BULK QUEUEING MODELS AND THEIR APPLICATIONS

by 45

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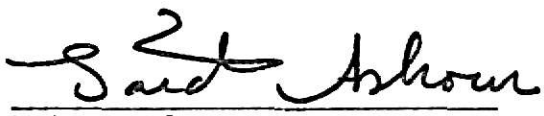
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CHAPTER I

INTRODUCTION

Management problems commonly arise from operational systems where men, machines or materials form a queue for some type of servicing. One of the principal characteristics of such systems is the uncertainity of randomness which is generally associated with arrival flow, service times, or both. In order to design such a dynamic system, usually referred to as queueing system, for optimal performance, it is essential to consider the effect of any random elements that may be present.

Queueing theory is concerned with the development of mathematical models to predict the behavior of systems that provides service for randomly arising demands. It has been applied to various problems. Queueing theory has formed a prominent place among modern analytical techniques of operations research, since it has considerable potential as a means for decision making in industry.

1.1. Bulk Queueing Theory

In a queueing process, customers arrive at the service station. They form a queue if the service station is busy. After being served, the customers depart from the service station. Customers may arrive singly or in batch (bulk). They may also be served in bulk. In case of the bulk arrival and/or bulk service, such systems are referred to as bulk queueing processes. For example, several people may go to a restaurant as a group, and obtain service together. Another example is that an elevator serves a group of people at the same time.

It is obvious that we are considering a more general type of queueing system which can yield the single arrival and single server as special cases. Bulk queueing theory has been often applied to a variety of industrial problems. To specify a queueing process, the distribution function and arrival pattern must be known. Also the service pattern, discipline and service time distribution must be described.

This chapter includes the reviewed literature of bulk queueing theory. Chapter II discusses the basic concepts of queueing theory in general. It covers the basic structure of queueing processes, types of queueing processes and methods of solutions. Chapter III is devoted to the analysis of various bulk queueing models. Some of these models are derived mathematically for the transient and steady-state conditions. Chapter IV discusses the applications of bulk queueing.

1.2. Literature Review

Considerable research has been done on bulk queueing processes. Most of this work is theoretical. Bulk queueing literature are reviewed below.

Bailey [4] has investigated a simple queueing process in which customers arrive at random, form a single queue in order of arrival, and are served in batches. The size of each batch has a fixed maximum. In most of the results obtained the time intervals between successive service are assumed to be independent χ^2 distribution with an even number of degrees of freedom. The equilibrium distribution of queue length has been studied by the imbedded markov chain method. Expressions for the mean and variance of queue length, and the mean waiting time are given. A useful inequality for the latter is also available in the special case of

a constant service time. An application has been made to hospital outpatient departments, with possible extensions to other situations such as elevators and buses.

Downton [16] has considered a process similar to that of Bailey [4]. In this process, the size of the batch is either a fixed number or equals the size of the queue, whichever is the smaller. Assuming that the queue is in equilibrium, the Laplace transform of the waiting time distribution is derived from the probability generating function of the queue length distribution. Expressions are obtained for the mean and variance of the waiting time. The computations of these quantities when the time intervals between successive epochs of service having independent χ^2 distribution with an even number of degrees of freedom, are briefly discussed. A short table of numerical results is given.

Gaver [21] has considered imbedded markov chain analysis of a queueing process with continuous time. In this analysis, customers arrive in batches and are served singly. A simple model for instantaneous defection is also analyzed. Using renewal theory, the properties of queue lengths and waiting times are discussed for the general process in which idle and busy periods recur.

Miller [44] has considered a problem where customers arrive in groups and are served in batch. It has been assumed that a service period starts only if there is at least one customer waiting in the queue in front of the server, and that quota does not have to be filled. Two models have been considered. In model A, all subsequent arrivals have to wait for the next service period. However, in model B, the addition of any subsequent arrivals is allowed to the group in service until a fixed quota has been

filled or the service period ends.

Jaiswal [32] has considered Bailey's queueing process [4] and solved it by a different method. The reason for this investigation has been two fold: (1) to modify suitably the method used by Gaver [22], Luchak [41, 42] and others, to fit the bulk service model; and (2) to investigate a method which may be capable of giving a time dependent solution to remove the drawback of Kendall's method [37, 38] as observed by Conolly [12]. In two papers [31, 32], Jaiswal has used a slight modification of this method and obtained steady-state probabilities. Furthermore, the time dependent solution of the same problem has been obtained. It should be pointed out that in Jaiswal's model [31] a unit is served immediately if, on arrival, there is no one waiting or being serviced, while in Bailey's model [4] the unit waits for the next service epoch.

Foster and Nyunt [20] have considered a case in which the units are serviced in batches of size k exactly by a single server in the order of their arrival. They have derived the stationary distribution function of the number of customers in the system just after their departure. In another paper Foster [18] has assumed that customers arrive in groups of a constant size but are served singly.

Craven [14] has considered the more general case of the bulk service queueing process introduced by Bailey [4]. The results are not restricted to negative exponential distributions of service times.

Foster [19] has considered two cases. In case A, arrivals are in single units and departures in batch. In case B, batch arrivals and departure in single units. Comparison has been made between the queue

sizes for the following: (1) just before arrivals; (2) just after departures; and (3) all the time.

Bhat [7] has considered time as a discrete variable and analyzed the transient behavior of a bulk service queueing system. The number of customers arriving within a fixed time interval follows a binomial probability distribution. The service times are assumed to be identically distributed and statistically independent. The distributions of the busy period and queue length have been obtained.

Arora [3] has analyzed a bulk service queueing system with two service stations. The Poisson arrival and exponential service time distributions have been considered. The model has been analyzed for two cases: (1) at least one service channel is busy, and (2) both service channels are busy. Various measures of effectiveness such as the mean queue length and the distribution of the busy period are computed.

Gupta [25] has studied the transient behavior of a batch arrival queueing model with poisson distribution. The batches are not necessarily of equal size. The model has been analyzed for a general class of service time distributions. Three particular cases of the model have been studied: (1) exponential service; (2) batches of equal size; and (3) steady-state solution. Various measures of effectiveness such as the mean service length and the waiting time distribution are computed.

Neuts [48] has considered a bulk service queueing system. The customers are served in a fixed number, or in any number n where $n < m$. The distribution of the busy period for a queue with poisson input have been studied. The busy period have been proved to be equal to the time between successive visits to state zero in an imbedded semi-markov processes,

associated with the queueing process.

Novaes and Frankel [49] have considered a bulk service queueing system. It has been assumed that the arrivals (cargo) have the same size from handling viewpoint. Special cases such as reneging, balking and consequent availability of alternative service have been considered. The model have been simulated for a two-part trade route with homogeneous equipment. This model may be used for the design of short to medium distance container, roll-on roll-off, or all palletized cargo operations and the related terminal requirements.

Roes [53] has considered a queueing system with bulk service and n servers. The waiting time distributions for three cases: (1) Service in order of arrival; (2) service in random order and (3) service in inverse order have been determined.

Giffin [23] has considered a simplified model for bulk service in a series of stations where the customers are served by a common fixed capacity carrier. Service measures are developed and computed by existing table of the poisson distribution. Example problems are included.

Soriano [57] has considered a queueing process in which the arrival of customers is in batch and service is single. He has considered the general types of batch interarrival time and service time distributions. Two models are analyzed in the transient and steady-state situations: (1) a random batch size; and (2) a periodic batch size. Some particular cases have also been considered. This model has been used to compare two scheduling systems.

Kashyap [35] has developed a double ended queueing model with bulk service and limited waiting space. The customers and taxis are considered at a taxi stand. The customers are served in bulk. The arrivals of customers and taxis are taken as general and poisson distributions respectively. Cases of Erlangian and Poisson arrivals and single seated taxis have been discussed as special cases.

Murari [45] has studied the transient behavior of a first come-first served with service in batches of variable size, poisson input, and exponential service time distributions. When the queue length increases to a fixed preassigned positive number, N, a special service channel capable of serving a group of N units is made available. However, it is cancelled at the termination of service, if the queue length becomes less than N. A limited waiting space is also assumed for the queue. Finally, some particular cases are discussed.

CHAPTER II

BASIC CONCEPTS OF QUEUEING THEORY

This chapter is devoted to discuss the basic structure of queueing processes, classify, and illustrate their elements. The queueing models are classified into deterministic and probabilistic models. Methods of solutions are also included to differentiate between analytical and simulation procedures. In analytical procedure, the transient and steady-state solutions are discussed.

2.1. Basic Structure of Queueing Processes*

The basic process assumed by most queueing models is the following: Units requiring service are generated over time by an input source. These units enter the queueing system and join a queue. At a certain time, a unit waiting in the queue, is selected for service by some rule referred to as service discipline. The required service is then performed for the unit by the service mechanism, after which the unit leaves the queueing system as an output. This queueing process and its elements are shown in Figures 1 and 2, respectively.

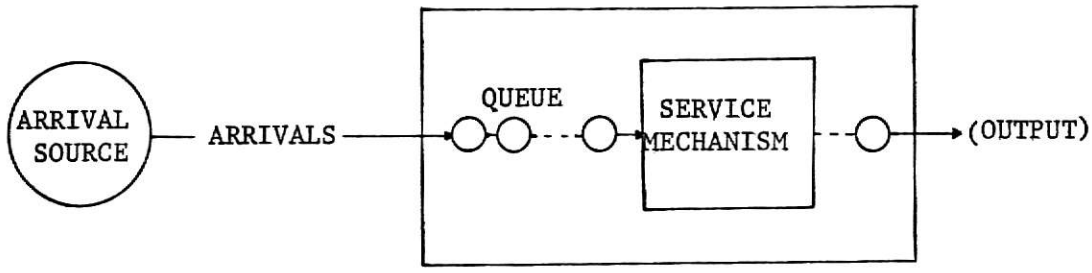


Figure 1. The Basic Queueing Process

*Adopted from Ashour, S., "Introduction to Queueing Theory", Unpublished Report, Kansas State University, 1967.

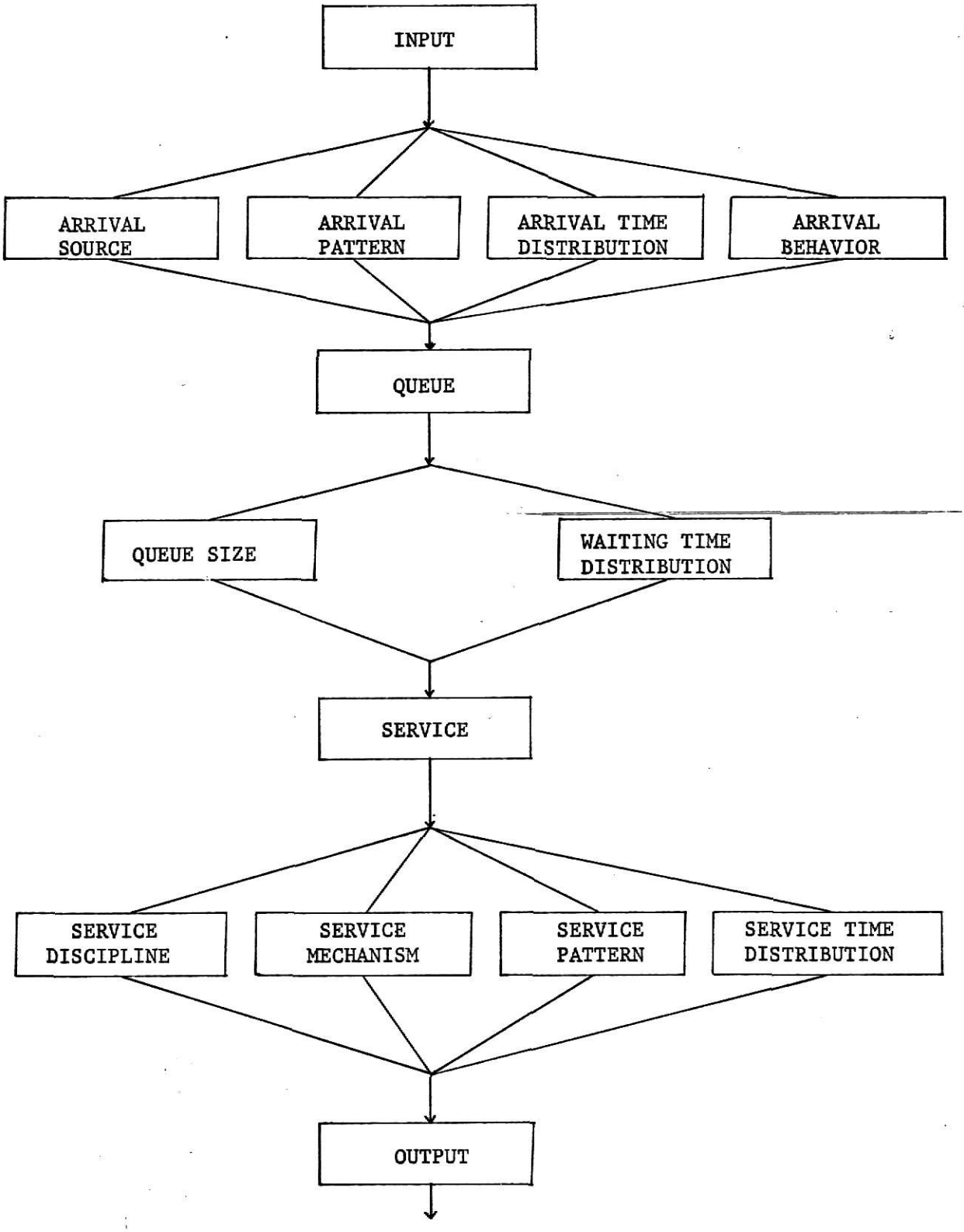
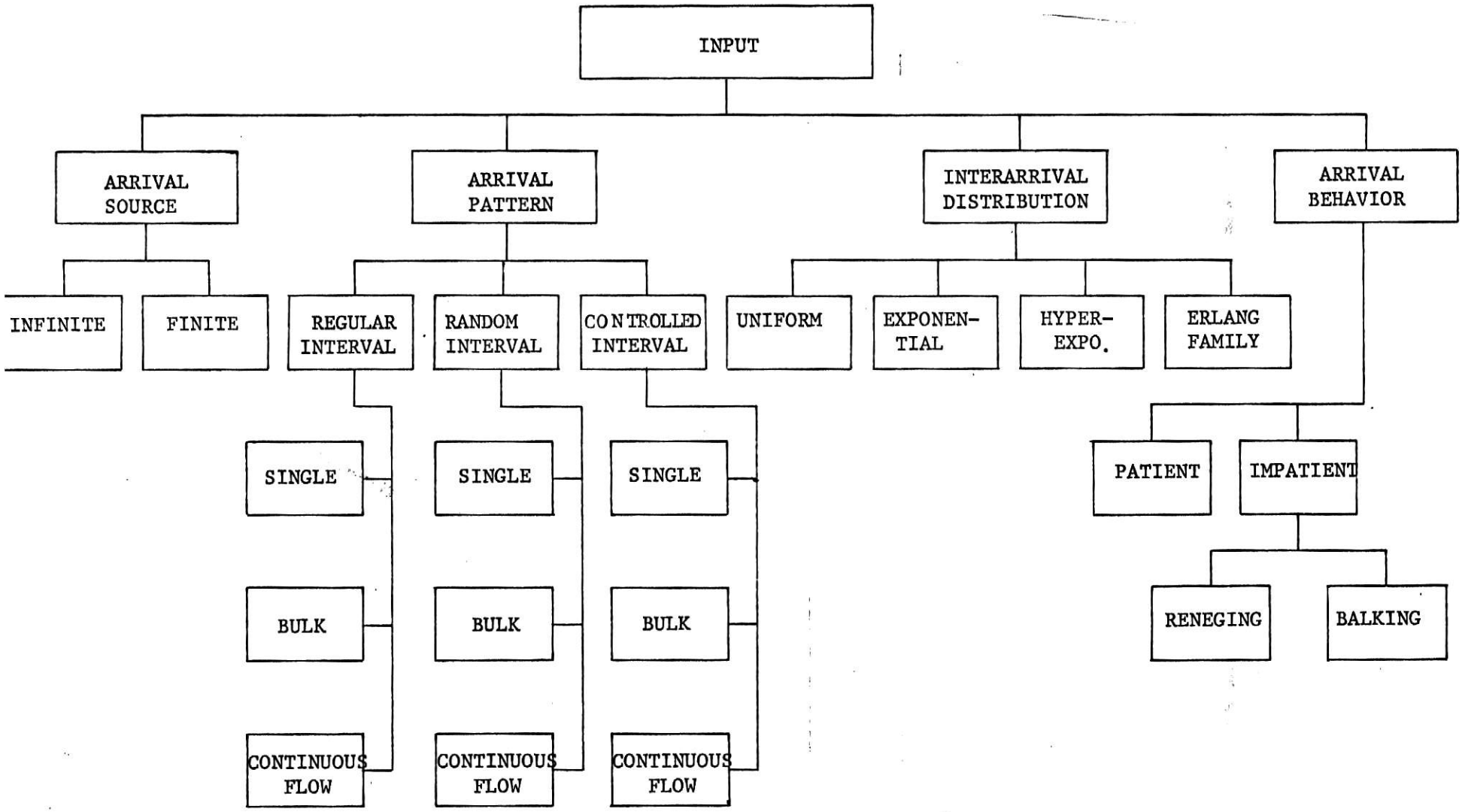


Figure 2. Elements of Queueing Processes


INPUT
ARRIVAL SOURCE
ARRIVAL PATTERN
INTERARRIVAL DISTRIBUTION
ARRIVAL BEHAVIOR
INFINITE
FINITE
REGULAR INTERVAL
RANDOM INTERVAL
CONTROLLED INTERVAL
UNIFORM
EXPONEN-TIAL
HYPER-EXPO.
ERLANG FAMILY
SINGLE
SINGLE
SINGLE
BULK
BULK
BULK
CONTINUOUS FLOW
CONTINUOUS FLOW
CONTINUOUS FLOW
PATIENT
IMPATIENT
RENEGING
BALKING

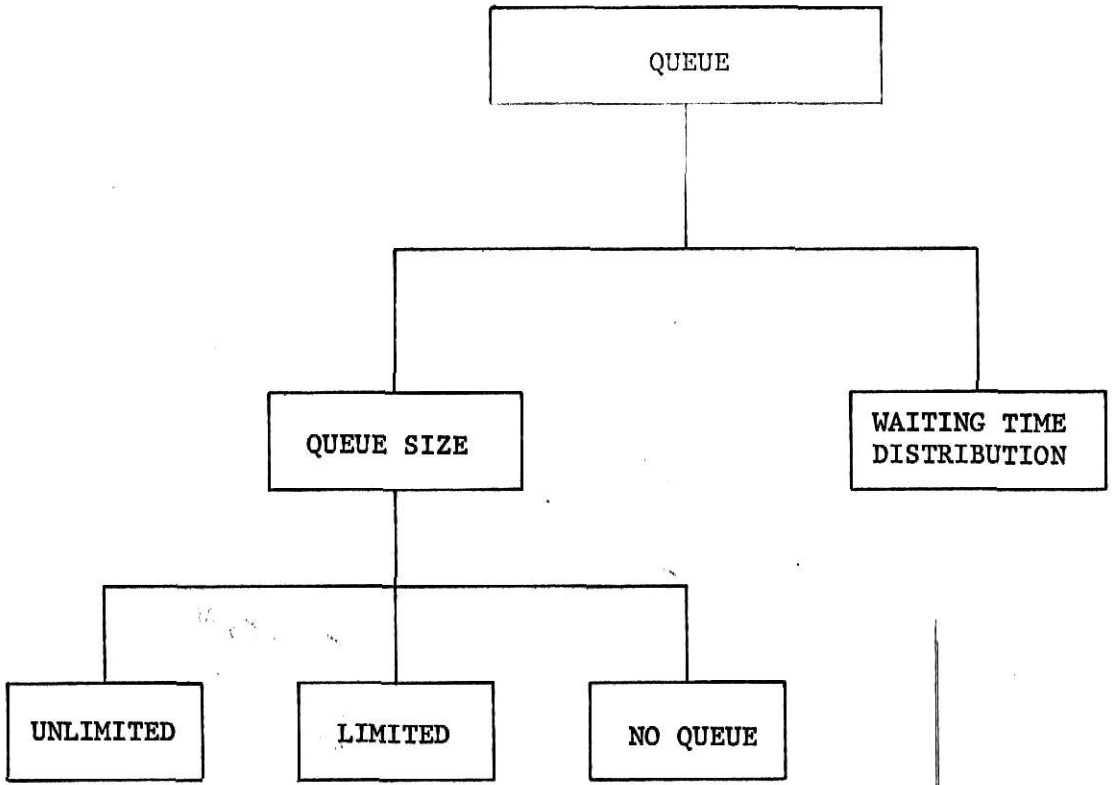


Figure 4. Elements of Queue

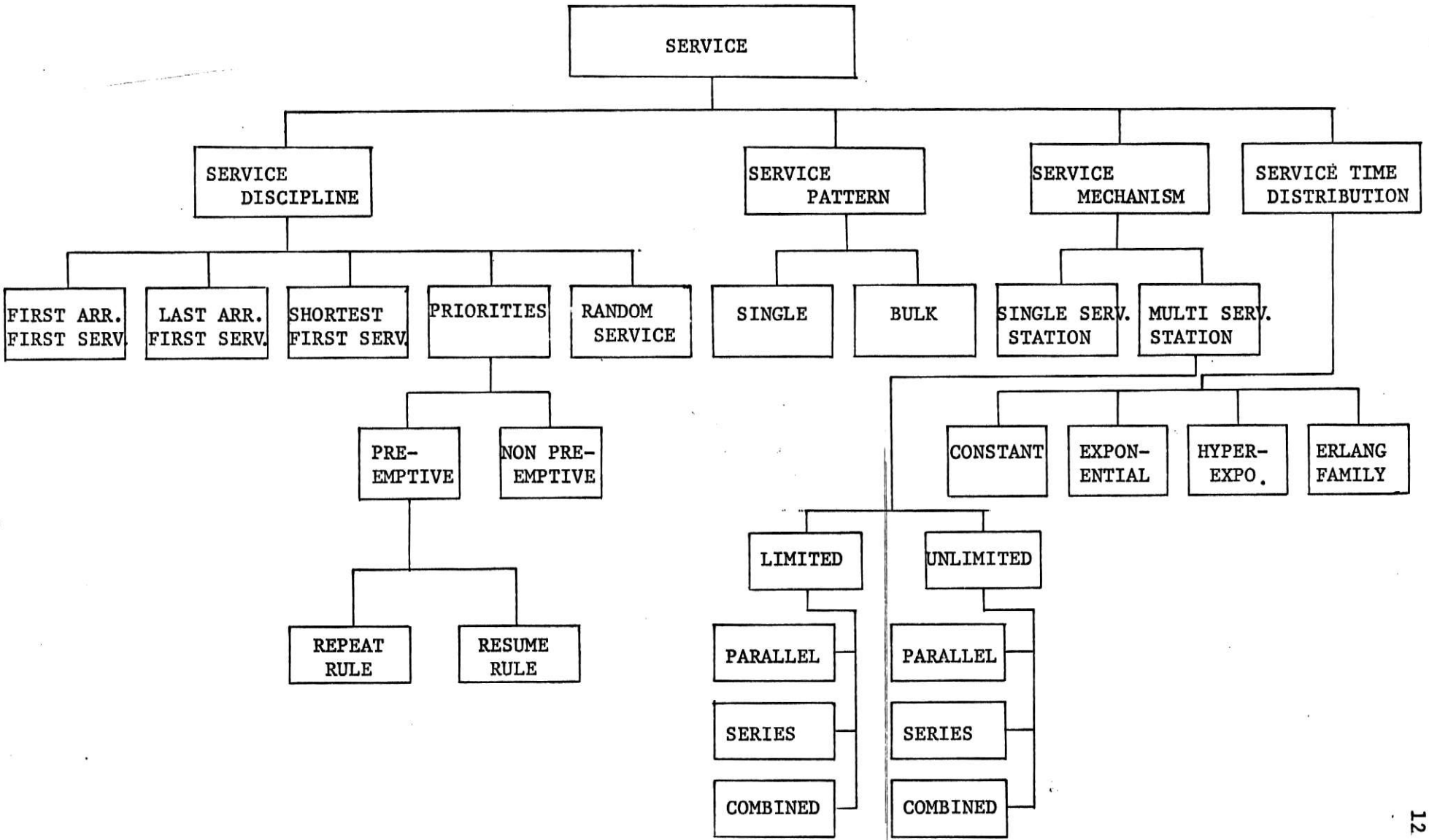


Figure 5. Elements of Service

Therefore, the most important characteristics in queueing theory are:

