^{|A|} \tilde{\nu}_k(u) x_k \frac{\partial}{\partial x_k} \qquad \text{which we can also write as} \\ &= \sum_{k=1}^{|A|} u(|x_k|) x_k \frac{\partial}{\partial x_k} \end{aligned}$$

we have $\tilde{\nu}_i = \delta^\vee(e_i^\vee)$ from before. The tangent space of the algebraic torus here is isomorphic to $\mathbb{C} \otimes L_A$ from taking the exponential. Any element of L_A is a linear combination of $\{\frac{\partial}{\partial t_i}\}$. X_i corresponds to the $i-$th basis vector in L_A. So we can identify elements of L_A as tangent vectors to the algebraic torus, but really $L_A \otimes \mathbb{C}$ is the tangent space. This shows that Ξ is what was asserted in previous sections. We want to find which Kähler differentials vanish on Ξ.

Note that $F(z_v) = \sum_{a \in A(v)} c_a x^{f_{v,a}} dx_a$ is well-defined from dx_a being a basis for Kähler differentials on $\mathbb{A}^n$.

2.3.14 Background on polytopes

Definition 1: Let $L(P)$ be the face lattice of a polytope (partially ordered by inclusion). For F a face of P, we define

$$N_F := \{u \in N_{\mathbb{R}}^{\vee} = M_{\mathbb{R}} \mid F \subset \{x \in P \mid \langle x, u \rangle = \min_{\tilde{x} \in P} \langle \tilde{x}, u \rangle\}\}$$

and the normal fan[18]:

$$\mathcal{N}_P := \{N_F \mid F \in L(P) \setminus \emptyset\}.$$

If $\mathcal{F}$ and $\mathcal{G}$ are both fans in the same space $\mathbb{R}^d$, then we define their common refinement as

$$\mathcal{F} \bigwedge \mathcal{G} := \{C \cap C' : C \in \mathcal{F}, C' \in \mathcal{G}\}.$$

In general, the pullback of the common refinement of two fans with support in the same space is the common refinement of the pullbacks, as follows:

Proposition: For linear spaces V, W and a linear map $V \xrightarrow{T} W$, if $\mathcal{F}, \mathcal{G}$ are fans with support in W, then $T^*\mathcal{F} \wedge T^*\mathcal{G} = T^*(\mathcal{F} \wedge \mathcal{G})$.

Proof. For $C_1 \in \mathcal{F}, C_2 \in \mathcal{G}$,

$$\begin{aligned}
T^*(C_1 \cap C_2) &:= \{v \in V \mid T(v) \in (C_1 \cap C_2)\} \\
&= \{v \in V \mid T(v) \in C_1, T(v) \in C_2\} \\
&= T^*C_1 \cap T^*C_2.
\end{aligned}$$

Since an arbitrary cone $D \in \mathcal{F} \wedge \mathcal{G}$ can be written $D = C_1 \cap C_2$ where $C_1 \in \mathcal{F}, C_2 \in \mathcal{G}$, we have

$$\begin{aligned}
T^*(\mathcal{F} \wedge \mathcal{G}) &= \{T^*D \mid D \in \mathcal{F} \wedge \mathcal{G}\} \\
&= \{T^*(C_1 \cap C_2) \mid C_1 \in \mathcal{F}, C_2 \in \mathcal{G}\}
\end{aligned}$$

since any cone $D \in \mathcal{F} \wedge \mathcal{G}$ can be written $D = C_1 \cap C_2$ where $C_1 \in \mathcal{F}, C_2 \in \mathcal{G}$. Continuing, we have

$$\begin{aligned}
T^*(\mathcal{F} \wedge \mathcal{G}) &= \{T^*C_1 \cap T^*C_2 \mid C_1 \in \mathcal{F}, C_2 \in \mathcal{G}\} \qquad \text{since } T^*(C_1 \cap C_2) = T^*C_1 \cap T^*C_2 \\
&= \{D \cap E \mid D \in T^*\mathcal{F}, E \in T^*\mathcal{G}\} \\
&=: T^*\mathcal{F} \wedge T^*\mathcal{G}.
\end{aligned}$$

$\square$

Claim: $\mathcal{F} = (\gamma^\vee)^*(\mathcal{OF})$ is normal fan to $\mathrm{gr}(Q, A)$. $\mathrm{gr}(Q) = \sum_{a \in \mathcal{A}}[0, a]$, zonotope.

Proof. $\Sigma_{[0,a]} = \{H_a^-, H_a, H_a^+\}$ so that [after repeating the process of common refinement over all elements of A, taken two at a time]:

$$N_{\sum_{a \in A}[0,a]} = \left\{ \bigcap_{a \in A} C_a \mid C_a \in \{H_a^-, H_a, H_a^+\} \right\}.$$

Now, the orthant fan is the common refinement of coordinate half-spaces, and is the common refinement to the coordinate intervals:

$$\begin{aligned}\mathcal{OF} &= \bigwedge_{i=1}^{n} \{H_{e_i}^{+}, H_{e_i}, H_{e_i}^{-}\} \\ &= \bigwedge_{i=1}^{n} \mathcal{N}_{[0,e_i]}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{F} &= (\gamma^{\vee})^{*}(\mathcal{OF}) \\ &= (\gamma^{\vee})^{*}(\bigwedge_{i=1}^{n} \mathcal{N}_{[0,e_i]}) \\ &= \bigwedge_{i=1}^{n} (\gamma^{\vee})^{*}(\mathcal{N}_{[0,e_i]}) \\ &= \bigwedge_{a\in A} \mathcal{N}_{[0,a]} \\ &= \mathcal{N}_{\sum_{a\in A}[0,a]} \\ &= \mathcal{N}_{\mathrm{gr}(Q)}.\end{aligned}$$

□

Definition: Given a cone σ whose support is in $N_{\mathbb{R}}$ and a linear map $f : N_{\mathbb{R}} \to \tilde{N}_{\mathbb{R}}$, then

$$f^{*}\sigma := \{u \in N_{\mathbb{R}} \mid f(u) \in \sigma\}.$$

Claim: $f^{*}\sigma$ is a cone in $N_{\mathbb{R}}$.

Proof. Let $\sigma = \bigcap_{i=1}^{s} H_{m_i}^{+}$. Then

$$
\begin{aligned}
\sigma &= \{u \in \tilde{N}_{\mathbb{R}} \mid \ < u, m_i > \geq 0 \ \forall \ 1 \leq i \leq s\} \\
&= \{u \in \tilde{N}_{\mathbb{R}} \mid m_i^{\vee}(u) \geq 0 \ \forall \ 1 \leq i \leq s\} && \text{so that} \\
f^{*}\sigma &= \{u \in N_{\mathbb{R}} \mid \ < f(u), m_i > \geq 0 \ \forall \ 1 \leq i \leq s\} \\
&= \{u \in N_{\mathbb{R}} \mid m_i^{\vee}(f(u)) \geq 0 \ \forall \ 1 \leq i \leq s\} \\
&= \{u \in N_{\mathbb{R}} \mid f^{\vee}(m_i^{\vee})(u) \geq 0 \ \forall \ 1 \leq i \leq s\} \\
&= \bigcap_{i=1}^{s} H^{+}_{f^{\vee}(m_i)}.
\end{aligned}
$$

$\square$

Since $f^{*}\sigma$ is the intersection of finitely-many half-spaces, $f^{*}\sigma$ is a also polyhedral cone.

Claim (Part II): We can describe $(\gamma^\vee + (\delta^\vee)^{-1}(\chi))^*(\mathcal{OF})$ as tropical polyhedral decomposition associated to a specific valuation on $\mathrm{gr}(Q)$. [Take maximal cells as well].

Proof. **Particular Case**: $A = \{0, e_1, \dots, e_n\}$ so that $\gamma =$ Identity and $\gamma^\vee$ is composition with identity. (2): $\mathrm{Trop}_\eta = (\gamma^\vee + \eta)^*(\mathcal{N}_\Delta)$.
Setup:

$$0 \to L_A \to \mathbb{Z}^A \xrightarrow{\gamma} N \to 0$$

$$0 \to M \xrightarrow{\gamma^\vee} \mathbb{Z}^A \to L_A^\vee \to 0$$

and

$$\begin{array}{ccc} M_\mathbb{R} & \xrightarrow{\gamma_\mathbb{R}^\vee} & (\mathbb{R}^A)^\vee \\ \cup & & \cup \\ (\gamma^\vee + \eta)^*(\mathcal{N}_\Delta) & & \mathcal{N}_\Delta \end{array}$$

and for $\eta \in (\mathbb{R}^A)^\vee, u \in M$ we have

$$\begin{array}{ccc} (\mathbb{R}^A) & \xrightarrow{\gamma} & N \\ \eta \downarrow & & \downarrow u \\ \mathbb{R} & \subset & \mathbb{C}^* \end{array}$$

so that for $x \in \mathbb{R}^A$, $(\gamma^\vee + \eta)(u)(x) := u(\gamma(x)) + \eta(x)$.

Now, let $F \preceq \mathcal{N}_{\Delta_n}$. Then $F = N_G$ for some $G \preceq \Delta_n$. Now

$$\begin{aligned} (\gamma^\vee + \eta)^*(F) &= (\gamma^\vee + \eta)^*(N_G) \\ &= \{u \in M \mid (\gamma^\vee + \eta)(u) \in N_G\} \\ &= \{u \in M \mid G \subset \{x \in \Delta_n \mid (\gamma^\vee + \eta)(u)(x) = \min_{\tilde{x} \in \Delta_n}\{(\gamma^\vee + \eta)(u)(\tilde{x}))\}\}\} \\ &= \{u \in M \mid G \subset \{x \in \Delta_n \mid u(x) + \eta(x) = \min_{\tilde{x} \in \Delta_n}\{u(\tilde{x} + \eta(\tilde{x})\}\}\} \end{aligned}$$

since $\gamma = \mathrm{Id}$ implies $\gamma^\vee$ is composition with identity. In general, $F \preceq P$ a face of a polytope with $P \subset \mathbb{R}^n$ implies $\dim(F) + \dim(N_F) = n$. In particular, $\dim(F) + \dim(G) = n$ here. Now, $\mathrm{trop}_\eta(m) := \min_{a \in A}\{< m, a > +\eta(a)\}$ has minimum assumed at least twice among elements a_{i_1}, a_{i_2} of A iff

$$G := \mathrm{Conv}(a_{i_1}, a_{i_2}) \subset \{x \in \Delta_n \mid (\gamma^\vee + \eta)(m)(x) = \min_{\tilde{x} \in \Delta_n}\{(\gamma^\vee + u)(m)(\tilde{x})\}\}$$

by linearity of $\gamma^\vee$ and η. [The same is true if η is an affine function since an equal shift is applied to all elements of Δ_n]. Therefore, it suffices to consider the case where G is 1-dimensional, which gives that F is a wall of Trop_η, and also a an $n-1$-dimensional cone $N_G \preceq \mathcal{N}_{\Delta_n}$. This is because if the minimum in trop_η is assumed more than twice at some $\{a_{i_1}, \ldots, a_{i_\ell}\} \in A$, then

$G = \mathrm{Conv}(a_{i_1}, \ldots, a_{i_\ell}) \subset \{x \in \Delta_n \mid (\gamma^\vee + \eta)(m)(x) = \min_{\tilde{x} \in \Delta_n}\{(\gamma^\vee + \eta)(m)(\tilde{x})\}\}$

and N_G is contained in some wall of Trop_η. Therefore, Trop_η, together with the maximal-dimensional cells of the polyhedral subdivision of $M_\mathbb{R}$ equals $(\gamma^\vee + \eta)^*(\mathcal{N}_{\Delta_n})$. □

General Case:

Proof. P any polytope, with

$$\begin{array}{ccccccccc} & & & & P & \twoheadrightarrow & Q & & \\ & & & & \cap & & \cap & & \\ 0 & \to & U & \to & V & \overset{\beta}{\twoheadrightarrow} & W & 0 & \end{array}$$

gives

$$U^\vee \twoheadleftarrow V^\vee \hookleftarrow W^\vee \leftarrow 0$$

$$\tilde{\eta} \leftarrow \eta.$$

with $\tilde{\eta} \in U^\vee, \eta \in V^\vee$.Now, $A := \{\beta(v) : v \text{ a vertex of } P\}$. $\eta : A \to \mathbb{R}$ and $\eta(\beta)(v) = \eta(v)$. Also, $\overline{\eta}(\beta(v)) := \eta(v)$ so that the same proof carries through as above: using an extra variable and projecting, $N_{Q_\mathbb{R}} \to \mathbb{R}^\vee$. We also have $\overline{\eta}(\beta(v)) := \eta(v)$, which makes the proof the same as above. **Note**: Evaluation on the vertices is given by η. □

To construct a **fiber polytope**[19], we take a marked polytope Q with a set of points A such that $\mathrm{Conv}(A) = Q$, and map the standard simplex to Q by map γ. From this map, we construct the fiber polytope via:

$\Sigma(A) = \Sigma(P, Q)$. $\Delta \to Q$. $\mathbb{R}^A \xrightarrow{\gamma} N_\mathbb{R} \to 0$ where $e_a \mapsto a$.

$$\begin{array}{ccc} P & \twoheadrightarrow & T(P) \\ \cap & & \cap \\ \mathbb{R}^n & \xrightarrow{T} & \mathbb{R}^m \end{array}$$

where T is a linear map.

Proposition: Let T, W be linear spaces and $T : V \to W$ a linear map. Let $P \subset V$ so that $P \twoheadrightarrow T(P) \subset W$. Then for a face $F \preceq T(P)$, $T^{-1}(F)$ is a face of P.

Proof. By the definition of a face of a polytope, $F \subset T(P)$ can be written $F = m^{-1}(0) \cap P$ for some affine function m subject to the condition that all $p \in T(P)$ satisfy $m(p) \geq 0$. Now,

$$\{v \in P \text{ such that } T^\vee(m(v)) = m(T(v)) = 0\}$$
$$= \{v \in P \mid T(v) \in F\}$$
$$= T^{-1}F$$
$$= P \cap (T^\vee(m))^{-1}(0)$$

is a face of P. $\square$

Claim: $\mathcal{N}_{T(P)} = (T^\vee)^*(\mathcal{N}_P)$.

Proof. Let $F \preceq T(P)$ be a face. By the proposition, $T^{-1}F$ is a face of P.

Now, $\mathcal{N}_{T(P)} = \{N_F \mid F \preceq T(P)\}$ and

$$N_F := \{u \in (\mathbb{R}^m)^\vee \mid F \subset \{x \in T(P) \text{ such that } u(x) = \ \min{}_{\tilde{x} \in T(P)}\{u(\tilde{x})\}\}\}$$
$$= \{u \in (\mathbb{R}^m)^\vee \mid F \subset \{T(v) \in T(P) \text{ such that } u(T(v)) = (T^\vee(u))(v) = \min_{T(\tilde{v}) \in T(P)} T^\vee(u)(\tilde{v})\}\}$$
$$= \{u \in (\mathbb{R}^m)^\vee \mid T^{-1}F \subset \{v \in P \ T^\vee u(v) = \min_{\tilde{v} \in P}\{T^\vee u(\tilde{v})\}\}\}$$
$$= (T^\vee)^* N_{T^{-1}F}$$

so that

$$\mathcal{N}_{T(P)} = \{(T^\vee)^* N_{T^{-1}F} \mid F \preceq T(P)\}$$
$$= \{(T^\vee)^* D \mid D \in \mathcal{N}_P\}$$

since if $G \preceq P$ is not the pre-image of any face in $T(P), T^\vee(\text{Relint})(G) = \emptyset$

$$= (T^\vee)^* \mathcal{N}_P.$$

$\square$

2.3.15 Koszul compexes and $Gr(Q), \Sigma(A)$

Sturmfels-Kapranov-Zelevinsky[20] relate fiber polytopes (Chow polytopes) to Koszul complexes. Here, we recall some constructions relating to a Koszul complex.

For R a ring and for the central element $x \in Z(R)$, we define[21] the Koszul complex $\mathbf{K}(x)$ to be

$$0 \to R \xrightarrow{\cdot x} R \to 0$$

where the first copy of R is in degree 1, the second in degree 0. If R is a ring and $\mathbf{x} = (x_1, \ldots, x_n)$ a finite sequence of central elements in R, then $K(\mathbf{x})$ is the total tensor product complex

$$K(x_1) \otimes_R K(x_2) \otimes_R \cdots \otimes_R K(x_n).$$

$K(\mathbf{x})$ has a grading, and the degree p part of $K(\mathbf{x})$ is generated by the symbols

$$e_{i_1} \wedge \cdots \wedge e_{i_p} = 1 \otimes \cdots \otimes 1 \otimes e_{x_{i_1}} \otimes \cdots \otimes e_{x_{i_p}} \otimes \cdots \otimes 1 \qquad (i_1 < \cdots < i_p).$$

We also have the

Koszul Resolution: If $\mathbf{x}$ is a regular sequence in R, then $K(\mathbf{x})$ is a free resolution of R/I, $I = (x_1, \ldots, x_n)R$. That is, the following sequence is exact:

$$0 \to \Lambda^n(R^n) \to \cdots \to \Lambda^2(R^n) \to R^n \xrightarrow{\mathbf{x}} R \to R/I \to 0.$$

We will say more about Koszul complexes in Section 3.1.1.

2.3.16 Toric ideals and Gröbner bases

Definition: A Toric ideal[22] $\mathbb{I}_{\mathcal{A}}$ is $\ker(\hat{\pi}) : \mathbf{k}[x] := \mathbf{k}[x_1, \ldots, x_n] \to \mathbf{k}[t_1^{\pm}, \ldots, t_d^{\pm}]$ of map $x_i \mapsto t^{a_i}$ induced by $\mathbb{N}^n \to \mathbb{Z}^d$ where $(u_1, \ldots, u_n) \mapsto u_1 a_1 + \cdots + u_n a_n$ for some choice of $\{a_1, \ldots, a_n\} \subset \mathbb{Z}^d$. Each $a_i \in \mathbb{Z}^d$ is associated with $t^{a_i} \in \mathbf{k}[x_1^{\pm}, \ldots, x_d^{\pm}]$. Note that $V(\mathcal{I}_{\mathcal{A}})$ is not necessarily normal here which differs from Fulton[23].

An **infinite** generating set for $\mathcal{I}_{\mathcal{A}}$ is

$$\{x^u - x^v \mid \pi(u) = \pi(v)\}$$

for $u, v \in \mathbb{N}^n$. This raises the question of how to find a more optimal generating set.

Definition: The **state polytope** is a polytope whose normal fan is the Gröbner fan[22].

2.3.17 An example of the general combinatorial approach: $\mathbb{P}^1$

A reference for this section is GKZ[24]. For a case where the primitive ray generators of $\Sigma(1)$ do not lie in a hyperplane not containing the origin, we'll consider $\mathbb{P}^1 \times \mathbb{P}^1$ as the GIT quotient $\mathbb{C}^4/\chi^G$. P= $Conv(\{(0,0),(1,0),(0,1),(1,1)\}$. Here, the affine toric variety to consider is the orbit of the diagonal. Let $A = \Sigma(1)$ generators as a subset of N, and $G = \mathbb{C}^* \otimes L$. We have an exact sequence

$$\begin{array}{ccccccccc} & & & & \mathbb{Z}^4 & & \mathbb{Z}^2 & & \\ & & & & || & & || & & \\ 0 & \to & L & \to & \mathbb{Z}^A & \to & N & \to & 0 \end{array}$$

with G acting on $\mathbb{C}^4$ via:

$(\lambda_1, \lambda_2) \cdot (z_1, z_2, z_3, z_4) = (\lambda_1 z_1, \lambda_1 z_2, \lambda_2 z_3, \lambda_2 z_4)$ and $\chi = L^\vee \cong \mathrm{Hom}(\mathbb{C}^* \otimes L, \mathbb{C}^*)$. The following sequence is exact:

$$0 \leftarrow L^\vee \otimes \mathbb{R} \overset{\alpha^\vee}{\leftarrow} \mathbb{R}^A \leftarrow M_{\mathbb{R}} \leftarrow 0$$

and for $\chi \in L^\vee \otimes \mathbb{R}$, $M_{\mathbb{R}} \cong (\alpha^\vee)^{-1}(\chi) \mapsto \chi$. Here, take $P = (\alpha^\vee)^{-1}(\chi) \cap \mathbb{R}^A_{\geq 0}$ which is unique up to translation.

The set A of primitive ray generators in N of Σ give the exact sequence

$$0 \to L \to \mathbb{Z}^A \to N \to 0$$

Tensoring with $\mathbb{C}^*$ gives

$$1 \to \mathbb{C}^* \otimes L \to (\mathbb{C}^*)^A \to T_N \to 1$$

Given Σ a fan in $N_\mathbb{R}$, $X_\Sigma = U_\Sigma/G$ and

$$X_\Sigma \times X_\Sigma = (U_\Sigma \times U_\Sigma)/(G \times G)$$

where $U_\Sigma \times U_\Sigma \subset (\mathbb{C}^A)^2$. The diagonal $\Delta_{X_\Sigma} = Y/(G \times G)$ where $Y \subset (\mathbb{C}^A)^2$.

The affine toric variety $Y = \overline{\Phi(\mathbb{C}^* \otimes L_\Delta)}$ is homogeneous with respect to $G \times G$, and $R \subset L_\Delta \otimes \mathbb{R}$ is a polytope giving Y. Here, we have $B \subset L_\Delta^\vee \otimes \mathbb{R}$ giving $Y = X_B$. Here, Pic is generated by line bundles, and we have a homomorphism from an equivariant divisor to line bundles. There is a natural morphism from $\mathbb{Z}^A$ to $L^\vee$ which gives a correspondence from $\mathbb{Z}^A$ to equivariant divisors and $(\mathbb{Z}^A)^\vee$. As finite dimensional vector spaces over $\mathbb{C}$, $L \cong L^\vee$, and $L^\vee$ is congruent to the group of characters of the acting torus. This gives the exact sequence, which splits

$$0 \to L \to \mathbb{Z}^A \rightleftarrows N \to 0$$

if we choose a basis to get a splitting, s. Now $G = \alpha(\mathbb{C}^* \otimes L)$,

$X_\Sigma = \mathbb{C}^A/(\chi^G)$ or $(\mathbb{C}^A)^{ss}_\chi/G$ for χ in the interior

$$0 \to L \otimes L \to \mathbb{Z}^A \oplus \mathbb{Z}^A \to N \oplus N \to 0$$

$\mathbb{C}^A \xrightarrow{\Delta} \mathbb{C}^A \oplus \mathbb{C}^A$. In order to to describe the $G \times G$ orbit of Δ, we have:

$$
\begin{array}{ccccccccc}
0 & \to & L \oplus L & \to & \mathbb{Z}^A \oplus \mathbb{Z}^A & \to & N \oplus N & \to & 0 \\
 & & & & \searrow & & \downarrow (u,v) \mapsto u - v & & \\
 & & & & & & N & & \\
 & & & & & & \downarrow & \searrow & \\
 & & & & & & 0 & & 0
\end{array}
$$

with $X_\Sigma \cong \Delta \subset X_\Sigma \times X_\Sigma$. Here, the affine toric variety is the orbit of diagonal, associated to

$$
\begin{array}{ccccccccc}
0 & & & & & & & & \\
 & \searrow & & & & & & & \\
 & & L_\Delta \cong L \oplus L \oplus N & & & & & & \\
 & & & \overset{\Phi}{\searrow} & & & & & \\
0 \to & & L \oplus L \to & & \mathbb{Z}^A \oplus \mathbb{Z}^A & \to & N \oplus N & \to & 0 \\
 & & & & \downarrow \pi_i & \searrow & \downarrow & & \\
 & & & & \text{compose with } \Phi, \text{ get } B & & N & & \\
 & & & & & & \downarrow & \searrow & \\
 & & & & & & 0 & & 0
\end{array}
$$

For $Y \subset \mathbb{Z}^6,\ Y = \overline{[z_0 z_2, z_0 z_3, z_0, z_1 z_2, z_1 z_3, z_1]} : z_1, z_2, z_3, z_4 \in \mathbb{C}^*$.

All rows satisfy $x_1 + x_2 = 1$, which is to say elements of the row space are contained in a hyperplane in $\mathbb{R}^4$, which gives a $3-$dimensional polytope $Conv(B)$. The rows generate $\mathbb{R}^4$.

Another example of the general combinatorial approach: $\mathbb{P}^2$

For $\mathbb{P}^2$, we have $\mathcal{A} = \{(-1,-1),(1,0),(0,1)\}$.

$$0 \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z}^3 \to \mathbb{Z}^2$$

With $L = \mathbb{Z}$ and $N = \mathbb{Z}^2$. $\alpha(1) = (1,1,1)$ gives

$$0 \to L_\Delta \to \mathbb{Z}^3 \oplus \mathbb{Z}^3 \to \mathbb{Z}^2 \to 0$$

with $L_\Delta \cong \mathbb{Z}^4$. We have matrices

$$\begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1 & 1 \end{bmatrix}$$

from the map $(a,b) \mapsto a - b$, and another matrix whose rows give B:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

since $B \subset L_\Delta^\vee$ and $G = \mathbb{C}^*$. Here, choosing $1 \in L$ corresponds to $(1,1,1,\ldots,1)$.

As a dimension check: subtracting row 1 from all rows shows that the rank is 3. This shifted matrix with choice of basepoint given by row 1 is:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 0 \end{bmatrix}$$

with rank 3.

2.3.18 Syzygy resolution of $\mathbb{P}^n$ and interior points of $Gr(P_\Delta)$

The interior points of the graph polytope $Gr(P_\Delta)$ give the free ranks appearing in a syzygy resolution for $\mathbb{P}^n$ as follows. From $Gr(P_\Delta) = Conv(\sum_{\gamma \in I} \gamma : I \subset Vert(P_\Delta))$ and $Vert(P_\Delta) = \{e_0, e_1, \ldots, e_n, e_0 + f, e_1 + f, \ldots, e_n + f\}$ we see that, for $[z_1, z_2, \ldots, z_{n+2}]$ in $Gr(P_\Delta)$, we have that

$$0 \leq z_i \leq 2$$

for all i with $1 \leq i \leq n+1$ and

$$0 \leq z_{n+2} \leq n+1.$$

This implies $Int(Gr(P_\Delta)) = \{[1, 1, \ldots, 1, b] : 0 < b < n+1\} = \{e_0 + e_1 + \cdots + e_n + bf : 0 < b < n+1\}$. These points are all candidates for interior points based on inequalities above, and we can get a 0 or 2 in each of first $n+1$ components (while z_{n+2} varies), and have $[0,0,\ldots,0]$ and $[2,2,\ldots,2,n+1] \in Gr(P_\Delta)$ so that these are all actually interior points.

In the set T, this corresponds to a subset of $\{1,2,\ldots,n+1\}$ of size $n+1$ and since b can vary with $0 < b < n+1$, we have

$$n\binom{n+1}{n+1}$$

interior points in $Gr(P_\Delta)$ is $k\binom{n+1}{k+1}$ with $k = n$. In the syzygy resolution, this gives the free rank of $S^{n\binom{n+1}{k+1}}$ in

$$S^{n\binom{n+1}{n+1}} \to S^{(n-1)\binom{n+1}{n}} \to \cdots \to S^{k\binom{n+1}{k+1}} \to \cdots \to S^{2\binom{n+1}{3}} \to S^{\binom{n+1}{2}} \to S.$$

Chapter 3

Derived Categories $D(\mathcal{X})$

3.1 Resolutions of diagonal

By a "resolution of the diagonal," we mean: View $\mathcal{O}_\Delta$ as a complex of sheaves concentrated in degree 0 in $D^b_{\mathrm{Coh}}(X_\Sigma \times X_\Sigma)$ for toric variety X_Σ. Now, give a chain complex $C^\cdot$ acyclic (away from H^0) and a chain map $C^\cdot \to \mathcal{O}_\Delta$ which is a quasisomorphism. A locally free resolution $\mathcal{F}^\cdot \to \mathcal{A}$ of a sheaf $\mathcal{A}$ is a complex $0 \to \mathcal{F}^\cdot \to \mathcal{A} \to 0$ of locally free sheaves with $\mathcal{A}$ in degree 0 which induces a quasi-isomorphism of complexes[25]

$$\begin{array}{ccccccccccc}
\cdots & \to & 0 & \to & F_n & \to & \cdots & \stackrel{d}{\to} & F_0 & \to & 0 \\
 & & \downarrow 0 & & \downarrow 0 & & \downarrow 0 & & \downarrow \epsilon & & \downarrow 0 \\
\cdots & \to & 0 & \to & 0 & \to & \cdots & \to & \mathcal{A} & \to & 0
\end{array}$$

Now H^0 of the two complexes coincides: $H^0(\mathcal{F}^\cdot) = F_0/\mathrm{Im}\ d = F_0/\ker\epsilon \cong \mathcal{A} = H^0(\mathcal{A})$ since a resolution is exact.

3.1.1 Koszul complexes

For $x \in R$ a central element with R a commutative ring with 1, $K(x)$ denotes the complex

$$0 \to R \xrightarrow{\cdot x} R \to 0$$

in degrees 1 and 0. Here[21], $H_0(K(x)) = R/\langle x \rangle$ and $H_1(K(x)) = \{r \in R \mid rx = 0\}$. If we let $1 = e_x$ in degree 1, then $d(e_x) = d(1) = x$.

More generally, for $(x_1, \ldots, x_n)$ a finite sequence of central elements in R, the Koszul complex $K(x_1, \ldots, x_n)$ given by

$$K(x_1) \otimes_R K(x_2) \otimes \cdots \otimes_R K(x_n)$$

can be constructed by induction as the total tensor product

$$(K(x_1) \otimes_R \cdots \otimes_R K(x_{n-1})) \otimes_R K(x_n).$$

Here the total tensor product $Tot^{\oplus}((K(x_1) \otimes_R \cdots \otimes_R K(x_{n-1})) \otimes_R K(x_n))$ has $d^h = d \otimes 1$ and the sign trick for $d^v = (-1)^p \otimes d$. For R_1 the corresponding copy of R in degree 1 in $K(x_i)$, $n = 2$ gives

$$(R_1 \otimes R_1) \xrightarrow{d^h + d^v} ((R_1 \otimes R_0) \oplus (R_0 \otimes R_1)) \xrightarrow{d^h + d^v} (R_0 \otimes R_0)$$

in degrees $2, 1$, and 0, which comes from

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \to & (R_1 \otimes R_1) & \overset{\cdot x \otimes 1}{\to} & R_0 \otimes R_1 & \to & 0 \\
 & & -1 \otimes y \downarrow & & \downarrow 1 \otimes y & & \\
0 & \to & R_1 \otimes R_0 & \overset{\cdot x \otimes 1}{\to} & R_0 \otimes R_0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

Since $K_p(x_1, \ldots, x_n)$ is isomorphic to $\bigwedge^p R^n$ and has rank $\binom{n}{p}$, $K(x_1, \ldots, x_n)$ is often called the exterior algebra complex. The differential $d : K_p(x_1, \ldots, x_n) \to K_{p-1}(x_1, \ldots, x_n)$ sends

$$e_{i_1} \wedge \cdots \wedge e_{i_k} \mapsto \sum_{j=1}^{k} x_{i_j} (-1)^{j+1} e_{i_1} \wedge \cdots \wedge \hat{e_{i_j}} \wedge \cdots \wedge e_{i_k}.$$

For $n = 2$, we see that

$$e_x \otimes e_y \mapsto (-e_x \otimes y, x \otimes e_y)$$

$$e_x \otimes 1 \mapsto x \otimes 1,$$

$$1 \otimes e_y \mapsto 1 \otimes y$$

so we say that up to isomorphism,

$$
\begin{aligned}
e_x \wedge e_y &\mapsto -ye_x + xe_y \\
e_x &\mapsto x \\
e_y &\mapsto y
\end{aligned}
$$

This differential corresponds to contraction with the central sequence (x, y) in $\bigwedge^{\cdot} R^2$ when $R = \Bbbk[x, y]$ for $\Bbbk$ a field, for instance.

