Wall-crossing structures in Seiberg-Witten integrable systems

by

Qiang Wang

B.S., Xiamen University, Xiamen, China, 2008

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas

2019

Abstract

Wall-crossing structure (WCS) is a formalism proposed and studied by M.Kontsevich and Y.Soibelman that enables us to encode the Donaldson-Thomas (DT) invariants (BPS degeneracies in physics) and to control their "jumps" when certain walls (walls of marginal stability in physics) on the moduli space are being crossed. The celebrated *Kontsevich-Soibelman wall-crossing formulas* (KSWCF) are the essential ingredients of WCS.

WCS formalism is well adapted to the data coming from the complex integrable system. The famous Seiberg-Witten (SW) integrable system is an example. By considering certain gradient flows on the base of the integrable system called the *split attractor flows*, WCS can produce an algorithm for computing the DT-invariants inductively.

This dissertation is about applying the WCS to the SW integrable systems associated to the pure SU(2) and SU(3) supersymmetric gauge theories. We will see that the results via the WCS formalism match perfectly well with those obtained via physics approaches. The main ingredients of this algorithm are the use of *split attractor flows* and KSWCF. Besides the known BPS spectrum in pure SU(3) case, we obtain new family of BPS states with BPS-invariants equal to 2.

Wall-crossing structures in Seiberg-Witten integrable systems

by

Qiang Wang

B.S., Xiamen University, Xiamen, China, 2008

A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas

2019

Approved by:

Major Professor
Yan Soibelman

Copyright

Abstract

Wall-crossing structure ("WCS" for short) is a formalism proposed and studied by M.Kontsevich and Y.Soibelman that enables us to encode the Donaldson-Thomas (DT) invariants (BPS degeneracies in physics) and to control their "jumps" when certain walls (walls of marginal stability in physics) on the moduli space are being crossed. The celebrated *Kontsevich-Soibelman wall-crossing formulas* ("KSWCF" for short) are the essential ingredient of WCS.

WCS formalism is well adapted to the data coming from the complex integrable system. The famous Seiberg-Witten integrable system ("SW integrable system" for short) is an example. By considering certain gradient flows on the base of the integrable system called the *split attractor flows*, WCS can produce an algorithm for computing the corresponding DT-invariants inductively.

This dissertation is about applying the WCS to the SW integrable systems associated to the pure SU(2) and SU(3) supersymmetric gauge theories. We will see that the results via the WCS formalism match perfectly well with those obtained via physics approaches. The main ingredients of this algorithm are the use of *split attractor flows* and KSWCF. Besides the known BPS spectrum in pure SU(3) case, we obtain new family of BPS states with BPS-invariants equal to 2.

Contents

List of Figures

Acknowledgments

First and foremost, I would like to express my sincere gratitude to my advisor Yan Soibelman, for his constant support during the period of my Ph.D study. His insights into mathematics inspired me to appreciate more about the intricacies and excitements about the subject; his proper guidance helped me in overcoming the obstacles that I had encountered when studying mathematics, especially in the process of writing this thesis. It is definitely my great honour to have him as my advisor.

I thank professor Gabriel Kerr and professor Ilia Zharkov for their valuable comments when I gave seminar talks, which enabled me to realize where further improvements should be made. Besides, their brilliant presentations on various topics in algebraic geometry had helped me in bridging the gap between the abstract apparatus and the geometric intuition. Their efforts deserved my great admiration and appreciation.

I thank professor David Auckly for his guidance both for my study and for my teaching. I also benefited enormously from his reading course on Gromov-Witten invariants which was just fabulous.

I would especially like to thank professor Zongzhu Lin for his invaluable advice on both research as well as on my career. I also thank Professor Larry Weaver for being a member of my defense committee, and for the book on asymptotic expansion that he gave me as a gift on our first meeting.

I wish to thank my parents for their endless care and encouragement for all these years. Special thank goes to my wife, Zhenhua Sun, for her enduring love and accompanying, and also to our beloved daughter, Sienna, for the unspeakable happiness she had brought to our family. This human touch, rooted in family cohesion, is the perpetual momentum that pushes my life forward.

Chapter 1

Introduction

1.1 Outline of the thesis

The study of BPS spectra in 4d N=2 Super-symmetric field theory is the central theme in modern theoretical physics (see [Moo12b] for a review). During the last decades, there has been tremendous progress in understanding the spectra in a large class of N=2 theories. The key in these developments lies in the gradual understanding of the so-called *wall-crossing phenomena* for the BPS-invariants (corresponding to *Donaldson-Thomas invariants* in mathematics) associated to the BPS-states. In particularly, the celebrated *Kontseivich-Soibelman Wall-Crossing Formula* (see [KS08]) had been proved to be the essential ingredients in understanding the structures of the BPS spectra. Moreover, it is believed that the wall-crossing formula gives universal solutions for all problems involving wall-crossing phenomena.

It is suspected that any N=2 theory would give rise to a *complex integrable system* (see [Nei14a] for a short exposition). The famous Seiberg-Witten solution to the $N = 2, d = 4$ super Yang-Mills theory is a particular example of this scheme ([SW94]), whence came with the notion of *Sieberg-Witten Integrable System* that had been widely investigated in various perspectives (see example [Mar99] and [Ler98] for the physics background and [Don97] for a mathematical treatment). For any such a theory, we have a complex manifold $\mathcal{B}$, called the *moduli space of vacua* in the physics literature. Over $\mathcal{B}$, it is endowed with a local system of lattices $\underline{\Gamma}$, which is called the *charge lattice*. Most importantly, for the cases being considered in this thesis, we are given a *central charge function* Z, which is simply a linear map from $\underline{\Gamma}$ to $\underline{\mathbb{Z}}$ (understood as constant local system $\underline{\mathbb{Z}}$ over $\mathcal{B}$).

Thus, it would be natrual and desirable to have a formalism encoding the wall-crossing phenomena for Donaldson-Thomas invariants (DT invariants for short) that is compatible with the data coming from complex integrable system. This is exactly the *wall-crossing structures* (WCS for short) that had been established in [KS14] by Kontsevich and Soibelman. In fact, WCS generalizes the notion of *stability data* that had been thoroughly treated in [KS08].

WCS formalism, when applied to the Seiberg-Witten integrable systems, will produce an algorithm for computing the DT-invariants associated to the systems. The most important ingredients of the WCS are the celebrated Kontsevich-Soibelman Wall-Crossing Formulas (KSWCF for short).

Roughly speaking, we will consider certain gradient flow lines associated to the central charge function Z on the base $\mathcal{B}$ of the integrable system. Following the terminology of physicists, we call these gradient flow lines the *split attractor flows*. The term "split" here refers to the fact that these flow lines will split into several branches when hitting certain co-dimensional one walls on the base $\mathcal{B}$. These walls are called the *walls of the first kind* (or *walls of marginal stability* in physics literature). After (possibly infinite many times of) such splitting, the flow lines will form a rooted tree on the base. Then, we assign the DT-invariants to the terminal points of the tree, called *initial data of WCS*, and move toward to the root of the tree. Under the good condition that the number of attractor trees is finite, we use the KSWCF at the split points. In this way, we will eventually arrive at the value of DT-invariant at the root of the tree, as well as the charge vector associated to the root.

This thesis is mainly about the application of the WCS formalism to the Seiberg-Witten Integrable Systems (SW Integrable Systems for short) with pure gauge group $SU(3)$. Our results show that WCS formalism is proven to be a very effective way to compute the DT-invariants (or BPS invariants in physics) algebraically. Not only the large sector of the known BPS spectra existing in physics literature (see for examples: [Hol97][FH97][CDP11][Tay01]) can be recovered in this way, but also certain BPS states with invariants 2 could also be obtained as a by product.

The contents of this dissertation are outlined as bellow:

In chapter two, we will give a detailed exposition of the wall-crossing structure formalism following the original paper [KS14] of Kontsevich and Soibelman.

In section 3.1 of the chapter three, we introduce the notion of complex integrable systems including the construction of the action-angle coordinates and their relation to the central charge. We will focus more on the geometry of the base of the integrable system, and show that the base is naturally endowed with the S^1-family of the *integral affine structures*.

In section 3.2, we review the basics notion of split attractor flows, as well as the axiomatic treatment by Kontsevich and Soibelman in [KS14]. We will discuss its relation to WCS in sub-section 3.2.3. At the end of the chapter three, we will display the relationship between the attractor flows and the *Hesse flows* by performing the explicit computations in the frame work of the *special Kähler geometry*, and we point out its relevance in Mirror symmetry.

Chapter four is about the geometry of the Seiberg-Witten integrable system. We collect many discussions and treatments scattered in the physics literature and rearrange them in a consistent fashion. In particularly, we will show several equivalent descriptions of the Seiberg-Witten integrable systems in terms of the so called *Seiberg-Witten curves* in section 4.1. And in section 4.2, we study the *vanishing cycles* and the associated monodromies of the $SU(2)$ and $SU(3)$ Seiberg-Witten integrable systems from various points of views. These results will be used in chapter five to give the *initial data* of the corresponding Wall-crossing structures. This section contains a lot of materials that are irrelevant to the last chapter and the previous chapters, thus, except the results to be cited for later use, this section is largely independent of the rest of the thesis.

Finally, in chapter five, we will construct the WCS for the Seiberg-Witten integrable systems associated to $SU(2)$ and $SU(3)$ respectively. The ingredients for constructing the WCS in $SU(2)$ case are essentially known in physics literature, we just reformulate them in the framework of the WCS formalism in section 5.1. This section can also be viewed as a warm up excises for section 5.2 in which the WCS for the Seiberg-Witten integrable system in $SU(3)$ case was constructed. The WCS in this case was constructed by reducing the situation to the $SU(2)$ case that had already been constructed in 5.1.

There are two appendices intending to provide the readers with the necessary physics backgrounds that are relevant to the main content of this dissertation. Appendix A provides a short exposition of the notion of BPS states in $4d, N = 2$ supersymmetric Yang-Mills theories. In appendix B, we give some motivations for the introduction of the split attractor flows in supergravity.

1.2 Summary of the main results

The most important result obtained in this these is the construction of the WCS for SW integrable system with gauge group $SU(3)$. This consists two parts: the WCS in *strong coupling region* and WCS in *weak coupling region*. The WCS in *strong coupling region* is discussed in section 5.2.4, while that in *weak coupling region* is investigated in section 5.2.5.

We showed in proposition 5.2.2 in section 5.2.5. that the walls of the first kind $\mathcal{W}_k^1$, when restricted to the plane $\mathcal{H}$ that cut the base $\mathcal{B}$ of the integrable system, is topologically homeomorphic to the circle S^1, and plays the same role as the wall of stability in $SU(2)$ case (see figure 5.24 there for illustration).

Consequently, the construction of WCS in $SU(3)$ case can be reduced to the similar situation that had been constructed for the $SU(2)$ case in section 5.1.

In particularly, in proposition 5.2.3, we will obtain easily by using the WCS formalism the following three families of BPS states and the associated DT-invariants that match those obtained through physics approach (see for example [FH97]) :

$$\begin{cases} n\nu_k^+ + (n-1)\nu_k^-, \quad k = 1,2,3;\ n = 1,2,\cdots \\ \nu_k^+ + \nu_k^-, \quad k = 1,2,3; \\ (n-1)\nu_k^+ + n\nu_k^-, \quad k = 1,2,3;\ n = \cdots - 3, -2, -1, 1. \end{cases}$$

All states with DT-invariants $\Omega = 1$, except for the middle row states, which have DT-invariants $\Omega = -2$. The split attractor flows that "represent" these states are illustrated in Figure 5.26 and Figure 5.27.

Moreover, since in the weak coupling region, there exists another wall $\mathcal{W}^1_{\alpha_1,\alpha_2}$ (of the first kind) that had been pointed in [Hol97] and [Kuc08], the attractor flows will split when hitting this wall (see figure 5.28). By applying the WCS formalism to this situation, we proved in proposition 5.2.6 that there exists new BPS states (denoted by γ_3 in the below) with DT-invariants jumps from one to two when crossing this wall.

The relevant KSWCF to be used at the split point is given as below.

Using short notations:

$$\begin{cases} \gamma_1 := (\alpha_1, n\alpha_1) \\ \gamma_2 := (\alpha_2, (n-1)\alpha_2) \\ \gamma_3 := (\alpha_3, \alpha_1 + (n-1)\alpha_3) \end{cases}$$

where α_1, α_2 and α_3 denote the three positive roots of the Lie algebra $\mathfrak{su}_3$, we have that the KSWCF at the split point is given by:

$$\mathcal{K}_{\gamma_1}\mathcal{K}_{\gamma_3}\mathcal{K}_{\gamma_2} = \Xi^+\, \mathcal{K}^2_{\gamma_3}\mathcal{K}^{-2}_{2\gamma_3}\, \Xi^-\, \mathcal{K}_{\gamma_1}$$

where

$$\Xi^{+} := \prod_{n=1}^{\infty} \mathcal{K}_{n\gamma_2+(n-1)(\gamma_1+\gamma_3)}\, \mathcal{K}_{n\gamma_2+(n-1)(\gamma_1+\gamma_3)+\gamma_3}$$

$$\Xi^{-} := \prod_{n=\infty}^{1} \mathcal{K}_{(n-1)\gamma_2+n(\gamma_1+\gamma_3)+\gamma_3}\, \mathcal{K}_{(n-1)\gamma_2+n(\gamma_1+\gamma_3)}$$

We also pointed out in remark 5.2.6 that the jump of the DT-invariants γ_3 computed above by using KSWCF is compatible with the so called *"primitive wall-crossing formula"*:

$$\Delta\Omega(\gamma_3 \to \gamma_1 + \gamma_2) = (-1)^{\langle\gamma_1,\gamma_2\rangle+1} |\langle\gamma_1, \gamma_2\rangle|\, \Omega(\gamma_1)\, \Omega(\gamma_2) = 1$$

• In section 5.1.4 and section 5.1.5, we reformulated the scattered facts in physics literature in terms of the language of the WCS formalism, thus produced the WCS for the SW integrable system with gauge group $SU(2)$. The BPS states, as well as the associated DT-invariants, produced by using WCS formalism, turns out to match the results that had been obtained by physics approaches perfectly.

1.3 Summarize of miscellaneous results

• In proposition 2.2.1 of section 2.2.3, we displayed that the KSWCF can be formally interpreted as the *triviality of the monodromy* around any small loop (already pointed out in section 2.3 of [KS08]), i.e., for any short loop, the monodromy

$$\prod_{t_i}^{\circlearrowleft} \mathbb{S}(l_{\gamma_i}) = id$$

where the product is taken in the increasing order of elements t_i. And in lemma 2.3.5 of section 2.3.3, we generalize it to the nilpotent case.

• In section 3.1.4, we introduced the holomorphic coordinates $\{z_1, \cdots, z_{2n}\}$ for the complex integrable system that are the complex analogy of the action coordinates $\{I^1, \cdots, i^{2n}\}$ in the real case. In particular, we showed in proposition 3.1.21 the relation between these holomorphic coordinates and the action-angle coordinates (I^i, θ_i) reads as

Write $w_i = e^{z_i}$ and suppose that $dz_i = \alpha_i$, then we have that

$$I^i = Re(\log w_i) = log|w_i| \quad \theta_i = Im(\log w_i).$$

• In the proof of the proposition 3.2.17 of section 3.2.3, we explicitly described how the WCS on the base $\mathcal{B}$ of the integrable system gives rise to an local embedding $\mathcal{B}^0 \hookrightarrow Stab(\mathfrak{g}_b)$ for each $b \in \mathcal{B}^0$.

• In section 3.2.4, we introduced the notion of *Hesse flow* proposed in [Van12], and compare it with the notion of *split attractor flow*. To this end, we also introduced the notions of *dual attractor flow* and the *dual Hesse flow*, then we proved the following result:

Start with central charge function $Z(\gamma)$ associated to the charge γ, and use the $\mathbb{Z}$-affine coordinates on $\mathcal{B}$, we see that taking the real part of $Z(\gamma)$ gives us the Hesse flow

$$Re\left(e^{-i\theta} Z(\gamma)\right) = \widetilde{\nabla}\widehat{\mathscr{F}^y} \cdot \gamma$$

while taking the imaginary part gives us the dual Hesse flow

$$Im\left(e^{-i\theta} Z(\gamma)\right) = \widetilde{\nabla}\widehat{\mathscr{F}^x} \cdot \gamma$$

Next, denote $\theta = Arg\left(Z(\gamma)\right)$. We see that in the $\mathbb{Z}$-affine structure $\mathcal{B}_\theta$, the dual Hesse flow specializes into the attractor flow; while in the $\mathbb{Z}$-affine structure $\mathcal{B}_{\theta+\frac{\pi}{2}}$, the Hesse flow becomes the dual attractor flow.

• In section 4.1.2, we displayed in the proof of the proposition 4.1.4 how to construct the *Seiberg-Witten differentials* by using the Abelian differentials of the first kind (see for example the book [ACG11]):

$$\omega_k = \frac{x^{n-k}dx}{y} \qquad k = 2, \cdots, n$$

on the *Seiberg-Witten elliptic curves*. In particular, by choosing suitable linear combination of the Abelian differentials of the first kind, we reproduced the following form of the Seiberg-Witten differential as given in the physics papers (for example [Ler98],[Kle+94]):

$$\lambda_{SW} = constant \cdot \left(\frac{\partial}{\partial x} \mathcal{W}_{A_{n-1}}(x, u_i) \right) \frac{x\, dx}{y} \tag{1.3.1}$$

up to an addition of exact form.

• In section 4.1.3, we represented the Seiberg-Witten curves as weight diagram fibrations, and we showed in proposition 4.1.7 that in this case, the Seiberg-Witten differential 1.3.1 above reduces into the following form

$$\lambda_{SW} = -x \frac{dz}{z} \tag{1.3.2}$$

• In proposition 4.1.11 of section 4.1.3, we showed that the monodromy around the branch points $z_i^{\pm}$ is given by the fundamental Weyl reflection M_{α_i} associated to the simple roots α_i of the gauge group $SU(n)$.

• In section 4.1.4, we study the geometry of Seiberg-Witten systems in terms of the K3-fibrations by lifting the weight diagram fibrations into higher dimension. We performed explicit computations to show the compatibility of the two approaches, which can be summarized as saying that the SW geometry, being encoded by the fibration of weight diagram over S^2, which is linearized by the local system of lattices $H_0(\mathscr{M}, \mathbb{Z})$ over $\mathbb{C}P^1$, can be obtained by the degeneration of the ALE space fibration over S^2 by letting y and z equal to zero in the defining equation (in physics term, it is called integrate out the variables) of the K3-fibration. In other words, the local system of lattices $H_2(ALE, \mathbb{Z})$ over $\mathbb{C}P^1$ encodes the same information as that of $H_0(\mathscr{M}, \mathbb{Z})$. The essential information is the root lattice Λ_R of type A_{n-1}. Indeed, the intersection matrix of the vanishing cycles in both cases equal to the negative of the Cartan matrix for $SU(n)$.

Besides, in proposition 4.1.15, we computed that by integrating the unique (up to scalar multiplication) holomorphic volume form

$$\Omega = \frac{dz}{z} \wedge \frac{dx \wedge dy}{\partial_w \mathcal{W}^{ALE}_{A_{n-1}}(x, y, w, z)} \in H^{3,0}(X, \mathbb{C})$$

over the vanishing 2-spheres S_{ij}, we will produce essentially the Seiberg-Witten differential given by 1.3.2 above, namely we have that up to multiplication by some constant, we have that

$$\iint_{S_{ij}} \Omega = (e_{\lambda_i} - e_{\lambda_j}) \frac{dz}{z}$$

In proposition 4.1.18, we clarified the relation between the periods of the CY 3-fold defined by the K3-fibrations and the periods of the Seiberg-Witten curves, i.e., for $\gamma \in H_3(X, \mathbb{Z})$, we have the following identity

$$Z^{CY}(\gamma) = \int_\gamma \Omega = \int_{\phi(\gamma)} \lambda_{SW} = Z(\phi(\gamma))$$

where the map $\phi : H_3(X,\mathbb{Z}) \longrightarrow H_1(\mathcal{C},\mathbb{Z})$ is the pushed forward map on the level of homology.

• In the $SU(2)$ case, the base of the integrable system can be identified with $\mathbb{C}P^1 \cong S^2$, we endow the base with the metric given by

$$ds^2 = |\lambda_{SW}|^2 = g_{z\bar{z}}\, dz d\bar{z} = (x(z)/z)^2\, dz d\bar{z}$$

Introducing the flat coordinate on the base given by $z \equiv \int^z \lambda_{SW}$, then the geodesic line equation in terms of the coordinate z reads

$$\sqrt{2u - z - \frac{\Lambda^2}{z}}\, z^{-1} \frac{\partial z(t)}{\partial t} = \alpha \tag{1.3.3}$$

Then we have the the following observation given in proposition 4.1.19:

Let $Z : \underline{\Gamma}_b := H_1(\mathcal{C}_b,\mathbb{Z}) \to \mathbb{Z}$ be the central charge function of the SW integrable system. Consider the wall of second kind defined by

$$Im(Z_b(\gamma)) = 0$$

for some charge $\gamma \in \underline{\Gamma}_b$. If the above equation is viewed as defining a curve in the base $\mathcal{B}$ of the SW integrable system, then it gives the attractor flow equation. However, if we present the SW curve as the fibration of weight diagram over $\mathbb{C}P^1$, then in terms of the base coordinate z, the wall equation above gives the geodesic line equation 1.3.3 on $\mathbb{C}P^1$.

• In section 4.1.5, we introduced the notion of *spectral network*, and compared it with the notion of split attractor flow in $SU(n)$ case when the base can be identified with $\mathbb{C}P^1$. We argued in proposition 4.1.21, that the spectral network on the base contains the same amount of information as that of the split attractor flows so long as the determination of the BPS charges (states) are concerned. We also proved the *balancing condition* for spectral network in proposition 4.1.22.

• In section 4.2.1, by using the expression of the *one-loop prepotential function*

$$\mathcal{F}_{\text{1-loop}} = \frac{i}{6\pi} \sum_{i<j} u \log\left((e_i - e_j)^2/\Lambda^2\right)$$

we performed the explicit computation to verify the known *semi-classical monodromies* in $SU(3)$ case ([KTL95]). See equation 4.2.12 and equation 4.2.16 in that section.

• In section 4.2.2, we gave very through treatment for the *quantum monodromy* for the $SU(2)$ case. For example, by using the Seiberg-Witten curve and Seiberg-Witten differential in the form of weight diagram fibration (see equation 4.2.21 and equation 4.2.22), we reproduced the Semi-classical monodromy by using the prepotential function. And we computed in the proof of the proposition 4.2.1 that the period integrals have the desired semi-classical approximation. Further more, we also computed the vanishing cycle ν_∞ at ∞ of the curve analytically by comparing the two equivalent forms of the Seiberg-Witten elliptic curve in $SU(2)$ case.

• In section 4.2.2, we displayed the compatibility of the Picard-Lefschetz formula with the monodromies computed in terms of the prepotential. See proposition 4.2.8 in that section for more details. We also give two proofs of the fact that the monodromies associated to two non-intersecting vanishing cycles commute with each other (see proposition 4.2.9), one by using Picard-Lefschetz formula directly, the other by manipulating algebraically the equations involving the tensor product of matrices.

• In section 4.2.3, we used the *Zariski-Van Kampen theorem* to compute the fundamental group of the complement of the cups curve (i.e., the discriminant locus) in $\mathbb{C}^2$ in proposition 4.2.21, which is used in section 4.2.4 to exhibit the relations among the quantum monodromy matrices in $SU(3)$ case, see equation 4.2.97 and remark 4.2.21 for more details.

Chapter 2

Introducing Wall-Crossing Structures

Wall-Crossing Structure ("WCS" for short), which had been proposed and studied by Kontseivich and Soibelman in [KS14], arises in the study of $N = 2$ super-symmetric quantum field theories in physics. On the mathematics side, it gives a natural framework for dealing with the so-called *Donaldson-Thomas invariants* ("DT-invariants" for short), which in the physics literature, corresponds to the *BPS-invariants* (see Appendix A for more information).

This chapter is organized as below.

We begin by reviewing the concept of *stability data* of [KS08] in section 2.1. This serves to motivate the concept of wall-crossing structure, as well as to introduce the necessary terminologies that will be used later. We will see how the celebrated *Kontsevich-Soibelman Wall-Crossing Formula* ("KSWCF" for short) could be written down explicitly in the formalism of WCS.

After this, we can generalize the concept of stability data to that of the *Wall-crossing structure* in section 2.3, which would be our main concern in this thesis. Moreover, we will see how KSWCF could be reinterpreted formally as kind of *cocycle condition* in the WCS formalism. The description of WCS in terms of sheaves will be discussed in section 2.4, which is useful when considering in section 2.5 WCS on general topological space.

Finally, in section 2.6, we give some examples of WCS.

2.1 Stability data

Roughly speaking, a stability data on a graded Lie algebra $\mathfrak{g}$ gives us a natural way to encode the set of Donaldson-Thomas (DT) invariants, and the space of stability data (called *stability data* on $\mathfrak{g}$) would enable us to describe how these invariants behave (KSWCF) when the central charge is being varied. The treatment in this subsection follows [KS08].

2.1.1 Donaldson-Thomas invariants

Fix a finite rank free abelian group $\Gamma \cong \mathbb{Z}^{\oplus n}$, called the **charge lattice**. We endow it with a skew symmetric bilinear form

$$\langle \cdot, \cdot \rangle : \Gamma \times \Gamma \longrightarrow \mathbb{Z} \tag{2.1.1}$$

The **BPS invariants** (or BPS degeneracies in physics) will given a map between sets

$$\Omega : \Gamma \longrightarrow \mathbb{Q} \tag{2.1.2}$$

The **central charge** is a abelian group homomorphism

$$Z : \Gamma \longrightarrow \mathbb{C} \tag{2.1.3}$$

which satisfies the following support properties:

Support property: By endowing a Euclidean norm $\|\cdot\|$ on $\Gamma \otimes_{\mathbb{Z}} \mathbb{R}$, then there exists a positive constant C such that for any $\gamma \in \Gamma$, with $\Omega(\gamma) \neq 0$, we have that

$$|Z(\gamma)| > C \cdot \|\gamma\| . \tag{2.1.4}$$

Remark 2.1.1. *From the support property, we can infer that in any bounded region of the complex plane $\mathbb{C} \cong \mathbb{R}^2$, there are only finitely many points of the form $Z(\gamma)$ with $\gamma \in \Gamma$ such that $\Omega(\gamma) \neq 0$. In fact, we have the estimation of the number of such points inside the disc of radius R as follows*

$$\# \left(Z(\gamma) \cap \{ z \in \mathbb{C} : |z| \leq R \} \right) = O(R^n).$$

Definition 2.1.1. *The **Donaldson-Thomas (DT) invariants** of the "charge" γ is defined via*

$$DT(\gamma) = \sum_{k|\gamma, k\geq 1} \frac{\Omega(\gamma/k)}{k^2} \in \mathbb{Q}, \tag{2.1.5}$$

Remark 2.1.2. *By Möbius inversion, we see that BPS invariants can also be expressed in terms of DT invariants as follows*

$$\Omega(\gamma) = \sum_{k|\gamma, k\geq 1} \frac{\mu(k)}{k^2} DT(\gamma/k) \tag{2.1.6}$$

where $\mu(k)$ is the Möbius function. So we see that the two invariants are given in terms of each other. For this reason, we also call $\Omega(\gamma)$ the DT-invariants. These invariants are conjectured to be integer-valued.

2.1.2 The torus Lie algebra $\mathfrak{g}$

Given the charge lattice $\Gamma \cong \mathbb{Z}^n$, we consider the associated complexified algebraic torus

$$\widetilde{\mathbb{T}} = \widetilde{\mathbb{T}}_{\Gamma,\langle\cdot,\cdot\rangle} := Hom_{\mathbb{Z}}(\Gamma, \mathbb{C}^*) \cong (\mathbb{C}^*)^n$$

Choosing a set of basis $\{\gamma_i\}_{i=1}^n$ of Γ and the set of dual basis $\{\check{\gamma_i}\}$ in the sense that $\check{\gamma_i}(\gamma_j) = \delta_{ij}$. Let X_{γ_i} be the character associated to γ_i, i.e., the functions on Γ through

$$X_{\gamma_i} : \sum_k \theta_k \gamma_k \longmapsto \exp\left(\sum_k \theta_k \check{\gamma_i}(\gamma_k)\right)$$

Then the coordinate ring $\mathbb{C}[\widetilde{\mathbb{T}}]$ of $\widetilde{\mathbb{T}}$ can be identified with the ring of Laurent polynomials:

$$\mathbb{C}[\widetilde{\mathbb{T}}] = \mathbb{C}[X_{\gamma_1}^{\pm}, \cdots, X_{\gamma_n}^{\pm}]$$

$\widetilde{\mathbb{T}}$ can be made into a symplectic manifold by endowing it with the following symplectic form

$$\omega = \frac{1}{2}\sum_{i\neq j} \langle\gamma_i,\gamma_j\rangle^{-1} d\log(X_{\gamma_i}) \wedge d\log(X_{\gamma_j}) \tag{2.1.7}$$

which induces the Poisson structure on $\widetilde{\mathbb{T}}$ with Poisson bracket given as

$$\{f,g\} := \omega^{-1}(df,dg) \text{ for } f,g \in \mathbb{C}[\widetilde{\mathbb{T}}]$$

.

We see that the bracket acts on the characters as

$$\{X_{\gamma_i}, X_{\gamma_j}\} = \langle\gamma_i,\gamma_j\rangle\, X_{\gamma_i} \cdot X_{\gamma_j} \tag{2.1.8}$$

Next, we consider the $\widetilde{\mathbb{T}}$-torsor, called the **twisted torus** given as

$$\mathbb{T} := \{\sigma : \Gamma \to \mathbb{C}^* : \sigma(\gamma_1+\gamma_2) = (-1)^{\langle\gamma_1,\gamma_2\rangle}\,\sigma(\gamma_1)\,\sigma(\gamma_2)\}$$

where $\widetilde{\mathbb{T}}$ acts on it by

$$(\mu\cdot\sigma)(\gamma) = \mu(\gamma)\cdot\sigma(\gamma) \in \mathbb{C}^*, \mu \in \widetilde{\mathbb{T}},\ \sigma \in \mathbb{T}$$

Choosing a base point $\sigma_0 \in \mathbb{T}$, we get the following identification between $\widetilde{\mathbb{T}}$ and $\mathbb{T}$ via

$$\lambda_{\sigma_0} : \widetilde{\mathbb{T}} \to \mathbb{T} \qquad \mu \longmapsto \mu\cdot\sigma_0$$

By this identification, we can give $\mathbb{T}$ the structure of algebraic variety, which is independent of the choice of the base point σ_0 as the translation map is algebraic. Moreover, since the Poisson structure (2.1.8) on $\widetilde{\mathbb{T}}$ is invariant under translation, we transfer it to the twisted torus $\mathbb{T}$ as follows

$$\{e_{\gamma_i}, e_{\gamma_j}\} = \langle\gamma_i,\gamma_j\rangle\cdot e_{\gamma_i}\cdot e_{\gamma_j} = (-1)^{\langle\gamma_i,\gamma_j\rangle}\,\langle\gamma_i,\gamma_j\rangle\cdot e_{\gamma_i+\gamma_j} \tag{2.1.9}$$

The coordinate ring $\mathbb{C}[\mathbb{T}]$, as a vector space, is spanned by the following set of **twisted characters** defined as

$$e_{\gamma_i} : \mathbb{T} \to \mathbb{C}^* \qquad e_{\gamma_i}(\sigma) := \sigma(\gamma_i) \in \mathbb{C}^*$$

We see easily that these twisted characters satisfy

$$e_{\gamma_i} \cdot e_{\gamma_j} = (-1)^{\langle \gamma_i, \gamma_j \rangle} \cdot e_{\gamma_i + \gamma_j} \tag{2.1.10}$$

Further more, we can endow $\mathbb{C}[\mathbb{T}]$ with the Lie algebra structure by defining the lie bracket simply as $[a, b] := \{a, b\}$, i.e.,

$$[e_{\gamma_i}, e_{\gamma_j}] = (-1)^{\langle \gamma_i, \gamma_j \rangle} \langle \gamma_i, \gamma_j \rangle \cdot e_{\gamma_i + \gamma_j} \tag{2.1.11}$$

We denote this Lie algebra by $\mathfrak{g} = \mathfrak{g}_{\Gamma, \langle \cdot, \cdot \rangle} := (\mathbb{C}[\mathbb{T}], [\cdot, \cdot])$, and call it the **torus Lie algebra** associated to Γ. This Lie algebra will be of central importance in formulating the stability data and wall crossing structure later. From the definition, it is easy to see that it admits the following decomposition

$$\mathfrak{g} = \bigoplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma \tag{2.1.12}$$

Remark 2.1.3. *It is pointed out in [KS08] that the Lie algebra $\mathfrak{g}$, when equipped with the associative product defined by (2.1.10), becomes a commutative algebra, and its spectrum is the twisted torus $\mathbb{T}$.*

2.1.3 Kontseivich-Soibelman transformation

The torus Lie algebra $\mathfrak{g}$ acts on the twisted torus $\mathbb{T}$ by Hamiltonian vector fields. For the generator e_{γ_i}, we see that $\{e_{\gamma_i}, \cdot\} = [e_{\gamma_i}, \cdot]$ is a derivation by Jacobi identity, thus there exists a vector field $\mathfrak{e}_{\gamma_i} \in Vect(\mathbb{T})$, which acts as

$$\mathfrak{e}_{\gamma_i} \cdot e_\gamma = \{e_{\gamma_i}, e_\gamma\}$$

The time-1 Hamiltonian flow of the vector field $\mathfrak{e}_{\gamma_i}$ gives a Poisson automorphism of $\mathbb{C}[\mathbb{T}]$. More precisely, by identifying $\mathfrak{e}_{\gamma_i}$ with e_{γ_i}, we see that $e_{\gamma_i} \in \mathfrak{g}$ acts on $\mathbb{C}[\mathbb{T}]$ via the Poisson derivation

$$e_{\gamma_i} \cdot e_{\gamma_j} = \{e_{\gamma_i}, e_{\gamma_j}\}$$

We associate to $\gamma \in \Gamma$ a ray $l_\gamma := \mathbb{R}_{>0} \cdot Z(\gamma) \subset \mathbb{R}^2$. Then, We attach to this ray a formal sum of twisted characters

$$DT(l_\gamma) = \sum_{\gamma \in \Gamma : Z(\gamma) \in l_\gamma} DT(\gamma) \cdot e_\gamma \in \mathfrak{g} \tag{2.1.13}$$

We view it as a formal function defined on the twisted torus $\mathbb{T}$. The associated time-1 Hamiltonian flow gives us a formal Poisson automorphism

$$\mathbb{S}(l_\gamma) \in Aut(\mathbb{T}) \tag{2.1.14}$$

We call it the **transformation** associated to the ray l_γ.

Remark 2.1.4. *We see that since the sum in (2.1.12) could be infinite, $DT(l_\gamma)$ may not be a well defined function on $\mathbb{T}$, and the transformation $\mathbb{S}(l_\gamma)$ would cease to be well defined. In the unpublished paper [KS13] by Kontsevich and Soibelman, technical works had been down to make these automorphisms rigorously defined.*

For a ray l_γ that contains only a single element $\gamma \in \Gamma$ with $\Omega(\gamma) = 1$, we check that

$$DT(l_\gamma) = \sum_{n\geq 1} \frac{e_{n\gamma}}{n^2} = \sum_{n\geq 1} \frac{e_\gamma^n}{n^2} = Li_2(e_\gamma) \tag{2.1.15}$$

where $Li_2(x) := \sum_{n\geq 1} \frac{x^n}{n^2}$ is the **classical dilogarithm function**. Indeed, by (2.1.12), we have in this case that

$$\begin{aligned} DT(\gamma) &= \sum_{Z(\gamma)\in l_\gamma} DT(\gamma)\cdot e_\gamma = \sum_{n\geq 1} DT(n\gamma)\cdot e_{n\gamma} \\ &= \sum_{n\geq 1}\sum_{k\geq 1, k|n} \frac{\Omega(n\gamma/k)}{k^2}\cdot e_{n\gamma} = \sum_{n\geq 1} \frac{\Omega(\gamma)}{n^2}\cdot e_{n\gamma} = \sum_{n\geq 1}\frac{e_\gamma^n}{n^2} \end{aligned}$$

Under the assumption that the associated time-1 Hamiltonian flow $\mathbb{S}(l_\gamma)$ is well defined on some open subset of $\mathbb{T}$, we compute formally as

$$\mathbb{S}(l_\gamma)(e_\kappa) = \exp\left\{\sum_{n\geq 1}\frac{e_\gamma^n}{n^2}, \bullet\right\}(e_\kappa)$$

It follows that

$$\begin{aligned} \log(\mathbb{S}(l_\gamma))(e_\kappa) &= \left\{\sum_{n\geq 1}\frac{e_\gamma^n}{n^2}, \bullet\right\}(e_\kappa) = \sum_{n\geq 1}\frac{1}{n^2}\{e_\gamma^n, e_\kappa\} \\ &= \sum_{n\geq 1}\frac{1}{n^2}\{e_{n\gamma}, e_\kappa\} = \sum_{n\geq 1}\frac{1}{n^2} n\langle\gamma,\kappa\rangle e_\gamma^n\cdot e_\kappa \\ &= \sum_{n\geq 1}\frac{\langle\gamma,\kappa\rangle}{n} e_\gamma^n\cdot e_\kappa = \langle\gamma,\kappa\rangle\sum_{n\geq 1}\frac{e_\gamma^n}{n}\cdot e_\kappa \\ &= \langle\gamma,\kappa\rangle\ \log(1-e_\gamma)\cdot e_\kappa = \log\left(1-e_\gamma\right)^{\langle\gamma,\kappa\rangle}\cdot e_\kappa \end{aligned}$$

Consequently

$$\mathbb{S}(l_\gamma)\left(e_\kappa\right) = \left(1-e_\gamma\right)^{\langle\gamma,\kappa\rangle}\cdot e_\kappa \tag{2.1.16}$$

The above automorphism, first observed by Kontsevich and Soibelman (see section 2.5 of [KS14]), is of central importance in encoding the wall-crossing phenomena in physics. And it is called **Kontseivich-Soibelman transformation** ("KS transformations") in physics—a terminology we adopt here.

We use the symbol $\mathcal{K}_\gamma$ to denote the KS transformation. i.e.,

$$\mathcal{K}_\gamma := \exp\left\{Li_2\left(e_\gamma\right), \bullet\right\} \tag{2.1.17}$$

which acts on the generator e_κ as

$$e_\kappa \longmapsto \left(1-e_\gamma\right)^{\langle\gamma,\kappa\rangle}\cdot e_\kappa \tag{2.1.18}$$

Remark 2.1.5. *$\mathcal{K}_\gamma$ extends holomorphically to the Zariski open subset of $\mathbb{T}$ which is the complement of the divisor $\{e_\gamma = 1\}$. Therefore, it defines a birational transformation of the twisted torus $\mathbb{T}$.*

For general ray l_γ, again, under assumption that the automorphism $\mathbb{S}(l)$ is well defined on a suitable open subset of $\mathbb{T}$, we have the following

Proposition 2.1.1. *Assuming that for the ray l_γ, there are only finitely many $\gamma \in \Gamma$ with integer-valued $\Omega(\gamma) \neq 0$ such that $Z(\gamma) \in l_\gamma$, then $\mathbb{S}(l_\gamma)$ extends to a birational automorphism of $\mathbb{T}$, with action on the generator e_κ given by*

$$\mathbb{S}(l_\gamma)\,(e_\kappa) = \prod_{Z(\gamma)\in l_\gamma} (1-e_\gamma)^{\Omega(\gamma)\cdot\langle\gamma,\kappa\rangle}\cdot e_\kappa \tag{2.1.19}$$

Proof.

$$\log \mathbb{S}(l_\gamma)\,(e_\kappa) = \left\{ \sum_{Z(\gamma)\in l_\gamma} \sum_{n\geq 1} \Omega(\gamma)\,\frac{e_\gamma^n}{n^2}, \quad \bullet \right\}(e_\kappa)$$

$$= \sum_{Z(\gamma)\in l_\gamma} \sum_{n\geq 1} \frac{\Omega(\gamma)}{n^2}\{e_\gamma^n, e_\kappa\} = \sum_{Z(\gamma)\in l_\gamma} \sum_{n\geq 1} \frac{\Omega(\gamma)\, n\langle\gamma,\kappa\rangle\, e_\gamma^n\cdot e_\kappa}{n^2}$$

$$= \sum_{Z(\gamma)\in l_\gamma} \langle\gamma,\kappa\rangle\cdot\Omega(\gamma)\sum_{n\geq 1}\frac{e_\gamma^n}{n}\cdot e_\kappa = \sum_{Z(\gamma)\in l} \langle\gamma,\kappa\rangle\cdot\Omega(\gamma)\log(1-e_\gamma)\cdot e_\kappa$$

$$= \sum_{Z(\gamma)\in l_\gamma} \log(1-e_\gamma)^{\Omega(\gamma)\cdot\langle\gamma,\kappa\rangle}\cdot e_\kappa$$

We see that when $\Omega(\gamma)$ are integers, $\mathbb{S}(l_\gamma)$ is a birational automorphism. □

Before closing this section, we note that by using the KS-transformation $\mathcal{K}_\gamma$, we can rewrite the transformation $\mathbb{S}(l_\gamma)$ more compactly as

$$\mathbb{S}(l_\gamma) = \prod_{Z(\gamma)\in l_\gamma} \mathcal{K}_\gamma^{\Omega(\gamma)} = \prod_{\gamma'\|\gamma} \mathcal{K}_{\gamma'}^{\Omega(\gamma')} = \prod_{\substack{Z(\mu)\in l_\gamma\\ m\geq 1}} \mathcal{K}_{m\mu}^{\Omega(m\mu)} \tag{2.1.20}$$

where we have denoted by μ the primitive charge vector of the ray l_γ. From the above formula, we see that for the ray l_γ with $\Omega(\gamma)\neq 0$, one has

$$\log\,\mathbb{S}(l_\gamma) \in \bigoplus_{\gamma'\|\gamma} \mathfrak{g}_{\gamma'} \tag{2.1.21}$$

2.1.4 Factorization Property

In order to establish the celebrated *Kontseivich-Soibelman Wall Crossing Formula* ("KSWCF"), not only we need to associate transformation to l_γ, but also need to associate transformation to any strict sector $\Delta \subset \mathbb{C}^* \cong \mathbb{R}^2$ (i.e., less than 180°).

Consider the pronilpotent Lie algebra $\mathfrak{g}_\Delta := \prod_{\gamma\in\Delta\cap\Gamma\setminus\{0\}} \mathfrak{g}_\gamma$, as Δ is strict, there exists $\phi \in \Gamma^*_{\mathbb{R}}$ such that its restriction to Δ gives a proper map to $\mathbb{R}_{\geq 0}$. Then for $N>0$, consider the quotient $\mathfrak{g}_{\Delta,N} = \mathfrak{g}/\mathfrak{g}_{\Delta,\geq N}$, where $\mathfrak{g}_{\Delta,\geq N} \subset \mathfrak{g}_\Delta$ is the ideal consisting of elements with $\phi(\gamma) > N$ for $\gamma \in \Delta\cap\Gamma\setminus\{0\}$. We get $\mathfrak{g}_\Delta = \lim_{\overleftarrow{N}} \mathfrak{g}_{\Delta,N}$. Denote by G_Δ and $G_{\Delta,N}$ the corresponding Lie groups.

Given $N > 0$, let us consider the truncation

$$\mathbb{S}(l_\gamma)_{<N} = \exp\left(DT(l_\gamma)_{<N}\right) \in G_{\Delta,N}.$$

Assume that there are only finitely many rays in Δ that carry charge with $\phi(\gamma) < N$, we form the finite product

$$\mathbb{S}(\Delta)_{<N} := \overrightarrow{\prod_{l_\gamma \subset \Delta}} \mathbb{S}(l_\gamma)_{<N} \in G_{\Delta,N}. \tag{2.1.22}$$

where the right arrow above the product sign indicates that the product is taken over all rays $l_\gamma \subset \Delta$ in the clockwise order.

Thus, by taking the limit as $N \to \infty$, we get a well-defined group element

$$\mathbb{S}(\Delta) = \overrightarrow{\prod_{l_\gamma \subset \Delta}} \mathbb{S}(l_\gamma) \in G_\Delta \tag{2.1.23}$$

This will be the desired BPS automorphism attached to the strict sector Δ. Also notice that the right hand side could possibly be an infinite product.

Observe that if $\Delta_1 \subset \Delta_2$, then there is an embedding of the corresponding torus Lie algebra $\mathfrak{g}_{\Delta_1} \subset \mathfrak{g}_{\Delta_2}$, which further induces the injective homomorphism on the corresponding nilpotent Lie group level $G_{\Delta_1} \hookrightarrow G_{\Delta_2}$.

Thus if an strict sector Δ can be decomposed into the disjoint union of two sub-sectors in the clockwise order (where we assume that Δ_- proceeds Δ_+ in the clockwise order), namely: $\Delta = \Delta_+ \sqcup \Delta_-$, then on the Lie algebra level, this produces the following decomposition: $\mathfrak{g}_\Delta = \mathfrak{g}_{\Delta_+} \oplus \mathbf{g}_{\Delta_-}$, while on the Lie group level, we get: $G_\Delta = G_{\Delta_+} \cdot G_{\Delta_-}$. Consequently, we have the following **factorization property**

$$\mathbb{S}(\Delta) = \mathbb{S}_{\Delta_+} \cdot \mathbb{S}_{\Delta_-} \tag{2.1.24}$$

Proposition 2.1.2. *The group element:*

$$\mathbb{S}(\Delta) = \overrightarrow{\prod_{l_\gamma \subset \Delta}} \mathbb{S}(l_\gamma) \in G_\Delta$$

determines the invariants $\Omega(\gamma)$ for all $\gamma \in \Gamma$ with $Z(\gamma) \in \Delta$.

Proof. Recall that $\mathbb{S}(l_\gamma)$ encodes $\Omega(\gamma)$ for $\gamma \in \Gamma$ such that $Z(\gamma) \in \Gamma$. Thus, we only need to show that $\mathbb{S}(\Delta)$ determines all $\mathbb{S}(l)$ for $l \subset \Delta$.

To this end, we want to decompose Δ into a finite many sub-sectors, each of which contains only a single l_γ that carries charges $\phi(\gamma) < N$. This is possible and is equivalent to working in the group $G_{\Delta,N}$. In this group, we have a finite product and since this decomposition is unique, we infer that $\mathbb{S}(\Delta)_{<N}$ determines all $\mathbb{S}(l_\gamma)_{<N}$ for $l_\gamma \subset \Delta$.

Since this holds for any $N > 0$, by taking the limit as $N \to \infty$, the desired result follows. $\square$

2.1.5 KSWCF

Proposition 2.1.2 above tells us that the DT-invariants $\Omega(\gamma)$ with $Z(\gamma) \in \Delta$ are encoded in the group element $\mathbb{S}(\Delta) \in G_\Delta$. In practice, we are concerned with how these invariants change if we deform the central charge function Z. Indeed, as can be seen from the definition of the group G_Δ, it depends highly non trivially on the central charge $Z(\gamma)$. In particular, it will "jump" as the central charge being deformed such that $Z(\gamma)$ crosses the boundary of Δ.

KSWCF gives an universal solution to this kind of wall crossing phenomena.

We will show that as Z varies, there exists a collection of hypersurfaces in $\Gamma^*_{\mathbb{R}}$, such that in side the chambers determined by the hypersurfaces, the wall crossing group G_Δ stays constant, so are those DT-invariants $\Omega(\gamma)$. Therefor, we expect the jumps to occur only when crossing these hypersurfaces. These hypersurfaces will be called **walls** associated to $\mathfrak{g}$, and we denote it by $Wall_{\mathfrak{g}}$.

Note that every hypersurface in $\Gamma^*_{\mathbb{R}}$ is associated to some class $\gamma \in \Gamma$, i.e., $\gamma^\perp$. and call it the "**wall associated with** γ". Thus

$$Wall_{\mathfrak{g}} = \bigcup_{\text{countably many } \gamma} \gamma^\perp$$

Further more, in view of the support property given in section 2.1.1, we require that for an strict sector Δ, the following constraints are satisfied:

$$Z(\gamma) \in \Delta, \qquad |Z(\gamma)| > C \cdot \|\gamma\|$$

Consequently, in stead of working with the charge lattice Γ directly, we can work with the subset $\Gamma_{(\Delta,C)}$ of Γ that consists of non-negative integral combinations of basis elements satisfying the above two constraints coming from the central charge Z. It is easy to see that $\Gamma_{(\Delta,C)}$ is closed under addition, thus all previous constructions associated to Γ can be carried on for $\Gamma_{(\Delta,C)}$. For example we have the Poisson subalgebra

$$\mathbb{C}_{(\Delta,C)}[\mathbb{T}] := \bigoplus_{\gamma \in \Gamma_{(\Delta,C)}} \mathbb{C} \cdot e_\gamma \subset \mathbb{C}_\Delta[\mathbb{T}]$$

as well as the associated Lie algebra and the corresponding Lie group

$$\exp : \mathfrak{g}_{(\Delta,C)} \longrightarrow G_{(\Delta,C)}$$

Again, all constructions and identities involving these objects in the last section still hold in this setting. Thus, for given $N > 0$, we work in the quotient algebra $\mathbb{C}_{(\Delta,C)}[\mathbb{T}]_{<N}$, and the corresponding *wall-crossing group* $G_{(\Delta,C)<N}$ depends only on the finite subset of Γ, namely

$$\Gamma_{(\Delta,C)<N} := \{\gamma \in \Gamma : \gamma \in \Gamma_{(\Delta,C)},\ Z(\gamma) < N\} \tag{2.1.25}$$

It is easy to see then that when we deform Z inside the space Γ^*, as long as $Z(\gamma)$ does not cross the boundary $\partial\Delta$ of the sector, the above finite set would stay constant under the deformation. Since $G_{(\Delta,C)<N}$ depends only on it, it also stays constant.

In other words, when deforming Z, the **wall crossing group** $G_{(\Delta,C)<N}$ does not change outside a finite collection of hypersurfaces. i.e.,

$$\Gamma^*_{\mathbb{R}} \setminus Wall_{\mathfrak{g}}$$

where each connected component of it is called a **chamber**.

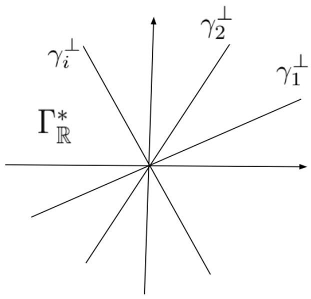

Figure 2.1: Walls and chambers

Later, we will see that for each γ that define the boundary ray l_γ of an strict sector, the wall $\gamma^\perp$ associated to it is the hypersurface of the following form

$$\mathcal{W}^2_\gamma = \gamma^\perp := \{Z \in \Gamma^*_{\mathbb{R}} : Im(Z(\gamma)) = 0\} \tag{2.1.26}$$

where $Im(Z(\gamma))$ denotes the imaginary part of $Z(\gamma)$. And we will call this the **wall of second kind** associated to γ, and denote it by $\mathcal{W}^2_\gamma$.

Note that different γ can give the same wall, i.e.,

$$\gamma_1^\perp = \gamma_2^\perp \Longleftrightarrow \gamma_1 \parallel \gamma_2$$

With the above preparation, we can state the **KSWCF**: Given any strict sector Δ, the associated group element

$$\mathbb{S}(\Delta)_{<N} = \overrightarrow{\prod_{l_\gamma \subset \Delta}} \mathbb{S}(l_\gamma)_{<N} \in G_{(\Delta,C)<N} \tag{2.1.27}$$

remains constant as long as Z varies only inside a given chamber.

In other words, the group element above jumps only when Z crosses some wall. In practice, the deformation of the central charge Z inside Γ^* is governed by certain parameter space $\mathcal{B}$, which is usually a finite dimensional complex manifold. Physicists call such space the *moduli space* of the theory. We will encounter such situation when we study *complex integrable system* later.

Thus, suppose our moduli space is a complex manifold $\mathcal{B}$, then for any $b \in \mathcal{B}$, we have a charge lattice Γ_b, in general, we assume that the family of lattice form a local system of abelian groups, denoted by $\underline{\Gamma}$, with the intersection forms being covariantly constant with respect to *Gauss-Manin connection.*

We also assume that the central charge Z_b depends on b holomorphically. Further more, we impose the following **uniform support property**:

Fix a covariantly constant family of Euclidean norms

$$\|\cdot\|_b : \Gamma_{\mathbb{R},b} \longrightarrow \mathbb{R}_{>0}.$$

then for any compact subset $K \subset \mathcal{B}$ there exists a constant $C > 0$ such that

$$\Omega_b(\gamma) \neq 0 \text{ for some } b \in K \;\Rightarrow\; |Z_b(\gamma_b)| > C \cdot \|\gamma\|_b$$

Under these assumptions, we give an equivalent formulation of KSWCF. Given a contractible open subset $U \subset \mathcal{B}$, a constant $N > 0$, as well as an strict sector Δ, assume that we can trivialize the local system Γ_b over U and hence identify Γ_b with a fixed lattice Γ. Take $C > 0$ as in above assumption and assume that the subset 2.1.25 as defined in the last subsection is constant near $b \in \mathcal{B}$. Then the group element in the wall crossing group

$$\mathbb{S}(\Delta)_{<N} = \overrightarrow{\prod_{l \subset \Delta}} \mathbb{S}(l)_{<N} \in \widehat{G}_{(\Delta,C)<N}$$

stays constant as b varies in U.

Indeed, we see that the assumptions above ensure that when b varies in U, the central charge $Z_b(\gamma)$ stays inside the chamber specified by Δ, and by the uniform support property, the constant C can be chosen uniformally for $b \in U$, while the constant N appearing is already fixed before hand.

Remark 2.1.6. $\underline{\Gamma}$ *can always be trivialized by either passing to the universal cover of* $\mathcal{B}$*, or by restricting to a contractible open subset* $U \subset \mathcal{B}$*.*

Proposition 2.1.3. *Under the assumption that the local system of charge lattices can be trivilized, then for any fixed charge with class* $\gamma \in \Gamma$*, there is a locally finite collection of walls as codimension one real submanifolds, which divide* $\mathcal{B}$ *into connected components as open submanifolds called chambers, such that the invariants* $\Omega(\gamma)$ *stay constant inside each chamber.*

Proof. Locally near $b \in \mathcal{B}$, We consider the composition of maps

$$\mathcal{B} \xrightarrow{\tilde{\pi}} Hom_{\mathbb{Z}}(\Gamma, \mathbb{C}) \cong \mathbb{C}^n \xrightarrow{Z} \mathbb{C}^* \xrightarrow{\Im} \mathbb{R} \tag{2.1.28}$$

$$b \longmapsto Z_b \longmapsto Z_b(\gamma) \longmapsto Im(Z_b(\gamma))$$

where the composition map $Z \circ \Im$ defines an element in $\Gamma^*_{\mathbb{R}}$. From this one sees easily that a locally finite collection of hypersurfaces in $\Gamma^*_{\mathbb{R}}$ pull back via the map $\tilde{\pi}$ to a collection of codimension one real submanifolds in $\mathcal{B}$, which we also call walls. The proposition then follows from KSWCF. $\square$

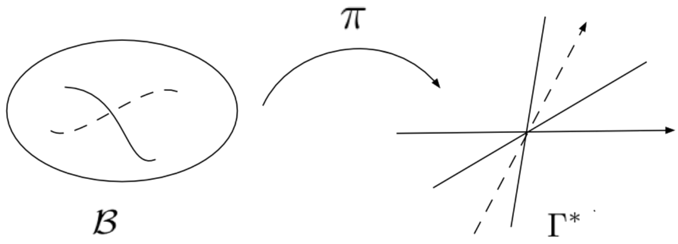

Figure 2.2: The induced walls and chambers on the manifold $\mathcal{B}$.

2.1.6 Stability data on $\mathfrak{g}$

We now review the concept of **stability data on a graded Lie algebra** introduced by Kontsevich and Soibelman ([KS08]). As before, we associate to Γ the torus Lie algebra

$$\mathfrak{g} = \bigoplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma$$

For a charge $\gamma \in \Gamma$ with $\Omega(\gamma) \neq 0$, the corresponding transformation $\mathbb{S}(l_\gamma)$ encodes the invariants $\Omega(\gamma')$ for $\gamma' \parallel \gamma$. We see previously that $\Omega(\gamma)$ can be expressed in terms of DT-invariants $DT(\gamma)$ through the formula (2.1.4), while $\mathbb{S}(l_\gamma)$ is the time-1 Hamiltonian flow associated to

$$DT(l_\gamma) = \sum_{Z(\gamma) \in l_\gamma} DT(\gamma) \cdot e_\gamma \in \mathfrak{g}$$

Thus, $DT(l_\gamma)$ is seen to be a collection of elements

$$a = \{a(\gamma)\}_{\gamma \in \Gamma \backslash \{0\}} := \{DT(\gamma) \cdot e_\gamma \in \mathfrak{g}_\gamma\}$$

Denote by $Supp\, a$ the set of all $\gamma \in \Gamma \backslash \{0\}$ such that $a(\gamma) \neq 0$. We see that $Supp\, a$ is the same as the support of $\mathfrak{g}$ defined as

$$Supp\, \mathfrak{g} := \{\gamma \in \Gamma : \mathfrak{g}_\gamma \neq 0\} \subset \Gamma \tag{2.1.29}$$

By convention, $\mathfrak{g}_0$ is set to be zero.

Definition 2.1.2. *A* ***stability data*** *on $\mathfrak{g}$ is a pair $\sigma = (Z, a)$, consisting of a central charge and a collection of elements $a(\gamma) \in \mathfrak{g}_\gamma$, which satisfies the support property.*

We denote by $Stab(\mathfrak{g})$ the set of all stability data on $\mathfrak{g}$. We remark that the support property is equivalent to the following condition:

There exists a quadratic form Q on $\Gamma_{\mathbb{R}} := \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ such that

- 1) $Q_{|\ker Z} < 0$;
- 2) $Supp\, a \subset \{\gamma \in \Gamma \backslash \{0\}\} : Q(\gamma) \geq 0\}$

To see this, we just spell out that the relation between the quadratic form Q and the norm $\| \bullet \|$ on $\Gamma_{\mathbb{R}}$ is given by

$$Q(\gamma) = -\|\gamma\|^2 + C_1 |Z(\gamma)|^2$$

for some sufficiently large constant $C_1 > 0$.

Reformulation of the stability data

We work in the same setting as above, and denote by $\mathcal{S}$ the set of all strict sectors in $\mathbb{R}^2$. Note that it may contains degenerate sectors, i.e., rays. Define another set $\widehat{Stab}(\mathfrak{g})$ as the set of pairs $(Z, \mathbb{S})$, where Z is a central charge function, and $\mathbb{S}$ a collection of elements $(\mathbb{S}_\Delta)_{\Delta\in\mathcal{S}}$, with elements $\mathbb{S}_\Delta$ belonging to the wall crossing group G_Δ. Again, we impose the support property.

We show that the set $\widehat{Stab}(\mathfrak{g})$ (also called the stability data on $\mathfrak{g}$) is actually equivalent to $Stab(\mathfrak{g})$.

Indeed, given a stability data (Z, a), we determine a pair $(Z, \mathbb{S})$ as follows

Recall that for every $l_\gamma \subset \Delta$, the transformation associated to it is given by

$$\mathbb{S}(l_\gamma) := \exp\{DT(l_\gamma), \bullet\} = \exp\left\{\sum_{Z(\gamma)\in l_\gamma} DT(\gamma)\cdot e_\gamma, \bullet\right\}$$

$$= \exp\left\{\sum_{Z(\gamma)\in l_\gamma} a(\gamma), \bullet\right\}$$

and by the Factorization Property, we have that

$$\mathbb{S}(\Delta) = \overrightarrow{\prod_{l_\gamma\subset\Delta}} \mathbb{S}(l_\gamma) \in G_\Delta$$

Conversely, given a pair $(Z, \mathbb{S})$, we take the same Z for $Stab\,(\mathfrak{g})$, and construct the elements $a(\gamma)$ as follows:

Set $a(\gamma) = 0$ if $Z(\gamma) = 0$. Let us assume that $Z(\gamma) \neq 0$. Consider the automorphism $\mathbb{S}(l)$ associated to the ray

$$l_\gamma = \mathbb{R}_{>0}\cdot Z(\gamma)$$

Then recall that we have established (see equation 2.1.21) that

$$\log\,\mathbb{S}(l_\gamma) \in \bigoplus_{\gamma'\|\gamma} \mathfrak{g}_{\gamma'}$$

From it we denote by $a(\gamma)$ the component of $\log\,\mathbb{S}(l)$ which belongs to $\mathfrak{g}_\gamma$. And we get the desired Lie algebra element.

2.2 More on KSWCF

We introduce a topology on the space of stability data $Stab(\mathfrak{g})$ and discuss its relation to KSWCF. Some explicit expressions of KSWCF will be given. The treatment in this section is based on the paper [KS08].

2.2.1 Topology on $Stab(\mathfrak{g})$ and KSWCF

Define the *forgetting map* from the stability data to $\mathbb{C}^n$ as follows

$$\begin{aligned} Stab(\mathfrak{g}) &\longrightarrow Hom_{\mathbb{Z}}(\Gamma, \mathbb{C}) \cong \mathbb{C}^n \\ (Z, a) &\longmapsto Z \end{aligned} \tag{2.2.1}$$

The central idea is that we can impose a topology on $Stab(\mathfrak{g})$ so to make the above map a local homeomorphism. Consequently, the complex manifold structure on $Stab(\mathfrak{g})$ can be obtained by pulling back the standard complex structure on $\mathbb{C}^n$ to $Stab(\mathfrak{g})$.

More precisely, we impose the following topology by specifying the notion of a *continuous family of points* in $Stab(\mathfrak{g})$.

Definition 2.2.1. *Let $(Z_b, a_b(\gamma))$ be a family parametrized by $b \in \mathcal{B}$, and let $b_0 \in \mathcal{B}$ be fixed, this family of points in $Stab(\mathfrak{g})$ is said to be continuous at b_0 if*

- *a) The forgetting map (2.2.1) is continuous at $b = b_0$.*
- *b) There exists open neighborhood $U_0 \ni b$, such that the constant $C > 0$ in the support property for $Stab(\mathbf{g})_b$ for $b \in U_0$ can be chosen uniformally.*
- *c) Near b_0, we choose a constant $C > 0$ as in b), and give an closed strict sector $\Delta \subset \mathbb{C}^*$ such that $Z(Supp\, a_{b_0}) \cap \partial\Delta = \emptyset$, then the map*

$$b \longmapsto \log \mathbb{S}(\Delta)_b \in \mathfrak{g}_{\Delta,b} \subset \prod_{\gamma \in \Gamma} \mathfrak{g}_\gamma$$

is continuous at $b = b_0$. Here the vector space $\prod_{\gamma\in\Gamma} \mathfrak{g}_\gamma$ is endowed with the product topology of discrete sets, and $\mathbb{S}(\Delta)_b \in G_{\Delta,b}$ is the group element associated with (Z_b, a_b) as well as the sector Δ.

The relation between this topology and KSWCF can be summarized in the following remark:

Remark 2.2.1. *Note that the continuity of the map in c) means that the group element $\log \mathbb{S}(\Delta)_b$ does not depend on $b \in \mathcal{B}$ as long as there is no element $\gamma \in Supp\, a_b$ such that $Z_b(\gamma)$ crosses the boundary $\partial\Delta$, i.e., it is locally constant as a function in b near b_0. And consequently, the invariants $\Omega_b(\gamma)$ is also a locally constant function in b, while the jumps of these invariants are controlled by KSWCF given in the next subsection.*

We remark without proving that that the topology imposed on $Stab(\mathfrak{g})$ is Hausdorff. The proof can be found in [KS08] (section 2.3. proposition 1 there).

2.2.2 Crossing the wall of the 1^{st} kind

For $\gamma_1, \gamma_2 \in \Gamma \backslash \{0\}$, two $\mathbb{Q}$-linear independent elements, define the set

$$\mathcal{W}_{\gamma_1,\gamma_2} := \{b \in \mathcal{B} : \mathbb{R}_{>0} \cdot Z_b(\gamma_1) = \mathbb{R}_{>0} \cdot Z_b(\gamma_2)\}$$
$$= \left\{ b \in \mathcal{B} : Im\left(\frac{Z_b(\gamma_1)}{Z_b(\gamma_2}\right) = 0 \right\}$$

Then the *wall of the first kind* associated to $\gamma \in \Gamma$ is defined as

$$\mathcal{W}^1_\gamma := \bigcup_{\gamma=\gamma_1+\gamma_2} \mathcal{W}_{\gamma_1,\gamma_2} \tag{2.2.2}$$

We note that the condition is equivalent to the rays $l_{\gamma_1}, l_{\gamma_2}$ are parallel to each other at the wall. Also, the totality of all these walls will by denoted by

$$\mathcal{W}^1 := \bigcup_\gamma \mathcal{W}^1_\gamma$$

and call it the set of **wall of the first kind**. When b_0 belongs to $\mathcal{W}^1_\gamma$, this corresponds to that the strict sector Δ bounded by $l_{\gamma_1}, l_{\gamma_2}$ becomes degenerate. Thus, wee see that if we consider a curve $b(t) \in \mathcal{B}$, $0 \leq t \leq 1$ such that b_0 corresponds to t_0, then on the left hand side of b_0, i.e., $b_- < b_0$, we have that $l_{\gamma_1} < l_{\gamma_2}$, say l_{γ_1} proceeds l_{γ_2} in the clockwise order. Similarly, on the right hand side of b_0 where $b_+ > b_0$, we have that $l_{\gamma_1} > l_{\gamma_2}$.