1. Input: The customers desiring service, arrive from a certain source at the queueing system. The arrival source, arrival pattern, arrival distribution and arrival behavior are basic elements of the input, see Figure 3.
2. Queue: The customers who have to wait for service, form a waiting line (queue). The elements of the queue are queue size and waiting time distribution, see Figure 4.
3. Service: The customers who arrive at the system are getting the service offered. The elements of service are service discipline, service pattern, service mechanism and service time distribution, see Figure 5.
4. Output: The customer is discharged from the system after being served. It is obvious that the output distribution may be studied in the cases where the service stations are on series or in cyclic queueing systems. In these cases, the output of a service station is the input of the successive one.

2.2. Types of Queueing Models*

The study of queueing systems requires information about the arrival rate, service discipline, and service time. There are two types of queueing models. First is deterministic, when the customers arrive exactly at known times and receive service of exact known length. In deterministic models, all interarrivals are known, so the service inter-

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*Adopted from Saaty, T.L., "Elements of Queueing Theory," McGraw-Hill Book Company, Inc. New York, 1961.

vals too. In practice there is no such regularity in queueing systems. The other model is that where there are variations in the interarrivals and/or in the service times. Sometimes more arrivals than the service facility can immediately accommodate, and a queue of varying length will form. Sometimes service channels will be idle. Accordingly, the discharges vary in a random fashion and each measurable quantity associated with the process is considered a stochastic variable, fluctuating with time.

The random variables describe the interarrivals, waiting times and service times of a customer. With a set of customers, there are a family of random variables which depend on time. Each random variable may have a different probability distribution for each point of time. These random variables form stochastic processes. A stochastic process may be defined as a family of random variables that depend on time.

The study of queueing theory is concerned with finding probability distributions of these random variables when enough information is given about them. The main problem is then:

1. to relate the variables correctly in order to describe a queueing problem,
2. to derive the associated distributions and determine their actual form by statistical measurement,
3. to use the distributions for deriving useful measures, and
4. to apply these measures to the improvement of a system

The mathematical model required to analyze such a system is referred to as a probabilistic queueing model.

2.3. Solutions of Queueing Models

To obtain a solution to a queueing model, two approaches are usually considered: (1) analytical approach, and (2) simulation approach. In the former approach, the arrival and service time distributions must be determined by being approximated to one of the well known probability distribution functions. From the structure of the system, the state probabilities are constructed in terms of difference-differential equations, depending on the number of units in the system, n, and time, t. These equations, when solved, provides a transient solution. However, if the queueing system is considered in equilibrium state, the above equations, independent on time, may be solved in the steady-state condition. The transient condition implies that the probability of finding the system in any given state will not be the same during the first hour of operation as, say, the last hour. It should be pointed out that the steady-state condition is that where the probabilities are independent of time but not that the system becomes deterministic. The transient solution is important if we are interested in the effect of sudden changes in system parameters, or such questions as the distribution of the length of a busy period. In general, transient solutions are difficult to obtain because it involves a large set of differential equations. Mostly, steady-state solutions of models have been derived.

Simulation approach is known as Monte Carlo method which depends on simulated data concerning arrivals and service times. This approach is valuable when a computer is used. Each simulated period of system operation is developed using a sample set of arrivals drawn from a specified input and service time distributions. This set of arrivals generates a sample

waiting time distribution for this period. The degree of variation among input samples and the degree of accuracy required in final results determine how many periods, or how long the period is for which the system operation must be simulated. The simulation approach is used in preference to the mathematical approach when the queueing system cannot be easily solved by mathematical means. Furthermore, simulated operating data can be generated for any variety of arrival and service-time distributions and queue discipline.

CHAPTER III

BULK QUEUEING MODELS

The different methods of solutions expressed in chapter II is a basis of analyzing the queueing models. It is the purpose of this chapter to analyze various bulk queueing models. The following five models have been analyzed:

1. Bulk service queueing models with an additional service station.
2. Queueing model for bulk service at a series of stations.
3. Double ended queue with bulk service queueing model.
4. Multiphase bulk service queueing model.
5. Batch arrival queueing model.

The first four models are of bulk service type and the last is of bulk arrival type model. The models are analyzed under the transient and steady-state conditions. The fourth model is a simplified model which does not require the evaluation of roots of the polynomials in the analysis.

3.1. Bulk Service Queueing Models with an Additional Service Station

This queueing model which has been developed by Murari [45] describes a system with a limited queue and one service station. The customers who arrive singly, are served in bulk. If the number of customers exceeds the capacity of the service channel, those remaining will not be turned down; however, special service station will be added. Two models are developed: (1) Model A represents the situation where an ordinary service channel will remain idle if queue size is zero and wait for a customer to arrive. (2) Model B represents the situation where the ordinary service channel will serve a batch of zero customers. Both models are analyzed for transient and steady-state conditions. Two special cases are also discussed. The first case is that when the size of the queue approaches infinity with additional channel if needed. The other case is similar to the first one but without additional channel.

Model Specification:

Arrival Source	:	Infinite
Arrival Pattern	:	Single
Arrival Behavior	:	Patient
Arrival Distribution	:	Poisson
Queue Size	:	Limited
Service Discipline	:	First arrive, first served

Service Pattern	:	Bulk
Service Mechanism	:	Single Channel, single phase, with additional service station
Service time distribution	:	Exponential
Number of Servers	:	One

Practical Problem: Passengers wait at the bus stand forming a queue with a limited size in order to get into the bus. The bus is considered as a service channel. If the number of passengers is less than the maximum capacity of the ordinary bus they ride in. Otherwise, an additional bus will be called. This situation arises often during rush hours.

Notation: The following is the notation used in this model:

λ = Mean arrival rate

μ = Mean Service rate of Ordinary Service Channel (OSC)

ν = Mean Service rate of Special Service Channel (SSC)

N = Number of customers served by Special Service Channel

M = Capacity of Ordinary Service Channel

$Q_E(t)$ = Probability that at time t, there is no customer in the system, the OSC is idle, and the SSC is not added

$Q_B(t)$ = Probability that at time t, there is no customer in the system, the OSC is idle, and the SSC is added

$P_{n,E}(t)$ = Probability that at time t, there are n customers in the system, the OSC is busy and the SSC is not added

$P_{n,B}(t)$ = Probability that at time t, there are n customers in the system, the OSC is busy and the SSC is added

$P_n(t)$ = Probability that there are n customers in the queue at time t

$F(\alpha)$ = Laplace Transform of $F(t)$

$G_E(t,x)$ = Generating Function of $P_{n,E}(t)$

$G_B(t,x)$ = Generating function of $P_{n,B}(t)$

$G(t,x)$ = Generating function of $P_n(t)$

Analysis: The service facility consists of two service channels, namely ordinary service channel (OSC) and special service channel (SSC). The capacity of ordinary service channel j, is determined at the completion of each service and is a random variable with probability p_j such that

$$\sum_{j=0}^{M} p_j = 1, \qquad p_j > 0.$$

If at any instant the queue length is n, $n > 0$, and the capacity is j then the OSC will take in a batch of min [n,j] units for service. However, if at any instant the queue length is n, $n \geq 0$, and the capacity is zero, then the OSC will take in a batch of zero units for service. We will consider two models:

1. Model A - If at any instant queue length is zero and the capacity is j, $j > 0$, then the OSC will remain idle and wait for a customer to arrive.

2. Model B - The OSC will take in a batch of zero units for service.

Assumptions:

1. The SSC serves customers in batches of fixed size N, $N > M$. If at the completion of service at the SSC the queue length is n, $n \geq N$, then the SSC will take in a batch of N units for service. If at the completion of service at the SSC the queue length is n, $n < N$, the SSC will stop operating; and as soon as the queue length increases to N, the SSC will start operating and take in the batch of N units for service.

2. Let the service times μ_k of the successive batches at the OSC are independently, identically distributed with

$$P(\nu_k \leq x) = 1 - e^{-\nu x}, \quad \begin{matrix} k = 1, 2, 3, \ldots \\ \nu > 0 \end{matrix}$$

3. If at any time queue length is equal to K, $K > N$, the arriving customers will be considered lost. When SSC is not operating queue length cannot be greater than N. However, when the SSC is operating the queue length is allowed to grow to K.

4. The stochastic processes involved namely: interarrival times of customers, service times of batches at the OSC, and service times of batches at the SSC, are independent of each other.

Solution of Model A

1. Transient Condition: The equations representing this model under the transient conditions are

1.1. In the case where the SSC is not operating

$$\frac{dQ_E(t)}{dt} + \lambda Q_E(t) = \sum_{j=1}^{M} \mu p_j P_{0,E}(t) + \nu Q_B(t) \tag{1}$$

$$\frac{dP_{0,E}(t)}{dt} + (\lambda+\mu)\, P_{0,E}(t) = \mu p_0 P_{0,E}(t) +$$

$$\sum_{j=1}^{M} \sum_{r=1}^{j} \mu p_j P_{r,E}(t) + \lambda Q_E(t) + \nu P_{0,B}(t) \tag{2}$$

$$\frac{dP_{n,E}(t)}{dt} + (\lambda+\mu)\, P_{n,E}(t) = \mu p_0 P_{n,E}(t) +$$

$$\sum_{j=1}^{M} \mu p_j P_{n+j,E}(t) + \lambda P_{n-1,E}(t) +$$

$$\nu P_{n,B}(t), \qquad 0 < n < N \tag{3}$$

1.2. In the case where the SSC is operating

$$\frac{dQ_B(t)}{dt} + (\lambda+\nu)\, Q_B(t) = \sum_{j=1}^{M} \mu p_j P_{0,B}(t) \tag{4}$$

$$\frac{dP_{0,B}(t)}{dt} + (\lambda+\mu+\nu)\, P_{0,B}(t) = \mu p_0 P_{0,B}(t) +$$

$$\sum_{j=1}^{M} \sum_{r=1}^{j} \mu p_j P_{r,B}(t) + \lambda Q_B(t) +$$

$$\nu P_{N,B}(t) + \lambda P_{N-1,E}(t) \tag{5}$$

$$\frac{dP_{n,B}(t)}{dt} + (\lambda+\mu+\nu)\, P_{n,B}(t) = \mu p_0 P_{n,B}(t) +$$

$$\sum_{j=1}^{M} \mu p_j P_{n+j,B}(t) + \lambda P_{n-1,B}(t) +$$

$$\nu P_{n+N,B}(t) , \qquad 0 < n < K \tag{6}$$

$$\frac{dP_{K,B}(t)}{dt} + (\mu+\nu)\, P_{K,B}(t) = \mu p_0 P_{K,B}(t) +$$

$$\lambda P_{K-1,B}(t) \quad , \tag{7}$$

$$P_{n,E}(t) = 0 \quad , \qquad n \geq N \tag{8}$$

$$P_{n,B}(t) = 0 \quad , \qquad n > K \tag{9}$$

$$P_n(t) = P_{n,E}(t) + P_{n,B}(t) \quad , \qquad n > 0 \tag{10}$$

$$P_0(t) = P_{0,E}(t) + P_{0,B}(t) + Q_E(t) + Q_B(t) \tag{11}$$

The record time from the instant when the queue length is equal to zero, the OSC is idle, and the SSC is not operating such that initial condition becomes

$$Q_E(0) = 1 \tag{12}$$

Multiplying equation (2) by x^M, such that

$$x^M \frac{dP_{0,E}(t)}{dt} + (\lambda+\mu)\, x^M P_{0,E}(t) = \mu x^M p_0 P_{0,E}(t) +$$

$$x^M \sum_{j=1}^{M} \sum_{r=1}^{j} \mu p_j P_{r,E}(t) + \lambda x^M Q_E(t) + \nu x^M P_{0,B}(t) \,, \qquad (13)$$

Multiplying equation (3) by x^{M+n} and summing over n, n = 1, 2, . . ., N-1,

$$\sum_{n=1}^{N-1} x^{M+n} \frac{dP_{n,E}(t)}{dt} + \sum_{n=1}^{N-1} (\lambda+\mu)\, x^{M+n} P_{n,E}(t) =$$

$$\sum_{n=1}^{N-1} \left[\mu p_0 x^{M+n} P_{n,E}(t) + x^{(M+n)} \sum_{j=1}^{M} \mu p_j P_{n+j,E}(t) + \right.$$

$$\left. \lambda x^{M+n} P_{n-1,E}(t) + \nu x^{M+n} P_{n,B}(t) \right] \,, \qquad (14)$$

then adding equations (13) and (14), and using equation (8),

$$\sum_{n=0}^{\infty} x^{M+n} \frac{dP_{n,E}(t)}{dt} + \sum_{n=0}^{\infty} (\lambda+\mu)\, x^{M+n} P_{n,E}(t) =$$

$$\sum_{n=0}^{\infty} \mu x^{M+n} P_0 P_{n,E}(t) + \lambda x^M Q_E(t) +$$

$$\sum_{j=1}^{M} x^M \left[\sum_{r=1}^{j} \mu p_j P_{r,E}(t) + \sum_{n=1}^{N-1} x^n \mu p_j P_{n+j,E}(t) \right] +$$

$$\sum_{n=1}^{N-1} \lambda x^{M+n} P_{n-1,E}(t) + \sum_{n=0}^{N-1} \nu x^{M+n} P_{n,B}(t) \ . \qquad (15)$$

Using the generating functions

$$G_E(t,x) = \sum_{n=0}^{\infty} x^n P_{n,E}(t) \ ,$$

and

$$G(t,x) = \sum_{n=0}^{\infty} x^n P_n(t) \ ,$$

Equation (15) becomes

$$x^M \frac{dG_E(t,x)}{dt} + x^M(\lambda+\mu) G_E(t,x) = \mu P_0 x^M G_E(t,x) +$$

$$\lambda x^M Q_E(t) + \sum_{j=1}^{M} \mu p_j \sum_{n=0}^{\infty} x^{M+n} P_{n+j,E}(t) +$$

$$\sum_{j=1}^{M}\sum_{r=1}^{j-1} x^{M}\mu p_j P_{r,E}(t) + \sum_{n=1}^{N-1} \lambda x^{M+n} P_{n-1,E}(t) +$$

$$\sum_{n=0}^{N-1} \nu x^{M+n} P_{n,B}(t) \; . \qquad (16)$$

Where the third term in the R.H.S. of equation (16) is simplified to

$$\sum_{j=1}^{M} \mu p_j \sum_{n=0}^{\infty} x^{M+n} P_{n+j,E}(t) = \sum_{j=1}^{M} \mu p_j \sum_{r=j}^{\infty} x^{M+r-j} P_{r,E}(t) =$$

$$\sum_{j=1}^{M} \mu p_j \sum_{r=0}^{\infty} x^{M+r-j} P_{r,E}(t) - \sum_{j=1}^{M} \mu p_j \sum_{r=0}^{j-1} x^{M+r-j} P_{r,E}(t) =$$

$$\sum_{j=1}^{M} \mu p_j x^{M-j} G_E(t,x) - \sum_{j=1}^{M} \mu p_j \sum_{r=0}^{j-1} x^{M+r-j} P_{r,E}(t) \; ,$$

the fourth term is simplified to

$$\sum_{j=1}^{M}\sum_{r=1}^{j-1} x^{M}\mu p_j P_{r,E}(t) = \sum_{j=1}^{M}\sum_{r=0}^{j-1} x^{M}\mu p_j P_{r,E}(t) -$$

$$- \sum_{j=1}^{M} x^M \mu p_j P_{0,E}(t) = \sum_{j=1}^{M} \sum_{r=0}^{j-1} x^M \mu p_j P_{r,E}(t) -$$

$$x^M \mu P_{0,E}(t) \sum_{j=1}^{M} p_j \quad ,$$

but

$$\sum_{j=0}^{M} p_j = 1 \quad ,$$

or

$$\sum_{j=1}^{M} p_j = 1 - p_0 \quad ,$$

thus

$$\sum_{j=1}^{M} \sum_{r=1}^{j-1} x^M \mu p_j P_{r,E}(t) = \sum_{j=1}^{M} \sum_{r=0}^{j-1} x^M \mu p_j P_{r,E}(t) -$$

$$\mu x^M (1-p_0) P_{0,E}(t) \quad ;$$

and let n-1 = n', the fifth term is simplified to

$$\sum_{n=1}^{N-1} \lambda x^{M+n} P_{n-1,E}(t) = \sum_{n'=0}^{N-2} \lambda x^{M+n'+1} P_{n',E}(t) =$$

$$\sum_{n'=0}^{N-1} \lambda x^{M+1+n'} P_{n',E}(t) - \lambda x^{M+N} P_{N-1,E}(t) =$$

$$\lambda x^{M+1} G_E(t,x) - \lambda x^{M+N} P_{N-1,E}(t) .$$

Using the above results, equation (16) becomes

$$x^M \frac{dG_E(t,x)}{dt} + x^M(\lambda+\mu) G_E(t,x) = \mu x^M p_0 G_E(t,x) +$$

$$\lambda x^M Q_E(t) + \sum_{j=1}^{M} \mu p_j x^{M-j} G_E(t,x) -$$

$$\sum_{j=1}^{M} \sum_{r=0}^{j-1} \mu p_j x^{M+r-j} P_{r,E}(t) +$$

$$\sum_{j=1}^{M} \sum_{r=0}^{j-1} \mu x^M \mu p_j P_{r,E}(t) - \mu x^M (1-p_0) P_{0,E}(t) +$$

$$\lambda x^{M+1} G_E(t,x) - \lambda x^{M+N} P_{N-1,E}(t) + \sum_{n=0}^{N-1} \nu x^{M+n} P_{n,B}(t) ,$$

which may be again simplified such that

$$x^M \frac{dG_E(t,x)}{dt} + \left[x^M(\lambda+\mu-\mu p_0) - \sum_{j=1}^{M} \mu p_j x^{M-j} - \lambda x^{M+1}\right] G_E(t,x) =$$

$$\lambda x^M Q_E(t) + \sum_{j=1}^{M} \sum_{r=0}^{j-1} \mu p_j \left[x^M - x^{M+r-j}\right] P_{r,E}(t) -$$

$$\mu x^M (1-p_0) P_{0,E}(t) - \lambda x^{M+N} P_{N-1,E}(t) +$$

$$\sum_{n=0}^{N-1} \nu x^{M+n} P_{n,B}(t) .$$

Taking laplace transform of both sides,

$$x^M[\alpha G_E(\alpha,x) - 0] + \left[x^M(\lambda+\mu-\lambda x) - \sum_{j=0}^{M} \mu p_j x^{M-j}\right] G_E(\alpha,x) =$$

$$\lambda x^M Q_E(\alpha) + \sum_{j=1}^{M} \sum_{r=0}^{j-1} \mu p_j [x^M - x^{M+r-j}] P_{r,E}(\alpha) -$$

$$\mu x^{M}(1-p_0)P_{0,E}(\alpha) - \lambda x^{M+N}P_{N-1,E}(\alpha) +$$

$$\sum_{n=0}^{N-1} \nu x^{M+n}P_{n,B}(\alpha) ,$$

and rearranging the terms, we obtain

$$G_E(\alpha,x)\left[(\alpha+\lambda+\mu-\lambda x)x^{M} - \sum_{j=0}^{M} \mu p_j x^{M-j}\right] =$$

$$x^{M}\left[\lambda Q_E(\alpha) - \mu(1-p_0)P_{0,E}(\alpha) - \lambda x^{N}P_{N-1,E}(\alpha)\right] +$$

$$\sum_{j=1}^{M}\sum_{r=0}^{j-1} \mu p_j [x^{M}-x^{M+r-j}]\, P_{r,E}(\alpha) + \sum_{n=0}^{N-1} \nu x^{M+n}P_{n,B}(\alpha) .$$