Here, we also see that

$$
\begin{aligned}
(d^h + d^v) \circ (d^h + d^v)(1 \otimes 1) &= (d^h + d^v)((-1 \otimes y) \oplus (x \otimes 1)) \\
&= 0 \oplus (-x \otimes y + x \otimes y) \oplus 0 \\
&= 0
\end{aligned}
$$

so that $d \circ d = (d^h + d^v) \circ (d^h + d^v) = 0$ in all degrees.

For $(x_1, x_2, x_3) = (x, y, z)$ in $\Bbbk[x, y, z]$, we see that $K(x, y) \otimes K(z)$ is given by

$$
\begin{aligned}
0 \to R \xrightarrow{\cdot x} R \to 0 \\
0 \to R \xrightarrow{\cdot y} R \to 0 \\
0 \to R \xrightarrow{\cdot z} R \to 0
\end{aligned}
$$

where $R_1 \ni e_x$ in the first complex, $R_1 \ni e_y$ in the second, and $R_1 \ni e_z$ in the third. Furthermore,

$$
\begin{array}{ccccc}
(R_1 \otimes R_1) \otimes R_1 & \stackrel{d^h}{\to} & (R_1 \otimes R_0 \oplus R_0 \otimes R_1) \otimes R_1 & \stackrel{d^h}{\to} & (R_0 \otimes R_0) \otimes R_1 \\
d^v \downarrow \cdot 1 \otimes 1 \otimes z & & \downarrow d^v & & \downarrow d^v \\
(R_1 \otimes R_1) \otimes R_0 & \stackrel{d^h}{\to} & (R_1 \otimes R_0 \oplus R_0 \otimes R_1) \otimes R_0 & \stackrel{d^h}{\to} & R_0 \otimes R_0 \otimes R_0
\end{array}
$$

shows that

$$
\begin{aligned}
e_x \wedge e_y \wedge e_z &\mapsto z \cdot e_x \wedge e_y - y e_x \wedge e_z + x e_y \wedge e_z \\
e_x \wedge e_y &\mapsto -y e_x + x e_y && \text{as before} \\
e_x \wedge e_z &\mapsto x e_z - z e_x \\
e_y \wedge e_z &\mapsto y e_z - z e_y \\
e_x &\mapsto x, \\
e_y &\mapsto y, \\
e_z &\mapsto z.
\end{aligned}
$$

In general, $K(x_1) \otimes K(x_2) \otimes \cdots \otimes K(x_n)$ inherits signs from those previously in $K(x_1) \otimes \cdots \otimes K(x_{n-1})$, and sign conventions from $K(x_n)$.

3.1.2 Koszul complexes from a homomorphism $\phi : E \to R$

The Koszul complex arising from a homomorphism $\phi : E \to R$ where R is a commutative ring and $E \in R - Mod$ is free with basis $(e_1, \ldots, e_n)$ is given as follows. For each q, the map $E^q \to \bigwedge^{q-1} E$ given by

$$
(e_1, \ldots, e_q) \mapsto \sum_i \phi(e_i)(-1)^{i+1} e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_q
$$

is $q-$linear and alternating, hence factors through a homomorphism $\phi_q : \bigwedge^q E \to \bigwedge^{q-1} E$.

Identity 1: We verify that if $a \in \bigwedge^r E$ and $b \in \bigwedge^{q-r} E$, then

$$\phi_q(a \wedge b) = \phi_r(a) \wedge b + (-1)^r a \wedge \phi(b)$$

as follows. The left-hand side gives

$$\begin{aligned}\phi_q(a \wedge b) = &\left(\sum_{i=1}^{r} \phi(e_i)(-1)^{i+1} e_1 \wedge \cdots \wedge \hat{e_i} \wedge \cdots \wedge e_r\right) \wedge b + \dots \\ &\cdots + (-1)^r a \wedge \left(\sum_{j=1}^{q-r} \phi(e_j)(-1)^{j+1} e_1 \wedge \cdots \wedge \hat{e_j} \wedge \cdots \wedge e_{q-r}\right).\end{aligned}$$

Then we prove by induction on q that $\phi_{q-1}\phi_q = 0$ as follows:

Proof. (I): **Base case:** For $q = 0, 1$ the statement holds since ϕ_k is defined to be 0 for $k < 0$.

(II): **Induction step:** Suppose $\phi_{k-1}\phi_k = 0$ for $k \in \mathbb{Z}_{>0}$. Write $a \in E^{k+1}$ as $\sum_i a_{k_i} \wedge b_i$ where $a_{k_i} \in \bigwedge^k E, b_i \in \bigwedge^1 E$. Then by R-linearity of $R-$module homomorphisms, it suffices to show $\phi_k\phi_{k+1} = 0$ for a given summand. Fix i. Let $a_k = a_{k_i}, b = b_i$. Then

$$\begin{aligned}\phi_k\phi_{k+1}(a_k \wedge b) &= \phi_k \circ (\phi_k(a) \wedge b + (-1)^k a \wedge \phi_1(b)) \\ &= \phi_k(\phi_k(a) \wedge b) + (-1)^k \phi_k(a \wedge \phi_1(b)) \\ &= \phi_k \circ \phi_k(a) \wedge b + (-1)^{k-1}\phi_k(a) \wedge \phi_1(b) + (-1)^k(\phi_k(a) \wedge \phi_1(b) + (-1)^k a \wedge \phi \circ \phi(b)) \\ &= (-1)^{k-1}\phi_k(a) \wedge \phi_1(b) + (-1)^k \phi_k(a) \wedge \phi_1(b) \\ &= 0\end{aligned}$$

by Identity 1 and the Base Case. $\square$

Therefore, we get a chain complex $K.(\phi) := \{(K_q, d_q) := (\bigwedge^q E, \phi_q)\}$.

3.2 Kähler differentials

Let $B \in A-$alg for A a commutative ring with 1, $B \otimes_A B$ a left $B-$module via multiplication on the left and $f : B \otimes_A B \to B$ a homomorphism of B modules gives by $f(b \otimes b') = bb'$. Let $I = \ker(f)$. Then I/I^2 inherits the structure of a B-module[26]. Define $d : B \to I/I^2$ by $d(b) := 1 \otimes b - b \otimes 1$. Then $\langle I/I^2, d\rangle$ is a module of relative differentials for B/A. Note that we've implied that $B \otimes_A B$ is also an $A-$algebra by writing I^2.

On $\mathbb{P}^n_{\Bbbk}$, $\mathcal{O}_{\mathbb{P}^n}$ is a free $\mathcal{O}_{\mathbb{P}^n}$ module, generated by 1. It has rank 1. Let $S = \Bbbk[x_0, \dots, x_n]$ denote the homogeneous coordinate ring of projective space, and $E := S(-1)^{n+1}$.

$\Omega_{\mathbb{P}^n}$ is defined by the Euler sequence

$$0 \to \Omega_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0$$

from the corresponding exact sequence of modules

$$0 \to M \to S(-1)^{n+1} \xrightarrow{f} S$$

where $f(e_i) = x_i$. If we localize at x_i, then $E_{x_i} \ni \frac{x_j}{x_i}$ and M_{x_i} is generated by $\{e_j - \frac{x_j}{x_i}e_i \mid j \neq i\}$ and $\Omega|_{U_i}$ is generated by elements in degree zero: $\{(\frac{1}{x_i})e_j - \frac{x_j}{x_i^2}e_i \mid j \neq i\}$. Since $U_i = Spec\ \Bbbk[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}]$ is affine, $\Omega|_{U_i}$ is a free $\mathcal{O}|_{U_i}$-module generated by $d(\frac{x_0}{x_i}), \dots, d(\frac{x_n}{x_i})$. Then the isomorphism $\Phi_i : \Omega_{U_i} \to \tilde{M}|_{U_i}$ is given by $\phi(d(\frac{x_j}{x_i})) = \frac{1}{x_i^2}(x_i e_j - x_j e_i)$. To show that the maps ϕ_i glue together, on $U_i \cap U_j$ we have

$$\frac{x_k}{x_i} = \frac{x_k}{x_j} \cdot \frac{x_j}{x_i}$$

implies by the Leibniz Rule that

$$\begin{aligned} d(\frac{x_k}{x_i}) &= d(\frac{x_k}{x_j})\frac{x_j}{x_i} + \frac{x_k}{x_j}d(\frac{x_j}{x_i}) \qquad \text{so} \\ d(\frac{x_k}{x_i}) - \frac{x_k}{x_j}d(\frac{x_j}{x_i}) &= \frac{x_j}{x_i}d(\frac{x_k}{x_j}) \end{aligned}$$

Now Φ_i of the left-hand side gives

$$\begin{aligned} \Phi_i(d(\frac{x_k}{x_i}) - \frac{x_k}{x_j}d(\frac{x_j}{x_i})) &= \frac{1}{x_i^2}(x_i e_k - x_k e_i) - \frac{x_k}{x_j}\frac{1}{x_i^2}(x_i e_j - x_j e_i)) \\ &= \frac{e_k}{x_i} - \frac{x_k e_i}{x_i^2} - \frac{x_k e_j}{x_i x_j} + \frac{x_k e_i}{x_i^2} \\ &= \frac{e_k}{x_i} - \frac{x_k e_j}{x_i x_j} \end{aligned}$$

and Φ_j of the right-hand side gives

$$\frac{x_j}{x_i}\frac{1}{x_j^2}(x_j e_k - x_k e_j) = \frac{e_k}{x_i} - \frac{x_k e_j}{x_i x_j}$$

so these are equal and the Φ_i glue together.

To summarize: On U_i, $\tilde{M}|_{U_i} = \left\langle \frac{1}{x_i}e_j - \frac{x_j}{x_i^2}e_i \text{ for } j \neq i \right\rangle$.
$\Omega|_{U_i} = \left\langle d(\frac{x_j}{x_i}) \text{ for } j \neq i \right\rangle$, subject to

$$d(\frac{x_k}{x_i}+\frac{x_j}{x_i})=d(\frac{x_k}{x_i})+d(\frac{x_j}{x_i}),$$
$$d(r)=0 \text{ for all } r\in\Bbbk,$$
$$d(\frac{x_j}{x_i}\frac{x_k}{x_i})=d(\frac{x_j}{x_i})\frac{x_k}{x_i}+\frac{x_j}{x_i}d(\frac{x_k}{x_i}).$$

This is equivalent to the construction of $\Omega|_{U_i}$ as $\langle I/I^2, d(b)=1\otimes b-b\otimes 1(\mod I^2)\rangle$ for $b\in\mathcal{O}|_{U_i}$ since: Here, $B=\mathcal{O}|_{U_i}, A=\Bbbk$.

To check the Leibniz rule[27], we note that

$$\begin{aligned} d(bb')-(d(b)b'+bd(b')) &= 1\otimes bb'-bb'\otimes 1-((1\otimes b-b\otimes 1)b'+b(1\otimes b'-b'\otimes 1)) \\ &= 1\otimes bb'-bb'\otimes 1-b'\otimes b+bb'\otimes 1-b\otimes b'+bb'\otimes 1 \\ &= (1\otimes b'-b'\otimes 1)(1\otimes b-b\otimes 1)\in I^2. \end{aligned}$$

To show $\Omega|_{U_i}\cong\left\langle 1\otimes\frac{x_j}{x_i}-\frac{x_j}{x_i}\otimes 1 \mid j\neq i\right\rangle$, we construct $g:\Omega|_{U_i}\to I/I^2$, show that g is surjective, then construct $h: I/I^2\to\Omega|_{U_i}$, show that h is well-defined, and show that $h\circ g:\Omega|_{U_i}\to U_i$ is the identity.

Proof. ($\Rightarrow$): Define $g:\Omega|_{U_i}\to I/I^2$ by $g(d(\frac{x_j}{x_i}))=1\otimes\frac{x_j}{x_i}-\frac{x_j}{x_i}\otimes 1$. Then g is surjective (mod I^2), since if for $z_i, w_i, z_j, w_j'\in\mathcal{O}_{U_i}$,

$\sum_i z_i\otimes w_i\in I \implies \sum z_i w_i=0$ so that $\sum_i z_i\otimes w_i=\sum z_i(1\otimes w_i-w_i\otimes 1)$.

($\Leftarrow$): Define $h: I/I^2\to\Omega|_{U_i}$ by $z\otimes w\mapsto zdw$. To show h is well-defined, we show that elements of I^2 are sent to 0:

$$\left(\sum z_i \otimes w_i\right)\left(\sum_j z_j' \otimes wj'\right) = \sum_{i,j} z_i z_{j'} \otimes w_i w_j'$$

must map to 0 when

$$\sum z_i w_i = \sum z_j' w_j' = 0.$$

But by the Leibniz rule,

$$\begin{aligned}
h\left(\sum_{i,j} z_i z_j' \otimes w_i w_j'\right) &= \sum_{i,j} z_i z_j' d(w_i w_j') \\
&= \sum_{i,j} z_i z_j' d(w_i) w_j' + \sum_{i,j} z_i z_j' w_i d(w_j') \\
&= \left(\sum_i z_i dw_i\right) \cdot \left(\sum_j z_j' w_j'\right) + \left(\sum_i z_i w_i\right) \cdot \left(\sum_j z_j' dw_j'\right) \\
&= 0.
\end{aligned}$$

Now $h \circ g : \Omega|_{U_i} \to \Omega|_{U_i}$ is the identity, as

$$\begin{aligned}
h \circ g(d(\frac{x_j}{x_i})) &= h(1 \otimes \frac{x_j}{x_i} - \frac{x_j}{x_i} \otimes 1) \\
&= d(\frac{x_j}{x_i}) - \frac{x_j}{x_i} \cdot 0 \\
&= d(\frac{x_j}{x_i})
\end{aligned}$$

as desired. We can now see the morphisms ϕ_i in the sequence

$$I/I^2 \xrightarrow{h} \Omega|_{U_i} \xrightarrow{\Phi_i} \mathcal{O}(-1)^{n+1}|_{U_i}$$

by

$$\Phi_i \circ h(1 \otimes \frac{x_j}{x_i} - \frac{x_j}{x_i} \otimes 1) = \Phi_i(d(\frac{x_j}{x_i}))$$

as before. $\square$

In complex geometry[28], the dual to the Euler sequence given above at $x = \pi(X)$ on fibers is given by

$$0 \to \mathbb{C} \cdot (X_0, \ldots, X_n) \to (H^{\oplus n+1})_x \to T'_x(\mathbb{P}^n) \to 0$$

where H is the hyperplane bundle defined to be dual to $\mathcal{O}(-1)$. That is, fibers of H are given by linear functions $\mathbb{C}\{X\}$ on the fiber of the tautological bundle $\mathcal{O}(-1)$ on $\mathbb{P}^n$. Here, linear functionals descend to give vector fields on $\mathbb{P}^n$, and $\sigma = (\sigma_0, \ldots, \sigma_n) \mapsto \pi_* \left(\sum \sigma_i(X) \frac{\partial}{\partial X_i} \right)$.

The corresponding exact sequence in algebraic geometry is given by

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{n+1} \to \mathcal{T}\mathbb{P}^n \to 0$$

and the dual sequence (tensored with $\mathcal{O}_{\mathbb{P}^n}(1)$) is given by

$$0 \to \Omega(1) \to \mathcal{O}^{n+1} \to \mathcal{O}(1) \to 0$$

where as before $e_i \mapsto x_i$ and $\tilde{M}(1)|_{U_i} = \left\langle e_j - (\frac{x_j}{x_i})e_i \mid j \neq i \right\rangle$.

3.2.1 Example: Kähler differentials on $\mathbb{P}^2$, showing $\mathbb{P}^2$ is a Fano toric variety

Let $\Sigma \subset N$ be the fan for $\mathbb{P}^2$ with rays ρ_0, ρ_1, ρ_2 with $v_0 = (-1, 1), v_1 = (1, 0), v_2 = (0, 1)$ primitive ray generators in N. Then the affine semigroup $\sigma_0^\vee$ has generators $\langle x, y \rangle$,

$\sigma_1^\vee = \langle x^{-1}, x^{-1}y\rangle$, and $\sigma_2^\vee = \langle xy^{-1}, y^{-1}\rangle$ so that $U_0 = Spec(\mathbb{C}[x,y])$ has coordinates (x,y), $U_1 = Spec(\mathbb{C}[x^{-1}, x^{-1}y])$ has coordinates $(x^{-1}, x^{-1}y)$ and $U_2 = Spec(\mathbb{C}[xy^{-1}, y^{-1}]$ has coordinates (xy^{-1}, y^{-1}). Since $\mathbb{P}^2$ is smooth, $\Omega_{\mathbb{P}^2}$ restricted to any affine open subset is locally free. In particular, $\Omega_{\mathbb{P}^2}\Big|_{U_0} \cong \langle dx, dy\rangle$ as a free $\mathbb{C}[x,y]$-module.

Let $\phi_{20} : U_0 \to U_2$ be the transition function given by $(x,y) \mapsto (xy^{-1}, y^{-1})$. Call the coordinates $(xy^{-1}, y^{-1}) = (a_1, a_2)$ on U_2. Then the Jacobian matrix

$$\text{Jac } \phi_{20} = \begin{pmatrix} y^{-1} & \frac{-x}{y^2} \\ 0 & -y^{-2} \end{pmatrix}$$

implies that $da_1 = y^{-1}dx - \frac{x}{y^2}dy$ and $da_2 = -y^{-2}dy$ in terms of (x,y)-coordinates on U_0. The transition functions for Ω from U_0 to U_1 are given similarly by Jac ϕ_{10}. Then the cocycle condition $\phi_{12} \circ \phi_{20} = \phi_{10}$ implies that $Jac\phi_{12} \circ Jac\phi_{20} = Jac_{10}$, and the transition functions on $\bigwedge^2 \Omega_{\mathbb{P}^2} = \omega_{\mathbb{P}^2} \cong \mathcal{O}\left(\sum_{\rho_i \in \Sigma(1)} -D_i\right)$ are given by

$$\begin{aligned} da_1 \wedge da_2 &= (y^{-1}dx - \frac{x}{y^2}dy) \wedge (-y^{-2}dy) \\ &= -y^{-3}dx \wedge dy \end{aligned}$$

and $dw_1 \wedge dw_2 = -x^{-3}dx \wedge dy$ for $(x^{-1}, x^{-1}y) = (w_1, w_2)$ on U_1. This shows[23] that $\bigwedge^2 \Omega_{\mathbb{P}^2} = \omega_{\mathbb{P}^2} \cong \mathcal{O}(-3)$.

Now the anticanonical divisor $-K_{\mathbb{P}^2} := \sum_{\rho_i \in \Sigma(1)} D_i$ gives the polytope

$$\begin{aligned} P_{-K_{\mathbb{P}^2}} :&= \{u \in M \mid \langle u, v_i\rangle \geq -a_i \ \forall\ i\} \\ &= Conv((-1,-1), (-1,2), (2,-1)) \end{aligned}$$

with $Card(P_{-K_{\mathbb{P}^2}} \cap M) = 10$ gives 10 global sections of $-K_{\mathbb{P}^2}$ which do not all simultaneously vanish on $\mathbb{P}^2$, and hence a well-defined map to $\mathbb{P}^{10-1} = \mathbb{P}^9$. That is, $\mathbb{P}^2$ is a Fano toric variety.

3.2.2 Beilinson resolution as a Koszul complex

In the case of complex projective spaces $\mathbb{P}^n_{\mathbb{C}}$, the resolution of diagonal is given by a Koszul complex corresponding to the morphism[29] $\mathcal{O}(-1) \otimes \Omega(1) \to \mathcal{O}_{X\times X}$. The relevant exterior algebra is then $\bigwedge^k(p^*\mathcal{O}(-1) \otimes q^*\Omega(1)) \simeq p^*(\mathcal{O}(-k)) \boxtimes q^*(\Omega(k))$. Dually[30], we can also say that the Koszul resolution for $\mathbb{P}^n$ is constructed using a section s of $\mathcal{O}(1) \boxtimes T(-1)$ on $\mathbb{P}^n \times \mathbb{P}^n$ which vanishes exactly on the diagonal[31;32]. Griffiths-Harris[28] also give a Koszul resolution for $\mathcal{O}/I$ as a module over $\mathcal{O}$. Beilinson's resolution of the diagonal also implies[33] that

$$
\begin{aligned}
D^b_{\mathrm{Coh}}(\mathbb{P}^n) &\cong \langle \mathcal{O}(a), \ldots, \mathcal{O}(a+n)\rangle \qquad\qquad \text{for any } a \in \mathbb{Z} \\
&\cong \left\langle \mathcal{O} = \Omega^0(0), \Omega^1(1), \ldots, \Omega^n(n)\right\rangle
\end{aligned}
$$

form a strong full exceptional collection for $D^b(\mathbb{P}^n)$.

$$
D^b_{\mathrm{Coh}}(\mathbb{P}^n) = \langle \mathcal{O}(-n), \mathcal{O}(-n+1), \ldots, \mathcal{O}\rangle .
$$

The Koszul complex corresponding to the morphism $\mathcal{O}(-1) \otimes \Omega(1) \to \mathcal{O}_{X\times X}$ from the first description given above is given as follows[29]. On the open set $U_i \times V_k$, $\mathcal{O}(-1) \boxtimes \Omega(1)$ is generated by $\{\frac{1}{x_i} \otimes (e_j - \frac{y_j}{y_k} e_k)\}_{j\neq k}$. The map Ψ is given by

$$
\frac{1}{x_i} \otimes (e_j - \frac{y_j}{y_k} e_k) \mapsto \frac{1}{x_i}(x_j - \frac{y_j}{y_k} x_k) = \frac{x_j}{x_i} - \frac{y_j x_k}{y_k x_i} \in \mathcal{O}_{\mathbb{P}\times\mathbb{P}}|_{U_i \times V_k}
$$

from the chain of morphisms

$$
q^*(\Omega_Y(1)) \to q^*(\mathcal{O}_Y^{n+1}) \xrightarrow{\sim} (\mathcal{O}_{X\times Y})^{n+1} \xrightarrow{\sim} p^*(\mathcal{O}_X^{n+1}) \to p^*(\mathcal{O}_X(1)).
$$

The composition resulting from such chain, tensored by $p^*(\mathcal{O}_X(-1))$, yields a morphism

$$p^*\mathcal{O}_X(-1) \otimes q^*\Omega_Y(1) \to p^*\mathcal{O}_X(-1) \otimes p^*\mathcal{O}_X(1).$$

Together with the chain of natural isomorphisms

$$\begin{aligned} p^*\mathcal{O}_X(-1) \otimes p^*\mathcal{O}_X(1) &\simeq p^*(\mathcal{O}_X(-1) \otimes \mathcal{O}_X(1)) \\ &\simeq p^*(\mathcal{O}_X) \\ &\simeq \mathcal{O}_{X\times Y} \end{aligned}$$

gives a morphism

$$p^*\mathcal{O}_X(-1) \otimes q^*\Omega_Y(1) \xrightarrow{\Psi} \mathcal{O}_{X\times Y}.$$

Ψ vanishes precisely on the diagonal $\{y_j = x_j \mid 0 \leq j \leq n\}$ of $\mathbb{P}^n \times \mathbb{P}^n$ and is minimal with respect to this property by a degree argument, hence $Cok(\Psi) = \mathcal{O}_\Delta$ and

$$\begin{array}{ccccc} \mathcal{O}(-1) \boxtimes \Omega(1) & \stackrel{\Psi}{\to} & \mathcal{O}_{\mathbb{P}\times\mathbb{P}} & \to & 0 \\ & & \downarrow & & \\ & & \mathcal{O}_\Delta & \to & 0 \end{array}$$

is exact at $\mathcal{O}_{\mathbb{P}\times\mathbb{P}}$. Now, we can form the Koszul complex associated to the map Ψ. Note that $\mathcal{O}(-1) \boxtimes \Omega(1)$ is locally free of rank given by the product $(\mathrm{rank}\mathcal{O}(-1)) \cdot (\text{ rank } \Omega(1)) = 1 \cdot n = n$. This Koszul complex is given by

$$
\begin{array}{l}
0 \to \bigwedge^{n}(\mathcal{O}(-1) \boxtimes \Omega(1)) \to \bigwedge^{n-1}(\mathcal{O}(-1) \boxtimes \Omega(1)) \to \cdots \to \mathcal{O}(-1) \boxtimes \Omega(1) \xrightarrow{\Psi} \mathcal{O}_{\mathbb{P}\times\mathbb{P}} \to 0 \\
\hspace{28em} \downarrow \\
\hspace{27em} \mathcal{O}_{\Delta} \to 0
\end{array}
$$

with $d_k : e_{i_1} \wedge \cdots \wedge e_{i_k} \mapsto \sum_\ell \Psi(e_{i_\ell})(-1)^{\ell+1} e_{i_1} \wedge \cdots \wedge \hat{e_{i_\ell}} \wedge \cdots \wedge e_{i_k}$, which gives[34] a locally free resolution of $\mathcal{O}_\Delta$.

Note that this same resolution will not work for a general toric variety X_Σ, as $Pic(X_\Sigma)$ need not be isomorphic to $\mathbb{Z}$. Therefore, $\mathcal{O}(-1)$ and $\Omega(1)$ need not be defined for a general toric variety X_Σ. Since $\{u_\rho \mid \rho \in \Sigma(1)\}$ span $N_\mathbb{R}$ in the case of $\mathbb{P}^n$, the fundamental exact sequence[3] for $\mathbb{P}^n$

$$0 \to M \to \mathrm{CDiv}_{T_N}(X_\Sigma) \to \mathrm{Pic}(X_\Sigma) \to 0$$

shows $Pic(\mathbb{P}^n) \cong \mathbb{Z}$. Here, the first map sends $m \mapsto \mathrm{div}(\chi^m)$ and the second sends a T_N-invariant divisor to its divisor class in $\mathrm{Pic}(X_\Sigma)$.

For a more general toric variety X_Σ, we have the generalized Euler-Jacewski sequence[17;35]

$$0 \to \Omega_X \to \bigoplus_{\rho \in \Sigma(1)} \mathcal{O}(-D_\rho) \to \mathcal{H} \to 0$$

where $\mathcal{H} = H \otimes \mathcal{O}(1)$, $H = H^1(X, \Omega_X)$. Here, $\mathcal{H}$ is isomorphic[35] to the Weil divisor class group of X (modulo linear equivalence) tensored with $\mathcal{O}_X$.

Generalizing Koszul resolutions for $\mathbb{P}^n$

Taylor resolutions[30] generalize Koszul resolutions, and are examples of cellular resolutions[36] of monomial ideals in the case that X is a $d-1$ simplex where d monomial generators label the vertices of X. This description of a Taylor resolution implies that Koszul resolutions can be viewed as the cellular resolution of an $d-1$ simplex. This is why the binomial coefficients appear as the free ranks in a Taylor resolution: for every subset of $0, 1, 2, \ldots, d-1$ of size $k \leq d$, there exists a face of X, the $d-1$ simplex with this subset as vertices.

However, toric ideals are binomial ideals (which can be viewed as lattice ideals[12;36;37]) rather than monomial ideals, so resolutions of toric ideals in general come from resolutions of lattice ideals.

Syzygy resolution for the lattice ideal from orbit of diagonal of $\mathbb{P}^2$

For $X_\Sigma = \mathbb{P}^2$, we have the set $\mathcal{A} = \{(1,0), (0,1), (-1,-1)\}$ of primitive ray generators with $|A| = 3$ giving the exact sequence

$$0 \to \xrightarrow{\alpha} \mathbb{Z}^{|A|} \xrightarrow{\gamma} N \to 0$$

with $L \cong \mathbb{Z}, N \cong \mathbb{Z}^2$. Here, the matrix γ is given by

$$\gamma = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

and we can view

$$L = \ker_{\mathbb{Z}} \gamma \cong \mathbb{Z} \cdot \left\langle \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \right\rangle .$$

Here, $\alpha : 1 \mapsto (-1, -1, -1)$. For the orbit of $\Delta \subset X \times X$, we have the following setup:

$$\begin{array}{ccccccccc}
0 & & & & & & & & \\
& \searrow & & & & & & & \\
& & L \oplus L \oplus N & \simeq L_{\Delta} & & & & & \\
& & & \searrow \Phi & & & & & \\
0 & \to & L \oplus L & \to & \mathbb{Z}^{|A|} \oplus \mathbb{Z}^{|A|} & \overset{\gamma \oplus \gamma}{\to} & N \oplus N & \to & 0 \\
& & & & & \searrow & \downarrow \; (a,b) \mapsto a - b & & \\
& & & & & & N & & \\
& & & & & & \downarrow & \searrow & \\
& & & & & & 0 & & 0
\end{array}$$

with Φ given by

$$\phi = \left(\begin{array}{c|c|c} \alpha & \mathbf{0} & \begin{array}{c} I_n \\ \hline 0 \end{array} \\ \hline \mathbf{0} & \alpha & \begin{array}{c} I_n \\ \hline 0 \end{array} \end{array} \right) = \left(\begin{array}{c|c|cc} -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ \hline 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{array} \right)$$

satisfies the hyperplane condition that $-(x_1 + x_2) = 1$. For a_i given by row i of Φ the unique minimal generators of the affine semigroup Q, we have the nonzero Betti numbers[36]

$\beta_{i,\mathbf{b}}$ of the number of minimal ith syzygies of the lattice ideal I_L in degree $\mathbf{b}$ are:

$\mathbf{b}$	$\vec{\beta}_{\mathbf{b}}$
$\begin{bmatrix} -1 & -1 & 1 & 0 \end{bmatrix}$	$(1,0,0,\dots)$
$\begin{bmatrix} -1 & -1 & 1 & 1 \end{bmatrix}$	$(1,0,0,\dots)$
$\begin{bmatrix} -1 & -1 & 0 & 1 \end{bmatrix}$	$(1,0,0,\dots)$
$\begin{bmatrix} 0 & -1 & 1 & 1 \end{bmatrix}$	$(0,1,0,0,\dots)$
$\begin{bmatrix} -1 & -2 & 1 & 1 \end{bmatrix}$	$(0,1,0,0,\dots)$

These correspond to the syzygy resolution

$$0 \leftarrow I_L \leftarrow \bigoplus_{\mathbf{b}\in Q} S(-\mathbf{b})^{\beta_{0,\mathbf{b}}} \leftarrow \bigoplus_{\mathbf{b}\in Q} S(-\mathbf{b})^{\beta_{1,\mathbf{b}}} \leftarrow \cdots \leftarrow \bigoplus_{\mathbf{b}\in Q} S(-\mathbf{b})^{\beta_{r,\mathbf{b}}} \leftarrow 0$$

and $\beta_{j,\mathbf{b}}$ of I_L equals[36] the dimension over k of the j^{th} reduced homology group $\tilde{H}_j(\Delta_{\mathbf{b}}; \Bbbk)$ where for any $\mathbf{b} \in Q$, $\Delta_{\mathbf{b}}$ is the simplicial complex defined on the vertex set $\{1, \dots, n\}$ by

$$\Delta_{\mathbf{b}} = \{I \subseteq \{1,\dots,n\} \mid \mathbf{b} - \sum_{i\in I} \mathbf{a}_i \text{ lies in } Q\}.$$

For this particular case of the orbit of the diagonal in $\mathbb{P}^2 \times \mathbb{P}^2$, we have zero-dimensional reduced homology generators from $\mathbf{b} = a_1 + a_5 = a_2 + a_4, a_1 + a_6 = a_4 + a_3$, and $a_2 + a_6 = a_5 + a_3$ in degree 2, with $\beta_{\mathbf{b}} = (1,0,0,\dots)$ for each. The free ranks of $k\binom{n+1}{k+1}$ appearing in Section 2.3.18 come from $k = 1$, from choosing (only listing one element of each equivalence class here, to avoid overcounting):

$$\{\mathbf{b} = a_i + a_j \mid 1 \le i \le n, n+2 \le j \le 2n+2, j > i+n+1\}$$

gives

$$
\begin{aligned}
n + (n-1) + \cdots + 2 + 1 &= \frac{n(n+1)}{2} \\
&= \binom{n+1}{2}
\end{aligned}
$$

generators for $\tilde{H}_0(\Delta_{\mathbf{b}}, \Bbbk)$, each with rank 1. We can see that $\Delta_{\mathbf{b}}$ here is the disjoint union of two line intervals here by considering the first two columns of Φ.

