AN APPROACH TO OPTIMIZING ECONOMIC FACTORS FOR A PROBABALISTIC CONVEYOR LOADING MODEL

by

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INTRODUCTION

> A large number of problems in production management are of the type where the question asked is, "how much?," "how many?," "when?," "how long?," "where?," and the like. Essentially the problem requires a single decision which often can be characterized by one number or a range of numbers. Such problems as those which involve inventory levels, scale of operations, number of maintenance personnel, and equipment investment are of this type. Almost always these problems involve some measure of uncertainty. (1)

The topic of this paper is one such problem. Specifically, the subject discussed is the economic analysis of conveyor system design and operation through the derivation of a mathematical model.

Conveyorization has received increased attention from industrial engineers and managers in recent years. As the marginal utility of improved methods of fabrication and processing materials gradually declines, it has become profitable to study the opportunities for cost reduction through mechanization of materials handling, delivery, and storage. Mechanization of this function has often taken the form of elaborate and expensive conveyor systems.

Because of the inherent variability of loading and unloading a typical conveyor system, it is often difficult to specify optimum conveyor speeds, capacities and similar parameters. As a result, many systems have been installed on the basis of such qualitative measures as

1. the achievements of competitors
2. the recommendation of trusted equipment salesmen

3. the amount of capital investment which can be obtained from higher management.

Recent advances in theories of operations research, coupled with the tremendous strides made in electronic digital computation, have made an analytic solution possible for many previously intractable conveyor analysis problems.

In this paper the emphasis is on development of a mathematical model to describe a random pattern of loading a conveyor from a series of independent stations. Analysis of this mathematical model helps in considering the effect of varying the capacity of the system, the arrangement and sequence of stations, the desirability of adopting various decision rules to cope with conveyor congestion, and most important, the basic question of whether the conveyor is to be installed at all. The primary emphasis is on the economic aspect of the conveyor system; that is, how it relates to the cost and profit objectives of the firm, rather than on its physical characteristics. Further, attention is directed towards a broad, theoretical analysis, rather than the description of specific types of conveyor systems or specific installations.

THE DEVELOPMENT OF CONVEYOR THEORY

For many years conveyors have been installed and designed for specific applications without too much concern for any theoretical concepts which might exist among them. However, beginning in about 1958, several engineers have started formulating general theories which unify the characteristics and problems that most conveyor systems have in common.

Several of the aspects of conveyor system design now under investigation have been listed by Morris (11):

1. Uses
    a. Delivery
    b. Storage and delivery
2. Loading stations
    a. Single
    b. Multiple
3. Unloading stations
    a. Single
    b. Multiple
4. Loading schedule
    a. Continuous uninterrupted
    b. Discontinuous interrupted
5. Unloading schedule
    a. Continuous uninterrupted
    b. Discontinuous interrupted
6. Loading station inventory
    a. Unlimited
    b. Truncated
7. Unloading station inventory
    a. Unlimited
    b. Truncated

8. Loading station arrival pattern
    a. Periodic
    b. Poisson
    c. Erlangian
    d. Batches
9. Unloading station arrival pattern
    a. Periodic
    b. Poisson
    c. Erlangian
    d. Batches
10. Carrier spaces
    a. Single
    b. Multiple.

The one conveyor type which has received the most theoretical attention has been the closed-loop, irreversible overhead trolley conveyor system. This is the type of conveyor system commonly found in plants manufacturing complex assemblies such as household appliances and automobiles. Properly designed and installed, it has proven to be an excellent investment. As opposed to the open-loop or power-and-free system it is much simpler and lower in cost per unit capacity, but not nearly so flexible.

Kwo (7) in 1958 analyzed many of the aspects of this closed-loop irreversible system. He formulated three basic principles which must be observed in order that such a conveyor function satisfactorily. They were entitled the Speed Rule, the Uniformity Principle, and the Capacity Constraint. The Speed Rule is that every conveyor system has some maximum and minimum speed between which it must be operated. The Uniformity Principle is concerned primarily with conveyor systems which serve intermittently operated producing and receiving systems. In this case the conveyor performs a storage function as well as a delivery function. Violation of the Uniformity Principle has led to the anomaly of previously loaded carriers passing through the loading area at a time when empty carriers are desired; and empty carriers passing through the receiving area when parts are desired. By relating the periodic schedule of the loading and receiving areas with the total revolution time of the conveyor system, a fairly uniform distribution of parts on the conveyor system can be achieved.

Because of its relationship to the analysis in this paper, the Capacity Constraint is presented as follows:

$$mqv/L = mq/W = qv/s = K$$

where:

- m = total number of carriers on the conveyor
- q = capacity of a carrier (number of parts one carrier can accommodate)
- v = speed of conveyor
- L = length of conveyor
- W = revolution time of conveyor
- s = spacing between carriers
- K = accommodations required per unit time; a constant determined by the revolution time of conveyor, the operating capacity, and the total reserve capacity to be provided

Upon examination of this equation it can be seen that the heart of the problem is an accurate determination of K, the accommodations required per unit time. Kwo used a numerical method to determine the value K, which considered the requirements of reserve capacity, revolution time, loading rate, and unloading rate. However this method does not explicitly account for the independence of loading stations, which results in interference, and requires additional reserve capacity.

The problem of interference between stations has been dealt with explicitly by Mayer (9) and Schneider (17). Mayer showed that, in general, the binomial probability distribution describes the manner in which parts are furnished to the conveyor. Among several restrictive assumptions made in (9) was the unrealistic one that in attempting to load output onto the conveyor a station would check one and only one loading space; and upon finding a loading space filled would set the unit of output aside for later rehandling. On this basis, he developed a criteria for comparing alternative proposals, called a "Measure-of-Demerit," which was the ratio of output refused by the conveyor to the total output of the series of loading stations. The model for the Measure-of-Demerit was:

$$D = F/N = 1 - H/N(1 - q^n)$$

where:

D = Measure-of-Demerit

F = number of units not loaded on the conveyor during a given time period

H = total number of hooks in a given time period

N = total number of units produced during the same time period

q = the probability that no attempt is made to load a hook at a given work station

n = the number of stations through which the conveyor passes

Mayer's Measure-of-Demerit was an average ratio for the entire system; however, as he pointed out, the quantity requiring rehandling at the last loading station would be "about" twice the average quantity. Actually, as will be shown later, it will be somewhat greater than twice the average, due to the accumulative effects of interference.

While Mayer was the first to explicitly consider conveyor loading in a probabalistic concept, his study did little to aid the engineer in applying this theory to an actual operating situation. The problem of utilization of the theory was attacked by Schneider (17), who developed the concept of a "risk that K[1] or more stations would want to load the conveyor at any given time." He constructed Conveyor Decision Charts which compared the risk values of various numbers of work stations between 5 and 100 with the probability that any one station would attempt to load a given hook. Reis and Schneider (15) extended this concept and pointed out some of the difficulties of which a potential user of Conveyor Decision Charts should be cognizant.

There have been several interesting approaches to development of conveyor theory within the framework of queueing theory.

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[1]Not related to K developed by Kwo

For example, Richman and Elmaghraby (16) have developed a queueing model to determine the space to allow between successive operations sharing a conveyor where it is desired that a bank be built up in front of each station, of such length that the variance in cycle time of the loading station and the receiving station will be absorbed by the bank. Their problem was two-fold:

1. to determine what the average size of the bank (or length of a queue) should be to insure that a receiving station would very rarely be without any input, and

2. what length of conveyor space should be allowed to accommodate this size bank to assure that a producing station would rarely be blocked by a full conveyor.

The authors built their mathematical model on the assumption of a Poisson process, which lends itself well to queueing analysis.

Morris (11) described a similar approach to this problem through the use of a truncated waiting line model, and extended the theory to the inclusion of a cost function, wherein he sought the minimum of the two costs of:

1. lengthening the conveyor as opposed to

2. permitting it to occasionally fill up and block the flow from the previous station.

Morris observed that the minimization of the resulting cost function was not easily reached by analytic methods, and recommended direct calculation of the function to reveal the minimum cost conveyor length.

The interaction between a series of loading stations and unloading stations has been analyzed in a probabalistic framework by Morris (11), who has also presented perhaps the most complete picture of the many aspects of conveyor theory discussed here.

A MATHEMATICAL CONVEYOR LOADING MODEL

Statement of the Problem

After a review of the work of pioneers in the field of conveyor theory, this writer and his colleagues felt that further work was justified in the analysis of random loading at individual stations. It was felt, too, that a useful broadening of the theory could be made by releasing the restrictive assumption that a station could attempt to load only one hook, or loading space, upon completion of a work cycle. In actuality, the opposite extreme seemed at least as valid: a station could be instructed to wait as long as necessary to load output, in which case some delay would be experienced.

Assuming validity at these extremes, one should be interested in the range of possibilities existing between the extremes. It was felt that if a model could be written describing the delay and the output set aside under the various conditions, then it should be possible to optimize the length of delay which would be tolerated.

In a like manner, it was observed that if successive stations face an ever-increasingly loaded conveyor there should be some point at which it would not be advisable to attempt to load the conveyor any further. This point would be reached at the load level which minimized the sum of the cost of providing reserve, or unused, capacity on the conveyor and the cost of experiencing delay or rejected output due to congestion.

In order to write mathematical expressions to describe the loading process, several definitions and assumptions are required. The full implications of the assumptions are discussed later.

First, it is most important to assume that the loading attempts of individual work stations are independent of each other and of the status of the conveyor. That is, the time a station will attempt to load the conveyor cannot be predicted as a function of the attempts made or successes achieved by any other station, nor by the availability of or lack of loading space on the conveyor. An attempt to load is that action which takes place at the end of each production cycle as a station tries to place output onto the conveyor; and a success in loading is the actual placing of a unit of output onto the conveyor.

While some variation between average cycle times of different stations is required in order to yield a random loading process, it is assumed that over a period of time, the cycle times of all stations can be averaged together to find an average cycle time which characterizes the entire system. This assumption may be stated symbolically:

$$\bar{w} = \frac{n}{(1/w_1 + 1/w_2 + \ldots + 1/w_n)} \qquad (1)$$

where:

$\bar{w}$ = the average station cycle time for the entire system

$w_1, w_2, \ldots, w_n$ are the cycle times of n stations.

n = the total number of stations.

The following relationship between an attempt to load the conveyor and a success in loading the conveyor is assumed: a loading range is defined as the maximum number of loading spaces which a station is allowed to examine in an attempt to load. If an empty loading space passes the station during this loading range, a success in loading occurs; if not, the unit is set aside. The loading range may be infinite, in which case the output set aside, or "rejected output", is zero. On the other hand, a finite loading range may be specified, in which case it is assumed that the station can examine all spaces in the range immediately upon an attempt to load. In this case, each attempt to load results in either:

1. a success in loading, and possible resultant delay in waiting for the selected loading space to pass, or

2. a failure to load the conveyor, resulting in a unit of rejected output but in no delay.

If a station is able to load the first space of a given loading range zero delay is experienced. This assumes that at any time there is one (and only one) loading space within immediate access.

Finally, it is assumed that the conveyor is empty as it approaches the first station in the system. For specific applications, this assumption can be relaxed, as will be shown later.