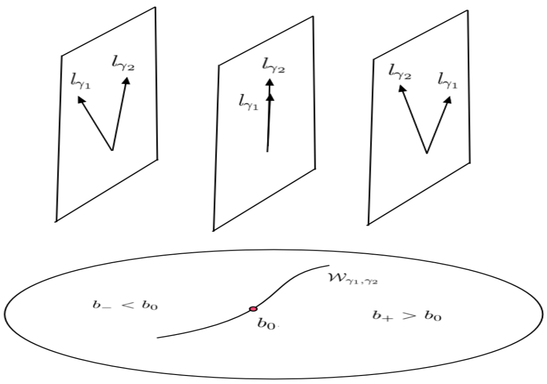

Figure 2.3: Wall of the first kind $\mathcal{W}_{\gamma_1,\gamma_2}$

When $t \to t_0$, i.e., the two rays $l_{\gamma_1}, l_{\gamma_2}$ coalesce, and change the order after crossing the wall. During this coalesce the rays and swapping the order process, the group element $\mathbb{S}(\Delta)$ stays constant by KSWCF. Thus, by taking the limit as $b \to b_0$ on both sides, we are able to describe exactly how DT-invariants $\Omega_b(\gamma)$ change when crossing the wall. First recall that (see section 2.1.3)

$$\mathbb{S}(\Delta) = \overrightarrow{\prod_{l_\gamma \subset \Delta}} \mathbb{S}(l_\gamma) = \overrightarrow{\prod_{l_\gamma \subset \Delta}} \exp\left(\sum_{Z(\gamma) \in l_\gamma} a(\gamma)\right)$$

and $\mathbb{S}(l_\gamma)$ can be further written as

$$\mathbb{S}(l_\gamma) = \overrightarrow{\prod_{\mu \in \Gamma^{prim}, Z(\mu) \in l_\gamma}} \exp\left(\sum_{n \geq 1} a(n\mu)\right)$$

where $\Gamma^{prim} \subset \Gamma$ denotes the set of primitive vectors. For $t \in [0,1]$, we define the limits

$$a_t^{\pm}(\gamma) = \lim_{\varepsilon \to 0, \varepsilon > 0} a_{t \pm \varepsilon}(\gamma)$$

where for $t = 0$ or $t = 1$, only one sided limit is well defined.

Then part c) of definition 2.2.1, i.e., the continuity of $\mathbb{S}(\Delta)$ near b implies the following identity, which we call it the **KS wall-crossing formula**

$$\overrightarrow{\prod_{\mu \in \Gamma^{prim}, Z_t(\mu) \in l_{\gamma,t}}} \exp\left(\sum_{n \geq 1} a_t^-(m\mu)\right) = \overrightarrow{\prod_{\mu \in \Gamma^{prim}, Z_t(\mu) \in l_{\gamma,t}}} \exp\left(\sum_{n \geq 1} a_t^+(m\mu)\right)$$

Or even more simply we can write it as

$$\overrightarrow{\prod_{\mu \in \Gamma^{prim}, Z_t(\mu) \in l_{\gamma,t}}} \exp\left(\sum_{n \geq 1} a_t(m\mu)\right) = \overleftarrow{\prod_{\mu \in \Gamma^{prim}, Z_t(\mu) \in l_{\gamma,t}}} \exp\left(\sum_{n \geq 1} a_t(m\mu)\right)$$

where the product on the left (right) hand side is taken in the clockwise (anti-clockwise) order of the rays $l_{\gamma,t}$.

We remark that $a_t^-(\gamma) = a_t^+(\gamma) = a_t(\gamma)$ unless $b(t)$ hits some wall $\mathcal{W}_\gamma^1$, i.e., there exist two non-zero γ_1, γ_2 such that $\gamma = \gamma_1 + \gamma_2$ and $b \in \mathcal{W}^1_{\gamma_1,\gamma_2}$. Indeed, as in this case the two rays l_{γ_1} and l_{γ_2} would get swapped when crossing the wall, thus the order of the products on both sides of the KSWCF also changes, which forces the DT-invariants to change in order to make the identity still valid. To see this more clearly, let us use KS-transformation introduced before to rewrite the above formula in which the change of the invariants $\Omega(\gamma)$ become visible.

Remark 2.2.2. *Informally speaking, the KSWCF says that for a very small strict sector Δ containing the rays $l_{\gamma,t}$, the corresponding group element $\mathbb{S}(\Delta)$, considered as function of the parameter t, stays constant in a neighborhood of t.*

By using (2.1.20), i.e.,

$$\mathbb{S}(l_\gamma) = \prod_{Z(\gamma) \in l_\gamma} \mathcal{K}_\gamma^{\Omega(\gamma)} = \prod_{\gamma' \| \gamma} \mathcal{K}_{\gamma'}^{\Omega(\gamma')} = \prod_{\substack{Z(\mu) \in l_\gamma \\ m \geq 1}} \mathcal{K}_{m\mu}^{\Omega(m\mu)}$$

where $\mathcal{K}_\gamma := \exp\left(Li_2\left(e_\gamma\right)\right)$ is the KS-transformation associated to γ. We rewrite the above KSWCFs as

$$\overrightarrow{\prod_{\mu \in \Gamma^{prim}, Z_t(\mu) \in l_{\gamma,t}}} \mathcal{K}_{m\mu}^{\Omega_t^-(m\mu)} = \overrightarrow{\prod_{\mu \in \Gamma^{prim}, Z_t(\mu) \in l_{\gamma,t}}} \mathcal{K}_{m\mu}^{\Omega_t^+(m\mu)} \tag{2.2.3}$$

$$\overrightarrow{\prod_{\mu \in \Gamma^{prim}, Z_t(\mu) \in l_{\gamma,t}}} \mathcal{K}_{m\mu}^{\Omega_t(m\mu)} = \overleftarrow{\prod_{\mu \in \Gamma^{prim}, Z_t(\mu) \in l_{\gamma,t}}} \mathcal{K}_{m\mu}^{\Omega_t(m\mu)} \tag{2.2.4}$$

More specifically, we consider the case when $b_0 \in \mathcal{W}^1_{\gamma_1,\gamma_2}$, and denote by $\Gamma_0 \subset \Gamma$ the sub-lattice generated by positive cone spaned by γ_1 and γ_2 as follows

$$\mathcal{C}_0 := \mathbb{Z}_{\geq 0} \cdot \gamma_1 \oplus \mathbb{Z}_{\geq 0} \cdot \gamma_2 = \{m\gamma_1 + n\gamma_2 : m, n \in \mathbb{Z}_{\geq 0}\}$$

Denote by $\Delta_{\gamma_1,\gamma_2}$ the acute sector bounded by the rays $l_{\gamma_1}, l_{\gamma_2}$. For a charge $\gamma = m\gamma_1 + n\gamma_2$, identify it simply with (m, n), where m, n are integers, not both zero.

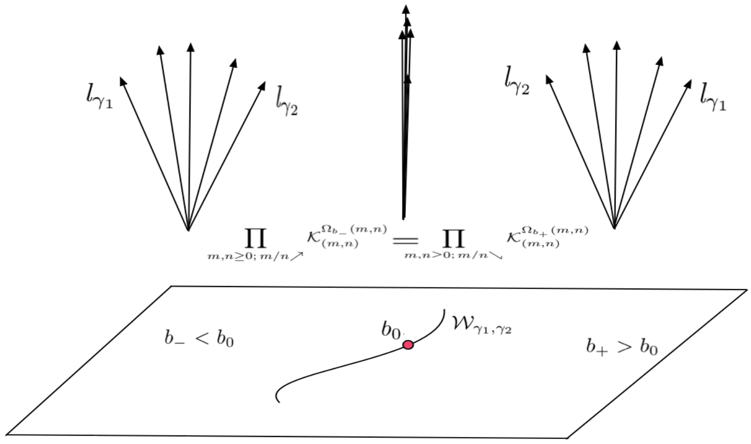

Figure 2.4: Illustration of wall crossing and KSWCF.

From the above figure, we see that b_- is the point over which $l_{\gamma_1} < l_{\gamma_2}$, i.e.,

$$Im\left(\frac{Z(\gamma_1)}{Z(\gamma_2)}\right) = Im(Z((\gamma_1) \cdot \overline{Z(\gamma_2)}) > 0$$

and b_+ is the point in $\mathcal{B}$ over which $l_{\gamma_1} > l_{\gamma_2}$, i.e.,

$$Im\left(\frac{Z(\gamma_1)}{Z(\gamma_2)}\right) = Im(Z(\gamma_1) \cdot \overline{Z(\gamma_2)}) < 0$$

And at the point b_0, we have that

$$Im\left(\frac{Z(\gamma_1)}{Z(\gamma_2)}\right) = Im(Z(\gamma_1) \cdot \overline{Z(\gamma_2)}) = 0$$

where the cone Γ_0 degenerate in to a ray. The KSWCF in this case then specializes into the following

$$\boxed{\overrightarrow{\prod_{l_\gamma \subset \Delta_{\gamma_1,\gamma_2}}} \mathbb{S}_{b_-}(l_\gamma) = \overrightarrow{\prod_{l_\gamma \subset \Delta_{\gamma_1,\gamma_2}}} \mathbb{S}_{b_+}(l_\gamma)} \tag{2.2.5}$$

Or by using the KS-transformation, this reads as

$$\boxed{\overrightarrow{\prod_{m,n\geq 0;\,(m,n)=1}} \mathcal{K}_{(km,kn)}^{\Omega_{b_-}(km,kn)} = \overrightarrow{\prod_{m,n\geq 0;\,(m,n)=1}} \mathcal{K}_{(km,kn)}^{\Omega_{b_+}(km,kn)}} \tag{2.2.6}$$

which is equivalent to the following

$$\boxed{\prod_{m,n\geq 0;\, m/n\nearrow} \mathcal{K}_{(m,n)}^{\Omega_{b_-}(m,n)} = \prod_{m,n\geq 0;\, m/n\searrow} \mathcal{K}_{(m,n)}^{\Omega_{b_+}(m,n)}} \tag{2.2.7}$$

where the product in the LHS is taken over all coprime m, n in the increasing order of $m/n \in \mathbb{Q}$, while the product in the RHS is taken over all coprime m, n in the decreasing order.

When trying to expand the above formulas, we find that it crucially depends on the torus Lie group $\mathfrak{g}$ associated to Γ_0, thus actually depends on the antisymmetric pairing $\langle\cdot,\cdot\rangle : \wedge^2 \Gamma \to \mathbb{Z}$.

In the following, we give some special cases of the formula (2.2.7).

We work in the field $\mathbb{Q}$, the field of rational numbers. This is sufficient for our purpose as the DT-invariants $\Omega(\gamma)$ are a priori given as rational numbers (though conjectured to be integers).

Now, suppose we are about to crossing a point $b_0 \in \mathcal{W}^1_{\gamma_1,\gamma_2}$, then the sublattice $\Gamma_0 \subset \Gamma$ sub-lattice generated by the positive cone $\mathcal{C}_0$ get mapped by Z_{b_0} into a line in $\mathbb{C}$. We denote by x the Lie algebra generator e_{γ_1}, and y that of e_{γ_2}. then the ring of the functions on $\mathbb{T}_{\Gamma_0}$ is given by

$$\mathcal{O}(\mathbb{T}_{\Gamma_0}) = \mathbb{Q}[[x, y]]$$

The antisymmetric pair on Γ_0 is the one induced from that on Γ. Denote by k its value on γ_1, γ_2, i.e., $k = \langle\gamma_1, \gamma_2\rangle$, then it reads for general charges as

$$\langle (x, y), (m, n)\rangle := k(xn - ym) \in \mathbb{Z}$$

for $(x, y) := x \cdot \gamma_1 + y \cdot \gamma_2 \in \Gamma_0$, and $(m, n) := m \cdot \gamma_1 + n \cdot \gamma_2 \in \Gamma_0$.

Also denote by $\mathcal{K}_{m,n}$ the KS transformation associated to the charge (m, n), then we claim that it specializes in this case into the following formula

$$\boxed{\mathcal{K}_{m,n} : (x, y) \mapsto \left(x\,(1 - (-1)^{kmn} x^m y^n)^{-kn},\ y\,(1 - (-1)^{kmn} x^m y^n)^{km}\right)} \tag{2.2.8}$$

Proof. By the formula (2.1.16), the computation goes as follow

$$\begin{aligned}
\mathcal{K}_{m,n} : e_{\gamma_1} &\mapsto (1 - e_{m\gamma_1+n\gamma_2})^{\langle m\gamma_1+n\gamma_2,\gamma_1\rangle} \cdot e_{\gamma_1}\\
&= \left(1 - (-1)^{\langle m\gamma_1,n\gamma_2\rangle}\, e^m_{\gamma_1} \cdot e^n_{\gamma_2}\right)^{-kn} \cdot e_{\gamma_1}\\
&= \left(1 - (-1)^{kmn}\, e^m_{\gamma_1} \cdot e^n_{\gamma_2}\right)^{-kn} \cdot e_{\gamma_1}
\end{aligned}$$

Similarly for the action on e_{γ_2}, we have

$$\begin{aligned}
\mathcal{K}_{m,n} : e_{\gamma_2} &\mapsto (1 - e_{m\gamma_1+n\gamma_2})^{\langle m\gamma_1+n\gamma_2,\gamma_2\rangle} \cdot e_{\gamma_2}\\
&= \left(1 - (-1)^{\langle m\gamma_1,n\gamma_2\rangle}\, e^m_{\gamma_1} \cdot e^n_{\gamma_2}\right)^{km} \cdot e_{\gamma_1}\\
&= \left(1 - (-1)^{kmn}\, e^m_{\gamma_2} \cdot e^n_{\gamma_2}\right)^{km} \cdot e_{\gamma_2}
\end{aligned}$$

Thus follows the formula (2.2.8). $\square$

Example 2.2.1: In the case when $k=1$, the KSWCF (2.2.7) specializes into the following **pentagon identity**

$$\mathcal{K}_{0,1}\,\mathcal{K}_{1,0}=\mathcal{K}_{1,0}\,\mathcal{K}_{1,1}\,\mathcal{K}_{0,1} \tag{2.2.9}$$

This means that on the one side of the wall, we have two "states", namely γ_1,γ_2 with DT-invariants $\Omega(\gamma_1)=\Omega(\gamma_2)=1$. And after crossing the wall, these two "states" still persist with the same DT-invariants, but a new state, namely the "bound" state $\gamma_1+\gamma_2$ with the DT-invariants $\Omega(\gamma_1+\gamma_2)=1$ appears.

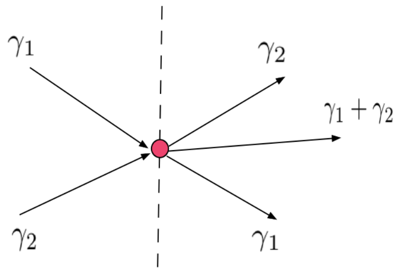

Figure 2.5: Illustration of the pentagon identity, after crossing the wall, the new "state" $\gamma_1+\gamma_2$ is created.

Example 2.2.2: In the case when $k=2$, we have the following KSWCF

$$\mathcal{K}_{2,-1}\,\mathcal{K}_{0,1}=(\mathcal{K}_{0,1}\,\mathcal{K}_{2,1}\,\mathcal{K}_{4,1}\,\cdots)\,\mathcal{K}_{2,0}^{-2}\,(\cdots\mathcal{K}_{6,-1}\,\mathcal{K}_{4,-1}\,\mathcal{K}_{2,-1}) \tag{2.2.10}$$

which is equivalent to

$$\mathcal{K}_{1,0},\mathcal{K}_{0,2}=(\mathcal{K}_{0,2}\,\mathcal{K}_{1,4}\,\mathcal{K}_{2,6}\,\cdots)\,\mathcal{K}_{1,2}^{-2}\,(\cdots\mathcal{K}_{3,4}\,\mathcal{K}_{2,2}\,\mathcal{K}_{1,0}) \tag{2.2.11}$$

or more generally, we have

$$\mathcal{K}_{\gamma_2}\,\mathcal{K}_{\gamma_1}=\left(\prod_{n=1}^{\infty}\mathcal{K}_{n\gamma_1+(n-1)\gamma_2}\right)\mathcal{K}_{\gamma_1+\gamma_2}^{-2}\left(\prod_{n=\infty}^{1}\mathcal{K}_{(n-1)\gamma_1+n\gamma_2}\right) \tag{2.2.12}$$

The interesting aspect of this formula is that it displays that when crossing the wall $\mathcal{W}_{\gamma_1,\gamma_2}$, there are infinitely many new states being created with DT-invariants all equal to one except for the bound state $\gamma_1+\gamma_2$, whose DT-invariant being -2.

As had been pointed out by Greg Moore and Frederik Denef, the factors in the formula gives the BPS spectrum of $N=2$, $d=4$ Super Yang-Mills theory studied by Seiberg and Witten in their seminar paper [SW94], we will come back to the formula later when discussing the wall-crossing structure for Seiberg-Witten integrable systems.

The formula (2.2.9) can be checked by hand easily (see below), while the direct verification of 2.2.10 is much harder (see for example [GMN10]).

LHS of (2.2.9) acts on (x,y) as

$$(x,y)\xrightarrow{\mathcal{K}_{1,0}}(x,y\,(1-y))\xrightarrow{\mathcal{K}_{0,1}}\left(\frac{x}{1-y\,(1-x)},y\,(1-x)\right)$$

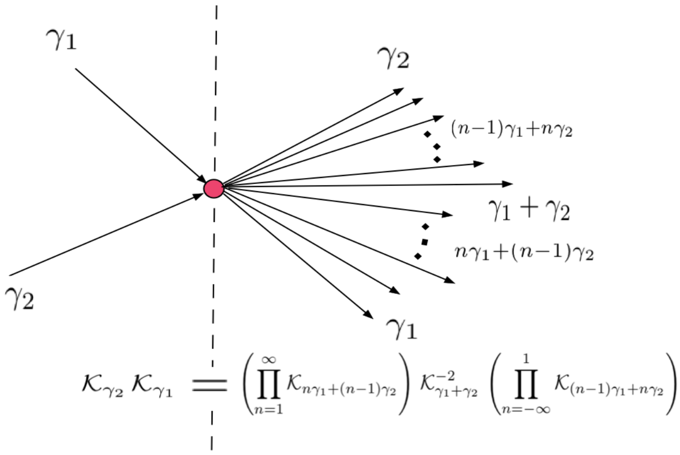

Figure 2.6: KSWCF in $k = 2$ case. After crossing the wall, there are infinitely many new "states" being created

while the RHS of (2.2.9) acts on (x, y) as

$$
\begin{aligned}
(x,y) \xrightarrow{\mathcal{K}_{0,1}} \left(\frac{x}{1-y}, y\right) &\xrightarrow{\mathcal{K}_{1,1}} \left(\frac{\frac{x}{1-y}}{1+\frac{xy}{1-y}}, y\left(1+\frac{xy}{1-y}\right)\right) \\
&= \left(\frac{x}{1-y+xy}, \frac{y\,(1-y+xy)}{1-y}\right) \\
&\xrightarrow{\mathcal{K}_{1,0}} \left(\frac{x}{1-y+xy}, \frac{y(1-y+xy)}{1-y}\left(1-\frac{x}{1-y+xy}\right)\right) \\
&= \left(\frac{x}{1-y\,(1-x)}, \frac{y(1-y+xy)}{1-y}\frac{1-y+xy-x}{1-y+xy}\right) \\
&= \left(\frac{x}{1-y\,(1-x)}, \frac{y(1-y+xy-x)}{1-y}\right) \\
&= \left(\frac{x}{1-y\,(1-x)}, y\,(1-x)\right)
\end{aligned}
$$

2.2.3 Crossing the wall of the 2^{nd} kind

In this subsection, we give a reformulation of the KSWCF in terms of the monodromy of the wall-crossing group elements around small loops. This will motivate the abstract formulation of the wall-crossing structure to be introduced in the next subsection.

First recall that the wall of second kind associated to a charge γ is given as (2.1.26) by the following

$$\mathcal{W}^2_\gamma = \gamma^\perp := \{Z \in \Gamma^*_\mathbb{R} : Im(Z(\gamma)) = 0\}$$

A path $\sigma = (Z_t)_{0\le t\le 1} \subset \Gamma^*$ is said to be *short* if the convexhull of the set:

$$\bigcup_{0\le t\le 1} Z_t\,(Supp\,(a))$$

is a strict sector Δ_σ in $\mathbb{C}^*$ (this definition is borrowed from [KS08], see section 3.4 there). Then we can see that for a generic short path $\sigma = (Z_t)_{0\le t\le 1}$, there exists no more than countably many $t_i \in [0,1]$ and the corresponding primitive $\gamma_i \in \Gamma\backslash\{0\}$ such that $Z_{t_i} \in \mathcal{W}^2_{\gamma_i}$.

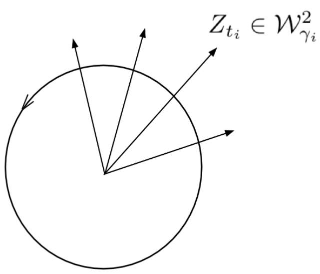

Figure 2.7: The monodromy around a short loop

Proposition 2.2.1. *For any short loop, the monodromy* $\prod^{\circlearrowleft}_{t_i} \mathbb{S}(l_{\gamma_i}) = id$*, where the product is taken in the increasing order of elements* t_i*.*

Proof. We consider an infinitesimal small short loop σ. If it does not circle a point Z_{b_0} such that $b_0 \in \mathcal{W}^1(\gamma_1, \gamma_2)$ for two $\mathbb{Q}$-linearly independent charges γ_1 and γ_2, then clearly the monodromy should be trivial, as can be seen from the case in which there were just two walls of second kind corresponding to γ_1, γ_2, namely, $\mathcal{W}^2_{\gamma_1}$ and $\mathcal{W}^2_{\gamma_2}$. The general case can be seen similarly.

More concretely, we see that in this case the monodromy becomes

$$\mathbb{S}(l_{\gamma_1})\,\mathbb{S}(l_{\gamma_2})\,\mathbb{S}^{-1}(l_{\gamma_2})\,\mathbb{S}^{-1}(l_{\gamma_1})$$

which is trivially the identity map.

Next, let us assume that it circles exactly one such Z_{b_0}. It is the intersection point of infinitely many walls of second kind, namely

$$\mathcal{W}^2_{m\gamma_1+n\gamma_2} \quad \text{for } m, n \ge 0,\ m+n \ge 1.$$

Since the loop is infinitesimal, we can replace the lattice Γ by Γ_0 in the computation (for example, we reduce the problem to rank 2 case), where at b_0, the sub-lattice Γ_0 is get mapped into a line, i.e., it degenerates into a line.

Thus, the computation of the monodromy is reduced to computing the product over the collection of BPS rays $l_{m\gamma_1+n\gamma_2}$.

By our assumption that the loop is short, we can assume that these rays fall into the union of two opposite strict sectors

$$\Delta_\sigma \cup (-\Delta_\sigma) \subset \mathbb{C}^*$$

Then it is clear that as we take the product in the increasing t_i order, we get that the product over the rays belonging to the sector Δ_σ can be identified with the following

$$\prod_{m,n\geq 0;\, m/n \nearrow} \mathbb{S}(l_{m\gamma_1+n\gamma_2})$$

while the product taken over the rays belonging to the opposite sector $-\Delta_\sigma$ becomes

$$\prod_{m,n\geq 0;\, m/n \searrow} \mathbb{S}^{-1}(l_{m\gamma_1+n\gamma_2})$$

Consequently the monodromy around σ is then given by

$$\prod_{m,n\geq 0;\, m/n \nearrow} \mathbb{S}(l_{m\gamma_1+n\gamma_2}) \prod_{m,n\geq 0;\, m/n \searrow} \mathbb{S}^{-1}(l_{m\gamma_1+n\gamma_2})$$

or by using KS transformation notation, it can be further written as

$$\prod_{m,n\geq 0;\, m/n \nearrow} \mathcal{K}_{(m,n)}^{\Omega_{b_-}(m,n)} \prod_{m,n\geq 0;\, m/n \searrow} \mathcal{K}_{(m,n)}^{-\Omega_{b_+}(m,n)}$$

which is the identity, i.e., the triviality of the monodromy around the small loop σ under the assumption that the KSWCF (2.2.7) holds. □

Remark 2.2.3. *In the case that the forgetting map:*

$$\pi : \mathcal{B} \longrightarrow \Gamma^* := Hom_{\mathbb{Z}}(\Gamma, \mathbb{C}) \cong \mathbb{C}^n, \quad b \longmapsto Z_b$$

is a local isomorphism, we can pull back various wall of second kind to the manifold $\mathcal{B}$ which are hypersurfaces in $\mathcal{B}$. Thus, by mimicking the above procedure, we can consider the monodromy around short loops in $\mathcal{B}$, and the triviality of monodromy result still holds in this setting.

2.3 Wall-crossing structure on a vector space

Now we have all that needed to formulate the notion of *wall crossing structure* ("WCS" for short). It generalize the stability data on $\mathfrak{g}$ discussed previously in the sense that the central charge function does not appear explicitly in this structure. This is useful because in practice, we have integrable systems without central charge, thus in order to deal with wall crossing phenomena associated to it, we need to adopt this WCS formalism.

2.3.1 Summarizing the previous facts

Let us first summarize what we have established so far. Given a charge lattice $\Gamma \cong \mathbb{Z}^{\oplus n}$ endowed with an antisymmetric bilinear form $\langle \cdot, \cdot \rangle : \wedge^2 \Gamma \to \mathbb{Z}$. Denote by $\Gamma_{\mathbb{R}} := \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ its associated real vector space. Again, let $\mathfrak{g} := \bigoplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma$ be the Γ-graded Lie algebra over $\mathbb{Q}$, with Lie bracket on it given by

$$[e_{\gamma_i}, e_{\gamma_j}] = (-1)^{\langle \gamma_i, \gamma_j \rangle} \langle \gamma_i, \gamma_j \rangle \cdot e_{\gamma_i + \gamma_j}$$

We have seen that given a stability data $(Z, a(\gamma))$ on $\mathfrak{g}$, the support of $a(\gamma)$, say $Supp\, a$ is the same as the support of $\mathfrak{g}$ defined in (2.1.29), namely

$$Supp\, \mathfrak{g} := \{\gamma \in \Gamma : \mathfrak{g}_\gamma \neq 0\} \subset \Gamma$$

We make the assumption that it is *finite and is contained in an open half-space* in $\Gamma_{\mathbb{R}}$. In this case the Lie algebra $\mathfrak{g}$ is *nilpotent*. Denote by G the corresponding nilpotent Lie group, which, under the exponential map, is bijective with $\mathfrak{g}$, i.e.,

$$\exp : \mathfrak{g} \to G$$

Also recall that the wall (of second kind) associated to γ is the hyerplanes $\gamma^\perp \subset \Gamma^*_{\mathbb{R}}$ given as

$$\mathcal{W}^2_\gamma = \gamma^\perp := \{Z \in \Gamma^*_{\mathbb{R}} : Im(Z(\gamma)) = 0\}$$

Denote by $Wall_{\mathfrak{g}}$ the collection of these hyperplanes. The complement of $Wall_{\mathfrak{g}}$ in $\Gamma^*_{\mathbb{R}}$ consists of a collection of connected components which are open convex domains in $\Gamma^*_{\mathbb{R}}$, called the chambers. These chambers are exactly open strata in the natural stratification of $\Gamma^*_{\mathbb{R}}$ associated with the collection of these hyperplanes $(\gamma^\perp)_{\gamma \in Supp \mathfrak{g}}$.

We know previously that the DT-invariants $\Omega(\gamma)$ are encoded in the group element $\mathbb{S}(\Delta)$ belonging to the wall crossing group $G_\Delta \subset G$, where Δ is an strict sector in $\mathbb{C}^*$. We allow it to degenerate into a ray l, in this case $\mathbb{S}(\Delta)$ becomes $\mathbb{S}(l)$ (see section 2.1.2 for the terminologies and definitions there).

A ray l_γ is associated to a class $\gamma \in \Gamma$ with $\Omega(\gamma) \neq 0$, i.e., $l_\gamma := Z(\gamma) \cdot \mathbb{R}_{>0}$ recall that we have established (see equation (2.1.21)) that

$$\log \mathbb{S}(l_\gamma) \in \bigoplus_{\gamma' \| \gamma} \mathfrak{g}_{\gamma'}.$$

Those group elements stay constant if the central charge moves only inside a given chamber.

But upon deforming the central charge, such that a wall of first kind, say

$$\mathcal{W}(\gamma_1, \gamma_2) = \left\{ Z \in Hom_{\mathbb{Z}}(\Gamma, \mathbb{C}) \right\} : Im\left(Z(\gamma_1) \cdot \overline{Z(\gamma_2)} \right) = 0 \Big\}$$

is crossed, then those DT-invariants $\Omega(\gamma)$ would jump. And its jump is controlled by the KSWCF (2.2.7), which is equivalent to the triviality of certain monodromy alone small short loops in the moduli space of central charge (see proposition 2.2.1). More precisely, if the loop is parametrized by the parameter t, and we have showed that it intersects at t_i with only countably many walls $\mathcal{W}^2_{\gamma_i}$ of second kind associated to the charges γ_i, then the monodromy around any small short loop σ is trivial

$$\prod_{t_i}^{\circlearrowleft} \mathbb{S}(l_{\gamma_i}) = id \tag{2.3.1}$$

2.3.2 Nilpotent case

Now we can formally state the definition of WCS from [KS14].

Definition 2.3.1. *A* ***(global) wall-crossing structure*** *("(global) WCS" for short) for* $\mathfrak{g}$ *is an assignment*

$$(y_1, y_2) \to g_{y_1,y_2} \in G$$

for any $y_1, y_2 \in \Gamma^*_{\mathbb{R}} \backslash Wall_{\mathfrak{g}}$*, which is locally constant in* y_1, y_2*. Further more, the assignment satisfies the following* ***cocycle condition***

$$g_{y_1,y_2} \cdot g_{y_2,y_3} = g_{y_1,y_3} \qquad \forall y_1, y_2, y_3 \in \Gamma^*_{\mathbb{R}} - Wall_{\mathfrak{g}} \tag{2.3.2}$$

such that in the case when the straight interval connecting y_1 *and* y_2 *intersects only with one wall* $\mathcal{W}^2_\gamma = \gamma^\perp$*, then we have that*

$$\log(g_{y_1,y_2}) \in \bigoplus_{\gamma' \| \gamma} \mathfrak{g}_{\gamma'} \tag{2.3.3}$$

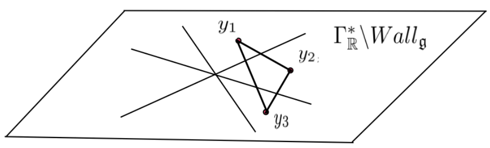

Figure 2.8: Wall-crossing structure on $\mathfrak{g}$

We denote the space of all wall-crossing structures on $\mathfrak{g}$ by $WCS_{\mathfrak{g}}$. We can deduce some easy consequences from the above definition.

Lemma 2.3.1. *If the the straight interval* $\overline{y_1 y_2}$ *does not intersect any walls in* $Wall_{\mathfrak{g}}$*, then* $g_{y_1,y_2} = id \in G$*.*

Proof. let us consider the situation when y_1 and y_2 are separated from y_3 by a single way $\gamma^\perp$ so that the straight intervals $\overline{y_1 y_3}$ ans $\overline{y_2 y_3}$ intersect only with the wall $\overline{y_1 y_3}$, and y_1 and y_2 lie in the same chamber, thus by the locally constant nature of the assignment, we see that $g_{y_1,y_2} = g_{y_1,y_3}$. Consequently, by the cocycle condition (2.3.2), we must have $g_{y_1,y_2} = id$. □

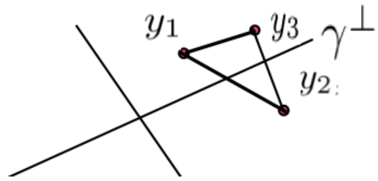

Figure 2.9: Illustration of the proof of the lemma 2.3.1 and lemma 2.3.2

Lemma 2.3.2. *For any $y_1, y_2 \in \Gamma^*_{\mathbb{R}} \backslash Wall_{\mathfrak{g}}$, we have that $g_{y_1,y_2} = g^{-1}_{y_2,y_1}$.*

Proof. First note that if y_1 and y_2 lie in the same chamber, then the straight interval $\overline{y_1 y_2}$ does not intersect any wall, so by lemma 2.3.1., we see that $g_{y_1,y_2} = id$, and the result is trivially true in this case. Thus let us consider a configuration in which y_1 and y_3, which lie in the same chamber, are separated from y_2 by a single wall $\gamma^\perp$. Since the straight interval $\overline{y_1 y_3}$ does not intersect any wall, by lemma 2.3.1 again, we have that $g_{y_1,y_3} = id$. Then in the cocycle condition

$$g_{y_1,y_2} \cdot g_{y_2,y_3} = g_{y_1,y_3}$$

Let y_3 approaches to y_1, we get that $g_{y_1,y_2} \cdot g_{y_2 y_1} = id$. The result follows. □

Lemma 2.3.3. *For any $y_1, y_2, \cdots, y_n \in \Gamma^*_{\mathbb{R}} \backslash Wall_{\mathfrak{g}}$, we have the following wall crossing formula*

$$\boxed{g_{y_1,y_2} \cdot g_{y_2,y_3} \cdots g_{y_{n-1},y_n} \cdot g_{y_n,y_1} = id} \tag{2.3.4}$$

Proof. The $n = 3$ case follows from the lemma 2.3.2. Indeed, the right multiplication by $g_{y_3,y_1} = g^{-1}_{y_1,y_3}$ on both sides of equation (2.3.2) gives us

$$g_{y_1,y_2} \cdot g_{y_2,y_3} \cdot g_{y_3,y_1} = id$$

The general case follows by induction. □

We see from the above discussion that for any codimension one stratum τ, which is an open domain in the hyperplane $\mathcal{W}^2_\gamma = \gamma^\perp$ for some $\gamma \in Supp\, \mathfrak{g}$, we can associate to it a jump

$$g_\tau := g_{y_1,y_2}$$

where points y_1 and y_2 are separated by the wall $\gamma^\perp$ i.e.,

$$y_1(\gamma) > 0, \;\; y_2(\gamma) < 0$$

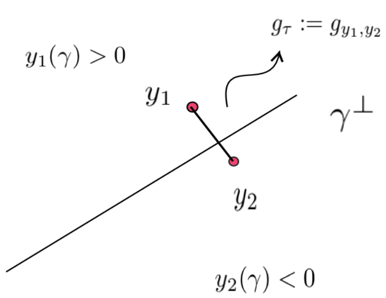

Figure 2.10: The jump associated to the wall

A WCS is uniquely determined by the collection of all jumps $(g_\tau)_{codim\tau=1}$, which satisfies the cocycle condition for each stratum of codimension 2.

More concretely, We can specialize these abstract (wall-crossing) group elements defined above by what we had already known.

Here the jump $g_\tau = g_{y_1,y_2}$ can be identified as the automorphism $\mathbb{S}(l_\gamma)$. And the condition (2.3.3) in the definition of WCS becomes exactly (2.1.21) in section 2.1.3. Besides, if the points y_1 and y_2 separated by $\gamma^\perp$ can be realized in terms of two central charges Z_1 and Z_2 (as two functionals $Im(Z_1)$ and $Im(Z_2)$) in such a way that

$$Im(Z_1)(\gamma) > 0, \;\; Im(Z_2)(\gamma) < 0.$$

Then we can reinterpret the formula (2.3.4) as the KSWCF.

Suppose that all the group elements appearing on the left hand side of 2.3.4 are non-trivial (if some of them are identity element, the result still holds). Thus, the straight intervals $\overline{y_i y_{i+1}}$ that connects y_i and y_{i+1} is assumed to intersect only with the wall $\gamma_i^\perp$, for $i = 1, 2, \cdots, n-1$. Here the straight intervals $\overline{y_n y_1}$ intersect only with another wall $\gamma_n^\perp$. Roughly, we may think that these straight intervals would patch together to form a loop in $\Gamma^*_{\mathbb{R}}$, namely the oriented loop $\sigma = \overrightarrow{y_1 y_2 \cdots y_n y_1}$. Since the assignment is locally constant, we can always move the points locally as long as they do not cross the wall, so we can connect y_i with y_{i+1} for $i = 1, 2, \cdots, n-1$ and y_n with y_1. Then the formula (2.3.4) becomes

$$\mathbb{S}(l_{\gamma_1})\,\mathbb{S}(l_{\gamma_2})\cdots\mathbb{S}(l_{\gamma_n}) = id \tag{2.3.5}$$

or more shortly

$$\prod_{\gamma_i}^{\circlearrowleft} \mathbb{S}(l_{\gamma_i}) = id \tag{2.3.6}$$

This is exactly the KSWCF (2.3.1) in the case when there are only finitely many walls in question (since we are dealing with nilpotent $\mathfrak{g}$ here). In order to recover the general KSWCF (2.3.1), we need to remove the restraints imposed on $Supp\,\mathfrak{g}$, this leads us to deal with the pronilpotent Lie algebra case, which is reminiscent in our discussion of the completion of Lie algebra in section 2.1.4.

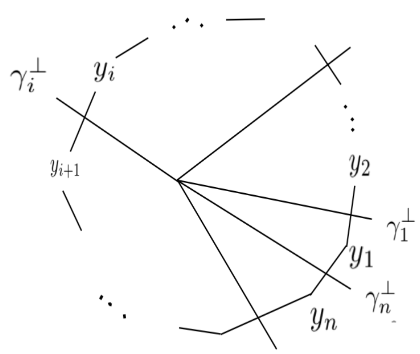

Figure 2.11: KSWCF in terms of cocycle condition

2.3.3 Pronilpotent case

Now we relax the restrictions on $Supp\,\mathfrak{g}$ in the nilpotent case. In particularly, we do not made the assumption that the support belongs to a half-space in $\Gamma_{\mathbb{R}}$.

Let Δ be a strict e sector in $\Gamma_{\mathbb{R}}$ (later when we have central charge function, this can be realized as the pull back of the acute sector in $\mathbb{C}^*$), we can consider as before (see section 2.2.3.) the following constructions

$$\mathfrak{g}_{\Delta} := \prod_{\gamma\in\Delta\cap\Gamma\backslash\{0\}} \mathfrak{g}_{\gamma}$$

which is an pronilpotent Lie algebra as we have an infinite product of Lie algebras. Under the exponential map, we get the corresponding pronilpotent Lie group G_{Δ}. We use without proof the following characterization of strict sectors.

Lemma 2.3.4. *If $\Delta \subset \Gamma_{\mathbb{R}}$ is an strict sector, then there exists a functional $\phi \in \Gamma_{\mathbb{R}}^*$ such that its restriction to Δ gives a proper map to $\mathbb{R}_{\geq 0}$.*

Remark 2.3.1. *In the special case of varing the stability data, ϕ here can be specified as the imaginary part of the central charge Z.*

Let us choose one such functional ϕ. Then for any $N > 0$, we consider the ideal $\mathfrak{g}_{\Delta,\geq N} \subset \mathfrak{g}_{\Delta}$ consisting of elements with $\phi(\gamma) \leq N$ for all $\gamma \in supp(\mathfrak{g})$. And we form the quotient algebra

$$\mathfrak{g}_{\Delta,N} = \mathfrak{g}/\mathfrak{g}_{\Delta,\geq N}$$

By taking the projective limit of the above system, we get the pronilpotent Lie algebra

$$\mathfrak{g}_{\Delta} = \lim_{\overleftarrow{N}} \mathfrak{g}_{\Delta,N}$$

And by exponentiating, we get the corresponding pronilpotent Lie group G_{Δ}. Then we have the natural projection of the groups

$$pr_{\Delta,N} : G_{\Delta} \to G_{\Delta,N} \tag{2.3.7}$$

Again, by the support property, wee see that for each $N > 0$, $Supp\,\mathfrak{g}_{\Delta}$ is finite, and by the assumption that Δ is chosen to be strict. It can be further made to lie in a half space in $\Gamma_{\mathbb{R}}$. Thus, the WCS is well defined on each $\mathfrak{g}_{\Delta,N}$.

Roughly speaking, the WCS on general $\mathfrak{g}$ to be defined later can be seen as the one induced from that on each $\mathfrak{g}_{\Delta,N}$, i.e., it should correspond to some sort of the "projective limit" of the WCS's on $\mathfrak{g}_{\Delta,N}$. Yet we cannot take this "projective limit" directly as the primitive definition of WCS is not suitable for this purpose. So we will find equivalent way of descriptions in the next subsection 2.4.

Anyway, let us assume that this can be done, so we have well defined WCS on the Lie algebra $\mathfrak{g}$, then the number of points in Lemma 2.3.3. can be assumed to be countably many, and for each $N > 0$, we can have a corresponding KSWCF (2.3.4). By taking the limit as $N \to \infty$, we will obtain the KSWCF in more general case, namely

Lemma 2.3.5. *For any* $y_1, y_2, \cdots, y_n, \cdots \in \Gamma^*_{\mathbb{R}} - Wall_{\mathfrak{g}}$, *we have the following wall crossing formula*

$$\left(\prod_{i=1}^{\infty} g_{y_i, y_{i+1}}\right) \cdot g_{y_\infty, y_1} = id \tag{2.3.8}$$

Or equivalently, suppose that the straight intervals $\overline{y_i y_{i+1}}$ *for* $i = 1, 2, \cdots, \infty$ *as well as* $\overline{y_\infty y_1}$ *together patch (by moving these points locally if necessary) to form a short oriented loop* $\sigma = \overrightarrow{y_1 y_2 \cdots y_n \cdots y_\infty y_1}$ *in* $\Gamma^*_{\mathbb{R}}$, *and assume it intersects only with the walls* $\gamma_i^{\perp}$, *for* $i = 1, 2, \cdots, \infty$, *then we have the following KSWCF (2.3.1).*

$$\prod_{1 \leq i \leq \infty}^{\circlearrowright} \mathbb{S}(l_{\gamma_i}) = id \tag{2.3.9}$$

where the product is taken in increasing i *order.*

2.4 Sheaf theoretic description

To begin with, let me make some heuristic remarks. Recall that a WCS on $\mathfrak{g}$ is an assignment for any pair $y_1, y_2 \in \Gamma^*_{\mathbb{R}} \backslash Wall_{\mathfrak{g}}$ a group element $g_{y_1,y_2} \in G$, which in locally constant and satisfies the cocycle condition (2.3.2). Now, instead of a pair of points, we want to assign group element for any given point $y \in \Gamma^*_{\mathbb{R}} \backslash \{0\}$. To this end, we need to consider sort of limit of g_{y_1,y_2} as $y_1 \to y_2$.

If $y \in \Gamma^*_{\mathbb{R}} \backslash \{0\}$ stays in a chamber, then we choose a nearby point y_1 lying in the same chamber, i.e., the straight interval $\overline{yy_1}$ does not intersect any wall. We know that in this case the group element $g_{y,y_1} = id$. So we can define g_y as the limit of g_{y,y_1} as y_1 approaches to y. Say

$$g_y := \lim_{y_1 \to y} g_{y,y_1} = id$$

On the other hand, if $y \in \gamma^\perp$ lies on a wall of second kind, in this case, in order to define g_y, we consider the following situation. Suppose that y is the intersection point of the straight interval $\overline{y_1 y_2}$ with only the wall $\gamma^\perp$. Then as g_{y_1,y_2} stays constant if both y_1 and y_2 do not cross the wall $\gamma^\perp$, so let y_1 and y_2 both approach to y individually within its own chamber, we can define g_y as the limit of g_{y_1,y_2}, which is the same since g_{y_1,y_2} stays constant in this limiting process. Thus we define g_{y_1,y_2} in this case such that

$$\log(g_y) \in \bigoplus_{\gamma' \| \gamma} \mathfrak{g}_{\gamma'} \quad \text{for } y \in \gamma^\perp$$

So we now know how to assign group element g_y for any $y \in \Gamma^*_{\mathbb{R}} \backslash \{0\}$. Later we will see that for $y = 0$, g_0 need to be identified with the entire Lie algebra $\mathfrak{g}$.

In summary, for any $y \in \Gamma^*_{\mathbb{R}} \backslash \{0\}$, the group element g_y is trivial unless y belongs to some wall $\gamma^\perp$, i.e. there exists some $\gamma \in \Gamma \backslash \{0\}$ such that $y(\gamma) = 0$.

We want to have a sheaf on $\Gamma^*_{\mathbb{R}}$ such that its stalk over $y \in \Gamma^*_{\mathbb{R}}$ equals to 0 if $y \in \Gamma^*_{\mathbb{R}} \backslash Wall_{\mathfrak{g}}$, but is a nontrivial element g_y such that $\log(g_y) \in \bigoplus_{\gamma' \| \gamma} \mathfrak{g}_{\gamma'}$ when y belongs to some wall.

It turned out that this can be done ([KS14]), for this we need some preparations.

2.4.1 Existence of the sheaf of WCS

Given $y \in \Gamma^*_{\mathbb{R}}$, let us define subsets in Γ as follows

$$\Delta_+ := \{\gamma \in \Gamma : y(\gamma) > 0\}$$

$$\Delta_0 := \{\gamma \in \Gamma : y(\gamma) = 0\}$$

$$\Delta_- := \{\gamma \in \Gamma : y(\gamma) < 0\}$$

Thus $\Gamma = \Delta_- \sqcup \Delta_0 \sqcup \Delta_+$, and we have the decomposition of the Lie algebra $\mathfrak{g}$ as follows

$$\mathfrak{g} = \mathfrak{g}_-^{(y)} \oplus \mathfrak{g}_0^{(y)} \oplus \mathfrak{g}_+^{(y)}$$

where $\mathfrak{g}_-^{(y)}$, $\mathfrak{g}_-^{(y)}$ and $\mathfrak{g}_+^{(y)}$ correspond to components $\mathfrak{g}_\gamma$ such that γ belongs to Δ_-, Δ_0 and Δ_+ respectively. On the Lie group level, this corresponds to the following decomposition

$$G = G_-^{(y)} \times G_0^{(y)} \times G_+^{(y)}$$

where $G_-^{(y)}$, $G_0^{(y)}$ and $G_+^{(y)}$ are the Lie groups corresponding to $\mathfrak{g}_-^{(y)}$, $\mathfrak{g}_0^{(y)}$ and $\mathfrak{g}_+^{(y)}$ respectively. Consequently, every element $g \in G$ can be decomposed uniquely as the product

$$g = g_-^{(y)}\, g_0^{(y)}\, g_+^{(y)}$$

Then we have the following canonical projection

$$\begin{aligned} \pi_y : G &\longrightarrow G_0^{(y)} = G_-^{(y)} \backslash G \,/ G_+^{(y)} \\ g &\longmapsto g_0^{(y)} \end{aligned} \tag{2.4.1}$$

Recall from the discussion given in the beginning of this subsection, we know that for any $y \in \Gamma_{\mathbb{R}}^*$, the group element g_y associated to y is nontrivial only if $y \in \gamma^\perp$, i.e.,

$$\log(g_y) \in G_0^{(y)}$$

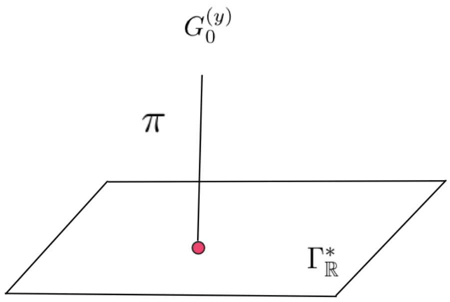

Figure 2.12: Sheaf of wall crossing structures

Claim: There exists a sheaf $\mathcal{WCS}_{\mathfrak{g}}$ of sets on $\Gamma_{\mathbb{R}}^*$ with the stalk over $y \in \Gamma_{\mathbb{R}}^*$ given by $G_0^{(y)}$, we call it the **sheaf of wall crossing structure**.

To show its existence, given a set S, recall that a sheaf of sets $\mathscr{S}$ over a topological space M with stalk at $m \in M$ being a set S_m is equivalent to a local homeomorphism from its étalé space $\mathscr{S}^{ét}$ consisting of pairs $\{(m, s) : m \in M, s \in S_m\}$ to the base M given by

$$\mathscr{S}^{ét} \longrightarrow M, \ (m, s) \longmapsto m \tag{2.4.2}$$

here the bases of the topology on *étalé* space $\mathscr{S}^{ét}$ is given by the sets

$$\mathscr{U}_{s,U} = \{(m, s') : m \in U, s' = \pi_m(s)\}$$

where s runs through the set S and U runs through the set of open subsets on M, and π_m is the surjection

$$\pi_m : S \longrightarrow S_m$$

We need the following well-known lemma presented in [KS14].

Lemma 2.4.1. *If for any $s_1, s_2 \in S$ the set $\{m \in M : \pi_m(s_1) = \pi_m(s_2)\}$ is open in M, then the projection (2.4.2) is a local homeomorphism.*

Given this lemma, and apply it to the case when $M = \Gamma^*_{\mathbb{R}}, S = G, S_y = G_0^{(y)}, y \in M$ and the surjection map π_m given as above, then the existence of the sheaf of wall crossing structure $\mathcal{WCS}_{\mathfrak{g}}$ amounts to verifying that for any $g_1, g_2 \in \Gamma^*_{\mathbb{R}}$, the following set

$$\{y \in \Gamma^*_{\mathbb{R}} : \pi_y(g_1) = \pi_y(g_2)\}$$

is open in $\Gamma^*_{\mathbb{R}}$, but this is obvious since the projection map π_m to the double coset is naturally continuous under the quotient topology. Later, we will show that space of sections $\Gamma(\Gamma^*_{\mathbb{R}}, \mathcal{WCS}_{\mathfrak{g}})$ of the sheaf $\mathcal{WCS}_{\mathfrak{g}}$ is isomorphic to the space of all wall-crossing structures on $\Gamma^*_{\mathbb{R}}$, namely the space $WCS_{\mathfrak{g}}$.

2.4.2 Equivalent descriptions of WCS

Given any open subset $U \in \Gamma^*_{\mathbb{R}}$, the space of section over U, denoted by $\Gamma(U, \mathcal{WCS}_{\mathfrak{g}})$, roughly speaking, can be identified as the set of locally constant maps from the set of connected components of intersections of codimension one strata with U to the corresponding subgroups of G, which satisfy the cocycle condition near points of strata of codimension two.

More precisely, we defines the convex hull of U by $\Delta(U) \subset \Gamma_{\mathbb{R}}$, and denote by $\Delta_{\pm}(U)$ the subsectors generated by those $\gamma \in \Gamma$ such that $\pm y_\gamma > 0$ for all $y \in U$. By our assumptions on *Supp* $\mathfrak{g}$, all these sectors are strict, and we can associate it with the corresponding nilpotent Lie groups as

$$G_{\pm}(U) = \exp\left(\bigoplus_{\gamma \in \Delta_{\pm}(U)} \mathfrak{g}_\gamma\right)$$

then we have that

$$\Gamma(U, \mathcal{WCS}_{\mathfrak{g}}) \cong G_-(U) \backslash G / G_+(U)$$

Given a WCS, i.e., a collection of jumps (g_τ) for all codimension one strata τ, we have already know how to associate it with a sheaf of wall crossing structure. Conversely, given a section $s \in \Gamma(\Gamma^*_{\mathbb{R}}, \mathcal{WCS}_{\mathfrak{g}})$, we want to associate it with an unique wall crossing structure as follows

For any $y_1, y_2 \in \Gamma^*_{\mathbb{R}}$, if the straight interval $\overline{y_1 y_2}$ lies in the same chamber, then g_{y_1,y_2} is deemed to be the identity. On the other hand, if $\overline{y_1 y_2}$ intersects only with the wall $\gamma^\perp$ at y_0, then the transformation g_{y_1,y_2} is given by $s(y_0) \in G_0^{(y_0)}$, which is non-trivial since $y_0(\gamma) = 0$.

We want to show that this assignment, which gives us a map from $\Gamma(\Gamma^*_{\mathbb{R}}, \mathcal{WCS}_{\mathfrak{g}})$ to $WCS_{\mathfrak{g}}$ is a well defined map.

Proposition 2.4.2. *The above assignment (map) is well defined.*

To proof this proposition, we need a lemma first.

Lemma 2.4.3. *$\Gamma^*_{\mathbb{R}}\backslash Wall_{\mathfrak{g}}$ contains two components $U_\pm$ characterized by that it consists of all points $y \in \Gamma^*_{\mathbb{R}}$ such that $y(\gamma) > 0$ (resp. $y(\gamma) < 0$) for all $\gamma \in Supp\,\mathfrak{g}$. Thus we can associate with any WCS $\sigma = (g_{y_1,y_2})$ an group element $g_{+,-} := g_{y_+,y_-} \in G$, for $y_\pm \in U_\pm$.*

Then consider $l \subset \Gamma^*_{\mathbb{R}}$ is a straight line intersecting both U_+ and U_-, endowed with the direction from U_+ to U_-. Assume that it does not intersect strata of codimension ≥ 2, and intersects with walls at $y_1, \cdots, y_n$, ordered according to the direction of l. Then, we have the following

a) the space of sections over l can be identified as

$$\Gamma(l, \mathcal{WCS}_{\mathfrak{g}}) \cong \prod_{i=1}^{n} G_0^{(y_i)} \tag{2.4.3}$$

b) further more, we have the following bijection of Lie algebras

$$\bigoplus_{i=1}^{n} \mathfrak{g}_0^{(y_i)} \cong \mathfrak{g} \tag{2.4.4}$$

c) and correspondingly a bijection among Lie groups

$$\prod_{i=1}^{n} G_0^{(y_i)} \cong G \tag{2.4.5}$$

Thus, any element $g \in G$ can be written uniquely as the ordered product of elements of $G_0^{(y_i)}$.

d) As a consequence of c), the space of sections can be further identified as

$$\Gamma(l, \mathcal{WCS}_{\mathfrak{g}}) \cong G \tag{2.4.6}$$

Proof. (of the lemma 2.4.3) a) is obvious. It suffices to prove b) as c).

Define the subsets $\Delta_i^0 \subset \Gamma^*_{\mathbb{R}}$ for $i = 1, \cdots, n$ as

$$\Delta_i^0 := \{\gamma \in Supp\,\mathfrak{g} : y_i(\gamma) = 0\}$$

then we show that we have the following decomposition

$$\Gamma_{\mathbb{R}} = \bigsqcup_{i=1}^{n} \Delta_i^0$$

Indeed, $\forall \gamma \in \Gamma_{\mathbb{R}}\backslash\{0\}$, since the straight line l goes through from U_+ to U_- with the only intersection points with wall at $y_1, \cdots, y_n$, wee see that the charge γ can be annihilated only at one of the above n points.

From the above the decomposition in a) follows, and b) also follows directly. $\square$

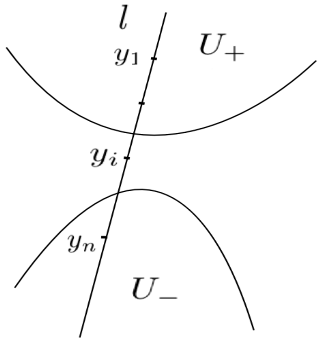

Figure 2.13: Illustration of the lemma 2.4.3

Proof. (of the proposition 2.4.2) We just need to check that the cocycle condition holds for all strata of codimension 2, that is, we need to check the triviality of monodromy around small loops. So let us consider such a strata κ which is being realized as the intersection of, say n hyperplanes $\gamma_i^\perp$ for $i = 1, 2, \cdots, n$ in general position, which are ordered in the increasing i. Thus, locally around κ, we have $2n$ codimension two open strata separated by $2n$ codimension one strata. Denote by $y_\pm$ the point in $\Gamma^*_{\mathbb{R}}$ such that $y_\pm(\gamma_i) = 0$, for all $i = 1, \cdots, n$. It is easy to see that these two points must lie in two separate ones among these $2n$ open strata.

In order to show the monodromy is trivial, we consider the paths connecting the two points going from y_+ to y_-. Up to homotopy, near the strata κ, there are two such paths, each of which intersects at these walls $\gamma^\perp$ at exactly one point. Without loss of generality, let us assume that y_+ lies in the strata neighboured by the walls $\gamma_1^\perp$ and $\gamma_2^\perp$, and y_- lies in the strata neighboured by the walls $\gamma_k^\perp$ and $\gamma_{k+1}^\perp$ for some k such that $1 < k < n$.

Under these assumptions ,we see that the first path σ_1, which intersects with the walls at the points ordered as $y_2^0 \to \cdots \to y_k^0$, while the second one σ_2 intersects with the walls at the points ordered as $y_1^0 \to y_n^0 \to y_{n-1}^0 \to \cdots \to y_{k+1}^0$, where $y_i^0 \in \gamma_i^\perp$ for each i.

Then by viewing these two special open strata which contains points $y_\pm$ as $U_\pm$ in Lemma 2.4.3, we can infer from the lemma that the composition of jumps along the path σ_1 is given by

$$g_0^{(y_2)} \cdots g_0^{(y_k)}$$

which equals to $g_{+,-}$ by lemma 2.4.3. And similarly the composition of jumps along the path σ_2 is given by

$$\left(g_0^{(y_1)}\right)^{-1} \left(g_0^{(y_n)}\right)^{-1} \cdots \left(g_0^{(y_{k+1})}\right)^{-1}$$

which also equals to $g_{+.-}$ by lemma 2.4.3. Thus we get that

$$g_0^{(y_2)} \cdots g_0^{(y_k)} = \left(g_0^{(y_1)}\right)^{-1} \left(g_0^{(y_n)}\right)^{-1} \cdots \left(g_0^{(y_{k+1})}\right)^{-1}$$

from which it follows that

$$\prod_k^{\circlearrowleft} g_0^{(y_k)} = id$$

This completes our proof of the triviality of monodromy. $\square$

It is not so obvious to establish the equivalence between $WCS_{\mathfrak{g}}$ and $\mathcal{WCF}_{\mathfrak{g}}$ directly, for this reason, we prove the following more general result.

Proposition 2.4.4. *We have the following canonical identification of sets*

$$G \cong \Gamma(\Gamma^*_{\mathbb{R}}, \mathcal{WCF}_{\mathfrak{g}}) \cong WCS_{\mathfrak{g}}$$

Before proving the proposition, we give some explanations. The map from G to $\Gamma(\Gamma^*_{\mathbb{R}}, \mathcal{WCF}_{\mathfrak{g}})$ is given by

$$\phi : G \longrightarrow \Gamma(\Gamma^*_{\mathbb{R}}, \mathcal{WCF}_{\mathfrak{g}}), \quad g \longmapsto \phi(g)(y) := \pi_y(g) = g_0^{(y)} \tag{2.4.7}$$

where the projection π_y is given as in (2.4.1). We have already defined the map from $\Gamma(\Gamma^*_{\mathbb{R}}, \mathcal{WCF}_{\mathfrak{g}})$ to $WCS_{\mathfrak{g}}$, and we denote this map by ψ. We can also define a map from $WCS_{\mathfrak{g}}$ to G by

$$\mu : WCS_{\mathfrak{g}} \longrightarrow G, \quad \sigma \longmapsto g_{+,-} := g_{+,-}(\sigma) \tag{2.4.8}$$

here the group element $g_{+,-}$ is defined as in lemma 2.4.3. All maps are well defined. And the proposition 2.4.4 is proved by the following three lemmas.

Lemma 2.4.5. *The map ϕ is a bijection.*

Proof. For any y, since every element $g \in G$ can be decomposed uniquely as the product

$$g = g_-^{(y)} g_0^{(y)} g_+^{(y)}$$

If $\phi(g)(y) = g_0^{(y)} = 0$, then $g = 0$, thus the map ϕ is injective. On the other hand, as $0 \in \Gamma^*_{\mathbb{R}}$ belongs to the closure of any stratum, therefore any section is uniquely determined by its value at 0, this shows the map ϕ is surjective. □

Lemma 2.4.6. *The composition of maps $\mu \circ \psi \circ \phi : G \to G$ is the identity map.*

Proof. Consider the situation as in lemma 2.4.3, by the decomposition of Lie group G (c.f., (2.4.5)), any $g \in G$ can be decomposed as the ordered product of $g_0^{(y_i)}$, but in this case $g = g_{+,-}$. Thus the lemma follows. □

Lemma 2.4.7. *The map μ is an injection.*

Proof. Recall that a WCS σ is determined by the jumps $g_0^{(y)}$ for $y \in \gamma^\perp$ and some $\gamma \in \Gamma$. We consider a straight line l as in lemma 2.4.3, which intersects the walls at $y_1, \cdots, y_n$. Without loss of generality, we can assume $y = y_k$ for $1 \leq k \leq n$. Then by (2.4.4), we infer that $g_{+,-}$ uniquely determines all jumps $g_0^{(y_i)}$. In particularly, the jump $g_0^{(y)} = g_0^{(y_k)}$. As this holds for all such y, we see that the WCS σ is uniquely determined by the element $g_{+,-}$. This finishes the proof of the lemma. □

Finally, we can give a proof of the proposition 2.4.4 as follows:

Proof. (of the Proposition 2.4.4) First, by lemma 2.4.6, we see that for any $g \in G$, we have that

$$g = \mu \circ \psi \circ \phi(g) = \mu \circ (\psi \circ \phi(g))$$

thus μ is an surjection. By combining it with the lemma 2.4.7, we conclude that μ is a bijection.

However, lemma 2.4.5 says that ϕ is also bijective, thus we have that the map $\psi = \mu^{-1} \circ \phi^{-1}$ is also a bijection. □

2.4.3 WCS for general $\mathfrak{g}$ and the support of the sections

We can use this sheaf theoretic description to define the $WCS_{\mathfrak{g}}$ for $\mathfrak{g}$ with general $Supp\,\mathfrak{g}$. Fix an strict sector $\Delta \in \Gamma_{\mathbb{R}}$, with the associated completed torus Lie algebra $\mathfrak{g}_{\Delta}$ as before, then **the sheaf of wall crossing structure** $\mathcal{WCS}_{g_{\Delta}}$ is defined as the projective limit of the sheaves $\mathcal{WCS}_{g_{\Delta,N}}$ for $N > 0$, i.e.

$$\mathcal{WCS}_{g_{\Delta}} := \lim_{\substack{\longleftarrow \\ N}} \mathcal{WCS}_{g_{\Delta,N}}$$

where we have used implicitly a map $\phi \in \Gamma_{\mathbb{R}}^{*}$ in order to obtain a filtration (this will be assumed in the following discussion). More generally, the strict sector Δ may depends on point $y \in \Gamma_{\mathbb{R}}^{*}$, in order to deal with such situation, we give the following definition.

Definition 2.4.1. ***(WCS for general*** $\mathfrak{g}$***)**: The sheaf $\mathcal{WCS}_{\mathfrak{g}}$ on $\Gamma_{\mathbb{R}}^{*}$ is described by the space of sections $\Gamma(U, \mathcal{WCS}_{\mathfrak{g}})$ over any open subset $U \subset \Gamma_{\mathbb{R}}^{*}$, which consists of a family of elements $a(y,\gamma) \in \mathfrak{g}_{\gamma}$ such that $y \in U, \gamma \in \Gamma\backslash\{0\}$ and $y(\gamma) = 0$ which satisfies the following conditions:*

a) for any $y \in U$, there exists a neighborhood U_y and strict sector $\Delta_{y,U_y} \subset \Gamma_{\mathbb{R}}$ such that for any $y' \in U_y$ the element $a(y',\gamma) \neq 0$ iff $\gamma \neq 0$ and $\gamma \in \Delta_{y,U_y}$.

b) For any $N > 0$, the image of the following projected elements (c.f., (2.3.7))

$$pr_{\Delta_{y,U_y},N}\left(\exp\left(\sum_{\gamma} a(y',\gamma)\right)\right)$$

for $y' \in U_y$, belong to the space of sections $\Gamma\left(U_y, \mathcal{WCS}_{\mathfrak{g}_{\Delta_{y,U_y},N}}\right)$.

Proposition 2.4.8. *Given a strict sector $\Delta \subset \Gamma_{\mathbb{R}}\backslash\{0\}$, we assume that $Supp\,\mathfrak{g} \subset \Delta$, then we have the following one to one correspondence*

$$\Gamma\left(\Gamma_{\mathbb{R}}^{*}, \mathcal{WCS}_{\mathfrak{g}}\right) \longleftrightarrow G_{\Delta}$$

Proof. By projecting a global section $\sigma = (a(y,\gamma))$ of $\mathcal{WCS}_{\mathfrak{g}}$ into the Lie subalgebra $\mathfrak{g}_{\Delta,N} \subset \mathfrak{g}_{\Delta}$, the one to one correspondence follows from proposition 2.4.4, and by letting $N \to \infty$, we get the desired correspondence. □

Definition 2.4.2. ***(support of a section)*** *Given a section $\sigma \in \Gamma(U, \mathcal{WCS}_{\mathfrak{g}})$ over an open subset $U \subset \Gamma_{\mathbb{R}}^{*}$, define its support, denoted by $Supp\,\sigma$ to be the minimal closed subset of $U \times \Gamma_{\mathbb{R}}$ such that it is conic in the direction of $\Gamma_{\mathbb{R}}$ and contains the set of pairs $(y,\gamma) \in U \times \Gamma_{\mathbb{R}}$, such that $y(\gamma) = 0$ as well as $\log\left(g_0^{(y)}\right)_{\gamma} \in \mathfrak{g}_{\gamma}\backslash\{0\}$.*

2.5 Wall-crossing structure on a topological space

Like in the case of continuous family of stability data, we would like the space of wall-crossing structure $WCS_{\mathfrak{g}}$, or equivalently the sheaf of wall crossing structure $\mathcal{WCS}_{\mathfrak{g}}$ to be parametrized by a topological manifold $\mathcal{B}$. This is useful in dealing with the wall crossing structures associated with complex integrable systems to be discussed in later chapters. For this purpose, we need to introduce the notion of *wall-crossing structure on* $\mathcal{B}$.

let us assume that $\mathcal{B}$ is a Hausdorff locally connected topological space endowed with a local system of finitely generated free abelian groups of finite rank

$$\pi : \underline{\Gamma} \longrightarrow \mathcal{B}$$

Besides, we need a local system of $\underline{\Gamma}-$graded Lie algebras over $\mathcal{B}$, i.e.,

$$\underline{\mathfrak{g}} = \bigoplus_{\gamma \in \underline{\Gamma}} \mathfrak{g}_{\gamma}$$

defined over the field of rational numbers $\mathbb{Q}$, i.e, $\mathfrak{g}_{\gamma} = \mathbb{Q}\langle e_{\gamma}\rangle$, with e_{γ} the generator of the Lie algebra. Under the assumption that the local system $\underline{\Gamma}$ is equipped with a covariatly constant system of antisymmetric bilinear pairs $\langle \cdot, \cdot \rangle_b$. Then the Lie bracket on $\underline{\mathfrak{g}}$ is induced from (2.1.9).

Now, given a homomorphism of sheaves of abelian groups

$$Y : \underline{\Gamma} \longrightarrow \mathscr{O}(\mathcal{B})$$

where $\mathscr{O}(\mathcal{B})$ denotes the sheaf of real valued continuous function on $\mathcal{B}$.