Thus, the generating function $G_E(\alpha,x)$ is

$$G_E(\alpha,x) = \frac{x^{M}[\lambda Q_E(\alpha)-\mu(1-p_0)P_{0,E}(\alpha)-\lambda x^{N}P_{N-1,E}(\alpha)]+\sum_{n=0}^{N-1} \nu x^{M+n}P_{n,B}(\alpha)}{\left|(\alpha+\lambda+\mu-\lambda x)x^{M}-\sum_{j=0}^{M} \mu p_j x^{M-j}\right|}$$

$$\frac{\sum_{j=1}^{M}\sum_{r=0}^{j-1} \mu p_j [x^M - x^{M+r-j}]\, P_{r,E}(\alpha)}{\left[(\alpha+\lambda+\mu-\lambda x)\, x^M - \sum_{j=0}^{M} \mu p_j x^{M-j}\right]} \tag{17}$$

Multiplying equation (5) by x^N,

$$x^N \frac{dP_{0,B}(t)}{dt} + (\lambda+\mu+\nu)x^N P_{0,B}(t) =$$

$$x^N \left[\mu p_0 P_{0,B}(t) + \sum_{j=1}^{M}\sum_{r=1}^{j} \mu p_j P_{r,B}(t) + \right.$$

$$\left. \lambda Q_B(t) + \nu P_{N,B}(t) + \lambda P_{N-1,E}(t)\right] , \tag{18}$$

multiplying equation (6) by x^{N+n} and summing over n, n = 1, 2, . . ., K-1,

$$\sum_{n=1}^{K-1} x^{N+n} \left[\frac{dP_{n,B}(t)}{dt} + (\lambda+\mu+\nu)P_{n,B}(t)\right] = \sum_{n=1}^{K-1} x^{N+n} \left[\mu p_0 P_{n,B}(t) + \right.$$

$$\left. \sum_{j=1}^{M} \mu p_j P_{n+j,B}(t) + \lambda P_{n-1,B}(t) + \nu P_{n+N,B}(t)\right] , \tag{19}$$

multiplying equation (7) by x^{N+K},

$$x^{N+K} \frac{dP_{K,B}(t)}{dt} + x^{N+K}(\mu+\nu)P_{K,B}(t) =$$

$$x^{N+K}[\mu p_0 P_{K,B}(t) + \lambda P_{K-1,B}(t)] , \qquad (20)$$

then adding equations (18), (19), and (20), and using equation (9) we get,

$$\sum_{n=0}^{\infty} x^{N+n} \frac{dP_{n,B}(t)}{dt} + \sum_{n=0}^{K-1} (\lambda+\mu+\nu)x^{N+n}P_{n,B}(t) +$$

$$(\mu+\nu)x^{N+K}P_{K,B}(t) = \sum_{n=0}^{\infty} x^{N+n}\mu p_0 P_{n,B}(t) +$$

$$\lambda x^N Q_B(t) + \sum_{j=1}^{M} \sum_{r=1}^{j} x^N \mu p_j P_{r,B}(t) +$$

$$\sum_{n=0}^{K-1} \nu P_{N+n,B}(t)x^{N+n} + \lambda x^N P_{N-1,E}(t) +$$

$$\sum_{n=1}^{K-1} x^{N+n} \sum_{j=1}^{M} \mu p_j P_{n+j,B}(t) + \lambda P_{n-1,B}(t) +$$

$$\lambda x^{N+K} P_{K-1,B}(t) . \qquad (21)$$

Using the generating functions

$$G_B(t,x) = \sum_{n=0}^{\infty} x^n P_{n,B}(t)$$

equation (21) becomes

$$x^N \frac{dG_B(t,x)}{dt} + (\lambda+\mu+\nu)x^N G_B(t,x) - \lambda x^{N+K} P_{K,B}(t) =$$

$$x^N \mu p_o G_B(t,x) + \lambda x^N Q_B(t) + \sum_{j=1}^{M} \sum_{r=1}^{j} x^N \mu p_j P_{r,B}(t) +$$

$$\sum_{n=0}^{K-1} \nu x^{N+n} P_{N+n,B}(t) + x^N P_{N-1,B}(t) +$$

$$\sum_{n=1}^{K-1} \sum_{j=1}^{M} x^{N+n} \mu p_j P_{n+j,B}(t) + \sum_{n=1}^{K-1} P_{n-1,B}(t) x^{N+n} +$$

$$\lambda x^{N+K} P_{K-1,B}(t) .$$

By using the simplified terms of equation (15) and the taking the laplace transform, we get

$$\{[\alpha+\lambda+\mu+\nu-\lambda x]x^N - \sum_{j=0}^{M} \mu p_j x^{N-j} - \nu\} G_B(\alpha,x) =$$

$$\sum_{j=1}^{M} \sum_{r=0}^{j-1} \mu p_j (x^N - x^{N+r-j}) P_{r,B}(\alpha) -$$

$$\sum_{n=0}^{N-1} \nu x^n P_{n,B}(\alpha) + x^N [\lambda Q_B(\alpha) - \mu(1-p_0)P_{0,B}(\alpha) +$$

$$\lambda P_{N-1,E}(\alpha) - \lambda x^K (x-1) P_{K,B}(\alpha)] \quad .$$

Thus, the generating function $G_B(\alpha,x)$ is

$$G_B(\alpha,x) =$$

$$\frac{\sum_{j=1}^{M} \sum_{r=0}^{j-1} \mu p_j (x^N - x^{N+r-j}) P_{r,B}(\alpha) - \sum_{n=0}^{N-1} \nu x^n P_{n,B}(\alpha) + x^N \lambda Q_B(\alpha)}{\{[\alpha+\lambda+\mu+\nu-\lambda x]x^N - \sum_{j=0}^{M} \mu p_j x^{N-j} - \nu\}} +$$

$$\frac{x^N[-\mu(1-p_0)P_{0,B}(\alpha) + \lambda P_{N-1,E}(\alpha) - \lambda x^K (x-1) P_{K,B}(\alpha)]}{\{[\alpha+\lambda+\mu+\nu-\lambda x]x^N - \sum_{j=0}^{M} \mu p_j x^{N-j} - \nu\}} . \quad (22)$$

Multiplying equation (10) by x^n and then summing over n, $n = 1, 2, \ldots, K$, we get

$$\sum_{n=1}^{K} x^n P_n(t) = \sum_{n=1}^{K} x^n P_{n,E}(t) + \sum_{n=1}^{K} x^n P_{n,B}(t) \ . \qquad (23)$$

Then adding equation (11) to (23) letting $K = \infty$, and then using equations (8) and (9), we get,

$$\sum_{n=0}^{\infty} x^n P_n(t) = \sum_{n=0}^{\infty} x^n P_{n,E}(t) + \sum_{n=0}^{\infty} x^n P_{n,B}(t) + Q_E(t) + Q_B(t) \ .$$

Using the generating functions

$$G(t,x) = G_E(t,x) + G_B(t,x) + Q_E(t) + Q_B(t)$$

Taking laplace transform of both sides,

$$G(\alpha,x) = G_E(\alpha,x) + G_B(\alpha,x) + Q_E(\alpha) + Q_B(\alpha) \ . \qquad (23)$$

Taking laplace transforms of equations (1) and (4), and then using equation (12) we get

$$(\alpha+\lambda)\ Q_E(\alpha) - 1 = \sum_{j=1}^{M} \mu p_j P_{0,E}(\alpha) + \nu Q_B(\alpha) \ . \qquad (24)$$

and

$$(\alpha+\lambda+\mu)\ Q_B(\alpha) = \sum_{j=1}^{M} \mu p_j P_{0,B}(\alpha) \ . \qquad (25)$$

The denominator in equation (17) is a polynomial of M+1 degree which has M+1 roots. Since $G_E(\alpha,x)$ is a polynomial, these roots must vanish the numerator in equation (17) giving rise to a set of M+1 equations.

Similarly the N+1 roots of the denominator in equation (22) give rise to a set of N+1 equations. Solving these two set of equations together with equations (24) and (25), we can determine the following M+N+4 unknowns,

$Q_E(\alpha)$,

$Q_B(\alpha)$,

$P_{r,E}(\alpha)$, r = 0, 1, 2, ... M-1,

$P_{N-1,E}(\alpha)$

$P_{r,B}(\alpha)$, r = 0, 1, 2, ... N-1,

$P_{K,B}(\alpha)$.

Once these variables are determined, $G(\alpha,x)$ is computed.

2. Steady State Condition: The steady state solution can be obtained by the well known property of the laplace transform.

$$\lim_{t\to\infty} F(t) = \lim_{\alpha\to 0} \alpha F(\alpha) \ , \tag{26}$$

if the limit on L.H.S. exists.

Thus if

$$\lim_{t\to\infty} P_{n,E}(t) = P_{n,E} \ ,$$

and

$$P_E(x) = \sum_{n=0}^{\infty} x^n P_{n,E} \ ,$$

we have

$$\lim_{\alpha\to 0} \alpha G_E(\alpha,x) = P_E(x) \ .$$

Applying equation (26) to equations (17), (22), (23), (24), and (25) we get

$$P(x) = P_E(x) + P_B(x) + Q_E + Q_B \quad ,$$

where

$$P_E(x) = \left[\sum_{j=1}^{M} \sum_{r=0}^{j-1} \mu p_j (x^M - x^{M+r-j}) P_{r,E} + \sum_{n=0}^{N-1} \nu x^{M+n} P_{n,B} + x^M [\lambda Q_E - \mu(1-p_0) P_{0,E} - \lambda x^N P_{N-1,E}] \right] / \{x^M(\lambda+\mu-\lambda x) - \sum_{j=0}^{M} \mu p_j x^{M-j}\} \quad ,$$

$$P_B(x) = \frac{\sum_{j=1}^{M} \sum_{r=0}^{j-1} \mu p_j (x^N - x^{N+r-j}) P_{r,B} - \sum_{n=0}^{N-1} \nu x^n P_{n,B} + x^N [\lambda Q_B - \mu(1-p_0) P_{0,B}]}{\{[\lambda+\mu+\nu-\lambda x] x^N - \sum_{j=0}^{M} \mu p_j x^{N-j} - \nu\}} +$$

$$\frac{x^N [\lambda P_{N-1,E} - \lambda x^K (x-1) P_{K,B}]}{\{[\lambda+\mu+\nu-\lambda x] x^N - \sum_{j=0}^{M} \mu p x^{N-j} - \nu\}} \quad ,$$

$$Q_E = (1/\lambda)[\sum_{j=1}^{M} \mu p_j P_{0,E} + \nu Q_B] ,$$

and

$$Q_B = [1/(\lambda+\nu)][\sum_{j=1}^{M} \mu p_j P_{0,B}] .$$

In this case, the unknowns are determined in the same way as in transient case.

Case I

Unlimited queue size and additional Service Channel: In this case the expression for the laplace transform of the probability generating function will be obtained by letting K tend to infinity in equations (17), (22), (23), (24), and (25). Equation (23) is

$$G(\alpha,x) = G_E(\alpha,x) + G_B(\alpha,x) + Q_E(\alpha) + Q_B(\alpha) .$$

In this $G_E(\alpha,x)$, $Q_E(\alpha)$, $Q_B(\alpha)$ are given by equations (17), (24), and (25) equations respectively, and $G_B(\alpha,x)$ given by

$$G_B(\alpha,x) =$$

$$\frac{\sum_{j=1}^{M}\sum_{r=0}^{j-1}\mu p_j(x^N-x^{N+r-j})P_{r,B}(\alpha)-\sum_{n=0}^{N-1}\nu x^n P_{n,B}(\alpha)+x^N\lambda Q_B(\alpha)}{\{(\alpha+\lambda+\mu+\nu-\lambda x)x^N-\sum_{j=0}^{M}\mu p_j x^{N-j}-\nu\}}+$$

$$\frac{x^N[-\mu(1-p_0)P_{0,B}(\alpha)+P_{N-1,E}(\alpha)]}{\{(\alpha+\lambda+\mu+\nu-\lambda x)x^N-\sum_{j=0}^{M}\mu p_j x^{N-j}-\nu\}} \qquad (27)$$

By applying Rouche's Theorem to the denominator in equation (27) we can prove that it has N roots inside the unit circle $|x| = 1$. Since $G_B(\alpha,x)$ is differentiable inside the unit circle these roots must vanish the numerator in equation (27) giving rise to a set of N equations. Solving this set of equations together with equations (24) and (25) we can determine all unknowns in equation (23).

Case II

Unlimited queue size and no additional service channel:
In this case the result is obtained by letting both K and N tends to infinity in equations (18), (22), (23), (24), and (25). Equation (23) reduces to

$$G(\alpha,x) = G_E(\alpha,x) + Q_E(\alpha) \ , \tag{26}$$

where

$$G_E(\alpha,x) =$$

$$\frac{\sum_{j=1}^{M}\sum_{r=0}^{j-1} \mu p_j (x^M - x^{M+r-j}) P_{r,E}(\alpha) + x^M[\lambda Q_E(\alpha) - \mu(1-p_0)P_{0,E}(\alpha)]}{\{(\alpha+\lambda+\mu-\lambda x)x^M - \sum_{j=0}^{M} \mu p_j x^{M-j}\}} \ ,$$

$$Q_E(\alpha) = [1/(\alpha+\lambda)][1 + \sum_{j=1}^{M} \mu p_j P_{0,E}(\alpha)].$$

The unknowns of the equation can be determined by the help of Rouche's Theorem.

Solution of Model B

1. Transient Condition: The equations representing the model, under the transient conditions, are

1. The case where the SSC is not operating

$$\frac{dP_{0,E}(t)}{dt} + (\lambda+\mu)P_{0,E}(t) = \mu p_0 P_{0,E}(t) +$$

$$\sum_{j=1}^{M}\sum_{r=1}^{j} \mu p_j P_{r,E}(t) + \nu P_{0,B}(t) \ , \tag{29}$$

$$\frac{dP_{n,E}(t)}{dt} + (\lambda+\mu)P_{n,E}(t) = \mu p_0 P_{n,E}(t) +$$

$$\sum_{j=1}^{M} \mu p_j P_{n+j,E}(t) + \lambda P_{n-1,E}(t) +$$

$$\nu P_{n,B}(t) , \qquad 0 < n < N \qquad (30)$$

2. The case where SSC is operating.

$$\frac{dP_{0,B}(t)}{dt} + (\lambda+\mu+\nu)P_{0,B}(t) = \mu p_0 P_{0,B}(t) +$$

$$\sum_{j=1}^{M} \sum_{r=1}^{j} \mu p_j P_{r,B}(t) + \nu P_{N,B}(t) + \lambda P_{N-1,E}(t) , \qquad (31)$$

$$\frac{dP_{n,B}(t)}{dt} + (\lambda+\mu+\nu)P_{n,B}(t) = \mu p_0 P_{n,B}(t) +$$

$$\sum_{j=1}^{M} \mu p_j P_{n+j,B}(t) + \lambda P_{n-1,B}(t) +$$

$$\nu P_{n+N,B}(t) , \qquad 0 < n < K \qquad (32)$$

$$\frac{dP_{K,B}(t)}{dt} + (\mu+\nu)P_{K,B}(t) = \mu p_0 P_{K,B}(t) + \lambda P_{K-1,B}(t) \ , \qquad (33)$$

$$P_{n,E}(t) = 0 \qquad n \geq N \qquad (34)$$

$$P_{n,B}(t) = 0 \ , \qquad n > K \qquad (35)$$

$$P_n(t) = P_{n,E}(t) + P_{n,B}(t) \ , \qquad n \geq 0 \qquad (36)$$

The above equations can be directly obtained by putting $Q_E(t) = 0$ and $Q_B(t) = 0$ in those of model A. The record time from the instant when the queue length is equal to zero and the SSC is not operating so that

$$P_{0,E}(0) = 1 \ . \qquad (37)$$

Multiplying equation (29) by x^M,

$$x^M \frac{dP_{0,E}(t)}{dt} + (\lambda+\mu)x^M P_{0,E}(t) = \mu x^M p_0 P_{0,E}(t) +$$

$$\sum_{j=1}^{M} \sum_{r=1}^{j} \mu p_j x^M P_{r,E}(t) + \nu x^M P_{0,B}(t) \ , \qquad (38)$$

multiplying equation (30) by x^{M+n} and then summing over n, n = 1, 2, . . ., N-1,

$$\sum_{n=1}^{N-1} x^{M+n} \frac{dP_{n,E}(t)}{dt} + \sum_{n=1}^{N-1} (\lambda+\mu) x^{M+n} P_{n,E}(t) = \sum_{n=1}^{N-1} x^{M+n} \cdot$$

$$\left[\mu p_0 P_{n,E}(t) + \lambda P_{n-1,E}(t) + \sum_{j=1}^{M} \mu p_j P_{n+j,E}(t) + \right.$$

$$\left. \nu P_{n,B}(t) \right] . \qquad (39)$$

Taking laplace transform of equations (38) and (39) and then using equations, (34), (35), and (37), we get

$$x^M [\alpha P_{0,E}(\alpha) - 1] + (\lambda+\mu) x^M P_{0,E}(\alpha) = \mu x^M p_0 P_{0,E}(\alpha) +$$

$$\sum_{j=1}^{M} \sum_{r=1}^{j} \mu p_j x^M P_{r,E}(\alpha) + \nu x^M P_{0,B}(\alpha) , \qquad (40)$$

$$\sum_{n=1}^{N-1} x^{M+n} [\alpha P_{n,E}(\alpha) - 0] + \sum_{n=1}^{N-1} (\lambda+\mu) x^{M+n} P_{n,E}(\alpha) =$$

$$\sum_{n=1}^{N-1} x^{M+n} \left[\mu p_0 P_{n,E}(\alpha) + \lambda P_{n-1,E}(\alpha) + \right.$$

$$\left. \sum_{j=1}^{M} \mu p_j P_{n+j,E}(\alpha) + \nu P_{n,B}(\alpha) \right] , \tag{41}$$

and adding equations (40) and (41) we get

$$\sum_{n=0}^{N-1} \alpha x^{M+n} P_{n,E}(\alpha) - x^M + \sum_{n=0}^{N-1} (\lambda+\mu) x^{M+n} P_{n,E}(\alpha) =$$

$$\sum_{n=0}^{N-1} \mu x^{M+n} p_0 P_{n,E}(\alpha) + \sum_{n=1}^{N-1} \lambda x^{M+n} P_{n-1,E}(\alpha) +$$

$$\sum_{n=0}^{N-1} \nu x^{M+n} P_{n,B}(\alpha) + \sum_{j=1}^{M} \mu p_j x^M \left[\sum_{r=1}^{j} P_{r,E}(\alpha) + \right.$$

$$\left. \sum_{n=1}^{N-1} x^n P_{n+j,E}(\alpha) \right] . \tag{42}$$

By using the simplified version of equation (15) and then applying condition

$$P_{0,E}(0) = 1$$

that is

$$\sum_{j=1}^{M} p_j \mu x^M P_{0,E}(t) = 0$$

equation (42) becomes

$$\sum_{n=0}^{N-1} \alpha x^{M+n} P_{n,E}(\alpha) - x^M + \sum_{n=0}^{N-1} (\lambda+\mu) x^{M+n} P_{n,E}(\alpha) =$$

$$\sum_{n=0}^{N-1} \mu x^{M+n} P_0 P_{n,E}(\alpha) + \sum_{n=0}^{N-1} \lambda x^{M+1+n} P_{n,E}(\alpha) -$$

$$\lambda x^{M+N} P_{N-1,E}(\alpha) + \sum_{n=0}^{N-1} \nu x^{M+n} P_{n,B}(\alpha) +$$

$$\sum_{j=1}^{M} \sum_{r=0}^{j-1} \mu x^M p_j P_{r,E}(\alpha) + \sum_{j=1}^{M} \sum_{n=0}^{N-1} \mu p_j x^{M+n} P_{n+j,E}(\alpha) ,$$

Where the last term in the R.H.S. is simplified to

$$\sum_{j=1}^{M} \mu p_j \sum_{n=0}^{N-1} x^{M+n} P_{n+j,E}(\alpha) = \sum_{j=1}^{M} \sum_{n=0}^{N-1} \mu p_j x^{M-j+n} P_{n,E}(\alpha) -$$

$$\sum_{j=1}^{M} \sum_{r=0}^{j-1} \mu p_j x^{M+r-j} \cdot P_{r,E}(\alpha)$$

Using the generating functions

$$\alpha x^M G_E(\alpha,x) - x^M + (\lambda+\mu) x^M G_E(\alpha,x) = \mu x^M p_0 G_E(\alpha,x) +$$

$$\lambda x^{M+1} G_E(\alpha,x) - \lambda x^{M+N} P_{N-1,E}(\alpha) +$$

$$\sum_{n=0}^{N-1} \nu x^{M+n} P_{n,B}(\alpha) + \sum_{j=1}^{M} \sum_{r=0}^{j-1} \mu p_j x^M P_{r,E}(\alpha) +$$

$$\sum_{j=1}^{M} \mu p_j x^{M-j} G_E(\alpha,x) - \sum_{j=1}^{M} \sum_{r=0}^{j-1} \mu p_j x^{M+r-j} P_{r,E}(\alpha) ,$$

or

$$\left[(\alpha+\lambda+\mu-\lambda x)x^M - \mu p_0 x^M - \sum_{j=1}^{M} \mu p_j x^{M-j}\right] P_E(\alpha,x) =$$

$$x^M\left[1 - \lambda x^N P_{N-1,E}(\alpha)\right] + \sum_{n=0}^{N-1} \nu x^{M+n} P_{n,B}(\alpha) +$$

$$\sum_{j=1}^{M} \sum_{r=0}^{j-1} \mu p_j (x^M - x^{M+r-j}) P_{r,E}(\alpha) .$$

Thus, the generating function $G_E(\alpha,x)$ is

$$G_E(\alpha,x) =$$

$$\frac{x^M[1-\lambda x^N P_{N-1,E}(\alpha)] + \sum_{n=0}^{N-1} \nu x^{M+n} P_{n,B}(\alpha) + \sum_{j=1}^{M} \sum_{r=0}^{j-1} \mu p_j (x^M - x^{M+r-j}) P_{r,E}(\alpha)}{\left[(\alpha+\lambda+\mu-\lambda x)x^M - \sum_{j=0}^{M} \mu p_j x^{M-j}\right]} \qquad (43)$$