The Fraction of Conveyor Capacity Removed by n Stations

As loading spaces of the conveyor pass through successive loading stations, each occasionally attempting to load its output onto the conveyor, the loaded fraction of its capacity gradually increases, and in an identically equal manner, the probability that any one loading space is filled increases. As a significant fraction of conveyor capacity becomes loaded, later stations will experience delays in loading the conveyor, and if a finite loading range is established, these stations will be forced to set some of their output aside. This rejected output, while an undesirable result of congestion, will tend to alleviate the congestion to some extent, in that less than 100% of the output of a station with rejected output goes onto the conveyor. It can be seen that in a like manner, the delay experienced by a station reduces its output exactly in proportion to the fraction of total time spent in delay. However, this fraction is always quite small in a realistic situation, and in this analysis it is assumed that a reasonable allowance for station delay has been made in determining average cycle times.

In order to quantify the conveyor loading process described above, the following relationship is established:

$$P = L/\overline{w} \qquad (2)$$

where:

L = the loading space time; the time elapsed during which successive loading spaces pass a given point

$\overline{w}$ = the average station cycle time

P = the loading ratio.

The use of the loading ratio as a fundamental parameter of the loading system frees the probabalistic analysis from concern with actual cycle times, conveyor speeds, and loading space density. These deterministic relationships are fairly obvious and have been fully developed by other authors, in particular Kwo (7). They will be introduced into illustrations as required.

It was observed that a finite loading range will result in the rejection of station output by the conveyor when it is congested, thus tending to reduce the fraction of capacity removed by later stations. The following two equations were derived to describe explicitly this removal of capacity:

$$f_{n;1} = P\left[1 - (F_{n-1;i})^{i}\right] \qquad (3)$$

and:

$$F_{n;i} = F_{n-1;i} + f_{n;i} \qquad (4)$$

where:

i = the loading range, i.e., the maximum number of loading spaces which a station is allowed to examine in an attempt to load

P = the loading ratio, i.e., the fraction of capacity removed by all the output of an average station

$f_{n;i}$ = the fraction of capacity removed by the n^{th} station, given a system loading range of i

$F_{n;i}$ = the total fraction of capacity removed by n stations, given a system loading range of i.

These two equations determine recursively the capacity removed by n stations; that is, the fraction removed by the first station is calculated, and this value is used to determine the fraction removed by the second station, which is then used in a like manner. During later work to derive a minimum cost functional equation, a series approximation method of determining $F_{n;i}$ was derived; this equation is shown in the appendix.

Equation (3) can best be interpreted by describing the probabilities which arise as loading spaces travel through the loading system. As the conveyor approaches the first station, it is completely empty; thus $f_{n-1;i} = 0$. The probability that station number 1 will attempt to load any given space is equal to P, the loading ratio. This fact depends upon the assumption made earlier that the loading process is a random one. Therefore, there is an equal chance that a station will attempt to load any given space.

While station number 1 will always succeed in its attempts to load because the conveyor is empty, such is not the case with succeeding stations. Each station after the first faces the possibility that its loading range will be filled, and its output rejected. The probability that a given loading space is filled as it approaches a station is equal to the fraction of capacity removed by all previous stations.

As a numerical illustration, assume ten stations have removed 0.15 of the capacity of the conveyor through their successful attempts to load. Station number 11 will face a constant probability of 0.15 that a given loading space is filled. However, if the station has a loading range of greater than one, the attempt to load may be distributed over all loading spaces in the range. If the range is $i = 3$, then by the multiplicative law of probability of independent events, the probability that <u>all three</u> spaces in his loading range are filled is (.15)(.15)(.15) or 0.003375.

The probability of success is equal to one minus the probability of failure. It has been shown that the probability of failure of the n^{th} station when attempting to load one of the spaces of a loading range of (i) is $(F_{n-1;i})^{i}$. Thus the probability that an attempt will succeed is $1 - (F_{n-1;i})^{i}$. The probability that an attempt will be made has been previously shown to be P. Thus the fraction of capacity actually removed by a station is a product of two probabilities:

1. that the station will attempt to load a given range of spaces, and

2. that the station will succeed in the attempt.

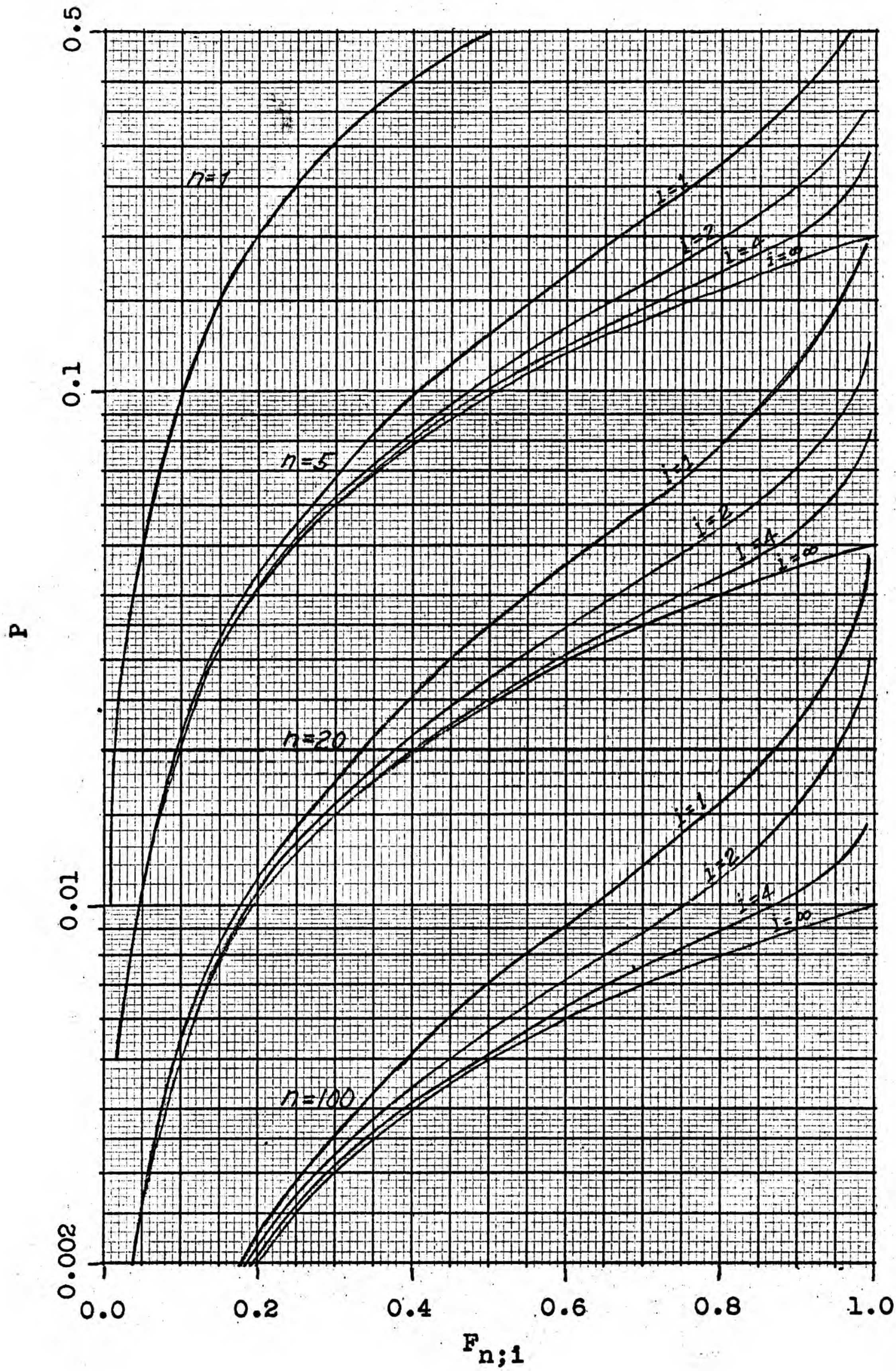
Equation (4) is a recursion statement of the fact that the total fraction of capacity removed by n stations is equal to the fraction removed by (n-1) stations plus the fraction removed by the n^{th} statement. While these two equations could be combined into a more complex equation involving a summation, they are stated separately here to aid in their explanation, Computer programs were written to calculate these equations for values of P from 0.001 to 0.5, values of i from 1 to 16, and values of n from 1 to 100, and are illustrated in the appendix. Plates I through V show the effect of selected loading ranges on the fraction of capacity removed by n stations.

EXPLANATION OF PLATE I

F as a Function of P, n, and i

These curves compare the effect of selected loading ranges, i = 1, 2, 4, and ∞, on the total fraction of conveyor capacity removed by selected numbers of stations, n = 1, 5, 20, and 100, for loading ratio P ranging from 0.0015 to 0.5.

PLATE I



EXPLANATION OF PLATES II, III, IV, & V

Plates II through V illustrate the effect of loading ranges 1, 2, 4, and 8 on the removal fraction for selected values of n.

PLATE II

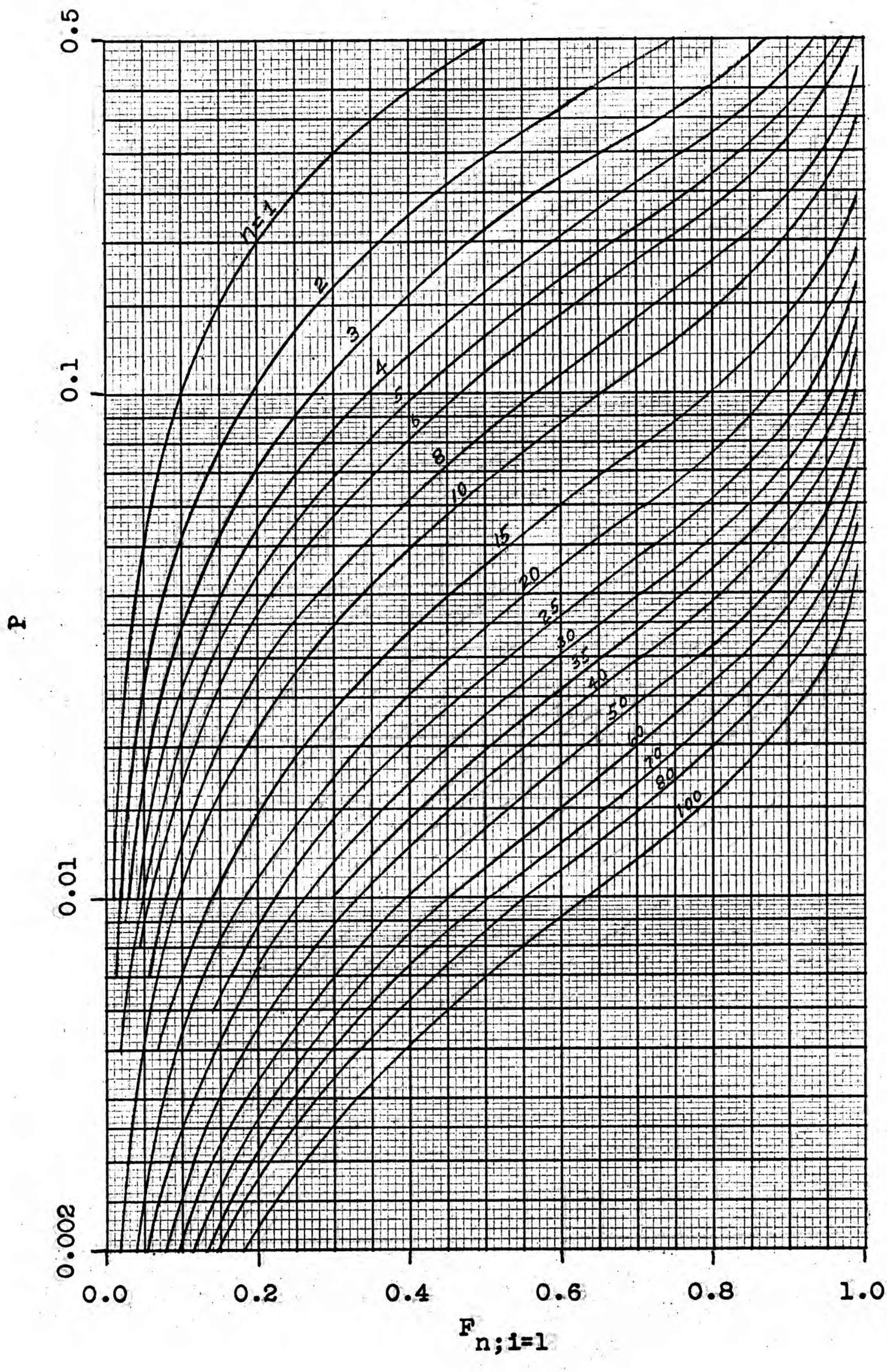

0.5
0.1
0.01
0.002
P
0.0
0.2
0.4
0.6
0.8
1.0
$F_{n;i=1}$
n=1
2
3
4
5
6
8
10
15
20
25
30
35
40
50
60
70
80
100