Note that the above map is equivalent, under duality to the following map (also denoted by Y), which is locally continuous

$$Y : \mathcal{B} \longrightarrow \Gamma^*_{\mathbb{R}} \tag{2.5.1}$$

Denote by $\mathcal{WCS}_{\underline{\mathfrak{g}},Y}$ the sheaf on $\mathcal{B}$ defined as the pull back of the sheaf of wall-crossing structure $\mathcal{WCS}_{\mathfrak{g}}$ on $\Gamma^*_{\mathbb{R}}$, i.e.,

$$\mathcal{WCS}_{\underline{\mathfrak{g}},Y} := Y^*(\mathcal{WCS}_{\mathfrak{g}}) \tag{2.5.2}$$

and we call this the *sheaf of wall-crossing structure* on $\mathcal{B}$ induced by the map Y, then we can give the following definition.

Definition 2.5.1. *(WCS on $\mathcal{B}$): A (global) wall crossing structure on $\mathcal{B}$ is a global section of the sheaf $\mathcal{WCS}_{\underline{\mathfrak{g}},Y}$.*

Remark 2.5.1. *Wee see that when our space $\mathcal{B}$ specializes into a single point $\{*\}$, then the sheaf of WCS on $\{*\}$ is nothing but the global WCS on $\Gamma^*_{\mathbb{R}}$. Thus, we can view $\mathcal{WCS}_{\mathfrak{g}}$ as the special case of $\mathcal{WCS}_{\underline{\mathfrak{g}},Y}$.*

In parallel with the definition 2.4.1 in the last section, we have the corresponding description of the sheaf $\mathcal{WCS}_{\underline{\mathfrak{g}},Y}$ as follows:

Definition 2.5.2. *The sheaf $\mathcal{WCS}_{\underline{\mathfrak{g}},Y}$ on $\mathcal{B}$ is described by the space of sections $\Gamma(U, \mathcal{WCS}_{\underline{\mathfrak{g}},Y})$ over any open subset $U \subset \mathcal{B}$, which consists of a family of elements $a(b,\gamma) \in \mathfrak{g}_{\gamma,b}$ such that $b \in U, \gamma \in \Gamma_b \backslash \{0\}$ and $Y(b)(\gamma) = 0$ that satisfies the following conditions:*

a) for $b \in U$, there exists a neighborhood U_b and a strict sector $\Delta_{b,U_b} \subset \Gamma_{\mathbb{R},b}$ such that for any $b' \in U_b$ the element $a(b',\gamma) \neq 0$ iff $\gamma \neq 0$ and $\gamma \in \Delta_{b,U_b}$.

b) For any $N > 0$, the image of the following projected elements (c.f., (2.3.7))

$$pr_{\Delta_{b,U_b},N}\left(\exp\left(\sum_{\gamma} a(b',\gamma)\right)\right)$$

for $b' \in U_b$, belong to the space of sections $\Gamma\left(U_b, \mathcal{WCS}_{\mathfrak{g}_{\Delta_{b,U_b},N}}\right)$.

Remark 2.5.2. *Clearly, the pairs $(b,\gamma) \in U \times \underline{\Gamma}_{\mathbb{R}}$ in the above definition that satisfies those constraints should be viewed as the support of the sections.*

More precisely, we have the following definition.

Definition 2.5.3. ***(Support of a section)*** *Given a WCS σ on $\mathcal{B}$, i.e., $\sigma \in \Gamma(\mathcal{WCS}_{\underline{\mathfrak{g}},Y})$, we define its support $Supp\,\sigma$ to be a closed subset of of $tot(\underline{\Gamma}_{\mathbb{R}})$ i.e. the total space of the sheaf $\underline{\Gamma}_{\mathbb{R}}$, whose fiber $Supp_b\,\sigma$ over any point $b \in \mathcal{B}$ is given by an strict sector $\Delta_{b,\sigma} \subset \underline{\Gamma}_{\mathbb{R},b}$, which equals to the support of the germ of $\mathcal{WCS}_{\underline{\mathfrak{g}},b}$ at the point $Y(b) \in \underline{\Gamma}^*_{\mathbb{R},b}$ associated with the section σ.*

By definition 2.4.2, we see that $Supp\,\sigma$ also admits the following concrete description.

Given a section $\sigma \in \Gamma(U, \mathcal{WCS}_{\underline{\mathfrak{g}},Y})$ over an open subset $U \subset \mathcal{B}$, $Supp\,\sigma$ is an minimal closed subset of $U \times \Gamma_{\mathbb{R}}$, which is conic in the direction of $\Gamma_{\mathbb{R}}$, and contains the set of pairs $(b,\gamma) \in U \times \Gamma_{\mathbb{R}}$, such that $Y(b)(\gamma) = 0$ and $\log\left(g_0^{(Y(b))}\right) \in \mathfrak{g}_\gamma \backslash \{0\}$.

Through the map Y in the definition, we can also define some hypersurfaces in the topological space $\mathcal{B}$, and we also call these the wall of the second kind. They are defined as the pull back of the walls in $\Gamma^*_{\mathbb{R}}$.

Definition 2.5.4. ***(walls of second kind in $\mathcal{B}$)*** *For every $\gamma \in \Gamma\backslash\{0\}$, the wall of second kind associated to it is defined as*

$$\mathcal{W}^2_\gamma = \gamma^\perp := \{b \in \mathcal{B} : Y(b)(\gamma) = 0\} \tag{2.5.3}$$

2.6 Examples of wall crossing structures

Example a): $Stab(\mathfrak{g})$ recovered.

Suppose we are given a free abelian group of finite rank Γ together with a Γ-graded Lie algebra $\mathfrak{g} = \bigoplus_{\gamma\in\Gamma} \mathfrak{g}_\gamma$, and a central charge function $Z : \Gamma \to \mathbb{Z}$.

Given such data, we know how to define the notion of stability data on $\mathfrak{g}$. We now show that $Stab(\mathfrak{g})$ can be recovered by considering the WCS on a circle.

Take $\mathcal{B} = S_\theta = \mathbb{R}/2\pi\mathbb{Z}$, and endow it with a local system of lattices $\underline{\Gamma}$ with fiber being Γ. Also, we equip it with a local system of $\underline{\Gamma}$-graded Lie algebra $\underline{\mathfrak{g}}$. We then consider the following map:

$$\begin{aligned} Y : S_\theta &\longrightarrow \Gamma^*_{\mathbb{R}} \\ \theta &\longmapsto Y_\theta(\gamma) := Im(e^{-i\theta} Z(\gamma)), \ \text{for}\, \theta\in\mathbb{R}/2\pi\mathbb{Z},\ \gamma\in\Gamma \end{aligned} \tag{2.6.1}$$

Proposition 2.6.1. *The wall-crossing structure on $\mathbb{R}/2\pi\mathbb{Z}$ induced by the above map Y is the same as a stability data: $\sigma = (Z, a(\gamma)) \in Stab\,(\mathfrak{g})$.*

Proof. Given a section $s \in \Gamma(\mathcal{WCF}_{\underline{\mathfrak{g}},Y})$, we want to associate it to a stability data $\sigma = (Z, a(\gamma)) \in Stab\,(\mathfrak{g})$. The central charge function Z is the same. In order to define $a(\gamma) \in \mathfrak{g}_\gamma$, we recall that the stalk of the wall-crossing sheaf on S_θ at a point $\theta \in S_\theta$ is given by the stalk of the wall-crossing structure sheaf on $\Gamma^*_{\mathbb{R}}$ at the point $Y(\theta)$, i.e.,

$$\left(\mathcal{WCS}_{\underline{\mathfrak{g}},Y}\right)_\theta = (\mathcal{WCS}_{\mathfrak{g}})_{Y(\theta)}$$

Thus, we see that if $Y(\theta)$ belongs to some wall of the second kind $\gamma^\perp$, i.e., if $Y(\theta)(\gamma) = 0$, then we can associate to it a "jump": $\mathbb{S}(l_\gamma) \in G$ such that

$$\log \mathbb{S}(l_\gamma) \in \bigoplus_{\gamma'\|\gamma} \mathfrak{g}_{\gamma'}$$

where $\mathbb{S}(l_\gamma)$ is the automorphism of the torus Lie algebra associated to the ray

$$l_\gamma = Z(\gamma)\cdot\mathbb{R}_{>0} \subset \mathbb{C}^* \cong \mathbb{R}^2$$

which by the condition that $Y(\theta)(\gamma) = Im(e^{-i\theta}Z(\gamma)) = 0$, can also be written as

$$l_\gamma = e^{i\theta}\cdot\mathbb{R}_{>0} \subset\cong \mathbb{R}^2 \tag{2.6.2}$$

Then, the components of $\log \mathbb{S}(l_\gamma)$ which belongs to g_γ are deemed as $a(\gamma)$ in the stability data $(Z, a(\gamma))$. Apparently, this association is unique.

Conversely, given a stability data $(Z, a(\gamma)) \in Stab\,(\mathfrak{g})$, we can have the group element $g_{+,-} \in G$ as defined in Lemma 2.4.3, then by the proposition 2.4.4, this determines uniquely a section of $\mathcal{WCS}_{\mathfrak{g}}$, which after pulling back through the map Y, gives arise to a section of $\mathcal{WCS}_{\underline{\mathfrak{g}},Y}$. This completes the proof. □

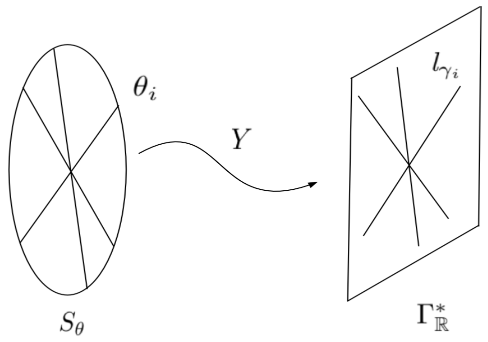

Figure 2.14: Stability data on $\mathfrak{g}$ as WCS

From the above proof and the definition 2.5.4, we see that the walls in this case are described as

$$\begin{aligned}\mathcal{W}^2_\gamma = \gamma^\perp &:= \{\theta \in S_\theta : Y(\theta)(\gamma) = Im(e^{-i\theta}Z(\gamma)) = 0\} \\ &= \{\theta \in S_\theta : \theta = arg\, Z(\gamma)\}\end{aligned}$$

Consider a collection of walls in S_θ, namely $\theta_1, \cdots, \theta_i, \cdots, \theta_n$ such that

$$\theta_i = arg\, Z(\gamma_i) \qquad \text{for all}\, i$$

We view the circle S_θ as the unit circle in the plan $\mathbb{R}^2$, and identify each θ_i as the ray passing through the origin in the direction specified by the angle θ_i, we note that this is nothing but the ray l_{γ_i} defined previously. Thus, given any loop σ circling around the origin, the triviality of monodromy in this case states

$$\prod_i^{\circlearrowright} \mathbb{S}(l_{\gamma_i}) = id$$

which is of course quite familiar to us. (c.f., proposition 2.2.1).

In order to recover $Stab\,(\mathfrak{g})$, the space of all stability conditions on $\mathfrak{g}$, we just need to consider the WCS on the space $\widehat{\mathcal{B}} := S_\theta \times Hom_{\mathbb{Z}}(\Gamma, \mathbb{C})$, endowed with a local system of charge lattice $\underline{\Gamma}$ and a local system of $\underline{\Gamma}$-graded Lie algebra $\underline{\mathfrak{g}}$.

The map Y in this case is given by

$$Y : S_\theta \times Hom_{\mathbb{Z}}(\Gamma, \mathbb{C}) \longrightarrow \Gamma^*_{\mathbb{R}}$$

$$(\theta, Z) \longmapsto Y(\theta, Z)(\gamma) := Im(e^{-i\theta}Z(\gamma)),\ \forall \gamma \in \Gamma \backslash \{0\}$$

Proposition 2.6.2. *A section of the sheaf $\mathcal{WCS}_{\underline{\mathfrak{g}},Y}$ on the space $\widehat{\mathcal{B}}$ is same as a family of stability data on $\mathfrak{g}$.*

Proof. If we fix $Z \in Hom_{\mathbb{Z}}(\Gamma, \mathbb{C})$, i.e., we consider the sub space $\mathcal{B}_Z = S_\theta \times \{Z\} \subset \widehat{\mathcal{B}}$, then by restricting the wall-crossing structure sheaf $\mathcal{WCS}_{\underline{\mathfrak{g}},Y}$ to this subspace, it is reduced to the situation considered in the proposition 2.6.1 above. Hence, a stability data can be associated. By moving Z in $Hom_{\mathbb{Z}}(\Gamma, \mathbb{C})$ locally, we get a stability data for nearby central charges. Since the circle S_θ is compact, we get a germ of universal family of stability data with central charges in a neighborhood of Z. $\square$

Example b): Continuous family of stability data recovered

In application, we usually use a complex manifold $\mathcal{B}$ to parametrize $Stab\,(\mathfrak{g})$, which have already been discussed in details in previous sections. Here, we want to realize it as the WCS on the space $\mathcal{B}_\theta := \mathcal{B} \times S_\theta$. Again, we endow this complex manifold $\mathcal{B}$ with a local system of charge lattice $\underline{\Gamma}$ and a local system of $\underline{\Gamma}$-graded Lie algebra $\underline{\mathfrak{g}}$. And consider the following map

$$\begin{aligned} Y : \mathcal{B} \times S_\theta &\longrightarrow \Gamma^*_{\mathbb{R}} \\ (b, \theta) &\longmapsto Y_\theta(b)(\gamma) := Im(e^{-i\theta} Z_b(\gamma)) \;\; \forall \gamma \in \Gamma \backslash \{0\} \end{aligned} \tag{2.6.3}$$

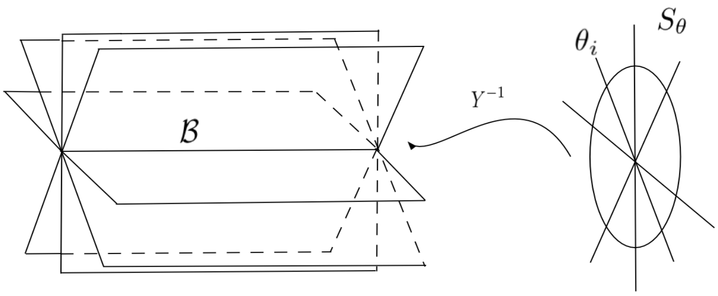

Figure 2.15: Continuous family of stability data on $\mathfrak{g}$

Assuming as before that the central charge function Z_b depends holomorphically in b, then we have the following

Proposition 2.6.3. *A section of the wall-crossing sheaf $\mathcal{WCS}_{\underline{\mathfrak{g}},Y}$ on the product space $\mathcal{B}_\theta$ is same as a WCS on the space $\mathcal{B}$.*

Proof. Fix $b_0 \in \mathcal{B}$ in (2.6.3), and consider the corresponding WCS, we will get a stability data on $\mathfrak{g}_{b_0}$ by proposition 2.6.1. As b varies locally near b_0, the corresponding central charges Z_b also varies locally around Z_{b_0} (since Z depends holomorphically on b). Thus we can use proposition 2.6.2 to get a germ of universal family of stability data near b_0. But this is the same as a germ of universal family of WCS near b_0. This finishes the proof of the proposition. □

Given a WCS σ on $\mathcal{B}_\theta$, which is a section of the corresponding wall-crossing sheaf, then by the above proposition, we can associate to it a locally constant map

$$\begin{aligned} a : tot\,(\underline{\Gamma}) &\longrightarrow tot(\underline{\mathfrak{g}}) \\ (b, \gamma) &\longmapsto a_b(\gamma) \in \underline{\mathfrak{g}}_{b,\gamma} \end{aligned} \tag{2.6.4}$$

where $a_b(\gamma)$ is the γ-component of the corresponding section of $\mathcal{WCS}_{\underline{\mathfrak{g}},Y}$ (see the proof of proposition 2.6.1). We see that $a_b(\gamma)$ is non-trivial only if there exists some $\theta \in S_\theta$ such that

$$Y_\theta(b)(\gamma) = 0$$

Thus we define a subset in $tot\,(\underline{\Gamma})$ as

$$\mathcal{B}' := \{(b, \gamma) \in tot\,(\underline{\Gamma}) : \exists\, \theta \in S_\theta,\, Y_\theta(b)(\gamma) = 0\} \tag{2.6.5}$$

Then we see that a restricts to a map on $\mathcal{B}'$, we know from before that this map encodes the DT-invariants $\Omega_b(\gamma)$.

Remark 2.6.1. *Since the map $a : \mathcal{B}' \to tot\,(\underline{\mathfrak{g}})$ is defined to depend only on the b-component of $\mathcal{B}_\theta$, so we will later just call a WCS on $\mathcal{B}_\theta$ as a wall-crossing structure on $\mathcal{B}$, with the role of the circle S_θ being understood implicitly.*

Recall from the definition 2.5.4, given $\gamma \in \Gamma$, the wall of second kind associated to it as

$$\begin{aligned}\mathcal{W}^2_\gamma &:= \gamma^\perp = \{(b,\theta) \in \mathcal{B}_\theta :\, Y_\theta(b)(\gamma) = 0\}\\ &= \left\{(b,\theta) \in \mathcal{B}_\theta :\, Im(e^{-i\theta} Z_b(\gamma) = 0\right\}\\ &= \{(b,\theta) \in \mathcal{B}_\theta :\, arg(Z_b(\gamma)) = \theta\}\end{aligned}$$

which is seen to be a hypersurface in $\mathcal{B}_\theta$—the pull back of the wall of second kind in $\Gamma^*_\mathbb{R}$ through the map Y. We can project this wall down to a hypersurface in $\mathcal{B}$, also denoted by $\mathcal{W}^2_\gamma$, i.e.

$$\mathcal{W}^2_\gamma = \{b \in \mathcal{B} :\, \exists\, \theta(\text{fixed})\ s.t.\ arg(Z_b(\gamma)) = \theta\} \tag{2.6.6}$$

Remark 2.6.2. *For a fixed point $b \in \mathcal{B}$, and given any $\gamma \in \underline{\Gamma}_b$, we can define the angle $\theta = arg\,(Z_b(\gamma))$, then we see that $Im(e^{-i\theta} Z_b(\gamma)) = 0$. From this, we can view the wall above as the rotated wall of second kind $\mathcal{W}^2_\gamma \subset \Gamma^*_\mathbb{R}$ defined before. In particularly, there are at most countably many*

$$\gamma_i \in Supp\, a_b := \{\gamma_i \in \underline{\Gamma}_b \backslash \{0\},\, a_b(\gamma_i) \neq 0\} = Supp\,\underline{\mathfrak{g}}_b$$

over a given point $b \in \mathcal{B}$. Thus, we can associate to it a collection of rays $l_{\gamma_i,b} \subset \mathbb{R}^2$ as before, and call these the rays associated to $b \in \mathcal{B}$, and denote this collection by ray_b

By the above remark, we get a collection of rays in $\mathbb{R}^2$ parametrized by points $b \in \mathcal{B}$. From our discussion on KSWCF, we infer that the set ray_b stays constant locally unless b crosses the wall of first kind defined below.

Definition 2.6.1. *Given two non-zero charges $\gamma_1, \gamma_2 \in \underline{\Gamma}$, define the* ***wall of first kind*** *associated to it as the following set*

$$\mathcal{W}_{\gamma_1,\gamma_2} = \left\{b \in \mathcal{B} :\, Im\left(Z_b(\gamma_1) \cdot \overline{Z_b(\gamma_2)}\right) = 0\right\} \tag{2.6.7}$$

And the wall of first kind associted to $\gamma \in \underline{\Gamma}$ is then defined to be

$$\mathcal{W}^1_\gamma := \bigcup_{\gamma = \gamma_1 + \gamma_2} \mathcal{W}_{\gamma_1,\gamma_2} \tag{2.6.8}$$

and these walls are locally finite hpersurfaces in $\mathcal{B}$ which hare locally a pull back of a $\mathbb{Z}PL$ hypersurface in $\Gamma^*_\mathbb{R}$.

We know that as b crosses the wall, say $\mathcal{W}_{\gamma_1,\gamma_2}$, the order of the two rays l_{γ_1} and l_{γ_2} are get swapped. Consequently the function $a_b(\gamma)$ becomes discontinuous at such $b \in \mathcal{W}^1_\gamma$, and its "jump" is governed by KSWCF (2.2.7), which can be recasted in the setting concerned here as follows:

The point $b \in \mathcal{W}_{\gamma_1,\gamma_2}$, at which the set of rays ray_b collapse into a single ray, can be viewed as the intersection of countably many walls of second kind, namely

$$\mathcal{W}^2_{m\gamma_1+n\gamma_2} \qquad \text{for } (m,n) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \;\text{ and }\; m+n > 0.$$

Thus by considering a small short loop σ circling around the point b, KSWCF predicts that the monodromy of the composite of the corresponding automorphisms along the loop should be trivial, i.e.,

$$\prod_{m/n\nearrow}^{\circlearrowright} \mathbb{S}(l_{m\gamma_1+n\gamma_2}) = id$$

Or by using KS transformation notation, the above is equivalent to

$$\prod_{m,n\geq 0;\, m/n\nearrow} \mathcal{K}_{(m,n)}^{\Omega_{b_-}(m,n)} \prod_{m,n\geq 0;\, m/n\searrow} \mathcal{K}_{(m,n)}^{-\Omega_{b_+}(m,n)} = id$$

and consequently, we get the KSWCF (2.2.7) in this situation:

$$\prod_{m,n\geq 0;\, m/n\nearrow} \mathcal{K}_{(m,n)}^{\Omega_{b_-}(m,n)} = \prod_{m,n\geq 0;\, m/n\searrow} \mathcal{K}_{(m,n)}^{\Omega_{b_+}(m,n)}$$

From the above KSWCF, we can deduce in principle the jumps of invariants $\Omega_b(\gamma)$ when b crosses the first kind wall $\mathcal{W}_\gamma^1$.

Chapter 3

Attractor flows, complex integrable systems and WCS

From the discussion of example b) in section 2.6. we see that the WCS on a topological space $\mathcal{B}$, (or by considering implicitly the product with the circle S_θ) enables us to study the variation of stability data parametrized by a complex manifold $\mathcal{B}$, i.e., by considering family of stability data. As had been displayed in the last chapter, this construction gives us a formalism to encode the DT-invariants $\Omega_b(\gamma)$ for $b \in \mathcal{B}$, as well as their "jumps" when crossing the wall of the first kind $\mathcal{W}^1_\gamma$.

But the question still remains as how to construct these DT-invariants in the first place. In this chapter, we will introduce necessary tools for dealing with this question. Roughly speaking, we will consider certain flows on the manifold $\mathcal{B}$, which for our purposes, should be realized as the base manifold of some complex integrable system. The combinatorial structures of the flow lines will give us an algorithm for computing the invariants for various charges at various points of $\mathcal{B}$. Consequently, we will produce by this algorithm a WCS on $\mathcal{B}$ that encodes these invariants.

For those willing to know the physics motivation of the attractor flow, they are encouraged to consult the Appendix B for further information. Here, we will follow the mathematical treatment of it as had been laid down in the foundational work [KS14] of M.Kontsevich and Y.Soibelman.

We will begin this chapter by giving a detailed exposition of the complex integrable system, then we will introduce the notion of (split) attractor flow and its connection to the WCS.

3.1 Geometry of complex integrable systems

In section 4 of [KS14], the concept of complex integrable systems was introduced and their geometric properties were discussed. Here, we will give a more detailed review based on the papers [B+09], [Bru],[Sep],[Fre99],[ACD02] as well as the first chapter of the book [BF04].

3.1.1 Definition and preparation

Definition 3.1.1. *A* ***complex integrable system*** *is a holomorphic surjective map $\pi : (X, \omega^{2,0}) \to \mathcal{B}$ with smooth fibers being holomorphic Lagrangian submanifolds. Here, $(X, \omega^{2,0})$ is a holomorphic symplectic manifold of complex dimension $2n$, and $\mathcal{B}$ is a complex manifold of dimension n.*

Denote by $\mathcal{B}^0 \subset \mathcal{B}$ the dense open subset over which the fibers of π being smooth, and by $\mathcal{B}^{sing} := \mathcal{B} - \mathcal{B}^0$ the discriminant locus. Consider the fibration (still denoted by π) over $\mathcal{B}^0$ by holomorphic Lagrangian submanifolds, i.e.,

$$\pi : (X^0, \omega^{2,0}) \longrightarrow \mathcal{B}^0 \tag{3.1.1}$$

In the following, we will discuss the geometry of the above complex integrable system near the points of $\mathcal{B}^0$, while the geometry near the discriminant locus $\mathcal{B}^{sing}$ is much more complicated and will be discussed later.

We make the assumption that the smooth fibres of π are compact, so that they are actually holomorphic Lagrangian tori (will be proved later in section 3.1.2).

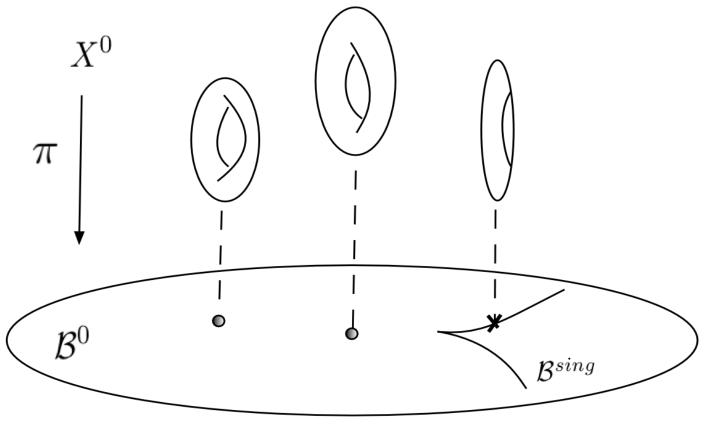

Figure 3.1: Complex integrable system as torus fibration

Symplectomorphism

To study the local geometry of the fibration π given by (3.1.1), we will consider certain diffeomorphism of X^0 induced by the differential one form on the base $\mathcal{B}^0$.

Denote by $Diff(X^0)$ the group of diffeomorphism of X^0, then its Lie algebra $\mathfrak{Diff}(X^0)$ can be identified with the Lie algebra of vector fields on X^0, denoted by $Vect(X^0)$. We can use the symplectic two form to identify $Vect(X^0)$ with the space of differential one form on X^0, denoted by $\Omega^1(X^0)$, as follows:

$$Vect(X^0) \xrightarrow{\simeq} \Omega^1(X^0)$$

$$\xi \longmapsto \iota_\xi \, \omega^{2,0}$$

where $\iota_\xi \, \omega^{2,0}$ denotes the interior product of the vector field ξ with the symplectic form $\omega^{2,0}$. The isomorphism follows from the fact that the symplectic two form is non-degenerate.

Recall that a symplectomorphism of X^0 is given by a diffeomorphism $\phi : X^0 \to X^0$ such that it preserves the symplectic structure, i.e. $\phi^*(\omega^{2,0}) = \omega^{2,0}$.

Suppose ϕ is such a symplectomorphism, with the associated vector field ξ, then the condition that it is a symplectomorphism can be written as: $\mathscr{L}_\xi \, \omega^{2,0} = 0$, where $\mathscr{L}_\xi$ is the Lie derivative with respect to the vector field ξ. By the Cartan's magic formula, we have:

$$\mathscr{L}_\xi \, \omega^{2,0} = d(\iota_\xi \omega^{2,0}) + \iota_\xi(d\omega^{2,0}) = d(\iota_\xi \omega^{2,0}) = 0$$

Thus, the corresponding one form $\iota_\xi \omega^{2,0}$ of ϕ is closed.

Denote by $\mathfrak{Symp}(X^0) \subset \mathfrak{Diff}(X^0)$ the Lie algebra of symplectomorphism of X^0, then we see from the above computation that it can be identified with the set of closed differential one form on X^0, i.e., $\mathfrak{Symp}(X^0) \simeq Z^1(X^0)$.

In particular, for a function $f \in C^\infty(X^{2,0})$, we have the associated Hamiltonian vector field ξ_f given as

$$\iota_{\xi_f} \, \omega^{2,0} = df \tag{3.1.2}$$

The diffeomorphism corresponding to the one form $\iota_{\xi_f} \, \omega^{2,0}$ is called *Hamiltonian diffeomorphism*, and denote the group formed by them by $Ham(X^0, \omega^{2,0})$. Form the above definition, we see that the corresponding Lie algebra $\mathfrak{Ham}(X^0, \omega^{2,0})$ can be identified with the set of exact one forms of X^0, i.e.,

$$\mathfrak{Ham}(X^0, \omega^{2,0}) \simeq B^1(X^0)$$

Consequently, $H^1_{dR}(X^0)$, the first deRham cohomology of X^0, parametrizes the set of infinitesimal symplectomorphism of X^0 modulo the action of infinitesimal Hamiltonian deformation.

Lie algebroids and Lie groupoids

We need the notion of Lie algebroid and the Lie groupoid to formalize the structures that appears when studying the local geometry of complex integrable system to be discussed in the next section. Here, I will give a rather sketching review, for more details, please see the references like [Rac04] and [MM03].

Definition 3.1.2. *Let M be a smooth manifold, a Lie algebroid over M is a vector bundle $V \to M$ such that the space $\Gamma(V)$ of smooth sections of V is endowed with the following antisymmetric bi-linear bracket*

$$[\cdot,\cdot] : \Gamma(V) \times \Gamma(V) \to \Gamma(V)$$

which satisfies the Jacobi-identity

$$[[\alpha,\beta],\gamma] + [[\beta,\gamma],\alpha] + [[\gamma,\alpha],\beta] = 0$$

Moreover, we have the anchor map $\rho : V \to TM$, which is a bundle map between V and the tangent bundle TM. This map induces, at the level of sections, a homomorphism (still denoted by ρ) $\rho : \Gamma(V) \to \Gamma(TM)$ taht satisfies the following Leibniz rule:

$$[\alpha, f\beta] = f[\alpha,\beta] + (\rho(\alpha)\cdot f)\beta$$

where f is any smooth function on M.

Lie algebroid generates the infinitesimal action on a manifold. More precisely, we have the following definition (see [Mok96]).

Definition 3.1.3. *Given a Lie algebroid (V,ρ) over M, the infinitesimal action of this Lie algebroid on a manifold X is defined to be a smooth map $\phi : X \to M$, together with a linear map $\mu : \Gamma(V) \to \Gamma(TX)$ such that*

a) $\mu([X,Y]) = [\mu(X),\mu(Y)]$, for $X, Y \in \Gamma(V)$;

b) $\mu(fX) = (f \circ \phi)\mu(X)$, for $X \in \Gamma(V), f \in C^\infty(M)$,

c) $\phi_*(\mu(X)(x)) = \rho(X)(\phi(x))$, for $X \in \Gamma(V)$, $x \in X$.

We can easily generalize this definition to the case when X is given by a fibered manifold $p : E \to M$ (the same base, we shall not give the definition for more general base). We have the following definition (c.f [KM02])

Definition 3.1.4. *Given a Lie algebroid (V,ρ) over M, the infinitesimal action of this Lie algebroid along $p : E \to M$ is given by linear map $\mu : \Gamma(V) \to \Gamma(TE)$ such that*

a) $\mu([X,Y]) = [\mu(X),\mu(Y)]$, for $X, Y \in \Gamma(V)$;

b) $\mu(fX) = (f \circ p)\mu(X)$, for $X \in \Gamma(V), f \in C^\infty(M)$,

c) For $X \in \Gamma(V)$, $\mu(X)$ is projectable to $\rho(X)$.

It is clear from the definition of the Lie algebroid that a Lie algebra is a Lie algebroid over a point. Besides, the tangent bundle TM of a smooth manifold M is seen to be a Lie algebroid by taking the bracket $[\cdot,\cdot]$ the usual Lie bracket on the space of vector fields, and ρ the identity map. More interesting Lie-algebroids are given by Poisson manifolds.

Definition 3.1.5. *A **Poisson structure** on a manifold M is given by an antisymmetric, bi-linear map (called Poisson bracket):*

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$$

which satisfies the Jacobi identity as well as the Leibniz rule, i.e,

$$\{fg, h\} = f\{g, h\} + \{f, h\}g$$

*A manifold equipped with a Poisson bracket is called a **Poisson manifold**. And a smooth map between two Poisson manifolds is called a **Poisson morphism** if it preserves the Poisson structures.*

Note that any symplectic manifold (M, ω) give rises to a Poisson manifold $(M, \{\cdot,\cdot\})$ with the Poisson bracket given as

$$\{f, g\} := \omega^{-1}(df, dg) = \omega(\xi_f, \xi_g)$$

Indeed, by the Leibniz rule, we see that $\{\cdot, h\}$ is a derivation on $C^\infty(M)$, i.e., there exists vector field ζ_h such that $\zeta_h(f) = \{f, h\}$. We call this vector field associated to h the Hamiltonian vector field of h. We show that it actually coincide with ξ_h defined before.

$$\{f, h\} = \zeta_h(f) = df(\zeta_h) = \iota_{\xi_f}\, \omega(\zeta_h) = \omega(\xi_f, \xi_h)$$

From the last equality above, we infer that $\zeta_h = \xi_h$. Thus, we we use the notation ξ_f to denote the Hamiltonian vector field associated to the function f.

For the complex integrable system $\pi : (X^0, \omega^0) \to \mathcal{B}^0$, by endowing B^0 with the trivial Poisson structure 0, we have the following:

Proposition 3.1.1. $\pi : (X^0, \omega^0) \longrightarrow (\mathcal{B}^0, 0)$ *is a Poisson morphism.*

The proof of this proposition can be found in [Vai12]. We now state the connection between Poisson structure and Lie algebroid.

Proposition 3.1.2. *A Poisson structure on M is the same as a Lie algebroid structure on the cotangent bundle $T^*M \to M$.*

Proof. (sketchy): Given a Poisson structure on M, the anchor map $\rho : T^*(M) \to TM$ is the natural one given by associating to a differential one form df the corresponding Hamiltonian vector field ξ_f. Then the antisymmetric bi-linear bracket on $\Gamma(T^*)$ is defined as

$$[df, dg] := d\langle dg, \rho(df)\rangle$$

where $\langle\cdot,\cdot\rangle$ denotes the natural pairing between the one form and vector field. The Jacobi identity and the Leibniz rule can then be easily verified.

Conversely, given a Lie algebroid structure $([\cdot,\cdot],\rho)$ on T^*M, one can associate the Poisson bracket $\{\cdot,\cdot\}$ on $C^\infty(M)$ as follows

$$\{f,g\} := \langle dg, \rho(df)\rangle = \rho(df)g$$

Then the antisymmetry of the bracket, Leibniz rule and Jacobi identity can be verified (details can be found in [Rac04]). $\square$

Roughly speaking, a Lie algebroid looks like Lie algebras in family.

Definition 3.1.6. *A **groupoid** G consists of a set G_0, called the set of objects; and G_1, the set of morphisms (arrows), equipped with the following maps*

a) $s,t: G_1 \rightrightarrows G_0$, called the source and target map respectively.

b) $m: G_{1s}\times_t G_1 \to G_1$ $(g,h)\mapsto gh$, called the multiplication map that satisfies $s(gh) = s(h)$, $t(gh)=t(g)$, and $(gh)k = g(hk)$, where $G_{1s}\times_t G_1$ is defined to be the set $\{(g,h)\in G_1\times G_1 : s(g)=t(h)\}$.

c) $e: G_0\to G_1$, called the identity section such that $e(t(g))g = g = ge(s(g))$.

d) $i: G_1\to G_1$, called the inverse section, and denote $i(g)$ by g^{-1} such that $g^{-1}g = e(s(g))$, $gg^{-1} = e(tg)$.

Definition 3.1.7. *A **Lie groupoid** is given by a groupoid G such that G_0 and G_1 are smooth manifold, and all group operations are smooth, besides, s and t are submersion.*

Again, a Lie group is just a Lie groupoid over a single point. i.e. $G_0=\{pt\}$.

If a Lie group H acts on a manifold X, we can form the so called *action groupoid* by viewing G_0 as X and G_1 as $H\times X$. Then the source and target maps are given respectively as

$$s(h,x) = h^{-1}\cdot x,\ t(h,x)=x$$

and the multiplication is given as $(h,x)\cdot(k,h^{-1}\cdot x) = (hk,x)$, while the inverse and identity maps are given respectively by

$$i(g,x) = (g^{-1}, g^{-1}\cdot x),\ e(x) = (id,x).$$

Definition 3.1.8. *A left action of a Lie groupoid G on a manifold X consists of the following*

a) a smooth map $\mu: X\to G_0$,

b) a smooth map $G_{1s}\times_\mu X\to X,\ (g,x)\mapsto g\cdot x$ where

$$G_{1s}\times_\mu X := \{(g,x) : s(g) = \mu(x)\}$$

such that the following conditions are satisfied

1) $\mu(g\cdot x) = t(g)$, where $(g,x)\in G_{1s}\times_\mu X$ and

2) $h\cdot(g\cdot x) = (h\cdot g)\cdot x$, when $s(h)=t(g)$.

Since for a Lie group, their is an associated Lie algebra, similarly, for a Lie groupoid, there is a corresponding Lie algebroid. For a Lie groupoid G, we now sketch how to associate it with a Lie algebroid, for the detailed construction, see the section 1.4 of [CF11].

As in the case of constructing the Lie algebra from the corresponding Lie group, we need to consider Left invariant vector fields on G. To this end, we need to construct the tangent space at unit of the group, but in our case, we have one unit for each point in G_0, so we will get a family of tangent spaces parametrized by G_0. Besides, since the left multiplication is only defined on the fiber of the target map $t : G_1 \to G_0$. Thus, we need to look at the vector fields that is tangent to the fiber of t, i.e., we consider the space

$$T^t G_1 := Ker(dt) \subset TG_1$$

which is a vector bundle on G_1. Under the identity map $e : G_0 \to G_1$, G_0 can be identified with its image in G_1. Denote by $V := Lie(G_1)$ the restriction of the bundle $T^t G_1$ on G_0, we claim that the vector bundle V carries the desired Lie algebroid structure associted to the Lie groupoid G.

To see this, we show that the space $\Gamma(V)$ of sections of the bundle V can be identified with the space of Left invariant vector fields on G_1, denoted by $Vect^t_{inv}(G_1)$, which is defined as

$$Vect^t_{inv}(G_1) := \{X \in \Gamma(T^t(G_1)) : X_{gh} = L_g(X_h),\ \forall (g, h) \in G_{1s} \times_t G_1\}$$

where L_g denotes the differential of left multiplication by g, i.e., $L_g : T^t_h G_1 \to T^t_{gh} G_1$. Then, given $\alpha \in \Gamma(V)$, we associate to it a vector field $\widetilde{\alpha}_g := L_g(\alpha_{s(g)})$, which is clearly left invariant. Conversely, given a left invariant vector field X on G_1, it is determined by its valued at points in G_0, i.e., for a point $g : x \to y$ in G_1, we have $X_g = L_g(X_x)$. Define $\alpha := X|_{G_0} \in \Gamma(v)$, then $X = \widetilde{\alpha}$.

In this way, we can establish the identification: $\Gamma(V) \xrightarrow{\simeq} Vect^t_{inv}(G_1)$, then the Lie bracket on $\Gamma(V)$ is the one induced by the above identification from that on the space of invariant vector fields, which can be easily seen to obey the Jacobi identity. The anchor map $\rho : \Gamma(V) \longrightarrow TG_0$, in our case is defined to be the differential of the source map s redistricted to V, i.e., $\rho = ds|_V$, which can be verified to be obey the Lebniz rule.

Before closing this subsection, let us recall the definition of a torsor.

Definition 3.1.9. *For a group G, a G-**torsor** (or a torsor over G), is a principal homogeneous G-space, i.e., a space P, together with an free and transitive G-action.*

Remark 3.1.1. *Each choice of a point $p \in P$ gives rise to an isomorphism between G and P, i.e., G is being identified with the orbit $\{G \cdot p\}$ of p. Thus, this isomorphism is not canonical. Morally speaking, a torsor is like a usual group that forges its identity element, any choice of a point $p \in P$ serves as the identity, which makes it an actual group.*

Remark 3.1.2. *We can easily generalize the above notion to the case when G is a Lie groupoid due to that we have good notion of the action of Lie groupoid on a space (see the definition 3.1.6).*

3.1.2 Local geometry near regular fibres and classification

Let us go back to the study of the complex integrable system near its regular fibers given by $\pi : (X^0, \omega^{2,0}) \to \mathcal{B}^0$. The moral is that all that had been encoded in the geometry of the base manifold $\mathcal{B}^0$, as we will see in the next section when discussion the notion of *special geometry*.

As is well known, the cotangent bundle $T^*\mathcal{B}^0$ of the base of the integrable system is naturally a symplectic manifold, endowed with the canonical symplectic form ω_{can}. We can endow the base $\mathcal{B}^0$ with the trivial Poisson structure, and consider by proposition 3.1.2 the induced Lie algebroid structure on the cotangent bundle

$$pr : (T^*\mathcal{B}^0, \omega_{can}) \longrightarrow \mathcal{B}^0$$

In this case, since the underlying Poisson structure is trivial, each fiber of the pr above can be viewed as an trivial abelian Lie algebra.

The ideal to study the local geometry of π is that the Lie algegbroid $T^*\mathcal{B}^0$ generate an infinitesimal diffeomorphism on X^0, and under certain assumption (to be discussed momentarily), can be integrated into a Lie groupoid action on X^0. The symmetry coming from this action enables us to find local model of the complex integrable system π that resembles to the cotangent bundle pr. In other words, we will show that pr is a good local model for π near regular fibers. The exposition in this section is mainly based on the papers [Sep] and [Bru]. We assume in the following the the fibers of π are connected and π is a surjective submersion.

Let $b \in \mathcal{B}^0$, and choose an open neighbourhood $U \subset \mathcal{B}^0$ around it, then we consider a differential one form α defined locally on U, i.e., a local section of $pr : T^*\mathcal{B}^0 \to \mathcal{B}^0$ near $b \in U$. As in the definition of Hamiltonian vector field of a function (see definition 3.1.2), we can associate to α a vector field ξ_α as follows:

$$\iota_{\xi_\alpha} \omega_{can} = \pi^* \alpha \tag{3.1.3}$$

This gives rise to the following map at the level of sections:

$$\mu : \Gamma(T^*U) \longrightarrow \Gamma(TX^0|_U), \quad \alpha \longmapsto \xi_\alpha \tag{3.1.4}$$

In application, we usually consider the case of symplectomorphism of π induced from the one form α through the map μ. By the discussion in section 3.1.1, wee see that this corresponds to the case when the one form α is being closed.

Roughly speaking, this means that given $b \in U$, and for any point $x \in \pi^{-1}(b)$, at the level of stalk, we get an infinitesimal action of $T_b^*\mathcal{B}^0$ (viwed as an abelian Lie algebra) on X^0. By utilizing the concept of the action of Lie algebroid (see definition 3.1.4), this can be formulated more elegantly as follows:

Proposition 3.1.3. *Suppose that pr admits a global section, i.e., we can have globally defined one form on $\mathcal{B}^0$, then the map μ defined above can be extended to $\mathcal{B}^0$ to give rise to an action of the Lie algebroid $pr : T^*\mathcal{B}^0 \to \mathcal{B}^0$ along the fibred map $\pi : X^0 \to \mathcal{B}^0$, where $\mathcal{B}^0$ is endowed with the trivial Poisson structure and the anchor map for the Lie algebroid is given by the isomorphism $\rho : T^*\mathcal{B}^0 \xrightarrow{\simeq} T\mathcal{B}^0$ induced by the canonical symplectic structure ω_{can} on the cotangent bundle.*

Proof. We need to verify the condition a), b) and c) in the definition 3.1.4 of the Lie algebroid action. As the Poisson structure on $\mathcal{B}^0$ is trivial, the bracket on $\Gamma(T^*\mathcal{B}^0)$ vanishes identically, thus the condition a) of the definition becomes

$$[\xi_\alpha, \xi_\beta] = 0 \qquad \forall \alpha, \beta \in \Gamma(T^*\mathcal{B}^0) \tag{3.1.5}$$

where the bracket above is the usual one actiong on vector fields.

Condition b) of the definition specializes in our case into the following

$$\xi_{f\alpha} = \pi^*(f)\, \xi_\alpha, \qquad \forall \alpha \in \Gamma(T^*\mathcal{B}^0),\ f \in C^\infty(\mathcal{B}^0) \tag{3.1.6}$$

while the condition c) becomes the following

$$\forall \alpha \in \Gamma(T^*\mathcal{B}^0) \qquad \xi_\alpha \in Ker\, \pi_* \tag{3.1.7}$$

Now we verify these conditions by performing local computations.

By the definition of the Hamiltonian vector field (3.1.3), the condition (3.1.6) follows straightforwardly. Near a point $b \in U$, we give the local coordinate $(q^1, \cdots, q^n) : U \longrightarrow \mathbb{C}^n$. Write $\alpha = \sum_{i=1}^n \alpha_i\, dq^i$, and $\beta = \sum_{j=1}^n \beta_j\, dq^j$, we first show that $\xi_\alpha \in Ker\, \pi_*$, i.e., ξ_α is a vertical vector field. Since

$$\xi_\alpha = \sum_{i=1}^n \pi^*(\alpha_i)\, \xi_{dq^i} = \sum_{i=1}^n \pi^*(\alpha_i)\, \xi_{q^i}$$

we just need to check the condition for ξ_{q^i}. Since q^i are is function on $\mathcal{B}^0$, it is constant along the fiber of π, thus we can consider it as the function $\pi \circ q^i$. In particularly, for and vertical vector field $\eta \in Ker\, \pi_*$, we have: $\eta(\pi \circ q^i) = 0$, but

$$\eta(\pi \circ q^i) = \pi^*(dq^i)(\eta) = \iota_{\xi_{q^i}}\, \omega_{can}(\eta) = \omega_{can}(\xi_{q^i}, \eta) = 0$$

Since this holds for all $\eta \in Ker\, \pi_*$, wee see that $\xi_{q^i} \in (Ker\, \pi_*)^\perp$, where " $\perp$ " denotes the symplectic orthogonal with the canonical symplectic form ω_{can}. As π is a Lagrangian fibration, we have that $(Ker\, \pi_*)^\perp = (Ker\, \pi_*)$, the desired result thus follows. Now we prove (3.1.5):

$$[\xi_\alpha, \xi_\beta] = \left[\sum_{i=1}^n \pi^*(\alpha_i)\, \xi_{q^i}, \sum_{j=1}^n \pi^*(\beta_j)\, \xi_{q^j} \right]$$

$$= \sum_{i,j=1}^n \left((\pi^*(\alpha_i))(\pi^*(\beta_j))[\xi_{q^i}, \xi_{q^j}] + (\pi^*(\alpha_i))\xi_{q^i}(\pi^*(\beta_j))\xi_{q^j} - (\pi^*(\beta_j))\xi_{q^j}(\pi^*(\alpha_j))\xi_{q^i} \right)$$

$$= \sum_{i,j=1}^n (\pi^*(\alpha_i))(\pi^*(\beta_j))[\xi_{q^i}, \xi_{q^j}] = \sum_{i,j=1}^n (\pi^*(\alpha_i))(\pi^*(\beta_j))\xi_{\{q^i, q^j\}}.$$

The last identity holds since each q^j is constant along the fiber, and ξ_{q^i} belongs to the vertical tangent vector space, thus

$$0 = \xi_{q^i}(q^j) = dq^j(\xi_{q^i}) = \iota_{\xi_{q^j}} \omega_{can}(\xi_{q^i}) = \omega_{can}(\xi_{q^i}, \xi_{q^j}) = \{q^i, q^j\}.$$

□

Now we "integrate" the above Lie algebroid action into a Lie groupoid action along π, for this we make a further assumption that for any compactly supported one form α on $\mathcal{B}^0$, the associated vector field ξ_α is complete. Under the above assumption, we consider the flow associated to the vector field ξ_α as

$$\phi_\alpha^t : \pi^{-1}(U) \to \pi^{-1}(U)$$

Since this vector field is vertical, the flow preserves the fiber $\pi^{-1}(b)$ for every $b \in U$. In order to see this more clearly, let us choose any compactly supported on form α defined over U such that $\alpha(b) = \alpha_b$, then the induced diffeomorphism of the fiber is given as

$$\phi_{\alpha_b}^t : \pi^{-1}(b) \to \pi^{-1}(b) \tag{3.1.8}$$

We can formulate more concisely by using the language of Lie groupoid (see definition 3.1.7) and its action on manifold (see definition 3.1.8).

The Lie groupoid to be considered is given by $T^*\mathcal{B}^0 \rightrightarrows \mathcal{B}^0$, with the source and target maps being identical and are given by the projection map pr. By viewing the fiber as vector space, the multiplication map is given by the fiber-wise addition of co-vectors. The unit elements are realized by taking zero section of pr, while the inversion is given by taking the negatives of covectors.

Proposition 3.1.4. *The action of Lie algebroid $pr : T^*\mathcal{B}^0 \to \mathcal{B}^0$ along the fibred map $\pi : (X^0, \omega^{2,0}) \to \mathcal{B}^0$ as in proposition 3.1.3 "integrates" into a Lie groupoid $pr : T^*\mathcal{B}^0 \rightrightarrows \mathcal{B}^0$ action along $\pi : (X^0, \omega^{2,0}) \to \mathcal{B}^0$.*

Proof. Consider the fibred product

$$T^*\mathcal{B}^0{}_{pr} \times_\pi X^0 := \{(\alpha, x) \in T^*\mathcal{B}^0 \times X^0 : pr(\alpha) = \pi(x)\}$$

Then the desired Lie groupoid action (c.f., definition 3.1.8) is given by

$$\mu : T^*\mathcal{B}^0{}_{pr} \times_\pi X^0 \to X^0 \;\; (\alpha, x) \mapsto \phi_\alpha^1(x) \tag{3.1.9}$$

where ϕ_α^1 is the diffeomorphism induced by the time one flow of the vector field ξ_α (see equation (3.1.4)). $\square$

Remark 3.1.3. *As an abelian Lie group, $T_b^*\mathcal{B}^0$ is isomorphic to $\mathbb{C}^n \cong \mathbb{R}^{2n}$.*

We will use the action μ (see 3.1.9) to analyze the local structure of the integrable system π. First, notice that as the symplectic form ω_{can} is non degenerate, the map in (3.1.4) is subjective, from which we infer that the action μ is transitive along the fiber $\pi^{-1}(b)$ for any $b \in \mathcal{B}^0$. Next, we compute the isotropy subgroup of the action $\phi_\alpha^1 : \pi^{-1}(b) \to \pi^{-1}(b)$. For $x, y \in \pi^{-1}(b)$, the corresponding isotropy subgroups H_x, H_y are conjugate to each other, but as the groups in question are abelian (see the remark above), they can be canonically identified. Thus, the isotropy subgroup at $b \in \mathcal{B}'$ can be computed as

$$H_b = \{\alpha_b \in T_b^*\mathcal{B}^0 : \exists\, x \in \pi^{-1}(b), s.t,\; \phi_{\alpha_b}^1(x) = x\} \tag{3.1.10}$$

It is clear that $dim\, T_b^*\mathcal{B}^0 = dim\, \pi^{-1}(b)$, $\forall\, b \in \mathcal{B}^0$, consequently $dim\, H_b = 0$.

As a discrete subgroup in $T_b^*\mathcal{B}^0 \cong \mathbb{C}^n \cong \mathbb{R}^{2n}$, H_b must be isomorphic to $\mathbb{Z}^k$ for some $k \leq 2n$. In the following, we will use the notation Λ_b to denote this lattice, and call it the **period lattice**. Thus, the fiber $\pi^{-1}(b)$ is isomorphic to

$$T_b^*\mathcal{B}^0/\Lambda_b \cong \mathbb{C}^n/\mathbb{Z}^k \cong \mathbb{R}^{2n}/\mathbb{Z}^k$$

The map π is assumed to be proper, so the fiber $\pi^{-1}(b)$ is a compact Lagrangian tori $\mathbb{T}^{2n}$. In this case, $k = 2n$. This is the case to be considered now. By using the notion of torsor (see definition 3.1.9), we can say that $\pi^{-1}(b)$ is a torsor over the tori $\mathbb{T}^{2n}$. We can generalize this to the global case. To this end, define

$$\Lambda := \bigsqcup_{b\in\mathcal{B}^0} \Lambda_b \subset T^*\mathcal{B}^0$$

which is called the **period net** associated to the complex integrable system $\pi : X^0 \to \mathcal{B}^0$. Following remark 3.1.1, we say that π is a torsor over the Lie groupoid $\widehat{pr} : T^*\mathcal{B}^0/\Lambda \rightrightarrows \mathcal{B}^0$ (with the Lie groupoid structure induced from that on $pr : T^*\mathcal{B}^0 \rightrightarrows \mathcal{B}^0$), which is topologically a torus bundle over $\mathcal{B}^0$. Indeed, the Lie groupoid action is the one induced from (3.1.9):

$$\widehat{\mu} : T^*\mathcal{B}^0/\Lambda \;{}_{\widehat{pr}}\times_{\pi} X^0 \to X^0 \quad ([\alpha], x) \mapsto \phi^1_\alpha(x) \tag{3.1.11}$$

where $T^*\mathcal{B}^0/\Lambda \;{}_{\widehat{pr}}\times_{\pi} X^0 \longrightarrow X^0$ is the fiber product defined by

$$T^*\mathcal{B}^0/\Lambda_{\widehat{pr}} \times_{\pi} X^0 := \left\{([\alpha], x) \in T^*\mathcal{B}^0 \times X^0 : \widehat{pr}([\alpha]) = \pi(x)\right\}$$

Here $[\alpha]$ denotes its image in the quotient.

Lemma 3.1.5. *The Lagrangian sections of $pr : T^*\mathcal{B}^0 \to \mathcal{B}^0$ correspond to the closed on forms on $\mathcal{B}^0$.*

Proof. Denote the standard coordinate system on $T^*\mathcal{B}^0$ by $(p_1, \cdots, p_n, q^1, \cdots, q^n)$. In these coordinates, the canonical symplectic form can be written as

$$\omega_{can} = \sum_{i=1}^n dp_i \wedge dq^i.$$

Then for a section $\eta = \sum_{i=1}^n \eta_i dq^i$ of pr, the restriction of ω_{can} on η is given by

$$\omega_{can}|_\eta = \sum_{i=1}^n d\eta_i \wedge dq^i = d\eta$$

thus the Lagrangian condition is equivalent to the closedness of η. □

Proposition 3.1.6. *The period net Λ is a Lagrangian submanifold of $T^*\mathcal{B}^0$.*

Proof. By the defining property of Λ, we see that it is an integral lattice locally spanned by the locally closed one forms α_b, and consequently, by the above lemma, we conclude that Λ corresponds to Lagrangian sections. □

A local section γ of $pr : T^*\mathcal{B}^0 \to \mathcal{B}^0$ induces a transformation along the fiber of pr by translation, i.e.,

$$T_b^*\mathcal{B}^0 \to T_b^*\mathcal{B}^0, \qquad \alpha_b \mapsto \alpha_b + \gamma_b \tag{3.1.12}$$

Lemma 3.1.7. *The above action along the fibers of pr given by translation is symplectic if and only if the section γ is Lagrangian.*

Proof. We see that ω_{can} is preserved by the action (3.1.1) if and only if the one form γ is closed, which by lemma 3.1.5 corresponds to a Lagrangian section. $\square$

Proposition 3.1.8. *The standard symplectic structure ω_{can} on $T^*\mathcal{B}^0$ induces a symplectic structure on the Lagrangian tori bundle $T^*\mathcal{B}^0/\Lambda$.*

Proof. By the definition of Λ (see 3.1.10), wee see that $T^*\mathcal{B}^0$ is invariant under the action of the period net Λ, thus by Lemma 3.1.7, this action is trivially symplectic, and consequently the canonical symplectic structure ω_{can} is preserved by the action of Λ. So ω_{can} descends to the quotient $T^*\mathcal{B}^0/\Lambda$. $\square$

The local structure of the complex integrable system $\pi : (X^0, \omega^{2,0}) \to \mathcal{B}^0$ can be summarized into the following proposition, which roughly says that π looks locally like (actually diffeomorphic to) the $\widehat{pr} : T^*\mathcal{B}^0/\Lambda \to \mathcal{B}^0$. Whether it is globally of such form or not depends on further topological information to be discussed momentarily.

Given a local section σ of the integrable system π, i.e., $\sigma : U \to \pi^{-1}(U)$ for $U \subset \mathcal{B}^0$. Then it induces the following map

$$\Psi_\sigma : T^*U \longrightarrow \pi^{-1}(U) \quad \alpha \longmapsto \phi^1_\alpha(\sigma \circ pr(\alpha)) \tag{3.1.13}$$

It further induces the following map at the quotient level:

$$\widehat{\Psi}_\sigma : T^*U/\Lambda|_U \longrightarrow \pi^{-1}(U) \quad [\alpha] \longmapsto \phi^1_\alpha(\sigma \circ \widehat{pr}([\alpha])) \tag{3.1.14}$$

Proposition 3.1.9. *The integrable system π can be locally trivialized by choosing local sections $\sigma : U \to X^0$ over open $U \subset \mathcal{B}^0$, while the trivialization is given by $\widehat{\Psi}_\sigma$ defined above.*

Proof. We need to show that $\widehat{\Psi}_\sigma$ is a local bijective diffeomorphism, which is amount to showing that the map Ψ_σ is a local diffeomorphism, since the bijectivity would follow from the dimension consideration.

As $dim\, T^*U = dim\, \pi^{-1}(U)$, to show that Ψ_σ is a diffeomorphism, by the implicit function theorem, we need to show that it is injective, i.e.,

$$Ker\, D\Psi_\sigma(\alpha) \neq 0,\ \forall \alpha \in T^*U$$

Consider now the map μ in (3.1.4), which can be viewed as the derivative of the diffeomorphism ϕ_t. Its image is the space of vertical vector fields for π, i.e., $Im\, \mu = Ker\, \pi_*$. Thus $D\Psi_\sigma(\alpha)(X) = 0$ if and only if $X = 0$ were $X \in T^{\text{vert}}_\alpha T^*U$, i.e., the vector field that is tangent to the fiber of pr. Thus, if we assume that there exists $X \in T_\alpha T^*U$ such that $D\Psi_\sigma(\alpha)(X) = 0$, then, it must not belong to the vertical subspace $T^{\text{vert}}_\alpha T^*U$. Consequently,

$$Dpr(\alpha)(X) = pr_*(\alpha)(X) \neq 0$$

As the local section σ is a submersion, $D\sigma \circ Dpr(\alpha)(X) \neq 0$. Then we have that

$$D\Psi_\sigma(\alpha)(X) = D\phi^1_\alpha \circ D\sigma \circ Dpr(\alpha)(X) = \mu \circ D\, pr(\alpha)(X) \neq 0$$

as X does not belong to $T^{vert}_\alpha T^*U$. This contradicts to our assumption. We conclude that $Ker\, D\Psi_\sigma(\alpha) \neq 0$. $\square$

Invariants and classifications

By the proposition 3.1.9, we see that the complex integrable system π near regular points looks like a cotangent bundle, and this has been realized through the trivialization map $\hat{\Psi}_\sigma$ (see 3.1.14) induced by Local section σ of π. Thus, in order to get global information, i.e., the invariants, we need to glue these local pieces together.

Consider two open subsets U_i, $U_j \subset \mathcal{B}^0$ near the regular point $b \in \mathcal{B}^0$ with non-empty overlapping $U_{ij} := U_i \cap U_j$, together with the local sections defined over them, i.e., $\sigma_i : U \to \pi^{-1}(U_i)$. Then we have the associated trivialization $\hat{\Psi}_{\sigma_i}$, and over the intersection U_{ij}, and the following *transition map*

$$\widehat{\Psi}_{\sigma_j}^{-1} \circ \widehat{\Psi}_{\sigma_i} : T_b^* U_{ij}/\Lambda|_{U_{ij}} \longrightarrow T_b^* U_{ij}/\Lambda|_{U_{ij}}$$

Suppose $\alpha \in \Lambda_b$, i.e., it represents the zero element in $T_b U_{ij}/\Lambda|_{ij}$, then by the formula (3.1.13), we see that under the above transitional map, it becomes the section $\widehat{\Psi}_{\sigma j}^{-1}(\sigma_i \circ \widehat{pr}(\alpha))$. By the transitivity of the Lie groupoid action ϕ^1 as had been noted after remark 3.1.3, we see that there exists unique section μ_{ji} of $pr : T_b^* U_{ij}/\Lambda|_{U_{ij}} \to U_{ij}$ such that $\phi^1_{\mu_{ji}}(\sigma_j) = \sigma_i$.

Lemma 3.1.10. *The transition map acts on $[\alpha] \in T_b^* U_{ij}/\Lambda|_{U_{ij}}$ in the following way*

$$\widehat{\Psi}_{\sigma_j}^{-1} \circ \widehat{\Psi}_{\sigma_i}([\alpha]) = [\alpha] + \widehat{\Psi}_{\sigma_j}^{-1}(\sigma_i \circ \widehat{pr})([\alpha])$$

Proof. By the formula (3.1.12), we see that under the transition map, α gets mapped to $\alpha + \mu_{ij}$. Besides, we have that

$$\widehat{\Psi}_{\sigma_j}^{-1} \circ \widehat{\Psi}_{\sigma_i}([\mu_{ij}]) = \widehat{\Psi}_{\sigma_j}^{-1} \circ \phi^1_{\mu_{ij}}(\sigma_i) = \widehat{\Psi}_{\sigma_j}^{-1}(\sigma_i),$$

from which the desired identity follows. □

Topological classification

From the above discussion, we see that $\mu_{ij} : U_{ij} \to T(U_{ij})/\Lambda|_{ij}$ defines a Čech 1-cocycle for the cohomology of $\mathcal{B}^0$ with coefficients in the sheaf $C^\infty(T^*\mathcal{B}^0/\Lambda)$ of smooth sections of $pr : T^*\mathcal{B}^0/\Lambda \to \mathcal{B}^0$, i.e. μ_{ij} s give rise to a cohomology class

$$\mu \in H^1(\mathcal{B}^0, C^\infty(T^*\mathcal{B}^0/\Lambda))$$

Then the class μ is the obstruction for $\pi : (X^0, \omega^{2,0}) \to \mathcal{B}^0$ to be globally diffeomorphic to the reference integrable system $pr : T^*\mathcal{B}^0 \to \mathcal{B}^0$. The obstruction is measured by the so- called Chern class to be defined below.

Denote by $C^\infty(\Lambda)$ the sheaf of smooth sections of $\Lambda \longrightarrow \mathcal{B}^0$, then we have the following short exact sequence

$$0 \longrightarrow C^\infty(\Lambda) \longrightarrow C^\infty(T^*\mathcal{B}^0) \longrightarrow C^\infty(T^*\mathcal{B}^0/\Lambda) \longrightarrow 0$$

from which we have the induced long exact sequence at the cohomology level

$$\cdots \longrightarrow H^1(\mathcal{B}^0, C^\infty(\Lambda)) \longrightarrow H^1(\mathcal{B}^0, C^\infty(T^*\mathcal{B}^0)) \longrightarrow H^1(\mathcal{B}^0, C^\infty(T^*\mathcal{B}^0/\Lambda))$$

$$\xrightarrow{\delta} H^2(\mathcal{B}^0, C^\infty(\Lambda)) \longrightarrow H^2(\mathcal{B}^0, C^\infty(T^*\mathcal{B}^0)) \longrightarrow \cdots$$

As $C^\infty(T^*\mathcal{B}^0)$ is a fine sheaf, $H^i(\mathcal{B}^0, C^\infty(T^*\mathcal{B}^0)) = 0$ for $i \geq 1$, we have in particular the following isomorphism

$$\delta : H^1(\mathcal{B}^0, C^\infty(T^*\mathcal{B}^0/\Lambda)) \xrightarrow{\cong} H^2(\mathcal{B}^0, C^\infty(\Lambda))$$

Define the **Chern class** of the complex integrable system π to be

$$\eta := \delta(\mu)$$

Thus, we see that $\eta = 0$ if and only if π is diffeomorphic fibre-wisely to pr, which is further equivalent to that π admits a global (smooth) section.

Symplectic classification

The trivialization map $\hat{\Psi}_\sigma$ is only a diffeomorphism (see proposition 3.1.9). If it is also a symplectomorphism, then the classification scheme in the smooth setting would yield a symplectic classification as well. Thus, we need to discuss the conditions for the trivialization maps to be symplectomorphism.

Proposition 3.1.11. *The trivialization $\hat{\Psi}_\sigma$ is a symplectomorphism if and only if the local section $\sigma : U \to \pi^{-1}(U)$ is a Lagrangian.*

Proof. By proposition 3.1.9, $\hat{\Psi}_\sigma$ is a local diffeomorphism from $T^*(U_{ij})/\Lambda|_{U_{ij}}$ to $\pi^{-1}(U_{ij})$, so it is sufficient to check the condition for the cotangent bundle.

Given a section $\alpha = \sum_i \alpha_i dq^i$, by (3.1.12), we see that it acts on $\beta = \sum_i \beta_i \, dq^i$ by

$$\beta_i \mapsto \beta_i + \alpha_i$$

where we have endowed T^*U_{ij} with the standard coordinates: $\{q^i, p_i\}$, so that the canonical symplectic form can be written as

$$\omega_{can} = \sum_i dp_i \wedge dq^i$$

Then we see that under the action, the symplectic form transforms according to

$$\omega_{can} = \sum_i dp_i \wedge dq^i \longmapsto \sum_i dp_i \wedge dq^i + \sum_i d\alpha_i \wedge dq^i$$

from which we see that ω_{can} is being preserved if and only if $d\alpha = 0$. However, lemma 3.1.5 tells us that the closed one form corresponds to the Lagrangian section. $\square$

Denote by $\mathcal{L}(M)$ the sheaf of Lagrangian sections of $M \to \mathcal{B}^0$, then as in the smooth case, we have the following short exact sequence

$$0 \to \mathcal{L}(\Lambda) \longrightarrow \mathcal{L}(T^*\mathcal{B}^0) \longrightarrow \mathcal{L}(T^*\mathcal{B}^0/\Lambda) \longrightarrow 0$$

from which the following long exact sequence being induced

$$\cdots \longrightarrow H^1(\mathcal{B}^0, \mathcal{L}(\Lambda)) \longrightarrow H^1(\mathcal{B}^0, \mathcal{L}(T^*\mathcal{B}^0)) \longrightarrow H^1(\mathcal{B}^0, \mathcal{L}(T^*\mathcal{B}^0/\Lambda))$$

$$\xrightarrow{\delta} H^2(\mathcal{B}^0, \mathcal{L}(\Lambda)) \longrightarrow H^2(\mathcal{B}^0, \mathcal{L}(T^*\mathcal{B}^0)) \longrightarrow \cdots$$

Again, since the sheaf $\mathcal{L}(T^*\mathcal{B}^0)$ is fine, we have the following isomorphism

$$\delta : H^1(\mathcal{B}^0, \mathcal{L}(T^*\mathcal{B}^0/\Lambda)) \xrightarrow{\cong} H^2(\mathcal{B}^0, \mathcal{L}(\Lambda))$$

By proposition 3.1.11, we have that

$$H^1(\mathcal{B}^0, \mathcal{L}(T^*\mathcal{B}^0/\Lambda)) \cong H^1(\mathcal{B}^0, Z^1(T^*\mathcal{B}^0/\Lambda)$$

where $Z^1(T^*\mathcal{B}^0/\Lambda)$ is the sheaf of closed one forms on $T^*\mathcal{B}^0/\Lambda$.