Multiplying equation (31) by x^N,

$$x^N \frac{dP_{0,B}(t)}{dt} + (\lambda+\mu+\nu)x^N P_{0,B}(t) =$$

$$x^N\left[\mu p_0 P_{0,B}(t) + \sum_{j=1}^{M}\sum_{r=1}^{j} \mu p_j P_{r,B}(t) + \right.$$

$$\left. \nu P_{N,B}(t) + \lambda P_{N-1,E}(t)\right] .$$

Taking the laplace transform of both sides,

$$x^N[\alpha P_{0,B}(\alpha) - 0] + (\lambda+\mu+\nu)x^N P_{0,B}(\alpha) =$$

$$x^N[\mu p_0 P_{0,B}(\alpha) + \sum_{j=1}^{M}\sum_{r=1}^{j} \mu p_j P_{r,B}(\alpha) +$$

$$\nu P_{N,B}(\alpha) + \lambda P_{N-1,E}(\alpha)] . \qquad (44)$$

Multiplying equation (30) by x^{N+n} and then summing over n, n = 1, 2, . . ., K-1,

$$\sum_{n=1}^{K-1} x^{N+n}\left[\frac{dP_{n,B}(t)}{dt} + (\lambda+\mu+\nu)P_{n,B}(t)\right] =$$

$$\sum_{n=1}^{K-1} x^{N+n}\left[\mu p_0 P_{n,B}(t) + \sum_{j=1}^{M} \mu p_j P_{n+j,B}(t) + \lambda P_{n-1,B}(t) + \nu P_{n+N,B}(t)\right] ,$$

and taking the laplace transform we get,

$$\sum_{n=1}^{K-1} x^{N+n}\left[\alpha P_{n,B}(\alpha) - 0 + (\lambda+\mu+\nu)P_{n,B}(\alpha)\right] =$$

$$\sum_{n=1}^{K-1} x^{N+n}\left[\mu P_0 P_{n,B}(\alpha) + \sum_{j=1}^{M} \mu p_j P_{n+j,B}(\alpha) + \lambda P_{n-1,B}(\alpha) + \nu P_{n+N,B}(\alpha)\right] , \quad (45)$$

Multiplying equation (33) by x^{K+N} and then taking the laplace transform,

$$x^{K+N}[\alpha P_{K,B}(\alpha) - 0] + (\mu+\nu)P_{K,B}(\alpha) =$$

$$[\mu p_0 P_{K,B}(\alpha) + \lambda P_{K-1,B}(\alpha)]x^{K+N} \quad (46)$$

Adding equations (44), (45), and (46) and then using equation (35) and the generating functions and simplified terms of equation (16) we get,

$$\left[(\alpha+\lambda+\mu+\nu-\lambda x)x^N - \sum_{j=0}^{M} \mu p_j x^{N-j} - \nu\right] P_B(\alpha,x) =$$

$$\sum_{j=1}^{M} \sum_{r=0}^{j-1} \mu p_j (x^N - x^{N+r-j}) P_{r,B}(\alpha) -$$

$$x^N[\lambda P_{N-1,E}(\alpha) - \lambda x^{K+1} P_{K,B}(\alpha)] - \sum_{n=0}^{N-1} \nu x^N P_{n,B}(\alpha) .$$

Thus, the generating function $G_B(\alpha,x)$ is

$$G_B(\alpha,x) =$$

$$\frac{\sum_{j=1}^{M} \sum_{r=0}^{j-1} \mu p_j (x^N - x^{N+r-j}) P_{r,B}(\alpha) - \sum_{n=0}^{N-1} \nu x^n P_{n,B}(\alpha) - x^N[\lambda P_{N-1,E}(\alpha)]}{\{(\alpha+\lambda+\mu+\nu-\lambda x)x^N - \sum_{j=0}^{M} \mu p_j x^{N-j} - \nu\}} +$$

$$\frac{\lambda x^{K+N+1} P_{K,B}(\alpha)}{\{(\alpha+\lambda+\mu+\nu-\lambda x)x^N - \sum_{j=0}^{M} \mu p_j x^{N-j} - \nu\}} . \qquad (47)$$

Multiplying equation (36) by x^n and then summing over n, n = 0, 1, 2, . . ., K,

$$\sum_{n=0}^{K} x^n P_n(t) = \sum_{n=0}^{K} x^n P_{n,E}(t) + \sum_{n=0}^{K} x^n P_{n,B}(t) ,$$

and using the generating functions and equations (34) and (35)

$$G(t,x) = G_E(t,x) + G_B(t,x) .$$

Taking the laplace transform of both sides

$$G(\alpha,x) = G_E(\alpha,x) + G_B(\alpha,x) . \qquad (48)$$

In equation (48) the values of $G_E(\alpha,x)$ and $G_B(\alpha,x)$ are given by equations (46) and (47). The procedure for solving these and finding roots is the same as that of model A.

2. Steady State Condition: Using the property of laplace transform given by equation (26) in equations (46), (47), and (48), we get

$$P(x) = P_E(x) + P_B(x)$$

where

$$P_E(x) =$$

$$\frac{\left[\sum_{j=1}^{M}\sum_{r=0}^{j-1}\mu p_j(x^M - x^{M+r-j})P_{r,E} + \sum_{n=0}^{N-1}\nu x^{M+n}P_{n,B} - \lambda x^{M+N}P_{N-1,E}\right]}{\{(\lambda+\mu-\lambda x)x^M - \sum_{j=0}^{M}\mu p_j x^{M-j}\}},$$

and

$$P_B(x) =$$

$$\frac{\sum_{j=1}^{M}\sum_{r=0}^{j-1}\mu p_j(x^N - x^{N+r-j})P_{r,B} - \sum_{n=0}^{N-1}\nu x^n P_{n,B} + x^N[\lambda P_{N-1,E} - \lambda x^{K+1}P_{K,B}]\}}{\{(\lambda+\mu+\nu-\lambda x)x^N - \sum_{j=0}^{M}\mu p_j x^{N-j} - \nu\}}$$

In this solution, the unknowns are determined as in the transient case.

Case 1:

Unlimited queue size and additional service channel: In this case K will tend to infinity in the previous derivations. The equations for this case will be

$$G(\alpha,x) = G_E(\alpha,x) + G_B(\alpha,x) \qquad (49)$$

where $G_E(\alpha,x)$ is the same as given by equation (43) and

$$G_B(\alpha,x)\left[(\alpha+\lambda+\mu+\nu-\lambda x)x^N - \sum_{j=0}^{M} \mu p_j x^{N-j} - \nu\right] =$$

$$\sum_{j=1}^{M} \sum_{r=0}^{j-1} \mu p_j (x^N - x^{N+r-j}) P_{r,B}(\alpha) -$$

$$\sum_{n=0}^{N-1} \nu x^n P_{n,B}(\alpha) + \lambda x^N P_{N-1,E}(\alpha) ,$$

or

$$G_B(\alpha,x) =$$

$$\frac{\sum_{j=1}^{M} \sum_{r=0}^{j-1} \mu p_j (x^N - x^{N+r-j}) P_{r,B}(\alpha) - \sum_{n=0}^{N-1} \nu x^n P_{n,B}(\alpha) + \lambda x^N P_{N-1,E}(\alpha)\}}{\{(\alpha+\lambda+\mu+\nu-\lambda x)x^N - \sum_{j=0}^{M} \mu p_j x^{N-j} - \nu\}} .$$

Case 2:

Unlimited queue size and no additional service channel: In this case in equation (49) $G_B(\alpha,x) = 0$ hence

$$G(\alpha,x) = G_E(\alpha,x)$$

where

$$G_E(\alpha,x) =$$

$$\frac{\left[\sum_{j=1}^{M}\sum_{r=0}^{j-1} \mu p_j (x^M - x^{M+r-j}) P_{r,E}(\alpha) + x^M\right]}{\{(\alpha+\lambda+\mu-\lambda x)x^M - \sum_{j=0}^{M} \mu p_j x^{M-j}\}} .$$

The unknowns of this case can be determined in the same way as in model A.

3.2. Queueing Model for Bulk Service at a Series of Stations

This queueing model which has been developed by Giffin [23] describes a system where customers are impatient. The customers who are not served whenever the service is available are lost. The arrivals are served in bulk at each of a number of stations placed in series. The analysis of this model offers rapid calculation for obtaining numerical solutions. Profit is one of the measures which directly determines the optimal solution.

Model Specification:

Arrival Source	:	Infinite
Arrival Pattern	:	Single
Arrival Behavior	:	Impatient
Arrival Distribution	:	Poisson
Queue Size	:	No Queue
Service Discipline	:	First arrive, first served
Service Pattern	:	Bulk
Service Mechanism	:	Single channel, single phase
Service Time Distribution	:	Uniform
Number of Servers	:	One

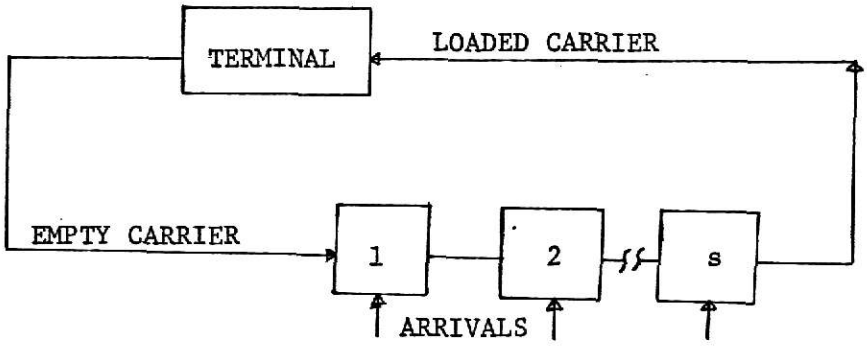


Figure 6. A Single Loop, Multiple Station System

Practical Problem: Goods arrive at any service station, S, and wait to be transported (served) to the terminal. The empty carrier is considered as one way transportation server. It passes through each station in order to load the goods waiting at each station.

Notation: The following is the notation used in this model:

t = Interarrival of customers

λ_i = Mean arrival rate seeking service at station i

s = Number of stations to be served

C = capacity of each carrier

$p_i(n, t)$ = Probability at station i of a demand n at time t

U = Number of units on board as carrier approaches a station (a random variable)

V = Number of units lost per carrier (a random variable)

W = Number of units equivalent to unused capacity of carrier (a random variable)

$f_i(U)$ = Probability density function for units on board as carrier approaches station i

$g_{s+1}(V)$ = Probability density function of units lost per carrier when carrier passed through s stations

$h_{s+1}(W)$ = Probability density function of unused capacity, W when carrier passed through s stations

C_1 = Cost per customer hour waiting

C_2 = Carrier operating cost per hour

C_3 = Cost of a lost customer

R = Route time for every carrier

v = Number of carriers in the system

I = Income from each served unit

Analysis: To design such system, the number and size of carriers needed to run the one-way transit operation prevalent during peak hours. A carrier leaving station 1 and approaching station 2 will have on board the number of units served at station 1. The number of units waiting at station i is U_i which is a function of the arrival rate for that station and the elapsed time since the preceding carrier departed. Consider such a system consisting of two stations only. The probability density function for units on board as a carrier approaches station 2 is

$$f_2(U) = \begin{cases} 0, & U < 0,\ U > C \\ p_1(U, t), & 0 \leq U < C \\ \sum_{j=C}^{\infty} p_1(j, t), & U = C \end{cases} \qquad (1)$$

To obtain the probability density for units on board along the route we will have to consider the distribution of a sum of random variables U_i, representing units present for service at each station thus far served.

Total units on the board as a carrier approaches the terminal, U_{s+1}, may be given as

$$U_{s+1} = \begin{cases} U_1 + U_2 + \ldots + U_s, & \sum_{i=1}^{s} U_i \leq C \\ C, & \sum_{i=1}^{s} U_i > C \end{cases} \tag{2}$$

$$\text{Unused carrier capacity} = C - U_{s+1}. \tag{3}$$

Since all U_i, $i = 1, 2, \ldots, s$ are independent and having poisson distribution, the summation $\sum_{i=1}^{s} U_i$ also has poisson distribution with mean $\sum_{i=1}^{s} \lambda_i t$.

Under Poisson arrival and constant spacing between carriers, the distribution functions for a variety of random variables may now be obtained. Each carrier will arrive at the terminal of an s stop system having U units on board with probability $f_{s+1}(U)$ given by

$$f_{s+1}(U) = \begin{cases} 0, & U < 0,\ U > C \\ p[U,\ t \sum_{i=1}^{s} \lambda_i], & 0 \leq U < C \\ p[C,\ t \sum_{i=1}^{s} \lambda_i], & U = C \end{cases} \tag{4}$$

Since the arrival behavior is of impatient type, some units are lost to the system whenever an approaching carrier is unable to provide service. This occurs everytime the total unit arrivals, during the passage of a carrier around the loop, exceeds the carrier capacity. The units lost per carrier, V are therefore distributed by

$$g_{s+1}(V) = \begin{cases} 0, & V < 0 \\ 1 - P[C+1, t \sum_{i=1}^{s} \lambda_i], & V = 0 \\ p[V+C, t \sum_{i=1}^{s} \lambda_i], & V > 0 \end{cases} \quad (5)$$

The unused capacity, W, on a carrier is represented by equation (3). This random variable is distributed by

$$h_{s+1}(W) = \begin{cases} 0, & W < 0, W > C \\ P[C, t \sum_{i=1}^{s} \lambda_i], & W = 0 \\ p[C-W, t \sum_{i=1}^{s} \lambda_i], & 0 < W \leq C \end{cases} \quad (6)$$

Measures of Effectiveness:

1. Mean number of units served per carrier

$$E(U) = \sum_{U} U f_{s+1}(U), = \sum_{U=0}^{C-1} U p(U, t \sum_{i=1}^{s} \lambda_i) + C P(C, t \sum_{i=1}^{s} \lambda_i). \quad (7)$$

But we know by identity due to Hadley and Whitin

$$\sum_{j=0}^{K} j p(j, m) = m[1 - P(k, m)] \quad (8)$$

Substituting in equation (7) such that

$$E(U) = (t \sum_{i=1}^{s} \lambda_i) [1 - P(C-1, t \sum_{i=1}^{s} \lambda_i)] + C P(C, t \sum_{i=1}^{s} \lambda_i). \quad (9)$$

On the average, system balance will occur in that units available for service by each carrier must equal units served plus units turned away.

2. Mean number of units lost per carrier:

$$E(V) = t \sum_{i=1}^{s} \lambda_i - E(U) = t \sum_{i=1}^{s} \lambda_i -$$

$$(t \sum_{i=1}^{s} \lambda_i)[1 - P(C-1, t \sum_{i-1}^{s} \lambda_i)] + CP(C, t \sum_{i=1}^{s} \lambda_i) =$$

$$(t \sum_{i=1}^{s} \lambda_i)\, P[C-1, t \sum_{i=1}^{s} \lambda_i] - CP[C, t \sum_{i=1}^{s} \lambda_i]. \qquad (10)$$

3. Load factor: It may be defined as the ratio of mean number of units served per carrier and capacity of carrier.

$$E(\text{load factor}) = \frac{E(U)}{C},$$

$$E(L.F.) = (t \sum_{i=1}^{s} \lambda_i/C)[1-P(C-1, t \sum_{i=1}^{s} \lambda_i)] + P(C, t \sum_{i=1}^{s} \lambda_i). \qquad (11)$$

4. Variance of units served per carrier:

In general,

$$Var(x) = E(x^2) - [E(x)]^2,$$

hence in this case

$$E(U^2) = \sum_U U^2 f_{s+1}(U) = \sum_{U=0}^{C-1} U^2 p(U, t \sum_{i=1}^{s} \lambda_i) + C^2 P(C, t \sum_{i=1}^{s} \lambda_i).$$

But by identity due to Hadley and Whitin we know

$$\sum_{j=0}^{k} j^2 p(j, m) = m[1 - P(k, m)] + m^2[1 - P(k - 1, m)]. \tag{12}$$

Using this result we obtain

$$E(U^2) = (t \sum_{i=1}^{s} \lambda_i)[1 - P(C-1, t \sum_{i=1}^{s} \lambda_i)] +$$

$$(t \sum_{i=1}^{s} \lambda_i)^2 [1 - P(C-2, t \sum_{i=1}^{s} \lambda_i)] + C^2 P(C, t \sum_{i=1}^{s} \lambda_i), \tag{13}$$

$$\text{Var}(U) = E(U^2) - [E(U)]^2.$$

Substituting values from equations (13) and (9) we get

$$\text{Var}(U) = (t \sum_{i=1}^{s} \lambda_i) \left[1 - P(C-1, t \sum_{i=1}^{s} \lambda_i) \right] +$$

$$(t \sum_{i=1}^{s} \lambda_i)^2 \left[1 - P(C-2, t \sum_{i=1}^{s} \lambda_i) \right] + C^2 P(C, t \sum_{i=1}^{s} \lambda_i) -$$

$$\left[\left(t \sum_{i=1}^{s} \lambda_i \right)\left[1 - P\left(C-1,\ t \sum_{i=1}^{s} \lambda_i\right)\right] + CP\left(C,\ t \sum_{i=1}^{s} \lambda_i\right) \right]^2 . \tag{14}$$

5. Variance of Units lost per Carrier:

$$\mathrm{Var}(V) = E(V^2) - [E(V)]^2 .$$

In a way similar to case (4)

$$\mathrm{Var}(V) = \left(t \sum_{i=1}^{s} \lambda_i \right)[1 - 2C]\, P\left[C,\ t \sum_{i=1}^{s} \lambda_i\right] +$$

$$\left[t \sum_{i=1}^{s} \lambda_i \right]^2 P\left[C-1,\ t \sum_{i=1}^{s} \lambda_i\right] + C^2 P\left(C+1,\ t \sum_{i=1}^{s} \lambda_i\right) -$$

$$\left[\left[t \sum_{i=1}^{s} \lambda_i \right] P\left[C-1,\ t \sum_{i=1}^{s} \lambda_i\right] - CP\left[C,\ t \sum_{i=1}^{s} \lambda_i\right] \right]^2 . \tag{15}$$

The above equations appear to be complex; however, numerical answers for any particular system are readily obtainable from tables of poisson probability density function p(k, m) and cumulative probability P(k, m) [5].

Profit as a measure of the system: There are variable associated costs, some of which are given such as: (1) with customer - waiting time in hours; (2) number of customers lost; and (3) carrier operations.

Gross income is directly proportional to the number of customers served.

Time between carriers is

$$t = \frac{R}{v} \quad . \tag{16}$$

Carriers per hour passing any station is

$$V_1 = \frac{v}{R} \quad . \tag{17}$$

The expected value of the total number of customers presenting themselves for service at any station i for each carrier is

$$E \text{ (customers)} = t\mu_i$$

$$= \frac{R\mu_i}{v} \quad . \tag{18}$$

Since any customer not served by an upcoming carrier is lost, the waiting cost per carrier at station i is

$$C_1(R/v)^2 \frac{\mu_i}{2} \quad , \tag{19}$$

and the waiting cost per hour at station i is

$$C_1(R/v)^2 \frac{\mu_i}{2} \left(\frac{v}{R}\right),$$

or

$$\frac{C_1 R\mu_i}{2v} . \tag{20}$$

Using the above equations we calculate

1. Expected waiting cost per hour is

$$\sum_{i=1}^{s} \frac{C_1 R\mu_i}{2v}$$

or

$$\frac{C_1 R \sum_{i=1}^{s} \mu_i}{2v} .$$

2. Expected hourly carrier cost is

$$C_2 v.$$

3. Expected hourly lost customer cost is

$$E\left(\begin{matrix}\text{Customers lost}\\ \text{per carrier}\end{matrix}\right) x \left(\begin{matrix}\text{Carriers}\\ \text{per hour}\end{matrix}\right) x \left(\begin{matrix}\text{Cost per}\\ \text{customer}\end{matrix}\right),$$

or

$$C_3 \left\{ \sum_{i=1}^{s} \mu_i \, P[C-1, \frac{R}{v} \sum_{i=1}^{s} \mu_i] - \frac{Cv}{R} P[C, \frac{R}{v} \sum_{i=1}^{s} \mu_i] \right\} .$$

4. Expected hourly income is

$$E\left(\begin{matrix}\text{Customers served}\\ \text{per carrier}\end{matrix}\right) \times \left(\begin{matrix}\text{Carriers}\\ \text{per hour}\end{matrix}\right) \times \left(\begin{matrix}\text{Income per}\\ \text{customer}\end{matrix}\right)$$

or

$$I\left[\sum_{i=1}^{s} \mu_i [1 - P[C-1, \frac{R}{v} \sum_{i=1}^{s} \mu_i]] + \frac{Cv}{R} P[C, \frac{R}{v} \sum_{i=1}^{s} \mu_i]\right].$$

Expected hourly profit is

$$I \sum_{i=1}^{s} \mu_i - (I + C_3)\left[(\sum_{i=1}^{s} \mu_i) P[C-1, \frac{R}{v} \sum_{i=1}^{s} \mu_i] - (\frac{Cv}{R}) P[C, \frac{R}{v} \sum_{i=1}^{s} \mu_i]\right] -$$

$$\frac{C_1 R}{2v} \sum_{i=1}^{s} \mu_i - C_2 v \, . \qquad (21)$$

Example: A system consists of a number of carriers serving customers from several stations to the terminal. The capacity of each carrier, C, is of 10 seats. Time taken by each carrier in one route, R, is one hour. Income per passenger served, I, is \$4.00. The mean number of customers per hour, $\sum_{i=1}^{s} \mu_i$, is 60. The different costs associated with the system are given below:

Cost per customer hour waiting, C_1, is \$5.00

Carrier operating cost per hour, C_2, is \$20.00

Cost of a lost customer, C_3, is \$10.00

The objective is to find the number of carriers such that the profit

is maximum.