Generators of $\tilde{H}_1(\Delta_{\mathbf{b}}, \Bbbk)$ live in degree 3. From the matrix Φ, augmented to include row labels as a_i:

$$
\left(\begin{array}{c|c|cccc|c}
1 & 0 & 1 & 0 & \cdots & 0 & a_1 \\ \hline
1 & 0 & 0 & 1 & \cdots & 0 & a_2 \\ \hline
\vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ \hline
1 & 0 & 0 & 0 & \cdots & 1 & a_n \\ \hline
1 & 0 & 0 & 0 & \cdots & 0 & a_{n+1} \\ \hline\hline
0 & 1 & 1 & 0 & \cdots & 0 & a_{n+2} \\ \hline
0 & 1 & 0 & 1 & \cdots & 0 & a_{n+3} \\ \hline
\vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\
0 & 1 & 0 & 0 & \cdots & 1 & a_{2n+1} \\
0 & 1 & 0 & 0 & \cdots & 0 & a_{2n+2}
\end{array}\right)
$$

We choose 3 elements from $\{1, 2, \ldots, n+1\}$. Now, homology generators for $\tilde{H}_1(\Delta_b, \Bbbk)$ come from $x_2 = 1, 2$. So for $i_1, i_2, i_3 \in \{1, 2, \ldots, n+1\}$, we have homology generators of the form

(i) $\mathbf{b} = a_{i_1} + a_{i_2} + a_{i_3+n+1}$ with $x_2 = 1$ and

(ii) $= a_{i_1} + a_{i_2+n+1} + a_{i_3+n+1}$ with $x_2 = 2$.

We can write (i) as

$$\begin{aligned} a_{i_1+n+1} + a_{i_2} + a_{i_3} &= a_{i_1} + a_{i_2+n+1} + a_{i_3} \\ &= a_{i_1} + a_{i_2} + a_{i_3+n+1} \end{aligned}$$

but not as

$$a_{i_1} + a_{i_2} + a_{i_3}$$

which contributes 1 free rank to $\tilde{H}_1(\Delta_{\mathbf{b}}, \Bbbk)$.

Similarly, for (ii) we have

$$\begin{aligned} a_{i_1+n+1} + a_{i_2+n+1} + a_{i_3} &= a_{i_1+n+1} + a_{i_2} + a_{i_3+n+1} \\ &= a_{i_1} + a_{i_2+n+1} + a_{i_3+n+1} \end{aligned}$$

satisfying $x_2 = 2$, which cannot be written as

$$a_{i_1+n+1} + a_{i_2+n+1} + a_{i_3+n+1}$$

since this satisfies $x_2 = 3$. Therefore, each element of (ii) contributes 1 free rank to $\tilde{H}_1(\Delta_{\mathbf{b}}, \Bbbk)$.

This case of $\mathbb{P}^2 \times \mathbb{P}^2$ is indicative of the homology generators for $\mathbb{P}^n \times \mathbb{P}^n$ in general. We have the following:

Claim: Homology generators of $\tilde{H}_{k-1}(\Delta_b, \Bbbk)$ for $\mathbb{P}^n \times \mathbb{P}^n$ come precisely from a subset of size $k+1$ from $\{1, 2, \ldots, n+1\}$, adjusted to be at height h where $1 \leq h \leq k$. These are values of $\mathbf{b}$ given by:

$$\left\{a_{i_1+n+1} + a_{i_2+n+1} + \cdots + a_{i_h+n+1} + a_{i_{h+1}} + a_{i_{h+2}} + \cdots a_{i_{k+1}} \mid 1 \leq h \leq k\right\}$$

An analogous proof to the case for $\mathbb{P}^2 \times \mathbb{P}^2$ shows that these are all distinct (for a given subset of size $k+1$, each distinct height $x_2 = h$ represents a distinct value for $\mathbf{b}$, and an analogous shuffle construction giving pairwise equality between the elements of the set

$$\left\{\sum_{j \in J} a_{i_j+n+1} + \sum_{j \in I \setminus J} a_{i_j} \mid J \subset I \text{ with } |J| = h\right\}$$

where I is the previous choice of $k+1$ elements in $\{1, 2, \ldots, n+1\}$ with no other ways to write this sum shows that each $\mathbf{b}$ contributes free rank 1 to $\tilde{H}_{k-1}(\Delta_{\mathbf{b}}, \Bbbk)$ for $k \in \{1, 2, \ldots, n\}$. To say more:

Claim: $\operatorname{rank}(\tilde{H}_{k-1}(\Delta_{\mathbf{b}}; \Bbbk)) = 1$ for

$$\mathbf{b} \in \left\{\sum_{i \in T} a_{i+n+1} + \sum_{i \in S \setminus T} a_i \;\middle|\; S \subset \{1, 2, \ldots, n+1\} \text{ with } |S| = k+1, T \subset S \text{ with } 1 \leq h = |T| \leq k\right\}.$$

Proof. Let $C = [0,1]^{k+1}$. C has 2^{k+1} vertices, since vertices come from either a 0 or a 1 in each component. If $h \in \{1, k\}$, then up to symmetry by set complement of $T \subset S$, it suffices to let $h = 1$. This gives that the set of $\mathbf{b}$ at $h = 1$ is

$$\left\{a_{j+n+1} + \sum_{i \in S \setminus \{j\}} a_i \;\middle|\; j \in S\right\}.$$

Let $\mathbf{b} = a_{i_{\alpha+n+1}} + \sum_{S \setminus i_\alpha} a_j$ for some $i_\alpha \in S$. Then $\Delta_{\mathbf{b}}$ has facets

$$\left\{\{a_i', \{a_j\}_{j \in S \setminus \{i_1\}}\} \;\middle|\; i_1 \in S\right\}$$

which retracts onto $\partial \Delta^k$, since there's a 1-1 correspondence between facets of $\Delta_{\mathbf{b}}$ and elements of S, with $|S| = k+1$ as the facets are

$$\{i_1', i_2, \ldots, i_{k+1}\}, \{i_1, i_2', \ldots, i_{k+1}\}, \ldots, \{i_1, i_2, \ldots, i_{k+1}'\}.$$

Since $\{i_1, i_2, \ldots, i_{k+1}\}$ is not a face of $\Delta_{\mathbf{b}}$, this shows that $\Delta_{\mathbf{b}}$ retracts onto $\partial\Delta^k$, the boundary of the standard k-simplex, so that $\operatorname{rank}(\tilde{H}_{k-1}(\Delta_{\mathbf{b}}; \Bbbk)) = 1$ in the case $h \in \{1, k\}$. Next, if $h = |T|$ satisfies $1 < h < k$, then for

$$\mathbf{b} \in \{\sum_T a_{j+n+1} + \sum_{S\backslash T} a_j \mid |T| = h\},$$

the facets of $\Delta_{\mathbf{b}}$ are given by

$$\left\{\{a_i'\}_{i\in T}, \{a_i\}_{i\in S\backslash T}\right\}.$$

Due to symmetry in T and $S\backslash T$, we can assume without loss of generality that $|T| \geq \frac{k+1}{2}$.

Let $C = \prod_{\alpha\in T}[0, e_\alpha]$ be the $k+1$-dimensional cube with half-space representation $\mathcal{H}rep$ from $\{e_i^\vee, 1 - e_i^\vee\}$:

$$C = \bigcap_{i\in S} \left(H^+_{e_i^\vee,0} \cap H^+_{-e_i^\vee,-1}\right).$$

C contains the hypersimplex

$$\Delta(h,k) = C \cap \left(\sum_{i\in S} e_i^\vee - h\right).$$

Facets of $\Delta_{\mathbf{b}}$ are

$$\left\{\{a_\alpha\}_{\alpha\in T}, \{a_\alpha\}_{\alpha\in S\backslash T}\right\}.$$

So there's a 1-1 correspondence:

$$\{\text{ Facets } F \text{ of } \Delta_{\mathbf{b}}\} \leftrightarrow \{T \subset S, |T| = h\}.$$

Consider the set map $\Psi : \{$ Facets of $\Delta_{\mathbf{b}}\} \to C$ given by $\Psi(F) = \sum_{\alpha \in T} e_\alpha$. Then $\mathrm{Im}\Psi \subset \Delta(h,k)$ and contains each vertex of $\Delta(h,k)$ since any subset $T \subset S$ with $|T| = h$ gives a way to write $\mathbf{b}$ as $\mathbf{b} = \sum_T a_i' + \sum_{S \setminus T} a_i$. That is, Ψ gives a bijection between facets of $\Delta_{\mathbf{b}}$ and vertices of $\Delta(h,k)$.

Now, the dual $\Delta(h,k)^\circ$ has a facet for each vertex of $\Delta(h,k)$. Combinatorially, $\Delta(h,k)^\circ$ is given by Conv($\{$inward pointing normals$\}$), so that facets of $\Delta(h,k)^\circ$ are given by the convex hull of all elements of a given set T.

That is, facets of $\Delta(h,k)^\circ$ are given by

$$\mathrm{Conv}\{v_\alpha : \alpha \in T\}.$$

Since $|T| \geq \frac{k+1}{2}$, each facet of Δ_b retracts onto the corresponding facet of $\Delta(h,k)^\circ$. Thus, $\Delta_{\mathbf{b}}$ retracts into $\partial\Delta(h,k)^\circ$, and surjectivity of the map Ψ shows that the retraction $\Delta_{\mathbf{b}} \hookrightarrow \partial\Delta(h,k)^\circ$ is onto.

This provides $k \cdot \binom{n+1}{k+1}$ values of $\mathbf{b}$ with free rank 1 of $\tilde{H}_{k-1}(\Delta_{\mathbf{b}}; \Bbbk)$. That is, distinct homology generators for $\tilde{H}_{k-1}(\Delta_{\mathbf{b}}; \Bbbk)$ come from

$$\mathbf{b} \in \left\{ \sum_{j=1}^{h} a_{i_j+n+1} + \sum_{j=h+1}^{k+1} a_{i_j} \mid 1 \leq h \leq k \right\}.$$

Since the free resolution of I_{X_Δ} computed by Macaulay2 is minimal and has free rank $k \cdot \binom{n+1}{k+1}$ for $1 \leq k \leq n$, these are indeed all homology generators for $\tilde{H}_{k-1}(\Delta_{\mathbf{b}}; \Bbbk)$. $\square$

3.3 Minimal resolution of unimodular Lawrence ideal J_L

For Σ a complete fan in the lattice $\mathbb{Z}^m$ and X_Σ the associated complete normal unimodular toric variety with $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\} \subset \Sigma(1)$ the primitive ray generators of Σ, and B the $n \times m$ matrix with row vectors $\mathbf{b}_i$, each $\mathbf{b}_i$ determines a torus-invariant Weil divisor D_i on X_i, and the group $Cl(X)$ of torus-invariant Weil divisors modulo linear equivalence is given by[12]

$$0 \to \mathbb{Z}^m \xrightarrow{B} \mathbb{Z}^n \xrightarrow{\pi} Cl(X) \to 0, \tag{3.1}$$

where π takes the i-th standard basis vector of $\mathbb{Z}^n$ to the linear equivalence class $[D_i]$ of the corresponding divisor D_i.

Definiton: The Lawrence ideal[12] J_L corresponding to L, a sublattice of $\mathbb{Z}^n$ is

$$J_L = \left\langle \mathbf{x}^{\mathbf{a}}\mathbf{y}^{\mathbf{b}} - \mathbf{x}^{\mathbf{b}}\mathbf{y}^{\mathbf{a}} \mid \mathbf{a} - \mathbf{b} \in L \right\rangle \subset S = k[x_1, \ldots, x_n, y_1, \ldots, y_n]$$

for k a field. Here $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ for $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$.

For X_Σ unimodular with homogeneous coordinate ring $R = \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)] \cong \mathbb{C}[x_1, \ldots, x_n]$, the toric variety $X_\Sigma \times X_\Sigma$ has homogeneous coordinate ring

$$S = R \otimes_k R = k[x_1, \ldots, x_n, y_1, \ldots, y_n].$$

The diagonal embedding $X_\Sigma \subset X_\Sigma \times X_\Sigma$ defines a closed subscheme, and is represented by a $Cl(X) \times Cl(X)$-graded ideal I_X in S. Here, $I_X = \ker(\psi)$ for $\psi : S = R \otimes_k R \to k[Cl(X)] \otimes R$ given by $\mathbf{x}^{\mathbf{u}}\mathbf{y}^{\mathbf{v}} = \mathbf{x}^{\mathbf{u}} \otimes \mathbf{x}^{\mathbf{v}} \mapsto [\mathbf{u}] \otimes \mathbf{x}^{\mathbf{u}+\mathbf{v}}$. That is,

$$I_X = \langle \mathbf{x}^{\mathbf{u}}\mathbf{y}^{\mathbf{v}} - \mathbf{x}^{\mathbf{v}}\mathbf{y}^{\mathbf{u}} \mid \pi(\mathbf{u}) = \pi(\mathbf{v}) \text{ in } Cl(X) \rangle \subset S.$$

Proof. First, we have the following

Lemma: Suppose $X = Spec(R)$ is an affine variety. Then the diagonal mapping $\Delta : V \to V \times V$ corresponds to the $\mathbb{C}$-algebra homomorphism $R \otimes R \to R$ given by $\sum r_1 \otimes r_2 \mapsto \sum r_1 r_2$, from the universal property of $X \times X$.

Next, locally, for $\Sigma \subset N_{\mathbb{R}}$, we have that the diagonal map $U_i \overset{\Delta}{\hookrightarrow} U_i \times U_i$ corresponds to

$$\mathbb{C}[\sigma_i^\vee \cap M] \otimes_k \mathbb{C}[\sigma_i^\vee \cap M] \to \mathbb{C}[\sigma_i^\vee \cap M] \text{ given by}$$
$$\sum r_1 \otimes r_2 \mapsto \sum r_1 r_2.$$

A choice of $\sigma \in \Sigma$ gives[3] the monomial

$$x^{\hat{\sigma}} = \prod_{\rho \notin \sigma(1)} x_\rho \in S$$

for S the homogeneous coordinate ring[38] of X. And the map

$$\chi^m \mapsto x^{<m>} = \prod_\rho x_\rho^{<m,u_\rho>}$$

induces the isomorphism

$$\pi_\sigma^* : \mathbb{C}[\sigma^\vee \cap M] \cong (S_{x^{\hat{\sigma}}})^G \text{ for } G \text{ corresponding to the GIT quotient of } \mathbb{A}_{ss}^{|\Sigma(1)|}$$
$$\cong (S_{x^{\hat{\sigma}}})_0$$

which are degree 0 in the $Cl(X)$-grading on S.

So locally, we have

$$(S_{x^{\hat{\sigma}}})_0 \otimes_k (S_{x^{\hat{\sigma}}})_0 \xrightarrow{\psi} (S_{x^{\hat{\sigma}}})_0 \text{ by}$$
$$\sum \mathbf{u}^{\mathbf{a}} \otimes_k \mathbf{u}^{\mathbf{b}} \mapsto \sum \mathbf{u}^{\mathbf{a}+\mathbf{b}}.$$

for $\mathbf{u}$ local coordinates in U_σ.

From

$$\begin{aligned} I_X &= \langle \mathbf{x}^{\mathbf{u}} \otimes \mathbf{x}^{\mathbf{v}} - \mathbf{x}^{\mathbf{v}} \otimes \mathbf{x}^{\mathbf{u}} \mid \pi(\mathbf{u}) = \pi(\mathbf{v}) \in Cl(X) \rangle \\ &= \ker(\psi) \end{aligned}$$

for $\psi : R \otimes_k R \to k[Cl(X)] \otimes_k R$ via $\mathbf{x}^{\mathbf{u}} \otimes \mathbf{x}^{\mathbf{v}} \mapsto [\mathbf{u}] \otimes \mathbf{x}^{\mathbf{u}+\mathbf{v}}$, we have that locally:

$$(I_\Delta)_{x^{\hat{\sigma}} \otimes_k x^{\hat{\sigma}}} = \ker(\mathbf{x}^{\mathbf{u}} \otimes_k \mathbf{x}^{\mathbf{v}} \mapsto [\mathbf{u}] \otimes_k \mathbf{x}^{\mathbf{u}+\mathbf{v}}) \subset R_{x^{\hat{\sigma}}} \otimes_k R_{x^{\hat{\sigma}}}$$

Now

$$R_{x^{\hat{\sigma}}} \cong k[x_1, \dots, x_r, w_1^{\pm}, \dots, w_\ell^{\pm}]$$

from getting powers of variables to cancel when localizing by $x^{\hat{\sigma}}$ for some r and ℓ, and

$$(R_{x^{\hat{\sigma}}})_0 = k[\frac{x_1}{\mathbf{w}^{\alpha_1}}, \dots, \frac{x_r}{\mathbf{w}^{\alpha_r}}]$$

from X unimodular implies that for each $b_k \in \sigma(1)$ there exists $m_k \in M$ such that $m_k \cdot b_i = \delta_i^k$ for all $b_k \in \sigma(1)$. Taking degree zero and considering the kernel gives

$$(I_\Delta)_{x^{\hat{\sigma}}\otimes_k x^{\hat{\sigma}}} = \left\langle \left\{ \frac{x_i}{w^{\alpha_i}} \otimes 1 - 1 \otimes \frac{x_i}{w^{\alpha_i}} \right\}_{i=1}^{r} \right\rangle$$

which we'll write as

$$\langle \{x^u - y^u\} \rangle$$

on affine charts. □

Now[12], the ideal $I_X \subset S$ defining the diagonal embedding $X \subset X \times X$ equals the Lawrence ideal J_L for the lattice $L = \mathrm{Im}(B) = \ker(\pi)$ of principal divisors from the sequence at the beginning of this section 3.3. This latice L is unimodular precisely when X_Σ is unimodular, and since the Lawrence lifting

$$\Lambda(L) = \{(u, -u) \mid u \in L\}$$

is isomorphic to L, this implies that $\Lambda(L)$ is unimodular as well[12].

Next, we'll require the following definitions:

Definition: Given a field k, we consider the Laurent polynomial ring $T = k[x_1^{\pm}, \ldots, x_n^{\pm}]$ as a module over the polynomial ring $S = k[x_1, \ldots, x_n]$. The module structure comes[37] from the natural inclusion of semigroup algebras $S = k[\mathbb{N}^n] \subset k[\mathbb{Z}^n] = T$.

Definition: A monomial module is an S-submodule of T which is generated by monomials $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}, \mathbf{a} \in \mathbb{Z}^n$.

Definition: When the unique generating set[37] of monomials of a monomial module M

forms a group under multiplication, M coincides with the lattice module

$$M_L := S\{\mathbf{x}^{\mathbf{a}} \mid \mathbf{a} \in L\} = k\{\mathbf{x}^{\mathbf{b}} \mid \mathbf{b} \in \mathbb{N}^n + L\} \subset T$$

for some lattice $L \subset \mathbb{Z}^n$ whose intersection with $\mathbb{N}^n$ is the origin $\mathbf{0} = (0, 0, \ldots, 0)$.

Definition: The $\mathbb{Z}^n/L$ graded lattice ideal I_L is[36]

$$I_L = \left\langle \mathbf{x}^{\mathbf{a}} - \mathbf{x}^{\mathbf{b}} \mid \mathbf{a} - \mathbf{b} \in L \right\rangle \subset S.$$

Claim: $I_{\Lambda(L)} = J_L$.

Proof.

$$\begin{aligned}
I_{\Lambda(L)} &= \left\langle \mathbb{x}^{\mathbf{a}} - \mathbb{x}^{\mathbf{b}} \mid \mathbf{a} - \mathbf{b} \in \Lambda(L) \right\rangle \\
&= \left\langle \mathbf{x}^{\mathbf{a}_1}\mathbf{y}^{\mathbf{a}_2} - \mathbf{x}^{\mathbf{b}_1}\mathbf{y}^{\mathbf{b}_2} \mid \mathbf{a} = (\mathbf{a}_1, \mathbf{a}_2), \mathbf{b} = (\mathbf{b}_1, \mathbf{b}_2), \mathbf{a} - \mathbf{b} \in \Lambda(L) \right\rangle \\
&= \left\langle \mathbf{x}^{\mathbf{a}_1}\mathbf{y}^{\mathbf{a}_2} - \mathbf{x}^{\mathbf{b}_1}\mathbf{y}^{\mathbf{b}_2} \mid (\mathbf{a}_1 - \mathbf{b}_1, \mathbf{a}_2 - \mathbf{b}_2) = (\mathbf{a}_1, \mathbf{a}_2) - (\mathbf{b}_1, \mathbf{b}_2) \in \{(\mathbf{u}, -\mathbf{u}) \mid \mathbf{u} \in L\} \right\rangle
\end{aligned}$$

where the rightmost constraint implies

$$\mathbf{a}_2 - \mathbf{b}_2 = -(\mathbf{a}_1 - \mathbf{b}_1) = \mathbf{b}_1 - \mathbf{a}_1 \text{ and } \mathbf{a}_1 - \mathbf{b}_1 \in L$$

gives

$$= \langle \mathbf{x}^{\mathbf{u}}\mathbf{y}^{\mathbf{v}} - \mathbf{x}^{\mathbf{v}}\mathbf{y}^{\mathbf{u}} \mid \mathbf{u} - \mathbf{v} \in L \rangle = J_L$$

□

While the hull resolution of the unimodular Lawrence ideal $I_{\Lambda(L)} = J_L$ is not necessarily minimal, the minimal resolution of $I_{\Lambda(L)}$ does come from a cellular resolution of $M_{\Lambda(L)}$ and is described by a combinatorial construction given by the infinite hyperplane arrangement [12] $\mathcal{H}_L$. Here, $\mathcal{H}_L$ comes from intersecting $\mathbb{R}L$, the real span of the lattice $L = \ker(\pi) \subset \mathbb{Z}^n$, with all lattice translates of the coordinate hyperplanes $\{x_i = j\}, 1 \leq i \leq n, j \in \mathbb{Z}$. Bayer-Popescu-Sturmfels[12] prove that each lattice point in L is a vertex of the affine hyperplane arrangement $\mathcal{H}_L$, and that there are no additional vertices in $\mathcal{H}_L$ iff L is unimodular. Furthermore, the quotient complex $\mathcal{H}_L/L$ of the hyperplane arrangement modulo L supports[12] the minimal S-free resolution of J_L, precisely because $\mathbb{R}L \cap \mathcal{H}_L = L$.

Example of cellular resolution of $\mathcal{H}_L/L$ for $\mathbb{P}^2$

For the unimodular toric variety $X_\Sigma = \mathbb{P}^2$, we have the fundamental toric exact sequence defining $Cl(X_\Sigma)$ via

$$0 \to \mathbb{Z}^2 \xrightarrow{B} \mathbb{Z}^3 \xrightarrow{\pi} \mathbb{Z} \to 0$$

with

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}$$

and $B(a, b)$ given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \\ -a-b \end{bmatrix}$$

so that $\ker(\pi) = \mathrm{Im}(B)$ contains

$$\{e_1 - e_2, e_1 - e_3, e_2 - e_3\}$$

but can be generated by

$$\{e_1 - e_2, e_2 - e_3\}\,.$$

Now $L = \mathbb{Z} \cdot \left\langle \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\rangle$ is rank 2.

If we label $\mathcal{H}_L$ with elements of the monomial module $M_{\Lambda(L)}$ and build the corresponding quotient cellular complex on $\mathcal{H}_L/L$ labeled with one up triangle and one choice of down triangle, subject to the orientation below, we see[12]:

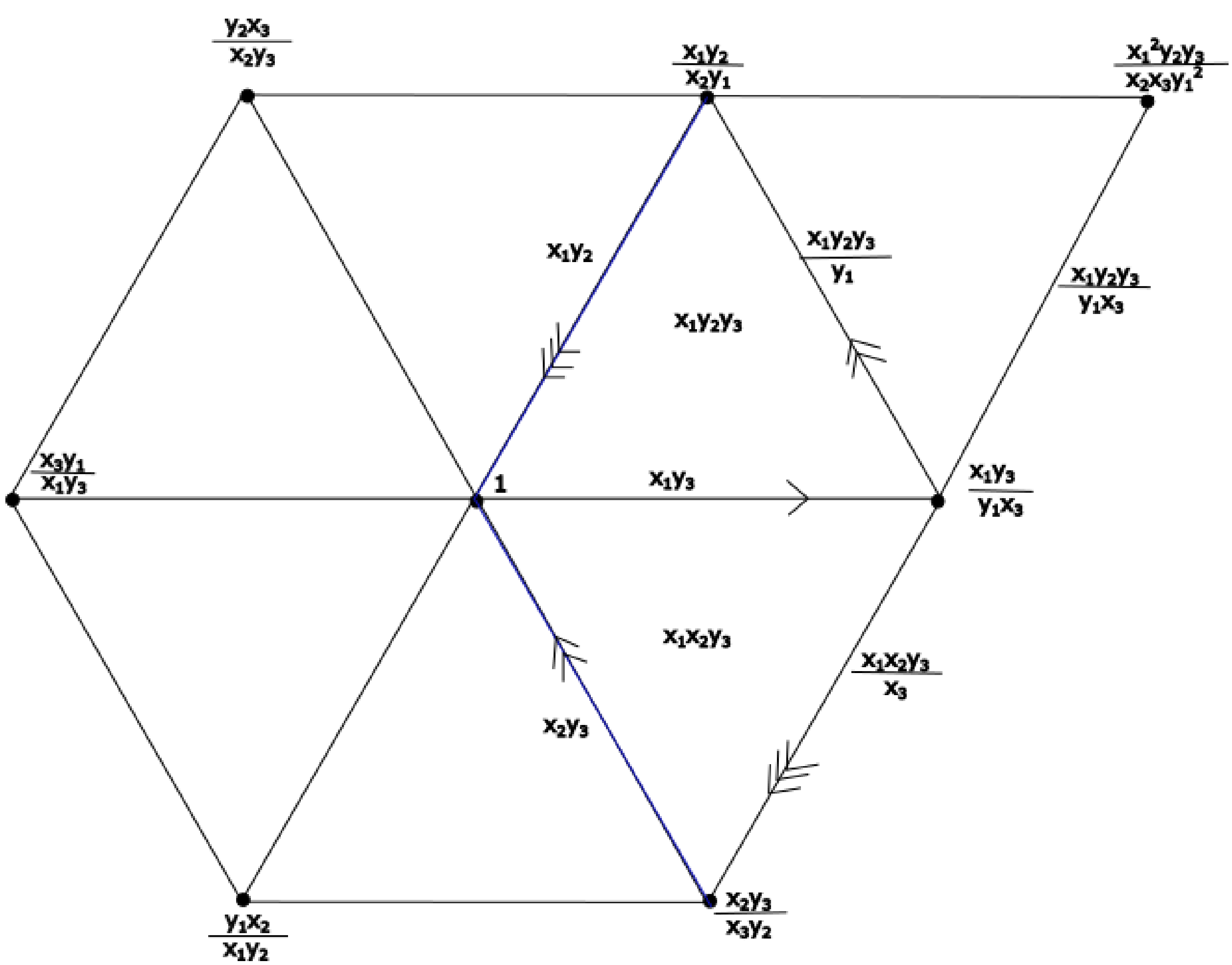

Figure 3.1: *Cellular Complex $\mathcal{H}_L/L$ for $\mathbb{P}^2$*

Here, the edges corresponding to the basis vectors $e_1 - e_2$ and $e_2 - e_3$ are labeled in blue, the edge ordering is given by the number of arrows (1-3), and the upward-pointing face of $\mathcal{H}_L/L$ is oriented counterclockwise while the downward-facing face is oriented clockwise in Figure 3.1. We order the 2-cells via {"up-triangle," "down-triangle"}. The cellular complex $\mathcal{H}_L/L$ has one vertex, three edges, and two 2-cells. This choice of fundamental domain and face ordering gives the S-free resolution[12] $(\mathbf{F}^{\bullet}_{\mathcal{H}_L/L}, \partial)$ resolution of S/J_L:

$$
\begin{array}{ccccccccc}
0 & \to & S^2 & \xrightarrow{\begin{pmatrix} y_2 & x_2 \\ y_1 & x_1 \\ y_3 & x_3 \end{pmatrix}} & S^3 & \xrightarrow{\begin{pmatrix} y_1x_3 - x_1y_3 \\ x_2y_3 - x_3y_2 \\ x_1y_2 - x_2y_1 \end{pmatrix}} & S^1 & \to & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & S/J_L & \to & 0
\end{array}
\tag{3.2}
$$

where

$$
\partial(F) = \sum_{\substack{F' \subset F, \\ codim(F', F) = 1}} \epsilon(F, F') \frac{m_F}{m_{F'}} \cdot F'
$$

for m_F the monomial label of the face F and $\epsilon(F, F') \in \{\pm 1\}$ with $\epsilon(F, F') = 1$ if the orientation of F' agrees with that induced by F. Here, m_F is the lowest common multiple of $m_{F'}$ over the facets of F, starting with a labeling of the vertices.

That is, the cellular resolution of S/J_L coming from $M_{\Lambda(L)}$ in Equation 3.2 resolves the ideal $I_{\mathbb{P}^2}$ of the diagonal $\mathbb{P}^2 \subset \mathbb{P}^2 \times \mathbb{P}^2$.

Discussion

Here, the cellular resolution for $M_{\Lambda(L)}$ works precisely because the vertices of a unimodular lattice are exactly the vertices of $\mathcal{H}_L$. In order to resolve the diagonal for simplicial toric varieties, we require the Morita theory of Section 3.5.5.

We generalize the Cellular Resolution of $\mathcal{H}_L/L$ for a simplicial toric variety in Section 3.5.7.

3.4 Discussion of combinatorial approach for differential forms on simplicial toric stack

Given the generalized Euler sequence from page 64, here we recast the previous combinatorial approach as follows: We have that $\Omega(D_i) = \Omega \otimes \mathcal{O}(D_i)$. For $A \subset N$ a $\mathbb{Z}-$linear spanning set, we have

$$0 \to L_A \xrightarrow{\delta} \mathbb{Z}^A \xrightarrow{\gamma} N \to 0$$

and dual sequence

$$0 \to M \xrightarrow{\gamma^\vee} \mathbb{Z}^A \xrightarrow{\delta^\vee} L_A^\vee \to 0$$

Let $R = \mathbb{C}[x_a \mid a \in A]$ graded by $L_A^\vee$ with $|x_a| = \delta^\vee(e_a^\vee)$. Let $\alpha : \bigoplus_{a\in A} R(-|x_a|) \to R \otimes L_A^\vee \to 0$ where $\bigoplus_{a\in A} R(-|x_a|) \ni u_a \mapsto \alpha(u_a) = x_a \otimes |x_a|$. Let $f_{v,a} = \sum_{\tilde{a}\in A(v),\tilde{a}\neq a} e_{\tilde{a}} \in \mathbb{Z}^A$.