PLATE III

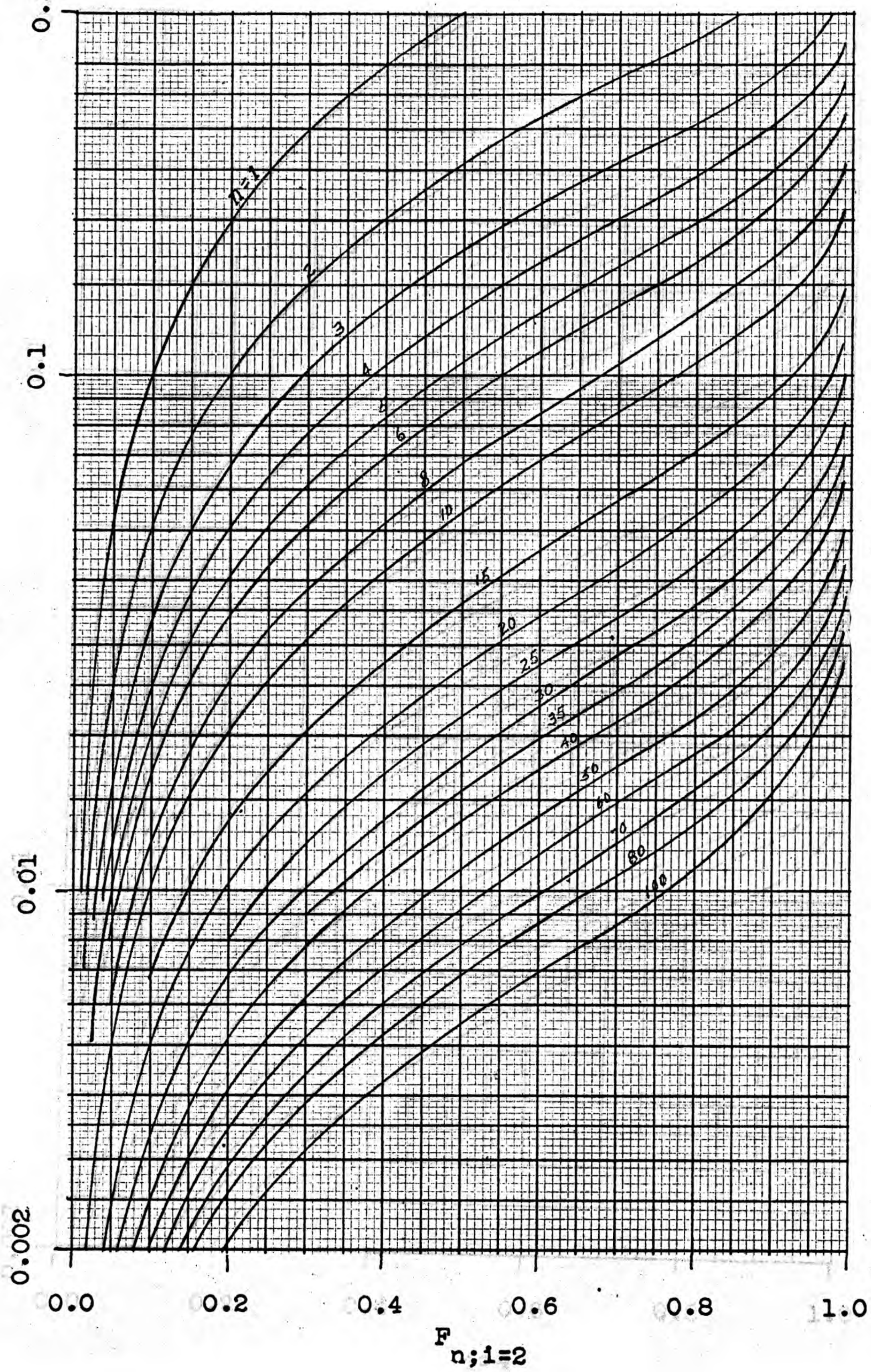

0.5
0.1
0.01
0.002
P
0.0
0.2
0.4
0.6
0.8
1.0
F n;i=2
n=1
2
3
4
5
6
8
10
15
20
25
30
35
40
50
60
70
80
100

PLATE IV

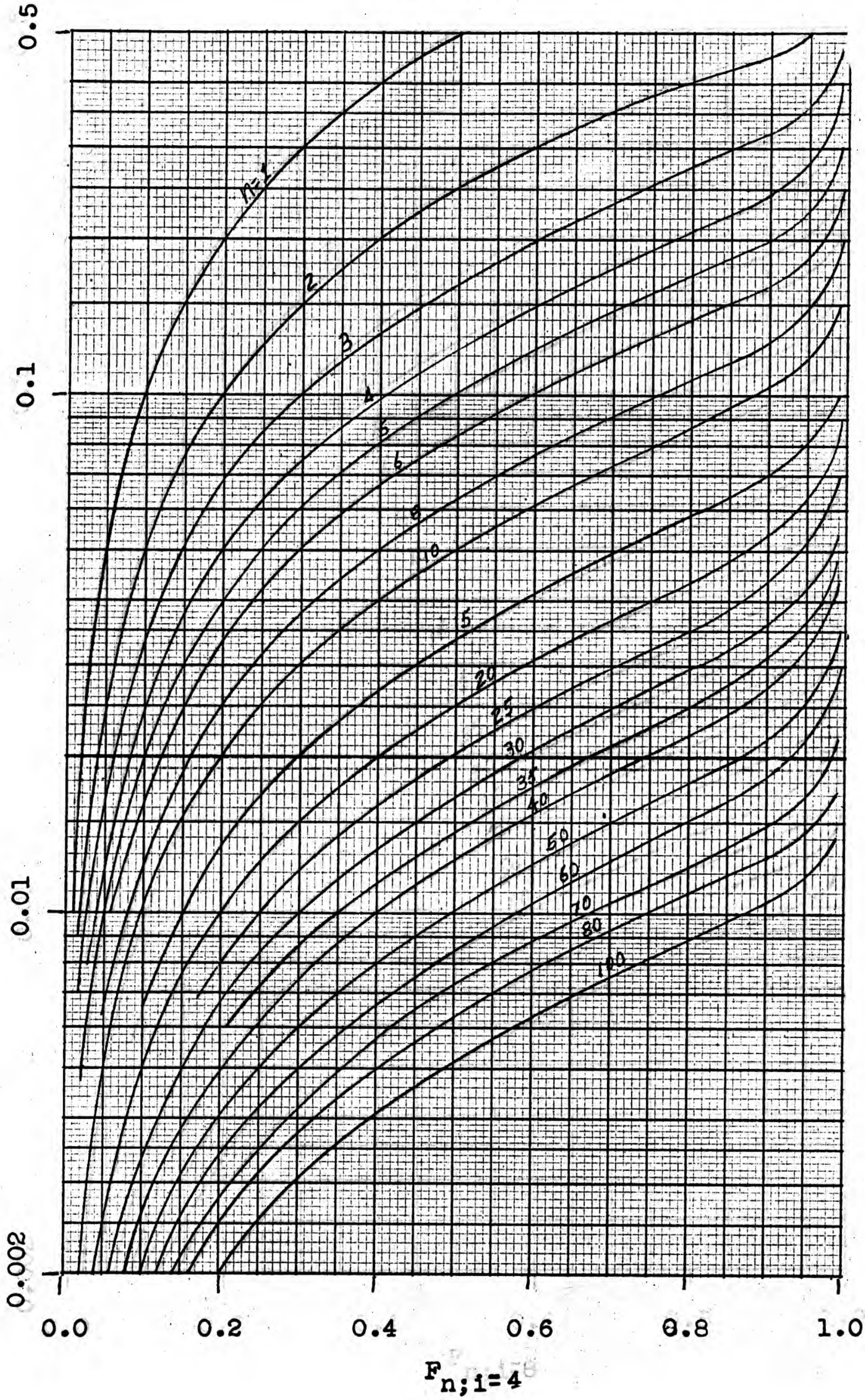
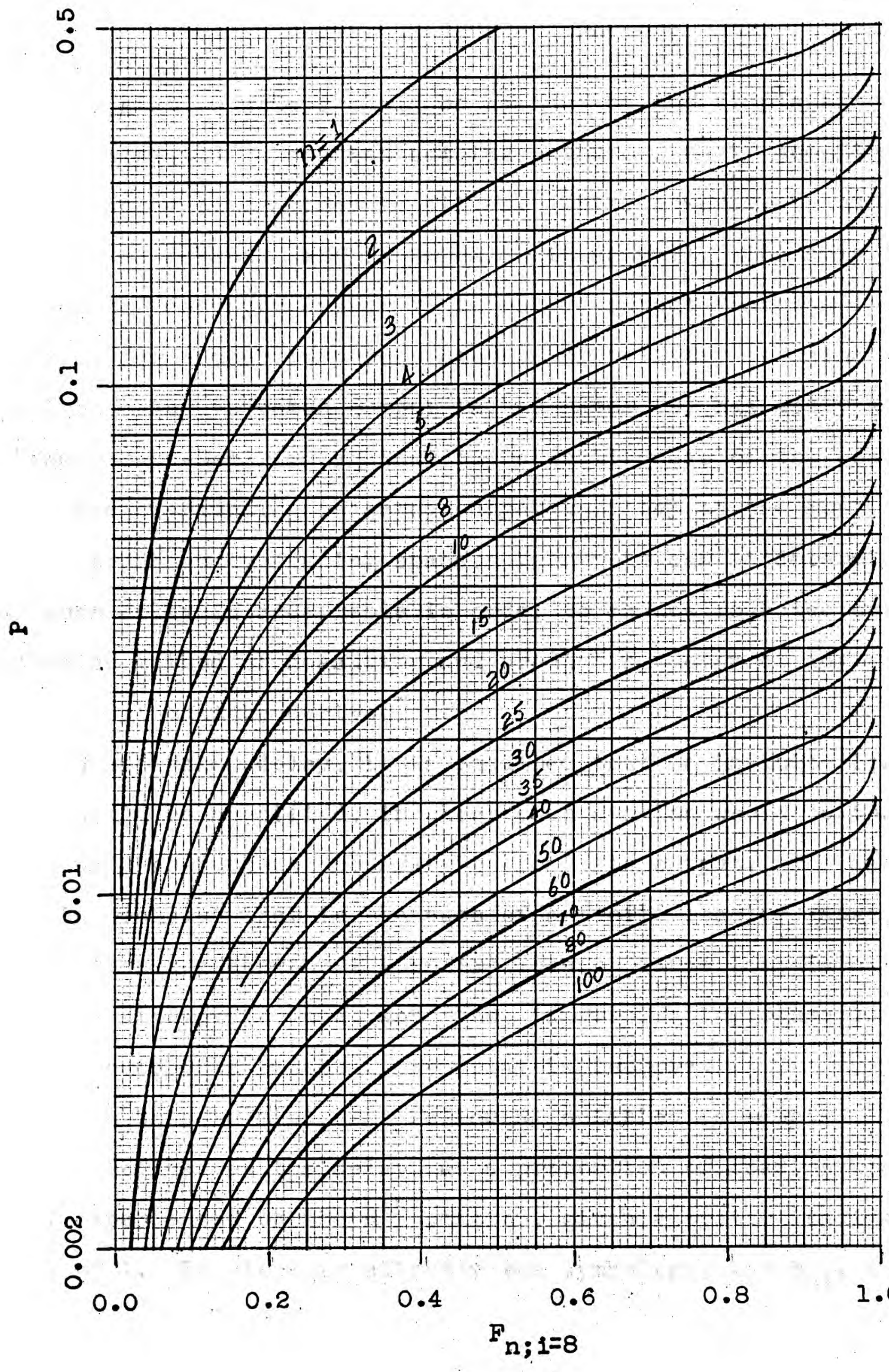