In order to find the number of carriers, the values of this example are substituted in equation (21).

$$E[P.F.] = 240 - 840\ P[9, \frac{60}{v}] + 140\ vP[10, \frac{60}{v}] - \frac{150}{v} - 20\ v.$$

For different probabilities obtained from tables [5], the following values of expected profit appear below:

v	$P[9, \frac{60}{v}]$	$P[10, \frac{60}{v}]$	E[P.F.]
3	.9979	.9950	- $290
4	.9626	.9301	- $174
5	.8450	.7576	- $ 70
6	.6672	.5421	- $ 9
7	.4906	.3600	$ 29
8	.3380	.2236	$ 27
9	.2327	.1404	$ 24
10	.1528	.0839	$ 13

Upon examining the above table, it appears that the maximum expected profit is $29. Consequently, the corresponding number of carriers is seven. Therefore, it is profitable to provide seven carriers to run the system.

3.3. Double Ended Queue with Bulk Service Queueing Model

This queueing model which has been developed by Kashyap [35] describes a system with a limited queue which may be formed by arriving customers as well as servers. Therefore, such model is referred to as double ended queue. The arrivals are served in bulk. The distributions of arriving customers and servers are considered as general and Poisson, respectively. Erlangian and Poisson distributions for the arrivals of customers and single seated taxis are considered as particular cases.

Model Specification:

Arrival Source	:	Infinite
Arrival Pattern	:	Single
Arrival Behavior	:	Patient
Arrival Distribution	:	General
Queue Size	:	Limited
Service Discipline	:	First arrive, first served
Service Pattern	:	Bulk
Service Mechanism	:	Single channel, single phase
Service time distribution	:	General
Number of Servers	:	One

Practical Problem: Passengers wait at the taxi stand in a limited queue in order to take an available taxi. The taxis are also waiting in a limited queue. They are considered as servers and serve the customers in bulk.

Notation: The following is the notation used in this model

λ = Mean arrival rate

M = Taxi Queue size

N = Customer queue size

A(x) = Probability density function of interarrival distribution

$P_n(x, t)$ = Probability of n customers waiting in the queue, at time t, and the elapsed time in service is x

$F(x, y, t)$ = generating function of $P_n(x, t)$ when n, n = -M, . . ., -1

$G(x, y, t)$ = generating function of $P_n(x, t)$ when n, n = 1, 2, . . ., N

$F(y, t)$ = generating function of $P_n(t)$ when n, n = -M, . . ., -1

$G(y, t)$ = generation function of $P_n(t)$ when n, n = 1, . . ., N

$P_n(t)$ = Probability that n customers (or -n taxis for negative n) are waiting at time t, irrespective of the value of x

$F(x, y, \alpha)$ = Laplace Transform of $F(x, y, t)$

$\Gamma(y, x)$ = Incomplete Gamma Function

δin = Kronecker delta

$\delta(x)$ = Dirac Delta function

Solution:

In order to derive the difference-differential equations for this model, we consider the continuity of the flow during a time interval $[t, t + \Delta t]$ which leads to the equations:

$$P_n(x + \Delta t, t + \Delta t) = P_n(x, t) \{1 - \lambda\Delta t\} \{1 - \mu(x)\Delta t\} +$$

$$P_{n+1}(x, t)\,\lambda(\Delta t),$$

or

$$P_n(x + \Delta t, t + \Delta t) = P_n(x, t) + \{-(\lambda + \mu(x))\,P_n(x, t) +$$

$$\lambda P_{n+1}(x, t)\}\Delta t\ ,$$

or

$$\frac{P_n(x + \Delta t, t + \Delta t) - P_n(x, t)}{\Delta t} = -(\lambda + \mu(x))\,P_n(x, t) + \lambda P_{n+1}(x, t),$$

$$\frac{P_n(x + \Delta t, t + \Delta t) - P_n(x + \Delta t, t) + P_n(x + \Delta t, t) - P_n(x, t)}{\Delta t} =$$

$$-(\lambda + \mu(x))\,P_n(x, t) + \lambda P_{n+1}(x, t).$$

Taking limit $\Delta t \to 0$, we get

$$\frac{\partial P_n(x, t)}{\partial t} + \frac{\partial P_n(x, t)}{\partial x} = -(\lambda + \mu(x))\,P_n(x, t) + \lambda P_{n+1}(x, t)$$

and

$$P_0\,(x + \Delta t, t + \Delta t) = P_0\,(x, t)\{1 - \lambda\Delta t\}\{1 - \mu(x)\Delta t\} + \lambda\Delta t \sum_{j=1}^{s} P_j(x, t),$$

$$P_0(x + \Delta t,\ t + \Delta t) - P_0(x,\ t) + \{\lambda + \mu(x)\}\ \Delta t\ P_0(x,\ t) = \lambda \Delta t \sum_{j=1}^{s} P_j(x,\ t),$$

or

$$\frac{P_0(x + \Delta t,\ t + \Delta t) - P_0(x + \Delta t,\ t) + P_0(x + \Delta t,\ t) - P_0(x,\ t)}{\Delta t} =$$

$$-(\lambda + \mu(x))\ P_0(x,\ t) + \lambda \sum_{j=1}^{s} P_j(x,\ t).$$

Taking limit $\Delta t \to 0$, we get

$$\frac{\partial P_0(x,\ t)}{\partial t} + \frac{\partial P_0(x,\ t)}{\partial t} = -(\lambda + \mu(x))\ P_0(x,\ t) + \lambda \sum_{j=1}^{s} P_j(x,\ t).$$

On the basis of these equations the differential equations are:

$$\frac{\partial P_{-M}(x,\ t)}{\partial t} + \frac{\partial P_{-M}(x,\ t)}{\partial x} = -\mu(x)\ P_{-M}(x,\ t) + \lambda P_{-M+1}(x,\ t), \qquad (1)$$

$$\frac{\partial P_n(x,\ t)}{\partial t} \quad \frac{\partial P_n(x,\ t)}{\partial x} = -(\lambda + \mu(x))\ P_n(x,\ t) + \lambda P_{n+1}(x,\ t),$$

$$-M + 1 \leq n \leq -1 \qquad (2)$$

$$\frac{\partial P_0(x,\ t)}{\partial t} + \frac{\partial P_0(x,\ t)}{\partial x} = -(\lambda + \mu(x))\ P_0(x,\ t) + \lambda \sum_{j=1}^{S} P_j(x,\ t), \qquad (3)$$

$$\frac{\partial P_n(x, t)}{\partial t} + \frac{\partial P_n(x, t)}{\partial x} = -(\lambda + \mu(x))\, P_n(x, t) + \lambda P_{n+S}(x, t),$$

$$1 \leq n \leq N - S \tag{4}$$

$$\frac{\partial P_n(x, t)}{\partial t} + \frac{\partial P_n(x, t)}{\partial x} = -(\lambda + \mu(x))\, P_n(x, t),$$

$$N - S+1 \leq n \leq N \tag{5}$$

These equations are subject to the following boundary conditions:

$$P_{-M}(0, t) = 0, \tag{6}$$

$$P_n(0, t) = \int_0^\infty P_{n-1}(x, t)\, \mu(x)dx, \qquad -M + 1 \leq n \leq N-1 \tag{7}$$

$$P_N(0, t) = \int_0^\infty [P_{N-1}(x, t) + P_N(x, t)]\, \mu(x)dx, \tag{8}$$

Consider

$$P_n(t) = \int_0^\infty P_n(x, t)dx, \tag{9}$$

$$F(y, t) = \int_0^\infty F(x, y, t)dx, \tag{10}$$

and

$$G(y, t) = \int_0^\infty G(x, y, t)dx. \tag{11}$$

Taking the transformations and then using the generating functions,

from equations (1) and (2) becomes

$$\frac{\partial F(x, y, t)}{\partial t} + \frac{\partial F(x, y, t)}{\partial x} + [\lambda + \mu(x) - \lambda/y]\, F(x, y, t) =$$

$$\lambda y^{-1} P_0(x, t) + \lambda y^{-M}(1 - 1/y)\, P_{-M}(x, t), \tag{12}$$

and equations (4) and (5) becomes

$$\frac{\partial G(x, y, t)}{\partial t} + \frac{\partial G(x, y, t)}{\partial x} + [\lambda + \mu(x) - \lambda y^{-S}]\, G(x, y, t) =$$

$$-\lambda y^{-S} \sum_{j=1}^{s} y^j P_j(x, t). \tag{13}$$

Similarly, equations (6) and (8) yield

$$F(0, y, t) = y \int_0^\infty F(x, y, t)\, \mu(x)dx - P_0(0, t), \tag{14}$$

and

$$G(0, y, t) = y \int_0^\infty G(x, y, t)\, \mu(x)dx + yP_1(0, t) +$$

$$y^N(1-y) \int_0^\infty P_N(x, t)\, \mu(x)dx. \tag{15}$$

Let the system start with the arrival of a customer that makes the number of units in the system equal to i, where $0 < i \leq S$; then

$$P_n(x, 0) = \delta in\ \delta(x), \tag{16}$$

therefore,

$$F(x, y, 0) = 0, \tag{17}$$

and

$$G(x, y, 0) = y^{i}\delta(x). \tag{18}$$

Taking laplace transforms of equations (12) and (13), and then using equations (17) and (18), we get

$$\frac{\partial F(x, y, \alpha)}{\partial x} + [\lambda + \alpha + \mu(x) - \lambda/y]\, F(x, y, \alpha) =$$

$$\lambda y^{-1} P_0(x, \alpha) + \lambda y^{-M}(1 - 1/y)\, P_{-M}(x, \alpha), \tag{19}$$

and

$$\frac{\partial G(x, y, \alpha)}{\partial x} + (\lambda + \alpha + \mu(x) - \lambda y^{-S})\, G(x, y, \alpha) =$$

$$-\lambda y^{-S} \sum_{j=1}^{S} y^{j} P_j(x, \alpha) + y^{i}\delta(x). \tag{20}$$

Equations (1) through (5) become

$$\frac{\partial P_{-M}(x, \alpha)}{\partial x} + [\alpha + \mu(x)]\, P_{-M}(x, \alpha) = \lambda P_{-M+1}(x, \alpha), \tag{21}$$

$$\frac{\partial P_n(x, \alpha)}{\partial x} + [\lambda + \alpha + \mu(x)]\, P_n(x, \alpha) = \lambda P_{n+1}(x, \alpha),$$

$$-M+1 \leq n \leq -1 \tag{22}$$

$$\frac{\partial P_0(x, \alpha)}{\partial x} + [\lambda + \alpha + \mu(x)]\, P_0(x, \alpha) = \lambda \sum_{j=1}^{S} P_j(x, \alpha), \tag{23}$$

$$\frac{\partial P_n(x, \alpha)}{\partial x} + [\lambda + \alpha + \mu(x)]\, P_n(x, \alpha) = \lambda\, P_{n+S}(x, \alpha) + \delta \text{in}\, \delta(x),$$

$$1 \leq n \leq N - S \tag{24}$$

$$\frac{\partial P_n(x, \alpha)}{\partial x} + [\lambda + \alpha + \mu(x)]\, P_n(x, \alpha) = 0, \quad N - S + 1 \leq n \leq N \tag{25}$$

Equations (6) through (8) become

$$P_{-M}(0, \alpha) = 0, \tag{26}$$

$$P_n(0, \alpha) = \int_0^\infty P_{n-1}(x, \alpha)\, \mu(x)dx, \quad -M+1 \leq n \leq N-1 \tag{27}$$

$$P_N(0, \alpha) = \int_0^\infty [P_{N-1}(x, \alpha) + P_N(x, \alpha)]\, \mu(x)dx, \tag{28}$$

$$F(0, y, \alpha) = y \int_0^\infty F(x, y, \alpha)\, \mu(x)dx - P_0(0, \alpha), \tag{29}$$

$$G(0, y, \alpha) = y \int_0^\infty G(x, y, \alpha)\, \mu(x)dx + yP_1(0, \alpha)$$

$$+ y^N(1-y) \int_0^\infty P_N(x, \alpha)\, \mu(x)dx. \tag{30}$$

Equations (21) to (25) can be solved recursively starting from the last equation backwards to give $P_n(x, \alpha)$, $n = N, N-1, \ldots -M$

On solving equation (25), we have

$$P_n(x, \alpha) = P_n(0, \alpha)\, \exp[-(\lambda + \alpha)x - \int_0^x \mu(u)du],$$

$$N - S + 1 \leq n \leq N \tag{31}$$

Equation (24) can now be solved to give

$$P_n(x, \alpha), \qquad n = N - S, N - S - 1, \ldots, 1$$

We observe that N may either be a multiple of S or it will lie between two consecutive multiples of S:

Let

$$C\,S \leq N < (C + 1)\,S, \tag{32}$$

Where C is a positive integer, then

$$0 \leq N - CS < S \leq N - (C-1)\,S < 2S. \tag{33}$$

From the solutions of the last S equations of (24), it can be shown

that

$$\sum_{j=1}^{S} P_j(x, \alpha) = \{\exp[-(\lambda + \alpha)\, x - \int_0^x \mu(u)du]\}$$

$$\{1 + \sum_{r=1}^{(C+1)} \sum_{j=(r-1)S+1}^{rS} \frac{(\lambda x)^{r-1}}{(r-1)!} P_j(0, \alpha)\} \qquad (34)$$

and

$$\sum_{j=1}^{S} y^j P_j(x, \alpha) = \{\exp[-(\lambda + \alpha)x - \int_0^x \mu(u)du]\}$$

$$\{y^j + \sum_{r=1}^{(C+1)} \sum_{j=(r-1)S+1}^{rS} \frac{(\lambda x y^{-S})^{r-1}}{(r-1)!} y^j P_j(0, \alpha)\} \qquad (35)$$

Solving equation (23) after substituting values from (34), we get

$$P_0(x, \alpha) = \{\exp[-(\lambda + \alpha)x - \int_0^x \mu(u)du]\}$$

$$\{P_0(0, \alpha) + \lambda x + \sum_{r=1}^{C+1} \sum_{j=(r-1)S+1}^{rS} \frac{(\lambda x)^r}{r!} P_j(0, \alpha)\} \qquad (36)$$

Thus, equation (21) gives

$$P_{-M}(x, \alpha) = \lambda\{\exp[-\alpha x - \int_0^x \mu(u)du]\} \{\int_0^x \frac{(\lambda z)^M}{M!} \exp(-\lambda z)dz +$$

$$\sum_{r=1}^{M} P_{r-M}(0, \alpha) \int_0^x \frac{(\lambda z)^{r-1}}{(r-1)!} \exp(-\lambda z)dz +$$

$$\sum_{r=1}^{(C+1)} \sum_{j=(r-1)S+1}^{rS} P_j(0, \alpha) \int_0^x \frac{(\lambda z)^{r+M-1}}{(r+M-1)!} \exp(-\lambda z)dz\}. \qquad (37)$$

Solving equation (19) after substituting values from (36) and (37), and then simplifying we get $F(x, y, \alpha)$.

In its solution we have to use Incomplete Gamma function defined as

$$\sqrt{(y, x)} = \int_0^x \exp(-z)z^{y-1}dz.$$

On substituting this value of $F(x, y, \alpha)$ in equation (29) and then simplifying we get the value of $F(0, y, \alpha)$.

Now substituting for $P_n(x, \alpha)$, $-M \leq n \leq N$, in equations (27) and (28) we get $M + N$ equations in $M + N$ unknowns $P_n(0, \alpha)$; $-M + 1 \leq n \leq N$. The value of all these unknowns can be determined.

Laplace transform of Probability for empty state at time t is given by

$$P_0(\alpha) = \int_0^\infty P_0(x, \alpha)dx,$$

where $P_0(x, \alpha)$ is given by equation (36).

Case 1:

K-Erlang interarrival of customers:

$$A(x) = \mu^K x^{K-1} \exp(-\mu x)/\Gamma K,$$

$$A(\alpha) = \left(\frac{\mu}{\mu+\alpha}\right)^K,$$

$$\mu(x) = \mu^K x^{K-1} \exp(-\mu x)/\Gamma(K, \mu x),$$

$$\exp\left(-\int_0^x \mu(u)du\right) = \frac{\Gamma(K, \mu x)}{\Gamma(K)}$$

where $\Gamma(K, \mu x)$ is incomplete Gamma function.

Using the above relations in the previous analysis we find $F(0, y, \alpha)$ and then substituting its value we find $F(x, y, \alpha)$. Furthermore, we calculate the value of the generating functions (its Laplace Transforms).

By following exactly the above procedure we find $G(0, y, \alpha)$ and then $G(x, y, \alpha)$. Then, the following can be shown

$$F(1, \alpha) + G(1, \alpha) + P_0(\alpha) = 1/\alpha$$

This equation on inversion yields total probability

$$F(1, t) + G(1, t) + P_0(t) = 1.$$

Case 2:

Poisson arrivals of customers: Results for this case are deduced from the above by setting $K = 1$

$$A(x) = \mu \exp(-\mu x),$$

$$A(\alpha) = \left(\frac{\mu}{\mu+\alpha}\right),$$

$$\mu(x) = \mu\exp(-\mu x)/\Gamma(1,\mu x),$$

But

$$\Gamma(1, \mu x) = \int_{\mu x}^{\infty} \exp(-z)dz = \exp(-\mu x),$$

$$\mu(x) = \mu,$$

$$\exp\left(-\int_0^x \mu(u)du\right) = \exp(-\mu x).$$

Using these relations in previous analysis we find $F(0, y, \alpha)$ and then substituting its value we find $F(x, y, \alpha)$. After this we calculate the value of generating functions. Exactly following above procedure we find $G(0, y, \alpha)$ and then $G(x, y, \alpha)$.

After making all these calculations we can show that

$$F(1, t) + G(1, t) + P_0(t) = 1.$$

Case 3:

Single Seated Taxis: In this case $S = 1$. Also $\sum_{r=1}^{C+1} \sum_{j=(r-1)S+1}^{rS}$ will be replaced by $\sum_{j=r=1}^{n}$.

Equation (36) becomes

$$P_0(x, \alpha) = \exp[-(\lambda + \alpha)x - \int_0^x \mu(u)du]$$

$$[P_0(0, \alpha) + \lambda x + \sum_{j=r=1}^{n} \frac{(\lambda x)^r}{r!} P_j(0, \alpha)].$$

Equation (37) becomes

$$P_{-M}(x, \alpha) = \lambda \exp[-\alpha x - \int_0^x \mu(u)du]x[\int_0^x \frac{(\lambda z)^M}{M!} \exp(-\lambda z)dz +$$

$$\sum_{r=1}^{M} P_{r-M}(0, \alpha) \int_0^x \frac{(\lambda z)^{r-1}}{(r-1)!} \exp(-\lambda z)dz +$$

$$\sum_{j=r=1}^{n} P_j(0, \alpha) \int_0^x \frac{(\lambda z)^{r+M-1}}{(r+M-1)} \exp(-\lambda z)dz].$$

On solving equation (19) with the help of above two equations we find $F(x, y, \alpha)$. Similarly we find $G(x, y, \alpha)$. Finally from these we determine total probability as in previous cases.

3.4. Multiphase Bulk Service Queueing Model

This queueing model which has been developed by Jaiswal [31] describes a system with an unlimited queue and multiphase service. The arrivals are served in bulk. A general multi phase service station problem is analyzed. Three particular cases are also discussed. Case 1 is that when there is one phase service station. Case 2 is that when the service time distribution is Erlang. Case 3 is that when the units are served singly. This model is analyzed for the transient and steady-state conditions.

Model Specifications:

Arrival Source	:	Infinite
Arrival Pattern	:	Single
Arrival Behavior	:	Patient
Arrival Distribution	:	Poisson
Queue Size	:	Unlimited
Service Discipline	:	First arrive, first served
Service Pattern	:	Bulk
Service Mechanism	:	Single channel, multiple phase
Service Time Distribution	:	Exponential for each phase
Number of Servers	:	r phases

Practical Problems 1. The passengers who have to take more than one bus in order to reach their destination, wait on the bus stand forming an unlimited queue. The buses are considered as service channels. The passengers change the buses at any station, where the service is in multiphase.

2. Races of ball bearings arrive at the service station to be grinded and finished in batch. They are grinded on grinder 1, first, and then on grinder 2, second. Finally, the grinded races receive their finishing process.