Define

$$z_v := \sum_{a\in A(v)_+} c_a x^{f_{v,a}} u_a - \sum_{a\in A(v)_-} c_a x^{f_{v,a}} u_a.$$

Then $\ker(\alpha) = \langle z_v \mid v \in M \rangle = \left\langle z_v : v \text{ is a ray generator of fan } \{C_{A_-,A_+}\} \right\rangle$.

This implies the graded module corresponding to $\Omega_X(D_i)$ is generated by

$$\{z_v \ : v \text{ is a ray generator of fan } \{C_{A_-,A_+}\}\}$$

with a grading shift given by $|\tilde{x_a}| = |x_a| - \delta^\vee(e_i^\vee)$.

If we naïvely attempt to mimic Beilinson's resolution of the diagonal for a general toric variety by taking a direct sum of the maps

$$p^*\mathcal{O}_X(-D_i) \otimes q^*\Omega_Y(D_i) \to \mathcal{O}_{X\times Y}$$

then the cokernel is no longer $\Delta \subset X_\Sigma \times X_\Sigma$. Since the direct sum of resolutions is the resolution of the direct sum, we do not resolve the diagonal for general X_Σ in this case. Explicitly, a map

$$\Omega_X(D_i) \boxtimes \mathcal{O}_Y(-D_i) \to \mathcal{O}_{X\times Y}$$

is given by a map of corresponding graded modules via the toric generalization of Serre's theorem giving an equivalence between coherent sheaves on X_Σ and graded modules over the homogeneous coordinate ring, modulo torsion.

The isomorphism modulo torsion of $\ker \alpha \to \Omega_{X_\Sigma}$ away from the unstable locus of $\mathbb{A}^{|\Sigma(1)|}$ is given by

$$\begin{aligned} F(z_v) &= \sum_{a\in A(v)} c_a x^{f_{v,a}} dx_a \\ &= \sum_{a\in A(v)} c_a x^{f_{v,a}} (1 \otimes x_a - x_a \otimes 1) \end{aligned}$$

So we look for a map

$$\left\langle \sum_{a\in A(v)} c_a x^{f_{v,a}} (1\otimes_k x_a - x_a \otimes_k 1) \right\rangle_{D_i \text{ in } \{x_a\}} \boxtimes \langle 1 \rangle_{-D_i \text{ in } \{y_a\}} \to \mathbb{C}[x_a, y_a : a \in A]$$
$$\cong \left\langle \sum_{a\in A(v)} c_a x^{f_{v,a}} dxa \right\rangle \otimes R(|x_{a_i}|) \otimes R(-|y_{a_i}|) \to \mathbb{C}[x_a, y_a : a \in A]$$

where the subscripts above denote grading shifts. Dually, if we look for sections of $\mathcal{T}_{X\times Y}$ of degree $-D_i$, we do not in general have as many sections as there are for $\mathbb{P}^n$. Thus, in general we do not have a non-trivial map

$$\left\langle \sum_{a\in A(v)} c_a x^{f_{v,a}} (1\otimes_k x_a - x_a \otimes_k 1) \right\rangle \otimes R(|x_{a_i}|) \otimes R(-|y_{a_i}|) \to \mathbb{C}[x_a, y_a : a \in A]$$

whose cokernel gives I_Δ.

3.5 Resolution of diagonal in bimodules

Let A be an algebra over R, a commutative algebra over k a field. A is not necessarily commutative. We consider $I_\Delta \leq A \otimes_k A$ as a submodule given by $\ker \mu : A \otimes_k A \to A$ of multiplication. Here, $A \otimes_k A \geq A$ contains an isomorphic copy of A. We consider $A \otimes_k A$ as an $A \otimes_k A^{op}$-bimodule, not as an algebra. Here,

$$A_\Delta = {}^{A\otimes_k A}\!\big/_{I_\Delta}$$

To consider derivations and differentials of elements in A, consider the map given by

$$\alpha \in A \mapsto \tilde{d}\alpha = \alpha \otimes 1 - 1 \otimes \alpha \in I_\Delta$$

We do not consider I/I^2 here.

Claim: $\ker \mu = I_\Delta \cong \left\langle \tilde{d}\alpha \mid \alpha \in A \right\rangle$ as $A \otimes A^{op}$-module.

Proof. ($\Rightarrow$): We construct $g : \ker \mu \to \left\langle \tilde{d}\alpha : \alpha \in A \right\rangle$ via $g\left(\sum_i z_i \otimes_k w_i\right) = \sum_i z_i \tilde{d} w_i$ as follows:

$$\sum_i z_i \otimes w_i \in \ker \mu \implies \sum_i z_i w_i = 0 \text{ so that}$$
$$\sum_i z_i \otimes w_i = \sum_i z_i(1 \otimes w_i - w_i \otimes 1) \in \left\langle \tilde{d}\alpha \mid \alpha \in A \right\rangle.$$

($\Leftarrow$): We clearly have that $\tilde{d}\alpha = 1 \otimes_k \alpha - \alpha \otimes_k 1 \in \ker \mu$. $\square$

Relative to R, if we suppose that A is finitely generated as an $R-$module over R by $\{f_1, \ldots, f_r\}$ then

$$\left\langle \tilde{d}a \mid a \in A \right\rangle \cong \left\langle \tilde{d}r \mid r \in R \right\rangle \otimes \{f_1, \ldots, f_r\}$$

so that this gives a finite set of generators for I_Δ.

Additionally, we observe that the Leibniz rule $\tilde{d}(\alpha\beta) = \tilde{d}(\alpha)\beta + \alpha\tilde{d}(\beta)$ holds here.

3.5.1 For $[\mathbb{A}^1/\mu_2]$

Now, for $\chi = \mathbb{A}^1/\mu_2$, we have that $\langle \mathcal{O}, \mathcal{O}(1) \rangle = D^b(\chi)$ since any graded module admits a direct sum decomposition over characters, and a direct sum of resolutions of submodules is a resolution of the direct sum of submodules. Similarly, $D^b([\mathbb{A}^1/\mu_n]) = \langle \mathcal{O}, \ldots, \mathcal{O}(n-1) \rangle$.

Here, $G = \mathcal{O} \oplus \mathcal{O}(1)$ is a generator for $D^b(\mathbb{A}^1/\mu_2)$ which is isomorphic to just $D^b(A-\text{ mod })$ (with no consideration of the grading) since we have a DG algebra concentrated in degree 0. G is projective, so just gives a degree 0 endomorphism algebra. So this DG endomorphism algebra

$$End^*(G) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a, d \in k[x]^{\text{even}} \text{ and } b, c \in k[x]^{\text{odd}} \right\}$$

admits a direct sum over $Hom(G, \mathcal{O}) \oplus Hom(G, \mathcal{O}(1))$ over the columns. Here, $k[x]^{\text{even}} = k[x^2]$ and $k[x]^{\text{odd}} = xk[x^2]$. Here, in the category of $\mathbb{Z}/2$-graded modules over $k[x]$ (graded by the characters of μ_2), we have an abelian category containing, for instance, $k[x]$ with 1 in degree 0 and another given by $k[x]$ with $|1| = \text{odd}$. (Here, $\mathcal{O} = k[x]$ corresponding to $\mathbb{A}^1$.)

On $\mathbb{P}^1$, there are no projective coherent sheaves, but for $\mathbb{P}^1$, we have torsion. Here, there is no torsion because we are working with affine stacks, so there are projective objects. Now, we fix $A = \text{End}^*(G), R = k[x]^{\mu_2} = k[x]^{\text{even}}$, the ring of invariants.

Here, $d\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)$ and $d\left(\begin{bmatrix} 0 & x \\ x & 0 \end{bmatrix}\right)$ generate I_Δ.

Note: $\tilde{d}(Id) = 0 = \tilde{d}1 = 0$ by construction.

By Lefevre-Hasegawa[39], A_Δ is cyclic in the following diagram, generated by $[Id \otimes_k Id] = 1 \in A$:

$$\begin{array}{ccccccc} & & & & I_\Delta & & \\ & & & \nearrow & \downarrow & & \\ \ldots \to & (A \otimes_k A)^2 & & \to & A \otimes A & \to & 0 \\ & & & & \downarrow & & \downarrow \\ & & & & A_\Delta & \to & 0 \end{array}$$

Notation: Let $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in A$. Then we have $e^2 = e$. For notation, let $E := e \otimes (1 - e) - (1 - e) \otimes e$. Now

$$
\begin{aligned}
\tilde{d}e &= e \otimes 1 - 1 \otimes e \\
&= e \otimes (1-e) - (1-e) \otimes e.
\end{aligned}
$$

Now,

$$
\begin{aligned}
(e \otimes_k A) \cdot (e \otimes (1-e) - (1-e) \otimes e) &= e \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} A \\
&= e \otimes \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}
\end{aligned}
$$

for $c \in k[x]^{\text{odd}}, d \in k[x]^{\text{even}}$, and

$$
\begin{aligned}
(A \otimes e) \cdot (e \otimes (1-e) - (1-e) \otimes e) &= -A \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes e \\
&\cong \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} \otimes e
\end{aligned}
$$

So we have projectors

$$
(1-e) \cdot A := Q_{bottom} = (e \otimes A) \cdot E = \left\langle e \otimes \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} \right\rangle \subset \langle E \rangle = \left\langle \tilde{d}e \right\rangle
$$

and

$$A \cdot e = Pe = (A \otimes e) \cdot E = \left\langle \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} \otimes e \right\rangle \subset \langle E \rangle = \left\langle \tilde{d}e \right\rangle$$

so that $< E >= \ell \otimes b \oplus r \otimes t$ (informally, "left tensor bottom plus right tensor top"). We'll refer to these as subspaces $P_i, Q_j \subset A$ where P_i comes from considering A as a left A-module, hence P_i consists of possibly nonzero elements in column 1. Similarly, Q_j as a vector subspace of A considered as a right A-module gives that Q_j has possibly nonzero entries only in row j.

Since P_i and Q_j are projective left- and right-submodules of A, respectively, if we tensor these together then $P_i \otimes Q_j$ is a projective $A \otimes A^{op}$-bimodule. We have that $A \cong \Delta \subset A \otimes A$, so to start the resolution we have that $A \otimes A \xrightarrow{\mu} A$ where A is unital so that $\ker = \Delta$. Since μ corresponds to matrix multiplication,

$$P_1 \otimes Q_2, P_2 \otimes Q_1 \subset \ker \mu = I_\Delta$$

but since these are projective subspaces of $A \otimes A^{op}$, we can split them off from $\ker(\mu)$. Based on whether we consider A as a left- or right- module over itself, we can say

$$A \cong \bigoplus P_i \cong \bigoplus Q_j.$$

Since $\left\langle \tilde{d}e \right\rangle$ gives a projective part of the kernel of multiplication, we can split this off.

To find the equivalence class of $[\tilde{d} \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix}] \in {}^{I_\Delta}\!/\!_{\left\langle \tilde{d}e \right\rangle}$, we note that

$$\tilde{d} \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} = \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} \otimes 1 - 1 \otimes \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \dots$$

$$\dots - \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \right)$$

where the first and fourth summands in the first line, and the second and third summands in the second line are in $\left\langle \tilde{d}e \right\rangle$. Therefore we are left with $[\tilde{d} \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix}] \in {}^{I_\Delta}\!/\!_{\left\langle \tilde{d}e \right\rangle}$ as given by

$$[\tilde{d} \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix}] = \left(\begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \right) + \dots$$

$$\dots + \left(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \right)$$

where

$$u := \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$$

and

$$v := \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}$$

where $u \in \ell \otimes t$ and $v \in r \otimes b$.

To find the kernel of the quotient map $\pi : I_\Delta \to {}^{I_\Delta}\!/\!_{\left\langle \tilde{d}e \right\rangle}$ which sends $\sum_i M_i \otimes N_i \mapsto [\sum_i M_i \otimes N_i]$, we consider $Ann_{A \otimes A^{op}}(u), Ann_{A \otimes A^{op}}(v)$ separately. Then, we note that since $u \in \ell \otimes t$ and $v \in r \otimes b$,

$$Ann_{A\otimes A^{op}}([\tilde{d}\begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix}]) \subset (Ann_{A\otimes A^{op}}(u) \cap Ann_{A\otimes A^{op}}(v))$$

as a set-theoretic intersection.

(i) If we first consider $Ann_{A\otimes A^{op}}(u)$, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \left(\begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \right) \cdot \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} = 0$$

$$\implies \begin{pmatrix} bx & 0 \\ dx & 0 \end{pmatrix} \otimes \begin{pmatrix} \tilde{a} & \tilde{b} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \otimes_k \begin{pmatrix} \tilde{c}x & \tilde{d}x \\ 0 & 0 \end{pmatrix}$$

Here, if we compare tensor product of matrices, we're left with four equations:

$$bx \otimes_k \tilde{a} = a \otimes \tilde{c}x \tag{3.3}$$

$$bx \otimes_k \tilde{b} = a \otimes \tilde{d}x \tag{3.4}$$

$$dx \otimes \tilde{a} = c \otimes \tilde{c}x \tag{3.5}$$

$$dx \otimes \tilde{b} = c \otimes \tilde{d}x \tag{3.6}$$

Since $a, b, c, d, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in k[x]$ and the tensors are over a field k, Equation (1) implies that there exists $\lambda \in k^*$ such that $\lambda bx = a, \lambda^{-1}\tilde{a} = \tilde{c}x$. Since $k[x]$ is an integral domain and UFD, we have that $x \mid b$ and that $a \neq 0 \iff b \neq 0$ since x is not a zero divisor in $k[x]$. Similarly for equations (2)-(4).

(ii) To consider $Ann_{A\otimes A^{op}}(v)$, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \left(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \right) \cdot \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} = 0$$

$$\implies \begin{pmatrix} 0 & ax \\ 0 & cx \end{pmatrix} \otimes_k \begin{pmatrix} 0 & 0 \\ \tilde{c} & \tilde{d} \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} \otimes_k \begin{pmatrix} 0 & 0 \\ \tilde{a}x & \tilde{b}x \end{pmatrix}$$

Again, we have four equations:

$$ax \otimes \tilde{c} = b \otimes \tilde{a}x \tag{3.7}$$

$$cx \otimes \tilde{c} = d \otimes \tilde{a}x \tag{3.8}$$

$$ax \otimes \tilde{d} = b \otimes \tilde{b}x \tag{3.9}$$

$$cx \otimes \tilde{d} = d \otimes \tilde{b}x \tag{3.10}$$

Here, (5) implies that there exists $\lambda \in k^*$ such that $\lambda ax = b$, $\lambda^{-1}\tilde{c} = \tilde{a}x$. Since x is not a zero-divisor in the integral domain and UFD $k[x]$ when k is a field, $a \neq 0 \iff b \neq 0$. Similarly for equations (6)-(8).

Now combining equations (1) and (5) gives that $a \neq 0 \iff b \neq 0$ and if either $a \neq 0$ or $b \neq 0$, then (1) gives $\deg(a) > \deg(b)$ while (5) implies $\deg(b) > \deg(a)$. Therefore, $a = b = 0$. Similarly for the relation between $\tilde{a}$ and $\tilde{c}$ from comparing (1) and (5).
Combining equations (2) and (7) yields that $\tilde{b} = \tilde{d} = 0$.
Combining equations (3) and (6) yields that $c = d = 0$. These also show $\tilde{a} = \tilde{c} = 0$.
Combining equations (4) and (8) also shows that $\tilde{b} = \tilde{d} = 0$.
Together, these show that $Ann_{A\otimes A^{op}}(u+v) = Ann_{A\otimes A^{op}}([\tilde{d}\begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix}]) = 0$. That is,

$[\tilde{d}\begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix}]$ is torsion-free in ${}^{I_\Delta}/_{\langle \tilde{d}e \rangle}$. Thus, we have the 2-term resolution of I_Δ for $[{}^{\mathbb{A}^1}/_{\mu_2}]$:

$$
\begin{array}{ccccccc}
 & & & & I_\Delta & & \\
 & & & \nearrow & \downarrow & & \\
0 \to & A \otimes_k A & \to & & A \otimes A & \to & 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & A_\Delta & \to & 0
\end{array}
$$

where the map $A \otimes A \to I_\Delta$ is given by $\tilde{d}\mathbf{X}$. Here $I_\Delta = \left\langle \tilde{d}e \right\rangle \oplus \left\langle [\tilde{d}\begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix}] \right\rangle$ where again, the second summand is taken in ${}^{I_\Delta}/_{\left\langle \tilde{d}e \right\rangle}$. Writing P_1 for left ("ℓ"), Q_2 for bottom ("b"), P_2 for right ("r"), and Q_1 for top ("t"), we note that $\langle E \rangle = P_1 \otimes Q_2 \oplus P_2 \otimes Q_1$. This gives, for $[{}^{\mathbb{A}^1}/_{\mu_2}]$,

$$
\begin{aligned}
I_\Delta &= \left\langle \tilde{d}e \right\rangle \oplus \langle u + v \rangle \\
&= \langle P_1 \otimes Q_2 \oplus P_2 \otimes Q_1 \rangle \oplus \langle u + v \rangle
\end{aligned}
$$

where $u \in P_1 \otimes Q_1$, $v \in P_2 \otimes Q_2$.

3.5.2 For $[{}^{\mathbb{A}^1}/_{\mu_n}]$

Recall that we have $D^b(\mathbb{A}^1/\mu_n) = \langle \mathcal{O}, \ldots, \mathcal{O}(n-1) \rangle$ so that we have
$G = \bigoplus_{\chi(g): g \in \mu_n} \mathcal{O}(\chi(g))$ and $A = End^*(G)$ a dg-algebra concentrated in degree 0 given by

$$
A = \left\{ M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \in \mathrm{Mat}_n(k[x]) \;\middle|\; a_{ij} \in \chi^{(i-j)}(G) = x^{i-j} \cdot k[x^n] \right\}
$$

from considering $A = \bigoplus_{g \in \mu_n} Hom(G_{\chi(g)}, G)$ and columns as the image of the j-th basis vector for G. Here, $i - j$ in $\chi^{(i-j)}$ is taken mod n. Note that
$Hom(G_i, G_j) = \langle x^{j-i} \cdot e_{ji} \rangle \in A = \bigoplus_{i,j \in \chi(\mu_n)} Hom(G_i, G_j)$ where $e_{ij} \in \mathrm{Mat}_n(k[x])$ is given by

$$(e_{ij})_{k\ell} = \begin{cases} 1 & k = i, \ell = j \\ 0 & \text{else.} \end{cases}$$

For general n, let

$$\mathbf{X} := \begin{pmatrix} 0 & 0 & \cdots & & x \\ x & 0 & \cdots & & 0 \\ 0 & x & \cdots & & 0 \\ 0 & 0 & \cdots & & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & x & 0 \end{pmatrix} \in A$$

Here, $R := \{r \cdot Id \mid r \in k[x^n]\}$ is a central subalgebra of A.

Claim: $\{\tilde{d}r \mid r \in R\} \subset \left\langle \tilde{d}\mathbf{X} \right\rangle$.

Proof. By linearity of $\tilde{d}$, it suffices to show that $\tilde{d}(x^n) \in \left\langle \tilde{d}\mathbf{X} \right\rangle$. But

$$\begin{aligned} \tilde{d}x^n &= \tilde{d}(\mathbf{X}^n) \\ &= \sum_{i=1}^{n} \mathbf{X}^{i-1} \cdot \tilde{d}\mathbf{X} \cdot \mathbf{X}^{n-i} && \text{[Leibniz Rule]} \\ &= \sum_{i=1}^{n} (\mathbf{X}^{i-1} \otimes 1) \cdot \tilde{d}\mathbf{X} \cdot (1 \otimes \mathbf{X}^{n-i}) \in \left\langle \tilde{d}\mathbf{X} \right\rangle \end{aligned}$$

□

From

$$\tilde{d}e_{ii} = e_{ii} \otimes 1 - 1 \otimes e_{ii} = e_{ii} \otimes (1 - e_{ii}) - (1 - e_{ii}) \otimes e_{ii} := E_{ii}$$

we have

$$(e_{ii} \otimes \tilde{A}) \cdot (E_{ii}) = e_{ii} \otimes \begin{pmatrix} \tilde{a}_{11} & 0 & 0 & \dots & 0 \\ 0 & \tilde{a}_{22} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \dots & \hat{\tilde{a}}_{ii} \dots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & & \cdots & \tilde{a}_{nn} \end{pmatrix}$$

with span

$$\begin{aligned} \langle e_{ii} \otimes (1 - e_{ii}) \rangle &= \left\langle e_{ii} \otimes \begin{pmatrix} \tilde{a}_{11} & 0 & 0 & \dots & 0 \\ 0 & \tilde{a}_{22} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \dots & \hat{\tilde{a}}_{ii} \dots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & & \cdots & \tilde{a}_{nn} \end{pmatrix} \right\rangle \\ &= \left\langle e_{ii} \otimes \sum_{j \neq i} \tilde{a}_{jj} e_{jj} \right\rangle \\ &= \langle \{P_i \otimes Q_j\}_{j \neq i} \rangle . \end{aligned}$$

so that $\langle E_{ii} \rangle \supset \langle \{P_i \otimes Q_j\}_{j \neq i} \rangle$ for a given value of i and

$\left\langle \{E_{ii}\}_{i=1}^{n-1} \right\rangle \supset \langle \{P_i \otimes Q_j\}_{1\leq i\leq n-1, j\neq i}\rangle$. Note: $\tilde{d}(Id) = \tilde{d}(1) = 0$ by definition, so we only need to take E_{ii} for $1 \leq 0 \leq n-2$.

To give a finite generating set for A over R as an R-algebra, we again consider:
$Hom(G_i, G_j) = \langle x^{j-i} \cdot e_{ji}\rangle$.
To generate I_Δ: we note that $I_\Delta = \left\langle \tilde{d}\alpha : \alpha \in A \right\rangle$. By the above, we have

$$\begin{aligned}\left\langle \tilde{d}\alpha \mid \alpha \in A \right\rangle &= \left\langle \tilde{d}r \mid r \in R \right\rangle \otimes \left\langle \{\tilde{d}f_i\}_{i=1}^{r} \right\rangle \\ &\subset \left\langle \tilde{d}\mathbf{X} \right\rangle + \left\langle \{\tilde{d}(x^{j-i} \cdot e_{ji}) \mid j \neq i\}, \{\tilde{d}(e_{ii})\}_{i=0}^{n-1} \right\rangle\end{aligned}$$

where A is generated over R as an R-algebra by $\{f_1, \ldots, f_r\}$. Here, note that $\mathbf{X} = \sum_{i=0}^{n} x \cdot e_{i(i-1)}$. Taking a quotient gives

$$\tilde{d}f_i / \tilde{d}\{e_{ii}\} = xe_{(i+1)i} \otimes e_{ii} - e_{(i+1)(i+1)} \otimes xe_{(i+1)i}$$

This implies

$$\{xe_{(i+1)i} \otimes e_{ii} - e_{(i+1)(i+1)} \otimes xe_{(i+1)i}\}_{i=0}^{n-1} \subset \bigoplus_{\chi(\mu_n)} P_i \otimes Q_i$$

generates $I_\Delta = \left\langle \tilde{d}\alpha \mid \alpha \in A \right\rangle$, if we only consider the $P_i \otimes Q_i$ parts of $\tilde{d}f_i$. Here, indices are considered (mod n).
Claim: $[\tilde{d}(\mathbf{X})]$ is torsion-free in ${}^{I_\Delta}\!/\!\left\langle \tilde{d}e_{ii} \right\rangle$.

Proof. To find $Ann_{A\otimes A^{op}}([\tilde{d}\mathbf{X}])$ for $[\tilde{d}\mathbf{X}] \in {}^{I_\Delta}\!/\!\left\langle \tilde{d}e_{ii} \right\rangle$, we note that based on the direct sum decomposition of A, we can consider the annihilator of the projected parts of $[\tilde{d}\mathbf{X}]$ separately to each of the P_i's and Q_j's, and then ask for an element in the intersection, analogously to the case of $[\mathbb{A}^1/\mu_2]$ above.

Considering the annihilator of $\tilde{d}\mathbf{X}$, we note that $\mathbf{X}$ is a multiple by x entrywise of a permutation matrix. Reading off from the rows, we have x multiplied in the entries by a cyclic permutation which yields the set of implications

$$a_{ij} \neq 0 \implies \deg(a_{ij}) = \deg(a_{ij}) + n, \tilde{a}_{ij} \neq 0 \implies \deg(\tilde{a}_{ij}) = \deg(\tilde{a}_{ij}) + n$$

for all i, j for $M \otimes \tilde{M} \in Ann_{A \otimes A^{op}} \tilde{d}(\mathbf{X})$ so that $Ann_{A \otimes A^{op}}([\tilde{d}\mathbf{X}]) = 0$ and $[\tilde{d}\mathbf{X}]$ is torsion-free. $\square$

So for general n, we have that I_Δ for ${\mathbb{A}^1}/{\mu_n}$ is resolved by a 2-step resolution with just the single element $[\tilde{d}\mathbf{X}]$:

$$\begin{tikzcd}
 & & I_\Delta \arrow[d] & \\
0 \arrow[r] & A \otimes_k A \arrow[r] \arrow[ur] & A \otimes A \arrow[r] \arrow[d] & 0 \arrow[d] \\
 & & A_\Delta \arrow[r] & 0
\end{tikzcd}$$

Note that Δ is codimension 1 here.

3.5.3 For $[{\mathbb{A}^2}/{\mu_2}]$

Here, we have $\mathcal{O} = k[z_1, z_2]$. $R = k[z_1^2, z_2^2] \cdot Id$. Let $\zeta_i = z_i \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

$D^b(\mathbb{A}^2/\mu_2) = \langle \mathcal{O}_0, \mathcal{O}_1 \rangle$ where $\mathcal{O}_0$ consists of even polynomials:

$$\mathcal{O}_0 = \left\langle k[z_1^2, z_2^2], z_1 z_2 k[z_1^2, z_2^2] \right\rangle$$

and $\mathcal{O}_1$ consists of odd polynomials:

$$\mathcal{O}_1 = \left\langle z_1 k[z_1^2, z_2^2], z_2 k[z_1^2, z_2^2] \right\rangle.$$

Here, $A = End^*(\mathcal{O} \oplus \mathcal{O}(1))$ denoted by

$$\begin{pmatrix} \mathcal{O}_0 & \mathcal{O}_1 \\ \mathcal{O}_1 & \mathcal{O}_0 \end{pmatrix}$$

From $\zeta_1^2 = z_1^2, \zeta_2^2 = z_2^2$, we have that $\tilde{d}R \subset \left\langle \tilde{d}\zeta_1, \tilde{d}\zeta_2 \right\rangle$.
To generate $\{\tilde{d}\alpha \mid \alpha \in A\}$ modulo $\left\langle \{\tilde{d}e_{ii}\} \right\rangle$, we have that $\tilde{d}1 = 0$,

$$\begin{aligned} \tilde{d}(xy \cdot e_{00}) = \tilde{d}(xy)e_{00} &= \tilde{d}(\zeta_1\zeta_2 \cdot e_{00}) \\ &= \tilde{d}(\zeta_1) \cdot \zeta_2 \cdot e_{00} + \zeta_1 \cdot \tilde{d}(\zeta_2) \cdot e_{00} \qquad \text{modulo } \left\langle \tilde{d}\{e_{ii}\} \right\rangle \end{aligned}$$

and similarly for $\tilde{d}(xy \cdot e_{11})$. For $\tilde{d}(z_1 \cdot e_{01})$, we have

$$\begin{aligned} \tilde{d}(z_1 \cdot e_{01}) &= \tilde{d}(\zeta_1 \cdot e_{01}) \\ &= \tilde{d}(\zeta_1 \cdot e_0 0) \\ &= \tilde{d}(\zeta_1) \cdot e_{00} \qquad \text{modulo } \left\langle \{\tilde{d}e_{ii}\} \right\rangle \\ &\in \left\langle \tilde{d}\zeta_1 \right\rangle \end{aligned}$$

and, for instance,

$$\begin{aligned} \tilde{d}(z_1 \cdot e_{10}) &= \tilde{d}(e_{11}\zeta_1) \\ &= e_{11} \cdot \tilde{d}(\zeta_1) \qquad \text{modulo } \left\langle \{\tilde{d}(e_{ii})\} \right\rangle \end{aligned}$$

Similarly for $\tilde{d}z_2e_{01}$, $\tilde{d}(z_2e_{10}) \in \left\langle \tilde{d}\zeta_1, \tilde{d}\zeta_2 \right\rangle$.

Toghether, these show

$$\left\langle \tilde{d}\alpha \mid \alpha \in A \right\rangle \in \left\langle \{\tilde{d}(\zeta_i)\}_{i=1}^2, \tilde{d}(e_{ii}) \right\rangle.$$

Thus, we have a resolution in terms of $\{\tilde{d}(\zeta_i)_{i=1}^2\}$, once we remove the projective part

$$P_1 \otimes Q_2 \oplus P_2 \otimes Q_1 = \left\langle \tilde{d}(e_{ii}) \right\rangle.$$

3.5.4 General resolution of diagonal for ${}^{\mathbb{A}^n}\!/_{G}$ for G a finite abelian group

Let $\hat{G} = Hom_{Ab}(G, \mathbb{G}_m), S = k[\mathbb{A}^n] = k[z_1, \ldots, z_n]$ the affine coordinate ring, and $F = S^n \in S-\text{Mod}$. Note: As a $k-$algebra, A is generated over k by $\langle \zeta_i, \{e_{ii}\}_{i\neq 0}\rangle$. Consider

$$\mathcal{M} := Mat_{|G|,|G|}(\bigwedge_S{}^{\bullet} F) \supset B = \{M \in \mathcal{M} \mid |z^\alpha| - (\rho_1 - \rho_2) = \chi_{i_1} + \chi_{i_2} + \cdots + \chi_{i_n}\}$$

which is $\mathbb{Z}_{\geq 0} \times \hat{G}$-graded, where the $\mathbb{Z}_{\geq 0}$-grading comes from the $\mathbb{Z}_{\geq 0}-$grading on $\bigwedge_S^{\cdot} F$, and the $\hat{G}$-grading comes from the sum of $\hat{G} \times \hat{G}$-grading on $\mathcal{A} = End^{\cdot}(\bigoplus_{i,j\in\hat{G}\times\hat{G}} G_i \oplus G_j)$ where G_j is the χ_j-graded part of S. As a module over k, $\mathcal{M}$ is generated by

$$\{z^\alpha e_{i_1} \wedge \cdots \wedge e_{i_k} e_{\rho_1\rho_2}\}$$

As an $S-$module, $\mathcal{M}$ has basis $\{e_1, \ldots, e_r\}$ where $\{e_i\}_{i\in I} = \{z^\alpha e_{\rho_a\rho_b}\}_{|z^\alpha|=\rho_a-\rho_b, a\neq b}$. In $\mathcal{M}$,

$$|z^\alpha e_{i_1} \wedge \cdots \wedge e_{i_\ell} e_{\rho_1\rho_2}| = (\ell, |z^\alpha| + (\rho_1 - \rho_2))$$

for z^α a monomial in S. Here,

$$
\begin{aligned}
B^{(*,0)} &= \mathcal{M}^{(*,0)} \\
&= \{M \in \mathcal{M} \mid |z^{\alpha}| = \rho_1 - \rho_2 = \chi_{i_1} + \cdots + \chi_{i_\ell}\} \\
&= M \in \mathcal{M} \mid \ \deg(M) = (*, 0)
\end{aligned}
$$

with no constraints on $*$. Here, imposing $|z_{\alpha}| = \rho_1 - \rho_2$ implies that we can view the columns as image of basis vectors (as left module) of an endomorphism of A, which is to say that columns are $\bigoplus_{\chi(g_i)} Hom(G_i, G)$.