0.5
0.1
0.01
0.002
P
0.0
0.2
0.4
0.6
0.8
1.0
$F_{n;i=4}$
n=1
2
3
4
5
6
8
10
15
20
25
30
35
40
50
60
70
80
100

PLATE V


0.5
0.1
0.01
0.002
P
0.0
0.2
0.4
0.6
0.8
1.0
$F_{n;i=8}$
n=1
2
3
4
5
6
8
10
15
20
25
30
35
40
50
60
70
80
100

Delay Experienced by n^{th} Station

All station delays could be avoided by the simple expedient of requiring that a station set its output aside if the first loading space it examines is filled. This, however, is undesirable for several reasons, and is unrealistic as well. The sole function of the conveyor system is to remove output from the loading stations for either temporary storage or delivery to using stations. Any rejected output is, as Mayer (9) has aptly said, a "Measure-of-Demerit" against the effectiveness of the conveyor. The price of reducing rejected output is delay as the station waits for an empty loading space. Before it can be determined how much delay is acceptable in order to reduce rejected output, a precise method of measuring and predicting delay at any station of interest must be derived.

In analyzing delay, it is recalled that an infinite loading range could be specified, in which each station would be instructed to wait as long as necessary to load its output. The assumption is recalled that in the case of a finite loading range, the station may examine the entire loading range immediately upon the attempt to load, and thus, if the loading range is filled and the output is rejected, no delay is incurred.

With this in mind, the following notation is adopted. Let $d_{n;i}$ be defined as the delay, as a proportion of loading space time, experienced by the n^{th} station, given a system loading range of i. To simplify slightly the symbology, let $E_{n;i}$ be the fraction of capacity remaining after n stations have removed the fraction $F_{n;i}$. Symbolically:

$$E_{n;i} = (1 - F_{n;i}) \qquad (5)$$

It is now possible to write an equation for $d_{n;i}$ as a function of F, E, and i:

$$d_{n;i} = (E_{n-1;i}) \sum_{u=0}^{i-1} u(F_{n-1;i})^u \qquad (6)$$

In many cases it is necessary to express delay as a fraction of station cycle time, although in general it is measured in terms of loading space time. Let $D_{n;i}$ be defined as the delay, expressed as a fraction of station cycle time, experienced by the n^{th} station, given a system loading range of i. Because the relationship between station cycle time and loading space time is P, the loading ratio, it is seen that:

$$D_{n;i} = (P)(d_{n;i}) \qquad (7)$$

Interpretation of equation (6), like that for equation (3) depends on the probability relationships it expresses. Upon completing a unit of output, a station attempts to load it. At this time there are (i + 1) mutually exclusive events which may occur. For example, he may load the first loading space, and experience a delay of zero. The probability of this event is equal to $E_{n-1;i}$, the fraction of the conveyor empty as the loading spaces approach station n. If the first loading space is filled, he may succeed in loading the second space. The probability that the first space is filled is $F_{n-1;i}$. The probability that he will be able to load the second space, given that he examines it at all, is the same as it was for the first space, since the loading pattern on the conveyor is randomly distributed. Loading the second space results in a delay of one loading space. The probability that both the first two spaces are filled and the third is examined is the product of the probabilities that each of them is filled, or $(F_{n-1;i})^2$. The probability that the third hook is empty remains equal to $(E_{n-1;i})$. The delay incurred in loading the third hook is 2 loading spaces.

The above reasoning process can be continued until the maximum permissible number of loading spaces has been examined without finding an empty one. The probability of such an event was described in the derivation of $F_{n;i}$, and is equal to $(F_{n-1;i})^i$. The delay associated with this event is zero, not i, because by definition, a failure to load results in rejected output but no delay.

The total expected delay is equal to the summation of each of the successive delays, multiplied by their probabilities of occurrence. Dropping the use of subscripts for simplicity:

$$d = (E)(0) + (F)(E)(1) + (F)^2(E)(2) + \ldots$$
$$\ldots(F)^{i-1}(E)(i-1) + (F^i)(0)$$

Combining these terms by the use of a summation, with u as the index of successive loading spaces examined, is the final step in the derivation of equation (6) as previously shown.

A numerical example may clarify the above discussion. Let $F_{n-1;i}$ be equal to 0.4 and $i = 3$. The delay experienced by station n can be computed as follows:

Event X	P(X)	P(Y;X)	P(Y)	d(Y)	E(d)
Examine space 1	1	.6	.6	0	0
Examine space 2	.4	.6	.24	1	.24
Examine space 3	.16	.6	.096	2	.192
Set unit aside	.064	1.0	.064	0	0
			1.000		.432

where:

X = the event indicated

P(S) = the probability of occurrence of event X

P(Y;X) = the probability that event X, having occurred, terminates the loading attempt

P(Y) = the probability that the load attempt is terminated by event X; $= [P(X)] \cdot [P(Y;X)]$

d(Y) = delay associated with P(Y)

E(d) = Expectation of delay; $= [P(Y)] \cdot [d(Y)]$.

In this example there are four possible events: a success on loading space one; success on loading space two; success on loading space three; and failure to load. The sum of these probabilities is shown to be equal to one, as it must for mutually exclusive events. The delays associated with these probabilities are 0, 1, 2, and 0. The expectation of delay is the product of each individual delay times its probability of occurrence, which is 0.432 as a fraction of loading space time. This figure would be multiplied by P to yield expected station delay as a fraction of station cycle time.

Values of d_n as a function of F_{n-1} and i were calculated on a desk calculator and are plotted on Plate VI. Examination of Plate VI shows that when a conveyor is loaded to less than one half its capacity, station delay increases approximately linearly as a function of F_{n-1}. It also points up the fact that while increasing i lengthens delay, it does so at a decreasing rate. At the 0.6 level, average delay is 0.25 for $i = 2$, 0.75 for $i = 4$, and 1.12 for $i = 6$. But at $i = 10$ it has increased only to 1.31. Finally it is noted that for any finite loading range, a peak is reached where the effect of zero delay for rejected output begins to counteract the greater delays associated with successes in loading.

Rejected Output Accumulated at the n^{th} Station

When the conveyor becomes congested, either a station is going to have to wait for a period of time in order to find an empty loading space or the output of the production cycle will not be placed on the conveyor, in which case it is said to be rejected.

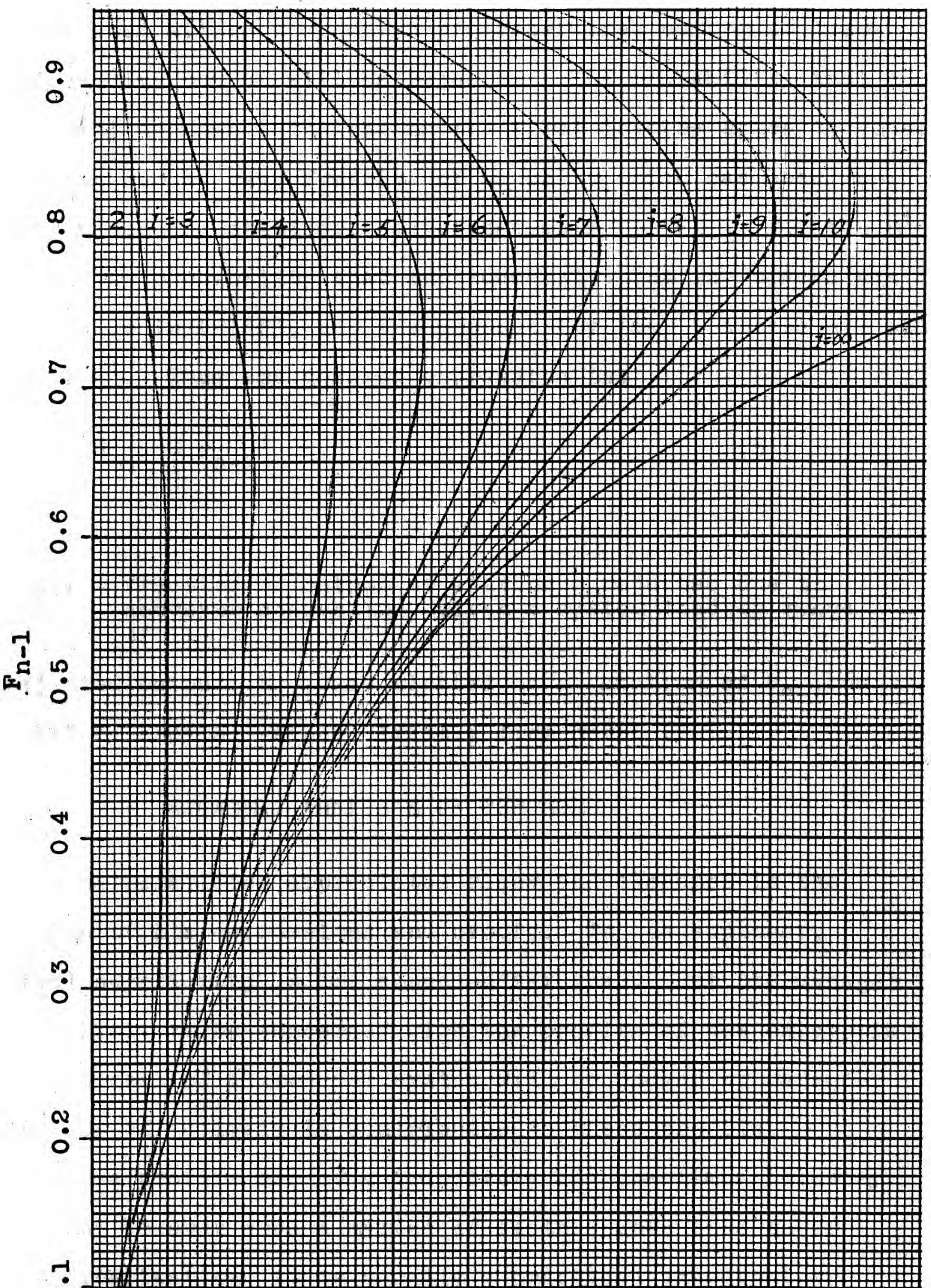
It has been stated as an assumption that if a station is

EXPLANATION OF PLATE VI

d_n as a function of F_{n-1} for various values of i

These curves compare the effect of loading ranges i from 1 to 10 on the delay experienced at a single station, expressed as a proportion of loading space time and as a function of F_{n-1}.