As in the smooth case, a cohomology class $\mu \in H^1(\mathcal{B}^0, Z^1(T^*\mathcal{B}^0/\Lambda)$ is associated to π, which is called the **Lagrangian Chern class** of π. It is a symplectic invariant of the complex integrable system π.

And the **Chern class** η of π is defined to be the image $\delta(\mu)$. It is the obstruction for $\pi : (X^0, \omega^{2,0}) \to \mathcal{B}^0$ to be symplectomorphic fibre-wisely to the reference integrable system $pr : T^*\mathcal{B}^0 \to \mathcal{B}^0$, which is equivalent to the existence of a global Lagrangian section of π.

3.1.3 Action-angle coordinates

In this subsection, let us endow the complex integrable system $\pi : (X^0, \omega^{2,0}) \to \mathcal{B}^0$ with the so-called *action-angle* coordinates. We first view the integrable system π as the real one by considering on X^0 the real symplectic form given by $\omega := Re(\omega^{2,0})$. The following exposition is based on the book[BF04].

The period net $\Lambda \subset T^*\mathcal{B}^0$ is a local system of lattice over $\mathcal{B}^0$, with the fiber at $b \in \mathcal{B}^0$ being $\Lambda_b \cong \mathbb{Z}^{2n}$. Near $b \in \mathcal{B}^0$, we choose an open neighbourhood $U \subset \mathcal{B}^0$ so small such that it is simply connected, i.e., $\pi_1(U) = 0$. Then we can choose a local section α of $\Lambda \to U$, that is, a set of closed one forms $(\alpha^1, \cdots, \alpha^{2n})$, such that $(\alpha^1(b), \cdots, \alpha^{2n}(b))$ form a $\mathbb{Z}$-basis of Λ_b, i.e., a $\mathbb{Z}$-basis for the first homology group:

$$H^1(\pi^{-1}(b), \mathbb{Z}) \cong H^1(T_b^*/\Lambda_b, \mathbb{Z}) \cong \Lambda_b \cong \mathbb{Z}^{2n}. \tag{3.1.15}$$

As U is simply connected, these one forms are locally exact, i.e. there exists local coordinate functions

$$(u^1, \cdots, u^{2n}) : U \longrightarrow \mathbb{R}^{2n}$$

such that $\alpha^i = du^i$. Since these coordinate functions are constant along the fiber of π, we can view them as coordinates on X^0,i.e., $u^i \circ \pi$ (we still denote these by u^i).

Lemma 3.1.12. *The coordinate functions u^i are in involution, i.e.,*

$$\{u^i, u^j\} = 0$$

Proof. See the last line in the proof of the proposition 3.1.3. □

We denote the Hamiltonian vector field associated to du^i by ξ_{u^i}. They are linearly independent and generate the space of vertical vector fields $Ker\, \pi_*$.

We know that the fiber $\pi^{-1}(b)$ is a tori $\mathbb{T}^{2n} = T_b^*U/\Lambda_b \cong \mathbb{C}^n/\mathbb{Z}^{2n}$.

Fix a reference point $x \in \pi^{-1}(b)$. Since the action of $T_b^*U \cong \mathbb{R}^{2n}$ on the fiber $\pi^{-1}(b)$ is transitive, for any point $y \in \pi^{-1}(b)$, there exists

$$\alpha = \sum_{i=1}^{2n} y_i\, \alpha^i \in T_b^*U$$

for $(y_1, \cdots, y_{2n}) \in \mathbb{R}^{2n}$, such that $\phi_\alpha^1(x) = y$. Then we can define the **angle coordinates** on the tori $\mathbb{T}^{2n}$ as follows:

$$\phi_i(y) := y_i\, (mod\, \mathbb{Z}), \;\; i = 1, \cdots, 2n. \tag{3.1.16}$$

Remark 3.1.4. *For the reference point x, there exists $\alpha = \sum_{i=1}^{2n} k_i\, \alpha^i \in \Lambda_b$ for $(k_1, \cdots, k_{2n}) \in \mathbb{Z}^{2n}$ such that $\phi_\alpha^1 = x$, then we see from the above definition that the angle coordinates for x is $(0, \cdots, 0)$.*

Together with $(u^1, \cdots, u^{2n})$ on the base, $(u^1, \cdots, u^{2n}, \phi_1, \cdots, \phi_{2n})$ gives a local coordinate system on $\pi : X^0 \to \mathcal{B}^0$.

Next, we shall find more canonical form of the base coordinates so that the (real) symplectic form on X^0 can be written in the form similar to that on the cotangent bundle. The vector fields $\frac{\partial}{\partial \phi_i}$ generated by angle coordinates are vertical fields and can be written in terms of $\{\xi_{u^i}\}$ as

$$\frac{\partial}{\partial \phi_i} = \sum_{k=1}^{2n} c_{ik}\, \xi_{u^k}$$

Denote by $\omega := Re(\omega^{2,0})$ the real symplectic form on X^0, and it can be expressed as

$$\omega = \sum_{i,j} \widetilde{c_{ij}}\, du^i \wedge d\phi_j + \sum_{i,j} a_{ij}\, d\phi_i \wedge d\phi_j + \sum_{i,j} b_{ij}\, du^i \wedge du^j$$

As the fiber $\mathbb{T}^{2n}_b$ is Lagrangian, the second term in the above expression for ω vanishes, consequently we have that

$$\omega = \sum_{i,j} \widetilde{c_{ij}}\, du^i \wedge d\phi_j + \sum_{i,j} b_{ij}\, du^i \wedge du^j \tag{3.1.17}$$

Lemma 3.1.13. $c_{ij} = \widetilde{c}_{ij}$.

Proof.

$$\begin{aligned}
\widetilde{c}_{ij} = \omega(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial \phi_j}) &= \omega\left(\frac{\partial}{\partial u^i}, \sum_k c_{jk}\, \xi_{u^k}\right) = \sum_k c_{kj}\, \omega\left(\frac{\partial}{\partial u^i}, \xi_{u^k}\right) \\
&= \sum_k c_{kj}\ (i_{\xi_{u^k}}\omega)\left(\frac{\partial}{\partial u^i}\right) = \sum_k c_{kj}\, du^k\left(\frac{\partial}{\partial u^i}\right) \\
&= \sum_{k,l} c_{kj}\, \frac{\partial u^k}{\partial u^l}\, du^l\left(\frac{\partial}{\partial u^i}\right) = c_{ij}
\end{aligned}$$

□

Lemma 3.1.14. *The coefficients b_{ij} does not depend on the coordinates ϕ_i.*

Proof. $d\omega = 0$ implies:

$$\frac{\partial b_{ij}}{\partial \phi_k} = \frac{\partial c_{kj}}{\partial u^i} - \frac{\partial c_{ki}}{\partial u^j}$$

The right hand side does not depend on the angle coordinates as c_{ij} does not. Now suppose that b_{ij} depends on ϕ_i, then it must be periodic, thus the right hand side would depend on ϕ_i, which is a contradiction. □

Now write $\omega_i = \sum_j c_{ij}\, du^i$ and $\beta = \sum_{i,j} b_{ij}\, du^i \wedge du^j$, they are forms on the base manifold $\mathcal{B}^0$, then the formula (3.1.17) can be rewritten as

$$\omega = \sum_i \omega_i \wedge d\phi^i + \beta \tag{3.1.18}$$

Lemma 3.1.15. *The forms ω_i and β are closed.*

Proof. $d\omega = \sum_i d\omega_i \wedge d\phi^i + d\beta = 0$, thus $d\beta = -\sum_i d\omega_i \wedge d\phi^i$. As both $d\omega_i$ and $d\beta$ are forms depending on u^i, the above identity between forms is possible only if both $d\omega_i$ and $d\beta$ vanish. □

Thus, near $b \in U$, from our assumption that $\pi_1(U) = 0$, by using the Poincaré lemma, we see that there exists a set of functions I^i and one form η defined locally near b such that $\omega_i = dI^i$ for $i = 1, \cdots, 2n$, and $\beta = d\eta$. Define the *action one form* α by

$$\alpha := \sum_i I^i \, d\phi_i + \eta \tag{3.1.19}$$

Then it is clear that $d\alpha = \omega$. We claim that $(I^1, \cdots, I^{2n})$ can serve as local coordinates near $b \in U$, called the **action coordinates**.

Lemma 3.1.16. *The set of functions $I^i = I^i(u^1, \cdots, u^{2n})$ are functionally independent.*

Proof. The matrix representation Ω of the symplectic form ω in $(u^1, \cdots, u^{2n}, \phi_1, \cdots, \phi_{2n})$ is given by

$$\begin{pmatrix} 0 & c_{ij} \\ -c_{ij} & b_{ij} \end{pmatrix}$$

where by definition

$$c_{ij} = \frac{\partial I^i}{\partial u^j}$$

from which we see that

$$det \, \Omega = (det \, C)^2 \neq 0$$

Consequently, the Jacobian of the transformation

$$I^i(u^1, \cdots, u^{2n}) = det \, (C) \neq 0$$

□

Write the one form η as $\eta = \sum_i g_i dI^i$, for functions g_i depending on $I^i, \; i = 1, \cdots, 2n$. Then define the new set of angle coordinates (which corresponds to a new choice of zero section of the affine tori $\mathbb{T}^{2n}$) by

$$\theta_i := \phi_i - g_i(I^1, \cdots, I^{2n}), \; i = 1, \cdots, 2n. \tag{3.1.20}$$

Definition 3.1.10. *The coordinate system $(I^1, \cdots, I^{2n}, \theta_1, \cdots, \theta_{2n})$ constructed above is called the* ***action-angle coordinates*** *for the complex integrable system $\pi : X^0 \to \mathcal{B}^0$.*

Proposition 3.1.17. *In the action-angle coordinates, the symplectic form ω can be written in the canonical form as*

$$\omega = \sum_i dI^i \wedge d\theta_i \tag{3.1.21}$$

Proof. By the formula (3.1.19), we compute as below:

$$\omega = d\alpha = \sum_i dI^i \wedge d\phi_i + d\eta = \sum_i dI^i \wedge (d\theta_i + dg_i) + \sum_i dg_i \wedge dI^i$$

$$= \sum_i dI^i \wedge d\theta_i + \sum_i dI^i \wedge dg_i - \sum_i dI^i \wedge dg_i = \sum_i dI^i \wedge d\theta_i$$

□

Remark 3.1.5. *We can define a canonical one form*

$$\alpha_{can} := \sum_i I^i d\theta_i$$

It can be viewed as the analogy of Liouville one form. It differs from α defined in (3.1.19) by an exact one form. Indeed, we have that $\alpha = \alpha_{can} + d(I^i g_i)$. Thus, they represent the same class in the de Rham cohomology group $H^1_{dR}(\mathcal{B}^0, \mathbb{R})$ of the base manifold $\mathcal{B}^0$.

Remark 3.1.6. *In terms of action-angle coordinate system, the fibration*

$$\pi : (X^0, \omega) \longrightarrow \mathcal{B}^0$$

is given by the projection:

$$\pi : (I^1, \cdots, I^{2n}, \theta_1, \cdots, \theta_{2n}) \longmapsto (I^1, \cdots, I^{2n}).$$

Actually, this projection can be viewed as the moment map for the action of of the affine tori (viewed as a torus Lie group) $\mathbb{T}^{2n} = T_b^*\mathcal{B}^0/\Lambda_b$ on the fiber $\pi^{-1}(b)$.

Lemma 3.1.18. *For $b \in \mathcal{B}^0$, the first homology group $H_1(\pi^{-1}(b), \mathbb{Z})$ can be identified with the period lattice Λ_b.*

Proof. By the isomorphism (3.1.15), $\Lambda_b \cong H^1(\pi^{-1}(b), \mathbb{Z})$, which is isomorphic to $H_1(\pi^{-1}(b), \mathbb{Z})$. More explicitly, the isomorphism can be realized as :

$$\lambda : H_1(\pi^{-1}(b), \mathbb{Z}) \longrightarrow \Lambda_b \qquad \gamma \longmapsto \lambda_\gamma \tag{3.1.22}$$

where λ_γ is the one form defined through:

$$\lambda_\gamma(v) := \oint_\gamma \iota_{\widetilde{v}}\, \omega \tag{3.1.23}$$

where $v \in T_b\mathcal{B}^0$, and $\widetilde{v}$ is its lifting to TX^0. □

By using the action-angle coordinates, we see that $\Lambda_b = Span_{\mathbb{Z}}\{dI^1, \cdots, dI^{2n}\}$, then under the above identification (3.1.22), there are corresponding basis elements $\{\gamma_1, \cdots, \gamma_{2n}\}$ for the lattice $H_1(\pi^{-1}(b), \mathbb{Z})$.

Lemma 3.1.19. *The one cycles γ_i above satisfy the relation*

$$\oint_{\gamma_i} d\theta_j = \delta_{ij} \tag{3.1.24}$$

where δ_{ij} denotes the Kronecker symbol, and θ_is the angle coordinates.

Proof. By definition, γ_i is the one cycle dual to the one form dI^i, thus under the isomorphism (3.1.22), we have that $\lambda_{\gamma_i} = dI^i$. Consequently from the formula (3.1.23), one compute as follows:

$$\delta_{ij} = dI^i(\partial_{I^j}) = \lambda_{\gamma_i}(\partial_{I^j}) = \oint_{\gamma_i} \iota_{\widetilde{\partial_{I^i}}}\, \omega = \oint_{\gamma_i} \iota_{\widetilde{\partial_{I^i}}} \left(\sum_k dI^k \wedge d\theta_k \right) = \oint_{\gamma_i} d\theta_i.$$

□

Lemma 3.1.20. *The action coordinate I^i can be written as*

$$I^i = \oint_{\gamma_i} \alpha_{can} \tag{3.1.25}$$

where $\alpha_{can} = \sum_i I^i d\theta_i$ is the canonical action one form defined in the remark 3.1.5.

Proof. By (3.1.24), we see that

$$\oint_{\gamma_i} \alpha_{can} = \oint_{\gamma_i} \sum_k I^k d\theta_k = \sum_k I^k \delta_{ik} = I^i$$

□

Remark 3.1.7. *By remark 3.1.5, we know that the action one form α as defined in (3.1.19) differs from α_{can} by an exact one form, thus the action coordinate I^i can also be written as*

$$I^i = \oint_{\gamma_i} \alpha \tag{3.1.26}$$

More generally, the ambiguity of the action one form tells us that the action coordinates are defined only up to an additive constant.

Indeed, suppose α and α' are two action forms, i.e. $d\alpha = d\alpha' = \omega$, then we can write $\alpha = \alpha' + \beta$ such that $d\beta = 0$, but then we have

$$I^{i\prime} = \oint_{\gamma_i} \alpha' = \oint_{\gamma_i} (\alpha + \beta) = \oint_{\gamma_i} \alpha + \oint_{\gamma_i} \beta = I^i + \oint_{\gamma_i} \beta$$

Definition 3.1.11. *We have a local system of lattice $\underline{\Gamma}$ over $\mathcal{B}^0$, with its "stalk" $\underline{\Gamma}_b$ at $b \in \mathcal{B}^0$ being the first homology group $H_1(\pi^{-1}(b), \mathbb{Z})$. This lattice will be called the **charge lattice**.*

Actually, we see from the above discussion that the action coordinates $\{I^i\}$ extends to a map from the local system of charge lattice $\underline{\Gamma}$ to $\mathbb{R}$. Indeed, the map $I : \underline{\Gamma} \to \mathbb{R}$ is defined through

$$I(\gamma) = \oint_{\gamma} \alpha \tag{3.1.27}$$

which is easily seen to be an abelian group homomorphism as the integration is additive in its domain. In particularly, we have that $I^i = I(\gamma_i)$, and for arbitrary

$$\gamma = \sum_i k_i \gamma_i \in H_1(\pi^{-1}(b), \mathbb{Z}) = \underline{\Gamma}_b$$

we have that $I(\gamma) = \sum_i k_i I^i$.

Remark 3.1.8. *We will show in the next subsection that the natural analogy of the functional I above in the complex case will be the central charge function of a complex integrable system.*

3.1.4 Holomorphic coordinates and Central charge

Motivated by the construction of action-angle coordinate system as in the last subsection, we want to find its complex analogy. This section is based largely on the papers [LTY14] and [Lin14].

Let us consider the complex integrable system $\pi : (X^0, \omega^{2,0}) \to \mathcal{B}^0$. Similar to the formula (3.1.23), by choosing a basis $\{\gamma_1, \cdots, \gamma_{2n}\}$ for the local system of lattice $\underline{\Gamma}$ near $b \in \mathcal{B}^0$, we can define the following set of holomorphic one forms

$$\alpha_i := \oint_{\gamma_i} \omega^{2,0} \qquad i = 1, \cdots, 2n. \tag{3.1.28}$$

Clearly, they are closed one forms. Thus, there exist holomorphic functions z_i defined locally near b such that $dz_i = \alpha_i$ for $1 \leq i \leq 2n$. These functions are defined only up to an additive constant.

The set of holomorphic functions $\{z_1, \cdots, z_{2n}\}$ are complex analogy of the action coordinates $\{I^1, \cdots, I^{2n}\}$ in the real case.

Proposition 3.1.21. *The holomorphic functions defined above are related to the action-angle coordinates (see definition 3.1.10) in the following way: write $w_i = e^{z_i}$ and suppose that $dz_i = \alpha_i$, then we have that*

$$I^i = Re(\log w_i) = log|w_i|, \quad \theta_i = Im(\log w_i).$$

Proof. Let us consider instead the holomorphic change of coordinates:

$$z_i = \log w_i, \quad \text{for } 1 \leq i \leq 2n$$

And assume that

$$dz_i = \alpha_i = \oint_{\gamma_i} \omega^{2,0}$$

Since $\log w_i = \log |w_i| + \sqrt{-1}\,\theta_i$, we have that

$$dRe(\log w_i) = d\log |w_i| = dRe(z_i) = \oint_{\gamma_i} Re(\omega^{2,0}) = dI^i$$

and $Im(\log w_i) = \theta_i$. $\square$

In contrast with the action coordinates, z_i s are not coordinates on the base $\mathcal{B}^0$, but they define an holomorphic map

$$(z_1, \cdots, z_{2n}) : U \longrightarrow \mathbb{C}^{2n} \tag{3.1.29}$$

where U is an open neighborhood of b.

Polarization

If the fibers of the complex integrable system are endowed with a covariantly constant integer polarization, we call such an integral system the **polarized complex integrable system**. Roughly speaking, a polarization gives rise to a covariantly constant skew-symmetric symplectic bilinear form (which is callde the DSZ paring in physics literature) on $\underline{\Gamma} \otimes \mathbb{Q}$, i.e.,

$$\langle \cdot, \cdot \rangle : \Lambda^2 \underline{\Gamma} \longrightarrow \underline{\mathbb{Z}}_{\mathcal{B}^0} \tag{3.1.30}$$

So the local system of charge lattices $\underline{\Gamma}$ is actually a local system of symplectic lattices over $\mathcal{B}^0$. Let ω_{ij} the value of $\langle \gamma_i, \gamma_j \rangle$, they form a matrix $(\omega_{i,j})$, and denote by (ω^{ij}) its inverse. Then we have the following

Proposition 3.1.22. *a)* $\sum_{i,j} \omega^{ij} dz_i \wedge dz_j = \sum_{i,j} \omega^{ij} \alpha_i \wedge \alpha_j = 0$.

$$b) \sqrt{-1} \sum_{i,j} \omega^{ij} dz_i \wedge d\bar{z}_j = \sqrt{-1} \sum_{i,j} \omega^{ij} \alpha_i \wedge \bar{\alpha}_j > 0.$$

Proposition 3.1.23. *The condition a) in the above proposition implies that the image of the map (3.1.29) is a Lagrangian sub-manifold, while the condition b) implies that the map is an immersion.*

Proof. Define the symplectic form on $\underline{\Gamma}^\vee \otimes \mathbb{C} \cong \mathbb{C}^{2n}$ by

$$\Omega = \sum_{i,j} \omega^{ij} du^i \wedge du^j$$

where $\{u^1, \cdots, u^{2n}\}$ are the chosen coordinates for $\mathbb{C}^{2n}$, then a) translates into the fact that the image of the holomorphic map (3.1.28) is Lagrangian w.r.t Ω. As we have Kähler metric space induced by the (1-1) form in b), thus the map in question can be viewed locally as a map between metric spaces, which must be injective. Consequently, the map is an immersion. □

We denote by $\mathcal{Z} : U \to \mathbb{C}^{2n}$ the map in (3.1.28), from the above proposition, we conclude that it defines holomorphic Lagrangian embedding of U to $\underline{\Gamma}^\vee \otimes \mathbb{C}$ defined up to a shift(as z^i s are defined only up to an additive constant). More explicitly, we have

$$\mathcal{Z} : U \to \underline{\Gamma}^\vee \otimes \mathbb{C}, \quad d\mathcal{Z}_b(\gamma) := \oint_\gamma \omega^{2,0} \tag{3.1.31}$$

where $b \in U$ and $\gamma \in \underline{\Gamma}_b$ can be locally continued to nearby fibers continuously.

Remark 3.1.9. *The above map is the complex version of the map I given in (3.1.27). Consequently, it can be viewed as the complex analogy of the action coordinates.*

The map $\mathcal{Z}$ is defined only locally. If such map is globally defined over $\mathcal{B}^0$, then we call it the *central charge* of the complex integrable system, and denote it by Z. More precisely, we have the following:

Definition 3.1.12. *The collection of one forms α_i defined in (3.1.27) give rise to an element $\delta \in H^1(\mathcal{B}^0, \underline{\Gamma}^\vee \otimes \mathbb{C})$, if δ is assumed to be zero, then there exists a section $Z \in \Gamma(\mathcal{B}^0, \underline{\Gamma} \otimes \mathscr{O}_{\mathcal{B}^0})$, called the **central charge** of the integrable system, which satisfies $dZ(\gamma_i) = \alpha_i$, for $1 \leq i \leq 2n$.*

Remark 3.1.10. *From the definition, we see that Z is an additive map from the local system of charge lattice $\underline{\Gamma}$ to $\mathbb{C}$.*

Definition 3.1.13. *If a complex integrable system $\pi : (X^0, \omega^{2,0}) \longrightarrow \mathcal{B}^0$ is endowed with a central charge function Z, then π is called the **integrable system of Seiberg-Witten type** (terminology borrowed from [KS14]).*

Remark 3.1.11. *In terms of central charge Z, the condition a) and b) in proposition 3.1.23 can be rewritten as*

$$a)' \;\; \langle dZ, dZ \rangle = 0 \;\; (transversality) \tag{3.1.32}$$

$$b)' \;\; \sqrt{-1}\, \langle dZ, d\bar{Z} \rangle > 0 \;\; (nondegeneracy) \tag{3.1.33}$$

Holomorphic coordinates induced from the central charge Z

By using the polarization given by the symplectic form (3.1.30), we can choose **symplectic basis** for $\{\alpha^i, \beta_j\}$, $1 \leq i, j \leq n$ for the charge lattice $\underline{\Gamma}$ near $b \in \mathcal{B}^0$ for every $b \in \mathcal{B}^0$. Here by "symplectic", we mean that the following relations among the basis elements should be satisfied

$$\langle \alpha^i, \alpha^j \rangle = \langle \beta_i, \beta_j \rangle = 0, \;\; \langle \alpha^i, \beta_j \rangle = \delta_i^j. \tag{3.1.34}$$

Accordingly, for the dual lattice $\underline{\Gamma}^\vee$ such that $\underline{\Gamma}_b^\vee \cong Hom_{\mathbb{Z}}(\underline{\Gamma}_b, \mathbb{Z})$, we can choose the basis $\{\alpha_i, \beta^j\}$ of it that are dual to $\{\alpha^i, \beta_j\}$ in the following sense:

$$\alpha_i(\beta_j) := \langle \beta_j, \alpha_i \rangle = \delta_j^i, \;\; \beta^j(\alpha^k) := \langle \alpha_k, \beta_j \rangle = \delta_k^j$$

where we have used the identification $\alpha \longmapsto \langle \cdot, \alpha \rangle$ from $\underline{\Gamma}$ to $\underline{\Gamma}^\vee$. Consequently, we can choose $\alpha_i := \beta_i$, and $\beta^j := -\alpha^j$. Under this basis, we can write the central charge function Z as

$$Z = \sum_{i=1}^n a^i \alpha_i + \sum_{i=1}^n a_{D,i} \beta^i \tag{3.1.35}$$

where a^i, $a_{D,i} \in \mathscr{O}_{\mathcal{B}^0}$ are a set of local defined holomorphic functions (depending on any given holomorphic coordinates $\{u^i\}_{1 \leq i \leq n}$ on $\mathcal{B}^0$) on the base $\mathcal{B}^0$. Then we see that the transversality condition (3.1.32) specializes into

$$d\left(\sum_i a_{D,i}\, da^i \right) = 0 \tag{3.1.36}$$

while the nondegeneracy condition (3.1.33) becomes

$$Im\left(\sum_i da_{D,i} \wedge d\bar{a}^i \right) > 0 \tag{3.1.37}$$

From (3.1.36), we have that $\sum_i a_{D,i}\, da^i = d\mathcal{F}$ for some holomorphic function $\mathcal{F}$, which is called the **prepotential**. Thus, we have the following relation

$$a_{D,i} = \frac{\partial \mathcal{F}}{\partial a^i}, \;\; i = 1, \cdots, n. \tag{3.1.38}$$

Besides, (3.1.37) implies that the two matrix $\left(\frac{\partial a^i}{\partial u^j}\right)$ and $\left(\frac{\partial a_{D,i}}{\partial u^j}\right)$ are invertible matrix, thus both a^i and $a_{D,i}$ can be considered as giving holomorphic coordinates on $\mathcal{B}^0$. These coordinates are called **special coordinates** (will be discussed in the next subsection), and by (3.1.38), we can interpret $a_{D,i}$ s as coordinates for $\mathcal{B}^0$ dual to a^i s.

Kähler metric on $\mathcal{B}^0$

By (3.1.37), we see that $\mathcal{B}^0$ carries the Kähler form

$$\omega^{1,1}_{\mathcal{B}^0} := \sqrt{-1}\, Im\left(\sum_i da_{D,i} \wedge d\bar{a}^i\right) \tag{3.1.39}$$

Denote by $g_{\mathcal{B}^0}$ the corresponding **Kähler metric** on $\mathcal{B}^0$, i.e.,

$$g_{\mathcal{B}^0} = Im\left(\sum_i da_{D,i}\, d\bar{a}^i\right) \tag{3.1.40}$$

By using (3.1.37) and $d\mathcal{F} = \sum_i a_{D,i}\, da^i$, it can be further written as

$$g_{\mathcal{B}^0} = Im\left(\sum_i d\left(\frac{\partial \mathcal{F}}{\partial a^i}\right) d\bar{a}^i\right) = Im\left(\sum_{i,j} \frac{\partial^2 \mathcal{F}}{\partial a^i \partial a^j}\, da^i d\bar{a}^j\right)$$

Defining the matrix $\tau = (\tau_{ij})$ by

$$\tau_{ij} := \frac{\partial a_{D,i}}{\partial a^j} = \frac{\partial^2 \mathcal{F}}{\partial a^i \partial a^j} \tag{3.1.41}$$

the Kähler metric above can be written more compactly as

$$g_{\mathcal{B}^0} = \sum_{i,j} Im(\tau_{ij})\, da^i d\bar{a}^j \tag{3.1.42}$$

with the matrix $Im(\tau_{ij})$ being positive definite by (3.1.36).

Remark 3.1.12. *It is clear that the* ***Kähler potential*** *for the metric is*

$$K = Im\left(\sum_i a_{D,i}\, \bar{a}^i\right) \tag{3.1.43}$$

which satisfies $\partial\bar{\partial}K = \omega^{1,1}_{\mathcal{B}_0}$.

Angle coordinates more explicitly

Polarization would enable us to find more canonical description of the angle coordinates (c.f., (3.1.16) and (3.1.20)). Inspired by lemma 3.1.19, we denote by θ_i, $1 \le i \le n$ the angle coordinates that satisfies the relations $\oint_{\alpha^i} d\theta_j = \delta^i_j$; and denote by θ_i $1 \le i \le n$ those that satisfy: $\oint_{\beta_i} d\theta^j = \delta^j_i$. So, the class $[d\theta^i]$ represents the coordinate β^i, i.e., the dual of β_i; and similarly, the class $[d\theta_i]$ represents the coordinate α_i, i.e., the dual of α^i.

Clearly, the class of $\sum_i d\theta^i \wedge d\theta_i$ would represent the symplectic pairing $\langle\cdot,\cdot\rangle$, i.e., the polarization.

As had been pointed out before that the analogy of the action coordinates I^i in complex case are given by the holomorphic function z_i, which by the formula (3.1.35), are realized in the symplectic basis as

$$Z(\alpha^i) = a^i \quad Z(\beta_i) = a_{D,i}, \;\; 1 \leq i \leq n. \tag{3.1.44}$$

From the definition of canonical action one form $\alpha_{can} = \sum_i I^i \, d\theta_i$ (also called Liouville one form) (see (3.1.25)), we find its complex analogy

$$\begin{aligned}\alpha_{Liouville} := \sum_i z_i \, d\theta^i &= \sum_i Z(\alpha^i)\, d\theta_i + \sum_i Z(\beta_i)\, d\theta^i \\ &= \sum_i a^i \, d\theta_i + \sum_i a_{D,i}\, d\theta^i \end{aligned} \tag{3.1.45}$$

Since in the real case, we have that

$$d\alpha_{can} = \omega = Re(\omega^{2,0})$$

while in the complex case, we have similarly that

$$\omega^{2,0} = d\alpha_{Liouville} = \sum_i da_{D,i} \wedge d\theta^i + \sum_i da^i \wedge d\theta_i \tag{3.1.46}$$

By the relation (3.1.41), the right hand side can be written as

$$\sum_{i,j} \tau_{ij} da^j \wedge d\theta^i + \sum_i da^i \wedge d\theta_i = \sum_{i,j} da^i \wedge (d\theta_i + \tau_{ij} d\theta^j)$$

Note that in the above computation, we used the symmetric property of τ_{ij}, i.e., $\tau_{ij} = \tau_{ji}$.

Thus we can introduce the canonical complex coordinates on the fiber of π defined as

$$w_i := \theta_i + \sum_j \tau_{ij}\, \theta^j \tag{3.1.47}$$

Together with the holomorphic coordinates a_i on the base, the holomorphic symplectic two form can be written in canonical form as

$$\omega^{2,0} = \sum_i da^i \wedge dw_i \tag{3.1.48}$$

Remark 3.1.13. *By using the complex coordinates w_i, the form $\sum_i d\theta^i \wedge d\theta_i$ that represents the symplectic paring becomes*

$$-\frac{\sqrt{-1}}{2} \sum_{i,j} ((Im\tau)^{-1})^{ij} dw_i \wedge d\bar{w}_j \tag{3.1.49}$$

which is manifestly negative definite (1-1)-form, defining the polarization. Thus the fiber is actually a principally polarized abelian variety, and our complex integrable system is called principally polarized.

Geometric interpretation of Z

Near $b \in \mathcal{B}^0$, let $\gamma = \sum_i q_i\alpha^i + \sum_j g^j\beta_j \in \underline{\Gamma}_b \cong H_1(\pi^{-1}(b), \mathbb{Z})$, where g^i, q_j s are integers, then by the formula , we have that the central charge of γ is

$$Z_b(\gamma) = \sum_i g^i a_{D,i} + \sum_j q_j a^j. \tag{3.1.50}$$

Now we want to give a geometric interpretation of the central charge function. To this end, let us denote the above local system of lattices by $\underline{\Gamma}^{symp}$, and by $\underline{\Gamma}$ the local system of lattices with fiber at $b \in \mathcal{B}^0$ being the second relative homology group $H_2((X^0, \pi^{-1}(b)), \mathbb{Z})$. Apparently, the boundary map

$$\partial : H_2((X^0, \pi^{-1}(b)), \mathbb{Z}) \longrightarrow H_1(\pi^{-1}(b), \mathbb{Z})$$

is a surjection, with the kernel being the second homology group $H_2(X^0, \mathbb{Z})$. As b varies, they form a local system of lattices denoted by $\underline{\Gamma}_0$. Consequently, we have the following short exact sequence of local systems:

$$0 \longrightarrow \underline{\Gamma}_0 \longrightarrow \underline{\Gamma} \longrightarrow \underline{\Gamma}^{symp} \longrightarrow 0 \tag{3.1.51}$$

Remark 3.1.14. *The non-degenerate integer symplectic bilinear form $\langle \cdot, \cdot \rangle$ on the symplectic lattice $\underline{\Gamma}^{symp}$ can be extended to $\underline{\Gamma}$ by*

$$\langle \sigma_1, \sigma_2 \rangle := \langle \partial\sigma_1, \partial\sigma_2 \rangle$$

for $\sigma_1, \sigma_2 \in H_2((X^0, \pi^{-1}(b), \mathbb{Z})$. It is clear that the extended pairing becomes degenerate and the kernel of it is exactly $\underline{\Gamma}_0$.

Proposition 3.1.24. *The central charge Z can be realized as the abelian group homomorphism from $\underline{\Gamma}$ to $\mathbb{C}$ given by*

$$Z_b(\sigma) := \oint_\sigma \omega^{2,0} \tag{3.1.52}$$

for each $\sigma \in H_2((X^0, \pi^{-1}(b), \mathbb{Z}) = \underline{\Gamma}_b$.

Proof. First, we show that the above map is well defined. Suppose we have two cycles σ, σ' that represent the same class in $H_2((X^0, \pi^{-1}(b), \mathbb{Z})$, i.e.,

$$\sigma - \sigma' \in \pi^{-1}(b)$$

Consequently, we have that

$$\oint_\sigma \omega^{2,0} - \oint_{\sigma'} \omega^{2,0} = \oint_{\sigma - \sigma'} \omega^{2,0} = 0$$

Note we have used the fact $\omega^{2,0}|_{\pi^{-1}(b)} = 0$ as the fibers are Lagrangian submanifolds. Recall that the formula says that

$$\omega^{2,0} = d\alpha_{Liouville} = \sum_i da_{D,i} \wedge d\theta^i + \sum_i da^i \wedge d\theta_i$$

Thus, for any $\gamma \in \underline{\Gamma}_b^{symp} = H_1(\pi^{-1}(b), \mathbb{Z})$, since the boundary map ∂ is surjective, we can choose σ such that $\partial\sigma = \gamma$, then we have that

$$Z_b(\sigma) = \oint_\sigma \omega^{2,0} = \oint_\sigma d\alpha_{Liouville} = \oint_{\partial\sigma} \alpha_{Liouville} = \oint_\gamma \alpha_{Liouville}$$

$$= \oint_\gamma \sum_i a_{D,i} \wedge d\theta^i + \oint_\gamma \sum_i a^i \wedge d\theta_i$$

Write $\gamma = \sum_i g^i \beta_i + \sum_j q_j \alpha^j$ for g^i and q_j s being integers. Recall further that

$$\oint_{\alpha^i} d\theta_j = \delta^i_j \quad \text{and} \quad \oint_{\beta_i} d\theta^j = \delta^j_i$$

we conclude that

$$Z_b(\sigma) = \sum_i g^i a_{D,i} + \sum_i q_i a^i = \sum_i g^i Z_b(\beta_i) + \sum_i q_i Z_b(\alpha^i)$$

$$= Z_b\left(\sum_i g^i \beta_i + q_i \alpha^i\right) = Z_b(\gamma)$$

Suppose that there is another choice σ' such that $\partial\sigma' = \gamma$, then it is obvious that $Z_b(\sigma) = Z_b(\sigma')$. □

Proposition 3.1.25. *The central charge Z is a holomorphic function on $\mathcal{B}^0$.*

Proof. Denote by J the complex structure on $\mathcal{B}^0$, i.e., a linear endomorphism J acting on the tangent space $T_b\mathcal{B}^0$ for each point $b \in \mathcal{B}^0$ such that $J^2 = -id$. Then the vectors in $T^{0,1}\mathcal{B}^0 \subset T\mathcal{B}^0$ are of the form $v + \sqrt{-1}Jv$ for some $v \in T_\mathbb{R}\mathcal{B}^0$. And by the formula , we compute as

$$(v + \sqrt{-1}Jv)Z(\gamma) = dZ(\gamma)(v + \sqrt{-1}Jv) = \oint_\gamma \iota_{v+\sqrt{-1}Jv}\,\omega^{2,0} = 0$$

since $\omega^{2,0}$ is a (2,0)-form, while $v + \sqrt{-1}Jv$ is a (0,1)-tangent vector. Thus, $\bar{\partial} Z_\gamma = 0$, i.e., $Z(\gamma)$ is holomorphic on $\mathcal{B}^0$ for every γ. □

Proposition 3.1.26. *For any $b \in \mathcal{B}^0$, $\{dZ_b(\gamma)\}_{\gamma \in \underline{\Gamma}_b}$ span $T^*_b\mathcal{B}^0$.*

Proof. Indeed, by the formula (3.1.50), we see that let γ runs through those "α"-cycles, i.e., α^j s, then $\{dZ_b(\gamma)\}$ span $T^*_b\mathcal{B}^0$ as $Z_b(\alpha^j) = a^j$ and a^j serves as local holomorphic coordinates. □

3.1.5 Affine, special Kähler and HyperKähler structures

$\mathbb{Z}$-affine structure (The main references are [Fre99][ACD02][GS03].)

Definition 3.1.14. *Recall that an n-dimensional topological manifold $\mathcal{B}$ is called an **(integral) affine manifold** if it is endowed with an (integral) affine structure that is given by a maximal atlas $\mathcal{A} = \{U_i, \Phi_i\}$ of coordinate charts such that the transition maps $\Phi_j \circ \Phi_i^{-1}$ belong to the group of (integrable) affine transformations of $\mathbb{R}^n$.*

Remark 3.1.15. *Denote by $Aff(\mathbb{R}^n) := GL(n;\mathbb{R}) \ltimes \mathbb{R}^n$ **the group of affine transformations** of $\mathbb{R}^n$, and by $Aff_{\mathbb{Z}}(\mathbb{R}^n) := GL(n;\mathbb{Z}) \ltimes \mathbb{R}^n$ the group of integer affine transformations of $\mathbb{R}^n$. We also use notation $\mathbb{Z}$-affine for the term "integral affine".*

Definition 3.1.15. *A map $f : \mathbb{R}^n \to \mathbb{R}^m$ is called affine ($\mathbb{Z}$-affine) if it belongs to $Hom(\mathbb{R}^n, \mathbb{R}^m) \times \mathbb{R}^m$ ($Hom(\mathbb{Z}^n, \mathbb{Z}^m) \times \mathbb{R}^m$). In the case when $n = m$, we can speak about affine ($\mathbb{Z}$-affine) diffeomorphism were the inverse f^{-1} exists.*

Definition 3.1.16. *For two affine ($\mathbb{Z}$-affine) manifolds $\mathcal{B}$ and $\mathcal{B}'$, a map $f : \mathcal{B} \to \mathcal{B}'$ between them is called affine ($\mathbb{Z}$-affine) if it is locally affine ($\mathbb{Z}$-affine),*

Proposition 3.1.27. *The base of the complex integrable system $\pi : (X^0, \omega^{2,0}) \to \mathcal{B}^0$ is a $\mathbb{Z}$-affine manifold of (real) dimension $2n$, where $n = dim_{\mathbb{C}}\, \mathcal{B}^0$.*

Proof. Cover $\mathcal{B}^0$ by open subsets $U_i \in \mathcal{B}^0$ such that the non empty intersections $U_{ij} := U_i \cap U_j$ are connected and each U_i s being simply connected, i.e., $\pi_1(U_i) = 0$. We also assume that above each U_i, the period net Λ is trivialized (this can always be made true by shrinking U_i s if necessary). Consequently, we have local sections $\{\alpha_i^1, \cdots, \alpha_i^{2n}\}$ of $\Lambda|_{U_i}$, such that for $b \in U_i$, $\{\alpha_i^1(b), \cdots, \alpha_i^{2n}(b)\}$ gives a $\mathbb{Z}$-basis for Λ_b. Then over each U_{ij}, the basis $\{\alpha_i^1, \cdots, \alpha_i^{2n}\}$ and $\{\alpha_j^1, \cdots, \alpha_j^{2n}\}$ are related by a $GL(2n, \mathbb{Z})$ transformation, i.e., $\exists!\ M = (m_{ij}) \in GL(n, \mathbb{Z})$ such that

$$\alpha_j^k(b) = \sum_l m_{jl}\, \alpha_i^l(b)$$

for $b \in U_{ij}$. Here as U_{ij} are chosen to be connected, the smooth matrix M is a constant matrix over each U_{ij}. As $\pi_1(U_i) = 0$ for each i, by Poincáre lemma, locally there exist smooth function $\{u_i^1, \cdots, u_i^{2n}\}$, such that $du_i^k = \alpha_i^k$, for $k = 1, \cdots, 2n$. Thus, we get the relation

$$du_j^k = \sum_l m_{jl}\, du_i^l = d\left(\sum_l m_{jl}\, u_i^l\right)$$

Consequently, there exists constant $c_{ji} \in \mathbb{R}$ such that $u_j^k = \sum_l m_{jl}\, u_i^l + c_{ji}$, which means that the transition maps for local coordinates u^k s are $\mathbb{Z}$-affine. $\square$

Special Kähler structure

We now give an equivalent characterization of the affine ($\mathbb{Z}$-affine) manifold, which leads us to the notion of *special geometry*.

Proposition 3.1.28. *Given an n-dimensional $\mathbb{Z}$-affine manifold $\mathcal{B}$, there is an torsion-free flat connection ∇ on the (real) tangent bundle $T_{\mathbb{R}}\mathcal{B}$, together with a maximal rank ∇-covariant lattice $T^{\mathbb{Z}}_{\mathbb{R}}\mathcal{B} \subset T_{\mathbb{R}}\mathcal{B}$. And conversely, given such a connection and the lattice, we can induce a $\mathbb{Z}$-affine structure on $\mathcal{B}$.*

Proof. Suppose we have a $\mathbb{Z}$-affine structure $\mathcal{A} = \{(U_i, \Phi_i)\}$ with $\mathbb{Z}$-affine coordinate u^k_i, then $\left\{\frac{\partial}{\partial u^k_i}\right\}$ is a local basis for $T_{\mathbb{R}}U_i$, from which we see that it transforms as

$$\frac{\partial}{\partial u^l_i} = \sum_k m_{kl} \frac{\partial}{\partial u^k_j}$$

where m_{kl} s are integers. Applying the differential d on both sides, we get that

$$d\left(\frac{\partial}{\partial u^l_i}\right) = \sum_k m_{kl}\, d\left(\frac{\partial}{\partial u^k_j}\right)$$

from which we get a connection ∇ on $T_{\mathbb{R}}\mathcal{B}$ given by d, i.e., $\nabla_{\frac{\partial}{\partial u^l}} \frac{\partial}{\partial u^k} = 0$. Clearly, in this particular coordinates, the connection coefficients Γ^k_{ij} s vanish, and in particularly, we have that $\Gamma^k_{ij} = \Gamma^k_{ji}$. Consequently, this is a flat, torsion-less connection. Besides, the full rank lattice $T^{\mathbb{Z}}_{\mathbb{R}}\mathcal{B}$ is spanned by $\left\{\frac{\partial}{\partial u^k_i}\right\}$, and it is clearly ∇-covariant. Conversely, if we are given a connection ∇ and a full rank lattice $T^{\mathbb{Z}}_{\mathbb{R}}\mathcal{B} \subset T_{\mathbb{R}}\mathcal{B}$, then we have a local system of lattice over $\mathcal{B}$, which is equivalent to a flat connection ∇ on $T_{\mathbb{R}}\mathcal{B}$ by well-known result. $\square$

Motivated by the above proposition, we give the following definitions.(c.f., [Fre99][ACD02])

Definition 3.1.17. *A complex manifold $(\mathcal{B}, J)$ is called a* ***special complex manifold*** *if $\mathcal{B}$ is endowed with a flat, torsionfree connection ∇ on its real tangent bundle $T_{\mathbb{R}}\mathcal{B}$ such that*

$$d_{\nabla} J = 0 \tag{3.1.53}$$

where J is the complex structure, and d_{∇} the covariant derivative associated to ∇. And by a ***special symplectic manifold****, we mean a special complex manifold $(\mathcal{B}, J, \nabla)$ together with ∇-parallel symplectic form ω. Finally, a* ***special Kähler manifold*** *is a special symplectic manifold $(\mathcal{B}, J, \nabla, \omega)$ for which ω is J-invariant, i.e., of type $(1,1)$. Then the induced metric $g(\cdot,\cdot) := \omega(J\cdot,\cdot)$ is called the* ***special Kähler metric*** *of the special Kähler manifold $(\mathcal{B}, J, \nabla, \omega)$.*

By the formula (3.1.42), we know that the base $\mathcal{B}^0$ of the complex integrable system $\pi : (X^0, \omega^{2,0}) \longrightarrow \mathcal{B}^0$ is a Kähler manifold, and thus from proposition 3.1.28, we conclude that $\mathcal{B}^0$ is a special Kähler manifold.

Now we deduce some consequences from the definition of Special Kähler manifold, and compare them with the descriptions of the complex integrable system that had been discussed in section 3.1.4.

Lemma 3.1.29. *The torsionfree condition of ∇ is equivalent to the following condition*

$$d_\nabla(id) = 0$$

where $id \in \Gamma(T^*\mathcal{B} \otimes T\mathcal{B})$ is the $T\mathcal{B}$-valued one form taking any tangent vector X to itself.

Proof. By definition, we have that

$$\begin{aligned} d_\nabla(id)(X,Y) &= (-1)^{1+1}\nabla_X id(Y) + (-1)^{1+2}\nabla_Y id(X) + (-1)^{1+2} id([X,Y]) \\ &= \nabla_X Y - \nabla_Y X - [X,Y] \end{aligned}$$

from which the lemma follows. □

Given a flat local framing $\{\xi_\alpha\}$ of $\mathcal{B}$, denote by $\{\eta^\beta\}$ the corresponding dual coframe, then the identity section is given by $id = \sum_\alpha \xi_\alpha \otimes \eta^\alpha$, then we see that

$$d_\nabla(id) = \sum_\alpha d_\nabla \xi_\alpha \otimes \eta^\alpha + \sum_\alpha \xi_\alpha \otimes d_\nabla \eta^\alpha$$

$$= \sum_\alpha \nabla \xi_\alpha \otimes \eta^\alpha + \sum_\alpha \xi_\alpha \otimes d_\nabla \eta^\alpha = \sum_\alpha \xi_\alpha \otimes d_\nabla \eta^\alpha$$

The last identity holds because for any ξ_β, we have that

$$\nabla \xi_\alpha(\xi_\beta) = \nabla_{\xi_\alpha} \xi_\beta = 0$$

Thus, we have that $d_\nabla \eta^\alpha = 0$ Then by Poicaré lemma, there exists local coordinates u^α such that $du^\alpha = \eta^\alpha$. Moreover, these local coordinates satisfy the flat condition $\nabla du^i = 0$. Such coordinates are therefore called **flat coordinates**, or by its connection to $\mathbb{Z}$-affine structure, they are also called **affine coordinates**.

Besides, the (real) symplectic form ω is preserved by the connection ∇, i.e., $\nabla\omega = 0$, thus, we can choose the local coordinates to be **Darboux** in the sense that there exists a set of local coordinates x^i, y_j s, for $1 \le i, j \le n = dim_{\mathbb{C}}\mathcal{B}$, such that

$$\omega = \sum_i dx^i \wedge dy_i \tag{3.1.54}$$

Remark 3.1.16. *The above Darboux coordinates are affine since by construction that the transition maps of these coordinates belong to $sp(2n, \mathbb{R}) \rtimes \mathbb{R}^{2n}$.*

Definition 3.1.18. *Let $(\mathcal{B}, J, \nabla, \omega)$ be a special Kähler manifold. An affine local coordinates (x^i, y_j) is called **real special coordinates** if (3.1.54) holds. Holomorphic coordinates $\{z_i\}$ is called **special** if its real part is real affine, i.e., $\nabla Re(dz^i) = 0$. We say that the special coordinates z_i and the real special coordinates $\{x^i, y_j\}$ are **adapted** if $Re(z^i) = x^i$. Further more, two set of special coordinates $\{z^i\}$ and $\{w_j\}$ are said to be **conjugate** if there exists real special coordinates $\{x^i, y_j\}$ such that $Re(z^i) = x^i$ and $Re(w_j) = y_j$.*

Next, we discuss the consequence of $d_\nabla J = 0$ (3.1.53) in the definition of special Kähler manifold.

Denote by $\pi^{1,0} \in \Omega^{1,0}(T_{\mathbb{C}}\mathcal{B})$ the $(1,0)$-form taking value in the complexified tangent space $T_{\mathbb{C}}$ that projects $X \in T_{\mathbb{C}}$ into its $(1,0)$-component. Then the condition (3.1.53) is equivalent to the condition

$$d_{\nabla}\pi^{1,0} = 0$$

By Poincaré lemma again, there exists locally a vector field $\xi \in T_{\mathbb{C}}$ up to a flat complex vector field such that

$$\nabla\xi = \pi^{1,0} \tag{3.1.55}$$

We then have the following proposition:

Proposition 3.1.30. *Expressing the complex vector field ξ in the real special coordinates $\{x^i, y_j\}$ as*

$$\xi = \sum_i z^i \frac{\partial}{\partial x^i} + \sum_j w_j \frac{\partial}{\partial y_j}$$

for some complex functions $\{z^i\}$ and $\{w_j\}$, then these are conjugate special coordinates that are adapted to the real special coordinates $\{x^i, y_j\}$.

Proof. As $\pi^{1,0}$ is a form of type $(1,0)$, we see that z^i, w_j s are holomorphic, and we have that

$$\pi^{1,0} = \nabla\xi = \sum_i dz^i \frac{\partial}{\partial x^i} + \sum_j dw_j \frac{\partial}{\partial y_j}$$

Taking the real part of the above equation, then the left hand side is just the identity map, i.e.,

$$id = \sum_i Re(dz^i)\frac{\partial}{\partial x^i} + \sum_j Re(dw_j)\frac{\partial}{\partial y_j}$$

From which we deduce that $Re(dz^i) = dx^i$ and $Re(dw_j) = dy_j$. $\square$

Proposition 3.1.31. *Their exists a local holomorphic function $\mathfrak{F}$, determined up to a constant, called the* ***holomorphic prepotential****, such that*

$$w_j = \frac{\partial\mathfrak{F}}{\partial z^j} \tag{3.1.56}$$

Denote then by τ_{ij} the quantity

$$\frac{\partial w_j}{\partial z^i} = \frac{\partial^2\mathfrak{F}}{\partial z^i \partial z^j}$$

Then it determines a Kähler potential is simply given by

$$K = Im(w_i \bar{z}^i) \tag{3.1.57}$$

while the corresponding Kähler form given through

$$\begin{aligned} \omega^{1,1} = \sqrt{-1}\partial\bar{\partial}K &= \sqrt{-1}\, Im\left(\frac{\partial^2\mathfrak{F}}{\partial z^i \partial z^j}\right) dz^i \wedge d\bar{z}^j \\ &= \sqrt{-1}\, Im(\tau_{i,j})\, dz^i \wedge d\bar{z}^j \end{aligned} \tag{3.1.58}$$

Proof. We write the real symplectic form ω in complex special coordinates as follows:

$$\omega = \sum_i dx^i \wedge dy_i = \sum_i \left(\frac{dz^i + d\bar{z}^i}{2}\right) \wedge \left(\frac{dw^i + d\bar{w}^i}{2}\right)$$
$$= \frac{1}{4}\sum_i \left(dz^i \wedge dw_i + dz^i \wedge d\bar{w}_i + d\bar{z}^i \wedge dw_i + d\bar{z}^i \wedge d\bar{w}_i\right)$$

as ω should be of type $(1,1)$, thus we have that

$$\sum_i dz^i \wedge dw_i = \sum_i d\bar{z}^i \wedge d\bar{w}_i = 0$$

In particular, we have

$$\sum_i \left(dz^i \wedge \sum_j \tau_{ji} dz^j\right) = 0$$

from which we deduce that $\tau_{ji} = \tau_{ij}$, that is:

$$\frac{\partial w^i}{\partial z^j} = \frac{\partial w^j}{\partial z^i}$$

Consequently, there must exit a holomorphic function $\mathfrak{F}$ such that (3.1.56) is satisfied. $\square$

With the above preparations, we see that the base $\mathcal{B}^0$ of the complex integrable system $\pi : (X^0, \omega^{2,0}) \longrightarrow \mathcal{B}^0$ is a special Kähler manifold. Recall that the formula (3.1.42) gives the Kähler metric of the base $\mathcal{B}^0$ of the integrable system π, namely

$$g_{\mathcal{B}^0} = Im\left(\sum_{i,j} \frac{\partial^2 \mathcal{F}}{\partial a^i \partial a^j} da^i d\bar{a}^j\right) \tag{3.1.59}$$

for $\{a^i\}$ special coordinates on $\mathcal{B}^0$. We have that the dual coordinates $a_{D,i}$ s are also special, and it is related to a^i by (3.1.38), i.e., $a_{D,i} = \frac{\partial \mathcal{F}}{\partial a^i}$. Then let

$$\begin{cases} x^i := Re(a^i) \\ y_j := Re(a_{D,j}) \end{cases}$$

We claim that these are real special coordinates. Indeed, by the constructions of a^i and $a_{D,j}$, we see that as their real parts are $\mathbb{Z}$-affine. To verify the condition (3.1.54), we note that since $\{x^i, y_j\}$ are local real affine coordinates of $\mathcal{B}^0$, the (canonical) real symplectic form ω of the base $\mathcal{B}^0$ is naturally given by

$$\omega = \sum_i dx^i \wedge dy_i$$

Remark 3.1.17. *Instead of the real part of the central charge Z, we can also use $\{Im\, a^i, Im\, a_{D,j}\}$ as $\mathbb{Z}$-affine coordinates on $\mathcal{B}^0$. These define the so called* ***dual affine structure*** *on $\mathcal{B}$. Roughly speaking, they differs from the adapted ones by a $\frac{\pi}{2}$-rotation of the central charge Z. i.e.,*

$$Re(e^{-\frac{i\pi}{2}} Z(\gamma_i)) = Im\, Z(\gamma_i).$$

More generally, we can consider the rotated central charge $e^{-i\theta} Z_\gamma$, and consequently the rotated holomorphic symplectic form $e^{-i\theta}\omega^{2,0}$ by arbitrary angle $\theta \in \mathbb{R}/\mathbb{Z}$, which would yield $\mathbb{Z}$-affine coordinates $\{Re(e^{-i\theta} Z(\gamma_i))\}$ or $\{Im(e^{-i\theta} Z(\gamma_i))\}$.

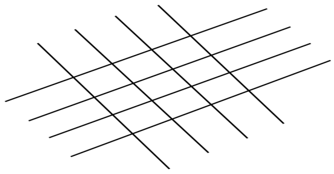

Figure 3.2: Affine structure and dual affine structure

We can also “rotate” the torsion-free, flat connection ∇ in the following way: Given $\theta \in \mathbb{R}/\mathbb{Z}$, and associate an endomorphism of tangent plane given by $e^{J\theta} = (\cos\theta)Id + (\sin\theta)J$, then define the rotated connection (c.f [ACD02]) by

$$\nabla^{\theta} X := e^{J\theta}\nabla(e^{-J\theta}X) \tag{3.1.60}$$

In particularly, for $\theta = \frac{\pi}{2}$, we get the so called **conjugate connection**

$$\nabla^{J} := \nabla^{\frac{\pi}{2}} = \nabla - J\nabla J \tag{3.1.61}$$

It is clear from the definition of rotated connection that if the rotated coordinates $Re(e^{-i\theta}Z(\gamma_i))$ are real affine with respect to ∇, i.e., $\nabla Re(e^{-i\theta}dZ(\gamma_i)) = 0$, then the coordinates $Re(Z(\gamma_i))$ are affine with respect to the rotated connection ∇^{θ}, i.e., $\nabla^{\theta}Re(dZ(\gamma_i)) = 0$. It is proved in [ACD02] that if $(\mathcal{B}, J, \nabla)$ is a special complex manifold, then $(\mathcal{B}, J, \nabla^{\theta})$ is a special complex manifold for any θ. And if $(\mathcal{B}, J, \nabla, \omega)$ is a special Kähler manifold, then $(\mathcal{B}, J, \nabla^{\theta}, \omega)$ is also a special Kähler manifold for any θ.

Hyperkähler structure

Good references for hyperk ahler manifolds with connection to integrable system includes [Hit91],[Hit+87b],[Fre99].[Hit97],[BM04], and our exposition is based on these references. Recall that for a manifold X endowed with a linear connection ∇, then the holonomoy of X at point $x \in X$ a representation

$$\rho : \pi_1(X, x) \longrightarrow GL(T_xX)$$

given by sending a loop γ to the parallel transport of a tangent vector along γ induced by ∇. The **holonomy group** $H_x(\nabla)$ (based at x) is defined to be the image of ρ.

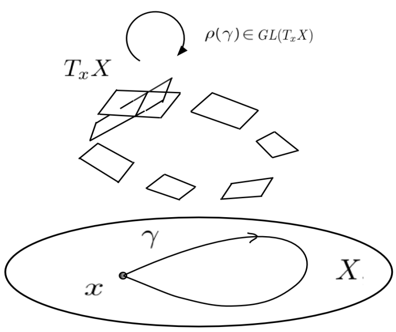

Figure 3.3: Holonomy group as representation of fundamental group on the tangent bundle

Definition 3.1.19. *A complex manifold M of dimension $2n$ is **hyperkähler** if its holonomy group can be reduced to $Sp(n)$, which is the group of $n \times n$ quaternionic unitary matrices.*

As $Sp(n) = U(2n) \cap Sp(2n, \mathbb{C})$, we see that hyperkähler manifolds are automatically Calabi-Yau, and admits a holomorphic symplectic form. Roughly speaking, hyperkähler geometry is more or less holomorphic symplectic geometry. We have the following equivalent definition of hyperkähler manifold:

Definition 3.1.20. *A hyperkähler manifold is a Riemannian manifold (M, g) endowed with three complex structures J_1, J_2, J_3 which satisfying: $J_1 J_2 = J_3$ (equivalent to quaternionic relations), and more over g is a Kähler metric for each J_i.*

Denote by ω_i the Kähler form corresponding t0 J_i, i.e., $\omega_i(\cdot, \cdot) = g(J_i \cdot, \cdot)$. Then we claim that $\omega^{2,0} := \omega_1 + i\,\omega_2$ is a **holomorphic symplectic form** with respect to the complex structure J_3. In the following, we denote by ω the Kähler form ω_3.

Proof of the claim: By definition, for $X, Y \in T_{\mathbb{C}}M$, we have the following:

$$\omega^{2,0}(X, Y) = \omega_1(X, Y) + i\,\omega_2(X, Y) = g(J_1 X, Y) + i\,g(J_2 X, Y)$$

Thus, we compute that

$$\omega^{2,0}(J_3 X, Y) = g(J_1 J_3 X, Y) + i\,g(J_2 J_3 X, Y)$$

$$= -g(J_2 X, Y) + i\,g(J_1 X, Y) = i(g(J_1 X, Y) + i\,g(J_2 X, Y)) = i\,\omega^{2,0}(J_3 X, Y)$$

Similarly, we have $\omega^{2,0}(X, J_3 Y) = i\,\omega^{2,0}(X, Y)$. Consequently, we see that $\omega^{2,0}$ is of type $(2, 0)$ with respect to the complex structure J_3. As ω_i s are closed, non-degenerate, it follows that $\omega^{2,0}$ is closed and non-degenerate, from which we infer that $\bar{\partial}\,\omega^{2,0} = 0$ by its type. $\square$

Besides, one can see easily that for any $\mathbf{u} = (a, b, c) \in S^2 \subset \mathbb{R}^3$, the endomorphism

$$I_{\mathbf{u}} := aI_1 + bI_2 + cI_3$$

of $T_{\mathbb{C}M}$ defines a complex structure, with the corresponding Kähler form

$$\omega_{\mathbf{u}} = a\omega_1 + b\omega_2 + c\omega_3$$

Consequently, we have a S^2 worth of complex structures on M.

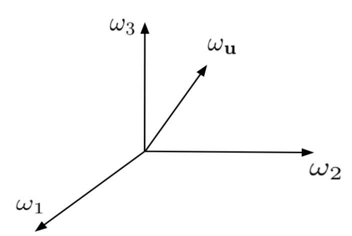

Figure 3.4: S^2 worth of complex structures on M

Identifying S^2 with the complex line $\mathbb{CP}^1$ (usually called the **twistor line**), we get the induced complex structure on S^2 with complex coordinate $\zeta \in \mathbb{CP}^1$. Then, it can be computed ([Lin17]) that the corresponding complex structure J_ζ and holomorphic symplectic form $\omega_\zeta^{2,0}$ are given respectively by

$$J_\zeta = \frac{i(-\zeta+\bar{\zeta})J_1 - (\zeta+\bar{\zeta})J_2 + (1-|\zeta|^2)J_3}{1+|\zeta|^2} \tag{3.1.62}$$

$$\omega_\zeta^{2,0} = -\frac{i}{2\zeta}\omega^{2,0} + \omega - \frac{i}{2}\zeta\,\bar{\omega}^{2,0} \tag{3.1.63}$$

It is clear that for $\zeta \neq 0, \infty$, J_ζ are all equivalent to J_1, while for $\zeta = 0, \infty$, the complex structure J_3 will be recovered. Besides, $\omega_\zeta^{2,0}$ is a holomorphic symplectic form in the complex structure J_ζ. The hyperkähler structures discussed above can be encoded by the so called **twistor space** defined as $Z = M \times S^2 \cong M \times \mathbb{CP}^1$. We have the following result due to Hitchin ([Hit91],[Hit+87b]).

Proposition 3.1.32. *Given an hyperkähler manifold (M, g), we have the following:*

1. *There exists a holomorphic fibration $p : Z \to \mathbb{CP}^1$ such that the fiber $M_\zeta = p^{-1}(\zeta)$ is complex in J_ζ.*

2. *There exits a holomorphic section $\omega^{2,0} \in \Gamma\left(\Omega^2_{Z/\mathbb{CP}^1} \otimes \mathscr{O}(2)\right)$ such that $\omega_\zeta^{2,0} = \omega^{2,0}|_{M_\zeta}$ is the holomorphic symplectic form on M_ζ.*

3. *There exists a anti-holomorphic map $\sigma : Z \to Z$ that covers the anti-involution $\zeta \mapsto -\frac{1}{\bar{\zeta}}$ on the twistor line $\mathbb{CP}^1$.*

4. *$\forall m \in M$, there exits a holomorphic section $s_m : \mathbb{CP}^1 \to Z$ with normal bundle $\mathscr{O}(1)^{\oplus 2r}$*

Conversely, we can construct a hyperkähler manifold from a twistor space Z satisfying the properties 1), 2), 3) *and* 4) *above.*

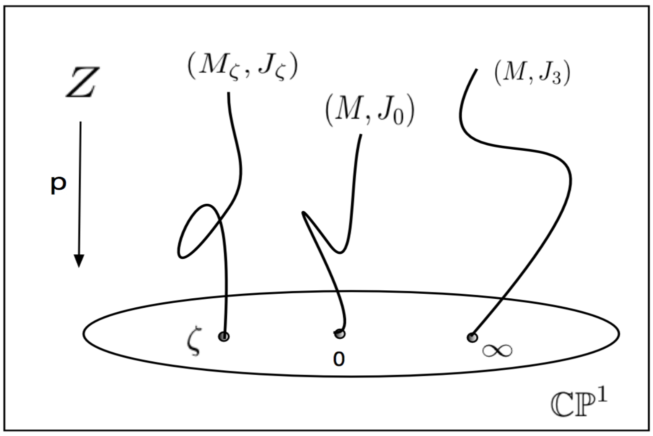

Figure 3.5: Hyperkähler manifold as twistor space

Hyperkähler rotation trick (c.f.,[Lin17])

We assume that the north and south pole of $\mathbb{CP}^1$ corresponds to the hyperkähler structure $(X, \omega, \omega^{2,0})$ and $(X, -\omega, \bar{\omega}^{2,0})$ respectively. Then the hyperkähler structures corresponding to the equator $\{e^{i\theta}\}$ are given by

$$\omega_\theta = -Im(e^{-i\theta}\omega^{2,0}) \tag{3.1.64}$$

$$\omega_\theta^{2,0} = \omega - iRe(e^{-i\theta}\omega^{2,0}) \tag{3.1.65}$$

Now, let us go back to the complex integrable system $\pi : (X^0, \omega^{2,0}) \longrightarrow \mathcal{B}^0$. We will show that X^0 is naturally a hyperkähler manifold. Thus there exists S^1_θ-family of complex structures J_θ, Kähler forms ω_θ, and holomorphic symplectic forms $\omega_\theta^{2,0}$ on X^0. And consequently, there would be S^1_θ-family of induced affine structures ∇^θ (see equation (3.1.60)) on $\mathcal{B}^0$, we denote the corresponding $\mathbb{Z}$-affine manifold by

$$\mathcal{B}^0_\theta := (\mathcal{B}^0, \nabla^\theta) \tag{3.1.66}$$

We now show that the induced affine coordinates coincide with the ones obtained before (see the remark 3.1.17). Indeed, take the negative of imaginary part of $\omega_\theta^{2,0}$, we get $Re(e^{-i\theta}\omega^{2,0})$. Denote the corresponding affine coordinates by I_γ, we then have

$$dI_\gamma = -\oint_\gamma Im\, \omega_\theta^{2,0} = \oint_\gamma Re\left(e^{-i\theta}\omega^{2,0}\right) = d\left(Re(e^{-i\theta}Z_\gamma)\right)$$

Consequently, we have that

$$I_\gamma = Re\left(e^{-i\theta}Z(\gamma)\right)$$

up to an addition of constant. Similarly, by rotating $\frac{\pi}{2}$ further, we would obtain the affine coordinate

$$I_\gamma = Im\left(e^{-i\theta}Z(\gamma)\right).$$

Proposition 3.1.33. *For the complex integrable system $\pi : (X^0, \omega^{2,0}) \longrightarrow \mathcal{B}^0$ defined as before, X^0 carries naturally a hyperkähler structure.*

Before giving the proof of this proposition, we give some preparations. As the base of an integrable system, $\mathcal{B}^0$ is special Kähler, thus it is endowed with a flat, torsion-free connection ∇ on $T_\mathbb{R}\mathcal{B}^0$, which is also called the **Gauss-Manin** connection. Given a local (Real) affine coordinate system $\{u^i\}$, it is characterized by

$$\nabla\frac{\partial}{\partial u^i} = 0$$

Recall (see definition 3.1.11) that the local system of charge lattice $\underline{\Gamma}$ is defined to be a locally constant shealf with stalk at $b \in \mathcal{B}^0$ given by $H_1(\pi^{-1}(b), \mathbb{Z})$. Thus, the dual local system $\underline{\Gamma}^\vee$, as a locally constant sheaf, can be identified with the direct image shealf $R^1\pi_*\mathbb{Z}$. By lemma 3.1.7, we have an isomorphism λ from $\underline{\Gamma}_b$ to the period lattice $\Lambda_b \subset T^*_b\mathcal{B}^0$ given as: for $\gamma \in \underline{\Gamma}_b$, we associated to it the following locally closed one form (see the formula (3.1.23))

$$\lambda_\gamma(v) = \oint_\gamma \iota_{\widetilde{v}}\,\omega^{2,0}$$

for $v \in T_b\mathcal{B}^0$. Consequently, we have the following isomorphism

$$T\mathcal{B}^0 \cong R^1\pi_*\mathbb{R} \otimes_{\mathbb{R}} \mathcal{C}^\infty_{\mathcal{B}^0} \tag{3.1.67}$$

The special Kähler structure is given by specifying the full rank lattice $T^{\mathbb{Z}}\mathcal{B}^0 \subset T\mathcal{B}^0$, which is locally spanned by $\{\frac{\partial}{\partial u^i}\}$. As a sheaf, it can be identified with $R^1\pi_*\mathbb{Z}$. By the isomorphism (3.1.67), we see that the Gauss-Manin connection induces a connection (also called the Gauss-Manin connection) on the local system $R^1\pi_*\mathbb{R}$. In fact, through this isomorphism, the Gauss-Manin connection on $T\mathcal{B}_0$ can be given explicitly as

$$\nabla = id \otimes d$$

for

$$\nabla : T\mathcal{B}^0 \to T\mathcal{B}^0 \otimes T^*\mathcal{B}^0$$

Consider the system $\pi : (X^0, \omega^{2,0}) \to \mathcal{B}^0$. First note that the space of vertical tangent vector fields on X^0, denoted by $Vert(TX^0)$, can be identified with $\pi^*T^*\mathcal{B}^0$ via the interior product of a vector field with symplectic form.