Notation: The following notation is used in this model:

λ = mean arrival rate

μ = mean service rate

s = Batch size

P_j = Probability that a batch is in phase j

$P_{n,j}(t)$ = Probability that n units are waiting in the queue and service is in phase j at time t

$P_0(t)$ = Probability that the system is empty at time t

$P_{n,j}(\alpha)$ = Laplace Transform of $P_{n,\,j}(t)$

$Q_n(x, \alpha)$ = Generating function of $P_{n,\,j}(\alpha)$

$F(x, y, \alpha)$ = Generating function of $Q_n(x, \alpha)$

Analysis: In order to derive the difference-differential equations for this model, we consider the continuity of the flow during a time interval $(t, t + \Delta t)$, which leads to the equations:

$$P_{n,j}(t + \Delta t) = (1 - \lambda\Delta t)(1 - \mu\Delta t)\, P_{n,\,j}(t) + \lambda\Delta t\, P_{n-1}(t) +$$

$$\mu\Delta t\, P_{n,\,j+1}(t) + \mu P_j \Delta t\, P_{n+s,\,1}(t)$$

or

$$\frac{P_{n,\,j}(t+\Delta t) - P_{n,\,j}(t)}{\Delta t} = -(\lambda + \mu)\, P_{n,\,j}(t) + \lambda P_{n-1}(t) +$$

$$\mu P_{n,\,j+1}(t) + \mu P_j\, P_{n+s,\,1}(t)$$

or

$$\frac{dP_{n,\,j}(t)}{dt} = -(\lambda + \mu)\, P_{n,\,j}(t) + \lambda P_{n-1,\,j}(t) +$$

$$\mu P_{n,\,j+1}(t) + \mu P_j\, P_{n+s,\,1}(t),$$

$$n > 0,\ 1 \leq j < r \qquad (1)$$

On the basis of this derivation, the difference-differential equations for other conditions are

$$\frac{dP_{n,\,r}(t)}{dt} = -(\lambda + \mu)\, P_{n,\,r}(t) + \lambda P_{n-1,\,r}(t) + \mu P_j\, P_{n+s,\,1}(t),$$

$$n > 0 \qquad (2)$$

$$\frac{dP_{0,\,j}(t)}{dt} = -(\lambda + \mu)\, P_{0,\,j}(t) + \lambda P_j\, P_0(t) + \mu P_j \sum_{m=1}^{s} P_{m,\,1}(t) +$$

$$P_{0,j+1}(t), \qquad 1 \leq j < r \qquad (3)$$

$$\frac{dP_{0,\,r}(t)}{dt} = -(\lambda + \mu)\, P_{0,\,r}(t) + \mu P_r \sum_{m=1}^{s} P_{m,\,1}(t) + \lambda P_r\, P_0(t), \qquad (4)$$

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_{0,\,1}(t). \qquad (5)$$

1. Transient Solution

The Laplace Transform of $P_{n,\,j}(t)$ is given by

$$P_{n,\,j}(\alpha) = \int_0^\infty P_{n,\,j}(t)e^{-\alpha t}\,dt$$

Since initially the process starts with no unit in the system, assume $P_0(0) = 1$.

The Laplace Transofrms of equations (1) to (5) are

$$-(\alpha+\lambda+\mu)\,P_{n,\,j}(\alpha) + \lambda P_{n-1,\,j}(\alpha) + \mu P_{n,\,j+1}(\alpha) + \mu P_j P_{n+s,\,1}(\alpha) = 0,$$

$$n > 0;\ 1 \le j < r \qquad (6)$$

$$-(\alpha+\lambda+\mu)\,P_{n,\,r}(\alpha) + \lambda P_{n-1,\,r}(\alpha) + \mu P_r\, P_{n+s,\,1}(\alpha) = 0,$$

$$n > 0 \qquad (7)$$

$$-(\alpha+\lambda+\mu)\,P_{0,\,j}(\alpha) + \lambda P_j\, P_0(\alpha) + \mu P_j \sum_{m=1}^{s} P_{m,\,1}(\alpha) + \mu P_{0,j+1}(\alpha) = 0$$

$$1 \le j < r \qquad (8)$$

$$-(\alpha+\lambda+\mu)\,P_{0,\,r}(\alpha) + \lambda P_r\, P_0(\alpha) + \mu P_r \sum_{m=1}^{s} P_{m,\,1}(\alpha) = 0, \qquad (9)$$

$$-(\alpha + \lambda)\, P_0(\alpha) + \mu P_{0,\,1}(\alpha) + 1 = 0. \tag{10}$$

For solving the above equations the following generating functions are used

$$Q_n(x, \alpha) = \sum_{j=1}^{r} x^j P_{n,\,j}(\alpha)$$

$$F(x, y, \alpha) = \sum_{n=0}^{\infty} y^n Q_n(x, \alpha)$$

Multiplying each of equations (6) and (8) by x^j and summing over j, j=1, 2, . . ., r - 1, we get

$$-(\alpha + \lambda + \mu) \sum_{j=1}^{r-1} x^j P_{n,\,j}(\alpha) + \sum_{j=1}^{r-1} \lambda x^j P_{n-1,\,j}(\alpha) + \sum_{j=1}^{r-1} \mu x^j P_{n,\,j+1}(\alpha) +$$

$$\sum_{j=1}^{r-1} \mu P_j x^j P_{n+s,\,1}(\alpha) = 0, \tag{11}$$

$$-(\alpha + \lambda + \mu) \sum_{j=1}^{r-1} x^j P_{0,\,j}(\alpha) + \lambda \sum_{j=1}^{r-1} P_j x^j P_0(\alpha) + \mu \sum_{j=1}^{r-1} P_j x^j \sum_{m=1}^{s} P_{m,1}(\alpha) +$$

$$\mu \sum_{j=1}^{r-1} x^j P_{0,\,j+1}(\alpha) = 0. \tag{12}$$

Multiplying each of equations (7) and (9) by x^r and then adding the results to equations (11) and (12) respectively. As a result equations (11) and (12) become

$$-(\alpha + \lambda + \mu) \sum_{j=1}^{r} x^j P_{n,\,j}(\alpha) + \sum_{j=1}^{r} \lambda x^j P_{n-1,\,j}(\alpha) + \sum_{j=1}^{r-1} \mu x^j P_{n,\,j+1}(\alpha) +$$

$$\sum_{j=1}^{r} \mu P_j x^j P_{n+s,\,1}(\alpha) = 0, \tag{13}$$

and

$$-(\alpha + \lambda + \mu) \sum_{j=1}^{r} x^j P_{0,\,j}(\alpha) + \lambda \sum_{j=1}^{r} P_j x^j P_0(\alpha) + \mu \sum_{j=1}^{r} P_j x^j \sum_{m=1}^{s} P_{m,\,1}(\alpha) +$$

$$\mu \sum_{j=1}^{r-1} x^j P_{0,\,j+1}(\alpha) = 0. \tag{14}$$

Let

$$A(x) = \sum_{j=1}^{r} P_j x^j,$$

equations (13) and (14) become

$$-(\alpha + \lambda + \mu) Q_n(x, \alpha) + \lambda Q_{n-1}(x, \alpha) + \mu A(x) P_{n+s,\,1}(\alpha) +$$

$$\sum_{j=1}^{r-1} \mu x^j P_{n,\,j+1}(\alpha) = 0, \tag{15}$$

and

$$-(\alpha + \lambda + \mu) Q_0(\alpha) + \lambda A(x) P_0(\alpha) + \mu A(x) \sum_{m=1}^{s} P_{m,\,1}(\alpha) + \mu \sum_{j=1}^{r-1} x^j P_{0,\,j+1}(\alpha) = 0. \tag{16}$$

Multiplying equation (15) by y^n, summing over n, n = 1, 2, . . ., ∞ and then adding equation (16) we get

$$-(\alpha + \lambda + \mu) \sum_{n=1}^{\infty} y^n Q_n(x, \alpha) + \lambda \sum_{n=1}^{\infty} y^n Q_{n-1}(x, \alpha) + \mu A(x) \sum_{n=1}^{\infty} y^n P_{n+s,1}(\alpha) +$$

$$\sum_{j=1}^{r-1} \mu x^j \sum_{n=1}^{\infty} y^n P_{n,\, j+1}(\alpha) - (\alpha + \lambda + \mu) Q_0(x, \alpha) +$$

$$\lambda A(x) P_0(\alpha) + \mu A(x) \sum_{m=1}^{s} P_{m,\, 1}(\alpha) + \mu \sum_{j=1}^{r-1} x^j P_{0,\, j+1}(\alpha) = 0,$$

or

$$-(\alpha + \lambda + \mu) \sum_{n=0}^{\infty} y^n Q_n(x, \alpha) + \lambda y \sum_{n-1=0}^{\infty} y^{n-1} Q_{n-1}(x, \alpha) +$$

$$\sum_{n=1}^{\infty} \mu \sum_{j=1}^{r-1} y^n x^j P_{n,j+1}(\alpha) +$$

$$\lambda A(x) P_0(\alpha) + \mu A(x) \left[\sum_{m=1}^{s} P_{m,1}(\alpha) + \sum_{n=1}^{\infty} y^n P_{n+s,1}(\alpha) \right] +$$

$$\mu \sum_{j=1}^{r-1} x^j P_{0,j+1}(\alpha) = 0. \qquad (17)$$

Now, let j+1 = n', the third term of the L.H.S. in equation (17) is simplified to

$$\sum_{n=1}^{\infty} \mu \sum_{j=1}^{r-1} y^n x^j P_{n,\, j+1}(\alpha) = \sum_{n=1}^{\infty} \mu \sum_{n'=2}^{r} y^n x^{n'-1} P_{n,n'}(\alpha) =$$

$$\sum_{n=1}^{\infty} \frac{\mu}{x} \sum_{n'=2}^{r} y^n x^{n'} P_{n,\, n'}(\alpha) =$$

$$\sum_{n=1}^{\infty} \left[\frac{\mu}{x} y^n Q_n(x, \alpha) - y^n P_{n,1}(\alpha) \right].$$

Using the above result, equation (17) becomes

$$-(\alpha + \lambda + \mu - \lambda y)F(x, y, \alpha) + \sum_{n=1}^{\infty} \left[\frac{\mu}{x} y^n Q_n(x, \alpha) - \mu y^n P_{n,1}(\alpha) \right] +$$

$$\lambda A(x) P_0(\alpha) + \mu A(x) \left[\sum_{m=1}^{s} P_{m,1}(\alpha) + \frac{1}{y^s} \sum_{n=1}^{\infty} y^{n+s} P_{n+s,1}(\alpha) \right] +$$

$$\frac{\mu}{x} Q_0(x,\alpha) - \mu P_{0,1}(\alpha) = 0,$$

$$-(\alpha + \lambda + \mu - \lambda y - \mu/x)F(x, y, \alpha) - \mu \sum_{n=0}^{\infty} y^n P_{n,1}(\alpha) + \mu A(x) P_0(\alpha) +$$

$$\frac{\mu A(x)}{y^s} \sum_{n=0}^{\infty} y^n P_{n,\, 1}(\alpha) + \sum_{m=1}^{s} P_{m,\, 1}(\alpha) \mu A(x) -$$

$$\frac{\mu A(x)}{y^s} \sum_{n=0}^{\infty} y^n P_{n,\, 1}(\alpha) + \mu A(x) \sum_{n=1}^{\infty} y^n P_{n+s,\, 1}(\alpha) = 0,$$

or

$$-(\alpha + \lambda + \mu - \lambda y - \mu/x)F(x, y, \alpha) - \mu\left[1 - \frac{A(x)}{y^s}\right]\sum_{n=0}^{\infty} y^n P_{n,1}(\alpha) +$$

$$\lambda A(x)P_0(\alpha) + \mu A(x)\left[-\frac{1}{y^s}\sum_{n=0}^{\infty} y^n P_{n,1}(\alpha) + \sum_{n=1}^{\infty} y^n P_{n+s,1}(\alpha) +\right.$$

$$\left.\sum_{m=1}^{s} P_{m,1}(\alpha)\right] = 0. \qquad (18)$$

Multiplying equation (10) by $\frac{A(x)}{y^s}$, and then adding the results to equation (18), we get

$$-(\alpha + \lambda + \mu - \lambda y - \mu/x)F(x, y, \alpha) - \mu\left[1 - \frac{A(x)}{y^s}\right]\sum_{n=0}^{\infty} y^n P_{n,1}(\alpha) +$$

$$\lambda A(x)P_0(\alpha) + \mu A(x)\left[-\frac{1}{y^s}\sum_{n=0}^{\infty} y^n P_{n,1}(\alpha) +\right.$$

$$\left.\sum_{n=1}^{\infty} y^n P_{n+s,1}(\alpha) + \sum_{m=1}^{s} P_{m,1}(\alpha)\right] - \frac{A(x)}{y^s}(\lambda + \alpha)P_0(\alpha) +$$

$$\frac{\mu A(x)}{y^s} P_{0,1}(\alpha) + \frac{A(x)}{y^s} = 0, \qquad (19)$$

Where the fourth and sixth terms in the L.H.S. of equation (19) is simplified to

$$\mu A(x)\left[-\frac{1}{y^s}\sum_{n=0}^{\infty} y^n P_{n,1}(\alpha) + \sum_{n=1}^{\infty} y^n P_{n+s,1}(\alpha) + \sum_{m=1}^{s} P_{m,1}(\alpha)\right] +$$

$$\frac{\mu A(x)}{y^s} P_{0,\,1}(\alpha) = \mu A(x)\left[- \frac{P_{0,1}(\alpha)}{y^s} - \sum_{n=1}^{s-1} \frac{y^n}{y^s} P_{n,1}(\alpha) - P_{s,1}(\alpha) - \right.$$

$$\sum_{n=s+1}^{\infty} \frac{y^n}{y^s} P_{n,1}(\alpha) + \sum_{n=1}^{\infty} y^n P_{n+s,1}(\alpha) + \sum_{m=1}^{s-1} P_{m,\,1}(\alpha) +$$

$$\left. P_{s,\,1}(\alpha) + \frac{P_{0,1}(\alpha)}{y^s} \right] = \mu A(x) \sum_{n=1}^{s-1} \left[\left(1 - \frac{y^n}{y^s}\right) P_{n,\,1}(\alpha) \right].$$

Using the above result, equation (19) becomes

$$-(\alpha + \lambda + \mu - \lambda y - \mu/x)F(x, y, \alpha) - \mu \left[1 - \frac{A(x)}{y^s}\right] \sum_{n=0}^{\infty} y^n P_{n,1}(\alpha) +$$

$$\lambda A(x) P_0(\alpha) + \mu A(x) \sum_{n=1}^{s-1} P_{n,1}(\alpha) \left[1 - \frac{y^n}{y^s}\right] -$$

$$\frac{A(x)}{y^s} \left[(\alpha + \lambda) P_0(\alpha) - 1\right] = 0,$$

or

$$\mu \left[1 - \frac{A(x)}{y^s}\right] \sum_{n=0}^{\infty} y^n P_{n,1}(\alpha) = \lambda A(x) P_0(\alpha) + \mu A(x) \sum_{n=1}^{s-1} P_{n,1}(\alpha) \left[1 - \frac{y^n}{y^s}\right] -$$

$$\frac{A(x)}{y^s} \left[(\alpha + \lambda) P_0(\alpha) - 1\right] - (\alpha + \lambda + \mu - \lambda y - \mu/x) F(x,y,\alpha). \tag{20}$$

Let

$$\frac{x}{\mu} = \frac{1}{(\lambda + \mu + \alpha - \lambda y)} \qquad \text{or } \frac{\mu}{x} = \mu + \lambda + \alpha - \lambda y.$$

Multiplying equation (20) by $\frac{y^s}{A(x)}$ and substituting the above relation we get

$$\mu\left[\frac{y^s}{A(x)} - 1\right]\sum_{n=0}^{\infty} y^n P_{n,1}(\alpha) = \lambda y^s P_0(\alpha) + \mu \sum_{n=1}^{s-1} P_{n,1}(\alpha)\left[y^s - y^n\right] - (\alpha+\lambda)P_0(\alpha) = 1,$$

or

$$G_j(y, \alpha) = \sum_{n=0}^{\infty} y^n P_{n,\,1}(\alpha)$$

$$= \frac{\mu \sum_{n=1}^{s-1} P_{n,1}(\alpha)[y^s - y^n] + [\lambda y^s - \alpha - \lambda]\, P_0(\alpha) + 1}{\mu\,[y^s/A(x) - 1\,]}, \qquad (21)$$

where

$$A(x) = \sum_{j=1}^{r} P_j x^j = \sum_{j=1}^{r} P_j [\mu/(\mu + \lambda + \alpha - \lambda y)]^j.$$

By Rouche's theorem, the denominator of equation (21) has s zeros inside the unit circle, for the functions

$$f(y) = y^s,$$

and

$$g(y) = y^s - \sum_{j=1}^{r} P_j [\mu/(\mu + \alpha + \lambda - \lambda y)]^j$$

have the same number of zeros inside the unit circle C,

$$y = e^{i\theta}$$

if (1) $f(y)$ and $g(y)$ are analytic inside C.

(2) $f(y) \neq 0$ on C

(3) $|f(y) - g(y)| < |f(y)|$ on C.

Let

$$\alpha = \sigma + i\tau$$

where $\sigma > 0$. Obviously $f(y)$ is analytic inside C and is not zero on C.

Also $g(y)$ has a simple pole at $y = \beta$ where

$$\beta = (\lambda + \mu + \sigma + i\tau)/\lambda.$$

Since

$$|\beta| > |\lambda + \mu + \sigma/\lambda|,$$

that is $|\beta| > 1$, $g(y)$ has no singularities inside C.

Now on C,

$$f(y) - g(y) = \sum_{j=1}^{r} P_j \left[\mu/(\mu + \lambda + \sigma + i\tau - \lambda e^{i\theta}\right]^j,$$

and since

$$|\lambda + \mu + \sigma + i\tau - \lambda e^{i\theta}| > |\lambda + \mu + \sigma| - \lambda = \mu + \sigma$$

or

$$|\mu/(\lambda + \mu + \sigma + i\tau - \lambda e^{i\theta})| < \mu/(\mu + \sigma)$$

and is therefore less than one, for σ, i. e. Real $\alpha > 0$.

Thus on C

$$|f(y) - g(y)| < 1,$$

and

$$|f(y)| = 1,$$

and hence on C,

$$|f(y) - g(y)| < |f(y)|.$$

Thus all the conditions of Rouche's theorem are satisfied for Real $\alpha > 0$. and therefore g(y) has s zeros inside the unit circle since f(y) has s zeros.

Since A(x), in equation (21), is regular inside and on the unit circle, $G(y, \alpha)$ will be regular for $|y| \leq 1$, if the s unknowns $P_0(\alpha)$, $P_{1,\,1}(\alpha)$, . . ., $P_{s-1,\,1}(\alpha)$ are chosen in such a way that the zeros in the numerator and denominator in the region coincide. As proved above, the denominator of equation (21) has s zeros inside the unit circle and the numerator must vanish at these s zeros, giving rise to s equations and s unknowns.

To determine $G(y, \alpha)$ uniquely it is necessary to show that these s equations are linearly independent. To prove this, substitute equation (21) in equation (20), we get the following after letting x = 1.

$$\sum_{n=0}^{\infty} y^n Q_n(1, \alpha) = [\mu(1/A(x) - 1)/(\alpha + \lambda - \lambda y)]G(y, \alpha), \tag{22}$$

where

$$Q_n(1, \alpha) = \sum_{j=1}^{r} P_{n,\, j}(\alpha).$$

Note that $Q_n(1, \alpha)$ is the Laplace transform of the probability that there are n units in queue at time t. Since $G(y, \alpha)$ is known, $Q_n(1, \alpha)$ can be calculated, the inverse transform of which will give the required time dependent probabilities.

Case 1:

Single Phase Bulk Service

When

$$P_j = 1, \; j = 1.$$

$$A(x) = \mu/(\mu + \lambda + \alpha - \lambda y)$$

Equation (21) becomes

$$G(y\ , \alpha) = \sum_{n=0}^{\infty} y^n P_{n,\, 1}(\alpha)$$

$$= \frac{\mu \sum_{n=1}^{s-1} P_{n,\, 1}(\alpha)[y^s - y^n] + (\lambda y^s - \alpha - \lambda)P_0(\alpha) + 1}{\mu[y^s/A(x) - 1]} \qquad (23)$$

and equation (20) becomes

$$\sum_{n=0}^{\infty} y^n Q_n(1, \alpha) = [\mu(1/A(x) - 1)/(\alpha + \lambda - \lambda y)] \;\; G(y, \alpha) \qquad (24)$$

$$Q_n(1, \alpha) = P_{n,\,1}(\alpha).$$

Case 2:

k-Erlang Service time Distribution

In this case

$$P_j = 1, \; j = k,$$

and

$$P_j = 0, \; j \neq k,$$

$$G(y, \alpha) = \frac{\mu \sum_{n=1}^{s-1} P_{n,\,1}(\alpha)[y^s - y^n] + P_0(\alpha)[\lambda y^s - \alpha - \lambda) + 1}{\mu[y^s(\mu + \alpha + \lambda - \lambda y/\mu)^k - 1]}, \tag{25}$$

and equation (20) becomes

$$\sum_{n=0}^{\infty} y^n Q_n(1, \alpha) = \frac{\mu[\mu + \alpha + \lambda - \lambda y/\mu)^k - 1]G(y, \alpha)}{[\alpha + \lambda - \lambda y]}. \tag{26}$$

We will first show that the denominator has s simple zeros inside the unit circle if $|\alpha| > \mu\ (\rho - 1)$,

where

$$\rho = \lambda k/s\mu.$$

For, the zeros of the denominator will be simple if the equations

$$y^s = [\mu/(\mu + \alpha + \lambda - \lambda y)]^k,$$

and

$$sy^{s-1} = (\lambda k/\mu)[\mu/(\mu + \alpha + \lambda - \lambda y)]^{k+1},$$

are not satisfied by any y whose modulus is less than one. Dividing we get

$y = (\alpha + \lambda + \mu)s/\lambda(s+k)$, the only possible repeated zero.

Therefore if $|(\alpha + \lambda + \mu)\ s/\lambda(s+k)| > 1$.

There will be no multiple zero inside and on the unit circle.

or

$$|(\alpha + \lambda + \mu)| > \frac{\lambda(s+k)}{s},$$

since

$$|(\alpha + \lambda + \mu)| \leq |\alpha| + (\lambda + \mu),$$

we have,

$$|\alpha| > \mu(\rho - 1) \quad \text{where } \rho = \frac{\lambda k}{s\mu}.$$

Hence if α is chosen to satisfy this condition, which is always possible since we only require that Real α should be greater than zero, the zeros inside the unit circle are simple.