PLATE VI



0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
F_{n-1}
0.0
0.5
1.0
1.5
2.0
2.5
$d_{n;i}$
2
i=3
i=4
i=5
i=6
i=7
i=8
i=9
i=10
i=∞

unable to load within a specific pre-determined number of loading spaces, its output will be rejected. The probability that a station will be unable to load on the first loading space is equal to the long run average fraction of capacity which has been removed by preceding stations, designated as F, or F_{n-1}, for the n^{th} station. The probability that two successive loading spaces are not available is (F)(F), and the probability that i spaces are all filled is F^i. Thus the desired expression is:

$$R_{n;i} = F_{n-1}{}^{i} \qquad (8)$$

where:

$R_{n;i}$ = fraction of output of station n rejected due to conveyor congestion, given a system loading range of i.

The rejected output, $R_{n;i}$ was plotted as a function of F_{n-1} on Plate VII by copying values directly from Morse (12).

ECONOMIC ANALYSIS OF THE MODEL

The mathematician appreciates rigor and elegance in the derivation of his expressions and proofs. To the engineer, technical superiority is the sought-after goal. But the manager is more likely to be concerned with the profits of the enterprise, and the success of a conveyor system will be judged by management on how effectively it contributes to these profits.

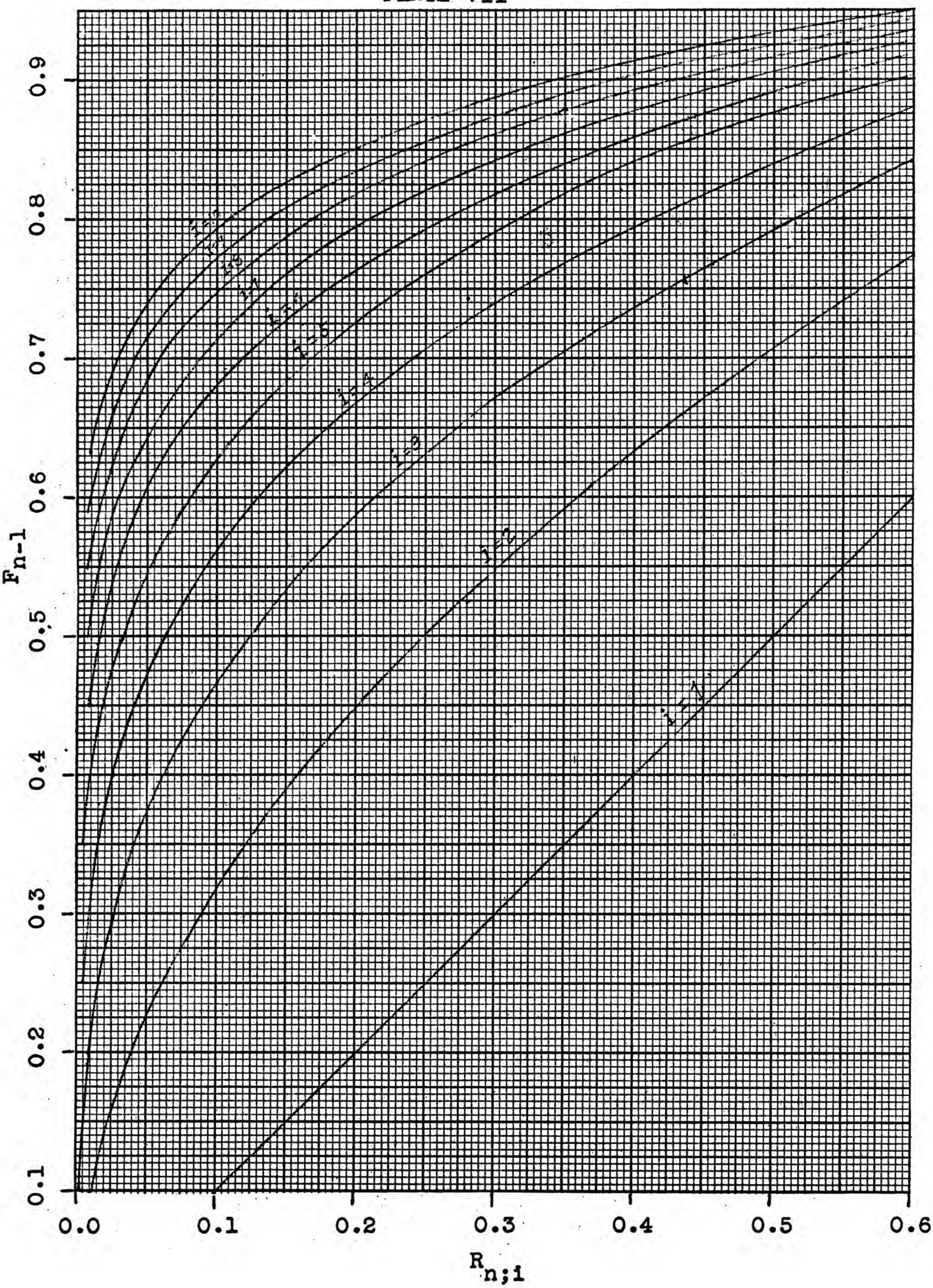
Because profits are of a transitory and sometimes illusory nature, and because such a myriad of factors goes into their eventual realization, it is usually a practical expedient to

EXPLANATION OF PLATE VII

$R_{n;i}$ as a function of F_{n-1} for various values of i

These curves compare the effect of loading ranges i from 1 to 10 on the fraction of the n^{th} station's output rejected by the conveyor, $R_{n;i}$, as a function of F_{n-1}.

PLATE VII



suboptimize to the extent of considering alternatives on the basis of minimization of cost, or on the maximization of the return on capital invested. At the engineering design level, cost minimization is the more logical measure of effectiveness, while at the capital budgeting or executive level, rate of return on investment is the more likely criteria (4).

In comparing alternative methods of materials handling, it is probably safe to say that the alternative which performs the desired delivery of material at minimum total variable cost would also lead to the highest rate of return on investment, since all the alternatives would presumably fulfill the same function, though in different ways, and with different capital costs and operating costs.

Attaining a functionally satisfactory design at minimum cost obviously involves many factors. Competing structural materials, power transmission devices, types of fixtures, and many other aspects of the conveyor would need to be evaluated in order to achieve the most suitable design at the lowest overall cost. Of particular interest in this discussion are those costs related to capacity, and some thought indicates that numerous costs are either directly or somewhat inversely related to capacity.

The job of the analyst is to systematically determine these costs, to find out how they are related to capacity, and then to derive expressions which will minimize their total. This general procedure will be illustrated for the mathematical conveyor loading model previously described.

Measurement of Costs

The accurate determination of costs in an industrial situation is an extremely elusive task. Grant (4) has observed,

> In most economy studies the calculations are much easier than the process by which the estimates used in the computations are obtained. Very frequently the internal sources of data have been set up for other purposes and consequently do not have readily available the information in the desired form. The estimator must be cautious regarding the information that he uses to be sure that it is relevant to his problem. For example, the books of account are kept according to a set of accounting rules that have been agreed upon by the interested management personnel. Frequently arbitrary groupings or allocations are made so that the estimator cannot use the accounting data for an economy study without first determining how he can eliminate the nonrelevant information.

The two most fundamental methods of collecting cost data have been called the "statistical approach" and the "engineering approach" (1). An illustration of the statistical approach would be to examine all available data from a company's records, choose that data which appeared most pertinent; assign all relevant costs to their appropriate causes, determine the values of cost parameters by multiple regression analysis; and test the results by multiple correlation analysis. On the other hand, the engineering approach is an attempt to analytically or experimentally predict future events without the benefit of collecting data on past experience.

These two methods are by no means mutually exclusive, and it is likely that in analyzing the costs of a conveyor system, both would be used to advantage. Some costs related to the increase in conveyor capacity, such as the cost of higher power, can be measured statistically. Others, such as the increase in strength requirements of structural members can best be determined analytically. The cost of delay can probably be determined very closely on an historical basis. The cost of rejected output will at best be a subjective figure, since so many intangibles enter into this cost.

In the analysis of this model it is assumed that the costs of these three factors -- capacity, delay, and rejected output -- bear a linear relationship to their quantity. It is undoubtedly true that some costs will exhibit a sort of step function, remaining constant over a given range and then increasing by some amount at a discrete point. Other costs might prove to be of a quadratic nature. However, in the absence of information to the contrary, linearity is a reasonable assumption, and it simplifies the analysis considerably. It should be mentioned, also, that only those costs which can be affected by the decision under consideration need be or should be considered in any given economy study. Authors on engineering economy (1)(4) have repeatedly cited examples of engineers including in their analysis costs which could not possibly be altered by the decision, and such inclusion naturally tends to distort the results.

For the purpose of the mathematical derivations to be made, the following notation is introduced: Let C_c be the annual variable cost of additional capacity, measured in dollars per loading space per hour. This cost coefficient will be made up largely of an annual charge against the incremental capital costs of larger motors, more floor space, stronger structural members, and similar capital factors. Increased power and maintenance expense will also enter into the cost. Let C_D be the annual cost of delay at an individual station, measured in dollars per station. In the typical case of an assembly worker at a loading station, this cost can probably be expressed as a direct function of the worker's hourly salary, taking into consideration fringe benefits, certain expenses allocated directly to hourly paid employees, such as the cost of operating tool cribs, cafeterias, dispensaries, the cost of safety equipment, and the like; and the number of hours per year which the station is expected to be used. Let C_R be the variable annual cost of rejected output, measured in dollars per unit of output. As was stated previously, considerable judgment will be required to arrive at a meaningful cost coefficient. Such aspects as the cost of rehandling, the allowance of temporary storage space, possible delays in subsequent operations due to a lack of needed parts, possible spoilage, damage, or pilferage may have to be considered. It may sometimes be the case that there is no linear coefficient, and either no rejected output can be accepted or some specified limit can be reached but not exceeded. A meaningful cost coefficient can best be assigned to rejected output when analyzing an existing system to aleviate the effects of congestion.