Also, recall that a connection on X^0 would give a splitting of the following **Atiyah sequence**

$$0 \longrightarrow VertTX^0 \longrightarrow TX^0 \longrightarrow \pi^*T\mathcal{B}^0 \longrightarrow 0$$

or equivalently, a splitting of the following sequence

$$0 \longrightarrow \pi^*T^*\mathcal{B}^0 \longrightarrow TX^0 \longrightarrow \pi^*T\mathcal{B}^0 \longrightarrow 0 \tag{3.1.68}$$

We will need the following lemma proved in [Bru].

Lemma 3.1.34. *Suppose that $\pi : (X^0, \omega^{2,0}) \to \mathcal{B}^0$ has a section, then the Gauss-Manin connection ∇ on the base $\mathcal{B}^0$ induces a connection on X^0.*

Proof. Consider the projective $pr : T^*\mathcal{B}^0 \to \mathcal{B}^0$, the Gauss-Manin connection ∇ on $\mathcal{B}^0$ also induces a connection on $T^*\mathcal{B}^0$, which provides a splitting of the following Atiyah sequence

$$0 \longrightarrow VertTT^*\mathcal{B}^0 \longrightarrow TT\mathcal{B}^0 \longrightarrow pr^*T\mathcal{B}^0 \longrightarrow 0$$

thus we have the isomorphism

$$TT\mathcal{B}^0 \cong VertTT^*\mathcal{B}^0 \oplus pr^*T\mathcal{B}^0$$

As π has a section $\sigma : \mathcal{B}^0 \to \mathcal{X}^0$, we have that $X^0 \cong T^*\mathcal{B}^0/\Lambda$. By composing pr with the section σ, we get the induced map $\rho : T^*\mathcal{B}^0 \to X^0$. It is easy to see that $\rho^*TX^0 \cong TT^*\mathcal{B}^0$, and $pr^* = \rho^* \circ \pi^*$. Consequently, we have the following

$$\rho^*TX^0 \cong \rho^*VertTX^0 \oplus (\rho^* \circ \pi^*)T\mathcal{B}^0 \cong (\rho^* \circ \pi^*)T^*\mathcal{B}^0 \oplus (\rho^* \circ \pi^*)T\mathcal{B}^0$$

where we have used the fact $Vert(TX^0) \cong \pi^*T^*\mathcal{B}^0$. Thus, we have the desired splitting of the sequence (3.1.68), which implies that there is a induced connection on X^0. $\square$

Proof. (of the proposition 3.1.33), the proof presented here is largely due to Daniel S. Freed [Fre99], though we borrowed some ideas from [BM04].

As the Gauss-Manini connection gives a splitting of the sequence (3.1.68), we have that

$$TX^0 \cong \pi^* T^* \mathcal{B}^0 \oplus \pi^* T\mathcal{B}^0 \tag{3.1.69}$$

We need to show that TX^0 is endowed with three complex structures I, J and K satisfying the quaternionic relations. Locally, the problem is reduced to the following linear algebra question.

Given a Hermitian vector space V endowed with the complex structure I, and the Hermitian metric $\langle \cdot, \cdot \rangle$ determines a real metric and symplectic form on $V_{\mathbb{R}}$ through

$$\langle u, v \rangle = g(u, v) + i\omega(u, v)$$

Consider $W = V \oplus \overline{V}$, we claim that it carries a constant hyperkähler structure. First, the complex structure I on V extends naturally to $V \oplus \overline{V}$., then define the following complex structure

$$J : V \otimes \overline{V} \to V \otimes \overline{V}$$

$$u \oplus \bar{v} \mapsto -v \oplus \bar{u}$$

Define $K = IJ$, then I, J, K satisfies the quaternionic relations.

Now, we apply the above construction to the (3.1.69) fiber wisely. we we get three almost complex structures I, J and K, with the corresponding Kähler form being ω_I, ω_J and ω_K. The integrability conditions for I, J and K is equivalent to the closedness of the corresponding Kähler forms.

Define the holomorphic symplectic form $\omega^{2,0} := \omega_J + i\omega_K$, which is holomorphic in I, then it can be shown that it coincides with the canonical form on $W = V \oplus \overline{V}$ defined as

$$\omega^{2,0}(v_1 \oplus l_1, v_2 \oplus l_2) = l_1(v_2) - l_2(v_1), \quad v_1, v_2 \in V, l_1, l_2 \in \overline{V} \cong V^*.$$

Consequently, the closedness of $\omega^{2,0}$ implies the closedness of ω_J and ω_K.

On X^0, we have the action angle coordinates $\{I^i, \theta_j\}$ (see definition 3.1.10). By using the symplectic basis of the charge lattice Γ, we can write I^i as $\{x^i, y_i\}$, and θ_i as $\{q_i, p^i\}$.

Then the symplectic (Kähler) form corresponding to I has the expression (see the formula (3.1.21))

$$\omega_I = \sum_i dI^i \wedge d\theta_i = \sum_i dx^i \wedge dq_i + \sum_i dy_i \wedge dp^i.$$

which is a closed two form.

□

3.1.6 Monodromy and the local model near discriminant

We have discussed in great details in previous sections the geometry of complex integrable system $\pi : (X^0, \omega^{2,0}) \longrightarrow \mathcal{B}^0$ away from the discriminant locus $\Delta = \mathcal{B}\backslash\mathcal{B}^0$. Near the discriminant locus, the Lagrangian fibration becomes degenerate, and some cycles, called the **vanishing cycles** shrink to zero. As we will see that this corresponds to the singularity of the $\mathbb{Z}$-affine structure.

Definition 3.1.21. *An $\mathbb{Z}$-affine manifold with singularities is a triple $(\mathcal{B}, \Delta, \mathcal{A})$ where $(\mathcal{B}\backslash\Delta, \mathcal{A})$ is an $\mathbb{Z}$-affine structure, and $\Delta \subset \mathcal{B}$ is a set which is locally a finite union of locally closed sub-manifolds of (real) codimension at least two.*

Given a $\mathbb{Z}$-affine structure with singularities on $\mathcal{B}$, it defines a special Kähler structure ∇ on $\mathcal{B}^0$ together with a local system of full rank lattice Λ in $T^*\mathcal{B}^0$, which by the isomorphism (3.1.15), is equivalent to the local system of charge lattices $\underline{\Gamma}$, with fiber being

$$\underline{\Gamma}_b := H_1(\pi^{-1}(b), \mathbb{Z})$$

By choosing a local basis $\{\gamma_i\}$ of $\underline{\Gamma}$ near b, there is a set of induced affine coordinates $\{u^i\}$, which generates a local system of full rank lattice $T^{\mathbb{Z}}\mathcal{B}^0$ inside $T\mathcal{B}^0$, which is locally spanned by $\left\{\frac{\partial}{\partial u^i}\right\}$.

In the following, we follow mainly the exposition given in [KS06],[KS11]. Recall that a $\mathbb{Z}$-affine structure gives rise to a representation of the fundamental group $\pi_1(\mathcal{B}^0, b)$ through the following

Monodromy representation

For the local system $\underline{\Gamma} \to \mathcal{B}^0$, the monodromy representation of $\pi_1(\mathcal{B}^0, b)$ into $GL(H_1(\pi^{-1}(b), \mathbb{Z})) \ltimes \mathbb{R}^{2n} \cong GL(2n, \mathbb{R}) \ltimes \mathbb{R}^{2n}$ is given through the map:

$$\begin{aligned} \widetilde{\rho} : \pi_1(\mathcal{B}^0, b) &\longrightarrow GL(2n, \mathbb{R}) \ltimes \mathbb{R}^{2n} \\ \gamma &\longmapsto \widetilde{\gamma}(1) \end{aligned} \tag{3.1.70}$$

where $\widetilde{\gamma}$ is the lifting of the loop γ based at b, i.e., $\gamma(0) = \gamma(1) = b$.

Definition 3.1.22. *The **monodromy group** based at b, denoted by $M_b(\widetilde{\rho})$, is defined to be the image of the above representation $\widetilde{\rho}$.*

Proposition 3.1.35. *The monodromy above coincides with the holonomy of the Gauss-Manin connection ∇, i.e., $M_b(\widetilde{\rho}) = H_b(\nabla)$.*

Proof. The monodromy representation $\widetilde{\rho}$ determines a local system $\underline{\Gamma}$ endowed with a flat connection ∇, and the holonomy of this connection is the monodromy by construction. □

Proposition 3.1.36. *The monodromy representation $\widetilde{\rho}$ is equivalent to the following data*
a) A local system over $\mathcal{B}^0$,
b) A cohomology class $\delta \in H^1(\mathcal{B}^0, T^{\mathbb{Z}}\mathcal{B}^0 \otimes_{\mathbb{Z}} \mathbb{R})$

Proof. $\widetilde{\rho}$ gives us a local system endowed with a flat connection ∇, which by the above proposition, is equivalent to the holonomy of the connection ∇. Denote by $T^{\nabla}\mathcal{B}^0$ the sheaf of flat sections with respect to ∇, they span the full rank lattice $T^{\mathbb{Z}}\mathcal{B}^0$ inside $T\mathcal{B}^0$, thus gives us a cohomology class

$$\delta \in H^1(\mathcal{B}^0, T^{\mathbb{Z}}\mathcal{B}^0 \otimes_{\mathbb{Z}} \mathbb{R}) \cong H^1(\mathcal{B}^0, T^{\nabla}\mathcal{B}^0)$$

where $T^{\mathbb{Z}}\mathcal{B}^0 \otimes_{\mathbb{Z}} \mathbb{R}$ is the abelian variety associated to the lattice $T^{\mathbb{Z}}\mathcal{B}^0$. Also note that the class δ represents a torsor over the abelian variety $T^{\mathbb{Z}}\mathcal{B}^0 \otimes \mathbb{R}$ which is in agree with our previous local construction of complex integrable system. □

Remark 3.1.18. *Since we can rotate the $\mathbb{Z}$-affine structure as in (3.1.60) or in (3.1.66), which is equivalent to rotating the tori $T^{\mathbb{Z}}\mathcal{B}^0 \otimes \mathbb{R}$, which can be encoded by the cohomology class in $H^1(\mathcal{B}^0, T^{\mathbb{Z}}\mathcal{B}^0 \otimes i\mathbb{R}/\mathbb{Z})$. (c.f. [KS11]).*

Remark 3.1.19. *Since the connection ∇ is flat, and the lattice $T^{\mathbb{Z}}\mathcal{B}^0$ is locally constant, the deRham representation of the cohomology class δ, still denoted by δ should be the identity section $id \in \Omega^1(\mathcal{B}^0, T\mathcal{B}^0)$, i.e., $\delta(v) = v$ for every $v \in T\mathcal{B}^0$, more explicitly, we have*

$$\delta = id = \sum_i \frac{\partial}{\partial u^i} \otimes du^i$$

which is obviously closed by the torsion-freeness of ∇ (see the lemma 3.1.29).

Geometry near the discriminant

We already know that for the full (polarized) complex integrable system $\pi : X \to \mathcal{B}$, outside the discriminant locus Δ, i.e., singularities of $\mathbb{Z}$-affine structure, we get a Lagrangian fibration $\pi : X^0 \to \mathcal{B}^0$, with the fibers being the algebraic torus $\mathbb{T}^{2n}$. Now, as we follow certain paths to the discriminant Δ, the tori would degenerate (some cycles on it shrink to zero), causing certain fibres to become singular over Δ. Roughly, we will get pinched torus over Δ (see Figure 3.1 at the beginning of chapter 3).

The geometry of the singular fibration over Δ concerning us here is encoded in its monodromy near the singular fibers. To this end, we need to make some assumptions on the nature of singularities. First we show that locally, the situation can be reduced to the real two dimension case. We know from (3.1.4) that the central charge Z defines a local embedding of $\mathcal{B}^0$ into $\underline{\Gamma}^{\vee} \otimes_{\mathbb{Z}} \mathbb{C}$, with the corresponding map on the tangent space level given by

$$dZ_b : T_b\mathcal{B}^0 \longrightarrow \underline{\Gamma}^{\vee} \otimes_{\mathbb{Z}} \mathbb{C} \tag{3.1.71}$$

By proposition 3.1.23, this induces an isomorphism

$$T_b\mathcal{B}^0 \cong \underline{\Gamma}^{\vee} \otimes_{\mathbb{Z}} \mathbb{C}$$

At $b \in \Delta$, we identify $\underline{\Gamma}_b$ with the nearby non-degenerate ones (this is possible since $\mathcal{B}^0$ is dense in $\mathcal{B}$). Then given a **special direction** specified by the co-vector $\mu^* \in \underline{\Gamma}_b^\vee$ near b which is invariant under the monodromy around b, we have $\mu \in \underline{\Gamma}_b$ with $(\mu^*, \mu) = 0$, where $(\cdot, \cdot)$ denotes the canonical pairing between $\Gamma^\vee$ and Γ. This means that we have special direction specified by μ^* together with a hyperplane (specified by μ) that is orthogonal to μ^*. Thus we get a hypersurface that is foliated by lines in special directions.

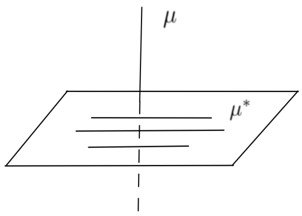

Figure 3.6: foliation of hyperplane in the special direction

Suppose we have two hypersurfaces that intersect transversely, by what had been said above, we should have $\mu_i^* \in \underline{\Gamma}_b^\vee$, $i = 1, 2$ and the corresponding orthogonal elements $\mu_i \in \underline{\Gamma}_b$, $i = 1, 2$, such that $(\mu_1^*, \mu_1) = (\mu_2^*, \mu_2) = 0$. Under the assumption that $(\mu_1^*, \mu_2) \neq 0$ and $(\mu_2^*, \mu_1) \neq 0$, we see that there is a splitting of the lattice $\underline{\Gamma}^\vee \otimes_{\mathbb{Z}} \mathbb{R}$ as

$$\underline{\Gamma}^\vee \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^2 \oplus \mathbb{R}^{2n-2} \tag{3.1.72}$$

where $\mathbb{R}^2$ is spanned by μ_1^* and μ_2^*.

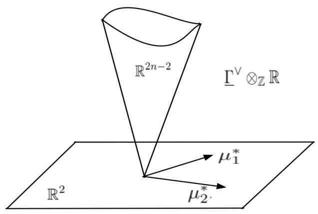

Figure 3.7: Reducing the the two dimension case

Above is the local picture for WCS to be considered later, we see that we can use the above procedure to reduce to the real dimension two case. Consequently, we will make the assumption that the **singular affine structure** in our situation corresponds to a two dimensional singularity times $\mathbb{R}^{2n-2}$

Focus-focus singularity

We assume that the singularity in dimension two is the simplest possible case, namely the so called A^1 singularity, or the focus-focus singularity, i.e., the singular fiber being the *pinched torus*. It can be modelled by the following *Ooguri-Vafa space* (see for example [Cha10]). The base $\mathcal{B}$ is the disc $\{u \in \mathbb{C} : |u| < \Lambda\}$ with a single singular point at the origin $\Delta = \{0\}$, thus, $\mathcal{B}^0 = \mathcal{B} \backslash \Delta$ is the punctured disc. The local system of charge lattices $\underline{\Gamma}$ is or rank two, and is spanned locally by sections $\{\gamma_m, \gamma_e\}$, with $\langle \gamma_e, \gamma_m \rangle = 1$. Then the monodromy around the origin is given by

$$\gamma_e(u) \longmapsto \gamma_e(u) \qquad \gamma_m(u) \longmapsto \gamma_m(u) + \gamma_e(u). \tag{3.1.73}$$

Equivalently, we say that the monodromy around the origin has matrix representation

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{3.1.74}$$

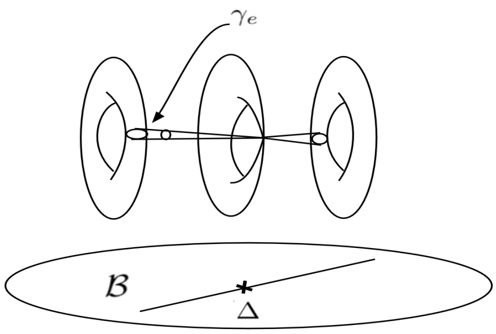

Figure 3.8: Ooguri-Vafa space

Proposition 3.1.37. *The central charge function $Z : \mathcal{B} \to \underline{\Gamma}^\vee \otimes_{\mathbb{Z}} \mathbb{C}$ for the Ooguri-Vafa space can be realized as*

$$Z_u(\gamma_e) = u \qquad Z_u(\gamma_m) = \frac{1}{2\pi i}\left(u \log \frac{u}{\Lambda} - u\right). \tag{3.1.75}$$

Proof. Since by definition, we see that Z is a homomorphism from $\underline{\Gamma}$ to $\mathbb{C}$, and from the monodromy (3.1.73) above, we conclude that $Z(\gamma_e)$ should stay the same after looping around the origin, while $Z(\gamma_m)$ will be shifted as

$$Z(\gamma_m) \longrightarrow Z(\gamma_m) + Z(\gamma_e)$$

By proposition 3.1.26, we know that Z is holomorphic in u, thus we should find the simplest holomorphic function in u that exhibits this monodromy property. We claim that the form given in 3.1.75 satisfies our purpose.

As $u \mapsto e^{2\pi i}u$, one easily sees that

$$Z(\gamma_e) = u \mapsto e^{2\pi i}u = u = Z(\gamma_e)$$

while for $Z(\gamma_m)$ we have

$$Z(\gamma_m) = \frac{1}{2\pi i}\left(u \log \frac{u}{\Lambda} - u\right) \mapsto \frac{1}{2\pi i}\left(e^{2\pi i}u \log \frac{e^{2\pi i}u}{\Lambda} - e^{2\pi i}u\right)$$

$$= \frac{1}{2\pi i}\left(u(2\pi i + \log(\frac{u}{\Lambda})) - u\right) = u + \frac{1}{2\pi i}\left(u \log \frac{u}{\Lambda} - u\right)$$
$$= Z(\gamma_e) + Z(\gamma_m).$$

□

Denote by $x = Re(Z(\gamma_m))$, $y = Re(Z(\gamma_e))$ the corresponding real affine coordinates, we see that they transform in the same way as in (3.1.72). Thus, the singular affine structure also has the monodromy (3.1.73).

More general type of singularities

More generally, we expect the following situation around the discriminant locus (see for example [Nei14b]). We won't discuss the more complicated type of singularities, since at one hand, for the most practical applications, the A^1-case is already sufficient, and at another hand, the more complicated situation is still being less understood.

We assume there there is a smooth component $\Delta_0 \subset \Delta$, such that for $b \in \Delta_0$, there is a primitive $\gamma_0 \in \underline{\Gamma}_b$, i.e, $\langle \gamma_0, \gamma \rangle = 1$, for some γ. Suppose $Z_b(\gamma_0) \to 0$ as b approaches Δ_0, i.e., γ_0 corresponds to a vanishing cycle. Then the monodromy of $\underline{\Gamma}$ around Δ_0 is given by the **Picard-Lefshetz type transformation**

$$\gamma \longmapsto \gamma + \langle \gamma, \gamma_0 \rangle \gamma_0. \tag{3.1.76}$$

We see that Ooguri-Vafa space is a special case in which γ_e is the vanishing cycle near the origin such that it corresponds to the monodromy invariant direction. As $\langle \gamma_e, \gamma_m \rangle = 1$, the above formula implies the monodromy given in the formula (3.1.73).

Local model near the discriminant

With these preparation, we can state the local model near the discriminant, which is essentially the A^1-**singularity assumption** given in the section 4.5 of [KS14]. We assume that $\Delta = \mathcal{B} \backslash \mathcal{B}^0$ is an analytic divisor, and there exists an analytic divisor $\Delta^1 \subset \Delta$ such that $dim \Delta^1 \leq dim \mathcal{B}^0 - 2$, and the complement $\Delta^0 := \Delta \backslash \Delta^1$ is smooth, then we have the following local model near Δ^0:

1. There exist local coordinates $\{z_1, \cdots, z_n\}$ near Δ^0 such that $\Delta^0 = \{z_1 = 0\}$

2. The central charge map $Z : \mathcal{B}^0 \longrightarrow \underline{\Gamma}^\vee \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}^{2n}$ is a multi-valued map given in coordinates by

$$(z_1, \cdots, z_n) \longmapsto (z_1, \cdots, z_n, \partial_1 F_0, \cdots, \partial_n F_0) \tag{3.1.77}$$

   where $\partial_i := \frac{\partial}{\partial z_i}$, and F_0 is given by

$$F_0 = \frac{1}{2\pi i} \frac{z_1^2}{2} \log z_1 + G(z_1, \cdots, z_n) \tag{3.1.78}$$

   where G is a holomorphic function. The function F_0 is called the **prepotential**, and satisfies the **positivity condition**

$$i \langle dZ, \overline{dZ} \rangle > 0 \tag{3.1.79}$$

3. The monodromy of the local system $\underline{\Gamma}$ about Δ_0 is given by the **Picard-Lefshetz type formula**

$$\gamma \longmapsto \gamma + \langle \gamma, \gamma_0 \rangle \gamma_0 \tag{3.1.80}$$

   where γ_0 is the vector such that $\langle \gamma_0, \cdot \rangle \in \underline{\Gamma}^\vee$ is a primitive covector.

Remark 3.1.20. *The Ooguri-Vafa space is easily seen to satisfy the A^1-singularity assumption above. Indeed, by integrating the second formula in (3.1.74), we get a potential of the form given in (3.1.77).*

3.1.7 Semi-polarized case

Previously, we have discussed in great details the geometry on polarized complex integrable system, now we can generalize the discussion to the semi-polarized case.

Definition 3.1.23. *A* ***semipolarized complex integrable system*** *is given by a holomorphic fibration of a complex analytic symplectic manifold $\pi : X^0 \to \mathcal{B}^0$ with fibers being the Lagrangian submanifolds which are semiabelian varieties (commutative group varieties which are extension of abelian varieties by torus) with polarized abelian quotients.*

Thus, we have on the local system of charge lattice $\underline{\Gamma} \to \mathcal{B}^0$ an integer valued skew-symmetric bilinear form (see the formula (3.1.30))

$$\langle \cdot, \cdot \rangle : \Lambda^2 \underline{\Gamma} \longrightarrow \underline{\mathbb{Z}}_{\mathcal{B}^0}$$

with the kernel $\underline{\Gamma}_0$. Denote by $\underline{\Gamma}^{symp}$ the symplectic quotient, then we have the following exact sequence of local systems

$$0 \longrightarrow \underline{\Gamma}_0 \longrightarrow \underline{\Gamma} \longrightarrow \underline{\Gamma}^{symp} \longrightarrow 0 \tag{3.1.81}$$

By (3.1.71), we have local embedding $Z : \mathcal{B}^0 \to \underline{\Gamma}^\vee \otimes_{\mathbb{Z}} \mathbb{C}$. Composing it with the natural map

$$\underline{\Gamma}^\vee \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow \underline{\Gamma}_0^\vee \otimes_{\mathbb{Z}} \mathbb{C} \tag{3.1.82}$$

gives us the following surjection

$$Z : \mathcal{B}^0 \longrightarrow \underline{\Gamma}_0^\vee \otimes_{\mathbb{Z}} \mathbb{C} \tag{3.1.83}$$

with fibers being complex Lagrangian submanifolds corresponding to the symplectic leaves in the Poisson manifold $\Gamma^\vee \otimes_{\mathbb{Z}} \mathbb{C}$, i.e., they are affine symplectic spaces parallel to the fibers of the map (3.1.22).

Remark 3.1.21. *It can be shown (c.f lemma 4.4.1 in [KS14]) that when the semipolarized complex integrable system is algebraic, the monodromy group G of $\underline{\Gamma}_0$ is finite. Consequently, in this case, we actually have the local embedding*

$$p_{\mathcal{B}_0} : \mathcal{B}^0 \hookrightarrow \left(\underline{\Gamma}_0^\vee \otimes_{\mathbb{Z}} \mathbb{C}\right)/G \tag{3.1.84}$$

Fix a non-singular point $Z_0 \in \left(\underline{\Gamma}_0^\vee \otimes_{\mathbb{Z}} \mathbb{C}\right)/G$, we let

$$M = \mathcal{B}^0_{Z_0} := p_{\mathcal{B}_0}^{-1}(Z_0) \tag{3.1.85}$$

then the local system $\underline{\Gamma}_0$, when restricted to M, becomes trivial. Besides, M is embedded as a symplectic leaf parallel to Γ^{symp} inside the Poisson manifold $\underline{\Gamma}^\vee \otimes_{\mathbb{Z}} \mathbb{C}$.

Remark 3.1.22. *Identify $\Gamma_0^\vee \otimes_{\mathbb{Z}} \mathbb{C}$ with $\mathbb{C}^m$, then $\Gamma^\vee \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}^{2n+m}$, while the symplectic quotient Γ^{symp} being identified with $\mathbb{C}^{2n}$, then the local model for the map is given by $\mathbb{C}^{2n+m} \to \mathbb{C}^m$ with fiber being $\mathbb{C}^{2n}$.*

The local model near the discriminant for the polarized case can be easily generalized to the semipolarized case.

A^1singularity assumption in the semipolarized case

We assume that $\Delta = \mathcal{B}\backslash\mathcal{B}^0$ is an analytic divisor, and there exists an analytic divisor $\Delta^1 \subset \Delta$ such that $dim\Delta^1 \leq dim\mathcal{B}^0 - 2$, and the complement $\Delta^0 := \Delta\backslash\Delta^1$ is smooth, then we have the following local model near Δ^0:

1. There exist local coordinates $\{z_1, \cdots, z_n, w_1, \cdots, w_m\}$ near Δ^0 such that

$$\Delta^0 = \{z_1 = 0\}$$

2. The central charge map $Z : \mathcal{B}^0 \longrightarrow \underline{\Gamma}^\vee \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}^{2n}$ is a multi-valued map given in coordinates by

$$(z_1, \cdots, z_n) \longmapsto (z_1, \cdots, z_n, \partial_1 F_0, \cdots, \partial_n F_0, w_1, \cdots, w_m) \tag{3.1.86}$$

where $\partial_i := \frac{\partial}{\partial z_i}$, and F_0 is given by

$$F_0 = \frac{1}{2\pi i}\frac{z_1^2}{2}\log z_1 + G(z_1, \cdots, z_n, w_1, \cdots, w_m) \tag{3.1.87}$$

where G is a holomorphic function, and the function F_0 is called the **prepotential**, and satisfies the **positivity condition**

$$i\langle dZ, \overline{dZ}\rangle > 0 \tag{3.1.88}$$

which is satisfied for the restriction of dZ to symplectic leaves

$$S_{c_1,\cdots,c_m} := \{(z_1, \cdots, z_n, w_1, \cdots, w_m) : w_i = c_i\}$$

Besides, the Poisson structure on $\mathbb{C}^{2n+m}$ is specified by the Poisson bivector field

$$\sum_{1\leq i\leq n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_{i+n}}$$

3. The monodromy of the local system $\underline{\Gamma}$ about Δ_0 is given by the **Picard-Lefshetz type formula**

$$\gamma \longmapsto \gamma + \langle\gamma, \gamma_0\rangle\gamma_0 \tag{3.1.89}$$

where γ_0 is the vector such that $\langle\gamma_0, \cdot\rangle \in \underline{\Gamma}^\vee$ is a primitive covector.

3.2 Attractor flows

In this section, we will introduce certain flows, called the attractor flows on the base $\mathcal{B}^0$ of the polarized complex integrable system $\pi : (X^0, \omega^{2,0}) \longrightarrow \mathcal{B}^0$. These flows first appeared in the study of $N = 2, d = 4$ supersymmetric gravity in superstring theory (see for examples [Den00],[Moo98]). The interested readers are encouraged to see the appendix B more information about the attractor flows on the physics side.

We will first introduce the concept of attractor flows from geometric point of view, which is more pertain to its physics origin. We will then introduce the axiomatic treatment by Kontsevich and Soibelman in [KS14]. After this, we will see how to use these flows to produce WCS in section 3.2.3.

3.2.1 Introducing the attractor flows

Recall that the base $\mathcal{B}^0$ of the complex integrable system is a special Käher manifold (see proposition 3.1.28). Choosing a symplectic basis (see (3.1.34)) $\{\alpha^i, \beta_j\}$, $1 \le i, j \le n$ for the charge lattice $\underline{\Gamma}$ near a point $b \in \mathcal{B}^0$, and with the corresponding dual basis $\{\alpha_i, \beta^j\}$, $1 \le i, j \le n$, then the central charge function can be written as (see (3.1.35))

$$Z = \sum_{i=1}^{n} a^i \alpha_i + \sum_{i=1}^{n} a_{D,i} \beta^i \tag{3.2.1}$$

where $a^i, a_{D,i}$ are holomorphic functions on $\mathcal{B}^0$. And these functions are related by (see (3.1.38))

$$a_{D,i} = \frac{\partial \mathcal{F}}{\partial a^i}$$

where $\mathcal{F}$ is the prepotential, and $a^i, a_{D,i}$ s are the special coordinates. It is clear that $Z(\alpha^i) = a^i$ and $Z(\beta_i) = a_{D,i}$.

We can take a^i, $1 \le i \le n$ the holomorphic coordinates for $\mathcal{B}^0$, then the Kähler metric on $\mathcal{B}^0$ is given as (see (3.1.42))

$$g_{\mathcal{B}^0} = \sum_{i,j} Im(\tau_{ij})\, da^i d\bar{a}^j \tag{3.2.2}$$

where $\tau = \tau_{ij}$ is the period matrix defined by $\tau_{ij} = \frac{\partial a_{D,i}}{\partial a^j} = \frac{\partial^2 \mathcal{F}}{\partial a^i \partial a^j}$, with corresponding Kähler potential given by

$$K = Im\left(\sum_i a_{D,i} \bar{a}^i\right) \tag{3.2.3}$$

By remark 3.1.17, the $\mathbb{Z}$- affine coordinates of the base $\mathcal{B}^0$ can be chosen to be

$$y^i = Im(a^i) \qquad y_j = Im(a_{D,j}) \tag{3.2.4}$$

Given $\gamma \in \underline{\Gamma}$ near $b_0 \in \mathcal{B}^0$, consider the function

$$F_\gamma(\mathbf{u}) := Re(Z_{\mathbf{u}}(\gamma))$$

for some complex coordinates $\mathbf{u}$ of $\mathcal{B}^0$. Then we have the following definition:

Definition 3.2.1. *The* ***attractor flow*** *associated to* $(b_0, \gamma) \in tot(\underline{\Gamma})$ *is given as the gradient flow of the function* $F_\gamma(\mathbf{u})$*, namely*

$$\dot{\mathbf{u}} + \nabla F_\gamma(\mathbf{u}) = 0 \tag{3.2.5}$$

where $\dot{\mathbf{u}}$ *denotes the derivative with respect to the "time" parameter* t*, and the gradient is taken with respect to the Kähler metric (3.2.2), i.e., its* i*-th component* ∇_i *is given by* $\sum_j g^{i\bar{j}}\,\overline{\partial}_j F_\gamma(\mathbf{u})$*. Note that by writing the Kähler metric tensor* $g_{i\bar{j}} = Im(\tau_{ij})$*, then its inverse is denoted by* $g^{i\bar{j}}$*.*

Remark 3.2.1. *The attractor flow in its original form ([Den00]), is written as the gradient flow associated to the function* $|Z_\gamma(\mathbf{u})|$*. Some authors ([Lin14]) also considered the function* $|Z_\gamma(\mathbf{u})|^2$ *to define the attractor flows. However, there is no essential difference among these choices, as we will show that they give the same gradient lines.*

The following is a basic result in complex analysis, and it generalizes easily to holomorphic functions in several variables.

Lemma 3.2.1. *For a holomorphic function* $f = u + iv$*, we have that* $\nabla u \cdot \nabla v = 0$*.*

From proposition 3.1.26, we know that $Z(\gamma)$ is a holomorphic function on $\mathcal{B}^0$. Applying the above lemma to the central charge function, we infer that

$$\nabla Re(Z(\gamma)) \cdot \nabla Im(Z(\gamma)) = 0$$

Consequently, we have the following

Proposition 3.2.2. *Along the attractor flow defined by (3.2.5),* $Im(Z(\gamma))$ *stays constant.*

From the above proposition, we see that the flow lines are straight lines in the $\mathbb{Z}$-affine coordinates given by (3.2.4). More precisely, suppose

$$\gamma = \sum_i g^i \beta_i + q_i \alpha^i$$

for $g^i, q_i \in \mathbb{Z}$ in symplectic basis, then we have that

$$Z(\gamma) = \sum_i g^i a_{D,i} + q_i a^i$$

Consequently, the attractor flow, in $\mathbb{Z}$-affine coordinates, lies in the hyperplane in $\mathcal{B}^0$ given by

$$\sum_i g^i\, y_i + q_i\, y^i = C, \qquad \text{where } C \text{ is a constant} \tag{3.2.6}$$

Thus the solution to the attractor flow equation (3.2.5) can be written as

$$\sum_i g^i\, y_i(t) + q_i\, y^i(t) = C, \quad \text{where } t \text{ denotes the "time" parameter.} \tag{3.2.7}$$

Remark 3.2.2. *The above discussion also implies that the flow lines of the attractor flow, though defined by using the Kähler metric on* $\mathcal{B}^0$*, turned out to be independent of the choice of Kähler metric, and is thus completely determined by the central charge function.*

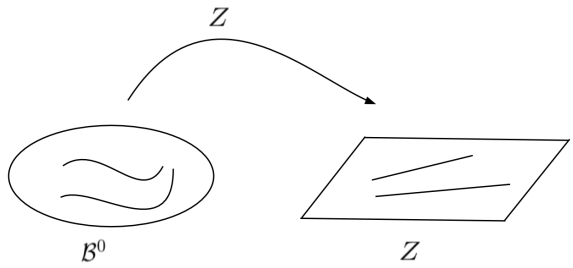

Figure 3.9: Attractor flows on $\mathcal{B}^0$ and Z-complex plane

As we have noted before (see (3.1.66)) that our base $\mathcal{B}^0$ is endowed with a S^1-family of $\mathbb{Z}$-affine structures. Given $\theta \in S^1$, the corresponding $\mathbb{Z}$-affine manifold is denoted by $\mathcal{B}^0_\theta$, with the adapted affine coordinates given by $Im\left(e^{-i\theta} Z(\gamma_i)\right)$. Then we see easily that in the affine structure corresponding to $\theta = ArgZ_{b_0}(\gamma)$, the flow line (3.2.7) is an affine line on $\mathcal{B}^0_\theta$ given by $Im\left(Z_{b(t)}(\gamma)\right) = Im\left(Z_{b_0}(\gamma)\right)$, which is equivalent to

$$Im\left(e^{-i\theta} Z_{b(t)}(\gamma)\right) = 0 \tag{3.2.8}$$

Thus in the $\mathbb{Z}$-affine coordinates on $\mathcal{B}^0_\theta$, the flow line (3.2.7) can be rewritten as

$$\sum_i g^i\, y_i(t) + q_i\, y^i(t) = 0 \tag{3.2.9}$$

Proposition 3.2.3. *The phase $ArgZ(\gamma)$ of is constant along the attractor flow.*

Proof. This is a direct consequence of the equation (3.2.8). □

Proposition 3.2.4. *The central function $Z(\gamma)$ has no critical points.*

Proof. Suppose to the contrary that b is a critical point of $Z(\gamma)$, then $dZ_b(\gamma) = 0$, then by (3.1.4), we see that for $v \in T_b\mathcal{B}^0$, we have that

$$dZ_b(\gamma)(v) = \oint_\gamma \iota_{\tilde{v}}\, \omega^{2,0} = 0$$

which is contradicts to the nondegeneracy of the holomorphic symplectic form $\omega^{2,0}$. Consequently, the central function is free from critical points. □

Proposition 3.2.5. *Away from the discriminant locus Δ, the function $F_\gamma(\mathbf{u})$ is decreasing along the flow line.*

Proof. By using the special kähler metric (see (3.2.2) above)

$$ds^2 = \sum_{i,j} Im(\tau_{ij})\, da^i d\bar{a}^j = \sum_{ij} g_{i\bar{j}}\, da^i d\bar{a}^j$$

the attractor flow equation (3.2.5) becomes

$$\dot{\mathbf{u}}^i = -\sum_j g^{i\bar{j}}\, \bar{\partial}_j F_\gamma(\mathbf{u})$$

from this we see that

$$\frac{d}{dt} F_\gamma(\mathbf{u}) = \sum_i \partial_i F_\gamma(\mathbf{u})\, \dot{\mathbf{u}}^i = -\sum_{ij} g^{i\bar{j}}\, \partial_i\, F_\gamma(\mathbf{u})\, \bar{\partial}_j\, F_\gamma(\mathbf{u}) < 0$$

□

Definition 3.2.2. *From the above proposition, the attractor flow would converge to the local minimum of the function $F_\gamma(\mathbf{u})$. These terminate points of the flow line will be called the* ***attractor points****.*

Remark 3.2.3. *By the equation (3.2.8), we see that $Im\left(e^{-i\theta}Z_b(\gamma)\right)$ vanishes identically along the flow line, consequently we must have that*

$$|Z_b(\gamma)| = Re\left(e^{-i\theta}Z_b(\gamma)\right)$$

This explains the rational of using $|Z(\gamma)|$ to define the attractor flows. Besides, this suggest that the attractor points would correspond to the minimum of the function $|Z(\gamma)|$, which has the meaning as "mass" for BPS particles in supersymmetric field theory (see the appendix A for further physics motivation). Thus, we also call the function $Re\left(e^{-i\theta}Z_b(\gamma)\right)$ as the ***mass function****.*

Remark 3.2.4. *Motivated by the above discussion, we can consider the attractor flow associated to the function $F_\gamma^\theta = Re\left(e^{-i\theta}Z_b(\gamma)\right)$, where $\theta = Arg(Z_{b_0}(\gamma))$, for $(b_0, \gamma) \in tot\,\underline{\Gamma}$ at the first place. This would be the case when we discuss its connection to the wall-crossing structure in section 3.2.3. In this "rotated case", the attractor line is seen to be given directly by the equation (3.2.8).*

The possible minimum of the mass function $|Z_b(\gamma)|$ are the zeros of the central charge. In particular, we see that near the discriminant locus Δ, certain homology cyce $\gamma \in \underline{\Gamma}_b \cong H_1(\pi^{-1}(b), \mathbb{Z})$ shrinks to the zero, which causes

$$Z_b(\gamma) \to 0 \quad \text{as } b \to \Delta$$

Thus we expect the attractor points lie in the discriminant Δ. So, we need to study how attractor flows behave near the discriminant.

Recall that from the A^1-singularity assumption in section 3.1.6, we see that the discriminant locus is given as $\Delta = \{z_1 = 0\}$, where $\{z_1, \cdots, z_n\}$ are local coordinates near Δ. Reducing to the two real dimensional case, and denote by γ_0 the vanishing cycle, and by γ_1 the remaining basis element, we see from (3.1.78) that near the singularity of $\mathbb{Z}$-affine structure $\{u = 0\}$, the central charge is of the following form

$$Z_u(\gamma_0) = u, \quad Z_u(\gamma_1) = \frac{1}{2\pi i}\left(u \log u + \frac{u}{2}\right) + \partial_u G(u)$$

Clearly, the cycle γ_0, under the identification $T\mathcal{B}^0 \cong \underline{\Gamma}$, corresponds to the **invariant direction** under the monodromy given by the Picard-Lefshetz formula (3.1.80).

Proposition 3.2.6. *Under the A^1-singularity assumption, the attractor flow line corresponding to the vanishing cycle γ_0 near the singularity of the affine structure, namely $\{u = 0\}$, terminates at this singularity which is thus an attractor point of the attractor flow, while for the charges γ other that the vanishing cycle, the attractor flow line can avoid this singular point.*

Proof. From the local expression of the central charge near singularity given above, we see that when we approach to the origin, $Z_u(\gamma_0) = u$ goes to zero, which is the minimum of $Re\,(Z_u(\gamma_0))$, thus by proposition 3.2.5 and the definition 3.2.2, we see that the flow line in this case would terminate at the attractor point, namely the origin.

On the other hand, if the attractor flow associated to $Re\,(Z_u(\gamma_1))$ terminates at the origin, then by proposition 3.2.3, the phase of the central charge function would keep constant value along the flow line, but this is impossible due to the particular form of $Z_u(\gamma_1)$ given above. $\square$

Split attractor flows

In general, the flow line would not flow directly into the attractor points as it would hit the walls on the base $\mathcal{B}^0$. Recall from the section 2.2.2, the **wall of the first kind** $\mathcal{W}^1_\gamma$ associated to the charge γ is defined as

$$\mathcal{W}^1_\gamma = \bigcup_{\gamma=\gamma_1+\gamma_2} \mathcal{W}_{\gamma_1,\gamma_2}$$

where $\mathcal{W}_{\gamma_1,\gamma_2}$ is the set defined as

$$\mathcal{W}_{\gamma_1,\gamma_2} := \left\{ b \in \mathcal{B} : Im\left(\frac{Z_b(\gamma_1)}{Z_b(\gamma_2}\right) = 0 \right\}$$

We want to understand what happens when the attractor flow lines hit this wall. Suppose that the flow line associated to the charge γ, i.e., gradient flow associated to the function $F_\gamma = Re\left(e^{-i\theta}Z(\gamma)\right)$, hits the wall $\mathcal{W}_{\gamma_1,\gamma_2}$ at which the charge splits as $\gamma = \gamma_1 + \gamma_2$. By definition, at the point of intersection of the flow line with the wall, we have the following condition

$$Arg(Z(\gamma)) = Arg(Z(\gamma_1)) = Arg(Z(\gamma_2)) \tag{3.2.10}$$

We know from proposition 3.2.3 that along the flow line $Arg(Z(\gamma))$ stays constant, consequently, at the intersection point, the flow line would split into two flow lines corresponding to $Re(Z(\gamma_1))$ and $Re(Z(\gamma_2))$ respectively. We denote by $\mathcal{L}_\gamma$ the flow line corresponding the the charge γ, thus at the intersection point of the flow line with the wall W^1_γ, we have schematically

$$\mathcal{L}_\gamma = \mathcal{L}_{\gamma_1} + \mathcal{L}_{\gamma_2}$$

which means that at this intersection point (called the **splitting point**), the flow line $\mathcal{L}_\gamma$ splits into two flow lines, namely $\mathcal{L}_{\gamma_1}$ and $\mathcal{L}_{\gamma_2}$ respectively.

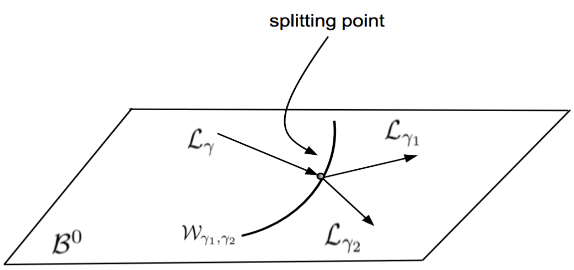

Figure 3.10: Split attractor flow and splitting point

Apparently, this process could be iterated until the resulting split attractor flow terminates at the attractor points. Namely, at certain stages, some charge, say γ_i splits at some wall into $\gamma_j + \gamma_k$ such that γ_j and γ_k corresponds to the vanishing cycles. Then by proposition 3.2.6, the flow lines $\mathcal{L}_{\gamma_j}$ and $\mathcal{L}_{\gamma_k}$ would terminate at the corresponding singularities at which the two cycles shrink to the zero. Clearly, this would give us a tree on the base $\mathcal{B}^0$ of the complex integrable system, we call it the **attractor tree** associated to the charge γ. We expect that this process would stop at finitely many steps and consequently the attractor tree would be finite. Since the mass function $|Z_b(\gamma)|$ is decreasing along the flow trees, which also implies the following proposition:

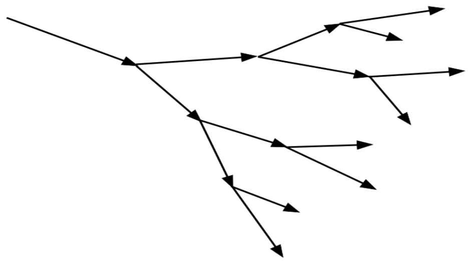

Figure 3.11: Attractor tree

Proposition 3.2.7. *The attractor trees associated to $(b, \gamma) \in tot\,\underline{\Gamma}$ form an acyclic graph.*

Proposition 3.2.8. *The flow line equation (3.2.9) can also be written as follows*

$$Z_{b(t)}(\gamma) = c(1-t) \tag{3.2.11}$$

where t is the real time parameter, and c is a complex constant depending on the initial condition of the attractor flow.

Proof. From (3.2.8), we see that the attractor flow is the constant phase curve for the central charge function, thus, it could be written in the form above. Suppose $Arg(Z_{b_0}(\gamma)) = \theta$, then c is a complex number with phase θ. When $t = 1$, the end point of the flow will land at the attractor point where the central charge vanishes. $\square$

Proposition 3.2.9. *There is another form of attractor follow equation*

$$\partial_t Im(e^{-i\theta}\alpha) = -\gamma \tag{3.2.12}$$

where α is the one form in defining the central charge (see definition 3.1.12, later will be identified with the Seiberg-Witten differential). Consequently, it can be further integrated into the following form

$$Im(e^{-i\theta}\alpha) = -\gamma t + Im(e^{-i\theta}\alpha)|_{t=0} \tag{3.2.13}$$

Proof. For the physics derivation of the equation (3.2.12), the readers are referred to section 3.4 of [Den00]. We present a mathematical proof here. First recall that the central charge Z is defined through

$$Z_b(\gamma) = \oint_\gamma \alpha =: \langle \alpha, \gamma \rangle$$

Thus equation (3.2.8) becomes

$$Im(e^{-i\theta}\langle \alpha, \gamma \rangle) = constant.$$

Thus we have that

$$\partial_t\, Im(e^{-i\theta}\langle\alpha,\gamma\rangle) = 0$$

As the flow line equation is linear in affine coordinates, by $\langle\gamma,\gamma\rangle = 0$, the above equation can be written further as

$$\partial_t\, Im(e^{-i\theta}\alpha) = -\gamma$$

It is easy to see that this integrates into (3.2.13). □

Proposition 3.2.10. *The attractor flow associated to the charge γ can intersect with the wall $\mathcal{W}^1_{\gamma_1,\gamma_2}$, where $\gamma = \gamma_1 + \gamma_2$ at most once. In particular, the flow line intersects with the wall transversely.*

Proof. We denote by b_* the intersection point with the corresponding time parameter being t_*, such that at b_*, we have that

$$Z_{b_*}(\gamma) = Z_{b_*}(\gamma_1) + Z_{b_*}(\gamma_2)$$

and

$$Arg(Z_{b_*}(\gamma)) = Arg(Z_{b_*}(\gamma_1)) = Arg(Z_{b_*}(\gamma_2)) = \theta$$

Then applying the equation (3.2.13) in our case, and take the intersection with γ_1, then we see that at the crossing point b_*, we have that

$$Im\left(e^{-i\theta}Z_{b_*}(\gamma_1)\right) = -\langle\gamma,\gamma_1\rangle\, t_* + Im\left(e^{-i\theta}Z_{b_0}(\gamma_1)\right)$$

The left hand side is seen to vanish, and under the assumption that $\langle\gamma_1,\gamma_2\rangle \neq 0$, we see that the above equation has an unique solution t_*, which completes the proof of the proposition. Indeed, we can give the solution as

$$t_* = \frac{Im\left(e^{-i\theta}Z_{b_0}(\gamma_1)\right)}{\langle\gamma_2,\gamma_1\rangle} = \frac{Im\left(\frac{\overline{Z_{b_0}(\gamma_1)+Z_{b_0}(\gamma_2)}}{|Z_{b_0}(\gamma_1)+Z_{b_0}(\gamma_2)|}\, Z_{b_0}(\gamma_1)\right)}{\langle\gamma_2,\gamma_1\rangle}$$

$$= \frac{1}{\langle\gamma_2,\gamma_1\rangle}\frac{Im\left(\overline{Z_{b_0}(\gamma_2)}Z_{b_0}(\gamma_1)\right)}{|Z_{b_0}(\gamma_1) + Z_{b_0}(\gamma_2)|}$$

□

From the computation done in the above proof, we can deduce that

$$-(t - t_*) = \frac{1}{\langle\gamma_1,\gamma_2\rangle}\frac{Im\left(\overline{Z_b(\gamma_1)}Z_b(\gamma_2)\right)}{|Z_b(\gamma_1) + Z_b(\gamma_2)|}$$

thus the following proposition holds.

Proposition 3.2.11. *Assuming $\langle\gamma_1,\gamma_2\rangle > 0$, then near the wall $\mathcal{W}^1_{\gamma_1,\gamma_2}$, the attractor flow would flows from the region where $Im\left(\overline{Z_b(\gamma_1)}Z_b(\gamma_2)\right) < 0$, i.e., where $Arg(Z_b(\gamma_1)) > Arg(Z_b(\gamma_2))$ into the region where $Im\left(\overline{Z_b(\gamma_1)}Z_b(\gamma_2)\right) > 0$, i.e., where $Arg(Z_b(\gamma_1)) < Arg(Z_b(\gamma_2))$;q1X while at the intersection point, we have that $Arg(Z_b(\gamma_1)) = Arg(Z_b(\gamma_2))$, which is compatible with the definition of the wall of the first kind.*

Finally, we can prove the following

Proposition 3.2.12. *The length of attractor flow line connecting $b \notin \mathcal{W}^1_{\gamma_1,\gamma_2}$ and b_* is bounded.*

Proof. Suppose the point b corresponds to the parameter 0, then we calculate as follows

$$\int_0^{t_*} \left(\frac{ds}{dt}\right)^2 dt = \int_0^{t_*} \sum_{ij} g_{i\bar{j}}\, \dot{u^i}\, \dot{\bar{u}^j} dt$$

$$= \int_0^{t_*} \sum_{ij} g_{i\bar{j}} \left(-\sum_k g^{i\bar{k}}\, \bar{\partial}_k |Z(\gamma)|\right)\left(-\sum_l g^{l\bar{j}}\, \partial_l |Z(\gamma)|\right) dt$$

$$= -\int_0^{t_*} \sum_{ij} \left(\partial_i |Z(\gamma)| + \bar{\partial}_j |Z(\gamma)|\right) dt = -\int_0^{t^*} d|Z(\gamma)| > 0$$

The last inequality holds because the mass function $|Z(\gamma)|$ is decreasing along the flow line (see proposition 3.2.5). Thus we infer that

$$\int_0^{t_*} \left(\frac{ds}{dt}\right)^2 dt = |Z_b(\gamma)| - |Z_{b_*}(\gamma)|$$

$$= |Z_b(\gamma)| - (|Z_{b_*}(\gamma_1)| + |Z_{b_*}(\gamma_2)|) \le |Z_b(\gamma)| < \infty$$

□

Remark 3.2.5. *The distance of attractor flow line between a point $b \notin \mathcal{W}^1$ and the attractor point may not be finite in general. However, under the A^1-singularity assumption, we see that the distance is still finite since by the formula (3.1.87), we deduce easily that in this case for γ being a vanishing cycle, $Z_b(\gamma)$ has a removable singularity at the origin, thus $Z_b(\gamma)$ and $dZ_b(\gamma)$ extends to zero at the singularity.*

3.2.2 Axiomatization

Motivated by the discussion of the attractor flow in previous subsection, we can now give the axiomatic treatment following Kontsevich and Soibelman. (see section 3 of [KS14]).

Over the base $\mathcal{B}^0$, we have the local system of lattices $\underline{\Gamma}$, endowed with the antisymmetric bilinear pairing $\langle\cdot,\cdot\rangle : \wedge^2\underline{\Gamma} \to \underline{\mathbb{Z}}$. Then this pairing gives rise to the following map

$$\iota : \underline{\Gamma} \longrightarrow \underline{\Gamma}^\vee \qquad \gamma \longmapsto \langle\gamma,\cdot\rangle$$

Denote by $\underline{\Gamma}_0$ the kernel of ι, i.e., the radical $Ann\langle\cdot,\cdot\rangle$ of the pairing, and by $\underline{\Gamma}^{symp}$ the symplectic quotient of $\underline{\Gamma}$ by $\underline{\Gamma}_0$.

Definition 3.2.3. *The* ***attractor flow*** *on* tot $\underline{\Gamma}$ *is defined to be*

$$\dot{b} = \iota(\gamma) \quad \dot{\gamma} = 0 \tag{3.2.14}$$

Here we used the identification $T_{\mathcal{B}^0} \cong \underline{\Gamma}^\vee$, and $Im(\iota)$ is an affine space in $\underline{\Gamma}^\vee$ parallel to $\underline{\Gamma}^{symp}$.

Remark 3.2.6. *The above definition is motivated by the attractor flow equation (3.2.9). The projection of the attractor flow (3.2.14) onto the base $\mathcal{B}^0$ would be the attractor flow in the usual sense.*

Remark 3.2.7. *The attractor flow can to extended to tot $\underline{\Gamma}_{\mathbb{R}}$ in the obvious manner.*

Recall that in the definition of the global wall-crossing structure on $\mathcal{B}^0$ (see definition 2.5.1), we have the continuous map $Y : \mathcal{B}^0 \to \underline{\Gamma}^*_{\mathbb{R}}$, and the walls of the second kind $\mathcal{W}^2_\gamma$ in $\mathcal{B}^0$ associated to γ (see definition 2.5.4), namely

$$\mathcal{W}^2_\gamma = \gamma^\perp := \{b \in \mathcal{B}^0 : Y(b)(\gamma) = 0\}$$

We consider the set

$$\mathcal{B}^{0\prime} := \{(b, v) \in tot\,\underline{\Gamma}_{\mathbb{R}} : Y(b)(v) = 0\}$$

and also its subset

$$\mathcal{B}^{0\prime}_{\mathbb{Z}} := \{(b, \gamma) \in tot\,\underline{\Gamma} : Y(b)(v) = 0\}$$

Clearly, $dim\,\mathcal{B}^{0\prime}_{\mathbb{Z}} = dim\,\mathcal{B}^0 - 1$, and $dim\,\mathcal{B}^{0\prime} = dim\,\mathcal{B}^0 + rk\Gamma - 1$.

Proposition 3.2.13. *The sets $\mathcal{B}^{0\prime}$ and $\mathcal{B}^{0\prime}_{\mathbb{Z}}$ are preserved by the attractor flow defined by equation (3.2.14).*

Proof. From the definition, the attractor flow line is locally given by

$$t \mapsto b_0 + \iota(\gamma)t$$

from which we see that

$$Y(b_0 + \iota(\gamma)\delta t)(\gamma) = Y(b_0)(\gamma) + Y'(b_0)\iota(\gamma)(\gamma)\delta t = Y'(b_0)\langle\gamma, \gamma\rangle\delta t = 0$$

□

Remark 3.2.8. *The proposition above is motivated by proposition 3.2.2 where we showed the the flow lines lie on the hypersurface that is defined by the constant phase condition (see equation (3.2.8)). Consequently, the attractor flow is actually defined on the two sets above.*

Definition 3.2.4. *The attractor flow (3.2.14) can be restricted to be a flow on $\mathcal{B}^{0\prime}$, which can be further restricted to $\mathcal{B}^{0\prime}_{\mathbb{Z}}$ and hence induces the* ***"integer" attractor flow*** *on it.*

Using this new definition, we can prove the proposition 3.2.10 more straightforwardly as follows. We want to show the attractor flow line intersects with the wall $\mathcal{W}^1_{\gamma_1,\gamma_2}$ transversely.

Proof. Near the intersection point, we have the spliting: $\gamma = \gamma_1 + \gamma_2$ such that

$$Y(b + \iota(\gamma)\delta t)(\gamma_i) = 0 \quad i = 1, 2.$$

that is

$$Y(b)(\gamma_i) + Y'(b)\langle\gamma, \gamma_i\rangle\delta t = 0 \quad i = 1, 2.$$

Consequently, we have that

$$Y(b)(\gamma_1) + Y'(b)\langle\gamma_2, \gamma_1\rangle(t - t_0) = 0$$

which is easily seen to have an unique solution in t. □

Remark 3.2.9. *In the semipolarized case, i.e., $\underline{\Gamma}_0$ being nontrivial, we see from section 3.1.7 that $\mathcal{B}^0$ is foliated by the fibers of the surjection*

$$p_{\mathcal{B}^0} : \mathcal{B}^0 \longrightarrow (\underline{\Gamma}_0 \otimes \mathbb{R})/G$$

where G denotes the finite group of monodromy of $\underline{\Gamma}_0$. We denote the symplectic leaf passing through $b \in \mathcal{B}^0$ by $M := M_b$, which can be identified via Y with an affine space over the vector space $\underline{\Gamma}^{symp}$. We can restrict the flow further to the symplectic leaves, which induces attractor flows on

$$M' := \{(m, v) \in \mathit{tot}\, \underline{\Gamma}_{\mathbb{R}} : Y(m)(v) = 0\}$$

and the integer flows on

$$M'_{\mathbb{Z}} := \{(m, \gamma) \in \mathit{tot}\, \underline{\Gamma} \backslash \mathit{tot}\, \underline{\Gamma}_0 : Y(m)(\gamma) = 0\}$$

Attractor trees

The attractor flows split when crossing the wall of the first kind, thus generate the trees on $\mathcal{B}^0$, called the **attractor trees** or **split attractor flows**. Now we cite its definition [KS14] (definition 3.2.1 there).

Definition 3.2.5. *An attractor tree is a metrized rooted tree T endowed with a continuous map $f : T \to M$ to a leaf $M \subset \mathcal{B}^0$ and a lift $f' : T - \{vertices\} \to M'_{\mathbb{Z}}$. We assume that f' maps edges of T to trajectories of the attractor flow, and the metric on each edge of T is given by $|dt|$, where t is the time parameter for attractor flow on its lifting. We assume that all tail edges are maximal positive trajectories of the corresponding internal vertices of T. We also assume that the* ***balancing condition***

$$\sum_i \gamma_i^{out} = \gamma^{in} \tag{3.2.15}$$

is satisfied at each internal vertex v. Here γ^{in} is the speed of the f' lift of the only edge incoming from v, and γ_i^{out} are speeds of the f' lifts of all outgoing edges. Furthermore, we assume that all γ_i^{out} are pairwise distinct and there exists i_1, i_2 such that $\langle \gamma_{i_1}^{out}, \gamma_{i_2}^{out} \rangle \neq 0$.

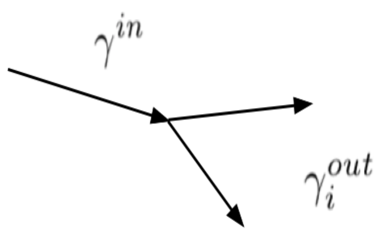

Figure 3.12: Balance condition

In the above definition, the term **maximal positive trajectory** means that for each point $(b, \Gamma) \in \mathcal{B}^{0\prime}$, there exits maximal possible

$$t_{max} := t_{max(b,\gamma)} \in (0, +\infty]$$

such that the attractor flow trajectory exists. The trajectory corresponding to $t \in [0, t_{max})$ is called the maximal positive trajectory of the point (b, γ). The **combinatorial type** of the attractor tree T rooted at (b, γ) consists of the abstract rooted tree $\mathcal{T}$ corresponding to T as well as a collection of velocities of all its edges, including its tails. The velocities can be treated as elements of $\underline{\Gamma}$ via the parallel transport along the edges. Varying (b, γ), the combinatorial types form a local system over $\mathcal{B}^{0\prime}_{\mathbb{Z}}$.

Motivated by proposition 3.2.12 in the last subsection, we infer that the inner edges of the attractor tree have finite length, thus we have the collection $\{l_e\}$ of lengths of its inner edges e. Then as the velocities is determined by the lengths and vice verse, we conclude that the combinatorial type of the attractor tree T at (b, γ) is uniquely determined by $\{l_e\}$. We have the following:

Proposition 3.2.14. *The collection $\{l_e\}$ and the vector $Y(b) \in \underline{\Gamma}^*_{b,\mathbb{R}}$ satisfy the following system of linear equations:*

1. *$Y(f(v))(f'(u)) = 0$, where v is a vertex and u is a point on an edge adjacent to v and sufficiently close to v;*
2. *For an inner edge e connecting vertices v_1 and v_2, we have that*
$$Y(f(v_2)) - Y(f(v_1)) = \iota(f'(u))\, l_e$$
where u is any point of e.

Proof. The first part of the proposition follows directly from the remark 3.2.8, while for 2), we consider the attractor flow which gives the edge e, namely:
$$\dot{b} = \iota(f'(u))$$
Suppose v_i corresponds to t_i respectively, $i = 1, 2$, then by integrating the attractor flow equation, we get
$$\int_{t_1}^{t_2} \dot{b}\, dt = \int_{t_1}^{t_2} \iota(f'(u))\, dt$$
Consequently, we get the desired identity. □

Given an attractor tree T rooted at (b, γ), consider the germ of the **universal deformation of** T, which consists of attractor trees with sufficiently close roots, combinatorial type and edge lengths. From the above characterization, we can see this this germ of universal deformation can be identified with an open domain in the vector subspace of the vector space $\underline{\Gamma}^*_{b,\mathbb{R}} \oplus \mathbb{R}^{inner\, edges}$ defined by the above systems of linear equations. In particular, we see that the set of roots of attractor trees which are close to T and have the same combinatorial type is locally an open domain in a vector subspace of $\underline{\Gamma}^*_{b,\mathbb{R}}$.

Definition 3.2.6. *The attractor tree T is said to be* ***locally planar*** *if for each internal vertex v, the corresponding vectors γ_i^{out} span a two dimensional vector subspace in $\underline{\Gamma} \otimes \mathbb{R}$.*

In [KS14] (proposition 3.2.3. contained therein), the following result was proved. We cite it here with its proof being omitted.

Proposition 3.2.15. *If T is locally planar, then the set of roots of all sufficiently close attractor trees of the same combinatorial type has codimension ≥ 1, and has codimension ≥ 2 were the tree is non-planar. Moreover in the formal case any sufficiently close attractor tree is uniquely determined by its root.*

3.2.3 Connection with WCS

In this subsection, we want to give a method about how to use the splitting attractor flows to produce WCS on the base $\mathcal{B}^0$ of the complex integrable system.

Suppose we have a closed subset $C^+ \subset \mathcal{B}^{0\prime} \subset tot\,\underline{\Gamma}_{\mathbb{R}}$ satisfying the following

1. Fibers of C^+ under the natural projection of $tot\,\underline{\Gamma}_{\mathbb{R}}$ to $\mathcal{B}^0$ are strict convex cones;
2. The set C^+ is preserved by the "inverse attractor flow": $\dot{b} = -\iota(v)\ \dot{v} = 0$, for $v \in \underline{\Gamma}_{b,\mathbb{R}}$.

Remark 3.2.10. *This set C^+ can usually be known by some apriori reason, and could serve as "upper bound" for the support of WCS to be discussed below.*

We define the **tail set** to be an open subset $\mathcal{T}_{\mathcal{B}^{0\prime}_{\mathbb{Z}}} \subset \mathcal{B}^{0\prime}_{\mathbb{Z}}$ consisting of points $(b, \gamma) \in C^+$ such that their maximal positive trajectories do not intersect the wall $\mathcal{W}^1_\gamma$, and belong to C^+. The same should hold for all nearby points (b', γ').