Now, to evaluate the S unknowns in the enumerator, the s equations are

$$\mu \sum_{n=1}^{s-1} P_{n,\,1}(\alpha)(y_i^s - y_i^n) + P_0(\alpha)\cdot(\lambda y_i^s - \alpha - \lambda) + 1 = 0,$$

$$i = 1, 2, \ldots, s \quad (27)$$

where y_i are the s simple zeros of the denominator which lie inside the unit circle. With the help of equation (10) the set of equation (27) can be written as

$$\sum_{n=1}^{s-1} [P_{n,\,1}(\alpha)/P_{0,\,1}(\alpha)](y_i^s - y_i^n) + (\lambda/\mu)y_i^s\, P_0(\alpha)/P_{0,\,1}(\alpha) - 1 = 0,$$

$$i = 1, 2, \ldots, s \qquad (28)$$

and these will be linearly independent if $\Delta \neq 0$, where

$$\Delta = \begin{bmatrix} y_1^s - y_1 & y_1^s - y_1^2 & \cdots & y_1^s - y_1^{s-1} & (\lambda/\mu)y_1^s \\ y_2^s - y_2 & y_2^s - y_2^2 & \cdots & y_2^s - y_2^{s-1} & (\lambda/\mu)y_2^s \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y_s^s - y_s & y_s^s - y_s^2 & \cdots & y_s^s - y_s^{s-1} & (\lambda/\mu)y_s^s \end{bmatrix}$$

$$\Delta = (\lambda/\mu)(-1)^{s-1} y_1 y_2 \ldots y_s \Pi_{j>i}(y_j - y_i).$$

Since $y_i \neq 0$ and all the y_i are different, $\Delta \neq 0$, so that using equation (10), $P_0(\alpha)$, $P_{0,\,1}(\alpha)$, $\ldots$ $P_{s-1,\,1}(\alpha)$, and hence $G(y, \alpha)$ is uniquely determined.

However, even in this case, explicit solution by inverting the generating function is difficult to obtain, so we restrict ourselves further to

the simplest case in which the service time distribution is also exponential. In this case,

$$G(y, \alpha) = c(\alpha)/(y_i - y)$$

where y_i is the root of modulus greater than one of the equation

$$\lambda y^{s+1} - (\mu + \alpha + \lambda)y^s + \mu = 0$$

and $c(\alpha)$ is to be determined.

Case 3:

Single Unit Service:

In this case s = 1, such that

$$G(y, \alpha) = [P_0(\alpha)(\lambda y - \alpha - \lambda) + 1]/\mu[y/B(y) - 1], \tag{29}$$

$$\sum_{n=0}^{\infty} y^n Q_n(1, \alpha) = \{\mu[1/B(y) - 1]/(\alpha + \lambda - \lambda y)\}\ G(y, \alpha), \tag{30}$$

where

$$B(y) = \sum_{j=1}^{r} P_j[\mu/(\mu + \alpha + \lambda - \lambda y)]^j.$$

From equation (28)

$$P_0(\alpha) = 1/(\alpha + \lambda - \lambda y_i),$$

where y_i is the root of

$$y - \sum_{j=1}^{r} P_j[\mu/(\mu + \alpha + \lambda - \lambda y)]^j = 0,$$

that lies inside the unit circle.

Substituting

$$x = \mu/(\mu + \alpha + \lambda - \lambda y),$$

we find that

$$P_0(\alpha) = x_i/(1 - x_i)$$

where x_i is the root inside the unit circle of the equation

$$x[\alpha + \lambda + \mu] - \mu - \lambda x \sum_{j=1}^{r} P_j x^j = 0.$$

This result is identical to that obtained by Luchak. $P_0(t)$ can be evaluated by his method. Substituting the value in equation (26), we directly obtain $Q_n(1, \alpha)$, inversion of which gives the probability that there are n units waiting at time t. Thus in place of mean phase length, we directly obtain the mean queue length by differentiating (26) and putting y = 1.

In particular, if the service time distribution is exponential, that is $P_j = 1$; $j = 1$, and $P_j = 0$, $j \neq 1$. We get

$$G(y, \alpha) = \sum_{n=0}^{\infty} y^n Q_n(1, \alpha),$$

$$= [1 - P_0(\alpha)(\alpha + \lambda - \lambda y)]/[\lambda y^2 - (\alpha + \lambda + \mu)y + \mu],$$

therefore

$$P_0(\alpha) = 1/(\alpha + \lambda - \lambda y_1).$$

where y_1 is a root of modulus less than unity of the equation

$$\lambda y^2 - (\alpha + \lambda + \mu)y + \mu = 0.$$

Hence

$$\sum_{n=0}^{\infty} y^n Q_n(1, \alpha) = [1/y_2(\alpha + \lambda - \lambda y_1)] \sum_{n=0}^{\infty} (y/y_2)^n,$$

so that

$$Q_{n-1}(1, \alpha) = 1/(\alpha + \lambda - \lambda y_1)\ y_2^n.$$

where y_2 is the root of modulus greater than unity such that $y_1 y_2 = \mu/\lambda$. After simplifying we get

$$Q_{n-1}(1, \alpha) = (1 - \rho)\ \rho^n \sum_{m=n+2}^{\infty} y_1^m/\sqrt{K_1} + \rho^n y_1^n \sqrt{K_1} + \rho^n y_1^{n+1}/\sqrt{K_1},$$

where $\rho = \lambda/\mu$, $K_1 = (\alpha + \lambda + \mu)^2 - 4\lambda\mu$.

Inverting above equation we get

$$Q_{n-1}(t) = \exp[-(\lambda + \mu)t][(1 - \rho)\rho^{n} \sum_{m=n+2}^{\infty} (\sqrt{\mu/\lambda}\, I_{m} (2\sqrt{\lambda\mu t})$$

$$+ (\sqrt{\mu/\lambda})^{-n} I_{-n} (2\sqrt{\lambda\mu t}) + (\sqrt{\mu/\lambda})^{1-n} I_{n+1} (2\sqrt{\lambda\mu t}). \qquad (31)$$

2. Steady State Solution: The steady state solution can be obtained by the well known property of Laplace Transform viz.

$$\lim_{t\to\infty} f(t) = \lim_{\alpha\to 0} f(\alpha),$$

if the limit on left exists.

Applying this to equation (21) and (22) we get

$$G(y) = \sum_{q=0}^{s-1} P_{q,1}(y^{s} - y^{q})/[y^{s}/K(y) - 1], \qquad (32)$$

$$\sum_{n=0}^{\infty} y^{n} Q_{n}(1) = [\mu\{1/K(y) - 1\}/\lambda(1 - y)]\, G(y), \qquad (33)$$

where

$$K(y) = \sum_{j=1}^{r} P_{j}[\mu/(\mu + \lambda - \lambda y)]^{j}.$$

To determine the value of constants in equation (32) if y_i, i = 1, 2, . . ., s - 1 are the roots of

$$y^s/K(y) - 1 = 0,$$

with modulus less than one, then

$$\sum_{q=0}^{s-1} P_{q,\,1}(y_i^s - y_i^q) = 0, \tag{34}$$

$$\sum_{q=0}^{s-1} P_{q,\,1}(s - q) = [s - K(1)]\} \sum_{n=0}^{\infty} P_{n,\,1}$$

$$= \{[s - K(1)]/\mu K(1)\} \sum_{n=0}^{\infty} Q_n(1), \tag{35}$$

here $\sum_{n=0}^{\infty} Q_n(1) = 1 - P_0$ hence we have

$$\mu P_{0,\,1} = \lambda P_0$$

and

$$\sum_{q=0}^{s-1} P_{q,\,1}(y_i^s - y_i^q) = 0,$$

$$\sum_{q=0}^{s-1} P_{q,\,1}(s - q) = \{\lambda[s - K(1)]/\mu K(1)\}(1 - P_0). \tag{36}$$

3.5 Batch Arrival Queueing Model

Most of the queueing systems studied by various authors are of the type of single arrivals and single service. In practical systems the arrivals are usually of batch type. The service may be of single or batch type. There are three different types of arrival batch given below:

1. Random Size Batch: where the size of the batches is random.
2. Periodic Size Batch: where the size of the batches is of a periodic pattern.
3. Constant Size Batch: where the size of the batches is constant.

Clearly the constant size batch type may be considered as a special case of that of the random batch size. Soriano [57] has studied the queueing systems for the general case of arrivals. In the analysis of queueing systems developed, the general waiting time distributions of any customers in an arrival batch have been derived under two conditions: (1) transient condition, (2) steady-state condition. The necessary and sufficient conditions for the convergence to equilibrium have also been investigated.

Model Specification:

Arrival Source	:	Infinite
Arrival Pattern	:	Batch
Arrival Behavior	:	Patient
Arrival Distribution	:	General
Queue Size	:	Limited
Service Discipline	:	First arrive, first served
Service Pattern	:	Single
Service Mechanism	:	Single channel, Single phase

Service time Distribution: : General

Number of Servers : One

Practical Problem: Patients wait at the hospital forming a queue to see the doctor. The patients are scheduled in batches. The doctor is considered as a service channel.

Notation: The following is the notation used in this model:

λ = Mean arrival rate

μ = Mean service rate

ρ = Load factor

T = Interarrival time between successive arrivals

S = Service period

M = Number of customers in a service group

$F_{rb}(x)$ = Waiting time distribution function of the packet r

W_{rb} = Waiting time of packet r

U_b = Waiting time of batch (a random variable)

S_{ri} = Service time of the customer i of the batch r

$G_S(x)$ = Service time distribution

$F_{ij}(x)$ = Steady state waiting time distribution function of the customer j of the batch i of a periodic pattern

k_{ri} = Number of customers of the periodic pattern r in the batch i

Assumptions

1. Arrivals of customers are in batch and served in groups.
2. The number of customers in an arrival group is an independent random variable.

3. The sizes of service groups and arrival groups need not necessarily to be the same.
4. Interarrival time and service times are respectively independent and identically distributed random variables.
5. Each arrival batch of customers is considered as a composite unit referred to as packet.

Analysis

According to the theorem in Soriano ([57], p. 35) the waiting time distribution function of the packet r, $F_{rb}(x)$, converges to the distribution function, $F_b(x)$ as r tends to infinity. This is true only if the ratio of the packet mean service time and mean interarrival time between successive packets is less than one.

The waiting distribution function of the packet r is nothing but the cumulative probability up to the waiting time of the packet r. In equation form

$$F_{rb}(x) = P(W_{rb} \leq x)$$

In limiting case when r tends to infinity, the distribution function $F_b(x)$ is given by the area under probability curve up to that point. It is of continuous type so can be summed up and given by

$$F_b(x) = \int_{u_b \leq x} F_b(x - u_b) dGu_b(u_b)$$

In order to derive the steady state waiting time distribution functions of each individual customer in an arrival batch, let $F_1(x)$, $F_2(x)$, ..., $F_n(x)$

denote the waiting time distribution of the customer to be serviced 1st, 2nd, . . ., nth, in their arrival batch, respectively. Note that the individual customers of the packet are considered as independent entities. Then,

$$F_1(x) = \int_{u_b \leq x} F_b(x - u_b) dG_{u_b}(u_b)$$

$$F_2(x) = \int_{y \leq x} F_1(x - y) dG_S(y)$$

$$\vdots$$

$$F_n(x) = \int_{y \leq x} F_{n-1}(x - y) dG_S(y),$$

In the periodic size batch type model the packet includes a group of batches while in the random size batch type model a packet includes only one batch. By defining k_{ri} as the number of customers in batch i of periodic pattern r. The packet r consists of a group of batches k_{r1}, k_{r2}, . . ., k_{rn}. As in the random size batch type model, the steady-state waiting time distribution function of the customer j of the batch i, $F_{ij}(x)$ is given ([57], pp. 40-51) such that

$$F_{11}(x) = \int_{y \leq x} F_{11}(x - y) dG_{u_p}(y)$$

Then,

$$F_{12}(x) = \int_{y \leq x} F_{11}(x - y) dG_S(y)$$

$$F_{13}(x) = \int_{y \le x} F_{12}(x - y)dG_S(y)$$

$$F_{1k_1}(x) = \int_{y \le x} F_{1,k_1-1}(x - y)dG_S(y)$$

Utilizing the above general formulae the waiting time distributions of the particular queueing systems may be determined. Two types of queueing systems are considered.

1. M/G/1 - Exponential interarrival times with general single service time distribution.
2. D/E/1 - Deterministic arrival and Erlangian service time distribution.

Case 1: Queueing System M/G/1

Let the arrival batch consists of two customers, service is of single unit type and the interarrival time between two successive arrival batches, T is not fixed.

Then

$$P(S_i \le x) = P(S \le x)$$

$$= G(x)$$

and

$$P(T \le x) = 1 - e^{-\lambda x}$$

$$= M(x)$$

Assuming that

$$\rho = (2/T)E[S] < 1$$

From the general formulae,

$$F_1(x) = \int_{u_2 \leq x} F_2(x - u_2)dG_2(u_2)$$

$$= \int_{-\infty}^{x} F_2(x - u_2)dG_2(u_2) \qquad (1)$$

and

$$F_2(x) = \int_{-\infty}^{x} F_1(x - u_1)dG_1(u_1)$$

where

$$G_2(u_2) = \int_0^{\infty} G(u_2 + t)dM(t)$$

$$= \int_0^{\infty} G(u_2 + t)\lambda e^{-\lambda t}dt$$

On substituting these values in equation (1) and then after simplifying we get

$$F_1(x) = \int_0^{\infty} F_2(y) \int_0^{\infty} g(v)\ \lambda e^{-\lambda(v+y-x)}d\ dy.$$

$$= c_1 e^{\lambda x}$$

where

$$c_1 = \lambda \int_0^{\infty} \int_0^{\infty} e^{-\lambda(v+y)} F_2(y)g(v)dvdy$$

In most general form $F_1^*(x)$ may be written as

$$F_1^*(x) = F_1(x), \quad x \geq 0$$

$$= c_1 e^{\lambda x}, \quad x < 0$$

For proof and computing procedure for the value of c_1 which is given by $c_1 = 1 - \frac{2\lambda}{\mu}$ see [57], pp. 53-61.

Case 2: Queueing System D/E/1.

In this model, as in case 1, the arrival batch consists of two customers and service is of single unit type. However, the interarrival time between two successive arrival batches, T is a fixed number.

Let T = 1,

Then,

$$\rho = 2E[S] < 1$$

and

$$P(S_i \leq y) = P(S \leq y),$$

$$= G(y).$$

Then, as before,

$$F_1(x) = \int_0^x F_b(x - u_b) dG_{u_b}(u_b), \tag{1}$$

and

$$F_2(x) = \int_0^x F_1(x - y) dG(y). \tag{2}$$

Since the service time distribution is of the Erlangian type, let probability distribution function is given by

$$g(y) = \frac{\mu^{n/2} y^{n/2-1} e^{-\mu y}}{(\frac{n}{2} - 1)!}, \qquad y \geq 0 \qquad (3)$$

$$= 0, \qquad y < 0$$

then

$$E[y] = \frac{n}{2\mu} . \qquad (4)$$

If we suppose $G_{S_b}(y)$ is denoted by $G_b(y)$ and then let probability distribution function is given by

$$g_b(y) = \frac{\mu^n y^{n-1} e^{-\mu y}}{(n-1)!}, \qquad y \geq 0 \qquad (5)$$

$$= 0 , \qquad y < 0$$

then

$$E[y] = \frac{n}{\mu} \qquad (6)$$

Here we note that in order for the steady-state equations to hold it is necessary and sufficient that $\mu > n$. To calculate $G_{u_b}(y)$ we proceed

$$\begin{aligned} G_{u_b}(y) &= P(S_b - T \leq y) \\ &= P(S_b \leq y + 1) \\ &= G_b(y + 1) \end{aligned} \qquad (7)$$

Then equation (1) becomes,

$$F_b(x) = \int_0^x F_b(x - y)dG_b(y + 1), \tag{8}$$

or

$$F_b(x - 1) = \int_0^x F_b(x - y)dG_b(y). \tag{9}$$

Substituting the value of $g_b(y)$ from equation (5) in equation (9).

$$F_b(x - 1) = \frac{\mu^n}{(n-1)!} \int_0^x F_b(x - y)y^{n-1}e^{-\mu y}dy \tag{10}$$

To solve equation (10) we can assume function $F_b(x)$ in such a way so the r.h.s. term may be integrated. Let its solution be of the form,

$$F_b(x) = e^{Zx}$$

where Z is some unknown constant. After substituting and integrating we will get the r.h.s. in a series form. But as per supposition l.h.s. $F_b(x - 1) = e^{-Z}e^{Zx}$. This will be true only if

$$e^{-Z} = \frac{\mu^n}{(\mu+Z)^n} \tag{11}$$

This equation is of nth degree for unknown Z. Root of the above equation is Z = 0.

Instead of assuming solution in the previous exponential form, if we assume the solution to be a linear combination of exponential terms,

that is,

$$F_b(x) = 1 + \sum_{i=1}^{n} c_i e^{Z_i x} \tag{12}$$

where

$$c_0 = 1,\ Z_0 = 0.$$

Then the r.h.s. of equation (5) may be simplified into a series which is equal to the l.h.s. only if

$$\sum_{i=0}^{n} \left[\frac{c_i}{(\mu+Z_i)^{r+1}} \right] = 0, \qquad r = 0, 1, 2, \ldots, n-1 \tag{13}$$

Relation (13) is a set of n equations for the n unknowns c_1, c_2, . . ., c_n.

Since

$$F_1(x) = F_b(x)$$

so the value of $F_2(x)$ can also be determined in a similar way. For its analysis refer to [57], pp. 66-68. To analyze the equations or to obtain the roots of equations the complex variable analysis have been used. The solution is given by

$$\frac{\theta - 2\pi a}{b \sin\theta} = e^{-b} e^{b\rho\cos\theta} \tag{14}$$

This equation (14) can be solved by the help of Newton Raphson iterative process. For its analysis refer to [57], pp. 104-110.

CHAPTER IV

BULK QUEUEING SYSTEMS

The purpose of this chapter is to present various applications of bulk queueing problems in industry and hospital. It includes application to iron and steel and glass industries. Furthermore, an elevator problem is included which has been simulated on IBM 360/50 computer using GPSS IV language.

4.1. A Queueing System in Iron and Steel Industry

In an integrated iron and steel plant, the flow of material among the main parts, namely, blast furnace, steel works, and rolling mill creates various problems. In such plants, usually, the pig iron produced in the blast furnace is stored in a large vessel called mixer. This molten metal is transformed into steel in the steel works, and then poured in the moulds to form ingots. These ingots are transported to the soaking pits for reheating to the rolling temperature. The volume of the soaking pits depends on the number of the ingots and the time of their arrivals. In case when the soaking pits are not free, the ingots often must wait for soaking. This results into their cooling off and it takes longer time to reheat them and make a line of waiting ingots.

This system can be reviewed as a queueing process where the arrival of ingots (customers) at soaking pits from the steel work is in batches. The number of waiting ingots in the

queue fluctuates according to the number of arrivals and the number of idle soaking pits (service stations). In this case the problem is how to schedule and control the material flow from steel work to the soaking pit. In the soaking pit, the ingots are served simultaneously. Therefore, the soaking pits can be divided into a number of service stations, each of which serves one ingot only. These service stations can be considered as a multichannel type.

In terms of queueing theory, this problem falls in the batch arrival multichannel single phase type queueing problem. The formulation of this type is given below:

The following are the assumptions made in formulating the above queueing system

(1) Each arrival consists of a group containing j number of ingots. The batches may not necessarily be of equal size. The interarrival times are independently distributed. The arrival at the service station is according to a poisson distribution with mean λ.

(2) All the ingots are served by a number, s, of service stations installed on parallel.

(3) The service time of the ingots in the soaking pits is a random variable, independently distributed.

(4) The interarrival and service times are statistically independent.

(5) The groups receive service in order of their arrival. The order of customers in a particular group is immaterial.

The following is the analysis of the problem on the basis of the above assumptions.

The following notations are used

λ = Mean arrival rate

μ = Mean service rate at each station

S = Total number of service stations

P_j = Probability that an arriving batch of ingots consists of j units

$P_n(t)$ = Probability that there are n units in the system at time t

$P_0(t)$ = Probability that there are zero units in the system at time t

The probability that there will be n, $n > 0$, units in the system at time $(t + \Delta t)$ may be expressed as the sum of four independent compound probabilities

1. The product of probabilities that
   - 1.1. There are n units in the system at time t — $P_n(t)$
   - 1.2. There are no arrivals during the Δt interval — $[1-\lambda\Delta t]$
   - 1.3. There are no units served during the Δt interval — $[1-n\mu\Delta t]$
2. The product of probabilities that
   - 2.1. There are n+1 units in the system at time t — $P_{n+1}(t)$
   - 2.2. There are no arrivals during the Δt interval — $[1-\lambda\Delta t]$
   - 2.3. There is one unit served during the Δt interval — $(n+1)\mu\Delta t$

3. The product of probabilities that

   3.1. There are n-j, units in the system at time t — $P_{n-j}(t)$

   3.2. There are j arrivals during the Δt interval — $P_j \lambda \Delta t$

   3.3. There are no units served during the Δt interval — $[1-n\mu\Delta t]$

4. The product of probabilities that

   4.1. There are n units in the system at time t — $P_n(t)$

   4.2. There are k arrivals during the Δt interval — $\lambda\Delta t$

   4.3. There are k units served during the Δt interval — $k\mu\Delta t$

The four probabilities may be transformed as follows:

$$P_n(t)[1-\lambda(\Delta t)][1-n\mu(\Delta t)] = [1-\lambda(\Delta t) - n\mu(\Delta t)]P_n(t) + 0(\Delta t)$$

$$P_{n+1}(t)\cdot(n+1)\mu\Delta t(1-\lambda\Delta t) = (n+1)\mu P_{n+1}(t)\ \Delta t + 0(\Delta t)$$

$$\sum_{j=1}^{n} P_{n-j}(t)P_j\lambda\Delta t(1-n\mu\Delta t) = \sum_{j=1}^{n} \lambda P_j P_{n-j}(t)\ \Delta t + 0(\Delta t)$$

$$P_n(t)(\lambda\Delta t)(k\mu\Delta t) = 0(\Delta t)$$

By adding these probabilities, we obtain the probability of n units in the system at time $(t + \Delta t)$.

$$P_n(t + \Delta t) = [1-\lambda\Delta t - n\mu\Delta t]P_n(t) + (n+1)\mu P_{n+1}(t)\Delta t +$$

$$O(\Delta t) + \sum_{j=1}^{n} \lambda P_j P_{n-j}(t)\Delta t$$

$$\frac{P_n(t + \Delta t) - P_n(t)}{\Delta t} = -(\lambda+n\mu)P_n(t) + (n+1)\mu P_{n+1}(t) +$$

$$\sum_{j=1}^{n} \lambda P_j P_{n-j}(t) \quad , \qquad 1 \leq n < s$$

$$\frac{dP_n(t)}{dt} = -(\lambda+n\mu)P_n(t) + (n+1)\mu P_{n+1}(t) +$$

$$\sum_{j=1}^{n} \lambda P_j P_{n-j}(t) \quad , \qquad 1 \leq n < s \qquad (1)$$

When $n \geq s$ in the above equation (1) n and n+1 will be replaced by s, then equation become

$$\frac{dP_n(t)}{dt} = -(\lambda+s\mu)P_n(t) + \lambda \sum_{j=1}^{n} P_j P_{n-j}(t) +$$

$$s\mu P_{n+1}(t) \quad , \qquad n \geq s \qquad (2)$$

In case when n = 0 i.e. zero units in the system at time t + Δt, the probability may be expressed as:

1. The product of probabilities that
   - 1.1. There are zero units in the system at time t — $P_o(t)$
   - 1.2. There are no arrivals during the Δt interval — $[1-\lambda\Delta t]$
2. The product of probabilities that
   - 2.1. There is one unit in the system at time t — $P_1(t)$
   - 2.2. There are no arrivals during the Δt interval — $[1-\lambda\Delta t]$
   - 2.3. There is one unit served during the Δt interval — $\mu\Delta t$

Upon adding these two independent probabilities, we obtain the probability of zero units in the system at time t + Δt.

$$P_0(t+\Delta t) = P_0(t)[1-\lambda\Delta t] + P_1(t)\mu\Delta t[1-\lambda\Delta t]$$

$$\frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = -\lambda P_0(t) + \mu P_1(t)$$

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_1(t) \qquad (3)$$

Equations (1), (2) and (3) represent the queueing system mentioned above. In the steady-state condition, equations (1),

(2) and (3) become

$$-(\lambda+n\mu)P_n + (n+1)\mu P_{n+1} + \sum_{j=1}^{n} \lambda P_j P_{n-j} = 0 \ , \quad 1 \leq n < s \qquad (4)$$

$$-(\lambda+s\mu)P_n + s\mu P_{n+1} + \sum_{j=1}^{n} \lambda P_j P_{n-j} = 0 \ , \quad n \geq s \qquad (5)$$

$$-\lambda P_0 + \mu P_1 = 0 \qquad (6)$$

Equations (4), (5) and (6) may be solved to obtain the values of the various probabilities, and then the measures of effectiveness can be computed.