An Optimum Loading Ratio for n Stations

If there were perfect harmony in the loading pattern of all stations loading a conveyor system, the optimum loading ratio would be the inverse of the number of stations using the system. For instance, 25 stations with an average cycle time of three minutes would be able to fully utilize a conveyor which provided 8.33 loading spaces per minute, since the total output would be (25 stations/3 minutes per unit of output/station) = 8.33. The loading ratio in this case would be $P = 1/25 = 0.04$ and the dynamic capacity could be obtained also from the relation $K = 1/P\overline{w}$. As the conveyor approached the 25^{th} station presumably every 25^{th} loading space would be empty and at the moment the station completed its output that one empty space would be approaching it. Then as the conveyor left the 25^{th} station and the loading system itself, all spaces would be filled and the conveyor would be completely utilized.

Because the loading attempts of stations within a loading system are often either partially or completely independent, it is useful to determine just how much this increases the desirable conveyor capacity. As before, complete randomness in loading attempts among stations is assumed. It is also assumed that in the design of a conveyor system it would not be desirable to deliberately plan for rejected output to occur. This is accounted for by letting $i = \infty$.

There are a number of approaches which might be taken in optimizing the conveyor capacity. One technique which has often been used is the direct calculation of a cost function through the range of possible values, and selection of the point yielding the lowest cost. A more general method, possible only when the mathematical model can be expressed as a continuous function, is to set the first derivative equal to zero and solve for the variable of interest. The second derivative could be checked to assure that a minimum, rather than a maximum, has been found. In most cost minimization problems, however, the nature of the function makes it possible for the analyst to determine by inspection that the extremal point is actually a minimum (4).

For the purpose of this analysis let the total variable cost of constructing and operating a conveyor system with n loading stations be approximated by the following function:

$$TVC = C_D D_n + C_c E_n \tag{9}$$

where:

TVC = total annual variable cost

C_D = the annual cost of delay at station n

D_n = the delay experienced at station n, as a fraction of station cycle time

C_c = the annual variable cost of additional capacity

E_n = the fraction of conveyor capacity empty as the conveyor leaves station n.

Substituting the results of equations (5), (6), and (7) into equation (9) makes it possible to express the total variable cost as a function of the loading ratio and the number of stations:

$$TVC = C_D(PE_{n-1}) \sum_{u=0}^{\infty} u(F_{n-1})^u + C_c(1 - F_n) \qquad (10)$$

It was previously observed that when n stations load all their output onto the conveyor, the resulting fraction of capacity removed is the product of the number of stations and the loading ratio. Thus:

$$F_{n-1} = P(n-1) \qquad (11)$$

and

$$F_n = (P)(n) \qquad (12)$$

It is recalled from elementary algebra that:

$$\sum_{u=0}^{\infty} uF^u = \frac{F}{(1 - F)^2} \quad \text{if } F < 1 \qquad (13)$$

Substituting equations (11)(12) and (13) into equation (10) and simplifying yields the total variable cost as a function of the two parameters, P and n:[1]

---

[1]This derivation and the one to follow are shown in full in the appendix.

$$TVC = \frac{C_D(P^2)(n-1)}{1 - P(n-1)} + C_c(1 - Pn) \qquad (14)$$

The first derivative of this function with respect to P is set equal to zero:

$$\frac{d(TVC)}{dP} = C_D \frac{2P(n-1) - 2(P)(n-1)^2}{(1 - P(n-1)^2} - C_E n = 0 \qquad (15)$$

Solving equation (15) for P gives the following minimum cost function:

$$P^* = \left[\frac{1}{n-1}\right]\left[1 - \frac{1}{\sqrt{1 + (C_c/C_D)n}}\right] \qquad (16)$$

where:

P^* = the optimum value of P

n = the number of stations

C_c/C_D = the ratio of the annual cost of additional capacity to the annual cost of delay, both considered at the n^{th} station.

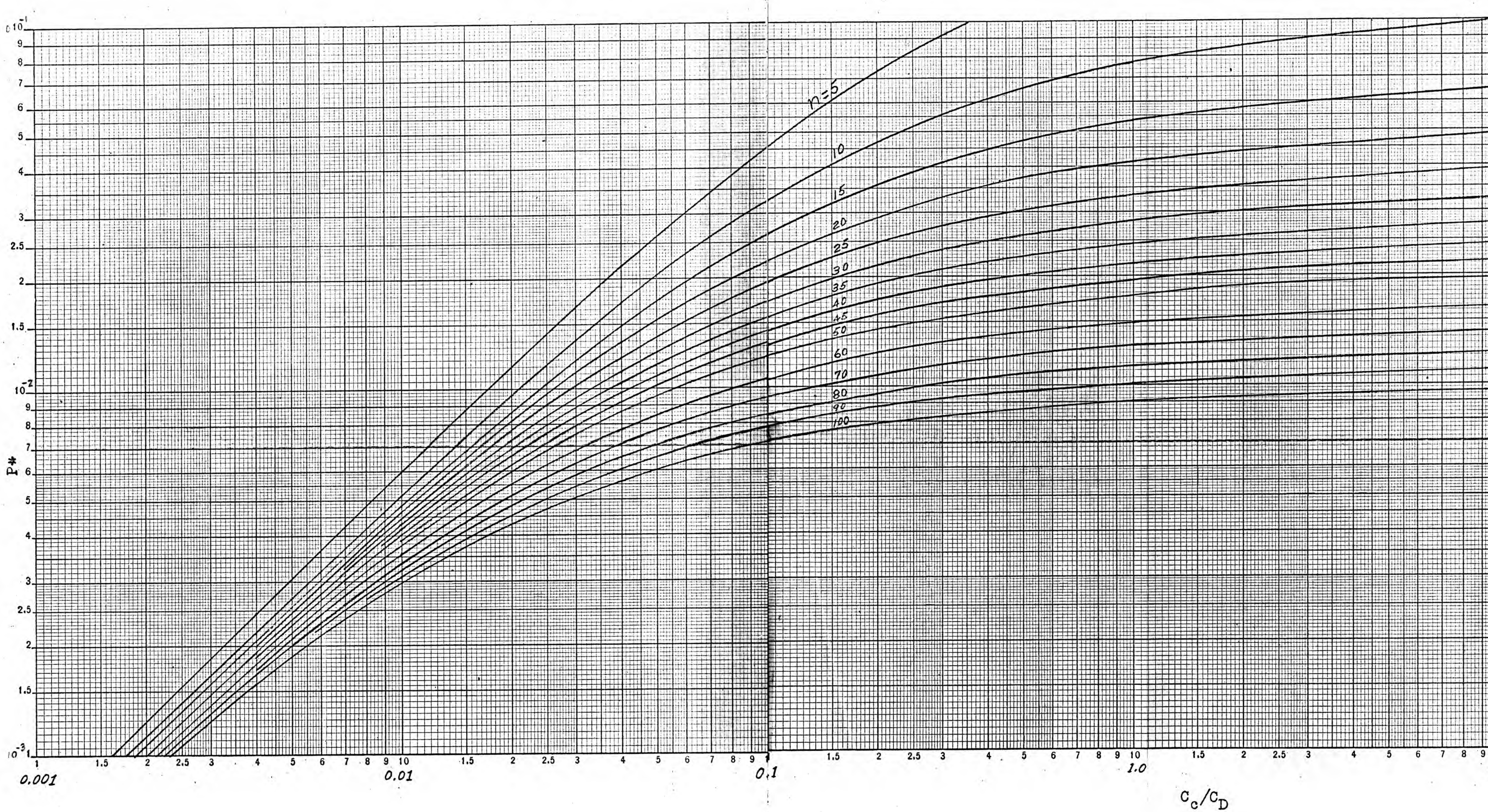
Values of this function were calculated on the IBM 1620 digital computer for values of the cost ratio C_c/C_D ranging from 0.001 to 10.0 and for n from 5 to 100 in increments of 5. The results of the calculation are plotted on Plate VIII. The computer program is shown in the appendix.

To demonstrate the usage of equation (16) and Plate VIII, assume that the ABC Corporation is planning to install a conveyor

EXPLANATION OF PLATE VIII

The optimal value of P, P*, is plotted as a function of the cost ratio C_c/C_D for a number of stations ranging from 5 to 100.

PLATE VIII


n=5
10
15
20
25
30
35
40
45
50
60
70
80
90
100
P*
10^{-1}
10^{-2}
10^{-3}
0.001
0.01
0.1
1.0
10
C_c/C_D

to serve a production line of 25 stations, all producing similar sized products with an overall average of 3 minutes per cycle. Management requests that an analyst determine the variable cost of capacity and the cost of station delay, and to specify an economical dynamic capacity.

After studying methods of increasing capacity, such as higher speed, greater density of carriers, multiple parts carriers, and the like, the analyst decides that the dynamic capacity can, in general, be increased at a capital cost of $60.00 per space per minute. He is informed that projects of this type are subject to a capital recovery factor of 0.25, which yields an annual capital cost of $15.00. There will be a moderate increase in the cost of power, maintenance, and repairs, amounting to $1.20 per year. Thus C_c = $15.00 + $1.20 = $16.20.

The annual cost of station delay is determined as a function of the wages earned by the production workers. The average hourly wage is found to be $2.00 and the accounting department estimates that an additional 30% should be charged to account for fringe benefits and employee services. Since the production line is to be manned on one shift only, 50 weeks per year, 40 hours per week, this figure is multiplied by 2000 to yield an annual cost of delay, C_D = $5200.

The cost ratio C_c/C_D is 0.0312. Entering Plate VIII with this value and proceeding to the ($N = 25$) line, P* is found to be 0.01. The dynamic capacity of the conveyor should be $1/P\overline{w} = 1/(0.01)(3) = 33.3$ spaces per minute.

An Optimum Loading Range

While a conveyor should be designed to obviate the need for rejected output, a conveyor already in operation may present a different problem. After completion it might prove highly impractical to change the loading ratio of the conveyor. If this already existing loading ratio is so large that a good deal of congestion results, a supervisor may be forced with one of several alternatives: to remove certain loading stations from the system entirely, and provide for the handling of their output by other means; to authorize delays greater than desired, or to permit only a certain level of delay to occur at each station, with resulting rejected output. For a given fraction of conveyor capacity removed, the average delay increases and the fraction of rejected output decreases as the loading range is increased. When cost coefficients are assigned to delay and rejected output in the manner previously described, the following total variable cost function results:

$$TVC = C_D D = C_R R \qquad (17)$$

where:

TVC = the total variable cost of operation at a given station

C_D = the annual cost of delay at the station

D = the delay experienced at the station, as a fraction of cycle time

C_R = the cost of rejected output at the station

R = the fraction of output rejected at the station.