Remainder on WCS

Given a local system of full rank lattice $\underline{\Gamma} \to \mathcal{B}^0$ over $\mathcal{B}^0$, and a local system of $\underline{\Gamma}$-graded Lie algebra $\underline{\mathfrak{g}} = \bigoplus_{\gamma \in \underline{\Gamma}} \mathfrak{g}_\gamma$ over $\mathcal{B}^0$. Moreover $\underline{\Gamma}$ is endowed with an integer valued symplectic paring $\langle \cdot, \cdot \rangle$ such that if $\langle \gamma_1, \gamma_2 \rangle = 0$, for $\gamma_i \in \underline{\Gamma}_b$, then the corresponding components of $\mathfrak{g}_b$ satisfies $[\mathfrak{g}_{b,\gamma_1}, \mathfrak{g}_{b,\gamma_2}] = 0$. We assume that for $\gamma \in \underline{\Gamma}_0$, $\mathfrak{g}_\gamma = 0$.

We have a locally continuous map $Y : \mathcal{B}^0 \to \Gamma^*_{\mathbb{R}}$. Then the WCS on $\mathcal{B}^0$ in section 2.5 would give us a map $a : tot\,\underline{\Gamma} \to \underline{\mathfrak{g}}$ such that for $b \in \mathcal{B}^0$, $\gamma \in \Gamma_b$ that satisfies $Y(b)(\gamma) = 0$, we have the element $a_b(\gamma) := a(b, \gamma) \in \underline{\mathfrak{g}}_{b,\gamma}$. It is related to the **DT-invariants** (thus the invariants $\Omega_b(\gamma)$ through the formula (2.1.3)) in the following way

$$a_b(\gamma) = DT(\gamma) \cdot e_\gamma \in \underline{\mathfrak{g}}_{b,\gamma}$$

The above map a is a locally constant function on $tot\,\underline{\Gamma}$ that becomes discontinuous at those (b, γ) such that $\gamma = \gamma_1 + \gamma_2$ and the phase of $Z_b(\gamma_1)$ and that of $Z_b(\gamma_2)$ get aligned. Clearly, the discontinuous variety projects to the wall of the first kind $\mathcal{W}^1_\gamma$ on $\mathcal{B}^0$, which is a locally-finite hypersurfaces in $\mathcal{B}^0$ pulled back through Y from a $\mathbb{Z}PL$ hypersurface in $\underline{\Gamma}^*_{\mathbb{R}}$. Also recall (see definition 2.5.3) that the **support** of a wall crossing structure σ, denoted by $Supp\sigma$ is defined to be a minimal closed subset of $tot\,\underline{\Gamma}_{\mathbb{R}}$ that is conic in the direction of $\underline{\Gamma}_{\mathbb{R}}$ such that it contains those points $(b, \gamma) \in tot\,\underline{\Gamma}_{\mathbb{R}}$ with $Y(b)(\gamma) = 0$ and $a_b(\gamma) \in \underline{\mathfrak{g}}_{b,\gamma} \backslash \{0\}$, i.e., those (b, γ) with nontrivial DT-invariants.

Finally, we recall that for each $\gamma \in \underline{\Gamma} \backslash \{0\}$, we have the associated wall of second kind (see definition 2.5.4)

$$\mathcal{W}^2_\gamma = \gamma^\perp := \left\{ b \in \mathcal{B}^0 : Y(b)(\gamma) = 0 \right\}$$

When the wall of the first kind $\mathcal{W}^1_\gamma$ is being crossed, the elements (that encode DT-invariants) $a_b(\gamma)$ would jump, and its jumps are governed by the KSWCF (2.2.5).

The use of split attractor flows

The algorithm for using attractor flows to construct WCS as proposed by Kontsevich and Soibelman in [KS14] goes roughly like the following:

In order to produce a WCS on the base $\mathcal{B}$ of the full integrable system $\pi : X \longrightarrow \mathcal{B}$, namely, a collection of elements $\left\{a_b(\gamma) \in \underline{\mathfrak{g}}_{b,\gamma}\right\}$ for $(b, \gamma) \in tot\,\underline{\Gamma}$ that satisfies KSWCF, we consider **all** attractor trees rooted at (b, γ) with tail edges hitting the discriminant locus Δ. Under certain conditions to be discussed momentarily, we expect that the number of such trees is finite. Then for every such tree we move from the tail vertices, i.e., attractor points which belong to Δ toward the root b, and apply the KSWCF at each internal vertex b_* lying on the wall of the first kind.

By assigning the **"initial data"** of WCS, i.e., the DT invariants at the discriminant locus Δ, we can construct by induction the DT invariant $a_b(\gamma)$.

Now we try to make the above algorithm more precise. Denote by $\underline{\mathfrak{g}}_{loc}$ the restriction of $\underline{\mathfrak{g}}$ to the tail set $\mathcal{T}_{\mathcal{B}^{0\prime}_{\mathbb{Z}}}$, then we define the initial data of a wall crossing structure as follows

Definition 3.2.7. *The **initial data** of a WCS bounded by C^+ is given by the restriction of the map a to $\mathcal{T}_{\mathcal{B}^{0\prime}_{\mathbb{Z}}}$.*

Remark 3.2.11. *From the A^1-singularity assumption, we expect that for $b \in \Delta$, there is only one special direction corresponding the the vanishing cycle over b,which is invariant under the monodromy around b. Consequently, the DT invariants at b in non-trivial only in this special direction, which means that the local system $\underline{\mathfrak{g}}_{loc}$ is typically trivial of rank one.*

Since we expect that for generic (b, γ), the attractor flow rooted at (b, γ) would terminate at the attractor points, thus we impose the following tail assumption:

Tail Assumption Given any open subset $U \subset \mathcal{B}^{0\prime}$, the subset of points $(b, \gamma) \in U$ such that their maximal positive trajectories intersect the tail set $\mathcal{T}_{\mathcal{B}^{0\prime}_{\mathbb{Z}}}$ is dense in U.

For an attractor tree T, denote by T^0 the tree obtained by deleting all its tail edges, then every edge of T^0 is joined by two vertices.

Compactness Assumption There exists an open dense subset $\mathcal{B}^{0\prime\prime}_{\mathbb{Z}} \subset \mathcal{B}^{0\prime}_{\mathbb{Z}}$ with the property that for every $(b, \gamma) \in \mathcal{B}^{0\prime\prime}_{\mathbb{Z}}$, there exists a compact subset $K_{(b,\gamma)} \subset \mathcal{B}^{0\prime}_{\mathbb{Z}}$ and an open neighborhood U of (b, γ) such that for every attractor tree T with the root and root edge in U the corresponding tree T^0 belongs to $K_{(b,\gamma)}$.

Mass Function Assumption There exists a function $X : \mathcal{B}^{0\prime}_{\mathbb{Z}} \longrightarrow \mathbb{R}$ that is decreasing along the attractor flow when restricted to C^+, and is strictly decreasing and strictly positive on the set $C^+ \backslash tot\,\underline{\Gamma}_{0,\mathbb{R}}$.

Remark 3.2.12. *The mass function assumption is motivated by the function $F_\gamma(b) = Re(e^{-i\theta} Z_b(\gamma))$ in the definition of the attractor tree.*

Consider the union of all attractor trees rooted at (b, γ), which is a graph denoted by $G(b, \gamma)$ for $(b, \gamma) \in \mathcal{B}_{\mathbb{Z}}^{0\prime\prime}$, then the Compactness assumption ensures that the graph $G(b, \gamma)$ is finite and the Mass function assumption implies that the graph is acyclic.

When the root b runs through the set of generic points, i.e., $Y(b)$ falls into the complement of the locally finite union of codimension ≥ 2 subspace in $\Gamma_{\mathbb{R}}^*$ such that $Y(b)(\gamma) = 0$, we infer from the proposition 3.2.15 that the graph $G(b, \gamma)$ is locally planar. Now we can state the following proposition which falls into our expectation.

Proposition 3.2.16. *The WCS with fixed $\mathfrak{g}$ and support belonging to $C_{\mathbb{Z}}^+ := C^+ \cap \mathcal{B}_{\mathbb{Z}}^{0\prime}$ is uniquely determined by its initial data.*

Proof. For a generic point $(b, \gamma) \in C_{\mathbb{Z}}^+$, the maximal acyclic graph $G(b, \gamma)$ above has all its tails belonging to $\mathcal{T}_{\mathcal{B}_{\mathbb{Z}}^{0\prime}}$ due to the tail assumption. Then the value of $a_b(\gamma)$ can be computed from the initial data by induction, namely, we have the value of DT-invariants at the attractor points, and then move toward the root along the edges of the graph. As the graph G is acyclic and locally planar, for any internal vertex (b_*, γ_*), $a_{b_*}(\gamma_*)$ is determined by the KSWCF in 2-dimensional case (see formula (2.2.7)). We continue moving until we meet the root (b, γ). By compactness assumption, there are only finitely many KSWCFs to be used. Thus, $a_b(\gamma)$ can be computed in finitely many steps. □

WCS for integrable system

The WCS on the base $\mathcal{B}^0$ is defined through the map

$$Y : \mathcal{B}^0 \to \Gamma_{\mathbb{R}}^* \quad b \mapsto Y_b := Im(e^{-i\theta} Z_b)$$

Recall that we have a local system of lattices $\underline{\Gamma} \to \mathcal{B}^0$, the radical $\underline{\Gamma}_0$ of $\underline{\Gamma}$ is assumed to have finite monodromy group G. We then have the submersion

$$p_{\mathcal{B}^0} : \mathcal{B}^0 \longrightarrow \Gamma_{0,\mathbb{C}}^*/G$$

and the fiber $M := \mathcal{B}_{Z_0}^0$ for non-singular point $Z_0 \in \Gamma_{0,\mathbb{C}}^*/G$ is a symplectic leaf in $\mathcal{B}^0$ and can be identified via the map Y above with an affine symplectic leaf parallel to $\Gamma_{\mathbb{R}}^{symp*}$.

We will consider the split attractor flows on $\mathcal{B}^0$, i.e., the projection of the attractor tree as defined in definition 3.2.5 onto the base. We can describe it by using the $\mathbb{Z}$-affine structure on $\mathcal{B}^0$ as follows:

Definition 3.2.8. *The **attractor tree** (or **tropical curve with stop**, see [Lin17]) on the affine manifold $\mathcal{B}^0$ with singularities being the focus-focus type (A^1-singularity assumption) is given by a rooted tree on $\mathcal{B}$ with root $b \in \mathcal{B}^0$ such that its edges are given by affine lines and each of its internal vertex is of valency at least two and the balancing condition is being satisfied there, while the tail edges terminating at Δ and is in the monodromy invariant direction.*

Remark 3.2.13. *We want the edges to be the affine line defined by the so called **good attractor tree** to be defined below.*

Definition 3.2.9. *An attractor tree is called **good** if it is locally planar in a fiber M of $p_{\mathcal{B}^0}$ such that its tail edges hit transversely the discriminant Δ, with the velocity of any tail edge being proportional to the corresponding vector γ (monodromy invariant direction).*

Remark 3.2.14. *From section 3.2.2, we conclude that the function $Re(e^{-i\theta}Z)$ in defining the attractor flow can serve as a mass function. And the initial data is given by assigning for $b \in \Delta$ and the primitive vanishing cycle γ the value*

$$a_b(k\gamma) := \frac{1}{k^2} e_{k\gamma}$$

for all $k \geq 1$, i.e., the DT-invariant $\Omega_b(\gamma) = 1$.

By the above remark, we can use the algorithm mentioned at the beginning to construct a WCS for the complex integrable system. In particularly, we have the following result due to Kontsevich and Soibelman (see section 2.7 of [KS08])

Proposition 3.2.17. *The WCS on the base $\mathcal{B}$ of the integrable system gives rise to an local embedding $\mathcal{B}^0 \hookrightarrow Stab(\mathfrak{g}_b)$ for each $b \in \mathcal{B}^0$.*

Proof. As b varies, we get a family of stability data on the graded Lie algebra $\mathfrak{g}_b$, recall that in proposition 2.6.3, we have shown that this can be identified with the global WCS on $\mathcal{B}_\theta := \mathcal{B} \times S_\theta$ via the map (see (2.6.3))

$$Y : \mathcal{B} \times \ S_\theta \to \Gamma^*_{\mathbb{R}}$$

$$(b, \theta) \mapsto Y_\theta(b)(\gamma) := Im(e^{-i\theta} Z_b(\gamma))$$

Given (b, γ), we consider all good attractor trees rooted at (b, γ), i.e., the split attractor flow associated to the function $Re(e^{-i\theta} Z_b(\gamma))$, where $\theta = Arg Z_b(\gamma)$. Since along the flow lines, the phase of the central charge stays constant, we see that a or $\Omega : tot\,\underline{\Gamma} \to \underline{\mathbb{Q}}$ restricts to $\mathcal{B}^{0\prime} := \{(b, \gamma) : Y_\theta(b)(\gamma) = 0\}$, which is compatible with the above Y.

At the discriminant locus Δ, motivated by remark 3.2.13, we assign the DT-invariant $\Omega_b(\gamma) = 1$ for γ corresponding to the monodromy invariant direction. As we moving forward toward the root (b, γ), the attractor flow lines lie on the co-dimensional one wall, the wall of second kind

$$\mathcal{W}^2_\gamma := \left\{ b \in \mathcal{B}^0 : Y_\theta(b)(\gamma) = 0 \right\}$$

And when the flow line hit the wall of first kind where $\gamma = \gamma_1 + \gamma_2$, i.e.,

$$\mathcal{W}^1_{\gamma_1, \gamma_2} := \left\{ b \in \mathcal{B}^0 : Im(Z_b(\gamma_1)\overline{Z_b(\gamma_2)}) = 0 \right\}$$

the flow line $\mathcal{L}_\gamma$ associated to γ will splits into two flow lines associated to the charge γ_1 and γ_2 respectively i.e.,

$$\mathcal{L}_\gamma = \mathcal{L}_{\gamma_1} + \mathcal{L}_{\gamma_2}$$

and the flow lines $\mathcal{L}_{\gamma_1}$ and $\mathcal{L}_{\gamma_2}$ sit inside the wall of second kind $\mathcal{W}^2_{\gamma_1}$ and $\mathcal{W}^2_{\gamma_2}$ respectively.

At each such splitting point, i.e., the intersection point of the flow line with the wall of first kind, we apply the KSWCF in two dimension case. By this procedure, we will finally compute $\Omega_b(\gamma)$ at finitely many steps of induction by our assumptions.

Thus, for each b inside $\mathcal{B}^0$ and $\gamma \in \underline{\Gamma}_b$, let $\theta = Arg Z_b(\gamma)$, we have obtained in this way the collection of DT-invariants $\Omega_b(\gamma)$ satisfying KSWCF. Note that the attractor trees on $\mathcal{B}$ can be lifted to attractor trees on $\mathcal{B} \times S_\theta$ in an obvious way.

We see in particularly that for each $b \in \mathcal{B}^0$, we have a WCS on $\{b\} \times S_\theta \cong S_\theta$, which by proposition 2.6.1, is the same as a stability data on the torus Lie algebra $\mathfrak{g}_b$. we thus get the desired embedding. $\square$

3.2.4 Attractor flows versus Hesse flows

We have already exhibited (see proposition 3.1.27) that the base $\mathcal{B}$ of a complex integrable system is naturally endowed with a $\mathbb{Z}$-affine structure with singularities at the discriminant locus Δ. On the smooth part of the base $\mathcal{B}^0$, it is endowed with a Kähler metric (see formulas (3.1.41) and (3.1.42))

$$g_{\mathcal{B}^0} = \sum_{ij} g_{ij}\, da^i d\bar{a}^j \tag{3.2.16}$$

where the metric metrix is given via the prepotential $\mathcal{F}$ and the dual coordinates $a_{D,j}$ through the following

$$g_{ij} = Im(\tau_{ij}) = Im\left(\sum_i da_{D,i}\, d\bar{a}^i\right) = \frac{\partial^2 \mathcal{F}}{\partial a^i \partial a^j} \tag{3.2.17}$$

and the Kähler potential of the metric is given by

$$K = Im\left(\sum_i a_{D,i}\, \bar{a}^i\right) \tag{3.2.18}$$

Next we recall the following definition of **Monge-Ampèr manifold**(c.f.[KS01]).

Definition 3.2.10. *A Monge-Ampère manifold is a triple (X, g, ∇), where (X, g) is a smooth Rieman manifold with metric g, and ∇ a flat connection on the tangent bundle TX such that*

- *∇ defines an affine structure on X.*
- *The metric g in local affine coordinates $(x_1, \cdots, x_n)$ can be expressed as*

$$g_{ij} = \frac{\partial^2 K}{\partial x_i \partial x_j} \tag{3.2.19}$$

  for some smooth real valued (potential) function K.
- *The following Monge-Ampère equation should be satisfied*

$$det\left(\frac{\partial^2 K}{\partial x_i \partial x_j}\right) = constant. \tag{3.2.20}$$

Obviously, the smooth part $\mathcal{B}^0$ of the base is a Monge-Ampère manifold.

Definition 3.2.11. *Let $U \subset \mathbb{R}^n$ be a convex open domain in $\mathbb{R}^n$ equipped with the standard affine coordinates $x_1, \cdots, x_n$, and $K : U \to \mathcal{R}$ a convex functions. Then the Legendre transform of the function K is defined as*

$$\widehat{K}(y_1, \cdots, y_n) := max_{x\in U}\left(\sum_i x_i y_i - K(x_1, \cdots, x_n)\right) \tag{3.2.21}$$

Remark 3.2.15. *In classical physics, the Legendre transform relates the Lagrangian formalism with the Hamiltonian formalism. Namely, given the Lagrangian $\mathcal{L}(q, \dot{q}, t)$ for the underlying dynamical system, we get the corresponding Hamiltonian $\mathcal{H}(p, q, t)$ through the Legendre transform*

$$H(p, q, t) = \widehat{\mathcal{L}} = p\dot{q} - \mathcal{L}(q, \dot{q}, t)$$

where p is the variable conjugate to q, i.e.,

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}}$$

Since Legendre transform is an involution, i.e., for a function K, we have that $\hat{\hat{K}} = K$. In our case, this means that the Lagrangian can also be expressed as the Legendre transform of the Hamiltonian, namely

$$\mathcal{L}(q, \dot{q}, t) = \widehat{\mathcal{H}} = \dot{q}p - H(p, q, t)$$

Then under the Legendre transform, the Lagrangian equations (of motion) in Lagrangian formalism get transformed into the Hamiltonian equations (of motion), that is

$$\dot{p} = \frac{\partial \mathcal{L}}{\partial q} \; \dot{p} = -\frac{\partial \mathcal{H}}{\partial q} \Longleftrightarrow \dot{q} = \frac{\partial \mathcal{H}}{\partial p} \; p = \frac{\partial \mathcal{L}}{\partial \dot{q}}$$

We have the following duality result for Monge-Ampère manifold (see c.f. lemma 1 in [KS01] for a short elegant proof).

Proposition 3.2.18. *If $K : U \to \mathbb{R}$ is a convex functions satisfying the Monge-Ampère equation,then its Legendre transform $\widehat{K}$ also satisfies the Monge-Ampère manifold.*

From the above proposition it follows that in the dual affine coordinates $(y_1, \cdots, y_n)$ on the base $\mathcal{B}^0$, we have another metric $\widehat{g}_{ij}$ (dual to g_{ij}) that is given by

$$\widehat{g}_{ij} = \frac{\partial^2 \widehat{K}}{\partial y_i \partial y_j} \tag{3.2.22}$$

Further more, we can state the following proposition which is given in [KS01].

Proposition 3.2.19. *For a given Monge-Ampère manifold (X, g, ∇) there is a canonically defined dual Monge-Ampère manifold $(\widehat{X}, \widehat{g}, \widehat{\nabla})$ such that (X, g) is identified with $(\widehat{X}, \widehat{g})$ as Riemann manifolds, and the local system $(T_{\widehat{X}}, \widehat{\nabla})$ is naturally isomorphic to the local system dual to (T_X, ∇). And if ∇ defines an $\mathbb{Z}$-affine structure on X, then $\widehat{\nabla}$ defines an $\mathbb{Z}$-affine structure on $\widehat{X}$.*

Remark 3.2.16. *For the relevance of the Monge-Ampère duality in dual torus fibration and mirror symmetry, see [KS01] for more information.*

Next, we will construct explicitly the dual Monge-Ampère manifold for the base of the complex integrable system. This part is based on the paper of Dieter Van der Bleeken[Van12].

Recall that on the base $\mathcal{B}^0$, we have the (rotated) special coordinates $\{e^{-i\theta}a^k\}$ and the dual coordinates $\{e^{-i\theta}a_{D,l}\}$. where $1 \leq k, l \leq n$. Consider the adapted real special affine coordinates (see definition 3.1.18 and remark 3.1.17 for the meaning of these terminologies), i.e.,

$$e^{-i\theta}a^k = x^k + iy^k \qquad e^{-i\theta}a_{D,l} = x_l + iy_l \tag{3.2.23}$$

We want to find the exact relations between these affine coordinates.

Since the dual coordinate $a_{D,l} = \partial\mathcal{F}/\partial a^l$ is defined through the prepotential $\mathcal{F}$, and $\mathcal{F}$ is a holomorphic function in a^k, we see that it can be viewed as a function in variables $\{x^k, y^k\}$. Consequently, we claim that

Proposition 3.2.20. *The affine variables x_l is conjugate to y^l, while y_l is conjugate to x^l, in the sense that there exists some holomorphic function $\mathscr{F}$ such that*

$$x_l = \frac{\partial \mathscr{F}}{\partial y^l} \qquad y_l = \frac{\partial \mathscr{F}}{\partial x^l} \tag{3.2.24}$$

Proof. We compute as follows:

$$\begin{aligned}
x_l + iy_l = e^{-i\theta}a_{D,l} = e^{-i\theta}\frac{\partial\mathcal{F}}{\partial a^l} &= e^{-i\theta}\left(\sum_k \frac{\partial\mathcal{F}}{\partial x^k}\frac{\partial x^k}{\partial a^l} + \frac{\partial\mathcal{F}}{\partial y^k}\frac{\partial y^k}{\partial a^l}\right) \\
= e^{-i\theta}\left(e^{-i\theta}\frac{\partial\mathcal{F}}{\partial x^l} - i\, e^{-i\theta}\frac{\partial\mathcal{F}}{\partial y^l}\right) &= \frac{\partial}{\partial x^l}\left(e^{-2i\theta}\mathcal{F}\right) - i\,\frac{\partial}{\partial y^l}\left(e^{-2i\theta}\mathcal{F}\right) \\
= \frac{\partial}{\partial x^l}\left(Re\left(e^{-2i\theta}\mathcal{F}\right) + i\, Im\left(e^{-2i\theta}\mathcal{F}\right)\right) &- i\,\frac{\partial}{\partial y^l}\left(Re\left(e^{-2i\theta}\mathcal{F}\right) + i\, Im\left(e^{-2i\theta}\mathcal{F}\right)\right)
\end{aligned}$$

which by applying the Cauchy-Riemann equation for the holomorphic function $e^{-2i\theta}\mathcal{F}$, equals to

$$2\left(\frac{\partial Im\left(e^{-2i\theta}\mathcal{F}\right)}{\partial y^l} + i\,\frac{\partial Im\left(e^{-2i\theta}\mathcal{F}\right)}{\partial x^l}\right)$$

By defining $\mathscr{F} := 2Im\left(e^{-2i\theta}\mathcal{F}\right)$, the relations (3.2.24) then follows from the above computations. $\square$

Since (x^k, y_l) is conjugate to the variables (y_k, x^l) by the above proposition, they are naturally related to each other through the Legendre transform of $\mathscr{F}$ with respect to the conjugate pairs $\{x^i, y_i\}$, then the function $\mathscr{F}$, which is given 'apriori ' as a function in the variables (x^i, y^i), now can be expressed as a function in the variables (y_i, y^i) through the Legendre transform with respect to the conjugate variables (x^i, y_i), namely

$$\widehat{\mathscr{F}^y}(y_i, y^i) = \sum_k y_k x^k - \mathscr{F}(x^i, y^i) \tag{3.2.25}$$

It is easy to see that

$$\frac{\partial\widehat{\mathscr{F}^y}}{\partial y^l} = -x_l \qquad \frac{\partial\widehat{\mathscr{F}^y}}{\partial y_k} = x^k \tag{3.2.26}$$

Thus the special coordinates $\{a^k\}$ as well as its dual $\{a_{D,l}\}$ can be expressed in terms of the real affine coordinates (y^i, y_i) as follows:

$$a^k = a^k(y_i, y^i) = e^{i\theta}(x^k + iy^k) = e^{i\theta}\left(\frac{\partial\widehat{\mathscr{F}^y}}{\partial y_k} + iy^k\right)$$

$$a_{D,l} = a_{D,l}(y_i, y^i) = e^{i\theta}(x_l + iy_l) = e^{i\theta}\left(-\frac{\partial \widehat{\mathscr{F}^y}}{\partial y^l} + iy_l\right) \tag{3.2.27}$$

Similarly, we can transform $\mathscr{F}$ into a function in variables (x_i, x^i) as

$$\widehat{\mathscr{F}^x}(x_i, x^i) = \sum_k x_k y^k - \mathscr{F}(x^i, y^i) \tag{3.2.28}$$

It is easy to see that

$$\frac{\partial \widehat{\mathscr{F}^x}}{\partial x^l} = -y_l \qquad \frac{\partial \widehat{\mathscr{F}^x}}{\partial x_k} = y^k \tag{3.2.29}$$

Thus, in terms of real affine coordinates (x_i, x^i), we have the following expressions for special coordinate and its dual

$$\begin{aligned} a^k &= a^k(x_i, x^i) = e^{i\theta}(x^k + iy^k) = e^{i\theta}\left(x^k + i\frac{\partial \widehat{\mathscr{F}^x}}{\partial x_k}\right) \\ a_{D,l} &= a_{D,l}(x_i, x^i) = e^{i\theta}(x_l + iy_l) = e^{i\theta}\left(x_l - i\frac{\partial \widehat{\mathscr{F}^x}}{\partial x^l}\right) \end{aligned} \tag{3.2.30}$$

Remark 3.2.17. *We can Legendre transform the function $\mathscr{F}$ as it is convex which follows from the holomorphicity of $\mathscr{F}$.*

The function $\widehat{\mathscr{F}^x}$ and $\widehat{\mathscr{F}^y}$ above are called the **Hesse potentials**, as the Kähler metric ds^2 (see (3.2.16)) on $\mathcal{B}^0$ can be expressed in terms of the Hessian matrices of the two potential functions.

To this end, let us first notice that the Kähler metric $g_{\mathcal{B}^0}$ can be written in the real affine coordinates in the following form

$$\begin{aligned} g_{\mathcal{B}^0} &= \sum_{ij} g_{ij} da^i \, d\bar{a}^j = \sum_{ij} Im(\tau_{ij})\, da^i d\bar{a}^j = \sum_k Im\left(da_{D,k}\, d\bar{a}^k\right) \\ &= \sum_k Im\left(e^{i\theta}(dx_k + idy_k)\cdot e^{-i\theta}(dx^k - idy^k)\right) \\ &= \sum_k dx^k \otimes dy_k - dx_k \otimes dy^k \end{aligned} \tag{3.2.31}$$

In terms of the $\mathbb{Z}$-affine structure on $\mathcal{B}^0$ given by the real affine coordinates (x_i, x^i), we can express the Kähler metric as

$$\begin{aligned} g_{\mathcal{B}^0} &= \sum_k dx^k \otimes dy_k - dx_k \otimes dy^k \\ &\underset{\text{by (3.2.29)}}{=\!=} -\sum_k dx^k \otimes \sum_l \frac{\partial^2 \widehat{\mathscr{F}^x}}{\partial x^k \partial x_l} dx_l - \sum_k dx_k \otimes \sum_l \frac{\partial^2 \widehat{\mathscr{F}^x}}{\partial x^k \partial x_l} dx_l \\ &= -2\sum_{k,l} \frac{\partial^2 \widehat{\mathscr{F}^x}}{\partial x^k \partial x_l} dx_k \otimes dx^l \end{aligned} \tag{3.2.32}$$

On the other hand, by utilizing another Hesse potential, we have the following

$$
\begin{aligned}
g_{\mathcal{B}^0} &= \sum_k dx^k \otimes dy_k - dx_k \otimes dy^k \\
&\xlongequal{\text{by (3.2.26)}} \sum_k \sum_l \frac{\partial^2 \widehat{\mathscr{F}^y}}{\partial y_k \partial y_l} dy_l \otimes dy_k + \sum_k \sum_l \frac{\partial^2 \widehat{\mathscr{F}^y}}{\partial y^k \partial y_l} dy_l \otimes dy^k \\
&= 2 \sum_{k,l} \frac{\partial^2 \widehat{\mathscr{F}^y}}{\partial y^k \partial y_l} dy_k \otimes dy^l
\end{aligned} \tag{3.2.33}
$$

Naturally, we would like to view the affine coordinates (x_i, x^i) and (y^i, y_i) as defining two $\mathbb{Z}$-affine structures on $\mathcal{B}^0$ which are dual to each other, correspondingly, the two metrics (3.2.32) and (3.2.33) above, can be viewed as dual to each other in the sense that we ought to make it a little bit precise.

Denote by $\mathcal{B}^0_x$ the affine manifold in the $\mathbb{Z}$-affine structure with coordinates (x_i, x^i), while by $\mathcal{B}^0_y$ the affine manifold in the $\mathbb{Z}$-affine structure with coordinates (y_i, y^i). Then by the above computation, we see that $\mathcal{B}^0_x$ is endowed with Riemann metric $g^x_{\mathcal{B}^0}$ given by the Hessian of the function $-2\widehat{\mathscr{F}^x}$, while the metric $g^y_{\mathcal{B}^0}$ on $\mathcal{B}^0_y$ is given through the Hessian of the function $2\widehat{\mathscr{F}^y}$. We want to show that the metric $g^x_{\mathcal{B}^0}$ is dual to $g^y_{\mathcal{B}^0}$ in the sense that their potentials are Legendre dual to each other. Indeed, we have the following proposition.

Proposition 3.2.21. *The two potential functions of the metric, namely, $-2\widehat{\mathscr{F}^x}$ and $2\widehat{\mathscr{F}^y}$ are Legendre transform of each other.*

Proof.

$$
\begin{aligned}
\widehat{\widehat{\mathscr{F}^x}}(y_i, y^i) &= \sum_k \left(x^k y_k + x_k y^k\right) - \widehat{\mathscr{F}^x}(x_i, x^i) \\
&= \sum_k \left(x^k y_k + x_k y^k\right) - \left(\sum_k x_k y^k - \mathscr{F}(x^i, y^i)\right) \\
&= \sum_k x^k y_k + \mathscr{F}(x^i, y^i) = \widehat{-\mathscr{F}^y}(x^i, y^i)
\end{aligned}
$$

Similarly, we have that

$$
\begin{aligned}
\widehat{\widehat{\mathscr{F}^y}}(x_i, x^i) &= \sum_k \left(x^k y_k + x_k y^k\right) - \widehat{\mathscr{F}^y}(x_i, x^i) \\
&= \sum_k \left(x^k y_k + x_k y^k\right) - \left(\sum_k x^k y_k - \mathscr{F}(x^i, y^i)\right) \\
&= \sum_k x_k y^k + \mathscr{F}(x^i, y^i) = \widehat{-\mathscr{F}^x}(x^i, y^i)
\end{aligned}
$$

□

Consequently, we see that the dual affine structures induced by (x^i, x_i) and (y^i, y_i), together with the corresponding dual metrics $g^x_{\mathcal{B}^0}$ and $g^y_{\mathcal{B}^0}$ make the base $\mathcal{B}^0$ a Monge-Ampère manifold in two ways which are dual to each other. This confirms the proposition 3.2.19 in our case.

Next, we rewrite the attractor flow equation introduced in section 3.2.1 in terms of two Monge-Ampère structures discussed above. First, recall that for the charge vector γ at $b_0 \in \mathcal{B}$ given by

$$\gamma = (q_k, g^k) = \sum_k q_k \alpha^k(\mathbf{u}) + \sum_k g^k \beta_k(\mathbf{u}) \in H_1(\pi^{-1}(b), \mathbb{Z})$$

It has central charge

$$Z_{b_0}(\gamma) = \sum_k q_k a^k(\mathbf{u}) + \sum_k g^k a_{D,k}(\mathbf{u})$$

The attractor flow line passing through b_0, in the direction γ is given by (see formulae (3.2.8) and (3.2.9))

$$Im\left(e^{-i\theta} Z_{b(t)}(\gamma)\right) = Im\left(\sum_k q_k\left(e^{-i\theta} a^k\right) + \sum_k g^k\left(e^{-i\theta} a_{D,k}\right)\right) = 0$$

where $\theta = Arg\, Z_{b_0}(\gamma)$. By using the real affine coordinates (y^i, y^i), the attractor flow equation can be rewritten as

$$\sum_k q_k y^k(t) + \sum_k g^k y_k(t) = 0 \tag{3.2.34}$$

which by using the relation (3.2.29) is equivalent to the following

$$\sum_k q_k \frac{\partial \widehat{\mathscr{F}^x}}{\partial x_k} - \sum_k g^k \frac{\partial \widehat{\mathscr{F}^x}}{\partial x^k} = 0 \tag{3.2.35}$$

Definition 3.2.12. *For a function $f(x_i, x^i)$ in affine structure given by (x_i, x^i), define its* ***gradient*** *to be the following vector field*

$$\widetilde{\nabla} f := \left(\frac{\partial f}{\partial x_k}, -\frac{\partial f}{\partial x^k}\right) \tag{3.2.36}$$

Then the equation (3.2.35) above can be written as

$$\widetilde{\nabla}\widehat{\mathscr{F}^x} \cdot \gamma = 0 \tag{3.2.37}$$

Consequently, we have the following proposition:

Proposition 3.2.22. *The attractor flow lines are equivalent to the flow lines on which the gradient of the Hesse potential $\widehat{\mathscr{F}^x}$ vanishes.*

Consider the dual of the attractor flow equation (3.2.27) above, namely

$$\widetilde{\nabla}\widehat{\mathscr{F}^y} \cdot \gamma = 0 \tag{3.2.38}$$

By duality, this equation should be interpreted as certain attractor flow called the **dual attractor flow**.

To this end, let us take the real part of the rotated central charge function $e^{-i\theta}Z_{\mathbf{u}(t)}(\gamma)$, which gives

$$\begin{aligned} Re\left(e^{-i\theta}Z_{\mathbf{u}(t)}(\gamma)\right) &= Re\left(\sum_k q_k\ \left(e^{-i\theta}a^k\right) + \sum_k g^k\ \left(e^{-i\theta}a_{D,k}\right)\right) \\ &= \sum_k q_k\, x^k(t) + \sum_k g^k\, x_k(t) \end{aligned}$$

By using the relations (3.2.26) and the definition 3.1.12., the right hand side of the above equation can be expressed as

$$\sum_k q_k \frac{\partial \widehat{\mathscr{F}^y}}{\partial y_k} - \sum_k g^k \frac{\partial \widehat{\mathscr{F}^y}}{\partial y^k} =: \widetilde{\nabla}\widehat{\mathscr{F}^y} \cdot \gamma \tag{3.2.39}$$

Consequently, we get the following relation between the central charge and Hesse potential

$$Re\left(e^{-i\theta}\, Z(\gamma)\right) = \widetilde{\nabla}\widehat{\mathscr{F}^y} \cdot \gamma \tag{3.2.40}$$

Equation above defines the so-called **Hesse flow** in [Van12]. However, as

$$Re\left(e^{-i\theta}\, Z(\gamma)\right) = |Z(\gamma)|$$

along the attractor flow line, it can not vanish identically. To make (3.2.40) compatible with (3.2.38), we observe that

$$Re\left(e^{-i(\theta+\frac{\pi}{2})}\, Z(\gamma)\right) \equiv 0$$

and this leads to the following

Proposition 3.2.23. *The attractor flow on the $\mathbb{Z}$-affine manifold $\mathcal{B}_\theta$ given by*

$$\widetilde{\nabla}\widehat{\mathscr{F}^x} \cdot \gamma = 0$$

corresponds to the dual attractor flow

$$\widetilde{\nabla}\widehat{\mathscr{F}^y} \cdot \gamma = 0$$

on the dual $\mathbb{Z}$-affine manifold $\mathcal{B}_{\theta+\frac{\pi}{2}}$.

Similarly, we introduce the notion of the **dual Hesse flow** described by the following equation:

$$Im\left(e^{-i\theta} Z(\gamma)\right) = \widetilde{\nabla}\widehat{\mathscr{F}^x} \cdot \gamma \tag{3.2.41}$$

which is seen to be nothing but the attractor flow equation when

$$\theta = Arg\, Z(\gamma)$$

Consequently, both the Hesse flow and its dual has very simple origin. Namely, by taking the real and imaginary of the rotated central charge function $e^{-i\theta}Z(\gamma)$ (i.e., the rotated special Kähler coordinate on $\mathcal{B}$), and then write them down by using the adapted $\mathbb{Z}$-affine coordinates on $\mathcal{B}$, we will get the Hesse flow and dual Hesse flow respectively.

3.2.5 Relation between the attractor flow and the Hesse flow

Motivated by the above discussion, we can treat both attractor flow and Hesse flow in the same framework as below:

We start with central charge function $Z(\gamma)$ associated to the charge γ, and use the $\mathbb{Z}$-affine coordinates on $\mathcal{B}$, we see that taking the real part of the central charge function gives us the Hesse flow

$$Re\left(e^{-i\theta}Z(\gamma)\right) = \widetilde{\nabla}\widehat{\mathscr{F}^y}\cdot\gamma \tag{3.2.42}$$

while taking the imaginary part gives us the dual Hesse flow

$$Im\left(e^{-i\theta}Z(\gamma)\right) = \widetilde{\nabla}\widehat{\mathscr{F}^x}\cdot\gamma \tag{3.2.43}$$

Next, denote $\theta = Arg\,(Z(\gamma))$. We see that in the $\mathbb{Z}$-affine structure $\mathcal{B}_\theta$, the dual Hesse flow (3.2.43) specializes into the attractor flow (3.2.27); while in the $\mathbb{Z}$-affine structure $\mathcal{B}_{\theta+\frac{\pi}{2}}$, the Hesse flow (3.2.42) becomes the dual attractor flow (3.2.43).

Remark 3.2.18. *It is known that the $\mathbb{Z}$-affine structures $\mathcal{B}_\theta$ and $\mathcal{B}_{\theta+\frac{\pi}{2}}$ correspond, on the mirror symmetry side, to the symplectic and complex affine structures respectively (for example, see [Lin17]). And this duality, heuristically speaking, manifests in the context of the above discussion as the operations of taking the imaginary and the real part, which, intuitively speaking, further corresponds to the operations of separating the "vertical" and the "horizontal" direction.*

Chapter 4

The Geometry of the Seiberg-Witten integrable system

The ideal of Seiberg-Witten integrable system ("SW integrable system" for short) first appeared in their seminar paper [SW94] by Seiberg and Witten in their efforts toward producing the exact solution to the $N = 2$ supersymmetric gauge field theory with gauge group $SU(2)$ by using the so-called Seiberg-Witten (hyper) elliptic curves. There is a short introduction in appendix A. The connection of Seiberg-Witten solution to the complex integrable system was further explored by Donagi and Witten in [DW96].

In section 4.1, we will give a detailed mathematical exposition of the geometry of SW integrable system with gauge group $SU(n)$. We will see that it fits into the general framework of complex integrable system as had been discussed in chapter 3. I will put together many results that had been scattered in the physics literature and try to present them in a mathematically self-contained manner. Occasionally, I also formulate certain propositions and prove them to fill the details in the existing literature.

In particularly, in subsection 4.1.2, we will see how to associate certain hyper-elliptic curves (the so called SW curves) with a complex integrable system. And in subsection 4.1.4, we will see that there is actually a high dimensional story lying behind, namely, the Seiberg-Witten geometry could be obtained from certain limit of the Calabi-Yau geometry. This, on the physics side, sheds lights on the desired scenery that the natural habitat of the Seiberg-Witten theory should be located in the 10 dim string theory scenery. Finally in subsection 4.1.5, we will view SW integrable system from more general Hitchin system perspective. And from this perspective, the split attractor flows on the base of the SW integrable system should correspond to the spectral networks living in the base of the Hitchin system. We will discuss this relation.

Then in section 4.2, we will focus on the computation of the vanishing cycles and the corresponding monodromies associated to the the discriminant locus of the base of the SW integrable system. These information will be used in chapter five in connection with the WCS for the $SU(2)$ and $SU(3)$ SW integrable system. Indeed, they will serve as the initial data (definition 3.2.7) of the corresponding WCS.

The materials and the computations in this chapter are based on the results in physics. The main references used in this chapter are [Kle+95][AF][Kle97][KTL95][Kle+94][AD95].

4.1 Geometry of SW integrable system for $SU(n)$

We review in this section the geometry of the SW integrable system for $SU(n)$ case. Along the way, we introduce the terminologies and the backgrounds for later use. Our exposition follows closely [Ler98]. For more details, the reader should consult the papers [Kle97],[Kle+94],[Kle+95],[AF] and the references contained therein. The theory to be discussed in this chapter applies for all A-D-E type simply laced group. However, for simplicity and for the purpose of this thesis, we just review the A_{n-1} case. For more general cases, the readers are advised to read the listed references above for further information.

4.1.1 Classical moduli space and A_{n-1} simple singularities

Denote by $\mathfrak{su}(\mathfrak{n})$ the Lie algebra associated to $SU(n)$, and by $\mathfrak{h}_n$ the Cartan sub-algebra of $\mathfrak{su}(\mathfrak{n})$.

Let $\{\mathbf{H}_i\}_{i=1}^{n-1}$ be the generators of the Cartan subalgebra $\mathfrak{h}_n$. Also, denote by Φ the set of roots of $SU(n)$, and by Φ_0 and Φ_+ the set of simple roots and positive roots of $SU(n)$ respectively.

Working in the fundamental matrix representation of $SU(n)$, we have that the basis for $SU(n)$ is given by the matrices $\mathbb{E}_{ij}$, where its kj-th entry is given by

$$\mathbb{E}_{ij}^{(kl)} = \delta_k^i \delta_l^j$$

Under this basis, a convenient choice of generators for Cartan-subalgebra is given by

$$\mathbf{H}_i = \mathbb{E}_{i\,i} - \mathbb{E}_{i+1\,i+1}, \ \ i = 1 \cdots n-1.$$

The dual basis $\mathbf{H}_i^*$ is defined through the relation $Tr(\mathbf{H}_i^* \mathbf{H}_j) = \delta_{ij}$, from which we deduce that

$$\mathbf{H}_i^* = \sum_{j=1}^{i} \mathbb{E}_{jj} - \frac{1}{n}\mathbb{I} \quad i = 1 \cdots n-1$$

where $\mathbb{I}$ denotes the identity $n \times n$ matrix.

The vectors $\alpha_i := diag(\mathbf{H}_i)$ form a basis for the root lattice Λ_R, while $\alpha_i^* := diag(\mathbf{H}_i^*)$ generate the weight lattice Λ_W.

Pick an element in $\mathfrak{h}_n$ given as follows

$$\phi = \mathbf{a} \cdot \mathbf{H} := \sum_{i=1}^{n-1} a_i \, \mathbf{H}_i$$

where a_i s are complex numbers. Following [Pet12], we will call them *electric coordinates.*

Let e_{λ_i} $i = 1 \cdots n$ denote the eigenvalues of the matrix ϕ, where λ_i labels the weights of the n-dimensional fundamental representation.

The Weyl group $\mathcal{W}_n$ of $SU(n)$ is given by the symmetric group on n letters S_n, which acts by permutation on the set of weights $\{\lambda_i\}$. Notice that the electric coordinates a_i are not invariant under the Weyl group action, so we need to construct out of them the Weyl invariant Casimirs $u_i(a)$. The construction goes as follows:

Consider the characteristic polynomial

$$det(x\mathbf{I} - \phi) = \prod_{i=1}^{n}(x - e_{\lambda_i}(a)) = x^n - \sum_{k=0}^{n-2} u_{k+2}(a)\, x^{n-2-k} \tag{4.1.1}$$

Note that $u_1 \equiv 0$ since $u_1 = tr(\phi) = 0$ by $\mathfrak{su}(\mathfrak{n})$ condition.

We see that

$$\begin{aligned} u_k &= (-1)^{k+1} \sum_{j_1 \neq j_2 \neq \cdots \neq j_k} e_{\lambda_{j1}} e_{\lambda_{j2}} \cdots e_{\lambda_{jk}}(a) \\ &\equiv \frac{1}{k} Tr(\phi^k) + \text{product of lower order} \end{aligned} \tag{4.1.2}$$

As u_ks are symmetric polynomials, they are invariant under S_n.

Definition 4.1.1. *The* ***electric charge vector*** $\mathbf{q} = (q_1 \cdots q_{n-1}) \in \mathbb{Z}^{n-1}$ *is defined to be the vectors on the root lattice* Λ_R*, while the* ***electric central charge*** *function* $Z(\mathbf{q})$ *is defined by* $Z(\mathbf{q}) = \mathbf{q}\cdot\mathbf{a}$*, where* $\mathbf{a}$ *is the electric coordinate defined above. Then the* ***electric mass function*** *for the charge vector* $\mathbf{q}$ *is defined through the relation:* $m^2 = |Z|^2$.

Remark 4.1.1. *As the root lattice* Λ_R *is spanned by* $\alpha_i = diag(\mathbf{H}_i)$*, we see that* $\mathbf{q}$ *can be written as linear combination of* α_i*s, i.e.,* $\mathbf{q} = \sum_{i=1}^{n-1} q_i\alpha_i$.

Remark 4.1.2. *Similar to the definition of electric charge vector, we can define* ***magnetic charge vector*** $\mathbf{g} = (g^1 \cdots g^{n-1}) \in \mathbb{Z}^{n-1}$ *to be the vectors on the weight lattice* Λ_w*, i.e.,* $\mathbf{g} = \sum_{i=1}^{n-1} g^i\alpha_i^*$.

Definition 4.1.2. *The* ***Classical Moduli Space*** $\mathscr{M}_c$ *associated to* $SU(n)$ *is defined to be the parameter space of the complexified Cartan sub-algebra modulo the Weyl group action.*

From the discussion above, we see that $\mathscr{M}_c$ parameterizes all possible eigenvalues $\{e_{\lambda_1} \cdots e_{\lambda_n}\}$ of the matrices ϕ in the Cartan sub-algebra, since $\sum e_{\lambda_i} = 0$, we conclude that

$$\mathscr{M}_c \cong \mathbb{C}^{n-1}/S_n$$

with invariant coordinates (under the Weyl group action) given by $\{u_k\}_{k=2}^{n}$.

By definition, we infer that $Z(\lambda_i^\vee) = e_{\lambda_i}$, where $\lambda_i^\vee$ is the root corresponding to the weight λ_i. Uder the identification between roots and weights, we can simply write this as:

$$Z(\lambda_i) = e_{\lambda_i} \tag{4.1.3}$$

From this, we know that there exists root $\alpha_{i,j} \in \Lambda_R$ such that

$$Z(\alpha_{i,j}) = e_{\lambda_i} - e_{\lambda_j} \tag{4.1.4}$$

for $i \neq j$. Apparently, $\alpha_{i,j}$ is the root dual to $\lambda_i - \lambda_j$.

In terms of Z, and the equation 4.1.2, we get

$$u_2(a) = \frac{1}{2n} \sum_{\alpha \in \Phi_+} (Z(\alpha))^2 = \frac{1}{2} \mathbf{a}^t \cdot C_{A_{n-1}} \cdot \mathbf{a}$$

where $C_{A_{n-1}}$ is the Cartan matrix of $SU(n)$. As well as the following

$$u_n(a) = (-1)^{n+1} \prod_{\text{fun.rep weights}\lambda} Z(\lambda)$$

Some Examples

$SU(2)$ case: In this case, the rank is one, the Cartan sub-algebra is generated by

$$\mathbf{H}_1 = \mathbb{E}_{1,1} - \mathbb{E}_{2,2} = diag(1, -1).$$

Thus, for $\phi = a\mathbf{H}_1 = diag(a, -a)$, the eigenvalues are given by $e_1 = a,\ \ e_2 = -a$. From this we see that the invariant coordinate (under $\mathbb{Z}_2$ action which sends a to $-a$) is given by

$$u = a^2 = \frac{1}{2} Tr(\phi^2) \tag{4.1.5}$$

Besides, we have that $Z(1) = e_1 = a$ and $Z(-1) = e_2 = -a$.

We now proceed to the more involved $SU(3)$ case.

In this case, the rank is two, and the Cartan sub-algebra is spanned by $\mathbf{H}_1$, and $\mathbf{H}_2$ given as:

$$\mathbf{H}_1 = \mathbb{E}_{1,1} - \mathbb{E}_{2,2} = diag(1, -1, 0)$$
$$\mathbf{H}_2 = \mathbb{E}_{2,2} - \mathbb{E}_{3,3} = diag(0, 1, -1)$$

Given $\phi = a_1\mathbf{H}_1 + a_2\mathbf{H}_2 = diag(a_1, a_2 - a_1, -a_2)$, we can identify

$$e_1 = a_1,\ \ e_2 = -a_2,\ \ e_3 = a_2 - a_1 \tag{4.1.6}$$

Through the formula 4.1.2, we compute that

$$u(a_1, a_2) \equiv u_2 = a_1^2 + a_2^2 - a_1 a_2$$
$$v(a_1, a_2) \equiv u_3 = a_1 a_2 (a_1 - a_2)$$

Remark 4.1.3. *From the above equation, we can solve a_1, a_2 in terms of u, v ([GR95])*

$$a_1(u, v) = \xi_+ + \xi_-$$
$$a_2(u, v) = e^{-2\pi i/6} \xi_+ + e^{2\pi i/6} \xi_- \tag{4.1.7}$$

where

$$\xi_\pm(u, v) = 2^{-1/3} \sqrt[3]{v \pm \sqrt{v^2 - \frac{4}{27} u^3}}$$

For the central charge function, we have the following

$$
\begin{aligned}
Z(1,0) &= a_1 = e_1 \\
Z(0,-1) &= -a_2 = e_2 \\
Z(-1,1) &= a_2 - a_1 = e_3
\end{aligned}
\tag{4.1.8}
$$

and

$$
\begin{aligned}
Z_1 \equiv Z(2,-1) &= 2a_1 - a_2 = e_1 - e_3 \\
Z_2 \equiv Z(-1,2) &= 2a_2 - a_1 = e_3 - e_2 \\
Z_3 \equiv Z(1,1) &= a_1 + a_2 = e_1 - e_2
\end{aligned}
\tag{4.1.9}
$$

Geometric Reformulation

Notice that the characteristic polynomial (4.1.1) is nothing but the *simple singularity* associated to the group $SU(n)$. The details of the theory relevent to this story can be found in Arnold'classical book on singularity theory [AGV].

Denote by

$$
\mathcal{W}_{A_{n-1}}(x,\mathbf{u}) \equiv x^n - \sum_{k=0}^{n-2} u_{k+2}(\mathbf{u})\, x^{n-2-k}
$$

the equation of simple singularities associated with $SU(n)$. And denote by $\mathscr{M}$ the variety associated to $\mathcal{W}_{A_{n-1}}$, then we see from (4.1.1) that the classical moduli space can be described as

$$
\mathscr{M}_c = \mathscr{M}/S_n = \big\{x : \mathcal{W}_{A_{n-1}}(x,\mathbf{u}) = 0\big\}/S_n = \{e_{\lambda_i}(\mathbf{u})\}/S_n = \{u_i\} \tag{4.1.10}
$$

Note that the classical moduli space $\mathscr{M}_c$ is singular precisely when two eigenvalues collide each other, i.e., $e_{\lambda_i} = e_{\lambda_j}$ for $i \neq j$. Consequently the singular locus (classical discriminant locus) Δ_c in $\mathscr{M}_c$ can be encoded by the discriminant of the polynomial $\mathcal{W}_{A_{n-1}}$. The discriminant is given by

$$
\delta(\mathcal{W}_{A_{n-1}}) := \prod_{i<j}^{n} \big(e_{\lambda_i}(\mathbf{u}) - e_{\lambda_j}(\mathbf{u})\big)^2 = \prod_{\alpha\in\Phi_+} (e_\alpha(\mathbf{u}))^2 \tag{4.1.11}
$$

Then the classical discriminant locus is given by $\Delta_c \equiv \big\{u_k : \delta(\mathcal{W}_{A_{n-1}}) = 0\big\}$, and denote by

$$
\mathscr{M}_c^0 := \mathscr{M}_c \setminus \Delta_c \tag{4.1.12}
$$

the smooth part of the classical moduli space.

Remark 4.1.4. *In singularity theory, the discriminant locus Δ_c is also called the **bifurcation sets** of the type A_{n-1} simple singularities.*

Remark 4.1.5. *The classical discriminant locus Δ_c is given by the intersection of complex codimension one hypersurfaces in $\mathscr{M}_c$. On each such hypersurface, $Z(\pm\boldsymbol{\alpha}) = 0$, for some pair of roots $\pm\boldsymbol{\alpha}$.*

In $SU(2)$ case, $\delta_c = 4a^2 = 4u$, thus the discriminant locus is just a point $\{u = 0\}$.

For $SU(3)$, by using the relation (4.1.6) and (4.1.7), we compute that

$$\begin{aligned}\delta_c = (a_1 + a_2)^2(2a_1 - a_2)^2(a_1 - 2a_2)^2 = (a_1 + a_2)^2 \left[2(a_1^2 + a_2^2) - 5a_1a_2\right]^2 \\ = (u + 3a_1a_2)(2u - 3a_1a_2)^2 = 4u^3 + 27(a_1a_2)^3 - 27u(a_1a_2)^2 \\ = 4u^3 + 27(a_1a_2)^2(a_1a_2 - u) = 4u^3 + 27(a_1a_2)^2 \left[a_1a_2 - (a_1^2 + a_2^2 - a_1a_2)\right] \\ = 4u^3 - 27(a_1a_2)^2(a_1 - a_2)^2 = 4u^3 - 27v^2\end{aligned}$$

So we get that

$$\delta_c = 4u^3 - 27v^2 \tag{4.1.13}$$

From it we see that the bifurcation set for A_2 singularity is the complex version of the cusp curve. See the picture below for the illustration of the classical moduli space in $SU(2)$ and $SU(3)$ cases.

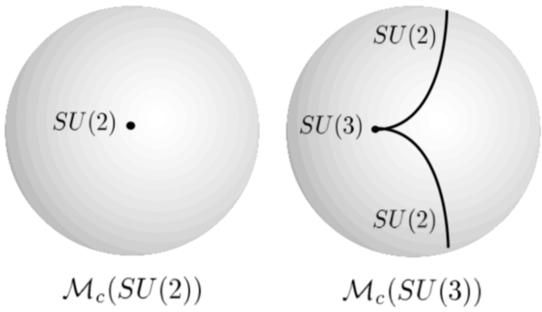

Figure 4.1: Classical moduli space in $SU(2)$ and $SU(3)$ cases, figure taken from [Ler98]

The space $\mathscr{M}$ depends on $\{u_i\}$, and it becomes singular at Δ_c in the sense that certain homology basis shrink to zero size as the moduli u approaches to the component of the discriminant locus.

Indeed, by definition, we have $\mathscr{M} = \{e_{\lambda_i}\}$, and Δ_c describes the location where e_{λ_i} collide with each other. In particularly, the basis $\left\{e_{\lambda_i} - e_{\lambda_j}\right\}_{i<j, i\neq j}$ vanish on Δ_c.

Denote by $\nu_{\alpha_{i,j}} := \left\{e_{\lambda_i} - e_{\lambda_j}\right\}$, i.e., the **vanishing** 0**- cycle** corresponding to the root $\alpha_{i,j}$.

It is clear that these cycles form a basis of $H_0(\mathscr{M}, \mathbb{Z})$, which can be identified with the root lattice Λ_R, i.e., $H_0(\mathscr{M}, \mathbb{Z}) \cong \Lambda_R$.

Formula (4.1.4) implies that for $u \in \Delta_c$,

$$Z(\alpha_{i,j}) = e_{\lambda_i} - e_{\lambda_j} \equiv 0$$

from which we see that the mass of the root $\alpha_{i,j}$ vanishes (c.f., definition 4.1.1).

We also have the following

$$\nu_{\alpha_i} \circ \nu_{\alpha_j} = \langle \alpha_i, \alpha_j \rangle \tag{4.1.14}$$

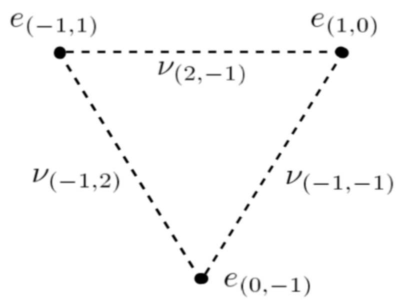

Figure 4.2: Weight diagram and vanishing cycles in $SU(3)$ case, figure taken from [Ler98]

where the left hand side is the intersection product of two vanishing cycle, while the right hand side is the inner product between root vectors.

Besides, the monodromy associated to the vanishing cycle ν is given by the Picard-Lefschetz formula

$$M_{\nu_{\alpha i}} : \gamma \longmapsto \gamma - (\gamma \circ \nu_{\alpha_i})\, \nu_{\alpha_i} \tag{4.1.15}$$

which can also be expressed in terms of the matrix acting on the weight space as

$$\begin{aligned} M_{\alpha_i} &= id - \alpha_i \otimes w_i \\ &= id - 2\,\frac{\langle \alpha_i, \cdot \rangle}{\langle \alpha_i, \alpha_i \rangle}\, \alpha_i \end{aligned} \tag{4.1.16}$$

where α_i and w_i are the simple roots and fundamental weights in the Dynkin basis.

Notice that (4.1.16) is nothing but the *fundamental Weyl reflection* associated with the simple root α_i.

Remark 4.1.6. *In the next section, we will see that in the quantum case, the above geometric picture is generalized to the case that the variety $\mathscr{M}$ is replaced by certain Riemann surface.*

4.1.2 Quantum moduli space and SW integrable system

In order to construct the Seiberg-Witten integrable system, we first introduce the following curve associated to the group $SU(n)$:

$$\mathcal{C}_{\mathbf{u}} : y^2 = P_n(x) = \left(\mathcal{W}_{A_{n-1}}(x, \mathbf{u})\right)^2 - \Lambda^{2n}, \quad x, y \in \mathbb{C} \tag{4.1.17}$$

Here, Λ is understood as a positive real parameter.

Since the degree of $P_n(x)$ equals to $2n$, which is larger or equal to 4 for $n \geq 2$, we see that the curve $\mathcal{C}$ is an hyper elliptic curve with genus $g = n - 1$.

The curve $\mathcal{C}_{\mathbf{u}}$ is called the **Seiberg-Witten curve** (SW curve for short) associated to $SU(n)$.

Geometrically, the curve $\mathcal{C}_u$ for each $\mathbf{u} \in \mathbb{C}^{n-1}$ can be described as a double covering of the complex plane $\mathbb{C}$, branched over the following discriminate locus Δ_Λ, which we call it *quantum discriminant locus* (in comparison with the classical discriminate locus defined previously).

Rewrite the right hand side of equation (4.1.17) as

$$P_n(x) = P_{n+}(x) \times P_{n-}(x) = \left(\mathcal{W}_{A_{n-1}}(x, \mathbf{u}) + \Lambda^n\right) \times \left(\mathcal{W}_{A_{n-1}}(x, \mathbf{u}) - \Lambda^n\right)$$

Denote by $e_i^{\pm}$ the roots of $P_{n\pm}$ respectively, then

$$P_n(x) = \prod_{i=1}^{n} \left(x - e_i^+\right)\left(x - e_i^-\right)$$

By the form of the polynomial $\mathcal{W}_{A_{n-1}}$ (see equation (4.1.1)), we see that

$$P_{n\pm} = \mathcal{W}_{A_{n-1}}(x; u_2, \cdots, u_{n-1}, u_n \mp \Lambda^n) \tag{4.1.18}$$

Notice that $P_{n+}(x) - P_{n-}(x) = 2\Lambda^n$, and $\Lambda \neq 0$, we see that the roots e_i^+ can not coincide with e_i^-. As a consequence, the *quantum determinant* δ_Λ can be computed as follows:

$$\delta_\Lambda = \delta(P_n) := \prod_{i<j} \left(e_i^{\pm} - e_j^{\pm}\right)^2 = \prod_{i<j} \left(e_i^+ - e_j^+\right)^2 \prod_{i<j} \left(e_i^- - e_j^-\right)^2 \prod_{i<j} \left(e_i^+ - e_j^-\right)^2$$

$$= \delta(P_{n+})\, \delta(P_{n-}) \prod_i \left(P_{n+}(e_i^-)\right)^2 = \delta(P_{n+})\, \delta(P_{n-})\, (2\Lambda^n)^{2n}$$

Thus, we have

$$\delta_\Lambda = 2^{2n}\Lambda^{2n^2}\delta(P_{n+})\, \delta(P_{n-}) \tag{4.1.19}$$

From this, we see that the *quantum discriminant locus* Δ_Λ, defined as the zero locus of the polynomial δ_λ, is given by the union of two hyper-surfaces which are the varieties associated to $\delta(P_{n\pm})$ respectively, i.e.,

$$\Delta_\Lambda = \Delta_+ \cup \Delta_- \tag{4.1.20}$$

where $\Delta_{\pm}$ denote the varieties associated to $\delta(P_{n\pm})$ respectively.

Notice also that from equation (4.1.19), we have that

$$\delta(P_{n\pm}) = \delta(\mathcal{W}_{A_{n-1}})(x; u_2, \cdots, u_{n-1}, u_n \mp \Lambda^n)$$

from which we conclude that Δ_Λ looks like two copies of classical discriminant locus shifted in the top Casimir variables u_n.

Next, we compute the quantum discriminant for $n = 2$ and $n = 3$ case.

$SU(2)$ **case**:

The SW curve is given by

$$\mathcal{C} : \ y^2 = \mathcal{W}_{A_1}^2 - \Lambda^4 = (x^2 - u)^2 - \Lambda^4 \tag{4.1.21}$$

from which, as well as the formula (4.1.19), we get

$$\delta_\Lambda = 2^4 \Lambda^8 4(u - \Lambda^2) 4(u + \Lambda^2) = 16^2 \Lambda^8 (u^2 - \Lambda^4) \tag{4.1.22}$$

Thus the discriminant locus in this case reads

$$\Delta_\Lambda = \{u : \delta_\Lambda = 0\} = \left\{-\Lambda^2, +\Lambda^2\right\} \tag{4.1.23}$$

$SU(3)$ **case**:

The SW curve is given by

$$\mathcal{C} : \ y^2 = \mathcal{W}_{A_2}^2 - \Lambda^6 = (x^3 - ux - v)^2 - \Lambda^6 \tag{4.1.24}$$

from which, as well as the expression of classical discriminant (4.1.13), we get

$$\begin{aligned} \delta_\Lambda &= 2^6 \Lambda^{18} \delta\left(\mathcal{W}_{A_2}\right)(x, u, v + \Lambda^3) \delta\left(\mathcal{W}_{A_2}\right)(x, u, v - \Lambda^3) \\ &= 2^6 \Lambda^{18} \left(4u^3 - 27(v + \Lambda^3)^2\right) \left(4u^3 - 27(v - \Lambda^3)^2\right) \end{aligned} \tag{4.1.25}$$

It follows that the discriminant locus Δ_Λ in this case is given by the union of two (shifted in v variable) cusp singularity curves in $\mathbb{C}^2$ with non trivial intersection (see the figure 4.3 below for illustration).

Remark 4.1.7. *We see that as $\Lambda \to 0$, Δ_Λ degenerates into the classical discriminant locus as defined in (4.1.13).*

Definition 4.1.3. *The quantum moduli space $\mathscr{M}_q$ is defined to be the parameter space of the family of hyperelliptic curves $\mathcal{C}_u$ as given in (4.1.17).*

Remark 4.1.8. *Clearly, the quantum moduli space $\mathscr{M}_q$ has much more complicated structure than the classical moduli space $\mathscr{M}_c$. In physics, $\mathscr{M}_q$ arises from the description of the $N = 2, d = 4$ supersymmetric Yang-Mills theoy, while $\mathscr{M}_c$ emerges from the classical limit of the theory. Indeed, as the parameter Λ (which controls the quantum effect) approaches to zero, the curve $\mathcal{C}_u$ degenerates from a double cover of the complex plane to a single cover of it branched at the zeros of $\mathcal{W}_{A_{n-1}}$ which can be characterized as the zero locus of the polynomial itself modulo the action of symmetric group S_n. This is nothing but the description of classical moduli space $\mathscr{M}_c$ (see (4.1.11)).*

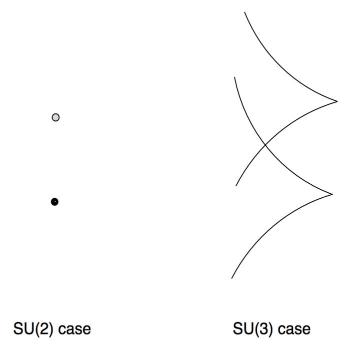

Figure 4.3: Quantum discriminant locus in $SU(2)$ and $SU(3)$ cases

Naively, as the parameters $\mathbf{u} \in \mathbb{C}^{n-1}$, $\mathscr{M}_q$ can be embedded into $\mathbf{C}^{n-1}$. When $\mathbf{u}$ approaches to the component of the quantum discriminant locus Δ_Λ, certain roots of the polynomial on the right hand of (4.1.17) coincide, and the curve $\mathcal{C}_{\mathbf{u}}$ becomes degenerate in the sense that certain middle dimensional homology cycles (called the **vanishing cycles**) on the curve shrinks to zero.

Thus, the points of Δ_Λ should be thought of as the singular points of the quantum moduli space. As will be shown, they play crucial role for the later discussion since they will play the role in determining the initial data of the WCS associated with the SW integrable system to be discussed in chapter five.

However, we will not concerned here with the structure of $\mathscr{M}_q$, we just need to associate to it a complex integrable system—-the so called the Seiberg-Witten integrable system, which will be used in producing the wall-crossing structure later.

SW integrable system associated to $SU(n)$

Recall from the definition 3.1.1, in order to specify a complex integrable system, we need have a holomorphic subjective map $\pi : (X, \omega^{2,0}) \to \mathcal{B}$ with generic fibers being Lagrangian sub-manifolds of X.

We show that from the quantum moduli space $\mathscr{M}_q$ given in definition 4.1.2, we can get a complex integrable system in the following manner.