We choose an equivariant basis of characters associated to G such that

$$g \cdot z_i = \chi_i(g) \cdot z_i$$

To generalize the constructions considered in previous cases, let

$$\zeta_i := z_i e_i \sum_{\rho \in \hat{G}} e_{(\rho+\chi_i)\rho}$$

Note: B is an algebra over k. B is also an $R-$algebra, where

$$R = \left\langle z^{\alpha} e_{\rho_i \rho_i} \mid \rho_i \in \hat{G}, |z^{\alpha}| = 0 \text{ in } F \right\rangle$$

Definition: Let $s \in B \otimes B$ be defined by

$$s = \sum_{i=1}^{n} \zeta_i \otimes 1 - 1 \otimes \zeta_i$$

Claim: Multiplication by s (on the right) defines a differential on the (character-deformed) Koszul complex $\bigwedge_S^{\cdot} F$, yielding a resolution of I_{Δ}.

Proof.

$$\begin{aligned}
s^2 &= \left(\sum_{i=1}^{n} \zeta_i \otimes 1 - 1 \otimes \zeta_i\right) \cdot \left(\sum_{i=1}^{n} \zeta_i \otimes 1 - 1 \otimes \zeta_i\right) \\
&= \left(\sum_{i=1}^{n} \zeta_i \otimes 1 - 1 \otimes \sum_{j=1}^{n} \zeta_j\right) \cdot \left(\sum_{k=0}^{n} \zeta_k \otimes 1 - 1 \otimes \sum_{\ell=1}^{n} \zeta_\ell\right) \\
&= \sum_{i,k} \zeta_i \zeta_k \otimes 1 - 2\sum_{i=1}^{n} \zeta_i \otimes \sum_{\ell=1}^{n} \zeta_\ell + 1 \otimes \sum_{j,\ell} \zeta_j \zeta_\ell \\
&= -2 \sum_{i,\ell} \zeta_i \otimes \zeta_\ell \\
&= 0
\end{aligned}$$

□

via the Koszul rule for signs, since s is in degree 1.

3.5.5 Another approach: Morita theory

Consider ${}^{\mathbb{A}^n}\!/\!_{\mu}$ where μ is a finite group of the torus. From the above, we understand $D({}^{\mathbb{A}^n}\!/\!_{\mu}) = D(k[z_1, \ldots, z_n] - \mathrm{Mod}^{\mu})$ where modules are considered with grading. The goal here is to understand $D({}^{\mathbb{A}^n}\!/\!_{\mu} \times {}^{\mathbb{A}^n}\!/\!_{\mu})$ in terms of $D({}^{\mathbb{A}^n}\!/\!_{\mu})$, and more specifically, to understand $\mathcal{O}_\Delta$ of $D({}^{\mathbb{A}^n}\!/\!_{\mu} \times {}^{\mathbb{A}^n}\!/\!_{\mu})$ in terms of $\mathcal{O}_\Delta$ on ${}^{\mathbb{A}^n}\!/\!_{\mu}$. That is, we would like a description of $\mathcal{O}_\Delta$ of ${}^{\mathbb{A}^n}\!/\!_{\mu} \times {}^{\mathbb{A}^n}\!/\!_{\mu}$ in terms of Δ for affine space. There is always the cohomological grading present, but now we also have an extra grading on $k[\mathbb{A}^n] - \mathrm{Mod}^{\mu}$. Here, let $S = k[\mathbb{A}^n]$. Note that $D({}^{\mathbb{A}^n}\!/\!_{\mu})$ is generated by line bundles $\mathcal{O}(\xi)$ for characters $\chi \in \mu^*$, so we have the generator

$$G = \sum_{\chi \in \mu^*} \mathcal{O}(\chi)$$

Note: $Hom(\mathcal{O}(\chi_i), \mathcal{O}(\chi_j))$ is S with the right grading. Ignoring grading, we have

$$G = \sum_{|\mu|} S$$

so that

$$End^{\bullet}(G) = Mat_{|\mu|}(S).$$

In order to incorporate the grading, we would then take the degree 0 part. Here we use Morita theory.

Background on Morita Equivalence

Note: Notation in this sub-subsection follows Weibel, and differs from notation used in later sections.

Definition: Two unital rings R, S are said to be **Morita equivalent** if there exists an $R-S$ bimodule P and $S-R$ bimodule Q such that $P \otimes_S Q \cong R$ as $R-R$ bimodules and $Q \otimes_R P \cong S$ as $S-S$ bimodules. Then

$$(-) \otimes_R P \text{ gives a functor from mod} - R \to \text{mod} - S \text{ and}$$
$$(-) \otimes_S Q \text{ gives a functor from mod} - S \to \text{mod} - R$$

which are inverse categorical equivalences, since for all $M \in$ mod $-R$, we have

$$M \otimes_R P \otimes_S Q \cong M \otimes_R (P \otimes_S Q) \cong M$$

and

$$M \otimes_S Q \otimes_R P \cong M \otimes_S (Q \otimes_R P) \cong M$$

for all $M \in \text{mod} - S$.

Next, we have[21;40] the following:

(a) Morita invariance is an equivalence relation.

Proof. [(i): Reflexivity] To show that R is Morita equivalent to R, we note that $P = Q = R$ gives

$$P \otimes_S Q = R \otimes_R R \cong R$$

as $R - R$ bimodules, and

$$Q \otimes_R P = R \otimes_R R \cong R = S$$

as $S - S$ bimodules, which are $R - R$ bimodules.

[(ii): Symmetric] $R \sim S \implies$ there exists $P \in R - S$ bimodules, $Q \in S - R$ bimodules such that

$$P \otimes_S Q \cong R$$

as $R - R$ bimodules, and

$$Q \otimes_R P \cong S$$

as $S - S$ bimodules. Now to show there exists a $P' \in S - R$ bimodules, $Q' \in R - S$ bimodules such that

$$P' \otimes_R Q' \cong S$$

as $S - S$ bimodules,

$$Q' \otimes_S P' \cong R$$

as $R - R$ bimodules, we note that $P' := Q, Q' := P$ give

$$P' \otimes_R Q' = Q \otimes_R P \cong S$$

as $S - S$ bimodules,

$$Q' \otimes_S P' = P \otimes_S Q \cong R$$

as $R - R$ bimodules.

[(iii): Transitivity] $R \sim S \implies$ there exists $P \in R - S$ bimod, $Q \in S - R$ bimod such that

$$P \otimes_S Q \cong R$$

as $R - R$ bimodules,

$$Q \otimes_R P \cong S$$

as $S - S$ bimodules. And $S \sim T \implies$ there exists $P' \in S - T$ bimodules, $Q' \in T - S$ bimodules such that

$$P' \otimes_T Q' \cong S$$

as $S - S$ bimodules,

$$Q' \otimes_S P' \cong T$$

as $T - T$ bimodules. Now

$$P \otimes_S P' \in R - T \text{ bimod },$$

$$Q' \otimes_S Q \in T - R \text{ bimod },$$

and

$$\begin{aligned}(P \otimes_S P') \otimes_T (Q' \otimes_S Q) &\cong P \otimes_S (P' \otimes_T Q') \otimes_S Q \\ &\cong P \otimes_S S \otimes_S Q \\ &\cong P \otimes_S Q \\ &\cong R\end{aligned}$$

and

$$\begin{aligned}(Q \otimes_S Q) \otimes_R (P \otimes_S P') &\cong Q' \otimes_S (Q \otimes_R P) \otimes_S P' \\ &\cong Q' \otimes_S S \otimes_S P' \\ &\cong Q' \otimes_S P' \\ &\cong T.\end{aligned}$$

□

(b) If R and S are Morita equivalent, so are R^{op} and S^{op}.

Proof. If $R \sim S$, then there exists $P \in R - S$ bimod, $Q \in S - R$ bimod such that

$$P \otimes_S Q \cong R$$

as $R - R$ bimodules,

$$Q \otimes_R P \cong S$$

as $S - S$ bimodules. Now, to show there exists a $P' \in R^{op} - S^{op}$ bimod,

$Q' \in S^{op} - R^{op}$-bimod such that

$$P' \otimes_{S^{op}} Q' \cong R^{op},$$

we note that $P' := Q, Q' := P$ gives

$$\begin{aligned} Q \otimes_{S^{op}} P &\cong P \otimes_S Q \\ &\cong R \end{aligned}$$

and

$$\begin{aligned} Q' \otimes_{R^{op}} P' &= P \otimes_{R^{op}} Q \\ &\cong Q \otimes_R P \\ &\cong S \end{aligned}$$

so that $R \sim S$ implies that $R^{op} \sim S^{op}$. Note that $R(^{op})^{op} \cong R, (S^{op})^{op} \cong S$ implies that $R \sim S \iff R^{op} \sim S^{op}$. $\square$

(c) If $R \sim S$, then $R - \mathrm{mod} - R$ is equivalent to $S - \mathrm{mod} - S$ via

$$Q \otimes_R (-) \otimes_R P$$

and

$$P \otimes_S (-) \otimes_S Q$$

from the following diagram

$$R - \mathrm{mod} - R \quad \overset{Q\otimes_R(-)\otimes_R P=\Phi}{\longrightarrow} \quad S - \mathrm{mod} - S$$
$$\underset{P \otimes_S (-) \otimes_S Q = \Psi}{\longleftarrow}$$

Proof. We know $R - \mathrm{mod} - R$ is equivalent to $S - \mathrm{mod} - S$ if $\Psi\Phi \cong Id_{R-\mathrm{mod}-R}$, and

$$\begin{aligned}
\Psi\Phi(M) &= \Psi(Q \otimes_R (M) \otimes_R P) \\
&\cong (P \otimes_S Q) \otimes_R M \otimes_R (P \otimes_S Q) \\
&\cong R \otimes_R M \otimes_R R \\
&\cong M
\end{aligned}$$

and

$$\begin{aligned}
\Phi\Psi(N) &= \Phi(P \otimes_S N \otimes_S Q) \\
&\cong (Q \otimes_R P) \otimes_S N \otimes_S (Q \otimes_R P) \\
&\cong S \otimes_S N \otimes_S S \\
&\cong N
\end{aligned}$$

so $R - \mathrm{mod} - R$ and $S - \mathrm{mod} - S$ are equivalent. □

- **Proposition**: For any $n \in \mathbb{Z}_{>0}$, the matrix rings R and $S := \mathrm{Mat}_n(R)$ are Morita equivalent.

  Proof. Lam[40] builds functors $G : \mathrm{mod} - R \to \mathrm{mod} - S$ and $F : \mathrm{mod} - S \to \mathrm{mod} - R$ and shows that they compose to the identity in each category, respectively. Weibel[21] takes a different approach, and uses P and Q as above to construct bimodule isomorphisms $P \otimes_S Q \to R$ and $Q \otimes_R P \to S$. We will give each argument below.

[(i): Lam] Let $S := M_n(R) \cong R$ for any $n \in \mathbb{Z}_{>0}$. We construct $G : \text{mod} - R \to \text{mod} - S$ as follows: Given $V \in \text{mod} - R$, we let $G(V) := V^{(n)} = (v_1, \ldots, v_n) \in \text{mod} - S$ and for $\alpha : V \to V'$,

$$(G\alpha)(v_1, \ldots, v_n) := (\alpha v_1, \ldots, \alpha v_n)$$

so that G defines a functor from $\text{mod} - R \to \text{mod} - S$.
Conversely, given $U \in \text{mod} - S$, let

$$F(U) := UE_{11}$$

(returns the first column in column 1) so that $UE_{11} \cdot s = U \cdot sE_{11} \in UE_{11}$ shows that $F(U) \in \text{mod} - S$ and given $\beta \in Hom_S(U, U')$, we have

$$\beta(UE_{11}) = (\beta U)E_{11} \subseteq U'$$

so we say $F(\beta) \in \text{Hom}_R(F(U), F(U'))$ is induced by β.
For $V \in \text{mod} - R$,

$$\begin{aligned}(F \circ G)(V) &= F(V^{(n)}) \\ &= F(v_1, \ldots, v_n)\end{aligned}$$

consists of row vectors of the form $(v_1, \ldots, 0)$. So $(F \circ G)(V) \cong Id_{mod-R}(V)$.
Next, For $U \in \text{mod} - S$, we have $F(U) = UE_{11} := V \in \text{mod} - R$ (returns the first column of U) and $(G \circ F)(U) = V^{(n)}$. To find a natural isomorphism ϵ_U from U to $V^{(n)} = (G \circ F)(U)$, let

$$\epsilon_U(u) := (uE_{11}, uE_{21}, \ldots, uE_{n1})$$

returns the i-th column in $\text{mod} - R$ as the first column of an $n \times n$ matrix. Now, $(uE_{i1}) \cdot E_{11} \in V = E_{11} = F(U)$ for all i. To check that ϵ_U is a homomorphism of right $S-$modules from U to $V^{(n)} = \{(u_1, \ldots, u_n) \mid u_i \in UE_{11} \text{ for all } i\}$. We have $\epsilon_U(u_1 + u_2) = \epsilon_U(u_1) + \epsilon_U(u_2)$ from distributing matrix multiplication across addition, and to check that $\epsilon_U(u \cdot s) = \epsilon_U(u) \cdot s$, we note that S is generated as an S module by the matrices E_{ij}. Since $s = \sum_{i,j} r_{ij}E_{ij}$, it suffices to check that $\epsilon_U(u \cdot rE_{ij}) = \epsilon_U(u) \cdot rE_{ij}$. Direct computation gives

$$\begin{aligned}
\epsilon_U(u \cdot rE_{ij}) &= (u \cdot rE_{ij}E_{11}, u \cdot rE_{ij}E_{21}, \ldots, u \cdot rE_{ij}E_{n1}) \\
&= (u \cdot r\delta_j^1 E_{i1}, u \cdot r\delta_j^2 E_{i2}, \ldots, u \cdot r\delta_j^n E_{i1}) \\
&= (0, 0, \ldots, u \cdot rE_{i1}, \ldots, 0)
\end{aligned}$$

with $u \cdot rE_{i1}$ in the j-th position. Additionally, we have

$$\begin{aligned}
\epsilon_U(u) \cdot rE_{ij} &= (uE_{11}, uE_{21}, \ldots, uE_{n1}) \cdot rE_{ij} \\
&= (0, 0, \ldots, u \cdot rE_{i1}, \ldots, 0)
\end{aligned}$$

with $u \cdot rE_{i1}$ in the j-th position. This concludes Lam's proof.

Following the proof method in Weibel, we note that for $S = \text{Mat}_n(R)$, $P = (r_1, \ldots, r_n) \in R - S$ bimod, and $Q = (r_1, \ldots, r_n)^T \in S - R$ bimod, we first consider the map $P \otimes_S Q \to R$ given by matrix multiplication μ: $p \otimes_S q \mapsto \sum_i p_i q_i$.

To show that this gives an isomorphism of $R-R$ bimodules, we show that μ has trivial kernel and is surjective.

(i) If $\mu(p \otimes q) = \sum_{i=1}^{n} p_i q_i = 0$, then

$$\begin{aligned}
(p_1, \ldots, p_n) \otimes_S \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix} &= (p_1, \ldots, p_n) \otimes_S \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix} \\
&= (p_1 \ldots p_n) \otimes_S \begin{pmatrix} q_1 & \cdots & 0 \\ q_2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ q_n & \ldots & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \\
&= (p_1 \ldots p_n) \cdot \begin{pmatrix} q_1 & \cdots & 0 \\ q_2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ q_n & \ldots & 0 \end{pmatrix} \otimes_S \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \\
&= \left(\sum_i p_i q_i, 0, \ldots, 0 \right) \otimes_S \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \\
&= (0 \ldots 0) \otimes_S \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \\
&= 0 \otimes_S 0
\end{aligned}$$

so that $\ker(\mu) = \{0 \otimes_S 0\}$ is trivial.

(ii) Since $R \ni 1$ is unital, we have that for fixed $r \in R$,

$$\mu\left(\begin{pmatrix} r & 0 & \cdots & 0 \end{pmatrix} \otimes_S \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right) = r$$

so that $\mu : P \otimes Q \to R$ is surjective. Therefore, μ induces an isomorphism of $R - R$ bimodules $P \otimes_S Q \overset{\cong}{\to} R$.

Next, we consider another map μ given by matrix multiplication: $Q \otimes_R P \to S$ and show that it is an isomorphism of $S - S$ bimodules by again showing it has trivial kernel and is surjective. This time, $\begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix} \otimes_R \begin{pmatrix} p_1 & p_2 & \cdots & p_n \end{pmatrix} \mapsto (q_i p_j) \in S$.

(i) If $q \otimes_R p \mapsto 0 = (q_i p_j)$, then $q_i p_j = 0\ \forall i, j$. This gives

$$\begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix} \otimes_R \begin{pmatrix} p_1 & p_2 & \cdots & p_n \end{pmatrix} = \sum_i \begin{pmatrix} 0 \\ 0 \\ \vdots \\ q_i \\ 0 \\ \vdots \\ 0 \end{pmatrix} \otimes_R \begin{pmatrix} p_1 & p_2 & \ldots & p_n \end{pmatrix}$$

$$= \sum_i \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} q_i \otimes_R \begin{pmatrix} p_1 & p_2 & \cdots & p_n \end{pmatrix}$$

$$= \sum_i \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \otimes_R q_i \begin{pmatrix} p_1 & p_2 & \cdots & p_n \end{pmatrix}$$

$$= \sum_i \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \otimes_R \begin{pmatrix} q_i p_1 & q_i p_2 & \cdots & q_i p_n \end{pmatrix}$$

$$= \sum_i \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \otimes_R \begin{pmatrix} 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$= 0 \otimes_R 0$$

so that $\ker(\mu) : Q \otimes_R P \to S$ is $\{0 \otimes_S 0\}$.

To show that $\mu : Q \otimes_R P \to S$ is a surjection: we note again that S is generated as an $S - S$ bimodule by $\{E_{ij} \mid 1 \leq i, j \leq n\}$ and for $\tilde{q}_i$ given by a 1 in position i and 0 elsewhere, and $\tilde{p}_j$ given by a 1 in position j and 0 elsewhere, we have

$$\tilde{q}_i \otimes_R \tilde{p}_j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \otimes_R \begin{pmatrix} 0 & \cdots & 1 & \cdots & 0 \end{pmatrix}$$
$$= E_{ij}$$

so the image generated by $Q \otimes_R P$ under μ is all of S. This concludes Weibel's proof. $\square$

Claim: The bimodule structures induce ring isomorphisms

$$\mathrm{End}_R(Q) \cong S \cong \mathrm{End}_R(P)^{op}$$

Proof. First, we show that P is a summand of S^m for some m as follows: Since $P \otimes_S Q \simeq R$, we have that $1 \in R$ can be written as

$$1 = \sum_i^m p_i \cdot q_i$$

for some m. Now let $e : P \to S^m$ be given by $p \mapsto (q_1 \cdot p, \ldots, q_m \cdot p) \in S^m$ where $(-) \cdot (-)$ here refers to matrix multiplication over tensors from $Q \otimes_R P$. Then e is a homomorphism of right S-modules. Next, construct $h : S^m \to P$ by $(s_1, \ldots, s_m) \mapsto \sum_{i=1}^m p_i s_i$ where here we view P as a right $S-$module. Then h is also a homomorphism of right S-modules, and

$$(h \circ e)(p) = \sum_i p_i(q_i \cdot p) = \sum (p_i \cdot q_i)p = p$$

from associativity of matrix multiplication. Thus P is a summand of some S^m in $\mathrm{mod} - S$. Now, since Morita equivalence is an equivalence relation (shown above), we have that Q is a summand of R^m in $\mathrm{mod} - R$ for some m. Therefore, when we view Q as an $S - R$ bimodule, we note that

$$S \cong \bigoplus_{\text{columns}} Hom(Q_i, Q)$$

Now, $\mathrm{End}_R(Q)$ is a ring with addition given by $(f + g)(q) = f(q) + g(q)$ and multiplication given by composition of functions. Here, $\mathrm{End}_R(Q)$ acts on the left of Q via $f \cdot q = f(q)$ gives $\mathrm{End}_R(Q) \cong S$. Since P can be expressed as a summand of R^n, and $P \in R - S - \mathrm{mod}$, we have that $S \cong \mathrm{End}_R(P)^{op}$ by a similar argument. $\square$

Morita equivalence for resolution of diagonal

This section resumes notation used in Sections 3.5.1-3.5.3 and 3.5.5. We have projectors P_i and Q_j as subspaces of $End^{\cdot}(G)$, based on whether we view $End^{\cdot}(G)$ as left- or right-$End^{\cdot}(G)$-modules, respectively. Viewing P as a column with $|\mu^*|$ entries and Q as a row, we have that

$$P \in (M_{|\mu|}S, S) - \text{ bimod },$$

$$Q \in (S, M_{|\mu|}S) - \text{ bimod}$$

Now $Q \otimes_{M_{|\mu|}(S)} P \simeq S_\Delta \cong S$ and $P \otimes_S Q = M_{|\mu|}(S)_\Delta \cong M_{|\mu|}(S)$. The diagonal Δ respects P, Q here. If we take $(-)^{op}$ of both $M_{|\mu|}S$ and of S, then we would reverse the role of P and Q. Ignoring this, we write

$$(S,S)-\ \text{bimod} \quad \underset{\overset{\leftarrow}{Q\otimes_{M_{|\mu|}S}(-)\otimes_{M_{|\mu|}S}P}}{\overset{P\otimes_S(-)\otimes_S Q}{\rightarrow}} \quad (M_{|\mu|}S, M_{|\mu|}S)-\ \text{bimod}$$

Starting with $M \in (S,S)-$bimod, we have that under the composition of arrows to the right and left, respectively:

$$\begin{aligned} M &\mapsto P\otimes_S M\otimes_S Q \\ &\mapsto Q\otimes_{M_{|\mu|}} P\otimes_S M\otimes_S Q\otimes_{M_{|\mu|}} P \cong (Q\otimes_{M_{|\mu|}} P)\otimes_S M\otimes_S (Q\otimes_{M_{|\mu|}} P) \\ &\cong S\otimes_S M\otimes_S S \\ &\cong M \end{aligned}$$

This gives an equivalence of monoidal categories, as a unit maps to a unit.

To get a projective resolution of Δ for $M_{|\mu|}S$, we note that units are mapped to units in $S_\Delta \mapsto M_{|\mu|}(S)_\Delta$ (these are monoidal functors). If we take a resolution $F.$ of S_Δ:

$$\begin{array}{c} \left(\bigwedge^{\bullet}(S\otimes_k S)^n, \iota_S\right) = F. \\ \downarrow \\ S_\Delta \end{array}$$

and get a projective resolution

$$\begin{array}{c} P\otimes_S F.\otimes_S Q \\ \downarrow \\ M_{|\mu|}(S)_\Delta \end{array}$$

Then we also need to compensate for the fact that the grading is not degree 0, which we will do in the next section.

3.5.6 Shifting the grading for Koszul complex of section 3.5.4, for ${}^{\mathbb{A}^n}\!/_G$ with G a finite group

Given that $s := \sum_{i=1}^{n}(x_i - y_i)e_i = \sum_{i=1}^{n}(x_i \otimes 1 - 1 \otimes x_i)e_i$, we have that the Koszul complex $(\bigwedge^{\bullet}(S \otimes_k S)^n, \iota_s)$ is isomorphic to the Koszul complex corresponding to the local sequence $(x_1 - y_1, \ldots, x_n - y_n)$ of $S \otimes_k S$. So

$$(\mathcal{F}^{\bullet}, \iota_s) := 0 \to \bigwedge^{n}(S \otimes_k S)^n \to \bigwedge^{n-1}(S \otimes_k S)^n \to \cdots \to (S \otimes_k S)^n \xrightarrow{s} S \otimes_k S \to 0$$

gives an isomorphic complex to

$$(\mathcal{F}^{\bullet}, d(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \sum_{j=1}^{k}(-1)^{j+1}\mathbb{x}_{i_j} e_{i_1} \wedge \cdots \wedge \hat{e}_{i_j} \wedge \cdots \wedge e_{i_k})$$

where $\mathbb{x} = (x_1 - y_1, \ldots, x_n - y_n)$ in the notation of Weibel. Here, we can see

$$(\bigwedge{}^{\bullet}(S \otimes_k S)^n, d) \cong {}^{S \otimes_k S}\!/_{I_\Delta} = A_\Delta \text{ in } D^b({}^{\mathbb{A}^n}\!/_G)$$

since $I_\Delta =< x_1 - y_1, \ldots, x_n - y_n >$. So $(\bigwedge^{\bullet}(S \otimes_k S)^n, \iota_s) \cong (\bigwedge^{\bullet}(S \otimes_k S)^n, d)$ gives a resolution of S_Δ.

Next, since $P \in (M_{|\mu|}S, S)-$bimod and $Q \in (S, M_{|\mu|}S)$-bimod are projective, they are flat. Now

$$(P \otimes_S \bigwedge^{\bullet}(S \otimes_k S)^n \otimes_S Q, 1 \otimes d \otimes 1)$$

has

$$H^i(P \otimes \dot{\bigwedge}(S \otimes_k S)^n \otimes Q, 1 \otimes d \otimes 1) = \begin{cases} P \otimes H^i(\bigwedge^{\bullet}(S \otimes_k S)^n, d) \otimes Q & i = 0, \\ 0 & \text{else .} \end{cases}$$

i.e. $P(\otimes_S \bigwedge^{\bullet}(S \otimes_k S)^n \otimes_S Q, 1 \otimes d \otimes 1)$ is a resolution of

$$\begin{aligned} P \otimes_S \langle x_1 - y_1, \ldots, x_n - y_n \rangle \otimes_S Q &\cong P \otimes_S I_{\Delta,S} \otimes_S Q \\ &\cong I_{\Delta, M_{|\mu|}} \subset M_{|\mu|} - \text{bimod.} \end{aligned}$$

We say that $(P \otimes (\bigwedge^{\bullet}(S \otimes_k S)^n) \otimes Q, 1 \otimes d \otimes 1)$ is a resolution of

$$P \otimes_S (S \otimes_k S) \otimes_S Q \big/ P \otimes I_{\Delta,S} \otimes Q$$

and

$$(M_{|\mu|}(S))_{\Delta} \cong (P \otimes_S \mathcal{F}^{\bullet} \otimes_S Q, 1 \otimes d \otimes 1)$$

For the grading, we turn S into a μ-graded ring by $\chi_1, \ldots, \chi_n$ and $g \cdot x_i = \chi(g) \cdot x_i$. Next, we shift $\mathcal{F}^{\bullet}$ by

$$\dot{\bigwedge}(\bigoplus_{i=1}^{n}(S \otimes_k S)(-\chi_i))$$

and turn the bi-grading on $S \otimes S$ into a grading by taking the sum of the bi-gradings. This gives a resolution of Δ in graded bimodules, and from our Morita equivalence functors,

$$P \otimes_S \dot{\bigwedge} \bigoplus_{i=1}^{n} (S \otimes_k S)(-\chi_i) \otimes_S Q$$

gives a resolution of $(M_{|\mu|}(S))_{\Delta}$ as a graded bimodule as follows: Here,

$$M_{|\mu|} \cong \mathrm{End}^\bullet(\bigoplus_\chi \mathcal{O}(\chi))$$

as an internal hom in graded S-mod where $G = \bigoplus_\chi \mathcal{O}(\chi)$ is a generator for $D^b(\mathbb{A}^n/G)$, so we can view

$$M_{|\mu|}(S) \cong \bigoplus_\chi \mathrm{Hom}^\bullet(\mathcal{O}(\chi), G)$$

as a direct sum over the columns $P \in (M_\mu, S)-$graded bi-modules and

$$M_\mu \cong \bigoplus_\chi Hom^\bullet(G, \mathcal{O}(\chi))$$

as a direct sum over the rows for $Q \in (S, M_{|\mu|}(S))-$bimodules. Here,

$$P = \mathrm{Hom}_{\text{gr-}S\text{-mod}}(\mathcal{O} \cong S, G = \bigoplus S(\chi))$$

is in right S-mod and

$$Q = \mathrm{Hom}_{\text{gr. } S-\text{mod}}(\bigoplus_\chi S(\chi), S)$$

is in left S-mod. Now, the components of $\mathcal{F}^\bullet$ are μ-bigraded $S - S$ bimodules and $P \in \mu - \text{gr } S\text{-mod}$, $Q \in \mu - \text{gr } S - \text{mod}$. Here, since the morphisms for $\mathcal{F}^\bullet$ are degree 0, we have a degree 0 differential on $\mathcal{F}^\bullet$. The degree 0 part of $\mathcal{F}^\bullet$ is not a graded module over $M_{|\mu|}(S)$ or over S, but only over $M^0_{|\mu|}$. Modules here are equivalent to graded modules over $\mathbb{A}^n/\mu$ as an affine stack. Here, $\mathcal{F}^0 \cong S \otimes_k S$ implies

$$\begin{aligned}
(P \otimes_S \mathcal{F}^0 \otimes_S Q)^0 &\cong (P \otimes_S (S \otimes_k S) \otimes_S Q)^0 \\
&\cong (P \otimes_k Q)^0 \\
&\cong \bigoplus_\chi P_\chi \otimes_k Q_{-\chi}
\end{aligned}$$

since $P \cong P \otimes_S S$ as an $(M_n, k)-$bimodule and $P \cong Hom(S, \bigoplus S(-\chi)$.

Now, $M^0_{|\mu|} = A$ and we want $P_\chi, Q_{-\chi}$ to be projective $A - A$ bimodules. To get P as an $A-$module: $P = \bigoplus P_\chi$,

$$P \otimes_k Q =_{\text{bimod}} \bigoplus_{\rho_1, \rho_2} P_{\rho_1} \otimes_k Q_{\rho_2}.$$

Now, we ask: What does a full resolution look like?

Since $\mathcal{F}^k = \bigwedge^k \bigoplus_{i=1}^n (S \otimes_k S)(-\chi_i)$ has basis $\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}$ with degree $\chi_{i_1} + \cdots + \chi_{i_k}$, we have

$$P \otimes_k Q = \bigoplus_{\rho_1, \rho_2} P_{\rho_1} \otimes_k P_{\rho_2}$$

so that

$$\mathcal{F}^k = \bigoplus_{i_1 < i_2 < \cdots < i_k} (S \otimes S)(-\sum_{j=1}^k \chi_{i_j})$$

as a free $S \otimes S$-graded bimodule. Note that $\mathcal{F}^1$ has basis $\{e_1, \ldots, e_n\}$ in degrees $\chi_1, \ldots, \chi_n$.