The use of subscripts $_n$ and $_{n-1}$ can be dropped in this expression, since all terms are based on the condition as the conveyor approaches the station. Substituting in the values of D and R derived in equations (6), (7), and (8):

$$TVC = C_D PE \sum_{u=0}^{i-1} uFu + C_R Fi \qquad (18)$$

where:

P = the loading ratio

E = the fraction of conveyor capacity unused as it enters the station

F = the fraction of conveyor capacity removed by previous stations

i = the loading range

u = an index of summation.

before this total variable cost function can be differentiated, an approximating substitution is necessary:

$$\sum_{u=0}^{i-1} uF^u \approx \int_0^{i-1} uF^u \, du$$

After the definite integral approximation is substituted for the summation, the following derivative is found and set equal to zero:

$$\frac{d(TVC)}{di} = C_D\ PE(i-1)F^{i-1} + C_R F^i \ln F = 0$$

Solving this equation for i* yields

$$i^* = 1 + \left[\frac{C_R}{C_D P}\right]\left[\frac{-F\ln F}{E}\right] \qquad (19)$$

where:

i^* = the optimum value of i

F = the fraction of conveyor capacity previously removed

E = the fraction of conveyor capacity remaining

C_R = the cost of rejected output

C_D = the cost of delay

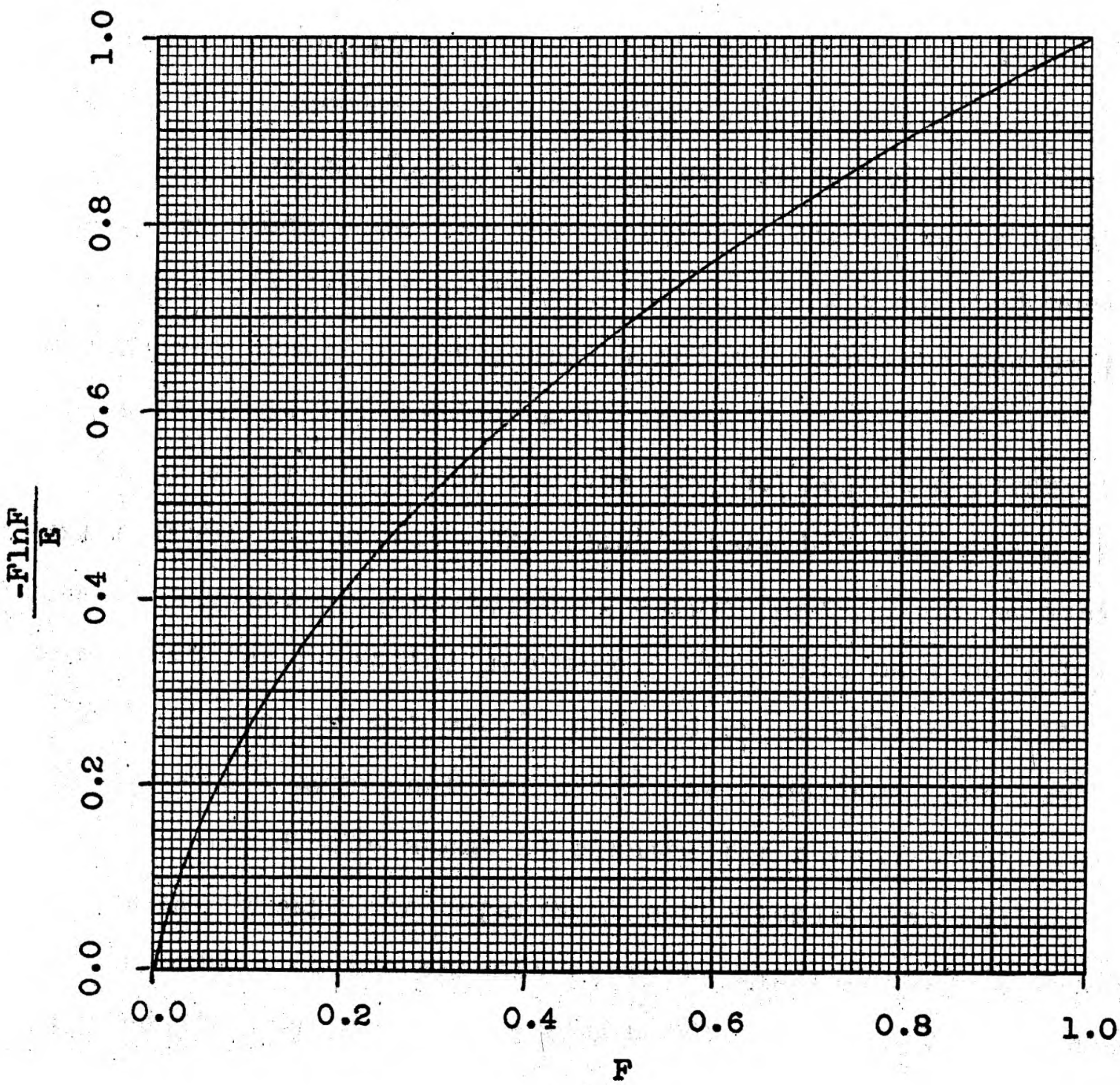
P = the loading ratio.

Interpreting equation (19) it is immediately apparent that as the ratio of cost of delay to cost of rejected output increases, C_R/C_D approaches zero and i* approaches 1 as a limit. In similar fashion, i* increases as the relative cost of rejected output increases, as the loading ratio decreases, and as the function ($-F\ln F/E$) increases. This function was computed on a desk calculator, and is plotted on Plate IX. Although not linear, it varies from 0 to 1 as F varies from 0 to 1. This result bolsters the intuitive feeling that as the conveyor becomes loaded to a higher level it is economical to permit a longer loading range.

EXPLANATION OF PLATE IX

$\frac{-F\ln F}{E}$ is plotted as a function of F

PLATE IX



To illustrate the use of equation (19), assume the following industrial situation: the coil winding department of the XYZ Company is composed of assembly workers who load their output onto a conveyor traveling to the chassis assembly department. The average time required to complete one assembly is six minutes and the conveyor has a dynamic capacity of ten loading spaces per minute. The loading ratio P is, therefore, (1)/(10)(6) = 0.00167. The foreman of the department has suspected that his workers who earn $2.50 per hour may be losing too much time waiting for empty space on the conveyor. The conveyor is also loaded by a previous department, and a worker stationed where the conveyor comes into the coil winding department has noticed that it is about 25% loaded as it comes in.

In an effort to determine the cost of rejected output the foreman first checks with the chassis assembly department, and finds that coils are not in short supply there; thus he assigns a zero cost for potential shortage. The coils have no value to any worker, and are not easily damaged by excessive handling, so he assigns a zero cost for pilferage, spoilage, and damage. However it would take about 20 minutes for a hand truck operator to come through the department periodically and deliver rejected output to the chassis assembly department. It is estimated that he could carry about 250 coils per trip, and earns $4.00 per hour.

The cost of delay at the first station is computed to be

1.4 times the cost of the worker's wages. On a daily basis:

$$C_D = (1.4)(2.50)(8) = \$28.00$$

The cost of rehandling output of the station is 1.4 times the hourly wages of the lift truck operator times the length of time spent by the lift truck operator in rehandling. Since the station produces one coil every six minutes, and the standards department estimates 450 productive minutes per day, the daily output of the station is 75 units. The cost of rehandling the entire output of the station is:

$$C_R = \frac{\$2.50 \text{ per hour x } 1.4 \text{ x } 1/3 \text{ hour/trip x } 75 \text{ parts/day}}{250 \text{ parts/trip}}$$

$$= \$0.35 \text{ per station}$$

For the first station, i* can now be calculated:

$$i^* = 1 + \left[\frac{C_R}{C_D P}\right]\left[\frac{-F\ln F}{E}\right] = 1 + \left[\frac{\$0.35}{(\$28)(.00167)}\right]\left[.46\right]$$

$$= 4.42$$

Similar calculations could be performed for other stations.

This calculation leads to the conclusion that some delay at the first station is preferable to immediately setting output aside and then rehandling it. A practical interpretation of ($i^* = 4.42$) would be an instruction from the foreman to the assembly worker: "If the first four or five spaces are filled, set aside your output; otherwise wait for an empty space and load it onto the conveyor." Consulting Plates VI and VII, it can be found that the average delay as a proportion of loading space time is .3 loading spaces, and the average fraction of output rejected is 0.005.

DISCUSSION

The results of mathematical analysis depend for their validity upon the assumptions on which they are based. In operating situations, one or more of the assumptions made in this paper might not be valid, and would have to be modified by the analyst. As Morris (11) has stated, "Analytical models of the sort we have discussed are rarely presented in 'ready-to-wear' form. They must be tailored to fit specific situations." In a given application it might be felt that there was too much variability among the cycle times of loading stations to justify a weighted average. The model expressed in equations (3), (4), (6), and (8) can easily be modified to account for variable cycle times, and a comparison could be made to determine the added accuracy derived. However, it would be more difficult to optimize such a model. Possibly direct calculation, rather than taking the first derivative, would be the preferred method. As another example, it might be quite unrealistic to assume, as was done here, that all stations in a loading system would use the same loading range. Another, more complex, model could probably be derived which would explicitly account for varying loading ranges.

From a research point of view, there is no limit to the added number of variables which might be accounted for in constructing a model. As a practical matter, however, a compromise must be made between the complexity of the real world and the simplicity of the model. No model will ever be absolutely "true." Thus, as Bowman and Fetter (1) have emphasized, the pragmatic criterion of usefulness should be the controlling one.

A simplifying assumption of a different type was made where a definite integral was substituted for a summation to make it possible to differentiate equation (18). It can be shown mathematically that such a substitution is not strictly true. The important point, however, is whether the results of the substitution lead to reasonable and economical decisions. Hansmann (5) has pointed out that:

> As long as overall economics is a criterion, the cost of using sophisticated tools is as real as any other cost and must be considered in appraising the merit of such tools. For this reason, mathematical complexity often forbids itself, even when it is realistic.

Thus it is felt that the value of the mathematical model can neither be established nor refuted by a purely mathematical approach. An important next step in the development of such a model as that one derived here would be the empirical validation or modification of these results. An interesting study could be based on the observation of several actual conveyor installations, using information collected by work sampling and study of production records to determine the average level of conveyor loading, the effect of congestion, and the apparent costs experienced in connection with these problems.

Another interesting and useful subject for further analysis would be the sensitivity of the model. Grant (4) has commented on sensitivity:

> Sensitivity refers to the relative magnitude of the change in one or more elements of a problem that will reverse a decision among alternatives. Thus if one particular element can be varied over a wide range of values without effecting the decision, the decision is said not to be sensitive to uncertainties regarding that particular element.

The sensitivity of any model can be roughly determined simply by solving the equations at various different points of possible interest. A thorough sensitivity analysis could be developed which would be much more revealing and also more efficient.

SUMMARY

A conveyor loading system of a series of stations independently placing their output onto the conveyor has been examined and modeled. The system has been characterized by three fundamental parameters:

1. A loading ratio, the relationship between the output of an average station and the capacity of the conveyor.

2. A loading range, the maximum number of spaces which a station is permitted to examine in an attempt to load the conveyor.

3. The number of stations in the system.

Based on the probabalistic relationships which result from assumption of independence among stations, expressions have been derived which describe three important and heretofore unspecified variables:

1. The fraction of conveyor capacity removed by a series of stations, all using some specific loading range.

2. The average delay which can be expected at any station, when a specific fraction of conveyor capacity has been removed by previous stations, and when that station consistently uses a given loading range.

3. The fraction of output "rejected" by the conveyor at any given station, when a specific fraction of conveyor capacity has been removed by previous stations, and when that station consistently uses a given loading range.

Emphasis has been placed on the economic analysis of the above model. A method of computing the cost of conveyor capacity, the cost of station delay, and the cost of station rejected output has been suggested. By using such a method to compute cost coefficients, it is possible to derive expressions to optimize, under certain limiting conditions, the three fundamental parameters mentioned above. Two minimum cost equations were derived:

1. An equation to minimize the sum of the costs of capacity and of station delay by choosing an optimum loading ratio.

2. An equation to minimize the cost of station delay and

station rejected output by choosing an optimum loading range.