Identifying the base $\mathcal{B}$ with $\mathbb{C}^{n-1}$. Then consider the torus fibration $\pi : X \to \mathcal{B}$ with fiber over $b \in \mathcal{B}$ being the Jacobian $Jac(\mathcal{C}_b)$ of the curve $\mathcal{C}_b$ (see formula (4.1.17)) sitting over b with local holomorphic coordinate near $b \in \mathcal{B}$ given by $\mathbf{u}$. Recall that the Jacobian of the curve $\mathcal{C}_b$ is defined as

$$Jac(\mathcal{C}_b) := H^0(\mathcal{C}_b, \Omega^1_{\mathcal{C}_b})/H_1(\mathcal{C}_b, \mathbb{Z}) \cong \mathbb{C}^g/\mathbb{Z}^{2g} \tag{4.1.26}$$

where $g = n - 1$ is the genus of the SW-curve $\mathcal{C}_b$, and $H^0(\mathcal{C}_b, \Omega^1_{\mathcal{C}_b})$ is the space of holomorphic one forms on the curve which is a g dimensional vector space.

Remark 4.1.9. *Note that the lattice $\Gamma_b := H_1(\mathcal{C}_b, \mathbb{Z}) \cong \mathbb{Z}^{2g}$ is embedded in $H^0(\mathcal{C}_b, \Omega^1_{\mathcal{C}_b})$ via the Abel-Jacobi map*

$$H_1(\mathcal{C}_b, \mathbb{Z}) \longrightarrow H^0(\mathcal{C}_b, \Omega^1_{\mathcal{C}_b})^\vee \tag{4.1.27}$$

which through the identification (3.1.22) and the isomorphism given in (3.1.15), can be identified with $H^1(\pi^{-1}(b), \mathbb{Z})$. Following the terminologies given in section 3.1.2, we call the lattice $\underline{\Gamma}_b$ the period lattice and the local system $\underline{\Gamma}$ formed by them the period net.

Definition 3.1.11 tells us that the local system of lattices $\underline{\Gamma}$ is also called the the local system of **charge lattice**.

Clearly, the fibration π is singular at the quantum discriminant locus Δ_Λ where the Jacobian variety degenerates due to the presence of vanishing cycles in $H_1(\mathcal{C}_b, \mathbb{Z})$ (See figure 4.4). Denote by $\mathcal{B}^0 := \mathcal{B}/\Delta_\Lambda$ the smooth part of the base,and consider the smooth torus fibration

$$\pi : X^0 \longrightarrow \mathcal{B}^0.$$

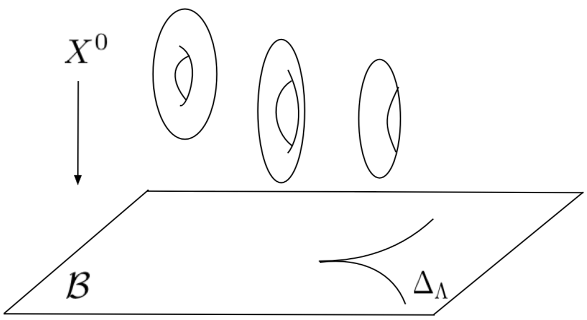

Figure 4.4: Torus fibration π over the base $\mathcal{B}$, torus fibers being degenerated over the discriminant locus Δ_Λ

We now show that the torus fibration π defined above is a complex integrable system of Seiberg-Witten type (see definition 3.1.13), which will be called the **SW integrable system** associated to $SU(n)$.

Let $\{\omega_1, \cdots, \omega_g\}$ be a set of basis of the space $H^0(\mathcal{C}_b, \Omega^1_{\mathcal{C}_b})$ of holomorphic differential one forms on the curve $\pi^{-1}(b) = \mathcal{C}_b$. Choosing symplectic basis $\{\alpha^i, \beta_j\}_{1\le i,j\le g}$ (see (3.1.34)) for the charge lattice $\underline{\Gamma}_b \cong H_1(\pi^{-1}(b), \mathbb{Z})$, i.e., the matrix of intersection product $\circ$ of these basis is the following symplectic matrix

$$\mathbf{J} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_g \\ -\mathbf{I}_g & \mathbf{0} \end{pmatrix} \tag{4.1.28}$$

Then the **period matrix** of the curve $\mathcal{C}_b$ is the $g \times 2g$ matrix $\mathbf{\Omega}$ defined as $\mathbf{\Omega} = (\mathbf{A}, \mathbf{B})$, where $\mathbf{A}$ and $\mathbf{B}$ are $g \times g$ matrix with entries given respectively by

$$\mathbf{A}_{ij} := \oint_{\alpha^i} \omega_j \qquad \mathbf{B}_{ij} := \oint_{\beta_i} \omega_j \tag{4.1.29}$$

The period matrix $\mathbf{\Omega}$ satisfies the following *Riemann bilinear relations*(see for example [Gri70]):

$$\text{Riemann } 1^{st} \text{ bilinear relation } \mathbf{\Omega J}^t\mathbf{\Omega} = 0$$
$$\text{Riemann } 2^{nd} \text{ bilinear relation } \sqrt{-1}\,\mathbf{\Omega J}^t\bar{\mathbf{\Omega}} > 0 \tag{4.1.30}$$

which follows from the following identities

$$\int_{\mathcal{C}_b} \omega_i \wedge \omega_j = 0 \qquad \sqrt{-1}\int_{\mathcal{C}_b} \omega_i \wedge \overline{\omega_j} > 0 \tag{4.1.31}$$

Definition 4.1.4. *Define the τ-matrix to be*

$$\tau = \mathbf{A}^{-1}\mathbf{B}$$

Riemann's 2^{nd} bilinear relation implies that the τ-matrix is symmetric and the imaginary part of it is positive definite, i.e., $Im\tau > 0$, which will be used to define the Riemann metric on $\mathcal{B}^0$.

Remark 4.1.10. *When $g = 1$ we have that*

$$\tau = \tau_{1,1} = \oint_\beta \frac{dx}{y} \Big/ \oint_\alpha \frac{dx}{y}$$

is the usual τ-function of elliptic curve which belongs to the upper half plane $\mathbb{H}$, i.e., $Im(\tau) > 0$.

As had been discussed in section 3.1.4, there exist angle coordinates $\{\theta^i, \theta_j\}_{1\le i,j\le g}$ on the torus fibers such that

$$\oint_{\alpha^i} d\theta_j = \delta^i_j \qquad \oint_{\beta_i} d\theta^j = \delta^i_j$$

Then introduce the canonical complex coordinates (see the formula (3.1.47)) on the fiber of π:

$$w_i := \theta_i + \sum_j \tau_{ij}\,\theta^j$$

together with the local holomorphic coordinates a^i on the base manifold $\mathcal{B}$, we can write the following holomorphic symplectic form (see the formula (3.1.48)) on the total space of fibration π as

$$\omega^{2,0} = \sum_i da^i \wedge dw_i \tag{4.1.32}$$

Remark 4.1.11. *By the formula (3.1.49) and the next proposition, we see that the complex integrable system defined by π is principally polarized.*

With these preparation, we can state the following proposition:

Proposition 4.1.1. *The torus fibration $\pi : X \to \mathcal{B}$ defines a complex integrable system of Seiberg-Witten type.*

Proof. By our construction, the generic fiber of π, being the Jacobian torus of the corresponding hyper-elliptic curve, is endowed with the angle coordinates θ_i (thus the complex coordinate w_i). This implies that the holomorphic symplectic form $\omega^{2,0}$ vanishes when restricted to the generic fiber. Consequently, the generic fiber of π is Lagrangian which completes the verification of the definition of complex integrable system.

To show that it is of Seiberg-Witten type, we need to construct a central charge function which is furnished by the following lemma. □

Lemma 4.1.2. *A central charge function Z as defined in definition 3.1.12 can be constructed for the complex integrable system $\pi : (X, \omega^{2,0}) \to \mathcal{B}$ constructed above.*

Proof. With the aid of the holomorphic symplectic form $\omega^{2,0}$ constructed above, the central charge function $Z \in \Gamma(\mathcal{B}^0, \underline{\Gamma} \otimes \mathscr{O}_{\mathcal{B}^0})$ is defined through (see definition 3.1.12)

$$dZ(\gamma) = \oint_\gamma \omega^{2,0} \tag{4.1.33}$$

which is clearly an additive map from $\underline{\Gamma}$ to $\mathbb{C}$. □

Choosing symplectic basis $\{\alpha^i, \beta_j\}_{1\le i,j\le g}$, then by (3.1.50), there exist locally defined holomorphic functions $\{a^i, a_{D,j})\}_{1\le i,j\le g}$ on $\mathcal{B}_0$ depending on any holomorphic coordinates $\mathbf{u}$, such that for

$$\gamma := (\mathbf{g}, \mathbf{q}) = (g^i, q_j) = \sum_i g^i \beta_i + \sum_j q_j \alpha^j \in \underline{\Gamma}_b \tag{4.1.34}$$

We have that

$$Z_b(\gamma) = \begin{pmatrix} \mathbf{g} & \mathbf{q} \end{pmatrix} \begin{pmatrix} \mathbf{a}_D \\ \mathbf{a} \end{pmatrix} = \sum_i g^i a_{D,i}(\mathbf{u}) + \sum_j q_j a^j(\mathbf{u}) \tag{4.1.35}$$

with $Z_b(\alpha^i) = a^i(\mathbf{u})$ and $Z_b(\beta_i) = a_{D,i}(\mathbf{u})$. Thus by the definition of the central charge function, we have that

$$da^i = \oint_{\alpha^i} \omega^{2,0} \qquad da_{D,j} = \oint_{\beta_j} \omega^{2,0} \tag{4.1.36}$$

The following proposition is pointed out in [NP12].

Proposition 4.1.3. *There exists Lagrangian sub-lattice $L_b \subset H_1(\pi^{-1}(b), \mathbb{Z})$.*

Proof. By remark 3.1.13, we see the symplectic form on $H_1(\pi^{-1}(b), \mathbb{Z})$, in terms of the angle coordinates, is given by

$$\sum_i d\theta^i \wedge d\theta_i$$

As the class $[d\theta_i]$ is dual to α^i, we see that the sub-lattice spanned by $\{\alpha^i\}_{1\le i\le g}$ is a Lagrangian sub lattice. □

By (3.1.38), their exits holomorphic function $\mathcal{F}$, called the prepotential, such that

$$a_{D,i} = \frac{\partial \mathcal{F}}{\partial a^i} \qquad i = 1, \cdots g = n - 1.$$

And recall that we have defined the symmetric matrix

$$\tau = (\tau_{ij}) = \left(\frac{\partial a_{D,i}}{\partial a^j}\right) = \left(\frac{\partial^2 \mathcal{F}}{\partial a^i \partial a^j}\right)$$

then we have the following:

Proposition 4.1.4. *The τ matrix coincide with that defined in definition 4.1.4 via the period matrix $\mathbf{\Omega}$.*

Proof. By formula (3.1.45), there exists action one form $\alpha_{\text{Liouville}}$ (also called Liouville one form) such that

$$d\alpha_{\text{Liouville}} = \omega^{2,0}$$

Using this, the relations (4.1.36) can be rewritten as

$$a^i = \oint_{\alpha^i} \alpha_{\text{Liouville}} \qquad a_{D,j} = \oint_{\beta_j} \alpha_{\text{Liouville}}$$

where $\alpha_{\text{Liouville}}$ is given by

$$\alpha_{\text{Liouville}} = \sum_i a_{D,i} \wedge d\theta^i + a^i \wedge d\theta_i$$

We will construct a meromorphic one form λ_{SW} with vanishing residues (the meaning of it will be explained in the lemma 4.1.6 below) in the cohomological class of $[\alpha_{\text{Liouville}}]$, which will be called the **Seiberg-Witten differential**, i.e., we should have

$$a^i = \oint_{\alpha^i} \lambda_{SW} \qquad a_{D,j} = \oint_{\beta_j} \lambda_{SW} \tag{4.1.37}$$

Then wee see that

$$\tau_{ij} = \frac{da_{D,i}}{da^j} = \sum_k \frac{\partial a_{D,i}}{\partial u_k} \Bigg/ \frac{\partial a^j}{\partial u_k} = \sum_k \oint_{\beta_i} \frac{\partial \lambda_{SW}}{\partial u_k} \Bigg/ \oint_{\alpha^i} \frac{\partial \lambda_{SW}}{\partial u_k}$$

from which we infer that in order for the τ matrix to match with that defined through the period matrix $\mathbf{\Omega}$, it is suffice for the following identities to be satisfied

$$\frac{\partial \lambda_{SW}}{\partial u_k} = f_k(\mathbf{u})\, \omega_k \quad k = 2, \cdots, n \tag{4.1.38}$$

for some holomorphic functions $f_k(\mathbf{u})$ on the base $\mathcal{B}^0$.

One sees that $\left\{ \frac{\partial \lambda_{SW}}{\partial u_k} \right\}_{1 \le k \le g}$ form a basis of $H^0\left(\mathcal{C}_{\mathbf{u}}, \Omega^1_{\mathcal{C}_u}\right)$ up to a scalar multiplication induced by holomorphic functions $f_k(\mathbf{u})$.

The simplest choice of scaling functions being $f_k(\mathbf{u}) \equiv 1$, $\forall k$. In this case, the SW differential form can be chosen to be

$$\lambda_{SW} = u_2\, \omega_2 + \cdots + u_n\, \omega_n \tag{4.1.39}$$

The basis of $H^0\left(\mathcal{C}_{\mathbf{u}}, \Omega^0_{\mathcal{C}_u}\right)$ can be chosen to be the following Abelian differentials of the first kind (see for example the book [ACG11]):

$$\omega_k = \frac{x^{n-k} dx}{y} \qquad k = 2, \cdots, n$$

Then (4.1.39) becomes

$$\begin{aligned} \lambda_{SW} &= \left(u_2\, x^{n-2} + u_3\, x^{n-3} + \cdots + u_n\right) \frac{dx}{y} \\ &= \left(x^n - \mathcal{W}_{A_{n-1}}(x, u_i)\right) \frac{dx}{y} \end{aligned} \tag{4.1.40}$$

$\square$

Following authors of the papers [Ler98],[Kle+94] e.t.c., we choose the following

Lemma 4.1.5. *The Seiberg-Witten differential can be chosen to be*

$$\lambda_{SW} = constant \cdot \left(\frac{\partial}{\partial x}\mathcal{W}_{A_{n-1}}(x, u_i)\right)\frac{x\,dx}{y} \tag{4.1.41}$$

up to addition of exact forms.

Proof. In the formula (4.1.38), we choose the scalar factors to be

$$f_k(\mathbf{u}) := -(n-k) \ \ k = 2, \cdots, n$$

which is being realized by letting λ_{SW} to be the linear combinations of ω_k's as follows

$$\lambda_{SW} = -(n-2)u_2\,\omega_2 - (n-3)u_3\,\omega_3 - \cdots - u_{n-1}\,\omega_{n-1} + nx^n$$
$$= \left(-(n-2)u_2\,x^{n-3} - (n-3)u_3\,x^{n-4} - \cdots - u_{n-1} + nx^{n-1}\right)\frac{x\,dx}{y}$$

which clearly equals

$$= \left(\frac{\partial}{\partial x}\mathcal{W}_{A_{n-1}}(x, u_i)\right)\frac{x\,dx}{y}$$

. ∎

Lemma 4.1.6. *The Seiberg-Witten differential λ_{SW} constructed above is residues free, thus when pairing with cycles, the value is invariant under continuous deformation of the cycle even if the cycle crosses the poles of λ_{SW}.*

Proof. By the proof of the lemma 4.1.5, we see that λ_{SW} is linear combination of basis $\{\omega_i\}$ of $H^1\left(\mathcal{C}_{\mathbf{u}}, \Omega^1_{\mathcal{C}_u}\right)$, which are holomorphic one forms, thus residues free. To prove the second part of the lemma, suppose the one cycle γ is being deformed continuously to another cycle γ' which encircles exactly one pole p of the differential λ_{SW}, then consider a small circle γ_p surrounding p with proper orientation such that $\gamma' = \gamma + \gamma_p$, then it is clear that

$$\int_{\gamma'} \lambda_{SW} = \int_{\gamma} \lambda_{SW} + \int_{\gamma_0} \lambda_{SW}$$
$$= \int_{\gamma} \lambda_{SW} + Res_p\left(\lambda_{SW}\right) = \int_{\gamma} \lambda_{SW}$$

∎

Before closing this subsection, let us compute the Seiberg-Witten differential form in $SU(2)$ and $SU(3)$ cases.

$\mathbf{SU(2)}$: By (4.1.21), we compute that up to a multiplication of a constant, we have that

$$\lambda_{SW} = \left(\frac{\partial}{\partial x}\mathcal{W}_{A_1}\right)\frac{x\,dx}{y} = \frac{\partial}{\partial x}(x^2 - u)\,\frac{x\,dx}{y}$$
$$= \frac{2x^2\,dx}{y} = \frac{2x^2\,dx}{\sqrt{(x^2-u)^2 - \Lambda^4}} \tag{4.1.42}$$

SU(3): By (4.1.24), we compute that up to a multiplication of a constant, we have that

$$\begin{aligned}\lambda_{SW} &= \left(\frac{\partial}{\partial x}\mathcal{W}_{A_2}\right)\frac{x\,dx}{y} = \frac{\partial}{\partial x}(x^3 - ux - v)\,\frac{x\,dx}{y} \\ &= \frac{(3x^2-u)\,x\,dx}{y} = \frac{(3x^2-u)\,x\,dx}{\sqrt{(x^3-ux-v)^2-\Lambda^6}}\end{aligned} \tag{4.1.43}$$

Using the Seiberg-Witten differential λ_{SW}, and by (4.1.34) and (4.1.36), we see that for the charge γ in (4.1.35), its central charge is given by

$$Z_b(\gamma) = \oint_\gamma \lambda_{SW} \tag{4.1.44}$$

Definition 4.1.5. *The mass $m(\gamma)$ of the charge γ is defined to be the absolute value of its central charge, i.e.,*

$$m_b(\gamma) := |Z_b(\gamma)| = \left|\oint_\gamma \lambda_{SW}\right| \tag{4.1.45}$$

Remark 4.1.12. *As had been mentioned in remark 4.1.8, when $\Lambda \to 0$, the quantum moduli space $\mathscr{M}_q$ collapses into the classical one $\mathscr{M}_c$, and the torus $\mathcal{C}_u$ degenerates into the level manifold $\mathscr{M}$ of the polynomial $\mathcal{W}_{A_{n-1}}$. This means that the homologically non-trivial one cycles γ on $\mathcal{C}_u$ would collapses into various contours encircling various zeros $e'_{\lambda_i}s$ of the polynomial $\mathcal{W}_{A_{n-1}}$.*

4.1.3 SW Curve as Weight Diagram Fibration

We make the transformation of variables as $y \mapsto z = y - \mathcal{W}_{A_{n-1}}(x, \mathbf{u})$ for the curve defined in the formula (4.1.17) as bellows

$$\mathcal{C}_{\mathbf{u}} : y^2 = P_n(x) = \mathcal{W}_{A_{n-1}}(x, \mathbf{u})^2 + \Lambda^{2n}, \ \ x, y \in \mathbb{C}$$

Then we get the following form of the curve in new variable z, still denoted by $\mathcal{C}_u$:

$$\mathcal{C}_{\mathbf{u}} : z + \frac{\Lambda^{2n}}{z} - 2\,\mathcal{W}_{A_{n-1}}(x, \mathbf{u}) = 0 \qquad x, z \in \mathbb{C} \tag{4.1.46}$$

This form of curves appear in physics literature (for example [MW96],[Mar99]) in connection with the Toda chain system [Tod12]. It also arises from the rigid limit of type II string theory on Calabi Yau 3-fold, (see for examples [KKV97],[LW98],[Kle+96],[Kle+96], [BS97],[May99]). The Seiberg-Witten curve written in this fashion also appears in the paper [KS08] (section 2.7) of Kontsevich and Soibelman.

Since the transformation is birational, $\mathcal{C}_{\mathbf{u}}$ is being transformed into the same curve. We first show what should be the form of Seiberg-Witten differential in terms of the new variables.

Proposition 4.1.7. *In terms of the new variable z, the Seiberg-Witten differential form λ_{SW} (see 4.1.42) (assuming the constant $\equiv 1$) can be written as*

$$\lambda_{SW} = -x\,\frac{dz}{z} \tag{4.1.47}$$

Proof. As $z = y + \mathcal{W}_{A_{n-1}}$, we have that

$$dz = dy - \frac{\partial \mathcal{W}_{A_{n-1}}}{\partial x}\,dx$$

then we compute as follows:

$$\begin{aligned} x\frac{dz}{z} &= \frac{x}{z}\left(dy + \frac{\partial \mathcal{W}_{A_{n-1}}}{\partial x}\,dx\right) = \frac{x}{z}\left(\frac{\mathcal{W}_{A_{n-1}}\frac{\partial \mathcal{W}_{A_{n-1}}}{\partial x}}{y}\,dx - \frac{\partial \mathcal{W}_{A_{n-1}}}{\partial x}\,dx\right) \\ &= \frac{x}{z}\,\frac{\partial \mathcal{W}_{A_{n-1}}}{\partial x}\,\frac{(\mathcal{W}_{A_{n-1}} - y)}{y}\,dx = \frac{x\,\frac{\partial \mathcal{W}_{A_{n-1}}}{\partial x}}{y - \mathcal{W}_{A_{n-1}}}\,\frac{\mathcal{W}_{A_{n-1}} - y}{y}\,dx \\ &= -\mathcal{W}_{A_{n-1}}\,\frac{x\,dx}{y} \end{aligned}$$

□

Compactifying $\mathbb{C}$ by adding point at ∞, we see that the equation (4.1.46) describes a fibration over the projective line $\mathbb{P}^1$ coordinated by z , with generic fibers being the *level manifold* associated to the singularity $\mathcal{W}_{A_{n-1}}(x, \mathbf{u})$. See figure 4.5 for the illustration in the rank one case.

Remark 4.1.13. *Consequently, we can roughly say that each fiber corresponds to a classical story as had been discussed in section 4.1.1, while its families give rise to the quantum deformation of the classical theory.*

Of course, the fibration is non-trivial only if the parameter $\Lambda \neq 0$, as when Λ approaches to zero, the fibration becomes

$$z + 2\,\mathcal{W}_{A_{n-1}}(x, \mathbf{u}) = 0$$

which gives nothing but the classical level surface

$$\mathscr{M} = \{x : \mathcal{W}_{A_{n-1}}(x, \mathbf{u}) = 0\}$$

This is the scenery to be expected from the remark above and the remark 4.1.8. Further more, we should also expect the central charge function as well as the mass function also degenerates into their classical counterparts when $\Lambda \to 0$. Indeed, we have the following result which appeared in [Mar99]

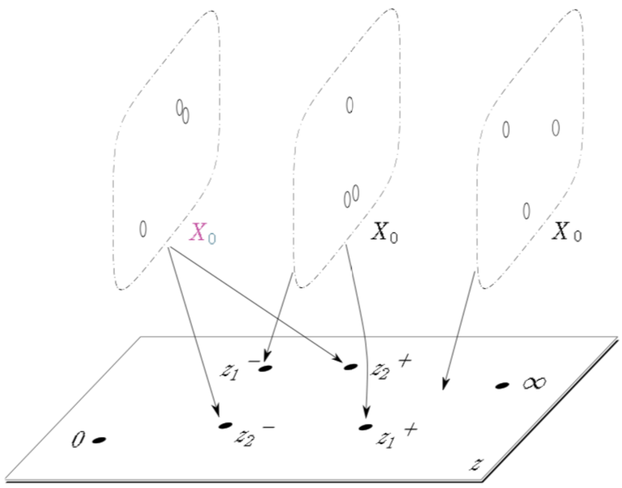

Figure 4.5: Illustration of weight diagram fibration in $SU(2)$ case, figure taken from [Ler98]

Proposition 4.1.8. *For the vanishing* 0 *cycle*

$$\nu_{i,j} = e_{\lambda_i} - e_{\lambda_i} \in H_0(\mathscr{M}, \mathbb{Z}) \cong \Lambda_R$$

corresponding to the root $\alpha_{i,j}$, we know from the formula (4.1.4) that

$$Z(\alpha_{i,j}) = e_{\lambda_i} - e_{\lambda_j}$$

By remark 4.1.12, we can lift a contour C_{ij} (∞- shaped) encircling the two roots e_{λ_i} and e_{λ_j} on the complex x-plane to a homology one cycle $\gamma_{ij} \in H_1(\mathcal{C}_{\mathbf{u}}, \mathbb{Z})$. Here, the moduli $\mathbf{u}$ lies near the components of Δ_Λ where the two roots collide. Then, the central charge of γ_{ij} equals to that of the root $\alpha_{i,j}$ given above, i.e.,

$$Z(\gamma_{ij}) = Z(\alpha_{i,j}) = e_{\lambda_i} - e_{\lambda_j} \tag{4.1.48}$$

Parallel result holds to the mass function.

Proof. When $\Lambda \to 0$, we see that

$$z \longrightarrow -2\mathcal{W}_{A_{n-1}}(x, \mathbf{u})$$

And by definition (see (4.1.44)), we compute as follows

$$Z(\gamma_{ij}) = \oint_{\gamma_{ij}} \lambda_{SW} = \oint_{\gamma_{ij}} x\,\frac{dz}{z} \xlongequal{\Lambda\to 0} \oint_{C_{ij}} x\,\frac{dz}{z} = \oint_{C_{ij}} x\,d\log z$$

Write $\mathcal{W}_{A_{n-1}}(x, \mathbf{u}) = \prod_k (x - e_{\lambda_k})$, we see that as $\Lambda \to 0$, we have that

$$Z(\gamma_{ij}) = \oint_{C_{ij}} x\,\frac{dz}{z} = \oint_{C_{ij}} x\,\frac{d\,(-2\,\mathcal{W}_{A_{n-1}})}{-2\,\mathcal{W}_{A_{n-1}}} = \oint_{C_{ij}} x\,d\log\left(\mathcal{W}_{A_{n-1}}\right) = \oint_{C_{ij}} x\,d\log\prod_k (x-e_{\lambda_k})$$

$$= \oint_{C_{ij}} x\,d\left(\sum_k \log(x - e_{\lambda_k})\right) = \oint_{C_{ij}} x \sum_k d\,\log(x - e_{\lambda_k}) = \oint_{C_{ij}} \sum_k \frac{x}{x - e_{\lambda_k}}\,dx$$

$$\xlongequal{C_{ij}\,is\,\infty\,shaped\,contour\,encircling\,two\,roots} Res_{e_{\lambda_i}}\left(\sum_k \frac{x}{x - e_{\lambda_k}}\right) - Res_{e_{\lambda_j}}\left(\sum_k \frac{x}{x - e_{\lambda_k}}\right)$$

$$= \lim_{x\to e_{\lambda_i}} (x - e_{\lambda_i}) \sum_k \frac{x}{x - e_{\lambda_k}} - \lim_{x\to e_{\lambda_j}} (x - e_{\lambda_j}) \sum_k \frac{x}{x - e_{\lambda_k}} = e_{\lambda_i} - e_{\lambda_j} = Z(\alpha_{i,j})$$

By equation (4.1.45), we see that

$$m_b(\gamma_{ij}) = |Z_b(\gamma_{ij})| = \left|\oint_{\gamma_{ij}} \lambda_{SW}\right| = |e_{\lambda_i} - e_{\lambda_j}| = |Z(\alpha_{i,j})|$$

□

By writing the SW differential form λ_{SW} in terms of z, we can easily prove the following proposition which justifies our expectation that the central charge of the charges represented by vanishing cycles should vanish.

Proposition 4.1.9. *The SW one form λ_{SW} is regular near the locus corresponding to vanishing cycles.*

Proof. Near the discriminant locus Δ_Λ, certain roots of $\mathcal{W}_{A_{n-1}}$ collide with each other, which means that whenever the complex moduli $\mathbf{u}$ hits Δ_Λ, we have

$$y \equiv 0, \quad and \quad \mathcal{W}_{A_{n-1}} = \pm\Lambda^n$$

Consequently, near the discriminant, we have

$$\lambda_{SW} = x\,d\log z = x\,d\log\left(y - \mathcal{W}_{A_{n-1}}\right) \sim x\,d\log(\mp\Lambda^n)$$

which is clearly regular. Thus, if ν is a vanishing cycle in the fiber over b, then

$$Z_b(\nu) = \oint_\nu \lambda_{SW} = 0$$

as expected. □

Geometric Reformulation

Given $z \in \mathbb{C}P^1$, there are generically n solutions to the equation (4.1.46) in x, thus $\mathcal{C}_{\mathbf{u}}$ could be realized as an n-sheeted cover of the projective line $\mathbb{C}P^1$. Denote these n solutions to the level surface equation of $\mathcal{W}_{A_{n-1}}(x, \mathbf{u})$ by

$$\{e_1(z, \mathbf{u}), \cdots, e_n(z, \mathbf{u})\}$$

Then each of the above n coordinates define the local coordinate on particular sheet of the covering, and each labeling a particular weight w_i of the fundamental representation. (Here $e_i(z, \mathbf{u})$ is just short for the previous notation $e_{\lambda_i}(z, \mathbf{u})$).

On the sheet corresponding to e_i, the SW differential form λ_{SW} lifts up to the form

$$\lambda_i = -e_i(z, \mathbf{u}) \frac{dz}{z} \tag{4.1.49}$$

which can also be viewed as the coordinate for the corresponding e_i-th sheet. The following proposition is inspired by the construction in section 11.2.1 of the book [Tac15].

Proposition 4.1.10. *The cycle $\{z : |z| = 1\} \subset \mathbb{C}P^1$ lifts up to n cycles on each sheet of the covering. Denote by γ_i the cycle living on the i-th sheet with coordinate given by λ_i, then the cycle $\gamma_{ij} \in H_1(\mathcal{C}_{\mathbf{u}}, \mathbb{Z})$ (as had been constructed in the proposition 4.1.8) can be identified with the the cycle represented by the difference cycle $\gamma_j - \gamma_i$.*

Proof. We compute that

$$\begin{aligned}
\frac{1}{2\pi i} \oint_{\gamma_j - \gamma_i} \lambda_{SW} &= \frac{1}{2\pi i} \oint_{\gamma_j} \lambda_{SW} - \frac{1}{2\pi i} \oint_{\gamma_i} \lambda_{SW} \\
&= \frac{1}{2\pi i} \oint_{\gamma_j} \lambda_j - \frac{1}{2\pi i} \oint_{\gamma_i} \lambda_i \\
&= \frac{1}{2\pi i} \oint_{\gamma_j} \left(-e_{\lambda_j}(z, \mathbf{u})\right) \frac{dz}{z} - \frac{1}{2\pi i} \oint_{\gamma_i} \left(-e_{\lambda_i}(z, \mathbf{u})\right) \frac{dz}{z} \\
&= \left(e_{\lambda_i}(z, \mathbf{u}) - e_{\lambda_j}(z, \mathbf{u})\right) \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} \\
&= e_{\lambda_i}(z, \mathbf{u}) - e_{\lambda_j}(z, \mathbf{u})
\end{aligned}$$

Comparing with the formula (4.1.49), we see that as homology classes in $H_1(\mathcal{C}_u, \mathbb{Z})$, we have that

$$[\gamma_{ij}] = [\gamma_j - \gamma_i]$$

$\square$

As a covering of the projective line $\mathbb{C}P^1 \cong S^2$, it is branched exactly when two or more roots e_i collide with each other, i.e., when the following discriminant vanishes (by viewing z as a constant):

$$\delta_z = \prod_{i<j} (e_i - e_j)^2 \tag{4.1.50}$$

Denote by Δ_z the vanishing locus of the polynomial equation δ_z.

Examples

SU(2) case: The equation of SW curve is given by

$$z + \frac{\Lambda^4}{z} + 2\left(x^2 - u\right) = 0 \tag{4.1.51}$$

In this case, the two roots corresponding to a given z are given as

$$e_\pm = \pm\sqrt{u - \frac{1}{2}\left(z + \frac{\Lambda^4}{z}\right)}$$

there for the discriminant is given by

$$\delta_z = 4\left(u - \frac{1}{2}\left(z + \frac{\Lambda^4}{z}\right)\right)^2$$

whence its vanishing locus (branching points) consists of the two points given as

$$z_\pm = u \pm \sqrt{u^2 - \Lambda^4} \tag{4.1.52}$$

Notice that for the moduli u to hit the quantum discriminant Δ_Λ, i.e. $u = \pm\Lambda^2$, the two branch points $z_\pm$ coincide with Δ_Λ.

SU(3) case: The equation of CW curve is given by

$$z + \frac{\Lambda^6}{z} + 2\left(x^3 - ux - v\right) = 0 \tag{4.1.53}$$

Using the discriminant for cubic equation, we compute that

$$\delta_z = 4u^3 - 27\left(\frac{1}{2}\left(z + \frac{\Lambda^6}{z}\right) - v\right)^2$$

from which its vanishing locus can be computed to consist two pairs of points on the sphere S^2, namely

$$\begin{aligned} z_1^\pm &= \frac{2u^{\frac{3}{2}} + 3\sqrt{3}v \pm \sqrt{\left(2u^{\frac{3}{2}} + 3\sqrt{3}v\right)^2 - 27\Lambda^6}}{3\sqrt{3}} \\ z_2^\pm &= \frac{-2u^{\frac{3}{2}} + 3\sqrt{3}v \pm \sqrt{\left(2u^{\frac{3}{2}} - 3\sqrt{3}v\right)^2 - 27\Lambda^6}}{3\sqrt{3}} \end{aligned} \tag{4.1.54}$$

In general, there are $n-1$ pairs of branch points $z_i^\pm$, $i = 1, \cdots, n-1$, each pair is related by the symmetry of the SW curve, namely

$$z \longmapsto \frac{\Lambda^{2n}}{z} \tag{4.1.55}$$

Remark 4.1.14. *Each pair of the branch points corresponds to a particular pair of roots, say e_i and e_j, colliding with each other, hence produces a vanishing cycle $\nu_{ij} = e_{\lambda_i} - e_{\lambda_j}$ in $H_0(\mathscr{M}, \mathbb{Z})$ (comparing with the corresponding discussion section 4.1.1).*

By the above remark, we see that the pair of branch points $z_i^{\pm}$ corresponds to the simple root $\alpha_i = \alpha_{i,i+1} \in \Lambda_R$, we then have the following

Proposition 4.1.11. *The monodromy around the branch points $z_i^{\pm}$ is given by the fundamental Weyl reflection M_{α_i} (see 4.1.16) associated to the simple roots α_i.*

Proof. By circling around the branch point z_i^+ (the same for z_i^-), the roots e_{λ_i} and $e_{\lambda_{i+1}}$ get exchanged, that is, the two sheets labeled by weights λ_i and λ_{i+1} get permutated which corresponds to a Weyl group action of the weights. Geometrically, the monodromy is given by Picard-Lefeshetz transformation (4.4.16) associated to the vanishing cycle $\nu_i = e_{\lambda_i} - e_{\lambda_{i+1}}$. Viewing it as action on the weight system, it is exactly the Weyl refection 4.1.16 in this case. □

Remark 4.1.15. *The new form of the curve (4.1.46), which is birational to the original one, defines a genus $g = n-1$ Riemann surface. To describe it by cut and pasting of complex plane $\mathbb{C}$, we add two more branch points, say $z_0 = 0$ and $z_\infty = \infty$, to the already existing $z_i^{\pm}$, $i = 1, \cdots, n-1$. Choosing the branch cuts from z_0 to each z_i^-, denoted by l_i^- as well as the cuts from z_∞ to each z_i^+, denoted by l_i^+, there are altogether $n-1$ pairs of cuts. Then by the standard Riemann surface construction (uniformization procedure), we play with two copies of $\mathbb{C}P^1 \cong S^2$, each of which is attached with the $n-1$ pair of branch cuts $l_i^{\pm}$. Gluing the two spheres together by pasting the cut $l_i^{\pm}$ cross-wisely with the cut $l_i^{\pm}$, the resulting surface would be the desired genus $g = n-1$ curve.*

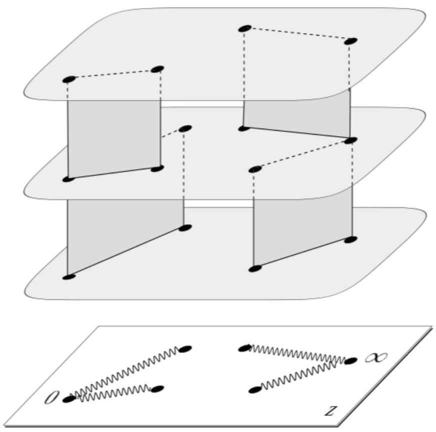

Figure 4.6: The Riemann surfaces through gluing and pasting, figure taken from [Ler98]

Monodromy action on the fiber

Considering a loop γ circling around components of the vanishing locus Δ_z, and by tracing along γ, we see that the individual roots e_{λ_i} may undergone change even though the totality of them are keep unchanged. Consequently, we get the **monodromy representation** associated to the covering $\mathcal{C} \to \mathbb{C}P^1$ as follows

$$\rho : \pi_1 \left(\mathbb{C}P^1 \backslash \Delta_z \right) \longrightarrow S_n \tag{4.1.56}$$

which associates a loop γ to a permutation induced by the parallel transport near a given fiber along the loop. The image of ρ generates the **monodromy group** $\mathcal{M}_\rho$ (also called the **Galois group**) associated to the covering.

Proposition 4.1.12. *The monodromy group of the covering associated to the SW curve (4.1.46) is the full permutation group S_n.*

Proof. As noted before, the sheets of the covering can be identified with the weights $\{\lambda_i\}$ of the n-dimensional fundamental representation $\underline{\mathbf{n}}$ of $SU(n)$. Tracing along a non-trivial loop (homologically non-trivial) in $\mathbb{C}P^1\backslash\Delta_z$, the sheets of the covering corresponding to the weights would get permutated while the characteristic polynomial

$$det(x\mathbf{I} - \phi) = \prod_i (x - e_{\lambda_i}(z, \mathbf{u}))$$

is invariant. This means that the roots $\{e_{\lambda_i}\}$ belong to the same conjugate class with its elements being related by the Weyl group of the corresponding Lie algebra $\mathfrak{su}_n$. As is well known that the Weyl group in this case is exactly S_n. $\square$

As an example, let us look at the $SU(2)$ case considered above. Similar to the simplest example $\sqrt{z}$, when we trace along a curve around the origin-the only branch point for $\sqrt{z}$, the two values $\{\sqrt{z}, -\sqrt{z}\}$ get exchanged. In our case, consider for example, a simple curve encircling only the branch point

$$z_+ = u + \sqrt{u^2 - \Lambda^2}$$

where e_+ collides with e_-, we see easily that the two roots would get exchanged. This verifies that the monodromy group in this case is given by the Weyl group of $\mathfrak{su}_2$, namely $S_2 \cong \mathbb{Z}_2$, which is also the Galois group for the equation $w = z^2$ defined over $\mathbb{Q}$.

For more involved aspects of the appearance of Galois groups in supersymmetric physics, see the paper of Frank Ferrari [Fer09] on Galois symmetry.

4.1.4 SW Geometry from K3 Fibration

From the perspective of physics, the appearance of Seiberg-Witten hyper elliptic curve in $N = 2, d = 4$ SYM is by no means accident, rather, it has a deep origin as certain limit of type IIA string compactification on Calabi-Yau 3-fold which could be realized as $K3$ fibration over the base $\mathbb{C}P^1 \cong S^2$ (see [May99],[Kle+96],[Ler98],[LW98],[Bil+98] for more details). Below, we review some mathematics that are relevant to this story.

Starting with the defining equation (4.1.46) of the SW curve, we consider its Morsification by adding extra quadratic pieces in the complex variables y and z which does not affect the singularity type

$$X_{\mathbf{u}} : z + \frac{\Lambda^{2n}}{z} + 2\mathcal{W}_{A_{n-1}}(x, \mathbf{u}) + 2y^2 + 2w^2 = 0 \tag{4.1.57}$$

Fixing $z \in \mathbb{C}P^1$, the above equation describes a local ALE geometry sitting in a K3 surface, i.e., we get a fibration over S^2 with generic fiber the non-compact ALE space of type A_{n-1} defined by the level manifold of the following polynomial equation in $\mathbb{C}^3$:

$$\mathcal{W}^{ALE}_{A_{n-1}} := \mathcal{W}_{A_{n-1}}(x, \mathbf{u}) + y^2 + w^2 \tag{4.1.58}$$

Here ALE (stands for *Asymptotically Locally Euclidean*) space of type A_{n-1} means a four dimensional Riemann manifold that at "∞" looks like $\mathbb{C}^2/\Gamma$, where $\Gamma \subset SU(2)$ is a finite group such that the quotient space has singularity of type A_{n-1}. It is proved in [Kro+89] that it is diffeomorphic to the minimal resolution of the Kleinian singularity $\mathbb{C}^2/\Gamma$.

Indeed, the parameters $\mathbf{u} = \{u_2, \cdots, u_n\}$ provide a minimal resolution of the singularity at the origin in $\mathbb{C}^3$. By blowing up the singular points, we arrive at a collection of exceptional spheres with self intersection 2, and they mutually interact with each other in a way that could be encoded by the corresponding Dynkin diagram of type A_{n-1}.

Figure 4.7: Dynkin diagram of type A_{n+1}

Proposition 4.1.13. *The critical points of the level manifold of $\mathcal{W}^{ALE}_{A_{n-1}}$ at the level l are the set*

$$\{(x_1, 0, 0), \cdots, (x_{n-1}, 0, 0)\}$$

where $\{x_1, \cdots, x_{n-1}\}$ is the set of critical points of the equation $\mathcal{W}_{A_{n-1}}$.

Proof. The critical points are the solutions of the following system of equations

$$\mathcal{W}^{ALE}_{A_{n-1}}(x, y, w) = l \qquad \frac{\partial}{\partial x}\mathcal{W}^{ALE}_{A_{n-1}} = \frac{\partial}{\partial x}\mathcal{W}_{A_{n-1}} = 0$$

$$\frac{\partial}{\partial y}\mathcal{W}^{ALE}_{A_{n-1}} = 2y = 0 \qquad \frac{\partial}{\partial z}\mathcal{W}^{ALE}_{A_{n-1}} = 2z = 0$$

which reduce to

$$\mathcal{W}_{A_{n-1}}(x,y,w) = l \qquad \frac{\partial}{\partial x}\mathcal{W}_{A_{n-1}} = 0$$

which is exactly the way to compute the critical points associated to the level manifold of A_{n-1} singularity. $\square$

By the above proposition, we see that the singularity type of the ALE space resembles to the one of the pre-classical moduli space $\mathscr{M}$ (see section 4.1.1) defined as the level manifold of the polynomial $\mathcal{W}_{A_{n-1}}$. Inspired by this observation, we obtain the following, which is the special case of **geometric Mckay correspondence** [GV83].

Proposition 4.1.14. *There are $n-1$ independent two spheres S_i, $i = 1, \cdots, n-1$ in each ALE space fiber, with intersection numbers given as*

$$S_i \circ S_i = 2, \;\; S_i \circ S_{i+1} = -1, \;\; S_i \circ S_j = 0, \; when\; ever |i-j| \geq 2 \tag{4.1.59}$$

which reflects exactly the Dynkin diagram of type A_{n-1}. These $n-1$ spheres consist of the vanishing cycles that generate the middle homology $H_2(X_{\mathbf{u}}, \mathbb{Z}) \subset H_2(K3, \mathbb{Z})$. Moreover, we have the following isomorphism of the lattices

$$H_2(X_{\mathbf{u}}, \mathbb{Z}) \cong H_2(\mathscr{M}, \mathbb{Z}) \cong \Lambda_R \tag{4.1.60}$$

where Λ_R is the root lattice for $SU(n)$.

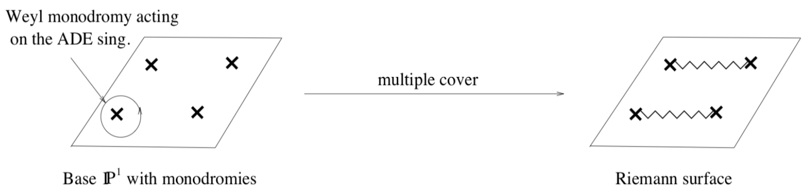

Figure 4.8: Correspondence between vanishing cycles in ALE fiber and that in level manifold, picture taken from[May99]

Proof. The $n-1$ spheres S_i are the exceptional fibers of the minimal resolution of the A_{n-1} singularity which can be described explicitly as follows:

Denote by l_i the critical value of $\mathcal{W}^{ALE}_{A_{n-1}}$ at the critical point $(x_i, 0, 0)$, i.e., we have that $l_i = \mathcal{W}_{A_{n-1}}(x_i, \mathbf{u})$. Near the critical point corresponding l_i, $\mathcal{W}^{ALE}_{A_{n-1}}$ can be put into the following standard form

$$\mathcal{W} \equiv \mathcal{W}^{ALE}_{A_{n-1}}(x,y,w) = l_i + x^2 + y^2 + w^2$$

Let the level $l \in \mathbb{C}$ trace along a small loop γ encircling only the critical value l_i. The equation of γ is then seen to be given by

$$\gamma : \; l(t) = l_i + \delta^2 e^{2\pi i t}$$

for δ small and $t \in [0,1]$.

Locally near the critical point, we have the situation described by the equation

$$x^2 + y^2 + w^2 = l - l_i = \delta^2 e^{2\pi i t}$$

Then the vanishing two sphere in this ALE-fiber is given by the real part of the above equation, i.e.,

$$\begin{aligned} S_i &= \left\{(x, y, z) : x^2 + y^2 + w^2 = \sqrt{l - l_i},\ Im(x) = Im(y) = Im(w) = 0\right\} \\ &= \sqrt{l - l_i}\, S^2 \end{aligned} \tag{4.1.61}$$

where S^2 denotes the unit 2-sphere.

Indeed, as we approach to the critical point $(x_i, 0, 0)$, $l \to l_i$, the Kähler class $\left[\sqrt{l - l_i}\right]$ measuring the size of the sphere S_i approaches to zero, justifying that the 2-sphere S_i is the vanishing two cycle desired.

Now we compute the intersection numbers between these vanishing two cycles. First, deforming the cycle S_i a litter bit by deforming the line segment $l - l_i$ to an arbitrary path but with the two end points retained, then it is clear that the deformed two cycle $\widetilde{S}_i$ intersect with S_i in exactly two points, i.e., the two end points. This shows that the self intersection of each S_i equals to two.

Let l be the line segment connecting the two critical points $(x_i, 0, 0)$ and $(x_{i+1}, 0, 0)$ and use this line segment to construct the two vanishing spheres S_i and S_{i+1} as above, then it becomes clear that by deforming the two sphere such that the sum of the radius of the two spheres equals to the length of the line l, we see that they intersect in exactly one point but with negative sign. Thus, $S_i \circ S_{i+1} = -1$.

Apparently, by the above reasoning $S_i \circ S_j = 0$ for $|i - j| \geq 2$. This finishes the proof of the first part of the proposition.

To prove the second part of the proposition, note that the critical point x_i of $\mathcal{W}_{A_{n-1}}(x, \mathbf{u})$ corresponds to the roots e_{λ_i} and $e_{\lambda_{i+1}}$ coinciding with each other, which by section 4.1.1 corresponds to the vanishing 0-cycle

$$\nu_i = e_{\lambda_i} - e_{\lambda_{i+1}} \in H_0(\mathscr{M}, \mathbb{Z})$$

As we know before that the $n - 1$ independent vanishing cycles ν_{ij} generates $H_0(\mathscr{M}, \mathbb{Z})$ and is isomorphic to the root lattice Λ_R. □

Remark 4.1.16. *By the above proposition, we infer that the SW geometry, encoded by the fibration of weight diagram over S^2, which is linearized by the local system of lattices $H_0(\mathscr{M}, \mathbb{Z})$ over $\mathbb{C}P^1$, can be obtained by the degeneration of the ALE space fibration over S^2 by letting y and z equal to zero in the equation 4.1.59 (in physics term, it is called integrate out the variables). In other words, the local system of lattices $H_2(ALE, \mathbb{Z})$ over $\mathbb{C}P^1$ encodes the same information as that of $H_0(\mathscr{M}, \mathbb{Z})$. The essential information is the root lattice Λ_R of type A_{n-1}. Indeed, the intersection matrix of the vanishing cycles in both cases equal to the negative of the Cartan matrix for $SU(n)$.*

In general, by remark 4.1.14, suppose that at $z_i^{\pm} \in \mathbb{C}P^1$, the two roots e_{λ_i} and e_{λ_j} collide with each other. Denote by S_{ij} the vanishing 2-sphere in $H_2(X, \mathbb{Z})$ corresponding to the vanishing 0-cycle $\nu_{ij} = e_{\lambda_i} - e_{\lambda_j}$ in $H_0(\mathscr{M}, \mathbb{Z})$.

From the proof of the above proposition, we see that the vanishing two spheres vanishes exactly when $\mathcal{W} \equiv \mathcal{W}^{ALE}_{A_{n-1}} = 0$, i.e., the points on the SW curve $\mathcal{C}_u$, that is when $x = e_{\lambda_i}$ for some i, which means that the radius of the vanishing two sphere S_i is measured by e_{λ_i}, and consequently that of the S_{ij} is measured by $e_{\lambda_i} - e_{\lambda_j}$. We will use this fact in the proof of the proposition below.

From the proposition and the remark above, we expect the SW one form λ_{SW} could also be deduced from the local CY geometry.

$X_{\mathbf{u}}$ should be viewed as the defining equation of a local Calabi-Yau 3-fold near the singular K3 surfaces fibred over $\mathbb{C}P^1$ with the singularity type A_{n-1}.

Denote by $\Omega \in H^{3,0}(X, \mathbb{C})$ the unique (up to scalar multiplication) holomorphic volume form (as ensured by Calabi-Yau condition: $K_X \equiv 0$), which is given by

$$\Omega = \frac{dz}{z} \wedge \frac{dx \wedge dy}{\partial_w \mathcal{W}^{ALE}_{A_{n-1}}(x, y, w, z)} \tag{4.1.62}$$

We want to produce a one form on $\mathbb{C}P^1$ by integrating the 3 form Ω over the vanishing 2-spheres S_{ij} constructed above.

Proposition 4.1.15. *Up to multiplication by some constant, we have that*

$$\iint_{S_{ij}} \Omega = (e_{\lambda_i} - e_{\lambda_j}) \frac{dz}{z} \tag{4.1.63}$$

Proof. As the two sphere S_{ij} corresponding to ν_{ij}, we see that locally S_{ij} is defined to be the real part of

$$x^2 + y^2 + w^2 = c^2 \cdot (e_{\lambda_i} - e_{\lambda_j})^2$$

where c is a constant, we then compute as follows

$$\begin{aligned}
\iint_{S_{ij}} \Omega = \iint_{S_{ij}} \frac{dz}{z} \wedge \frac{dx \wedge dy}{\partial_w \mathcal{W}^{ALE}_{A_{n-1}}(x, y, w, z)} &= \iint_{S_{ij}} \frac{dz}{z} \wedge \frac{dx \wedge dy}{2w} \\
= \frac{1}{2} \iint_{S_{ij}} \frac{dz}{z} \wedge \frac{dx \wedge dy}{\sqrt{(c(e_{\lambda_i} - e_{\lambda_j}))^2 - (x^2 + y^2)}} &\overset{x=\rho cos\theta,\, y=\rho sin\theta}{=\!=\!=} \\
= \frac{1}{2} \iint_{S_{ij}} \frac{dz}{z} \wedge \frac{d\rho \wedge d\theta}{\sqrt{(c(e_{\lambda_i} - e_{\lambda_j}))^2 - \rho^2}} \\
= \frac{2\pi}{2} \frac{dz}{z} \int_0^{(c(e_{\lambda_i} - e_{\lambda_j}))^2} \frac{d\rho}{\sqrt{(c(e_{\lambda_i} - e_{\lambda_j}))^2 - \rho^2}} \\
= 2\pi c \, (e_{\lambda_i} - e_{\lambda_j}) \frac{dz}{z}
\end{aligned}$$

$\square$

Proposition 4.1.16. *Up to multiplication by a constant, we have that*

$$\int \Omega = \lambda_{SW} = -x \frac{dz}{z} \tag{4.1.64}$$

Proof. By above proposition, we see that when integrated out the extra variables y and z over the vanishing 2-sphere S_{ij}, which corresponds to degenerate the lattice from $H_2(X, \mathbb{Z})$ to $H_0(\mathscr{M}, \mathbb{Z})$, the volume form Ω was reduced to the form

$$(e_{\lambda_i} - e_{\lambda_j}) \frac{dz}{z}$$

By (4.1.49), this form equals $\lambda_j - \lambda_i$, which vanishes on ν_{ij}. The claim then follows from proposition 4.1.10. □

Given the volume form Ω on a CY-3 fold X, we can define an analogous notion of central charge function in this case, namely

Definition 4.1.6. *The central charge Z^{CY} associated to a CY-3 fold $(X, \Omega)$is given through the following period map, which is an abelian group homomorphism:*

$$\begin{aligned} Z^{CY} \,:\, H_3(X,\, \mathbb{Z}) &\longrightarrow \mathbb{C} \\ \gamma \longmapsto Z^{CY}(\gamma) &:= \int_\gamma \Omega \end{aligned} \tag{4.1.65}$$

Following from the proposition 4.1.14 and proposition 4.1.15 above, we can show that the central charge function (periods of SW curve) associated to the SW integrable system can be induced from the central charge function (periods of CY 3-fold) associated with the CY 3-fold defined above.

By formula (4.1.64), the integration of the holomorphic form Ω over the vanishing two spheres S_{ij} in the local ALE fiber gives the SW form $\lambda_{SW} = -x\frac{dz}{z}$ restricted to the i-th and the j-th sheets of the weight diagram fibration. As the vanishing 2 sphere S_{ij} is associated with $\nu_{ij} = e_{\lambda_i} - e_{\lambda_j}$ that corresponds to the pair of branch points $z_{ij}^{\pm}$ over which the two roots coincide. Connect the two branch points (directed from z_{ij}^+ to z_{ij}^-) by a line segment l_{ij} (branch cut) and integrate over it, we get the period for Ω as

$$\begin{aligned} \int_{z_{ii}^-}^{z_{ij}^+} \iint_{S_{ij}} \Omega = (e_{\lambda_i} - e_{\lambda_j}) \int_{z_{ij}^-}^{z_{ij}^+} \frac{dz}{z} &= \int_{z_{ij}^-}^{z_{ij}^+} e_{\lambda_i} \frac{dz}{z} - \int_{z_{ij}^-}^{z_{ij}^+} e_{\lambda_j} \frac{dz}{z} \\ = \int_{l_{ij}} x \frac{dz}{z} + \int_{-l_{ij}} x \frac{dz}{z} &= \int_{\gamma_{ij}} x \frac{dz}{z} = \int_{\gamma_{ij}} \lambda_{SW} \end{aligned} \tag{4.1.66}$$

where $\gamma_{ij} \in H_1(\mathcal{C}, \mathbb{Z})$ denotes the lift of the line segment $l_{ij} \subset \mathbb{C}P^1$. The identity in the last line of the above equation follows from the fact that the SW curve C_u is presented as a Riemann surface by pasting two copies of complex line $\mathbb{C}P^1$ along the branch cuts l_{ij} with opposite direction. (see remark 4.1.15)

From the above computation, we see that when consider the three cycle $\gamma \in H_3(X, \mathbb{Z})$ that can be written as a cycle composed of a vanishing two sphere S_{ij} over the line segment l_{ij} lying the Riemann sphere $\mathbb{C}P^1$. Then we have that

$$Z^{CY}(\gamma) = \int_\gamma \Omega = \int_{\gamma_{ij}} \lambda_{SW} = Z(\gamma_{ij}) \tag{4.1.67}$$

Proposition 4.1.17. *The three cycle γ above is topologically homemorphic to the three sphere S^3.*

Proof. Clearly, the cycle γ presented above can be viewed as a two sphere fibration over the line segment l_{ij} with two end points z_{ij}^+ and z_{ij}^-. Since at the two end points, the sphere shrinks into zero size, we see that the topology type of γ is the same as that of S^3, i.e, by slicing S^3 into S^2 leaves with the north pole of S^3 being identified with the vanishing two sphere on one end, while the south pole that of the other end. □

The topology type of the non contractible connected one cycles on two sphere $\mathbb{C}P^1$, besides the line segments described above, there are also cycles of the topology type of the circle S^1. The natural three cycles lifted from this type on the base $\mathbb{C}P^1$ of the fibration to X is of topology type $S^2 \times S^1$ (see Figure 4.9).

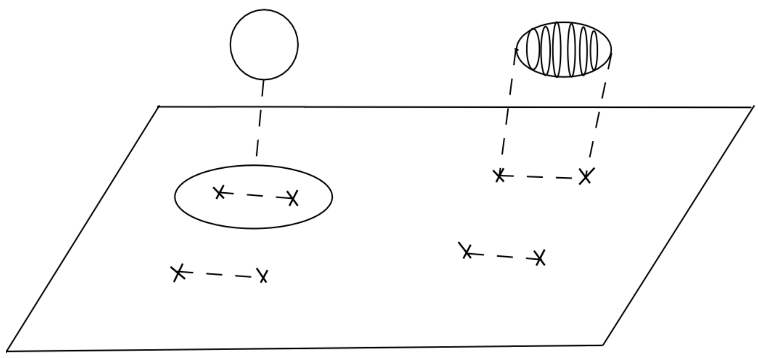

Figure 4.9: The topology types of the lifted one cycles, either of the type $S^2 \times S^1$ as shown in the left hand side of the figure, or of the type of the three sphere S^3 on the right hand side of the figure.

Under the projection $\pi : X \to \mathbb{C}P^1$, the three cycles in X were mapped to one cycles on $\mathbb{C}P^1$ which falls into the two types described above. As we showed in proposition 4.1.14 that the middle dimensional homology of the ALE-space fiber $X_{\mathbf{u}}$ is given by

$$H_2(X_{\mathbf{u}}, \mathbb{Z}) \cong \Lambda_R^{SU(n)}$$

which is generated by the set of vanishing two spheres S_{ij} corresponding to the collision of two roots e_{λ_i} and e_{λ_j} of A_{n-1} singularity $\mathcal{W}_{A_{n-1}}$. Thus, by considering the fibration of the vanishing two spheres over the two type of one cycles on the base of the fibration, the three cycles of X would be generated.

In conclusion, we get the following map on the homology level

$$\phi : H_3(X, \mathbb{Z}) \longrightarrow H_1(\mathcal{C}, \mathbb{Z}) \tag{4.1.68}$$

which is defined as the composition

$$\phi : H_3(X, \mathbb{Z}) \xrightarrow{\pi_*} H_1(\mathbb{C}P^1, \mathbb{Z}) \xrightarrow{lift} H_1(\mathcal{C}, \mathbb{Z}) \tag{4.1.69}$$

We have already noted above that if the three cycle γ is mapped into line segment l_{ij} connecting the pair of branch points $z_{ij}^{\pm}$ corresponding to the collision of two roots e_{λ_i} and e_{λ_j}, then ϕ would map γ to the lifted one cycle γ_{ij} in $H_1(\mathcal{C}, \mathbb{Z})$. On the other hand, if the three cycle γ is mapped under π_* into a circle $S^1 \subset S^2$, then we denote by γ^{S^1} its lift on the Riemann surface $\mathcal{C}$. Then it can be shown that these two types of cycles γ^{S^1} and γ_{ij} generates $H_1(\mathcal{C}, \mathbb{Z})$.

It is clear that we have the following property which relates the periods of CY 3-fold to the periods of SW curves $\mathcal{C}_{\mathbf{u}}$.

Proposition 4.1.18. *We have that for $\gamma \in H_3(X, \mathbb{Z})$ the following identity*

$$Z^{CY}(\gamma) = \int_\gamma \Omega = \int_{\phi(\gamma)} \lambda_{SW} = Z(\phi(\gamma)) \tag{4.1.70}$$

$H_3(X, \mathbb{Z})$ is endowed with the intersection form $\langle \cdot, \cdot \rangle_{CY}$, which is symplectic as the complex dimension of X is 3. Denote by $\langle \cdot, \cdot \rangle_{SW}$ the intersection form on $H_1(X, \mathbb{Z})$, then it was claimed in [Kle+96] that the map ϕ preserves the intersection form, i.e.,

$$\langle \gamma_1, \gamma_2 \rangle_{CY} = \langle \phi(\gamma_1), \phi(\gamma_2) \rangle_{SW} \tag{4.1.71}$$

Besides it is also claimed in [Kle+96] that

$$ker(\phi) = 0 \tag{4.1.72}$$

which, together with (4.1.71), means that the information about the intersection of three cycles on X could be encoded in the interaction pattern of the one cycles on the Riemann Surface $\mathcal{C}$.

Sketch of the proof of (4.1.71) and (4.1.72)

To show the validity of (4.1.71), notice that since we have presented the three cycles in $H_3(X, \mathbb{Z})$ as S_2 (which generates $H_2(X_{\mathbf{u}}, \mathbb{Z})$) fibration over the one cycles lying on $\mathbb{C}P^1$, while the one cycles $H_1(\mathcal{C}_u, \mathbb{Z})$ on the SW curve is generated by the one cycles on the base $\mathbb{C}P^1$ that either connects the two branch points of the ALE fibration or the closed loops circling around the branch points, thus under the push-forward map ϕ, the intersection product $\langle \cdot, \cdot \rangle_{CY}$ is determined by the induced intersection patterns on its ALE-fibers, namely that induced on $H_2(X_{\mathbf{u}}, \mathbb{Z})$. By proposition 4.1.14, this group is identified with $H_1(\mathcal{C}_u, \mathbb{Z})$ from which it follows that the intersection patterns of the three cycles on X coincide with that of one cycles on the SW curve, i.e., we have

$$\phi_\sharp \langle \cdot, \cdot \rangle_{CY} = \langle \cdot, \cdot \rangle_{SW}$$

Next, to show that the kernel of ϕ is trivial as homology class, we note that if $\phi(\gamma) = \partial c$ for some two cycle $c \in H_2(\mathcal{C}, \mathbb{Z})$, then we show that $\gamma = \partial \tilde{c}$. Indeed, by projecting the two cycle c on to the base sphere $\mathbb{C}P^1$, we will get a two cycle c^* on the base. Then we take the boundary ∂c^* and consider the S^2 fibration over it, we denote the three cycle thus obtained by $\tilde{c}$, which by the construction of the map ϕ described in (4.1.69), we see that $\partial \tilde{c} = \gamma$. $\square$

Remark 4.1.17. *The one cycles on the Riemann sphere $\mathbb{C}P^1 \cong S^2$ is called in physics literature as the self-dual strings (see [Kle+96][BS97][May99] for more information about this story). From the discussion above, it is expected that the BPS states in type IIB string theory compactified on CY 3 fold X, being represented as D3-brane wrapping three cycles on X (see appendix A), could be studied through the configuration of self-dual strings on complex z-plane. Of course, by supersymmetry the three branes wrapping three cycles should minimize its volume in its homology class. Remember that we have represented the three cycles as vanishing two spheres over the one cycle on $\mathbb{C}P^1$, as the two spheres are already of the minimum volume possible, we only need to minimize the length of the self-dual strings in question, i.e., the geodesics of curves on S^2 with the natural metric defined below.*

Definition 4.1.7. *The natural metric on the base $\mathbb{C}P^1$ is given by*

$$ds^2 = |\lambda_{SW}|^2 = g_{z\bar{z}}\, dz d\bar{z} = (x(z)/z)^2\, dz d\bar{z} \tag{4.1.73}$$

As the dependence of x on z is assumed to be holomorphic, we get that the curvature

$$\partial\overline{\partial} \log g_{z\bar{z}} \equiv 0$$

Consequently, by going to the flat coordinate where the flat metric is in canonical form, we see that the geodesics are straight affine lines in the affine structure induced by the flat metric (see section 3.1.5).

The canonical flat coordinate is given by

$$Z \equiv \int^z \lambda_{SW} \tag{4.1.74}$$

Indeed, in terms of the coordinate Z, the metric can be rewritten as

$$ds^2 = dZ d\overline{Z} = |\lambda_{SW}|^2 \tag{4.1.75}$$

which justifies the name "flat" of the coordinate Z. Using it, the equation of geodesics can be written in the following form

$$Z(t) = \alpha t + \beta \tag{4.1.76}$$

where t is the "time" parameter for the geodesic line, α and β two arbitrary complex valued constant. Apparently, α specifies the direction of the geodesic line in complex plane.

The above equation is the solution of the first order differential equation, i.e. the geodesic equation in canonical flat coordinate:

$$\frac{d}{dt} Z(t) = \alpha \tag{4.1.77}$$

which in our case reads

$$x(z) z^{-1} \frac{\partial z(t)}{\partial t} = \alpha \tag{4.1.78}$$

For the simplest $SU(2)$ case, the above equation specifies into the following

$$\sqrt{2u - z - \frac{\Lambda^2}{z}}\, z^{-1} \frac{\partial z(t)}{\partial t} = \alpha \tag{4.1.79}$$

Some trajectories of the above equation and the relation to the BPS states for $SU(2)$ supersymmetric Yang-Mills theory can be found for example in [Kle+96].

Remark 4.1.18. *Note that the form of equation (4.1.77) is very similar to the attractor flow equation (3.2.7) in affine coordinates in section 3.1.7 if the flat coordinate Z here could be interpreted as the central charge function. This is indeed the case, which on the physics side, supports the plausibility that the structures of the self dual strings should in principle encodes the BPS spectral of the theory lying behind.*

We now state the following proposition which clarifies the remark above.

Proposition 4.1.19. *Let $Z : \underline{\Gamma}_b := H_1(\mathcal{C}_b, \mathbb{Z}) \to \mathbb{Z}$ be the central charge function of the SW integrable system. Consider the wall of second kind defined by*

$$Im(Z_b(\gamma)) = 0$$

for some charge $\gamma \in \underline{\Gamma}_b$. If the above equation is viewed as defining a curve in the base $\mathcal{B}$ of the SW integrable system, then it gives the attractor flow equation (3.2.7) defined before. However, if we present the SW curve as the fibration of weight diagram over $\mathbb{C}P^1$, then in terms of the base coordinate z, the wall equation above gives the geodesic line equation (4.1.79) on $\mathbb{C}P^1$.