Now,

$$P \otimes_S \mathcal{F}^k \otimes_S Q = \bigoplus_{\substack{\rho_1, \rho_2, \\ i_1 < \cdots < i_k}} P_{\rho_1} \otimes (S \otimes S)(-\sum_{j=1}^k \chi_{i_j}) \otimes Q_{\rho_2}$$

which we can write as

$$= \bigoplus_{\substack{\rho_1,\rho_2,I\subset[n] \\ |I|=k \text{ and } \rho_1+\rho_2=\chi_I}} P_{\rho_1} \otimes_k Q_{\rho_2}$$
$$= (P \otimes \mathcal{F}^\bullet \otimes Q)^\bullet$$

3.5.7 Adapting the argument of section 3.3 for X_Σ simplicial

For X'_Σ a simplicial toric variety which is a global quotient X_Σ/G of a unimodular toric variety X_Σ with G a finite abelian group, we can choose an isomorphism of the lattice L of principal divisors of X_Σ which is unimodular such that $\tilde{L} \subset L$ is cofinite, and $L/\tilde{L} \cong G$. Furthermore, locally, $X'_\Sigma \cong \mathbb{A}^n/G$, so that locally we can consider $L \cong \mathbb{Z}^n$.

Diagonal Object in Mod-A

Here, let $R = k[\mathbb{A}^n]$ and $S = R \otimes_k R \cong k[x_1, \dots, x_n, y_1, \dots, y_n]$. In $D(\mathbb{A}^n/G)$, we have generator $H = \bigoplus_{\chi \in \hat{G}} \mathcal{O}(\chi)$, with endomorphism algebra $A = End(H)$. Now, $A \simeq \Delta \in A \otimes A^{op} - Mod$. We have functors given in Figure 3.2

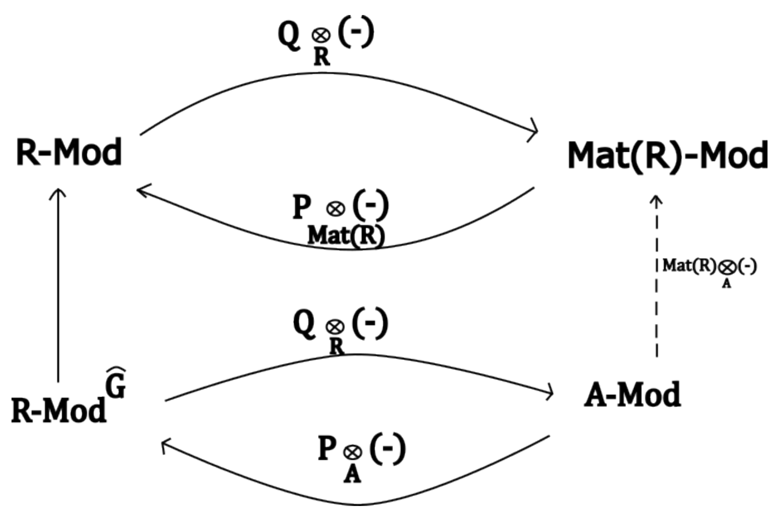

Figure 3.2: *Morita Equivalence Functors Between $R - Mod$ and $Mat(R) - Mod$, and their Graded and $\hat{G}$-Invariant Counterparts Between $R - Mod^{\hat{G}}$ and $A - Mod$*

Here, $P \in (R^{\hat{G}}, Mat(R)) - Mod$ is a row vector with grading on generators

$$P = \langle e_\chi \mid |e_\chi| = -\chi \rangle \, ,$$

$Q \in (Mat - R, R^{\hat{G}}) - Mod$ is a column vector with grading on generators

$$Q = \langle e_\chi \mid |e_\chi| = \chi \rangle$$

and we have

$$Hom_{Mat}(Q, -) \simeq P \otimes_{R^{\hat{G}}} (-)$$

For $P \otimes_A (-) : A - Mod \to R - Mod^{\hat{G}}$ applied to $\Delta \cong A \in Obj(A \otimes A^{op} - Mod)$, we have that

$$P \otimes_A A = \langle z^\alpha e_{\rho_1\rho_2} \mid |z^\alpha e_{\rho_1\rho_2}| = |z^\alpha| + \rho_1 - \rho_1 + \rho_2 \rangle$$

with grading

$$\begin{aligned} |z^\alpha e_{\rho_1\rho_2}| &= |z^\alpha| + \rho_1 - \rho_1 + \rho_2 \\ &= |z^\alpha| + \rho_2. \end{aligned}$$

For $\pi(M_{\Lambda(L)}) = M_{\Lambda(L)} \otimes_{S[\Lambda(\tilde{L})]} S$, we have

$$\pi(M_{\Lambda(L)}) = S \cdot \{\frac{x^u}{y^u} \mid u \in L\} \otimes_{S[\Lambda(\tilde{L})]} S$$

with $S[\Lambda(\tilde{L})] = \left\langle x^{u_1} y^{u_2} \otimes z^{(v,-v)} \mid (u_1, u_2) \in \mathbb{N}^{2n}, v \in \tilde{L} \right\rangle$ induces an ${}^{L}\!/\!{}_{\tilde{L}}$ grading left on, say, $\{x_1, \dots, x_n\}$ from cancelling y^u in the denominator. Here, the map

$$\phi : S[L] \to M_L \text{ via } x^{u_1} y^{u_2} \otimes z^{(v_1,v_2)} \mapsto x^{u_1+v_1} y^{u_2+v_2}$$

induces an $S[\Lambda(L)]$-module structure on M_L, hence an $S[\Lambda(\tilde{L})]$-structure on $M_{\Lambda(L)}$. In Bayer-Sturmfels[37], the map $\phi : S[L] \to M_L$ was used to show that

$$\pi(M_L) \cong {}^{S[L]}\!\diagup_{\ker\phi} \otimes_{S[L]} S \cong {}^{S}\!\diagup_{I_L}$$

by using the fact that $\pi(M_L)$ is cyclic when we only have the unimodular lattice L to consider.

We can also view

$$(R - Mod)^{\hat{G}} \to Mod - A$$

by

$$\bigoplus_{\rho\in\hat{G}} M_\rho \cong M. \mapsto Hom(\bigoplus_{\rho\in\hat{G}} R(-\rho), M) \cong \bigoplus_{\rho} Hom(R(-\rho), M) \cong \bigoplus_{\rho} M_\rho$$

where $\bigoplus_\rho M_\rho$ contains idempotents $e_{\rho\rho} \in A$ on the right-hand side.
We also have $\pi(M_{\Lambda(L)}) \in (S - Mod)^{\hat{G}\times\hat{G}}$ with

$$\pi(M_{\Lambda(L)}) \simeq +_{\overline{u}\in\hat{G}} M_{\tilde{L}} \cdot \tilde{z}^{\overline{u}}$$

is a sum which is not necessarily direct, and

$$S \cdot \left\{ \frac{x^u}{y^u} \mid u \in L \right\} \subset S_{(x_1\dots x_n y_1 \cdots y_n)} \cong k[\mathbb{Z}^{2n}].$$

Here, the localized ring $S_{(x_1\dots x_n y_1\dots y_n)}$ admits a $\mathbb{Z}^{2n}$ grading descending to an ${}^{L}\!\diagup_{\tilde{L}} \oplus {}^{L}\!\diagup_{\tilde{L}}$ grading. While $\pi(M_{\Lambda(L)})$ is no longer cyclic, we do have

$$S[L] \cong \bigoplus_{\overline{u}\in\hat{G}} S[\tilde{L}] \cdot z^{\overline{u}}$$

and

$$S[\Lambda(L)] \cong \bigoplus_{u\in\hat{G}} S[\Lambda(\tilde{L})] \cdot z^{(u,-u)}$$

carries an anti-diagonal grading. Now

$$S[L] = \langle x^u \otimes z^v \mid v \in L\rangle$$

and in

$$S[\Lambda(L)] = \left\langle x^{u_1}y^{u_2} \otimes z_1^u z_2^{-u} \mid u \in L\right\rangle$$

S has an $L \times L$ grading since $L \cong \mathbb{Z}^n$, and taking the quotient of L by $\tilde{L}$ induces a quotient on $\Lambda(L)$ by $\Lambda(\tilde{L})$ which gives an ${}^{L}\!/_{\tilde{L}} \times {}^{L}\!/_{\tilde{L}}$-grading on $S[L]$. Now $S[\tilde{L}]$ has a degree 0 grading on z's with respect to ${}^{L}\!/_{\tilde{L}}$.

For $\phi : S[L] \to M_L$ via $x^u \otimes z^v \mapsto x^{u+v}$, we have that ϕ is surjective,
$\ker\phi = \langle x^u \otimes z^v - x^{u+v} \otimes 1\rangle$ and

$$I_L = \langle x^u - x^v \mid u - v \in L\rangle \leq S$$

comes from $\ker\phi$ by setting $z's$ equal to 1. Here, we have that

$$M_L \cong {}^{S[L]}\!/_{\ker\phi}$$

as $S[L]$ modules. As $S[\tilde{L}]$-modules, we still have

$$S[L] \cong \bigoplus_{\overline{u} \in \hat{G}} S[\tilde{L}] \cdot z^{\overline{u}}$$

For $\phi : S[\Lambda(L)] \to M_{\Lambda(L)}$, we have that

$$\ker \phi = \left\langle y_i \otimes z^{(e_i, -e_i)} - x_i \right\rangle$$

is homogeneous of degree $(1,0)$ is one generating set, and $\ker \phi = \left\langle x_i \otimes z^{(-e_i, e_i)} - y_i \right\rangle$ is another generating set of homogeneous elements of degree $(0,1) \in (\hat{G}, \hat{G})$. Furthermore, we have that

$$I_{\Lambda(L)} = \langle x_j - y_j \rangle$$

since $L \cong \mathbb{Z}^n$, and considering ϕ as a map of $S[\tilde{L}]$-modules gives that $\ker \phi$ still specializes to $I_{\Lambda(L)}$ when we set the z's equal to 1.

To show that

$$\begin{array}{c} A \cong \quad \pi(M_{\Lambda(L)}) = M_{\Lambda(L)} \otimes_{S[\Lambda(\tilde{L})]} S \in (S - Mod)^{\hat{G} \times \hat{G}} \\ || \\ M_L \otimes_{S[\tilde{L}]} S \end{array}$$

we note that A is generated over k by

$$A/k = \langle w^{\alpha} e_{\rho_1 \rho_2} \mid |\alpha| = \rho_1 - \rho_2 \rangle$$

so we construct a map $\Psi : A \to \pi(M_{\Lambda(L)})$ by describing its image on generators of A/k:

$$\psi(w^\alpha e_{\rho_1\rho_2}) = x^\alpha \otimes z^{\Lambda(\rho_2)}.$$

Given $M \in R-Mod$, we can use the projectors $e_{\rho_i\rho_i}$ to get

$$e_{\rho_i\rho_i}M = M_{\rho_i} \in k-v.s.$$

and

$$M = \bigoplus_i M_{\rho_i}$$

as a graded $R-$module.

To get a $\hat{G}\times\hat{G}$-grading on $A \in (A\otimes A^{op})-Mod$, we use projectors

$$e_{\rho\rho}Ae_{\rho'\rho'}$$

to pick up the $(\rho,-\rho')$-homogeneous component of A in the $\hat{G}\times\hat{G}$-grading.

To get a $\hat{G}\times\hat{G}$-grading on $\pi(M_L)$, we have that

$$|x^{\alpha_1}y^{\alpha_2}\otimes z_1^\beta z_2^{-\beta}| = (\alpha_1+\beta, \alpha_2-\beta).$$

Here, the grading on generators of A over k comes from Figure 3.3.

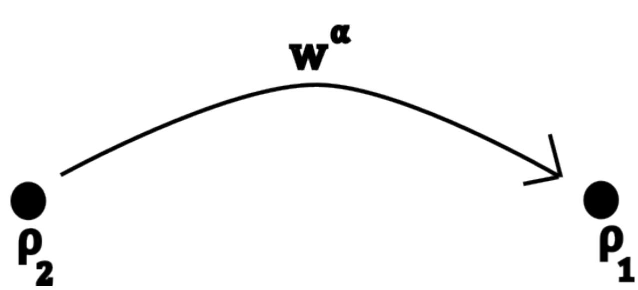

Figure 3.3: *Quiver associated to* $w^\alpha e_{\rho_1\rho_2}$

Since $\pi(M_{\Lambda(L)}) \in (S - Mod)^{\hat{G}\times\hat{G}}$ and $A \simeq \Delta \in (A \otimes A^{op}) - Mod$, we put an $(S - Mod)^{\hat{G}\times\hat{G}}$ structure on A via

$$S = R \otimes_k R$$

and use the matrix representation representing multiplication by $r \in R$ to give $R \hookrightarrow A$ so that both A and $\pi(M_{\Lambda(L)}) \in (S - Mod)^{\hat{G}\times\hat{G}}$.

Now to check that ψ is compatible with the $(S-\text{ Bimod })^{\hat{G}\times\hat{G}}$ structure, we note that

$$x^\beta w^\alpha e_{\rho_1\rho_2} = w^{\alpha+\beta} e_{\rho_1+|\beta|,\rho_2} \qquad \text{Left action}$$

and

$$y^\beta w^\alpha e_{\rho_1\rho_2} = w^{\alpha+\beta} e_{\rho_1,\rho_2-|\beta|} \qquad \text{Right Action}$$

gives that we have an $S - Mod$ homomorphism:

$$f(s \cdot m) = s \cdot f(m)$$

since

$$
\begin{aligned}
\psi(x^\beta w^\alpha e_{\rho_1\rho_2}) &= \psi(w^{\alpha+\beta} e_{\rho_1+|\beta|,\rho_2}) \\
&= x^{\alpha+\beta} \otimes z^{\lambda(\rho_2)} \\
&= x^\beta \cdot (x^\alpha \otimes z^{\lambda(\rho_2)} \\
&= x^\beta \psi(w^\alpha e_{\rho_1\rho_2})
\end{aligned}
$$

and

$$
\begin{aligned}
\psi(y^\beta w^\alpha e_{\rho_1\rho_2}) &= \psi(w^{\alpha+\beta} e_{\rho_1,\rho_2-|\beta|}) \\
&= x^{\alpha+\beta} \otimes z^{(\rho_2-|\beta|,-\rho_2+|\beta|)} \\
&= (x^\beta \otimes z^{(-\beta,\beta)}) \cdot (x^\alpha \otimes z^{\lambda(\rho_2)}) \\
&= y^\beta (x^\alpha \otimes z^{\lambda(\rho_2)}) \\
&= y^\beta \psi(w^\alpha e_{\rho_1\rho_2})
\end{aligned}
$$

Lastly, to show that this is an isomorphism, we construct an inverse to Ψ. Since $\pi(M_{\Lambda(L)}) \cong R[\hat{G}]$ as $k-v.s.$, we construct

$$
\Phi : x^\alpha \otimes z^{\overline{v}} \mapsto w^\alpha e_{\alpha+\overline{v},\overline{v}}
$$

as an inverse to ψ.

This shows that for global quotients, we tensor over $S[\Lambda(\tilde{L})]$ rather than over $S[\Lambda(L)]$ to get Δ.

3.5.8 Global diagonal object and resolution of the diagonal for $X'_\Sigma = {}^{X_\Sigma}\!/_G$ a global quotient of unimodular X_Σ by G a finite abelian group

Here, we'll give a global diagonal object for the example of ${}^{\mathbb{P}^n}\!/_G$ and a proof for why this is a global diagonal object for ${}^{\mathbb{P}^n}\!/_G$ which depends only on the fact that $\mathbb{P}^n$ is unimodular, hence giving a global diagonal object for the more general case of $X'_\Sigma = {}^{X_\Sigma}\!/_G$ a global quotient of unimodular X_Σ by G a finite abelian group.

Claim: We have a global diagonal object for ${}^{\mathbb{P}^n}\!/_G$ for G a finite abelian group.

Proof. Locally, ${}^{\mathbb{P}^n}\!/_G$ is isomorphic to ${}^{\mathbb{A}^n}\!/_G$. So locally, we have $U'_\sigma = {}^{U_\sigma}\!/_G$ for $U_\sigma = \mathbb{A}^n$ open in $X_\Sigma = \mathbb{P}^n$. For $\mathbb{A}^n$, we have presentation of $Cl(\mathbb{A}^n) = 0$ via

$$0 \to M \xrightarrow{B} \mathbb{Z}^{\Sigma(1)} \xrightarrow{\pi} Cl(X) \to 0$$

gives

$$0 \to \mathbb{Z}^n \xrightarrow{Id} \mathbb{Z}^n \xrightarrow{\pi} 0 \to 0$$

and $L = \ker(\pi) = Im(B) \cong \mathbb{Z}^n$. This gives $\Lambda(L) = \{(u, -u) \mid u \in \mathbb{Z}^n\}$. We have Δ for ${}^{\mathbb{A}^n}\!/_G$ as

$$\begin{aligned}
\Delta \cong \pi(M_{\Lambda(L)}) &= {}^{S[\Lambda(L)]}\!/_{\ker\phi} \otimes_{S[\Lambda(\tilde{L})]} S \\
&= {}^{k[\mathbb{A}^{2n}] \otimes \left\langle z^{\Lambda(e_i)\pm} \right\rangle}\!/_{\left\langle y_i - x_i \otimes z^{\Lambda(e_i)} \right\rangle} \otimes_{S[\Lambda(\tilde{L})]} k[\mathbb{A}^{2n}] \\
&\cong R[\hat{G}]
\end{aligned}$$

where $R[\hat{G}]$ is the group-ring with variables w_i and $G \cong {}^{L}\!/_{\tilde{L}}$. Now the transition maps for coordinates x_i on local charts of $\mathbb{A}^n = U_\sigma \subset X_\Sigma = \mathbb{P}^n$ induce transition maps on transition maps for $U_\sigma \times U_\sigma$, and hence on $U'_\sigma \times U'_\sigma = {}^{U_\sigma}\!/_G \times {}^{U_\sigma}\!/_G$. This induces transition maps on the $z^{\Lambda(e_i)}$'s from locally writing $x_i = y_i \otimes z^{\Lambda(e_i)}$ in M_L in U'_σ. This implies that the transition maps for the $z^{\Lambda(e_i)}$'s glue together, and the diagonal object Δ for ${}^{\mathbb{A}^n}\!\diagup_{G}$ glues together to give a global diagonal object for ${}^{\mathbb{P}^n}\!\diagup_{G}$. □

Note: This proof is highly dependent on the fact that locally, $\mathbb{P}^n$ is isomorphic to $\mathbb{A}^n$ in the sense that $\mathbb{P}^n = \cup_i U_{\sigma_i}$ with each $U_{\sigma_i} \cong \mathbb{A}^n$. Since this also holds for any unimodular toric variety by the **Lemma** below, an analogous argument gives a global diagonal object for $X'_\Sigma = {}^{X_\Sigma}\!\diagup_{G}$ with X_Σ unimodular.

Lemma: A complete unimodular toric variety X_Σ has a cover by open charts which are all isomorphic to $\mathbb{A}^n$.

Proof. Let the complete unimodular toric variety X_Σ have fan Σ with maximal-dimensional cone $\sigma \in \Sigma(n)$. Then $\sigma^\vee \cap M$ is generated as a semigroup by normal vectors to the facets of σ, which is to say by the ray generators of $\sigma^\vee$. But there are n such normal vectors, and X_Σ unimodular impies that σ is unimodular, so subsets of $\rho_i \in \sigma(1)$ up to size n are linearly independent over $\mathbb{Z}$. This implies that the n ray generators of $\sigma^\vee$ are also linearly independent over $\mathbb{Z}$. So $U_\sigma \cong \mathbb{A}^n$. □

Claim: $\pi(M_{\Lambda(L)})$ is resolved by the cellular complex associated to ${}^{\mathcal{H}_L}\!\diagup_{\tilde{L}}$, which we view as

$$ {}^{\mathcal{H}_L}\!\diagup_{\tilde{L}} \cong \bigoplus_{u \in G} {}^{\mathcal{H}_L}\!\diagup_{L} $$

together with a covering map of

$$ \begin{array}{c} {}^{\mathcal{H}_L}\!\diagup_{\tilde{L}} \\ \downarrow \\ {}^{\mathcal{H}_L}\!\diagup_{L} \end{array} $$

Proof. Since $M_{\Lambda(L)}$ is a monomial module, $\mathcal{H}_L$ resolves $M_{\Lambda(L)}$. The complex $(\mathcal{F}^{\bullet}_{\mathcal{H}_L}, \partial)$ is not S-finite, but has finite length $m = \operatorname{rank}(L)$. It is also a minimal $\mathbb{Z}^{2n}$-graded free S-resolution of the lattice module $M_{\Lambda(L)}$[12]. Since ${\mathcal{H}_L}/{L}$ resolves

$$M_{\Lambda(L)} \otimes_{S[\Lambda(L)]} S$$

we have that ${\mathcal{H}_L}/{\tilde{L}}$ resolves

$$\pi(M_{\Lambda(L)}) = M_L \otimes_{S[\Lambda(\tilde{L})]} S.$$

□

3.5.9 Example of resolution for ${\mathbb{P}^1}/{\mu_6}$

Recall that for $\mathbb{P}^1$ we have a presentation of $Cl(\mathbb{P}^1) \cong \mathbb{Z}$ given by

$$\begin{array}{ccccccccc} 0 & \to & M & \xrightarrow{B} & \mathbb{Z}^{|\Sigma(1)|} & \xrightarrow{\pi} & Cl(\mathbb{P}^1) & \to 0 \\ & & \| \wr & & \| \wr & & \| \wr & & \\ 0 & \to & \mathbb{Z} & \to & \mathbb{Z}^2 & \to & \mathbb{Z} & \to & 0 \end{array}$$

with maps B and π given by

$$0 \to \mathbb{Z} \xrightarrow{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow{(1\ 1)} \mathbb{Z} \to 0$$

so that $L = \ker(11)$ with $L = \mathbb{Z}\left\langle \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\rangle \subset \mathbb{Z}^2$. If we fix $L \cong \mathbb{Z}$, then $\tilde{L} = 6\mathbb{Z}$ and ${L}/{\tilde{L}} \cong \mu_6 = G$. Now $\mathbb{P}^1$ has monomial labels on ${\mathcal{H}_L}/{L}$ given by Figure 3.4:
For $G = \mu_6$, we have the cellular complex ${\mathcal{H}_L}/{\tilde{L}}$ and is given by Figure 3.5.

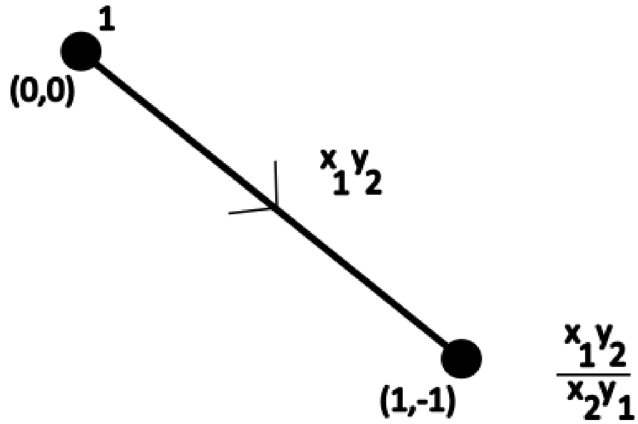

Figure 3.4: ${\mathcal{H}_L}\diagup_{L}$ *for* $\mathbb{P}^1$

The decomposition

$$ {\mathcal{H}_L}\diagup_{\tilde{L}} = \bigoplus_{u \in G} {\mathcal{H}_L}\diagup_{L} $$

and corresponding covering map h is given by Figure 3.6
with corresponding complex and resolution

$$
\begin{array}{ccccc}
S^6 & \xrightarrow{\begin{pmatrix} -x_1y_2 & 0 & \cdots & x_2y_1 \\ x_2y_1 & -x_1y_2 & \cdots & 0 \\ 0 & x_2y_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -x_1y_2 \end{pmatrix}} & S^6 & \rightarrow & 0 \\
 & & \downarrow & & \\
 & & \pi(M_{\Lambda(L)}) & \rightarrow & 0
\end{array}
$$

The covering map h corresponds to a subdivision of the maximal torus of $\mathbb{P}^n$ by G.

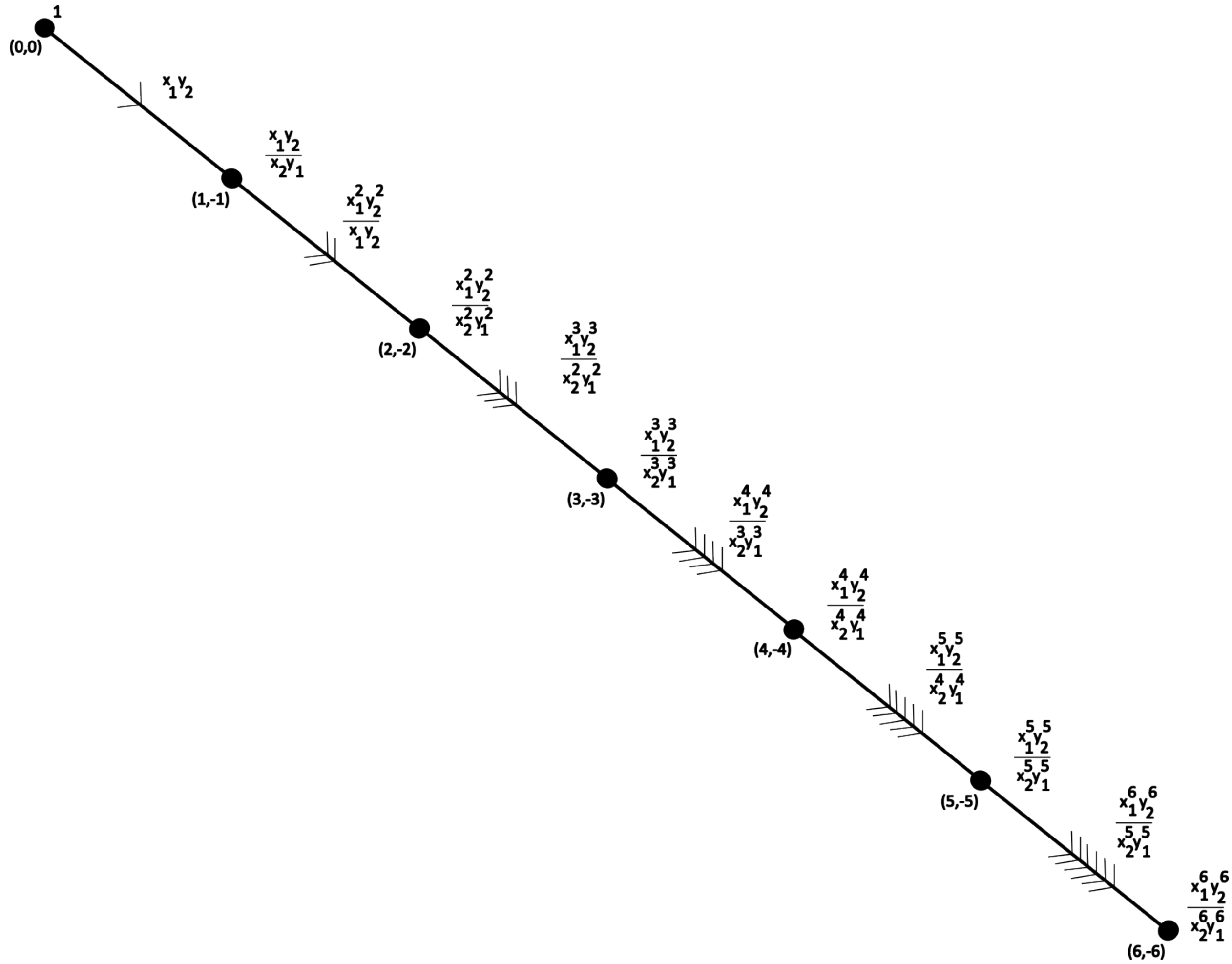

Figure 3.5: $\mathcal{H}_{L}/_{\tilde{L}}$ *for* $\mathbb{P}^1/_{\mu_6}$

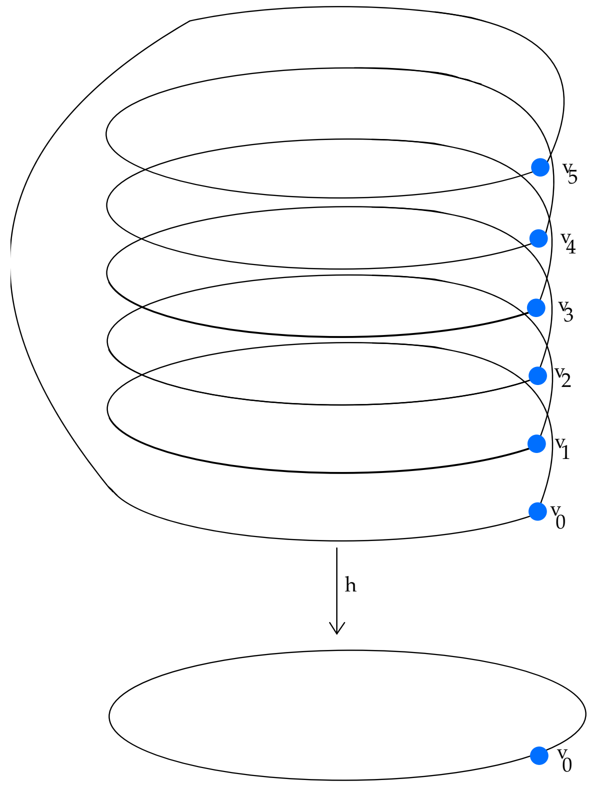

Figure 3.6: $\mathcal{H}_L \big/ \tilde{L}$ *for* $\mathbb{P}^1 \big/ \mu_6$

3.6 Upgrading the grading on $(S-Mod)^{\hat{G}\times\hat{G}}$

If we add in a $(\mathbb{Z}\oplus\hat{G})\times(\mathbb{Z}\oplus\hat{G})$-grading to $(S-Mod)^{\hat{G}\times\hat{G}}$, and view $\pi(M_{\Lambda(L)}) \in Ob\ (S-Mod)^{\hat{G}\times\hat{G}}$, for instance, for $\mathbb{P}^1\big/\mu_4$ with $R=\mathbb{C}[x_0,x_1]$, exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 4 & 4 \end{pmatrix}} \mathbb{Z} \to G \to 0$$

with map

$$\mathbb{Z} \oplus {}^{\mathbb{Z}}\!/_{4\mathbb{Z}} \leftarrow \mathbb{Z}^2$$

$$(a+b, \overline{a}) \leftarrow (a, b)$$

and

$$|x_0^{a_0} x_1^{a_1}| = (a_0 + a_1, \overline{a_0})$$

with $a_0 = -a_1$, $a_0 = 4b$ from $L = Im(B) \subset \mathbb{Z}^2$ gives a $\mathbb{Z} \oplus G$-grading on R, inducing a $(\mathbb{Z} \oplus G) \times (\mathbb{Z} \oplus G)$-grading on S.

Now, with

$$S[\Lambda(L)] \cong \bigoplus_{\overline{u} \in \hat{G}} S[\Lambda(\tilde{L})] \cdot z^{\Lambda(\overline{u})}$$

allows us to write the resolution for ${}^{\mathbb{P}^1}\!/_{\mu^4}$ as

$$\begin{array}{ccccc}
0 \to S^4 & \xrightarrow{\begin{pmatrix} -x_1y_2 & 0 & 0 & x_2y_1 \\ x_2y_1 & -x_1y_2 & 0 & 0 \\ 0 & x_2y_1 & -x_1y_2 & 0 \\ 0 & 0 & x_2y_1 & -x_1y_2 \end{pmatrix}} & \bigoplus_{\chi \in \hat{G}} \mathcal{O}(\chi, -\chi) & \to & 0 \\
 & & \downarrow & & \\
 & & \Delta & \to & 0
\end{array}$$

where the vertical map downwards takes the basis element e_χ for $\mathcal{O}(\chi, -\chi)$ to $z^{\Lambda(\overline{u})} \in \pi(M_{\Lambda(L)})$.