4.2. A Queueing System in Glass Industry

In glass industry, different articles are produced from silica sand. The articles, after being cast, are cooled slowly to the room temperature so that it may not break due to sudden change of the temperature. This cooling operation is processed in the seasoning room. The volume of the seasoning rooms depends on the number of articles and the time of their arrivals.

This system can be reviewed as a queueing process where the arrival of articles (customers) at the seasoning rooms (service stations) is in batch, however the articles enter

the seasoning room singly. The seasoning room is divided into a number of rooms on the basis of the temperature. They move from one room to the other which has lower temperature. The seasoning room can be considered as a multiphase single channel type service station. Each article passes through all the service stations in general. There might be cases when the article may not pass through all the service stations but might be taken out early. However, here we will consider the general case. The size of the seasoning rooms shall be such that the articles waiting outside are minimum in number. In terms of queueing theory, this type of problem falls in the batch arrival single channel multiphase queueing system. The formulation of this problem is given below:

The following are the assumptions made in formulating the above queueing system

(1) Each arrival consists of a group containing j number of articles. The batch may not necessarily of equal size. The arrival at a service station is according to a poisson distribution. The interarrival times are independently distributed.

(2) The seasoning room has been assumed equivalent to a single channel multiphase type service station

(3) The service time of the articles in the seasoning room is a random variable, independently distributed.

(4) The interarrival and service times are statistically independent.

(5) The groups receive service in order of their arrival. The order of customers in a particular group is immaterial.

The following is the analysis of the problem on the basis of the above assumptions. The following notations are used.

λ = Mean arrival rate

μ = Mean service rate

r = Total number of service phases

P_j = Probability that an arriving batch of products is of j units

$P_{n,k}(t)$ = Probability that there are n units in the system and service is in k th phase at time t

$P_0(t)$ = Probability that there are no units in the system at time t

Q_j = Probability the unit requires service in j phases , $1 \leq j \leq r$

The probability that there will be n, $n > 0$, units in the system at time $(t + \Delta t)$ may be expressed as the sum of five independent compound probabilities:

1. The product of probabilities that

   1.1. There are n units in the system and the service is in kth phase at time t — $P_{n,k}(t)$

   1.2. There are no arrivals during the Δt interval — $[1-\lambda\Delta t]$

   1.3. There are no units served during the Δt interval — $[1-\mu\Delta t]$

2. The product of probabilities that

   2.1. There are n+1 units in the system at time t — $P_{n+1,k}(t)$

2.2. There are no arrivals during the Δt interval $[1-\lambda\Delta t]$

2.3. There is one unit served during the Δt interval $Q_k \mu \Delta t$

3. The product of probabilities that

3.1. There are n-j units in the system at time t $P_{n-j,k}(t)$

3.2. There are j arrivals during the Δt interval $P_j \lambda \Delta t$

3.3. There are no units served during the Δt interval $[1-\mu\Delta t]$

4. The product of probabilities that

4.1. There are n units in the system and service is in (k+1)th phase at time t $P_{n,k+1}(t)$

4.2. There are no arrivals during the Δt interval $[1-\lambda\Delta t]$

4.3. There is one phase service during the Δt interval $\mu\Delta t$

5. The product of probabilities that

5.1. There are zero units in the system at time t $P_0(t)$

5.2. There are n arrivals during the Δt interval $Q_k P_n \lambda \Delta t$

5.3. There are no units served during the Δt interval $[1-\mu\Delta t]$

The five probabilities may be transformed as follows:

$$P_{n,k}(t)[1-\lambda\Delta t][1-\mu\Delta t] = [1-\lambda\Delta t - \mu\Delta t]P_{n,k}(t) + 0(\Delta t)$$

$$P_{n+1,k}(t)[1-\lambda\Delta t]Q_k\mu\Delta t = Q_k\mu P_{n+1,k}(t)\Delta t + 0(\Delta t)$$

$$\sum_{j=1}^{n} P_{n-j,k}(t)P_j\lambda\Delta t[1-\mu\Delta t] = \sum_{j=1}^{n} \lambda P_j P_{n-j,k}(t)\Delta t + 0(\Delta t)$$

$$P_{n,k+1}(t)[1-\lambda\Delta t]\mu\Delta t = \mu P_{n,k+1}(t)\Delta t + 0(\Delta t)$$

$$P_0(t)Q_kP_n\lambda\Delta t[1-\mu\Delta t] = \lambda Q_kP_nP_0(t)\Delta t + 0(\Delta t)$$

By adding these probabilities, we obtain the probability of n units in the system and service in kth phase at time $(t + \Delta t)$.

$$P_{n,k}(t+\Delta t) = [1-\lambda\Delta t-\mu\Delta t]P_{n,k}(t) + Q_k\mu P_{n+1,k}(t)\Delta t +$$

$$\sum_{j=1}^{n} \lambda P_j P_{n-j,k}(t)\Delta t + \mu P_{n,k+1}(t)\Delta t +$$

$$\lambda Q_kP_nP_0(t)\Delta t + 0(\Delta t)$$

$$\frac{P_{n,k}(t+\Delta t) - P_{n,k}(t)}{\Delta t} = -(\lambda+\mu)P_{n,k}(t) + \mu Q_kP_{n+1,k}(t) +$$

$$\sum_{j=1}^{n} \lambda P_j P_{n-j,k}(t) + \mu P_{n,k+1}(t) + \lambda Q_k P_n P_0(t)$$

$$\frac{dP_{n,k}(t)}{dt} = -(\lambda+\mu)P_{n,k}(t) + \mu Q_k P_{n+1,k}(t) +$$

$$\sum_{j=1}^{n} \lambda P_j P_{n-j,k}(t) + \mu P_{n,k+1}(t) + \lambda Q_k P_n P_0(t),$$

$$n \geq 1 \qquad (7)$$

In case when zero units in the system at time t + Δt, the probability may be expressed as:

1. The product of probabilities that
   - 1.1. There are zero units in the system at time t — $P_0(t)$
   - 1.2. There are no arrivals during the Δt interval — $[1-\lambda\Delta t]$
2. The product of probabilities that
   - 2.1. There is one unit in the system at time t — $P_{1,1}(t)$
   - 2.2. There are no arrivals during the Δt interval — $[1-\lambda\Delta t]$
   - 2.3. There is one unit served during the Δt interval — $\mu\Delta t$

Upon adding these two independent probabilities we obtain the probability of zero units in the system at time t + Δt

$$P_0(t+\Delta t) = P_0(t)[1-\lambda\Delta t] + P_{1,1}(t)[1-\lambda\Delta t]\mu\Delta t$$

$$\frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = -\lambda P_0(t) + \mu P_{1,1}(t)$$

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_1(t) \qquad (8)$$

Equations (7) and (8) represent the queueing system mentioned above. In the steady-state condition, equations (7) and (8) become

$$-(\lambda+\mu)P_{n,k} + \mu Q_k P_{n+1,k} + \sum_{j=1}^{n} \lambda P_j P_{n-j,k} + \mu P_{n,k+1} +$$

$$\lambda Q_k P_n P_0 = 0 , \qquad n \geq 1 \qquad (9)$$

$$-\lambda P_0 + \mu P_1 = 0 \qquad (10)$$

Equations (9) and (10) may be solved to obtain the values of the various probabilities, and then the measures of effectiveness can be computed.

4.3. A Queueing System in Hospital

In hospitals, a scheduling problem, particularly the appointment system for the outpatients of a hospital can be reviewed as a queueing system. Appointment systems can be classified into four general types.

(1) Pure block appointment systems

(2) Individual appointment systems

(3) Mixed block-individual appointment systems

(4) Other appointment systems.

A block appointment system arranges for an initial group of patients to arrive at the clinic at the beginning of the clinic session. The rest of the patients, scheduled to be treated during the same clinic session, are arranged to arrive at the clinic at equally spaced intervals. In individual appointment systems patients are given appointments on an individual basis. The appointments are equally spaced in time and the interval between successive appointments known as the appointment interval, is equal to the mean consultation time.

Here two different appointment systems are considered;

System I - Individual appointment system

System II - Two at a time appointment system

The purpose of this application is to compare corresponding steady-state waiting time distribution functions of the outpatients under above two different arrival patterns. For these two systems Soriano [57] has investigated under different load factors, ρ given below:

ρ =	0.5	0.6	0.7	0.8	0.85	0.9	0.95	0.99

On the basis of batch arrival queueing model discussed in Chapter III, the waiting time distributions for systems I and II are discussed.

Assumptions:

1. Patients arrive on time

2. Medical consultation times of various outpatient clinics were distributed according to gamma distribution.

3. Appointment time will be used as the unit in the scale.

Using equation (12) in section 3.5, the waiting time distribution for system I is written such that

$$F(x) = 1 + \sum_{j=1}^{4} c_j \, e^{z_j x}$$

where the value of c_j and z_j are calculated from

$$\frac{(\mu/2)^4}{[\mu/2 + z'_j]^4} = e^{-z'_j}$$

and

$$\sum_{j=0}^{4} \left\{ \frac{c_j}{[\mu/2 + z'_j]^k} \right\} = 0 \,, \quad k = 1, 2, 3, 4$$

where

$$c_o = 1, \qquad z_o' = 0$$

and

$$z_j = 2z_j'$$

Similarly, for system II, the waiting time distributions of the 1st and 2nd patients in a batch, $F_1(x)$, $F_2(x)$, respectively are given by

$$F_1(x) = 1 + \sum_{j=1}^{8} c_j e^{z_j x} \quad ,$$

and

$$F_2(x) = \frac{\mu^4}{3!} \int_o^x F_1(x-y) y^3 e^{-\mu y} dy \quad .$$

where the value of c_j and z_j are calculated from

$$\frac{\mu^8}{(\mu+z_j)^8} = e^{-z_j}$$

and

$$\sum_{j=0}^{8} \left\{ \frac{c_j}{[\mu + z_j]^k} \right\} = 0 , \qquad K = 1, 2, \ldots 8$$

where $c_o = 1$ and $z_o = 0$.

For the analysis of the above, refer to [57], p. 104. Soriano [57] has tabulated and plotted the values of $F(x)$, $F_1(x)$, and $F_2(x)$ in Tables 5-12 and Figures 2-9 for the different load factors, respectively. For example, the table showing the values of $F(x)$, $F_1(x)$ and $F_2(x)$ for load factor, $\rho = 0.70$ is given in Table 4.1.

Table 4.1.
Waiting Time Distributions for the $\rho = 0.70$ Case

x	$F_1(x)$	$F_2(x)$	$F(x)$
0.0	0.852	0.0	0.742
0.1	0.909	0.042	0.848
0.2	0.946	0.187	0.913
0.3	0.968	0.407	0.951
0.4	0.982	0.618	0.973
0.5	0.990	0.775	0.985
0.6	0.994	0.876	0.992
0.7	0.997	0.929	1.000
0.8	0.998	0.962	
0.9	0.999	0.980	
1.0	1.000	0.991	
1.5		1.000	

Generally speaking, there are not so much difference between the values of the waiting time distribution functions $F(x)$ and $F_1(x)$. $F(x)$ remains always between $F_1(x)$ and $F_2(x)$ as it must from the

theoretical point of view. As the load factor, , is increased the difference among $F(x)$, $F_1(x)$, and $F_2(x)$ decreases. Waiting times of the corresponding cases increase as the load factor increases. For more details refer to [57], pp. 81-94.

4.4. A Queueing System for an Elevator

The purpose of elevator is to transport goods or customers from one level to other level. There are some restrictions due to the capacity of elevator. Also sometimes there may not be any customers or goods for transporting. In this case owner would like to select a way so that the cost incurred is minimum. Also the space allotted for waiting goods or customers on each level is optimum.

In this case we see that service is in bulk. Service time from one level to other level is constant. Arrival of the goods or customers may be of single or batch type but in general it is of single type. This problem can be analyzed by the help of queueing.

This can also be simulated to give the optimum number of goods which shall be transported from one level to other level. For example, manufacturer uses a freight elevator to transport certain parts from the first to the second floor and different parts from the second to the first floor. The arrival time distribution for the parts destained for the second floor is exponential with a mean of 2.8 minutes. The arrival time distribution for the parts destained for the first floor is exponential with a mean of 3.3 minutes.

The cost associated with waiting parts is $2.00 per hour per part for those to be transferred from the first to the second floor and $2.25 per hour per part for those to be transferred from the second to the first floor.

Elevator capacity is 12 parts which are loaded on the first floor and 8 parts which are loaded on the second floor.

Elevator cost $0.25 to run one way.

Service time = one minute for a trip.

In this example we would like to determine the optimal number of parts to load on each floor before transporting them to other floor so that cost is minimum.

This example is simulated on 360/50 computer using GPSS IV language.

Upon examining the table 4.2., it appears that the minimum transportation cost is $95.05. Consequently, the corresponding number of parts loaded on the first and the second floors are 2, 2 respectively.

Table 4.2.

Cost Incurred for Various Combinations

S. No.	Parts Loaded on First Floor	Parts Loaded on Second Floor	Cost in $
1	1	1	111.40
2	1	2	104.80
3	1	3	116.80
4	1	4	164.20
5	1	5	242.80
6	2	1	101.40
7	2	2	95.05
8	2	3	114.90
9	2	4	144.80
10	2	7	127.00
11	2	8	460.40
12	3	1	118.60
13	3	2	101.50
14	3	3	95.80
15	3	4	145.90
16	3	8	458.60
17	4	2	102.70
18	4	3	134.90
19	4	4	186.90
20	5	4	134.90
21	5	5	152.40
22	5	6	309.60
23	5	7	119.80
24	5	8	452.60
25	6	5	148.00
26	6	6	273.40
27	6	7	113.60
28	6	8	449.70
29	7	6	273.40
30	7	7	117.70
31	8	2	438.60
32	8	6	443.80
33	9	5	892.20
34	8	7	148.70
35	10	7	284.00
36	11	7	586.50
37	12	8	984.20

APPENDIX A

COMPUTER PROGRAM

This appendix includes the computer program of an elevator process presented in subsection 4.4, as a queueing system. The computer program has been written on IBM 360/50 computer using GPSS IV language.

```
BLOCK
NUMBER   *LOC    OPERATION  A,B,C,D,E,F,G                COMMENTS
                SIMULATE
         *  TIME IN MINUTES
            ABC TABLE      MP1,0,1,20                   WAITING TIME ON FIRST FLOOR
            DEF TABLE      MP2,0,1,20                   WAITING TIME ON SECOND FLOOR
              1 FUNCTION   RN1,C24
         0      0     .1    .104  .2    .222  .3    .355  .4    .509  .5    .69
         .6     .915  .7    1.2   .75   1.38  .8    1.6   .84   1.87  .88   2.12
         .9     2.3   .92   2.52  .94   2.81  .95   2.99  .96   3.2   .97   3.5
         .98    3.9   .99   4.6   .995  5.3   .998  6.2   .999  7.0   .9997 8.0
                INITIAL    X1,2/X2,2/X3,50
         *      COSTS ARE MULTIPLIED BY 1000
              1 FVARIABLE  333*MP1          COST FOR TRANSFERRING TO 2
              2 FVARIABLE  373*MP2          COST FOR TRANSFERRING TO 1
         *BOTH OF ABOVE COSTS ARE MULTIPLIED BY 1000
              3 VARIABLE   X3+X4
              1 STORAGE    12
              2 STORAGE    8
1                GENERATE   3,FN1
2                MARK       1
3                QUEUE      1,1
4                GATE LR    1
5                ENTER      1,1
6                TEST GE    S1,X1,LAX
7                TEST NE    S1,Q1,MAN
8                TEST NE    S1,K12,MAN
9           LAX  TERMINATE
10          MAN  LOGIC I    1
11               DEPART     1,S1
12               TABULATE   ABC
         *       SAVEVALUE  3 -INITIALIZED TO INCLUDE COST OF 200 TRIPS = 50
13               SAVEVALUE  3+,V1
14               ADVANCE    1
15               LEAVE      1,S1
16               LOGIC I    2
17               TERMINATE
18               GENERATE   4,FN1
19               MARK       2
20               QUEUE      2,1
21               GATE LS    2
22               ENTER      2,1
23               TEST GE    S2,X2,BOT
24               TEST NE    S2,Q2,TOT
25               TEST NE    S2,K8,TOT
26          BOT  TERMINATE
27          TOT  LOGIC I    2
28               DEPART     2,S2
29               TABULATE   DEF
30               SAVEVALUE  4+,V2
31               SAVEVALUE  5,V3
32               ADVANCE    1
33               LEAVE      2,S2
34               LOGIC I    1
35               TERMINATE  1
                 START      200
                 REPORT
```

BLOCK NUMBER	SYMBOL	REFERENCES BY CARD NUMBER	
26	BOT	41	
9	LAX	23	
10	MAN	24	25
27	TOT	42	43

ABLE SYMBOLS AND CORRESPONDING NUMBERS

1	ABC
2	DEF

```
* TIME IN MINUTES
1       TABLE       MP1   0      1      20
2       TABLE       MP2   0      1      20
1       FUNCTION    RN1   C24
0       0     .1    .104  .2     .222   .3     .355   .4     .509   .5     .69
.6      .915  .7    1.2   .75    1.38   .8     1.6    .84    1.87   .88    2.12
.9      2.3   .92   2.52  .94    2.81   .95    2.99   .96    3.2    .97    3.5
.98     3.9   .99   4.6   .995   5.3    .998   6.2    .999   7.0    .9997  8.0
        INITIAL     X1,2/X2,2/X3,50
*       COSTS ARE MULTIPLIED BY 1000
1       FVARIABLE   333*MP1
2       FVARIABLE   373*MP2
*BOTH OF ABOVE COSTS ARE MULTIPLIED BY 1000
3       VARIABLE    X3+X4
1       STORAGE     12
2       STORAGE     8
1       GENERATE    3       FN1
2       MARK        1
3       QUEUE       1       1
4       GATE LR     1
5       ENTER       1       1
6       TEST GE     S1      X1      9
7       TEST NE     S1      Q1      10
8       TEST NE     S1      K12     10
9       TERMINATE
10      LOGIC1      1
11      DEPART      1       S1
12      TABULATE    1
*       SAVEVALUE 3 -INITIALIZED TO INCLUDE COST OF 200 TRIPS = 50
13      SAVEVALUE   3+      V1
14      ADVANCE     1
15      LEAVE       1       S1
16      LOGIC1      2
17      TERMINATE
18      GENERATE    4       FN1
19      MARK        2
20      QUEUE       2       1
21      GATE LS     2
22      ENTER       2       1
23      TEST GE     S2      X2      26
24      TEST NE     S2      Q2      27
25      TEST NE     S2      K8      27
26      TERMINATE
27      LOGIC1      2
28      DEPART      2       S2
29      TABULATE    2
30      SAVEVALUE   4+      V2
31      SAVEVALUE   5       V3
32      ADVANCE     1
33      LEAVE       2       S2
34      LOGIC1      1
35      TERMINATE   1
        START       200
```

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BULK QUEUEING MODELS AND THEIR APPLICATIONS

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AN ABSTRACT OF A MASTER'S REPORT

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This report is concerned with the bulk queueing models and their applications. The basic concepts of queueing theory are discussed. The structure of the queueing processes is presented. Various bulk queueing models are analyzed. Four models are of bulk service type, and one is of bulk arrival type. The models are analyzed under transient and steady-state conditions. Several applications to bulk queueing theory such as systems in iron and steel industry, glass industry, and hospitals are discussed. An elevator process is analyzed as a queueing system and then simulated on IBM 360/50 computer using GPSS IV language.