In evaluating the model it was observed that it, like any mathematical formula, is completely dependent on the assumptions from which it is derived, and on the data which is collected by the analyst that uses it. Thus it should continually be borne in mind that use of the model does not solve any real problem, but only an artificial problem which, it is hoped, closely approximates this real problem. The final value of the model is in its successful use in aiding the solution of real problems.

ACKNOWLEDGMENTS

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APPENDIX

Glossary

Special Terms. Some special terms discussed in this paper are defined as follows:

Attempt to load. The action which occurs at the end of each production cycle as the station tries to load output onto the conveyor.

Conveyor. A mechanism passing by a series of loading stations at a constant rate, designed specifically to carry away the output of these stations.

Loading space. The amount of space occupied on the conveyor by one unit of output.

Loading station. An integral producing entity within the loading system, which operates independently of other stations in the system.

Success in loading. The actual placing of a unit of output onto the conveyor.

Unit of output. The physical quantity resulting from one production cycle.

Symbols. The algebraic symbols used in mathematical formulas are defined as follows:

C_c = cost coefficient of additional capacity

C_D = cost coefficient of delay

C_R = cost coefficient of rejected output

d_n = delay experienced by n^{th} station, as a proportion of loading space time

D_n = delay experienced by n^{th} station, as a fraction of production cycle time of the n^{th} station

E_n = fraction of conveyor capacity remaining after passing through n stations

f_n = fraction of conveyor capacity removed by the n^{th} station

F_n = fraction of conveyor capacity removed by n stations

i = loading range, the maximum number of loading spaces which a station is permitted to examine in attempting to load a unit of output

K = dynamic capacity, loading spaces per minute, $= \frac{1}{L}$

L = loading space time, minutes; the elapsed time which successive loading spaces pass a station. If two or more loading spaces pass a station simultaneously, as in the case of multiple baskets, it would be the time between successive spaces, divided by the number of simultaneous spaces.

n = a station numbered in sequence from the start, which may or may not be the last station

P = loading ratio, the reciprocal of the number of loading spaces passing through the system during the time span of an average production cycle

R_n = rejected output of the n^{th} station, as a fraction of total output of the n^{th} station

w_n = production cycle time of n^{th} station, minutes; the time required at the n^{th} station to produce one unit of output; the time between successive attempts to load at the n^{th} station

$\overline{w}$ = average cycle time, minutes; the weighted average of the production cycle times of all stations under consideration

Computer Programs

Several Fortransit IBM 650 programs were written to compute the fraction of capacity removed by n stations using a loading range of i and a loading ratio of P. At the outset of the research it was not known what ranges of values would be of the most interest, and as each program was run, possibilities for improvement of the program and for additional values of the variables appeared. The following program proved to be an efficient one for calculating the removal fraction. In this program, the letter M represented the loading range, N the station number and the letter J indexed the PUNCH statement to punch values for every tenth station. Statements 000080 and 000090 in the program correspond to equations (3) and (4). Data cards with appropriate values of P were prepared to accompany this program.

```
000010 READ P
000020 M = 1
000030 J = 1
000050 N = 1
000060 SIG = P
000070 PUNCH, SIG, N, M, P
000080 U = P*(1 - (SIG**M))
000090 SIG = SIG + U
000100 IF (N-101) 11, 17, 17
000110 N = N + 1
```

```
000120 J = J + 1
000130 IF (J-10) 16, 14, 15
000140 PUNCH, SIG, N, P
000150 J = J - 10
000160 GO TO 8
000170 IF (M-10) 18, 20, 20
000180 M = 2*M
000190 GO TO 3
000200 GO TO 1
000210 END
```

A computer program was written for the IBM 1620 Fortran system to determine the optimum values of P for n stations as C_c/C_D ranged from 0.0001 to 40.0. The number of stations was taken in increments of five from five to 100 and the cost ratio was computed at points 0.0001, 0.0004, 0.001, 0.004, 0.01, 0.04, 0.1, 0.4, 1, 4, 10, and 40. By the use of these points it was possible to fair smooth curves onto the semilogarithmic graph paper of Plate VIII. In the program shown, the letter S represented the number of stations, T the cost ratio, and P the optimum value of P. Statements number 000030 and 000040 correspond to equation (16).

```
000880 FORMAT (E10.4, F3.0)
000990 FORMAT (E10.3)
000010 T = .0001
       TYPE 99, T
```

```
S = 5
U = SQRT (1. + T*S)
P = (1./(S-1.))*(1. - 1./U)
TYPE 88, P, S
IF (S - 100.) 7, 9, 9
S = S + 5.
GO TO 3
IF (T - 10.) 10, 12, 14
T = 10.*T
TYPE 99, T
GO TO 2
T = .0004
TYPE 99, T
GO TO 2
END
```

A Series Approximation to F

During the course of research to develop an expression for the optimum value of P, a series method of calculating F was developed. Although this approach was not used in any of the work shown here, it might be of value to others in the development of conveyor theory.

If $i = 1$,

$$F_{n;1} = nP - \frac{(n)(n-1)}{2!}P^2 + \frac{(n)(n-1)(n-2)}{}P^3 - \ldots + P^n$$

where:

n = the number of stations under consideration

P = the system loading ratio

This approximation was empirically tested and it was found to converge quite rapidly to the true value of F. Substituting in values of $n = 10$ and $P = 0.05$:

$$F = 10(0.05) - 45(0.0025) + 120(0.000125) - 210(0.00000625)$$
$$+ 252(.0000003125) - \ldots$$

The successive values of the approximation are, to four place accuracy,

0.50000, 0.3875, 0.40250, 0.40120, 0.40128.

The true value of F, calculated recursively as shown in equations (3) and (4), is 0.4012631. Thus, four terms of the approximation give results accurate enough for all practical purposes. Tests for other typical values yielded like results.

Some work was done to generalize the series approximation to account for any loading range. Although results were not conclusive, it appeared that a series of summations would be required in such a case.

Mathematical Derivations

Equation (10) expressed the total variable cost at the n^{th} station in terms of station delay and unused conveyor capacity:

$$TVC = C_D D_n + C_c E_n \qquad (9)$$

By means of elementary algebra and a differentiation, this equation was eventually transformed into a solution for the minimum-cost value of P. Although the key steps were shown in the text many algebraic steps were omitted. The complete derivation of the equations for TVC and P* follows. By substituting previously derived expressions for D_n and E_n:

$$TVC = C_D(PE_{n-1}) \sum_{u=0}^{\infty} u(F_{n-1})^u + C_c(1 - F_n) \qquad (10)$$

Substitute $\sum_{u=1}^{\infty} uF^u = \frac{F}{(1 - F)^2}$ and $E = 1 - F$:

$$TVC = C_D P \frac{F_{n-1}}{1-F_{n-1}} + C_c(1-F_n)$$

Substitute $F_{n-1} = P(n-1)$ and $F_n = Pn$:

$$TVC = \frac{C_D P^2(n-1)}{1-P(n-1)} + C_c(1-Pn) \qquad (14)$$

Substitute $m = n-1$

$$TVC = \frac{C_D P^2 m}{1-Pm} + C_c\ (1-Pm-P)$$

Take the first derivative:

$$\frac{d(TVC)}{dP} = \frac{C_D(1-Pm)(2Pm)-P^2m(-m)}{(1-Pm)^2} - C_c(m + 1)$$

Set the first derivative equal to zero:

$$\frac{(1-Pm)(2Pm) + P^2m^2}{(1-Pm)^2} - \frac{C_c}{C_D} = 0$$

Let $C_c/C_D = c$:

$$(1 - Pm)(2Pm) + P^2m^2 - c(m + 1)(1 - Pm)^2 = 0$$

$$2Pm - 2P^2m^2 + P^2m^2 - c(m + 1)(1 - 2Pm + P^2m^2) = 0$$

$$2P - P^2m - c + 2cPm - cP^2m - c/m + 2cP - cmP^2 = 0$$

$$(m + cm^2 + cm)P^2 - (2 + 2cm + 2c)P + c + c/m = 0$$

Let $1 + cm + c = z$:

$$mzP^2 - 2zP + c + c/m = 0$$

$$P^2 - (2/m)P = \frac{- c + c/m}{mz}$$

$$P^2 - (2/m)P + (1/m)^2 = (1/m)^2 - \frac{c + c/m}{mz}$$

$$(P - 1/m)^2 = \frac{1}{m^2} \cdot \frac{z - mc - c}{z}$$

But $m = n - 1$ and $z = 1 + cm + c = 1 + cn$:

$$P - \frac{1}{n-1} = \pm \frac{1}{n-1} \sqrt{\frac{1 + cn - (n-1)c - c}{1 + cn}}$$

Discard the extraneous root:

$$P = \left[\frac{1}{n-1}\right]\left[1 - \sqrt{\frac{1}{1 + cn}}\right]$$

Thus P has been solved for the point at which the tangent to TVC as a function of P is zero. This value is normally signified by P*. Substituting $c = C_c/C_D$:

$$P* = \left[\frac{1}{n-1}\right]\left[1 - \frac{1}{\sqrt{1 + (C_c/C_D)n}}\right] \qquad (16)$$

AN APPROACH TO OPTIMIZING ECONOMIC FACTORS
FOR A PROBABALISTIC CONVEYOR LOADING MODEL

by

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Industrial engineers have recently devoted increased attention to the theoretical analysis of conveyor systems. Although these systems have been successfully installed and operated repeatedly, the bases for many fundamental decisions in their design and operation have been experience, intuition, and amount of funding available. There has been a lack of accurate, reliable means of predicting the proper capacity of a conveyor and of specifying the type and level of loading restrictions that should be made.

In this paper, one segment of the overall theory of conveyors has been explored and developed. Specifically, a loading system composed of a series of independent loading stations independently and successively placing their output onto the conveyor has been examined and modeled. The system has been characterized by the following parameters:

1. A loading ratio, the relationship between the output of an individual station and the capacity of the conveyor
2. A loading range, which is the maximum number of spaces on the conveyor which any station is permitted to examine in an attempt to load a unit of output onto the conveyor
3. The number of stations in the system.

Based on the probabalistic relationships which result from the assumption of independence among stations, expressions have been derived which describe three important variables of the conveyor system:

1. The fraction of conveyor capacity removed by a series of stations, all using a given loading range

2. The expected delay at any station facing a conveyor loaded to some specific capacity, when that station uses a given loading range

3. The expected fraction of output of any station which will be "rejected" by the conveyor as a result of failure to load a unit of output, when that station is facing a conveyor loaded to some specific capacity and uses a given loading range.

Emphasis has been placed on the economic analysis of the above model. A method of computing the cost of conveyor capacity, the cost of station delay, and the cost of station rejected output has been suggested. By using such a method to compute cost coefficients, it is possible to derive expressions to optimize, under certain limiting conditions, the three fundamental parameters mentioned above. Two minimum cost equations were derived:

1. An equation to minimize the sum of the costs of capacity and of station delay, assuming an infinite loading range, by choosing an optimal loading ratio

2. An equation to minimize the sum of the costs of station delay and of station rejected output by choosing an optimal loading range.

Consideration was given to the implications of the derived mathematical models in an actual situation. Means by which the results of this study could be applied were discussed, and the restrictions and limitations placed on the model by certain assumptions were analyzed.

All important equations presented were graphically illustrated and, where appropriate, the IBM 650 and 1620 digital computer systems were programmed to aid in their solution.