3.7 Weighted projective space $\mathbb{P}(1,2)$

For $\mathcal{L}, \mathcal{L}'$ line bundles on normal toric variety X_Σ, if $\mathcal{L}$ has sheaf of sections $\mathcal{O}(D)$ and $\mathcal{L}'$ has sheaf of sections $\mathcal{O}(E)$, then to compute

$$Ext^i(\mathcal{O}_{X_\Sigma}(D), \mathcal{O}_{X_\Sigma}(E))$$

we note that for $i = 0$,

$$\mathcal{H}om(\mathcal{O}(D), \mathcal{O}(E)) \simeq \mathcal{O}(E - D)$$

and $Ext^i(\mathcal{O}(D), \mathcal{O}(E)) = R\Gamma(\mathcal{O}(E - D)) = H^*(X_\Sigma, \mathcal{O}(E - D))$.
Now, for

$$\mathbb{P}(1,2) = \left[\mathbb{C}^2 \setminus \{(0,0)\} \big/ \mathbb{C}^* \right]$$

where $\mathbb{C}^*$ acts via $\lambda \cdot (z_1, z_2) = (\lambda z_1, \lambda z_2)$, the set

$$\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)$$

is an exceptional collection.

Proof. Note that

$$\begin{aligned} H^*(\mathcal{O}(-1)) &= R\Gamma Hom(\mathcal{O}(2), \mathcal{O}(1)) \\ &= R\Gamma Hom(\mathcal{O}(1), \mathcal{O}). \end{aligned}$$

To find $H^*\mathcal{O}(n)$ on $\mathbb{P}(1,2)$, we form a Čech cover on $\mathbb{C}^2$, noting that z_1 has degree 1 and z_2 has degree 2. On U_1, we localize at z_1 to get $\mathcal{O}(U_1)$. In Figure 3.7 each lattice point is labeled by the total degree of monomial. On U_2, localizing at z_2 gives $\mathcal{O}(U_2)$. Here, lattice points in turqoise are again labeled by total degree of monomial. Level sets of total degree are highlighted in increments of 2. In the Čech cochain complex, $\check{C}^0(U_1 \cup U_2)$ is given by $\Gamma(U_1) \times \Gamma(U_2)$ in Figure 3.7. Note that both $\Gamma(U_1)$ and $\Gamma(U_2)$ contain $(0,0)$. For

$$H^* \left(\left[\mathbb{C}^2 \setminus \{(0,0)\} \big/ \mathbb{C}^* \right] \right)$$

we consider elements only of degree 0.

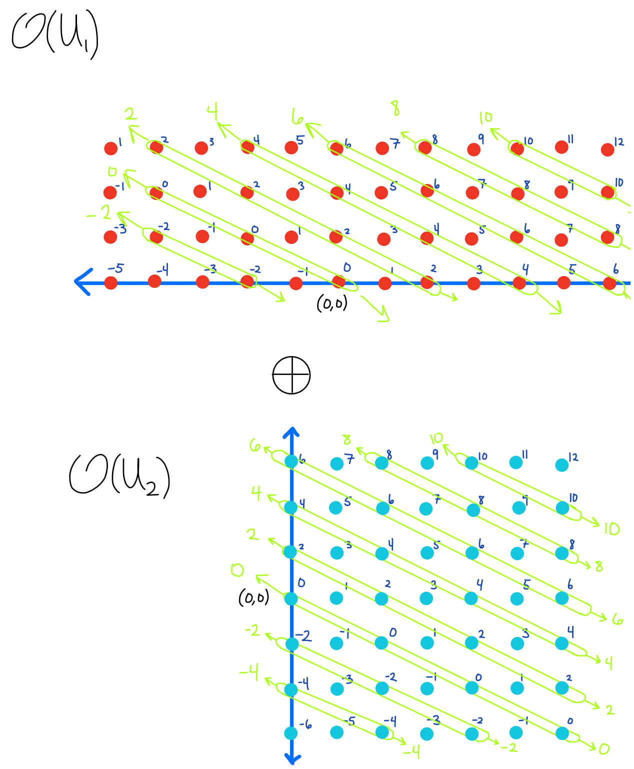

Figure 3.7: *$\mathcal{O}(U_1)$ and $\mathcal{O}(U_2)$ on $\mathbb{P}(1,2)$*

To build the Čech complex for $\mathcal{O}(n)$ on $\mathbb{P}(1,2)$, we map $\mathcal{O}(U_1)$ to $\mathcal{O}(U_2)$ using d^0, which shifts $\mathcal{O}(U_2)$ n steps to the left. For $\mathcal{O}(-1)$, and $H^*(\mathcal{O}(-1))$, $d^0(\check{C}^0)$ and $\check{C}^1$ are given in Figure 3.8. Here, H^1 is the kernel of d^1 by a dimension argument. Any lattice points labeled by both red and turquoise have elements in the kernel. But this does not occur for

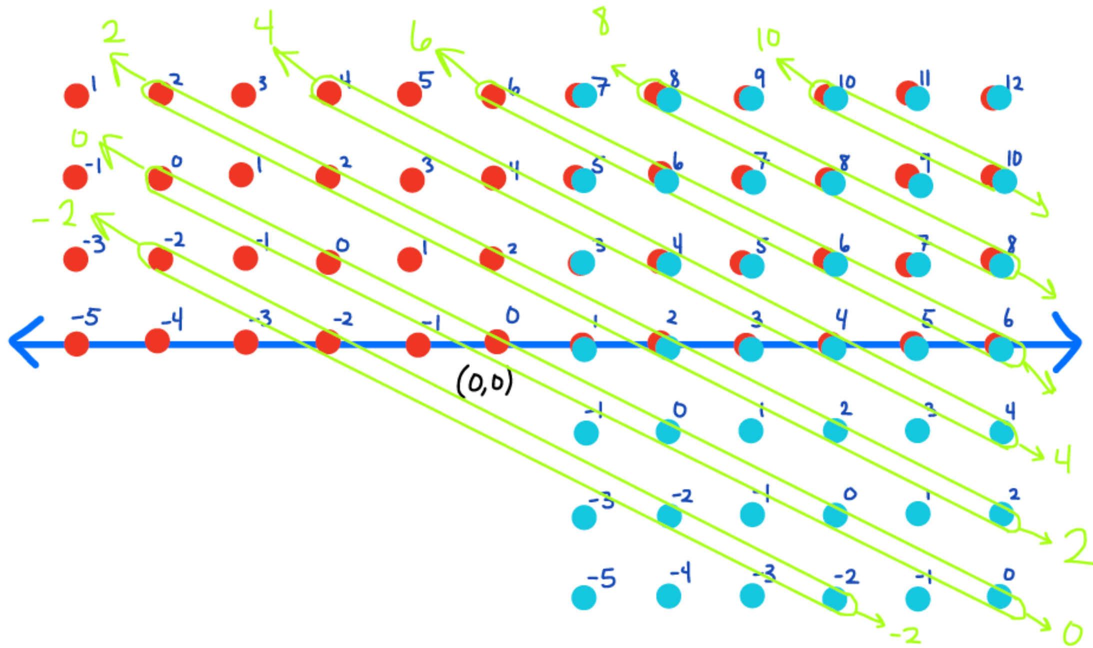

Figure 3.8: $d^0(\check{C}^0)$ *for* $\mathcal{O}(-1)$ *on* $\mathbb{P}(1,2)$

any points with degree 0, so $H^1(\mathbb{P}(1,2), \mathcal{O}(-1)) = 0$. Here, H^0 is the cokernel: but all degree 0 elements lie in the image of $\check{C}^0$, since each degree 0 lattice point is labeled either red or turquoise. Therefore,

$$H^i(\mathbb{P}(1,2), \mathcal{O}(-1)) = 0$$

for all i.

For $\mathcal{O}(-2)$, we would move the turquoise dots over to the right by 1 in Figure 3.8 (a total of 2 steps to the right). However, we still have no lattice points of degree 0 labeled by both red and turqoise lattice points, so $H^1(\mathbb{P}(1,2), \mathcal{O}(-2)) = 0$. Similarly, all degree 0 lattice points are labeled by either red or turqoise dots, so the cokernel is 0, and $H^0(\mathcal{O}(-2)) = 0$ on $\mathbb{P}(1,2)$.

Therefore

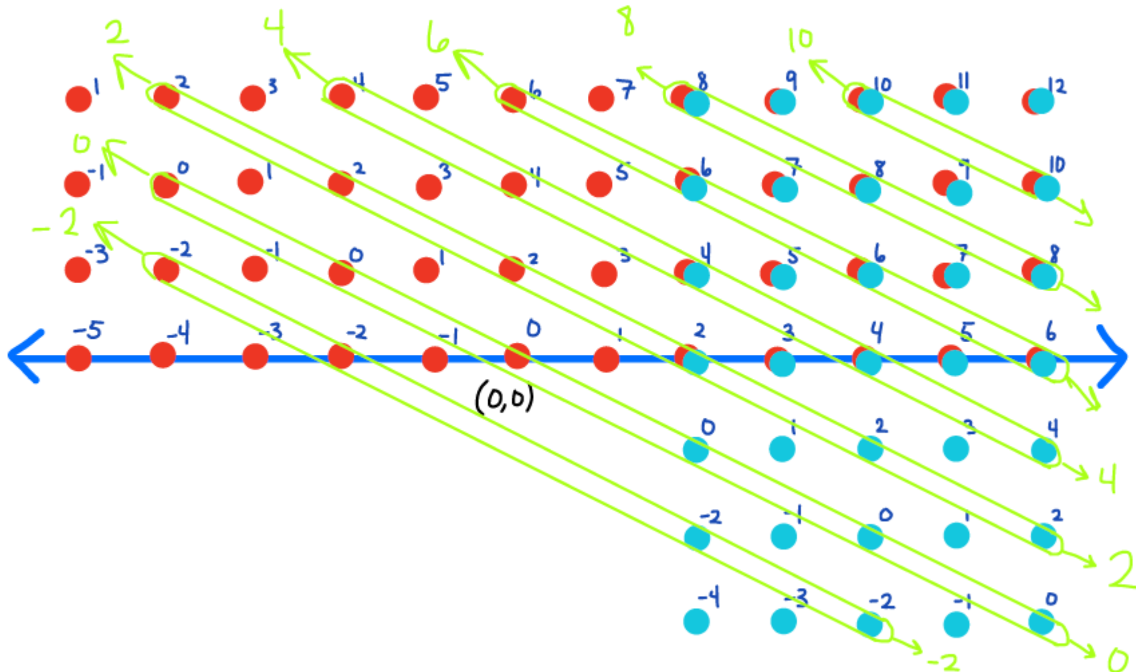

Figure 3.9: $d^0(\check{C}^0)$ *for* $\mathcal{O}(-2)$ *on* $\mathbb{P}(1,2)$

$$\begin{aligned} Ext(\mathcal{O}(2),\mathcal{O}(1)) \cong Ext(\mathcal{O}(1),\mathcal{O}) &\cong R\Gamma(\mathcal{O}(-1)) \\ &\cong H^*(\mathcal{O}(-1)) = 0 \end{aligned}$$

and

$$Ext(\mathcal{O}(2),\mathcal{O}) = R\Gamma(\mathcal{O}(-2)) = H^*(\mathcal{O}(-2)) = 0.$$

Note that $d^0(\check{C}^0)$ for $\mathcal{O}(-2)$ is given in Figure 3.9. To see that $\mathcal{O},\mathcal{O}(1),\mathcal{O}(2)$ generate $D^b(\mathbb{P}(a,b))$ for $a,b>0$ and coprime, we note exactness of

$$0 \to \mathcal{O}(-a-b) \to \mathcal{O}(-a) \oplus \mathcal{O}(-b) \to \mathcal{O} \to 0$$

on $\mathbb{P}(a,b)$. $\square$

3.8 $Bl_{[1:0:0]}(\mathbb{P}^2)$ and deforming to $\mathcal{H}_L^\epsilon$ for projective toric varieties which are smooth and non-unimodular in the Bayer-Popescu-Sturmfels sense

For toric varieties which are smooth and non-unimodular in the sense of Bayer-Popescu-Sturmfels, we'll need to deform the cellular complex $\mathcal{H}_L$ by an element $\epsilon \in \mathbb{R}^A$. We use ϵ to give transversal intersections at the vertices in $\mathcal{H}_L^\epsilon(0)$.
Here, we deform the cellular complex $\mathcal{H}_L$ to $\mathcal{H}_L^\epsilon$ for $Bl_{[1:0:0]}\mathbb{P}^2$ which is smooth, Fano, and unimodular as an illustrative example.

Recall that a sublattice $L \subset \mathbb{Z}^n$ is called **unimodular** if L is the image of an integer matrix B with linearly independent columns, such that all maximal minors of B lie in the set $\{0, 1, -1\}$[12]. If we blow-up the torus-invariant point $[1:0:0]$ of $\mathbb{P}^2$ (following Cox-Little-Schenck, I will call $X_\Sigma = Bl_{[1:0:0]}\mathbb{P}^2$ the toric variety associated to the fan in $N_\mathbb{R}$ given in Figure 3.10), then the lattice L given by the image of B in

$$0 \to M \xrightarrow{B} \mathbb{Z}^4 \xrightarrow{\pi} Cl(X_\Sigma) \to 0$$

where B^T is given by $\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$ is unimodular.
Here,

$$Cl(X_\Sigma) \cong \langle D_1, \dots, D_4 \rangle \big/ (D_1 + D_2 - D_4, D_2 + D_3 - D_4)$$

Since D_2 corresponds to the exceptional divisor in the blow-up, D_2 has self-intersection number -1:

$$D_2 \cdot D_2 = -1$$

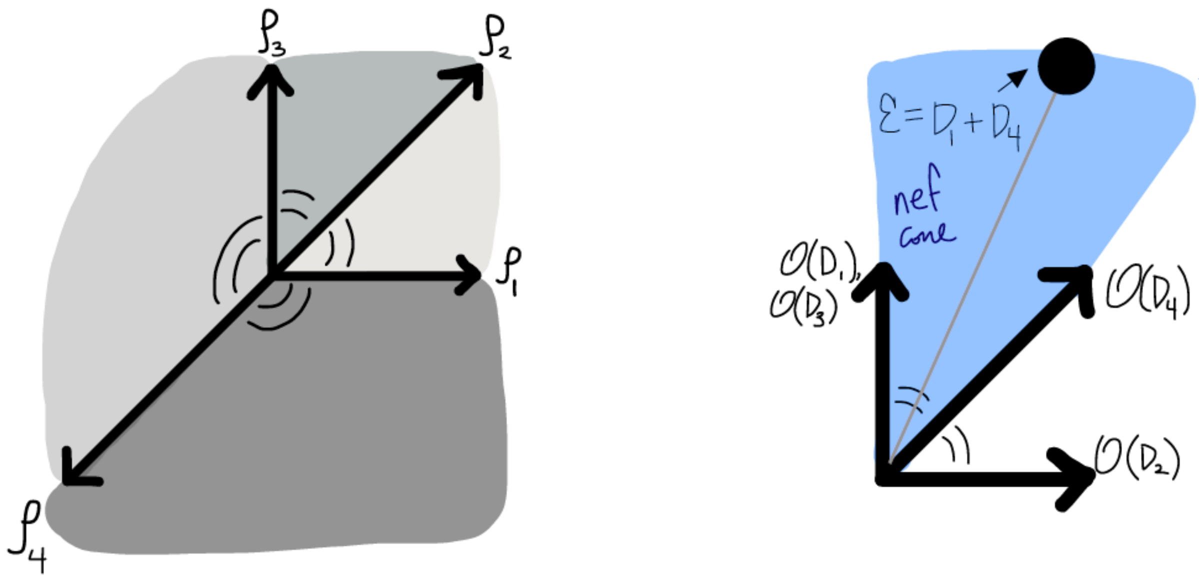

Figure 3.10: *The fan Σ and the nef cone for $Bl_{[1:0:0]}\mathbb{P}^2$*

while D_1, D_3, and D_4 have self-intersection number 1, so the nef cone of X_Σ is given by the span of D_1 and D_4. This description of the nef cone comes from a map

$$(\mathbb{R}^2)^\vee \xleftarrow{\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}} \mathbb{R}^4$$

corresponding to a fan in the first orthant of $\mathbb{R}^4$. Here, the element $D_1 + D_4$ in the nef cone corresponds to $\begin{bmatrix} \epsilon \\ 0 \\ 0 \\ \epsilon \end{bmatrix} = \epsilon \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^4$.

Deforming the cellular complex $\mathcal{H}_L$

Here, X_Σ is assumed to be smooth. If we consider $\mathbb{R}L = L \otimes_{\mathbb{Z}} \mathbb{R} \subset \mathbb{R}^4$ as before from the unimodular case, then the intersection of the infinite hyperplane arrangement $\mathcal{H}_L$ with $\mathbb{R}L$ will have more lattice points than just what appears in L[12]. That is, for X_Σ smooth and non-unimodular,

$$(\mathcal{H}_L \cap \mathbb{R}L) \setminus L \neq \emptyset.$$

$\mathbb{R}L \cap \mathcal{H}_L$ is given in Figure 3.11 for $Bl_{[1:0:0]}\mathbb{P}^2$, which is unimodular (hence no extra vertices appear). Here, vertical lines in yellow correspond to integer values of x_1, and x_1 increases in value as we move to the right in the diagram. Horizontal lines in black correspond to integer values of x_3, which increase as we move up. Diagonal lines in blue give integer values in x_2 and x_4, which increase up and to the right, and down and to the left, respectively.

Since more than two hyperplanes intersect in $\mathbb{R}L$ at each point, the cellular complex $\mathcal{H}_L$ does not have transversal intersections at each vertex.

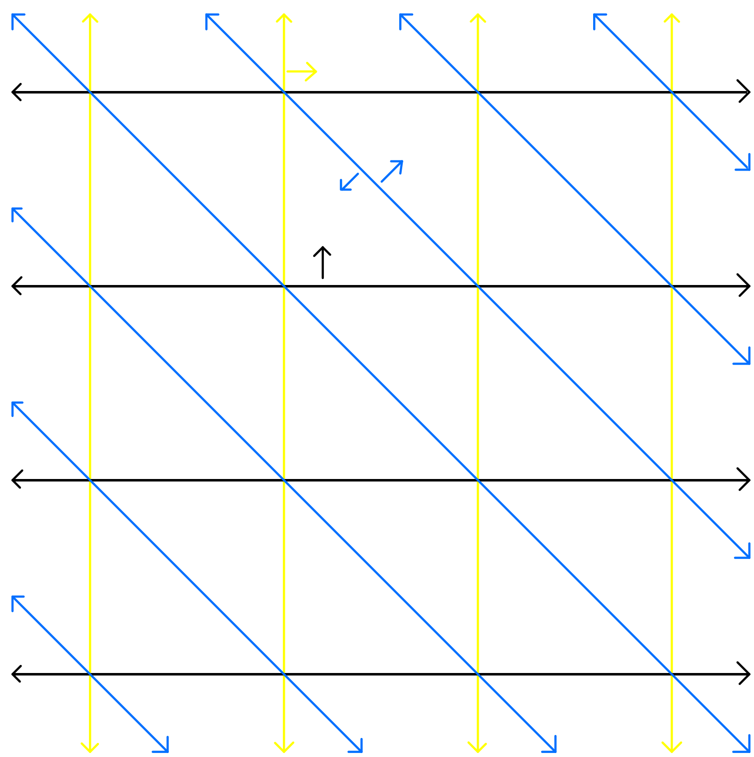

Figure 3.11: $\mathbb{R}L$, *showing* $\mathbb{R}L \cap \mathcal{H}_L \subset \mathbb{R}^4$

To remedy the lack of transversal intersection at each vertex $v \in \mathcal{H}_L(0)$, we adjust the infinite cellular complex $\mathcal{H}_L$ by deforming the hyperplane arrangement in a manner governed by our choice of ϵ in the nef cone, given by

$$x \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} \epsilon \\ 0 \\ 0 \\ \epsilon \end{bmatrix} = 0$$

corresponding to row 1 and row 2 of B, giving a basis for $L \subset \mathbb{R}^4$.

This gives the deformed hyperplane arrangement $\mathcal{H}_L^\epsilon$ corresponding to the shifts $e_1 + \epsilon \in \mathbb{Z}, e_4 + \epsilon \in \mathbb{Z}$ for $0 < \epsilon \ll 1$. We construct the Hull resolution $(\mathcal{F}^\bullet_{\mathcal{H}_L^\epsilon}, \partial^\epsilon)$ below. The deformed complex $\mathcal{H}_L^\epsilon$ is given in Figure 3.12, together with the monomial labelings given by taking the vertex labeling associated to the floor function for each coordinate of a given point in $\mathcal{H}_L^\epsilon \subset \mathbb{R}^4$. Vertices with common color labeling are given the same monomial vertex labeling in Figure 3.12. The arrows off of hyperplanes indicate direction of increase of x_i for $1 \leq i \leq 4$, as in Figure 3.11.

Here, we emphasize that $\mathbb{R}L$ is a plane in $\mathbb{R}^4$ along which we are considering the intersections with the hyperplane arrangement $\mathcal{H}_L = \{x_i \in \mathbb{Z} \mid 1 \leq i \leq 4\}$.

Taking the quotient of $\mathcal{H}_L^\epsilon$ by L and giving monomial labels from $M_{\Lambda(L)}$ given by the integer floor function in each component are given in Figure 3.13. Here, the differential

$$\partial_1^\epsilon : \bigoplus_{e \in \mathcal{H}_L^\epsilon(1)} S \to \bigoplus_{v \in \mathcal{H}_L^\epsilon(0)} S$$

is given by

$$\begin{array}{c|cccccccccc|}
v_4 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & x_3y_4 & 0 & -x_3y_1 \\
v_3 & 0 & 1 & -y_2 & 0 & 0 & 1 & 0 & 0 & -x_3y_1 & 0 \\
v_2 & 0 & 0 & 0 & 1 & -1 & 0 & -x_1y_4 & 0 & 0 & x_1y_3 \\
v_1 & y_2 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & x_1y_3 & 0 \\
v_0 & -x_2 & 0 & x_2 & 0 & 0 & 0 & x_4y_1 & -x_4y_3 & 0 & 0 \\
\hline
 & E_0 & E_1 & E_2 & E_3 & E_4 & E_5 & E_6 & E_7 & E_8 & E_9
\end{array}$$

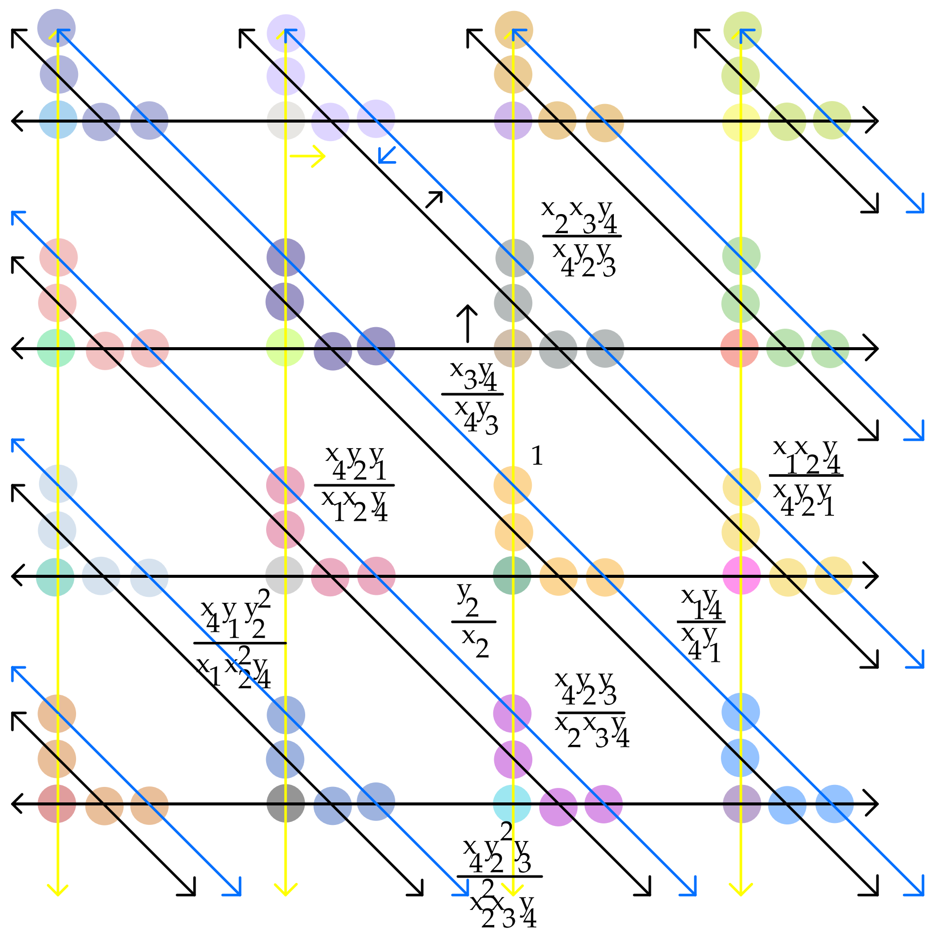

Figure 3.12: $\mathcal{H}_L^\epsilon$ *for* $Bl_{[1:0:0]}\mathbb{P}^2$ *deformed by* $\mathcal{O}(D_1) + \mathcal{O}(D_4)$

subject to the labeling in Figure 3.13. The color labeling of vertices and edges in Figure 3.13 corresponds to the color labeling in Figure 3.12. Exactness away from homological index $i = 0$ follows as before for the unimodular case by the same convexity argument used in Theorem 3.1 in Bayer-Popescu-Sturmfels[12]. We investigate the cokernel of ∂_0^ϵ in the following section.

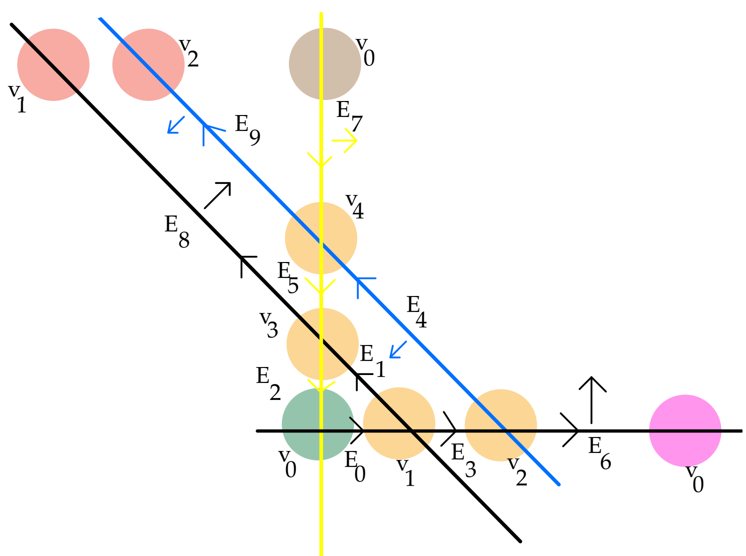

Figure 3.13: *The quotient cellular complex* $\mathcal{H}_L^\epsilon\!/_L$ *with monomial labelings from* $M_{\Lambda(L)}$

3.8.1 Cokernel of ∂_0^ϵ

Here, we consider the map

$$(\mathcal{F}_{\mathcal{H}_L^\epsilon}, \partial^\epsilon) \xrightarrow{f} S_{\prod x_i}$$

given by $\bigoplus_{v \in \mathcal{H}_L^\epsilon} S(-m_F) \to S_{\prod x_i}$ where a vertex in $\mathcal{H}_L^\epsilon$ maps to its vertex monomial label in $M_{\Lambda(L)}$.

For $Bl_{[1:0:0]}\mathbb{P}^2$, we have that $L = \left\langle \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\rangle$,

$$\begin{aligned} I_L &= \langle x^{v_+} - x^{v_-} \mid v \in L \rangle \\ &= \langle x_1x_2 - x_4, x_2x_3 - x_4, x_1 - x_3 \rangle, \end{aligned}$$

$$\begin{aligned} J_L &= \langle x^u y^v - x^v y^u \mid u - v \in L \rangle \\ &= \langle x_1x_2y_4 - x_4y_1y_2, x_2x_3y_4 - x_4y_2y_3, x_1y_3 - x_3y_1 \rangle \end{aligned}$$

and $M_{\Lambda(L)}$ as an $S-$submodule of T has the infinite generating set $\{x^u y^{-u} \mid u \in L\}$. In the quotient by L, the cellular complex $\mathcal{H}_L^\epsilon$ carries monomial labels given in Figure 3.14. When we consider monomial labels on vertices in the quotient $\mathcal{H}_L^\epsilon/L$, we only need to consider the monomial labels on vertices given in Figure 3.14 due to which vertices in $\mathcal{H}_L^\epsilon(0)$ carry the same monomial label.

Here, $L \cap \mathbb{N}^n = \{0\}$ implies that $M_{\Lambda(L)} \ni 1$, so that $M_{\Lambda(L)} \subset Im(f)$ above, by considering the action of L on the quotient from Figure 3.14. Modulo the action of L, there is one additional monomial $\frac{y_2}{x_2}$ in $Im(f)$ which is not contained in $M_{\Lambda(L)}$. Note that since

$$\{\frac{x_1x_2y_4}{x_4y_1y_2}, \frac{x_2x_3y_4}{x_4y_2y_3}, \frac{x_1y_3}{x_3y_1}\} \subset M_{\Lambda(L)},$$

we also have

$$\{\frac{x_4y_1y_2}{x_1x_2y_4}, \frac{x_4y_2y_3}{x_2x_3y_4}, \frac{x_3y_1}{x_1y_3}\} \subset M_{\Lambda(L)}.$$

Now for $X_\Sigma = Bl_{[1:0:0]}\mathbb{P}^2$,

$$I_{irr} = \langle x_3x_4, x_1x_4, x_1x_2, x_2x_3\rangle$$
$$= \langle x_1, x_3\rangle \cap \langle x_2, x_4\rangle$$

and I_{irr} for $X_\Sigma \times X_\Sigma$ is given by

$$\langle x_1, x_3\rangle \cap \langle x_2, x_4\rangle \cap \langle y_1, y_3\rangle \cap \langle y_2, y_4\rangle .$$

Now, for any monomial q in I_{irr} for $X_\Sigma \times X_\Sigma$, we must have that either x_2 or x_4 divides q. If x_2 divides q, then

$$x_2\left(\frac{y_2}{x_2}\right) = y_2 \in M_{\Lambda(L)}$$

since $M_{\Lambda(L)} \ni 1$ shows that $q \cdot \frac{y_2}{x_2} \in M_{\Lambda(L)}$. If $x_4|q$, then

$$y_3x_4 \cdot \left(\frac{y_2}{x_2}\right) = \left(\frac{x_4y_2y_3}{x_2x_3y_4}\right) \cdot x_3y_4 \in M_{\Lambda(L)}$$

shows that $q \cdot \left(\frac{y_2}{x_2}\right)$ is in the image of the action of S on $q \cdot \left(\frac{y_2}{x_2}\right)$. Therefore, the cokernel of the inclusion of $M_{\Lambda(L)} \hookrightarrow Im(f)$ is torsion with respect to I_{irr} for $X_\Sigma \times X_\Sigma$. In general, we say that an $R-$module M is I torsion iff there exists $k \in \mathbb{Z}_+$ such that $I^kM = 0$. Here, $k = 1$ suffices to show that the cokernel of the inclusion of $M_{\Lambda(L)} \hookrightarrow Im(f)$ is torsion with respect to I_{irr} for $X_\Sigma \times X_\Sigma$.

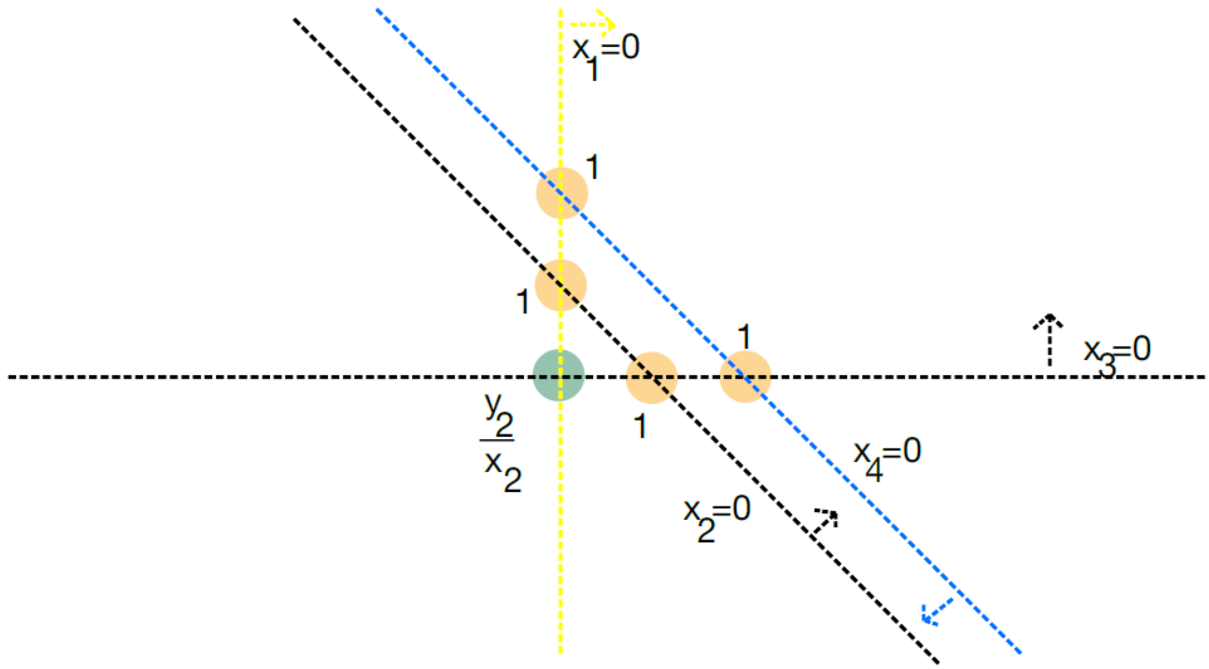

Figure 3.14: *Monomial labels on vertices $v \in \mathcal{H}_L^\epsilon(0)$ modulo L*

3.8.2 Moving around in the effective cone

Here we consider coker $(M_{\Lambda(L)} \hookrightarrow Im(f))$ for ϵ in the interior of the other chamber of the effective cone, spanned by $\mathcal{O}(D_2)$ and $\mathcal{O}(D_4)$. (Again, D_2 corresponds to the exceptional divisor of the blow-up of $Bl_{[1:0:0]}\mathbb{P}^2$.) Our previous discussion of coker$(M_{\Lambda(L)} \hookrightarrow Im(f))$ from Section 3.8.1 took place inside the nef cone of $N^1(X_\Sigma)$, with $N^1(X_\Sigma) \cong Cl(X_\Sigma) \otimes_{\mathbb{Z}} \mathbb{R}$ since X_Σ is smooth. More generally, we can ask about the cokernel of the inclusion of $M_{\Lambda(L)} \hookrightarrow Im(f)$ inside of the other chamber of the effective cone, given in Figure 3.15. Inside of this cone spanned by $\mathcal{O}(D_2)$ and $\mathcal{O}(D_4)$ in $Cl(X_\Sigma) \otimes_{\mathbb{Z}} \mathbb{R}$, we have a corresponding fan for which the ray ρ_2 is removed. This gives the fan for $\mathbb{P}^2$, though we retain the information that ρ_2 no longer lives in any maximal cone so that the irrelevant ideal for X_Σ becomes

$$I_{irr} = \langle x_1, x_3, x_4 \rangle \cap (x_2)$$

and the irrelevant ideal for $X_\Sigma \times X_\Sigma$ in this case is

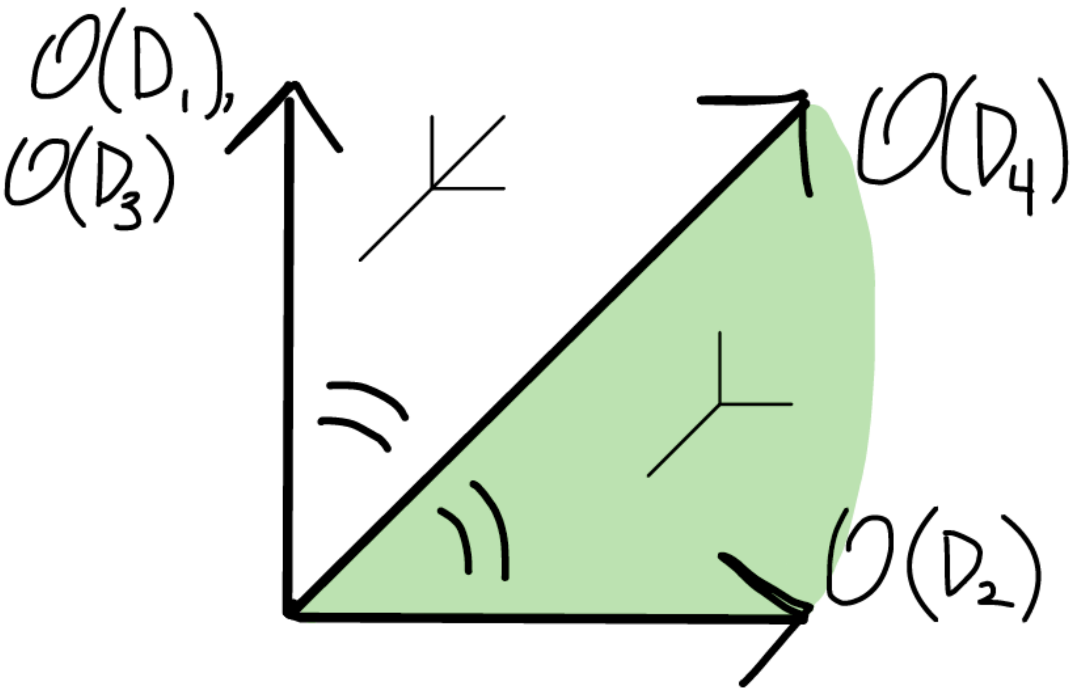

Figure 3.15: *Chambers of the Effective Cone*

$$I_{irr} = \langle x_1, x_3, x_4 \rangle \cap (x_2) \cap \langle y_1, y_3, y_4 \rangle \cap (y_2).$$

Taking $\epsilon = \begin{bmatrix} 0 \\ \epsilon \\ 0 \\ \epsilon \end{bmatrix}$ corresponding to $\mathcal{O}(D_2) + \mathcal{O}(D_4)$ in the interior of the chamber of the effective cone spanned by $\mathcal{O}(D_2)$ and $\mathcal{O}(D_4)$, we have the deformed cellular complex $\mathcal{H}_{\mathcal{L}}^{\epsilon}$ whose quotient by L is given in Figure 3.16. Here, $x_2 = 0$ and $x_4 = 0$ are labeled in blue and yellow, respectively, since those planes are deformed in $\mathbb{R}L$.

In this case, I_{irr} for $X_\Sigma \times X_\Sigma$ is given above and $\operatorname{coker}(M_{\Lambda(L)}) \ni \frac{y_1}{x_1}, \frac{y_3}{x_3}$ and is spanned by translates of these elements by L. Note that in this case, at least one of x_1, x_3, or x_4 must divide any monomial q in the irrelevant ideal I_{irr} of $X_\Sigma \times X_\Sigma$.

For L-translates of $\frac{y_1}{x_1}$, we note that if x_1 divides q, then

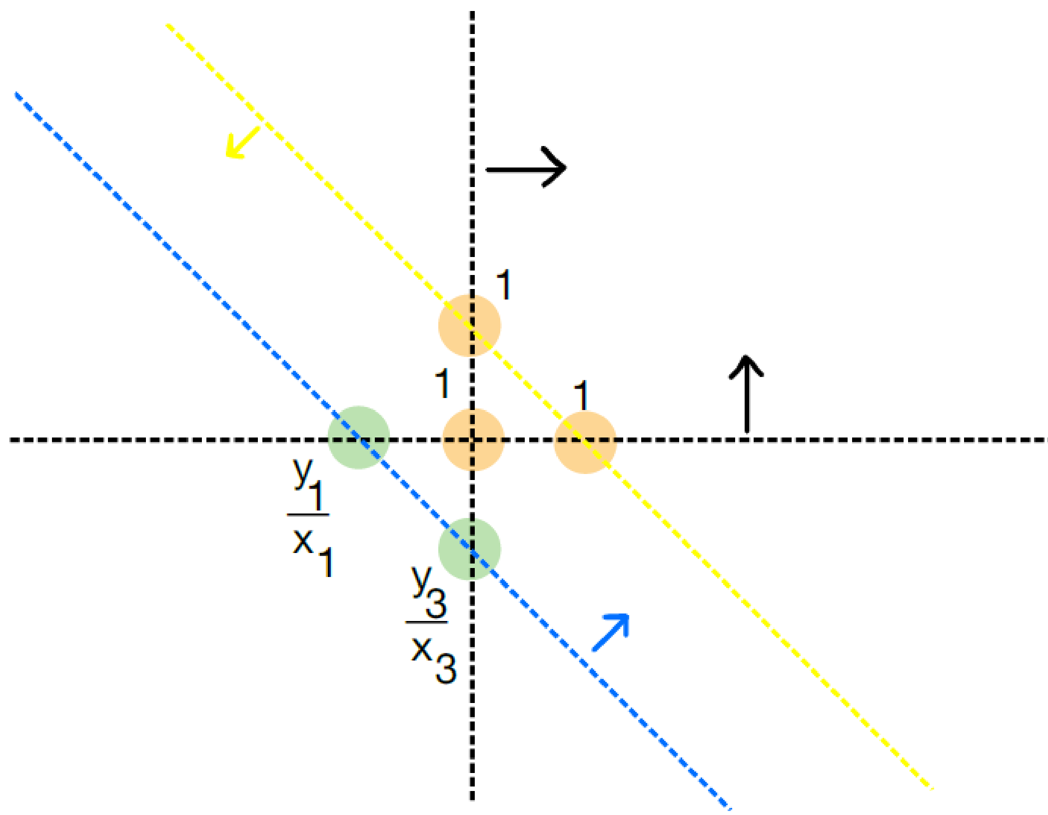

Figure 3.16: *Deformed complex* $\mathcal{H}_L^\epsilon$ *for* ϵ *corresponding to* $\mathcal{O}(D_2) + \mathcal{O}(D_4)$

$$x_1 \cdot (\frac{y_1}{x_1}) = y_1 \in M_{\Lambda(L)}$$

since $1 \in M_{\Lambda(L)}$, and $M_{\Lambda(L)}$ is an $S-$submodule of $T = k[x_1^{\pm}, \ldots, x_n^{\pm}, y_1^{\pm}, \ldots, y_n^{\pm}]$. If x_3 divides q, then

$$x_3 \cdot \frac{y_1}{x_1} = y_3 \cdot \left(\frac{x_3 y_1}{x_1 y_3}\right) \in M_{\Lambda(L)}.$$

If x_4 divides q, then

$$x_4 y_2 \left(\frac{y_1}{x_1}\right) = x_2 y_4 \left(\frac{x_4 y_1 y_2}{x_1 x_2 y_4}\right) \in M_{\Lambda(L)}$$

so that $\overline{\frac{y_1}{x_1}} = \overline{0} \in \text{coker}(M_{\Lambda(L)} \hookrightarrow Im(f))$.

Next, for ($L-$translates of) $\frac{y_3}{x_3}$, we again have that at least one of x_1, x_3, or x_4 divides any monomial q in I_{irr}, the irrelevant ideal for $X_\Sigma \times X_\Sigma$. If x_1 divides q, then

$$\frac{y_3}{x_3} \cdot x_1 = \left(\frac{x_1 y_3}{x_3 y_1}\right) \cdot y_1 \in M_{\Lambda(L)}$$

If x_3 divides q, then

$$\frac{y_3}{x_3} \cdot x_3 = y_3 \in M_{\Lambda(L)}$$

since $1 \in M_{\Lambda(L)}$. Lastly, if $x_4|q$, then

$$\left(\frac{y_3}{x_3}\right)x_4y_2 = \left(\frac{x_4y_2y_3}{x_2x_3y_4}\right)x_2y_4 \in M_{\Lambda(L)}.$$

This shows that $\overline{\frac{y_3}{x_3}} = 0 \in \text{coker}(M_{\Lambda(L)} \hookrightarrow Im(f))$. Hence,

$$\text{coker}(M_{\Lambda(L)} \hookrightarrow Im(f)) = 0$$

modulo I_{irr} for $X_\Sigma \times X_\Sigma$ for $\epsilon = \mathcal{O}(D_2) + \mathcal{O}(D_4)$ in the interior of the chamber spanned by $\mathcal{O}(D_2)$ and $\mathcal{O}(D_4)$ in the effective cone of the secondary fan.

3.8.3 General argument for coker$(M_{\Lambda(L)} \hookrightarrow Im(f))$ for ϵ in the effective cone

Given X_Σ smooth and projective with no torus factors, we have the following exact sequence:

$$\begin{array}{ccccccccc} 0 & \leftarrow & Cl(T)_{\mathbb{R}} & \leftarrow & \mathbb{R}^A & \leftarrow & L_{\mathbb{R}} & \leftarrow & 0 \\ & & & & \Cup & & & & \\ & & & & \epsilon & & & & \end{array}$$

where $\epsilon \mapsto \overline{\epsilon}$ in the effective cone of $\mathcal{F}_{\Sigma(A)}$. Now $\overline{\epsilon}$ gives a convex function $g_\epsilon : N_{\mathbb{R}} \to \mathbb{R}$ which is linear on all cones of a corresponding fan Σ_ϵ such that the maximal domains of linearity correspond to maximal cones in Σ_ϵ. So g_ϵ gives the fan Σ_ϵ with corresponding irrelevant ideal I_{irr} for $\Sigma_\epsilon \times \Sigma_\epsilon$ given by

$$\left\langle \prod_{i \in I_\emptyset} x_i \prod_{\rho \notin \sigma} x_\rho \mid \sigma \in \Sigma_{max} \right\rangle \cap \left\langle \prod_{i \in I_\emptyset} y_i \prod_{\rho \notin \sigma} y_\rho \ \sigma \in \Sigma_{max} \right\rangle$$

Continuing with the map $(\mathcal{F}^\bullet_{\mathcal{H}^\epsilon_L}, \partial^\bullet_\epsilon) \xrightarrow{f} S_{\prod x_i}$ and our consideration of $coker(M_{\Lambda(L)} \hookrightarrow Im(f))$ modulo I_{irr} for $\Sigma_\epsilon \times \Sigma_\epsilon$, we consider $p \in \mathbb{R}^{\mathcal{A}}$ such that

$p \mapsto \overline{p} \in \mathcal{H}_L^\epsilon(0)$ near $\overline{0}$ in $\mathbb{R}L$. We have previously referred to $\overline{p} \in \mathbb{R}L$ as p. Here, we take

$$\overline{p} = \bigcap_{\rho \in \mathcal{B}_p} \{x_\rho = 0\}$$

as a definition of $\mathcal{B}_p$. Now transversality of $\mathcal{H}_L^\epsilon$ implies that $\mathcal{B}_p$ gives a basis for $\mathbb{R}L$. Since p is near $\vec{0}$ in $\mathbb{R}^{\mathcal{A}}$, the monomial label m_p is squarefree, and $\epsilon \in \mathbb{R}^{\mathcal{A}}_{\geq 0}$ implies that m_p can be written $\frac{y_{i_1} \cdots y_{i_\ell}}{x_{i_1} \cdots x_{i_\ell}}$.

By the definition of $\mathcal{H}_L^\epsilon$, the monomial label on p is given by

$$m_p = \frac{x^{\vec{u}}}{y^{\vec{u}}} \mid \vec{u} = \lfloor p \rfloor.$$

We now adjust g_ϵ by subtracting a linear functional ℓ_p determined by $\ell_p(u_\rho) = \epsilon_\rho$ for $\rho \in \mathcal{B}_p$. ℓ_p is determined by this information, since $\mathcal{B}_p$ gives a basis for $\mathbb{R}L$ and for N. That is, we construct

$$\tilde{g}_\epsilon := g_\epsilon - \ell_p$$

so that we have

$$p = \tilde{g}_\epsilon(\{u_\rho\})$$

except for coordinates in $I_\emptyset$. Since any variable corresponding to an element of $I_\emptyset$ always appears in the support of any monomial in the irrelevant ideal, we may assume without loss of generality that $p = \tilde{g}_\epsilon$. To show that $coker(M_{\Lambda(L)} \hookrightarrow Im(f)) = 0$ modulo I_{irr}, let σ be any maximal cone in Σ_{max}. From convexity of g_ϵ, it follows that $\tilde{g}_\epsilon$ is also convex. This gives a subdivision of maximal cones in Σ such that either:

(i) $\sigma \subseteq \tilde{g}_\epsilon^{-1}(\mathbb{R}_{\geq 0})$

(ii) $\sigma \subseteq \tilde{g}_\epsilon^{-1}(\mathbb{R}_{<}0)$

(iii) Neither (i) nor (ii) holds

In case (i), since we only need to worry about what's not in $I_\emptyset$, we can suppose that $g_\epsilon(u_\rho) = \epsilon_\rho$ so that $p = \tilde{g}_\epsilon(\{u_\rho\})$. If we choose ϵ to be small so that $|\epsilon| << 1$, then ℓ_p

remains small as a continuous function of ϵ which gives that the coordinates of $p \in (-1,1)$ and the floor function of all coordinates is in the set $\{-1,0\}$. Therefore, the set of indices $\{i_j\}$ for which x_{i_j} is in the support of m_p are exactly the coordinates for which the floor function returns -1. Since $x^{\hat{\sigma}} = \prod_{\rho \notin \sigma} x_\rho$ is a generator of I_{irr} (ignoring $I_\emptyset$), we have that all variables cooresponding to rays of σ do not lie in the support of m_p, so that $x^{\hat{\sigma}}$ can clear denominators to give $x^{\hat{\sigma}} \cdot m_p \in M_{\Lambda(L)}$.

In case (ii), we have that $\sigma \subseteq \tilde{g}_\epsilon^{-1}(\mathbb{R}_{<0})$, $m_p = \frac{y_{i_1} \cdots y_{i_\ell}}{x_{i_1} \cdots x_{i_\ell}}$ and $x^{\hat{\sigma}} = \prod_{\rho \notin \sigma} x_\rho$. For p near $\vec{0}$ we show that there exists $k > 0$ such that $I_{irr}^k \cdot m_p \in M_{\Lambda(L)}$ and $coker(M_{\Lambda(L)} \hookrightarrow Im(f)) = 0$ by adding in the assumption that X_Σ is Fano. If X_Σ is Fano, then the anticanonical divisor $D = \sum_{\rho \in \Sigma(1)} D_\rho$ is very ample so that we can define $v \in L$ to be $v(u_\rho) = -1$ for $\rho \in \sigma$ and away from σ, $v(\rho') > -1$ for $\rho' \notin \sigma$. Since $v \in L$ is an element of the lattice L, $v(u_\rho) \in \mathbb{Z}$ for all $\rho \in \Sigma(1)$ implies $v(\rho') \geq 0$ for all $\rho' \notin \sigma$.

Now $\frac{x^{\vec{v}}}{y^{\vec{v}}} \in M_{\Lambda(L)}$ and we need only consider variables in the support of m_p for which $\rho \in \sigma$ from our previous description of I_{irr}. Here, there exists some $\vec{w} \in \mathbb{N}^{\mathcal{A}}$ with $y^{\vec{w}} \in S$ and some $\{s_\rho\} \subset \mathbb{N}$ so that

$$\prod_{\rho \notin \sigma} x_\rho^{s_\rho} \cdot m_p = y^w \cdot \frac{x^{\vec{v}}}{y^{\vec{v}}} \in M_{\Lambda(L)}$$

and

$$(x^{\hat{\sigma}})^{\max\{s_\rho\}} \cdot m_p = q \cdot \prod_{\rho \notin \sigma} x_\rho^{s_\rho} m_p \in M_{\Lambda(L)}$$

for some $q \in S$. Here we use the fact that $|\mathcal{A}| < \infty$ to take a maximum, and the fact that I_{irr} has a finite generating set since S is Noetherian.

We generalize statements (i)-(iii) in the next subsection without the Fano assumption and therefore omit consideration of (iii) here.

For $p \in \mathcal{H}_L^\epsilon(0)$ away from $\vec{0}$

For p away from $\vec{0}$ and X_Σ smooth (not assumed to be Fano here), we make use of the following lemmas:

Lemma 1: Given $p \in \mathcal{H}_L^\epsilon(0)$ and $\sigma_1, \sigma_2 \in \Sigma(n)$, there exists $k > 0$ with $(x^{\hat{\sigma}_1} y^{\hat{\sigma}_2})^k \cdot m_p \in M_{\Lambda(L)}$ iff there exists $v \in L$ and an $\ell > 0$ such that $(x^{\hat{\sigma}_1} y^{\hat{\sigma}_2})^\ell \cdot m_{p+v} \in M_{\Lambda(L)}$.

Proof. Any $v \in L$ has integer coordinates so that $\lfloor v \rfloor = v$ and $m_v = \frac{x^{\lfloor v \rfloor}}{y^{\lfloor v \rfloor}} = \frac{x^v}{y^v}$. This implies that adding $v \in L$ to p shifts all coordinates by only integer amounts, so that no cancellation occurs when we take a floor function:

$$\lfloor p + v \rfloor = \lfloor p \rfloor + v$$

In particular, we note that if both p and v are in L, then $m_{p+v} = m_p \cdot m_v$ since $p, v \in L \implies p + v \in L$. So if there exists $k > 0$ with $(x^{\hat{\sigma}_1} y^{\hat{\sigma}_2})^k \cdot m_p \in M_{\Lambda(L)}$, then $(x^{\hat{\sigma}_1} y^{\hat{\sigma}_2})^k \cdot m_{p+v} = (x^{\hat{\sigma}_1} y^{\hat{\sigma}_2})^k \cdot m_p m_v \in M_{\Lambda(L)}$ since $M_{\Lambda(L)}$ is closed under the action of $S[L]$. Now closure of $M_{\Lambda(L)}$ under the action of $S[L]$ also proves the reverse direction. $\square$

For the following **Claim**, we require the **Separation Lemma**[23], which separates convex sets by a hyperplane:

Separation Lemma: If σ and σ' are convex polyhedral cones whose intersection τ is a face of each, then there is a u in $\sigma^\vee \cap (-\sigma')^\vee$ with

$$\tau = \sigma \cap u^\perp = \sigma' \cap u^\perp.$$

Generalizing the previous cases (i)-(iii) from Section 3.8.3, we make the following claim.

Claim: Given $p \in \mathcal{H}_L^\epsilon(0)$, for all $\sigma_1, \sigma_2 \in \Sigma(n)$, there exists $u \in L, k \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{N}^{\Sigma(1)}$ such that $(x^{\hat{\sigma}_1} y^{\hat{\sigma}_2})^k m_p = x^\alpha y^\beta m_u \in M_{\Lambda(L)}$.

Proof. The **Claim** holds iff there exists $u \in L, k \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{N}^{\Sigma(1)}$ such that

$$(x^{\hat{\sigma}_1}y^{\hat{\sigma}_2})^k m_p m_{-u} = x^\alpha y^\beta \in S$$

Let $\lfloor p_{\rho_i} \rfloor := \tilde{a}_{\rho_i}$ for all i so that

$$\begin{aligned}\lfloor p \rfloor &= (\lfloor p_{\rho_1} \rfloor, \ldots, \lfloor p_{\rho_{\Sigma(1)}} \rfloor)\\ &= (\tilde{a}_{\rho_1}, \ldots, \tilde{a}_{\rho_{|\Sigma(1)|}})\end{aligned}$$

and define the sets A_+, A_- by

$$m_p = \frac{\prod_{A_+} x_\rho^{a_\rho} \prod_{A_-} y_\rho^{a_\rho}}{\prod_{A_+} y_\rho^{a_\rho} \prod_{A_-} x_\rho^{a_\rho}}$$

with $a_\rho = |\tilde{a_\rho}|$ for all ρ. Since X_Σ is smooth, there exists $u_1 \in L$ such that $u_1(e_\rho) = a_\rho$ for all $\rho \in (\sigma_1 \cap \sigma_2)(1) = \sigma_1(1) \cap \sigma_2(1)$. By the **Separation Lemma**, there exists $u_2 \in L$ such that

$$u_2\Big|_{\sigma_1 \cap \sigma_2} = 0, u_2\Big|_{\sigma_1 \backslash \sigma_2} > 0, u_2\Big|_{\sigma_2 \backslash \sigma_1} < 0.$$

Since $x^{\hat{\sigma}_1}y^{\hat{\sigma}_2}$ is a generator of I_{irr}, up to torsion by I_{irr} it suffices to consider only factors (over S) in the numerator and denominator of m_p for variables x_ρ and y_ρ for $\rho \in \sigma_1 \cup \sigma_2$. Now there exists $k_0 \in \mathbb{N}$ such that

$$\begin{aligned}(x^{\hat{\sigma}_1}y^{\hat{\sigma}_2})^{k_0} m_p &= \frac{f_1}{\prod_{\sigma_2 \cap A_+} y_\rho^{a_\rho} \prod_{\sigma_1 \cap A_-} x_\rho^{a_\rho}}\\ &= \frac{f_2}{\prod_{\sigma_1 \cap \sigma_2 \cap A_+} y_\rho^{a_\rho} \prod_{(\sigma_2 \backslash \sigma_1) \cap A_+} y_\rho^{a_\rho} \prod_{\sigma_1 \cap \sigma_2 \cap A_-} x_\rho^{a_\rho} \prod_{(\sigma_1 \backslash \sigma_2) \cap A_-} x_\rho^{a_\rho}} \quad \text{[subdivision of sets } A_+, A_-\text{]}\end{aligned}$$

for some monomials $f_1, f_2 \in S$. Now $m_p m_{-u_1} = m_{p-u_1}$ since $u_1 \in L$ gives that there exists

$k_1 \in \mathbb{N}$ such that

$$(x^{\hat{\sigma}_1}y^{\hat{\sigma}_2})^{k_1}m_{p-u_1} = \frac{f_3}{\prod_{(\sigma_2\backslash\sigma_1)\cap A_+} y_\rho^{b_\rho} \prod_{(\sigma_1\backslash\sigma_2)\cap A_-} x_\rho^{b_\rho}}$$

for some $\{b_\rho\} \subset \mathbb{N}$ and some monomial $f_3 \in S$, so that there exists $\ell \in \mathbb{N}$ such that $m_{p-u_1}m_{-\ell u_2} = m_{p-u_1-\ell u_2}$ since $\ell u_2 \in L$ gives

$$(x^{\hat{\sigma}_1}y^{\hat{\sigma}_2})^{k}m_{p-u_1-\ell u_2} = f_4$$

for some monomial $f_4 = x^\alpha y^\beta \in S$, for some $\alpha, \beta \in \mathbb{N}^{|\Sigma(1)|}$.

□

By Lemma 1, this implies there exists $k \in \mathbb{N}$ such that $(x^{\hat{\sigma}_1}y^{\hat{\sigma}_2})^k \cdot m_p \in M_{\Lambda(L)}$ for any choice of $\sigma_1, \sigma_2 \in \Sigma(n)$ so that $coker(M_{\Lambda(L)} \hookrightarrow Im(f)) = 0$ for ϵ in the Effective Cone.

In fact, this argument shows that for $p \in \mathcal{H}_L(0)$, $coker(M_{\Lambda(L)} \hookrightarrow Im(f)) = 0$ so that we have a resolution of the diagonal for any smooth toric variety X_Σ, without requiring the deformation $\mathcal{H}_L^\epsilon$.

This also strengthens the construction of the diagonal object giving the kernel of a Fourier-Mukai transformation inducing the identity in $D^b_{Coh}(X'_\Sigma)$ to the case that $X'_\Sigma = {X_\Sigma}/{\mu}$ is a global quotient of a smooth toric variety X_Σ by a finite abelian group μ. That is, since smooth toric varieties have an open cover by affine spaces:

$$X_\Sigma = \cup_{\sigma \preceq \Sigma(n)} U_\sigma$$

with $U_\sigma = \text{Spec } \mathbb{C}[\sigma^\vee \cap M] \cong \mathbb{A}^m$ for $m = \text{rank } (M) = \text{rank } (N)$, we have that the same diagonal object

$$\pi(M_{\Lambda(L)}) = M_{\Lambda(L)} \otimes_{S[\Lambda(\tilde{L})]} S$$

and corresponding resolution $(\mathcal{F}_{\mathcal{H}^\epsilon_L/\tilde{L}}, \partial)$ carry through in the case of a global quotient

$$X'_\Sigma = {}^{X_\Sigma}\!\diagup_{\mu}$$

with X_Σ smooth (not necessarily unimodular) and μ a finite abelian group.

Appendix A

Appendix

Lemma 2 from Section 3.8.3 for X_Σ smooth, Fano, not necessarily unimodular: For any $p \in \mathcal{H}_L^\epsilon(0)$ and $\sigma \in \Sigma(n)$, there exists $v_\sigma \in L$ with $(p+v_\sigma)_\rho < 0$ only on $\sigma(1)$, and $(p+v_\sigma)_\rho > 0$ for $\rho \notin \sigma(1)$.

Proof. We previously noted that since X_Σ is Fano, there exists $v \in L$ such that

$$\begin{cases} v(u_\rho) < 0 & \forall\ \rho \in \sigma(1), \\ v(u_{\rho'}) \geq 0 & \forall \rho' \notin \sigma(1) \end{cases}$$

Now there exists $r \in \mathbb{Z}$ such that $(r \cdot v)_\rho << 0$ for all coordinates with $\rho \in \sigma(1)$ so that all $\rho \in \sigma(1)$ become much less than any coordinate $\rho' \notin \sigma(1)$. Additionally, there exists a unique $\tilde{v} \in L$ such that

$$\begin{cases} (p+\tilde{v})(u_\rho) = 0 & \forall\ \rho \in \sigma(1) \\ (p+\tilde{v})(u_{\rho'}) > 0 & \forall\ \rho' \notin \sigma(1). \end{cases}$$

Now adding on sufficiently large multiples of v and $\tilde{v}$ ensure that there exists $r, k \in \mathbb{Z}$ such that $v_\sigma = rv + k\tilde{v}$ gives that $p + v_\sigma$ satisfies

$$\begin{cases} (p+v_\sigma)_\rho < 0 & \text{iff } \rho \in \sigma(1), \\ (p+v_\sigma)_\rho > 0 & \text{otherwise.} \end{cases}$$

□

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