Remark 4.1.19. *Since both forms of SW curve are equivalent, we see that attractor flows on the base of the SW integrable system and the self dual strings on the Riemann sphere essential contains the same amount of information as regards to determining the BPS-invariants (aka DT-invariants). Notice that in the $SU(2)$ case the base $\mathcal{B}$ actually coincides with the Riemann sphere S^2 after one point compactification of the complex plane.*

4.1.5 Connection with Hitchin System and Spectal Networks

The Seiberg-Witten curve is usually presented (for example the one given in [KS08]) in the form that is pertinent to the Hitchin integrable system (for more details see [Moo12a][Nei14a][Lu10] and the references contained therein).

Given a punctured Riemann surface S together with certain data describing the boundary conditions at the punctures, and let G be a simply laced compact Lie group (in the following, G is assumed to be $SU(n)$). Let E be a complex G bundle over S endowed with a G-connection of $(0,1)$ type, which can be written as

$$D = \partial_{\bar{z}} + A_{\bar{z}}$$

Consider $(1,0)$ form valued in $End(E)$ as follows

$$\phi \in \Omega^{(1,0)}\left(S,\, End(E)\right) = H^0\left(S,\, End(E) \otimes \mathcal{K}\right)$$

where $\mathcal{K}$ is the canonical bundle of S. The adjoint field ϕ is called a **Higgs field** if it gives solution to the following Hitchin equations:

$$\begin{cases} F_D + R^2[\phi, \bar{\phi}] = 0 \\ \partial_{\bar{z}}\phi + [A_{\bar{z}}, \phi] = 0 \end{cases} \tag{4.1.80}$$

where F_D is the curvature of the connection D, and R a positive real number.

The pair (E, ϕ) is then called a **Higgs bundle** over S. The space of all Higgs bundles modulo the group $\mathcal{G} := Aut(E)$ gives the **Hitchin moduli space** $\mathcal{M}_{Higgs}$ which carries hyperkähler structure (see the definition 3.1.20).

A point of $\mathcal{M}_{Higgs}$ corresponds to a cover of S through the spectral curve:

$$\mathcal{C} := \{\lambda \in T^*S :\, det(\lambda\mathbf{I} - \phi) = 0\} \subset T^*S \tag{4.1.81}$$

where in the fundamental representation of $SU(n)$, the determinant can be written (trace$\equiv 0$ by the $SU(n)$ condition) as

$$det(\lambda\mathbf{I} - \phi) = \lambda^n + \lambda^{n-2}\phi_2 + \cdots + (-1)^n\phi_n = 0 \tag{4.1.82}$$

Here the fields $\phi_i \in H^0\left(S,\, \mathcal{K}_S^{\otimes i}\right)$ may have singularities at the punctures. Clearly, by utilizing the spectral curve $\mathcal{C}$, we get a n-folded cover of the Riemann surface S

$$\pi :\, \mathcal{C} \xrightarrow{n:1} S \tag{4.1.83}$$

with the fiber over a generic $z \in S$ given by

$$\pi^{-1}(z) := \left\{\lambda(z) \in T^*S :\, \lambda^n + \sum_{i=2}^{n} (-1)^i\phi_i\,\lambda^{n-i} = 0\right\} \tag{4.1.84}$$

The fiber above generically consists of the n-roots of the equation (4.1.82). The covering π is branched at the discriminant locus Δ where two or more roots collide.

The **Coulomb branch** $\mathcal{B}$ consists of tuples $\{\phi_2, \cdots, \phi_n\}$ of meromorphic differentials, which are the Casimirs of the Higgs field ϕ. Thus

$$\mathcal{B} := \bigoplus_{i=2}^{n} H^0\left(S, \mathcal{K}_S^{\otimes i}\right) \tag{4.1.85}$$

Given a point of $\mathcal{B}$, i.e., by specifying $n-1$ tuples $\{\phi_i\}_{i=2}^{n}$, we will obtain a branched covering of S through the spectral curve construction (4.1.81). Moving points in $\mathcal{B}$ gives a family of covering (4.1.83).

Denote by $\mathcal{B}^{sing}$ the locus where this fibration is branched. And denote by $\mathcal{B}^0 = \mathcal{B}\backslash\mathcal{B}^{sing}$ the smooth part. Then we have the **Hitchin fibration**

$$h : \mathcal{M}_{Higgs} \longrightarrow \mathcal{B} = \bigoplus_{i=2}^{n} H^0\left(S, \mathcal{K}_S^{\otimes i}\right)$$

which maps a Higgs bundle (E, ϕ) to the characteristic polynomial of ϕ. It can be proved that away from the singular locus $\mathcal{B}^{sing}$, it gives a complex integrable system ([Hit+87a][Bot83]) in the sense that it is a holomorphic mapping with the generic fibers being the compact Lagrangian tori. Actually, in our case, the generic fiber $h^{-1}(b)$ is an abelian variety given by the **Prym variety** $J(\mathcal{C}_b)$ of the covering $\mathcal{C}_b \to S$, which in our A_{n-1} case can be identified with the Jacobi variety of $\mathcal{C}_b$.

The following picture due to M.Kontsevich vividly illustrates the geometry of the Hitchin fibration as a complex integrable system.

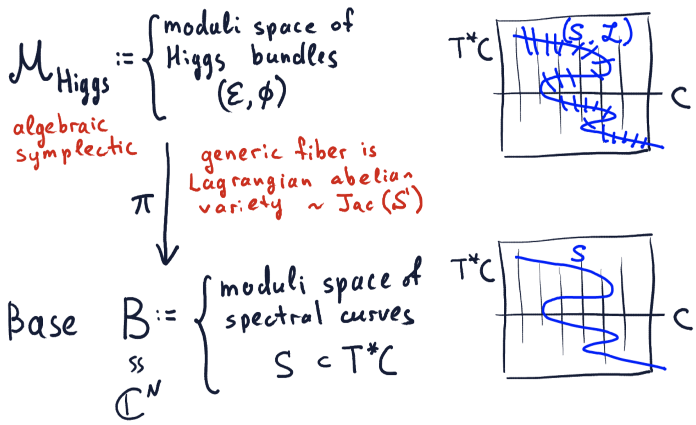

Figure 4.10: The geometry of Hitchin integrable system

Remark 4.1.20. *The Prym variety of the covering π is defined to be the kernel of the corresponding map on the Jacobian level, i.e.,*

$$J(\mathcal{C}) := \ker\left(Jac(\mathcal{C}) \to Jac(S)\right) \tag{4.1.86}$$

In the case when $S = \mathbb{C}P^1$ (to be focused on momentarily), as the Jacobian of S is trivial, the Prym variety of the covering coincide with the Jacobian of $\mathcal{C}$.

Remark 4.1.21. *Given coordinate $(z, x) \in T^*S$, the tautological one form (Liouville one form) on the cotangent space reads*

$$\lambda = xdz \tag{4.1.87}$$

while the canonical symplectic form on T^*S is given by (up to a normalizing constant)

$$\Omega^{2,0} = d\lambda = dx \wedge dz \tag{4.1.88}$$

The restriction of the tautological one form λ on the spectral surface $\mathcal{C}$ gives the so called Seiberg-Witten differential, while the surface $\mathcal{C}$ itself is termed as Seiberg-Witten curve. The rational of these terminologies will be justified in the following in the case for gauge group to be of type A_{n-1}.

Relation to SW curve for A_{n-1} case

We take the Riemann surface to be the Riemann sphere $S = \mathbb{C}P^1 \cong S^2$ with holomorphic coordinate z, while the complex bundle E is taken to be its cotangent bundle $T^*\mathbb{C}P^1$. Let x be the fiber coordinate of $T^*\mathbb{C}P^1$, then the tautological (Liouville) one form on the symplectic manifold $T^*\mathbb{C}P^1$ is given by

$$\lambda_{can} = x\,\frac{dz}{z} \tag{4.1.89}$$

Indeed, it is invariantly defined on $\mathbb{C}P^1$ since its form is kept (albeit differed by a negative sign which is inconsequential for our purpose) under the coordinate patch transformation on the projective line $\mathbb{C}P^1$, namely: $z \mapsto\ z^{-1}$.

The canonical symplectic form on $T^*\mathbb{C}P^1$ is given by

$$\omega^{2,0} = dx \wedge \frac{dz}{z} = d\left(x\,\frac{dz}{z}\right) = d\,\lambda_{can} \tag{4.1.90}$$

We claim that by substituting

$$\lambda = \lambda_{can} \qquad \phi_i = u_i \left(\frac{dz}{z}\right)^i$$

in the characteristic polynomial (4.1.82), the classical simple singularity of A_{n-1} type in the description of SW curve (4.1.46) will be recovered.

Indeed, the characteristic polynomial in this case becomes

$$det(\lambda\mathbf{I} - \phi) = \frac{x^n}{z^n}(dz)^n + \frac{x^{n-2}}{z^{n-2}}(dz)^{n-2}u_2\left(\frac{dz}{z}\right)^2 + \cdots + (-1)^n u_n \left(\frac{dz}{z}\right)^n$$

$$= \left(\frac{dz}{z}\right)^n \left(x^n + u_2\, x^{n-2} + \cdots + (-1)^n u_n\right) = \left(\frac{dz}{z}\right)^n \left(\mathcal{W}_{A_{n-1}}(x, \mathbf{u})\right)$$

Thus in terms of coordinates $(z, x) \in T^*\mathbb{C}P^1$, the equation defining the spectral curve can be written as $\mathcal{W}_{A_{n-1}}(x, \mathbf{u}) = 0$, which means that for each $z \in \mathbb{C}P^1$, we get a copy of the level manifold of $\mathcal{W}_{A_{n-1}}$, while the level itself can be fixed by judicious choice of λ.

Consequently we get the SW curve in the form of (4.1.46), i.e., as the weight diagram fibration over $\mathbb{C}P^1$. For this reason, the curve $\mathcal{C}$, viewed as a branched covering of $\mathbb{C}P^1$ is also called the Seiberg-Witten curve in the mathematics and physics literature once a fundamental representation of A_{n-1} has been fixed.

Let's take again the example of $SU(2)$ case as an illustration of the perspective presented in above. The spectral curve equation in this A_1 case reads

$$\lambda^2 + \phi_2 = 0$$

Make the following choice for the quadratic differential

$$\phi_2 = \left(-\frac{u}{z^2} - \frac{1}{2z} - \frac{\Lambda^4}{2z^3} \right) \otimes (dz)^2 \tag{4.1.91}$$

Then by substituting $\lambda = \lambda_{can} = x\frac{dz}{z}$, we see that the spectral curve equation becomes

$$x^2 \left(\frac{dz}{z} \right)^2 - \left(\frac{u}{z^2} + \frac{1}{2z} + \frac{\Lambda^4}{2z^3} \right) \otimes (dz)^2 = 0$$

which, after collecting the terms becomes

$$\left(x^2 - u - \frac{1}{2}\left(z + \frac{\Lambda^4}{z} \right) \right) \otimes \left(\frac{dz}{z} \right)^2 = 0$$

that is

$$z + \frac{\Lambda^4}{z} + 2\left(x^2 - u\right) = 0$$

And this is exactly the SW curve (4.1.51) for $SU(2)$ case.

Similarly, in $SU(3)$ case, the characteristic polynomial reads as

$$\lambda^3 + \lambda\,\phi_2 - \phi_3 = 0$$

Choose

$$\phi_2 = \frac{-u}{z^2} \otimes (dz)^2$$

and

$$\phi_3 = \left(\frac{v}{z^3} - \frac{1}{2}\left(\frac{1}{z^2} + \frac{\Lambda^6}{z^4} \right) \right) \otimes (dz)^3$$

Substituting them in the characteristic polynomial, together with $\lambda = x\frac{dz}{z}$, we will obtain the equation for SW curve (4.1.53) in this case, namely

$$z + \frac{\Lambda^6}{z} + 2(x^3 - ux - v) = 0$$

Remark 4.1.22. *From the above description, one sees easily that the SW differential on form for the SW curve $\mathcal{C}_u$ is exactly given by the minus of the restriction of the Liouville one form λ_{can} to $\mathcal{C}$, which is the expected scenery from the perspective of the complex integrable system.*

Split attractor flow versus spectral network

For a generic $z \in S$, the fiber of the spectral covering π (see 4.1.83) is given by the n roots of the characteristic equation (4.1.82), namely

$$\{\lambda_1, \cdots, \lambda_n\} \tag{4.1.92}$$

which correspond to the sheets of the covering. At the discriminant locus Δ, two or more roots coincide, and the corresponding sheets collapse. Following the terminologies in [GMN13b] and [GMN13a], we make the following definition.

Definition 4.1.8. *The **BPS ij-string** of phase $\theta \in S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$ labeled by pairs of sheets is defined to be an integral path on S obeying the following differential equation:*

$$\langle \lambda_i - \lambda_j, \partial_t \rangle \in e^{i\theta}\, \mathbb{R}_+ \tag{4.1.93}$$

where t is the "time" parameter for the path and ∂_t is the tangent vector along the path.

Locally, write $\lambda_i = f_i(z)dz$, then the equation (4.1.93) is equivalent to the following

$$\left(f_i(z(t)) - f_j(z(t))\right) \frac{dz}{dt} = e^{i\theta} \tag{4.1.94}$$

which, by introducing the $\mathbb{Z}$-affine coordinate

$$w^{ij} := \int^z \lambda_i - \lambda_j \tag{4.1.95}$$

the above equations is solved by straight lines in the rotated $\mathbb{Z}$ affine structure (see remark 3.1.17 in section 3.1.5), namely

$$Im\left(e^{-i\theta} w^{ij}\right) \equiv 0 \tag{4.1.96}$$

Remark 4.1.23. *For fixed i, j, the affine lines solving the above equation gives a foliation of the surface S.*

Denote by $\mathcal{P}_{ij}$ the path that solves the above differential equation. It can end on the branch point where

$$\lambda_{ij} := \lambda_i - \lambda_j = 0$$

or it can end on a **junction point** where a ij-string, an jk-string and an ki-string meet each other.

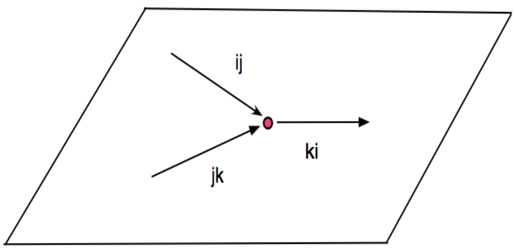

Figure 4.11: Junction point where a ij-string, a jk-string and a ki-string meet.

Definition 4.1.9. *The central charge of $\mathcal{P}_{ij}$ is defined through the following*

$$Z(\mathcal{P}_{ij}) = \int_{\mathcal{P}_{ij}} \lambda_{ij} \tag{4.1.97}$$

while the mass function is defined as

$$M(\mathcal{P}_{ij}) = \int_{\mathcal{P}_{ij}} |\lambda_{ij}| \tag{4.1.98}$$

Remark 4.1.24. *Clearly, we always have that $M(\mathcal{P}_{ij}) \geq |Z(\mathcal{P}_{ij})|$. Since BPS-states corresponds to those paths that saturate this inequality, we see that this is possible if and only if the phase of λ_{ij} stays constant along the path. In this case we have the following relation $Z(\mathcal{P}_{ij}) = e^{i\theta} M(\mathcal{P}_{ij})$ which motivates the introduction of the notion of BPS string.*

Lemma 4.1.20. *If an ij-string, jk-string and ki-string meet at a junction point, then the phases of the corresponding BPS strings coincide.*

Proof. Suppose the central charge Z has phase θ_{ij} along the path $\mathcal{P}_{ij}$, θ_{jk} along $\mathcal{P}_{jk}$, and θ_{ki} along $\mathcal{P}_{ki}$. Since p is the junction point where the three paths meet, then at this p, we must have $\theta_{ij} = \theta_{jk} = \theta_{ki}$, as the phases are preserved along the paths by the definition of the BPS-string, we conclude that the three paths share the same phase. $\square$

The ij-string $\mathcal{P}_{ij}$ can be endowed with an orientation by requiring

$$Re\left(e^{-i\theta} \int_{\mathcal{P}_{ij}} \lambda_{ij}\right) > 0 \tag{4.1.99}$$

along the path, while the opposite orientation gives an jk-BPS string.

Definition 4.1.10. *A* ***Spectral network*** *is defined to be a finite web consisting of BPS strings at a given value θ (which is justified by the above lemma 4.1.20). All strings in the web are assumed to have finite total central charge.*

An example of spectral network due to Andrew Neitzke (taken from the note of his talk in string math conference, 2013) is given below:

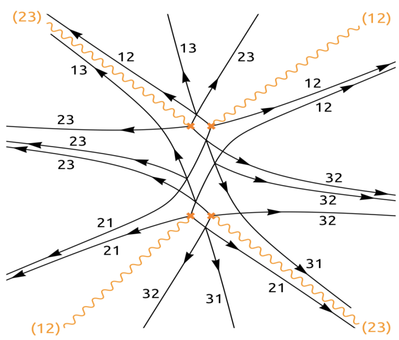

Figure 4.12: Example of spectral network. The wigged lines denote the new strings being created after the "scattering".

We now associate the **charge** to BPS-strings as well as that of in the charge lattice $\Gamma = H_1(\mathcal{C}, \mathbb{Z})$ by lifting the paths and webs from S to the SW curve $\mathcal{C}$. Thus, suppose that we are given a spectral network as a web of BPS strings

$$\mathcal{W} := \bigcup_{\text{finite}} \mathcal{P}_i \tag{4.1.100}$$

Each path $\mathcal{P}_i \subset S$ can be lifted to a closed one cycle $\gamma_i \in H_1(\mathcal{C}, \mathbb{Z})$ in $\mathcal{C}$ (see the discussion around (4.1.68) for the construction of the lift or see the relevant discussion in[GMN13a] for more details).

The lifted one cycle $\gamma_i \in H_1(\mathcal{C}, \mathbb{Z})$ is then called the **charge** associated to the path $\mathcal{P}_i$. As the union is finite, we can define the charge associated to the web $\mathcal{W}$ as

$$\gamma^{\mathcal{W}} := \sum_i \gamma_i \tag{4.1.101}$$

then the **central charge** of it is determined additively, i.e.,

$$Z(\gamma^{\mathcal{W}}) = \sum_i Z(\gamma_i) = \sum_i \oint_{\gamma_i} \lambda_{SW} \tag{4.1.102}$$

In terms of the charge and the central charge of the path, we can rewrite the equation of BPS string (see (4.1.96)) as

$$Im\left(e^{-i\theta} Z(\gamma)\right) = Im\left(e^{-i\theta} \oint_{\gamma} \lambda_{SW}\right) = 0 \tag{4.1.103}$$

were γ the charge associated to certain BPS string $\mathcal{P} \subset S$.

Specializing to the case of SW integrable system for A_{n-1}, i.e., by take $S = \mathbb{C}P^1$ as before, then the BPS string equation above is exactly the equation for self-dual string on $\mathbb{C}P^1$, which should be obvious on physics ground.

Proposition 4.1.21. *The spectral network on S associated with A_{n-1} contains the same amount of information as that of the split attractor flows introduced in section 3.2.1 so long as the determination of the BPS charges (states) are concerned.*

Proof. (**Sketch**) A single BPS string and a single attractor flow is two aspects of the same object which follows directly from proposition 4.1.19, while for the webs of the BPS strings, we just need to notice that the junction points of three BPS strings correspond exactly to the splitting point of the attractor flows by the lemma 4.1.20 proved above. Consequently, the WCS in terms of attractor flows (see section 3.2.3) can be transported to that in terms of the spectral network as had been defined and studied in great details in, for example [GMN13b],[GMN13a], and more recently[Lon18]. The proposition then follows from these considerations. □

Next we show that the **balance condition** for the split attractor flows (see definition 3.2.5) also holds in the case of spectral networks.

Proposition 4.1.22. *Consider the situation when an* ij*-BPS string* $\mathcal{P}_{ij}$*, an* jk*-BPS string* $\mathcal{P}_{jk}$ *and an* ik*-BPS string* $\mathcal{P}_{ik}$ *meet at the junction point* p*, with the corresponding charges of the three strings denoted by* γ_{ij}*,*γ_{jk} *and* γ_{ik} *respectively, then at the junction point* p*, we have that*

$$\gamma_{ik} = \gamma_{ij} + \gamma_{jk}$$

Proof. By lemma 4.1.20, the central charges of the three charges share the same phase, say θ. Consider then the following web defined as:

$$\mathcal{W}^{ijk} := \mathcal{P}_{ij} \cup \mathcal{P}_{jk} \cup \mathcal{P}_{ki}$$

with the corresponding charge $\gamma^{\mathcal{W}^{ijk}} \in H_1(\mathcal{C}, \mathbb{Z})$, where the path $\mathcal{P}_{ki}$ is the reverse path of $\mathcal{P}_{ik}$. Then we compute as follows

$$\begin{aligned}
m(\gamma_{ij}) + m(\gamma_{jk}) + m(\gamma_{ki}) &= \oint_{\gamma_{ij}} |\lambda_{SW}| + \oint_{\gamma_{jk}} |\lambda_{SW}| + \oint_{\gamma_{ki}} |\lambda_{SW}| \\
&= e^{-i\theta}\left(\oint_{\gamma_{ij}} \lambda_{SW} + \oint_{\gamma_{jk}} \lambda_{SW} + \oint_{\gamma_{ki}} \lambda_{SW}\right) \\
&= e^{-i\theta}\left(\oint_{\gamma_{ij}} (\lambda_i - \lambda_j) + \oint_{\gamma_{jk}} (\lambda_j - \lambda_k) + \oint_{\gamma_{ki}} (\lambda_k - \lambda_i)\right) \\
&\leq e^{-i\theta}\left(\oint_{\gamma^{\mathcal{W}^{ijk}}} (\lambda_i - \lambda_j) + \oint_{\gamma^{\mathcal{W}^{ijk}}} (\lambda_j - \lambda_k) + \oint_{\gamma^{\mathcal{W}^{ijk}}} (\lambda_k - \lambda_i)\right) \\
&= e^{-i\theta}\left(\oint_{\gamma^{\mathcal{W}^{ijk}}} (\lambda_i - \lambda_j) + (\lambda_j - \lambda_k) + (\lambda_k - \lambda_i)\right) \equiv 0
\end{aligned}$$

But as

$$m\left(\gamma^{\mathcal{W}^{ijk}}\right) = m(\gamma_{ij}) + m(\gamma_{jk}) + m(\gamma_{ki}) \geq 0$$

which forces it to vanish identically. As a consequence, we get that

$$Z\left(\gamma^{\mathcal{W}^{ijk}}\right) = Z(\gamma_{ij}) + Z(\gamma_{jk}) + Z(\gamma_{ki}) = Z(\gamma_{ij} + \gamma_{jk} + \gamma_{ki}) = 0$$

Since by proposition 4.1.9, the SW form λ_{SW} does not blow up, from which it follows that

$$\gamma_{ij} + \gamma_{jk} + \gamma_{ki} = 0$$

which, by our orientation convention, is equivalent to the following balancing condition at the junction point p, i.e.,

$$\gamma_{ij} + \gamma_{jk} = \gamma_{ik}$$

□

4.2 Vanishing Cycles and Monodromies

Recall that in section 4.1.2, we have already constructed SW integrable system associated to $SU(n)$. It is given through the family of SW curves (4.1.17) $\mathcal{C}_{\mathbf{u}}$ with the parameter $\mathbf{u} \in \mathcal{B} = \mathbb{C}^{n-1}$, which is identified with the base of the SW integrable system. The Lagrange tori fiber over $\mathbf{u}$ is given by the Jacobian $Jac(\mathcal{C}_{\mathbf{u}})$ of the curve $\mathcal{C}_u$, while the period lattice defining the Jacobian is identified with the first homology group with $\mathbb{Z}$ coefficients. Therefore, we get a local system of lattices over $\mathcal{B}$, with the charge lattice being defined as previously by: $\underline{\Gamma}_{\mathbf{u}} = H_1(\mathcal{C}_{\mathbf{u}}, \mathbb{Z})$. The SW integrable system is endowed with a central charge function

$$Z_{\mathbf{u}} : \underline{\Gamma}_{\mathbf{u}} = H_1(\mathcal{C}_{\mathbf{u}}, \mathbb{Z}) \to \mathbb{C}$$

which is an abelian group homomorphism defined via the SW form λ_{SW} by

$$Z_{\mathbf{u}}(\gamma) = \oint_{\gamma} \lambda_{SW} \qquad \text{for } \gamma \in \underline{\Gamma}_{\mathbf{u}}$$

We know that when the moduli $\mathbf{u}$ approaches the *discriminant locus* $\Delta \subset \mathcal{B}$, the torus fiber becomes degenerate (or singular) in the sense that certain nontrivial homology one cycles on it shrink to zero. These middle dimensional cycles are termed **vanishing cycles** and they are known to generate the middle dimensional homology $H_1(\mathcal{C}, \mathbb{Z})$ (see for example [AGV]).

Over $\mathcal{B}^0 := \mathcal{B} \backslash \Delta$, the Lagrangian torus fibration π is smooth. However, when circling around the components of Δ, a given fiber of π would usually not come back to itself, it undergoes the so called *monodromy* transformation which is given by the *Picard Lefschetz formula* (see the formula (3.1.89)).

In view of the connection to WCS (see section 3.2.3), the attractor flows, which are defined as the gradient flow lines of the real part of the (rotated) central charge function Z, will terminate at the points (attractor points) belonging to the discriminant locus Δ. The charges γ corresponding to the *monodromy invariant directions* is assigned the DT-invariants $\Omega_b(\gamma) = 1$ (see remark 3.2.14). In this way, we will obtain the *initial data* of the WCS.

The WCS can then be constructed, roughly speaking, by shooting rays (attractor flows in $\mathbb{Z}$-affine structure) from the discriminate locus Δ, and applying the KSWCF at the intersection points of these *tail edges* which lies on the *wall of the first kind* where the phases of the corresponding central charges align.

Since their central importance in construction WCS for SW integrable system, the vanishing cycles and the associated monodromies will be studied in the next few subsections. We focus here on the $SU(2)$ and $SU(3)$ cases.

4.2.1 Classical and Semi-classical Monodromy

The exposition in this subsection is based mainly on the papers [**klemm1996nonperturbative**] and [Kle+94].

Recall that the discriminant locus Δ_c in the classical moduli space $\mathscr{M}_c$ (see definition 4.1.2) is given by the intersection of certain hypersurfaces (see the formula (4.1.11)):

$$Z(\alpha_{ij}) = e_{\alpha_{ij}}(\mathbf{u}) = e_{\lambda_i} - e_{\lambda_j} \equiv 0$$

for some positive roots $\alpha \in \Phi_+$. That is, the discriminant locus Δ_c characterize the location of the vanishing 0 cycles in $\mathscr{M}_c$:

$$\nu_{\alpha_{i,j}} = e_{\lambda_i} - e_{\lambda_j}$$

We already noted in section 4.1.1 that the **classical monodromy** associated to the vanishing cycle $\nu_{\alpha_{i,j}}$, i.e., the effects on the coordinates $\mathbf{a}$ induced by tracing along a contractible loops around the hypersurface: $Z(\alpha_{ij}) = 0$ is given by the fundamental Weyl refection associated with the simple roots $\alpha_{i,j}$. We now illustrate this in $SU(2)$ and $SU(3)$ cases.

In the simplest $SU(2)$ case, we know that the discriminant $\delta_c = 4a^2 = 4u$. Thus, by looping around the point $\{u = 0\}$, i.e., $u \mapsto e^{2\pi i t}u$ for $0 \leq t \leq 1$, which amounts to $a^2 \mapsto (e^{\pi i t}a)^2$, from which we see that the effect on a is given by

$$a \mapsto -a$$

which is exactly $\mathbb{Z}_2$ transformation, i.e., the Weyl reflection in $SU(2)$ case.

Next, we consider the $SU(3)$ case. In this case, we have

$$u = \frac{1}{6}\sum_{\alpha\in\Phi_+} (Z(\alpha))^2 = \frac{1}{2}\mathbf{a}^t \cdot C_{A_2} \cdot \mathbf{a}$$

$$= \frac{1}{2}\begin{pmatrix} a_1 & a_2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1^2 + a_2^2 - a_1 a_2$$

where C_{A_2} is the Cartan matrix for A_2.

as well as

$$v = \prod_{\text{fund.rep.weights}\lambda} Z(\lambda) = e_1 e_2 e_3 = a_1(-a_2)(a_2 - a_1)$$

$$= a_1 a_2 (a_1 - a_2)$$

The discriminant is given by (see formula (4.1.12))

$$\delta_c = \prod_{i<j}^{n} \left(e_{\lambda_i}(\mathbf{u}) - e_{\lambda_j}(\mathbf{u})\right)^2 = \prod_{\alpha\in\Phi_+} (e_\alpha(\mathbf{u}))^2 = 4u^3 - 27v^2$$

It is involved to compute the monodromy directly as in the previous $SU(2)$ case. We just verify here that the monodromy is indeed given by the Weyl group action.

The Weyl group in this case reads S_3, i.e., the permutation of the three roots e_i

Around the locus Δ_c where $\nu_{1,2} = e_1 - e_2$ shrinks to zero, the monodromy is induced by the permutation $(12) \in S_3$, which exchanges e_1 and e_2 while leaving e_3 unchanged. Consequently, from the relation (4.1.9), we see that the effect on $a_i\,s$ is given by

$$a_1 \longmapsto -a_2,\; a_2 \longmapsto -a_1$$

Similarly, the transposition (13) induces the following monodromy

$$a_1 \longmapsto a_2 - a_1,\; a_2 \longmapsto a_2$$

And the transposition (23) induces the following monodromy

$$a_1 \longmapsto a_1,\; a_2 \longmapsto a_1 - a_2$$

As the permutation group S_3 is generated by these transpositions, thus the Weyl action is realized as the monodromy explicitly. Following the convention in [**klemm1996nonperturbative**], we denote by r_1 the Weyl refection induced by the transposition (13), and by r_2 that induced by (23) and r_3 that by (12). In matrix representation, the action of Weyl group on $(a_1\ a_2)^t$ is given by

$$r_1 = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \quad r_2 = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \quad r_3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \tag{4.2.1}$$

Besides the "electric coordinates" $\mathbf{a}$, we also have the dual coordinates (magnetic duals) $\mathbf{a}_D$, which is given through the prepotential function $\mathcal{F}(\mathbf{a})$ by

$$a_{D,i} \equiv \frac{\partial \mathcal{F}}{\partial a_i} \tag{4.2.2}$$

We claim that the induced monodromy on $\mathbf{a}_D$ is given by $(r^{-1})^t$ in terms of the matrix representation, where the uppercase t denotes the transpose of the matrix.

To see this, we compute as follows:

$$a_{D,i} \equiv \frac{\partial \mathcal{F}}{\partial a_i} = \sum_k \frac{\partial \mathcal{F}}{\partial u_k}\frac{\partial u_k}{\partial a_i}$$

As $\mathcal{F}$ is holomorphic and invariant under Weyl group action, and u_k s being invariant coordinates, thus the effect of the action r on $\mathbf{a}$ changes $\mathbf{a}_D$ only through the Jacobian matrix $\left(\frac{\partial u_k}{\partial a_i}\right)$ of the coordinate transformation. Denote by $\mathbf{a}'$ and $\mathbf{a}'_D$ the transformed forms of $\mathbf{a}$ and $\mathbf{a}_D$ respectively under the action of $r = (r_{ij})$. Then, we see that

$$a_{D,i} \equiv \frac{\partial \mathcal{F}}{\partial a_i} = \sum_k \frac{\partial \mathcal{F}}{\partial u_k}\frac{\partial u_k}{\partial a_i} = \sum_{k,j} \frac{\partial \mathcal{F}}{\partial u_k}\frac{\partial u_k}{\partial a'_j}\frac{\partial a'_j}{\partial a_i} = \sum_{k,j} \frac{\partial \mathcal{F}}{\partial u_k}\frac{\partial u_k}{\partial a'_j} r_{ji} = \sum_j r_{ji}\, a'_{D,j}$$

thus it follows that

$$\mathbf{a}_D^{t\prime} = (r^{-1})^t\, \mathbf{a}_D^t \tag{4.2.3}$$

Consequently, the **classical monodromy** action on $(\mathbf{a}_D)^t, \mathbf{a}$ is given in matrix forms as

$$\begin{pmatrix} \mathbf{a}'_D \\ \mathbf{a}' \end{pmatrix} = \begin{pmatrix} (r^{-1})^t & \mathbf{0} \\ \mathbf{0} & r \end{pmatrix} \begin{pmatrix} \mathbf{a}_D \\ \mathbf{a} \end{pmatrix} \tag{4.2.4}$$

For notational convenience, we denote by r^{class} the matrix that appeared in the above formula.

The above classical symmetry can be augmented to the **semi-classical symmetry** by considering the form of the prepotential $\mathcal{F}$ in the semi-classical regime, i.e., the regime where $|\mathbf{u}| \gg \Lambda^n$. In this region, the semi-classical contribution to the prepotential function is given by ([Kle+94])

$$\begin{aligned} \mathcal{F}_{\text{1-loop}} &= \frac{i}{2\pi n} \sum_{\alpha \in \Phi_+} (Z_\alpha)^2 \log\left[(Z_\alpha)^2/\Lambda^2\right] \\ &= \frac{i}{2\pi n} \sum_{i<j} (e_i - e_j)^2 \log\left[(e_i - e_j)^2/\Lambda^2\right] \end{aligned} \tag{4.2.5}$$

while the classical part of the prepotential function is given by

$$\mathcal{F}_{class} = \frac{1}{2} \tau\, \mathbf{a} \tag{4.2.6}$$

where τ is the matrix with entries $\frac{\partial a_{D,i}}{\partial a_j}$, which would be later (quantum moduli space case) identified with the period matrix of certain hyperelliptic curve.

The semi-classical contribution comes from the ambiguity in the logarithmic piece in $\mathcal{F}_{\text{1-loop}}$ when circling various locus $\{Z_\alpha = 0\}$.

$SU(2)$ **semi-classical monodromy**

In the simplest $SU(2)$ case, $e_1 = a$, $e_2 = -a$, then

$$\mathcal{F}_{\text{1-loop}} = \frac{i}{\pi} a^2 \log \frac{4a^2}{\Lambda^2} = \frac{i}{\pi}\, u \log \frac{4u}{\Lambda^2} = \frac{i}{\pi}\, u \log \frac{\delta_c}{\Lambda^2}$$

The semi-classical contribution to the monodromy action on $a_D = \frac{\partial \mathcal{F}_{\text{1-loop}}}{\partial a}$ when circling the discriminant locus $\Delta_c = \{\delta_c = 0\}$ is given by the nontrivial logarithmic piece in the expression a_D, i.e.,

$$a_D \sim \frac{2i}{\pi} a \log \frac{\delta_c}{\Lambda^2} = \frac{2i}{\pi} a \log \frac{4a^2}{\Lambda^2}$$

The actual contribution to the classical monodromy depends on the particular path circling around the singular locus $\Delta_c = \{u = 0\}$, for example, consider $u \mapsto e^{2\pi i t} u$, that is $a \mapsto e^{\pi i t} a$, for $0 \le t \le 1$, then we have that

$$\begin{aligned} a_D \mapsto \frac{2i}{\pi} e^{\pi i t} a \log \frac{4a^2 e^{2\pi i t}}{\Lambda^2} &= \frac{2i}{\pi} e^{\pi i t} a \log \frac{4a^2}{\Lambda^2} + \frac{2i}{\pi} e^{\pi i t} a(2\pi i t) \\ &= \frac{2i}{\pi} e^{\pi i t} a \log \frac{4a^2}{\Lambda^2} - 4t\, e^{\pi i t} a \end{aligned}$$

when $t = 1$, it specialize into

$$a_D \longmapsto -a_D + 4a$$

Consequently, the semi-classical monodromy acting on $(a_D, a)^t$ is given in matrix form as

$$\begin{pmatrix} a'_D \\ a' \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a_D \\ a \end{pmatrix}$$

By letting $a \mapsto \frac{1}{a}$, we see easily that the above monodromy coincide with the monodromy around $u = \infty$, for this reason, we denote by the above semi-classical monodromy matrix by $\mathcal{M}_\infty$, namely

$$\mathcal{M}_\infty = \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix} \tag{4.2.7}$$

$SU(3)$ semi-classical monodromy

Next, we consider the $SU(3)$ case. In this case, we have that

$$e_1 = a_1 \; e_2 = -a_2 \; e_3 = a_2 - a_1$$

Thus, at the branch where $Z_{\alpha_{1,3}} = e_1 - e_3 = a_2 - 2a_1 \to 0$, that is, the place where $a_2 \approx 2a_1$. Consequently, we have:

$$\begin{aligned} (e_1 - e_2)^2 = (a_1 + a_2)^2 \approx (3a_1)^2 \\ (e_2 - e_3)^2 = (a_1 - 2a_2)^2 \approx (-3a_1)^2 \end{aligned}$$

Besides, we have that

$$u = a_1^2 + a_2^2 - a_1 a_2 \approx 3a_1^2$$

Similarly, when $Z_{\alpha_{2,3}} = e_2 - e_3 = a_1 - 2a_2 \to 0$, that is, when $a_1 \approx 2a_2$, we have:

$$\begin{aligned} (e_1 - e_2)^2 = (a_1 + a_2)^2 \approx (3a_2)^2 \\ (e_1 - e_3)^2 = (a_2 - 2a_1)^2 \approx (-3a_2)^2 \end{aligned}$$

Thus

$$u = a_1^2 + a_2^2 - a_1 a_2 \approx 3a_2^2$$

Finally, when $Z_{\alpha_{1,2}} = e_1 - e_2 = a_1 + a_2 \to 0$, that is, when $a_1 \approx -a_2$, we have:

$$\begin{aligned} (e_2 - e_3)^2 = (a_1 - 2a_2)^2 \approx (3a_1)^2 \\ (e_1 - e_3)^2 = (a_2 - 2a_1)^2 \approx (-3a_1)^2 \end{aligned}$$

Thus

$$u = a_1^2 + a_2^2 - a_1 a_2 \approx 3a_1^2$$

In conclusion, we see that near the discriminant locus Δ_c, we can use u as local coordinate, in which the prepotential function (4.2.5) (can be expressed similarly as in $SU(2)$ case as

$$\begin{aligned}\mathcal{F}_{\text{1-loop}} &= \frac{i}{6\pi}\sum_{i<j} u\log\left((e_i-e_j)^2/\Lambda^2\right) \\ &= \frac{i}{6\pi}\,u\log\left(\prod_{i<j}(e_i-e_j)^2/\Lambda^6\right) = \frac{i}{6\pi}\,u\log\left(\delta_c/\Lambda^6\right)\end{aligned} \tag{4.2.8}$$

Using this, we can now determine the semi-classical contribution of the monodromy acting on $\mathbf{a}_D$.

First, the contribution to $a_{D,1}$ is determined by

$$\frac{\partial\mathcal{F}_{\text{1-loop}}}{\partial a_1} \sim \frac{i}{6\pi}\frac{\partial u}{\partial a_1}\log(\delta_c/\Lambda^6) = \frac{i}{6\pi}\left((2a_1-a_2)\log(\delta_c/\Lambda^6)\right) \tag{4.2.9}$$

while that of $a_{D,2}$ is determined by

$$\frac{\partial\mathcal{F}_{\text{1-loop}}}{\partial a_2} \sim \frac{i}{6\pi}\frac{\partial u}{\partial a_2}\log(\delta_c/\Lambda^6) = \frac{i}{6\pi}\left((2a_2-a_1)\log(\delta_c/\Lambda^6)\right) \tag{4.2.10}$$

Thus, tracing along the locus in which $e_1 \to e_3$, the monodromy on $(a_1\,a_2)^t$ is given by the action of the Weyl reflection r_1 which can be implemented by the following path

$$\begin{aligned} a_1(t) &= e^{i\pi t}a_1 + \frac{1}{2}(1-e^{i\pi t})a_2 \\ a_2(t) &\equiv a_2 \end{aligned}$$

Consequently, the coefficient $(2a_1-a_2)$ in (4.2.8) transform into

$$2(a_2-a_1)-a_2 = a_2-2a_1$$

Similarly, the coefficient $(2a_2-a_1)$ in (4.2.9) transform into

$$2a_2-(a_2-a_1) = a_1+a_2$$

Besides, as $u \approx 3a_1^2$ and $v \approx -2a_1^3$, we see that $\delta_c = 4u^3-27v^2$ is of homogeneous degree 6 in a_1, so we see easily that upon traversing along the curve given above, the logarithmic pieces in (4.2.8) and (4.2.9) will pick up a factor of $\log e^{6\pi it} = 6\pi it$, which when multiplied by the coefficient $\frac{i}{6\pi}$, gives us the monodromy factor -1 after evaluating at $t=1$.

In conclusion, we see that the semi-classical monodromy action on $\mathbf{a}_D$ is given by the following:

$$\begin{pmatrix} a'_{D,1} \\ a'_{D,2}\end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & 1\end{pmatrix}\begin{pmatrix} a_{D,1} \\ a_{D,2}\end{pmatrix} + \begin{pmatrix} 2 & -1 \\ -1 & -1\end{pmatrix}\begin{pmatrix} a_1 \\ a_2\end{pmatrix} \tag{4.2.11}$$

Combining with the monodromy action on $\mathbf{a}$, we conclude that the total semi-classical monodromy associated to the Weyl reflection r_1 is given by the following matrix which acts on the vector $(\mathbf{a}_D \quad \mathbf{a})^t$ as:

$$\mathcal{M}^c_{r_1} = \begin{pmatrix} -1 & 0 & 2 & -1 \\ 1 & 1 & -1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.2.12}$$

Which can be rewritten more succinctly in block form as follows:

$$\mathcal{M}^c_{r_1} = \begin{pmatrix} (r_1^{-1})^t & \mathbf{0} \\ \mathbf{0} & r_1 \end{pmatrix} \begin{pmatrix} \mathbf{1} & \tilde{C} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \tag{4.2.13}$$

where the matrix $\tilde{C}$ is determined by

$$(r_1^{-1})^t \, \tilde{C} = \begin{pmatrix} 2 & -1 \\ -1 & -1 \end{pmatrix}$$

from which we get that

$$\tilde{C} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} = - \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = -C$$

where C is the Cartan matrix of A_2.

Consequently, we get that

$$\mathcal{M}^c_{r_1} = \begin{pmatrix} (r_1^{-1})^t & \mathbf{0} \\ \mathbf{0} & r_1 \end{pmatrix} \begin{pmatrix} \mathbf{1} & -C \\ \mathbf{0} & \mathbf{1} \end{pmatrix} = \begin{pmatrix} (r_1^{-1})^t & \mathbf{0} \\ \mathbf{0} & r_1 \end{pmatrix} \begin{pmatrix} \mathbf{1} & C \\ \mathbf{0} & \mathbf{1} \end{pmatrix}^{-1}$$

Denote by T the matrix $\begin{pmatrix} \mathbf{1} & C \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$, which is called the **quantum monodromy** in physics literature.

Finally, we write the total semi-classical monodromy as

$$\mathcal{M}^c_{r_1} = r_1^{class} \, T^{-1} \tag{4.2.14}$$

Computing in exactly the same fashion as that for r_1 for the other two fundamental Weyl reflections r_2 and r_3, the same pattern appears. We conclude that the semi-classical monodromy is generated by

$$\mathcal{M}^c_{r_i} = r_i^{class} \, T^{-1}, \qquad i = 1, 2, 3. \tag{4.2.15}$$

By the matrix forms of r_2 and r_3 in the formula (4.2.1), we give the matrix representations of $\mathcal{M}^c_{r_2}$ and $\mathcal{M}^c_{r_2}$ as follows

$$\mathcal{M}^c_{r_2} = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & -1 & -1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \qquad \mathcal{M}^c_{r_3} = \begin{pmatrix} 0 & -1 & -1 & 2 \\ -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{4.2.16}$$

Monodromy in terms of the SW-curve

We want reproduce the monodromy matrix directly in terms of the geometry of the SW curve. The idea is that we choose a particular point $\mathbf{u} \in \mathcal{B}$ such that the monodromy would be easy to see in this case. The following exposition is based on the treatment given in [AF95].

We choose a special point $\mathbf{u} = (u_2, \cdots, u_{n-1}, u_n) = (0, \cdots, 0, u_n)$, which corresponds to the direction in moduli space in which the roots of $\mathcal{W}_{n-1}(x, \mathbf{u}) - \Lambda^{2n}$ are maximally separated. Indeed, in this case, the curve (4.1.17) becomes (for the sake of simplicity, in the following, we denote by $u \equiv u_n$)

$$y^2 = (x^n - u)^2 - \Lambda^{2n} = (x^n - u + \Lambda^n)(x^n - u - \Lambda^n)$$

Denote by $u_\pm := \sqrt[n]{u \mp \Lambda^n}$. Notice that in the semi-classical region, i.e., when $|u| \gg \Lambda^n$, we should have that $u_+ \approx u_-$. The roots of the right hand of the above equation then become

$$e_k^\pm = \omega_n^k \, u^\pm, \quad 1 \leq k \leq n.$$

where $\omega_n := e^{\frac{2\pi i}{n}}$ denotes the n-th root of unit. Consequently, among these $2n$-roots, n of them, namely $\{e_i^+\}$ (or $\{e_i^-\}$) are distributed evenly along a circle centered at the origin with radius $|u_+|$ (or $|u_-|$). The two circles become very close to each other when in the semi-classical region, i.e., in large radius region.

The cycles $\alpha^i \, (1 \leq i \leq n)$ is defined to be the (lifted up to the double cover of the x-complex plane) one encircling the pair of roots $e_i^\pm$, while the cycle $\beta_i \, (1 \leq i \leq n-1)$ the one that encircles the roots e_i^+ and e_n^+. Notice that the cycles α^i are not linearly independent since they are easily seen to satisfy the relation

$$\sum_i \alpha^i = 0$$

Clearly, the intersection numbers between α^i and β_i are given as $\langle \alpha^i, \beta_j \rangle = \delta^i_j$ after choosing proper orientations along the α and β cycles.

Now consider the monodromy around $u \gg \Lambda^n$, i.e., the monodromy around ∞ in the moduli space. Thus, take $u \mapsto e^{2\pi i\theta} \, u, \; 0 \leq \theta \leq 1$, we see that

$$e_k^\pm = \omega_n^k \, u_\pm \longmapsto \omega_n^k \, \sqrt[n]{e^{2\pi i\theta} \, u \mp \Lambda^n} \approx \omega_n^k \, \omega \, \sqrt[n]{u} \approx \omega_n^{k+1} \, u_\pm = e_{k+1}^\pm$$

which means that the cycle α^k is get transformed into α^{k+1}.

These transformations generate the permutation group S_n, which is exactly the Weyl group for $SU(n)$.

This further implies that the semi-classical monodromy acts on $\mathbf{a}$ by Weyl reflections, which justifies our previous results by using the perturbative behaviour of the prepotential function $\mathcal{F}$.

Thus, the monodromy acts on $\overrightarrow{\alpha} = (\alpha^1 \cdots \alpha^{n-1})$ by the permutation matrix P defined as

$$P^{ij} = \delta^{i+1,j} - \delta^{i+1,n}$$

The action on the β cycles is slightly more complicated, let us determine it now.

As had been displayed above, going around a large loop near ∞, we have that

$$e_k^+ \longmapsto e_{k+1}^+, \ 1 \leq k \leq n-1, \ \text{and}\, e_n^+ \longmapsto e_1^+$$

So the action corresponds to the permutation $(12\cdots n)$. As during the rotation process, the cycle β_k crosses the branch cut connecting e_k^+ and e_k^-, we expect that the transformed cycle $\tilde{\beta}$ will receive contribution from α^k.

To determine it, we assume that the monodromy action on $\boldsymbol{\beta} = (\beta_1 \ \cdots \beta_{n-1})^t$ is given in the following form:

$$\boldsymbol{\beta} \longmapsto A \cdot \boldsymbol{\beta} + B \cdot \boldsymbol{\alpha}$$

We claim that the matrix A is given by $A = (P^{-1})^t$. Indeed, under the monodromy transformation, the intersection numbers undergo no changes. Thus, we should have that

$$\left\langle \sum_k P^{ik} \cdot \alpha_k, \ \sum_l A_{jk} \cdot \beta^k + \sum_k B^{jl} \cdot \alpha_l \right\rangle = \langle \alpha^i, \beta_j \rangle$$

from which we get

$$\sum_k P^{ik} A^{jk} = \delta^i_j$$

Thus, $A = (P^{-1})^t$. To determine the "correction" matrix B, we first note that from the above discussion, the (semi-classical) monodromy (around ∞) acts on the $2(n-1)$ column vector $(\boldsymbol{\beta} \quad \boldsymbol{\alpha})^t$ by the matrix

$$M = \begin{pmatrix} (P^{-1})^t & B \\ \mathbf{0} & P \end{pmatrix} = \begin{pmatrix} (P^{-1})^t & \mathbf{0} \\ \mathbf{0} & P \end{pmatrix} \cdot \begin{pmatrix} \mathbb{I} & \widetilde{C} \\ \mathbf{0} & \mathbb{I} \end{pmatrix}$$

where $(P^{-1})^t \cdot \widetilde{C} = B$. We now make the "ansatz" such that

$$(P^{-1})^t \cdot \widetilde{C} = \widetilde{C} \cdot P$$

so that the two matrices on the right hand side of the above identity commute with each other, and consequently, we should have that

$$M^n = \begin{pmatrix} (P^{-1})^t & \mathbf{0} \\ \mathbf{0} & P \end{pmatrix}^n \cdot \begin{pmatrix} \mathbb{I} & \widetilde{C} \\ \mathbf{0} & \mathbb{I} \end{pmatrix}^n = \begin{pmatrix} \mathbb{I} & n\widetilde{C} \\ \mathbf{0} & \mathbb{I} \end{pmatrix}$$

Applying the monodromy M n-times, all roots are rotated back to the original place, thus all α cycles return to themselves as indicated by the entries of the second row of the above matrix.

One subtle point here is that even though the β cycles would go back to themselves, however, as β_k wounds n-times around the branch-cuts connecting $\{e_n^\pm\}$ and $\{e_k^\pm\}$ respectively, we see that it gets shifted by

$$n\alpha^{k-1} - 2n\alpha^k + n\alpha^{k+1} + 2n\alpha^n$$

From this observation, one can infer that

$$\beta_k \longmapsto \beta_k + n\sum_j \widetilde{C}_{kj}\,\alpha^k = \beta_k + n\alpha^{k-1} - 2n\alpha^k + \alpha^{k+1}$$

Thus the matrix $\widetilde{C}$ is given by

$$\widetilde{C}_{kj} = \delta_{k-1,j} - 2\,\delta_{k,j} + \delta_{k+1,j}$$

That is $\widetilde{C} = -C$, where C is the Cartan matrix for A_{n-1}.

This agrees with the previously computed (by using prepotential $\mathcal{F}$) results for the A_1 and A_2 case, and it generalizes to arbitrary $SU(n)$.

4.2.2 Quantum Monodromy: $SU(2)$ case

In the quantum situation, the classical discriminant locus Δ_c splits into two. That is, the quantum discriminant locus becomes: $\Delta_\Lambda = \Delta_+ \cup \Delta_-$, over which certain cycles on the torus fibers degenerate.

By the well known facts in the singularity theory (for example [AGV]), the monodromies generated by tracing paths around these discriminant are given by the *Picard-Lefschetz formula*, namely, for the component of Δ_Λ where ν is the associated vanishing cycle, then the associated monodromy $\mathscr{M}_\nu$ acts on the one cycle $\mu \in H_1(\mathcal{C}_{\mathbf{u}}, \mathbb{Z})$ is given as

$$\mathscr{M}_\nu : \mu \longmapsto \mu + \langle \nu, \mu \rangle \mu \tag{4.2.17}$$

where $\langle \cdot, \cdot \rangle$ denotes the intersection paring on the one cycles.

In this subsection, we compute the vanishing cycles and the associated monodromies in $SU(2)$ and $SU(3)$ cases, which would be our first step toward determining the WCS for both cases. To start with, let us reproduce the relevent results for the $SU(2)$ case as a warming us exercise toward the more involved $SU(3)$ case.

The methods and results presented in this subsection, while scattered around physics literature ([Kle97],[Ler98][SW94],[AD95],[Kle+94],[KTL95],[Kle+95],[AF] etc.) are all well-known, thus no originality is claimed, despite the fact that substantial revision and re-organization have been made on my part to suit for the main theme of this thesis.

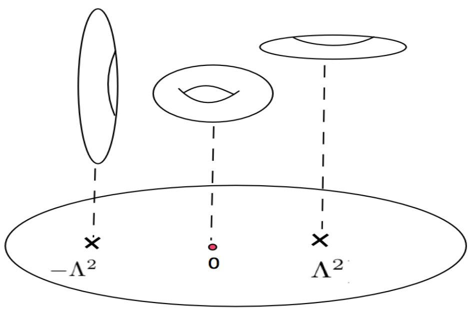

Figure 4.13: Seiberg-Witten integrable system associated to $SU(2)$.

The moduli space $\mathcal{B}$ is the complex plane $\mathbb{C}$ in this case. And the discriminant locus Δ_Λ consists of two points $\{-\Lambda^2, \Lambda^2\}$, over which the torus fibration degenerates. This is illustrated by the figure 4.13 above.

The fiber over smooth point $\mathbf{u} \in \mathcal{B}^0 := \mathcal{B} \backslash \Delta_\Lambda$ is given by (cf., (4.1.21))

$$\mathcal{C}_{\mathbf{u}} : y^2 = \mathcal{W}_{A_1}^2 - \Lambda^4 = (x^2 - u)^2 - \Lambda^4$$

The right hand side polynomial factors into the product of two polynomials, given as

$$P_\pm(u; x) = x^2 - u \pm \Lambda^2$$

Denote by $e_k^{\pm}$, $k = 1, 2$. the roots to the polynomial $P_{\pm}$ respectively, which are given as

$$\begin{aligned} e_1^+ &= \sqrt{u - \Lambda^2} \qquad & e_2^+ &= -\sqrt{u - \Lambda^2} \\ e_1^- &= \sqrt{u + \Lambda^2} \qquad & e_2^- &= -\sqrt{u + \Lambda^2} \end{aligned}$$

Now consider a small loop

$$\gamma_{\Lambda^2}(\delta) := \{u : u = \delta e^{i\theta} + \Lambda^2\}$$

of small radius δ around the singularity Λ^2 in $\mathcal{B}$. By tracing around the loop, i.e., by letting the parameter θ go from 0 to 2π, th two roots e_1^+, e_2^+ rotates around each other. Indeed, along the loop, the two roots have the following expression:

$$e_1^+ = \sqrt{\delta}\, e^{\frac{i\theta}{2}} \qquad e_2^+ = -\sqrt{\delta}\, e^{\frac{i\theta}{2}}$$

Thus, as θ goes from 0 to 2π, e_1^+ goes from $\sqrt{\delta}$ to $-\sqrt{\delta}$, while e_2^+, the accompanying root, goes from $-\sqrt{\delta}$ to $\sqrt{\delta}$.

By a similar argument, it is easy to see that when circling around the small loop $\gamma_{-\Lambda^2}(\delta)$ around the other singularity $-\Lambda^2$, the roots e_1^- and e_2^- will rotate around each other.

Then, the canonical basis for homology one cycles on the torus fibers are given as the lift of the two cycles circling around the two pairs of roots to its double cover. Namely, the cycle encircling around e_2^- and e_2^+ on the complex plane, when lifted, gives the cycle denoted by α, while the cycle encircling around e_2^+ and e_1^+ gives the beta cycle β. Choosing the orientation of the two cycles properly, we assume that their intersection number is given as $\langle \alpha, \beta \rangle = 1$.

As the moduli $\mathbf{u}$ approaches to the singularity Λ^2, the root e_2^+ collide with the root e_1^+, which means that the **vanishing cycle** $\nu_{+\Lambda^2}$, associated to the singularity $+\Lambda^2$ is just the cycle β.

Hence, we can compute the (quantum) monodromy $\mathcal{M}_{+\Lambda^2}$ by using the Picard-Lefschetz formula (4.2.17) as follows:

$$\begin{aligned} \alpha \longmapsto \alpha + \langle \beta, \alpha \rangle \beta = \alpha - \beta \\ \beta \mapsto \beta + \langle \beta, \beta \rangle \beta = \beta \end{aligned}$$

Recall that we have identified the special Kähler coordinates a and a_D with the period integral of the elliptic curve in question, namely

$$a = \oint_\alpha \lambda_{SW} \qquad a_D = \oint_\beta \lambda_{SW}$$

where the Seiberg-Witten differential in this $SU(2)$ case is given by (see (4.1.42))

$$\lambda_{SW} = \frac{2x^2 dx}{y}$$

We see that the monodromy acting on $(a_D\ a)^t$ is determined through

$$a_D = \oint_\beta \lambda_{SW} \longmapsto a_D$$

while for the period a, we have that

$$a = \oint_\alpha \lambda_{SW} \longmapsto \oint_{\alpha-\beta} \lambda_{SW} = \oint_\alpha \lambda_{SW} - \oint_\beta \lambda_{SW} = a - a_D$$

Thus, the quantum monodromy $\mathcal{M}_{+\Lambda^2}$ associated to the singularity $+\Lambda^2$ is given in matrix form (which acts on $(a_D\ a)^t$) on the left) as:

$$\mathcal{M}_{+\Lambda^2} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \tag{4.2.18}$$

Remark 4.2.1. *The monodromy above can be determined directly by computation. As β is the lifted cycle of that encircling the two roots e_2^+ and e_1^+ on the complex plane, we see that β in unchanged when tracing along a small loop around $+\Lambda^2$, consequently, the β-period a_D remains unchanged during the process. However, the α-period a changes according the following:*

$$\begin{aligned} a = \oint_\alpha \lambda_{SW} = 2\int_{e_2^-}^{e_2^+} \lambda_{SW} &\longmapsto 2\int_{e_2^-}^{e_1^+} \lambda_{SW} \\ &= 2\int_{e_2^-}^{e_2^+} \lambda_{SW} + 2\int_{e_2^+}^{e_1^+} \lambda_{SW} = a - a_D \end{aligned}$$

Similarly, when u approaches the singularity $-\Lambda^2$, the two roots e_1^- and e_2^- collide with each other, which causes the cycle that is the lift of the cycle encircling these two roots on $\mathbb{C}$ shrinking into zero size. Consequently, the vanishing cycle $\nu_{-\Lambda^2}$ can be chosen in the basis $\{\alpha, \beta\}$ as $\nu_{-\Lambda^2} = \beta - 2\alpha$, from which we compute the quantum monodromy associated by using Picard-Lefschtz formula as follows:

$$\begin{aligned} \mathcal{M}_{-\Lambda^2}(\alpha) = \alpha + \langle \nu_{-\Lambda^2}, \alpha\rangle\, \nu_{-\Lambda^2} = \alpha + \langle \beta - 2\alpha, \alpha\rangle\,(\beta - 2\alpha) = 3\alpha - \beta \\ \mathcal{M}_{-\Lambda^2}(\beta) = \beta + \langle \nu_{-\Lambda^2}, \beta\rangle\, \nu_{-\Lambda^2} = \beta + \langle \beta - 2\alpha, \beta\rangle\,(\beta - 2\alpha) = 4\alpha - \beta \end{aligned}$$

which means that $\mathcal{M}_{-\Lambda^2}$ acts on $(a_D\ a)^t$ as

$$\begin{aligned} a_D = \oint_\beta \lambda_{SW} &\longmapsto \oint_{4\alpha-\beta} \lambda_{SW} = 4a - a_D \\ a = \oint_\alpha \lambda_{SW} &\longmapsto \oint_{3\alpha-\beta} = 3a - a_D \end{aligned}$$

In matrix form it is given by

$$\mathcal{M}_{-\Lambda^2} = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix} \tag{4.2.19}$$

Remark 4.2.2. *It is easy to verify that the product of the two quantum monodromies $\mathcal{M}_{\pm\Lambda^2}$ coincide with the semi-classical monodromy $\mathcal{M}_\infty$ given in last subsection (see formula (4.2.7)), i.e., we have the following formula*

$$\mathcal{M}_{+\Lambda^2}\mathcal{M}_{-\Lambda^2} = \mathcal{M}_\infty \tag{4.2.20}$$

This relation should be anticipated in view of the fact that the monodromy action forms a representation of the fundamental group (see (3.1.70) and the related discussion there) of $\mathcal{B}^0 := \mathbb{C}\backslash\{-\Lambda^2, +\Lambda^2\}$ into $SL(2,\mathbb{Z})$ in our case (More precisely, $\mathcal{M}_{\pm\Lambda^2}$, together with $\mathcal{M}_\infty$ generates the subgroup $\Gamma_0(4) \subset SL(2,\mathbb{Z})$).

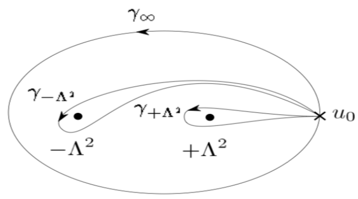

Figure 4.14: Monodromy relation

Apparently, the fundamental group $\pi_1(\mathcal{B}^0, u_0)$ is generated by two loops,namely, the closed loops $\gamma_{\pm\Lambda^2}$ based at u_0 encircling the two singularities $\pm\Lambda^2$ respectively. Tracing along the two loops gives the two monodromies $\mathcal{M}_{\pm\Lambda^2}$. However, by taking a loop that encompassing both loops, which is to be denoted by γ_∞, then with the proper orientations being assigned to the three loops, we have the equivalence of loops:

$$\gamma_\infty \approx \gamma_{+\Lambda^2} \cdot \gamma_{-\Lambda^2}$$

from which the above identity (4.2.20) follows.

Remark 4.2.3. *We notice that the monodromies $\mathcal{M}_{\pm\Lambda^2}$ computed above by using the Picard-Lefschtz formula associated to the vanishing cycles $\nu_{\pm\Lambda^2}$ coincide with that (see formulas (A.6.16) and (A.6.20) in appendix A) computed by using the perturbative expansion of the prepotential $\mathcal{F}_{1-loop}$ (4.2.5) in appendix A (except that in the appendix the two singularities $\pm\Lambda^2$ are normalized to be ± 1 respectively, but that does not alter the monodromies considered here). This provides rationality for using the Seiberg-Witten (hyper-elliptic) curve to encode the analytically information expressed in terms of the prepotential function $\mathcal{F}$, which is an infinite perturbative expansion (instanton expansion in physicist's terminology). This is wonderful since the curve, though a finite object, could encode the perturbation of prepotential function. This gives the so called "exact solutions" (non-perturbative aspect) to the physics question regarding $N = 2, d = 4$ supersymmetric Yang-Mills theory (see for example [SW94] for more information about this story).*

We give further rationality for the relation (4.2.20) by computing directly from the Seiberg-Witten curve that the semi-classical monodromy is indeed given by $\mathcal{M}_\infty$. To this end, we use the another form of Seiberg-Witten curve (as Weight diagram fibration; see section 4.1.3 for more details) to do the computation. This will proven to be much simpler than the curve given in terms of the A_1-singularity. Since we have proved the equivalence (bi-rational) of the two forms of curves, this does not affect the results.

First, recall that in $SU(2)$-case, the curve is given by (4.1.51) as

$$z + \frac{\Lambda^4}{z} + 2\left(x^2 - u\right) = 0 \tag{4.2.21}$$

while the Seiberg-Witten differential assumes the following form

$$\lambda_{SW} = -x\frac{dz}{z} \tag{4.2.22}$$

The branch points, besides 0 and ∞, had been computed to be

$$z_{\pm} = u \pm \sqrt{u^2 - \Lambda^4} \tag{4.2.23}$$

Then the semi-classical monodromy can be computed as follows:

The semi-classical region corresponds to the region where $|u| \gg \Lambda^2$, i.e., near ∞ in $\mathcal{B}$. The α cycle is obtained by lifting the circle $|z| = 1$ on $\mathcal{B} \cong \mathbb{C}$, while the β cycle can be obtained by lifting the cycle circling the two branch points $z_{\pm}$. Then in the semi-classical limit $|u| \gg \Lambda^2$, from equation (4.2.21), we see that near the circle $|z| = 1$, x is approximately $\sqrt{u} = a$, thus the α-period integral is computed as:

$$a = \oint_{\alpha} \lambda_{SW} = -\oint_{|z|=1} x\frac{dz}{z} \approx -\oint_{|z|=1} a\,\frac{dz}{z} \approx -2\pi i\sqrt{u}$$

From expression (4.2.23), we see easily that in the semi-classical limit, x can be approximated near the β-cycle as $x \approx \sqrt{u} = a$, thus the β-period a_D is computed as:

$$\begin{aligned} a_D = \oint_{\beta} \lambda_{SW} &= -2\int_{z_-}^{z_+} x\,\frac{dz}{z} = -2\int_{u-\sqrt{u^2-\Lambda^4}}^{u+\sqrt{u^2-\Lambda^4}} \sqrt{u}\,\frac{dz}{z} \\ &= -2\sqrt{u}\,\ln\frac{u+\sqrt{u^2-\Lambda^4}}{u-\sqrt{u^2-\Lambda^4}} = -2\sqrt{u}\,\ln\frac{(u+\sqrt{u^2-\Lambda^4})^2}{\Lambda^4} \\ &\approx -4\sqrt{u}\,\ln\frac{2u}{\Lambda^2} \end{aligned}$$

From above computations, then it is easy to see by taking a loop around $u = \infty$, $a \mapsto -a$, while $a_D \mapsto 4a - a_D$, which is indeed the semi-classical monodromy $\mathcal{M}_\infty$ given before (by formula (4.2.7)).

Remark 4.2.4. *Since the Seiberg-Witten curve (4.2.21) has four branch points, given respectively by $z_+, z_-, 0, \infty$, these are points where the double cover degenerates. This suggests to consider the following elliptic curve, which has ramification exactly at these four branch points:*

$$y^2 = x(x - z_+)(x - z_-) = x(x - u + \sqrt{u^2 - \Lambda^4})(x - u - \sqrt{u^2 - \Lambda^4})$$

That is

$$y^2 = x^3 - 2ux^2 + \Lambda^4 x \tag{4.2.24}$$

which has two branch cuts, one from 0 to z_+, and the other from z_- to ∞.

By the construction, it contains the same topological information as given by the form of curve (4.2.21) as well as the form given in terms of A_1 Singularity (see equation (4.1.21)), we may expect them all to be birational equivalent in the sense that they have the same j-invariant as elliptic curves, thus differ each other only by some birational transformations of the variables of defining equations. Indeed, the birational equivalence of the later two curves has been established at the beginning paragraph of the section 4.1.3, while the direct equivalence between (4.1.21) and (4.2.24) can be obtained by using projective transformation that maps the four zeros $e_{1,2}^{\pm}$ to the four branch points $0, \infty, z_{\pm}$ in some proper order, we omit the details here.

Before ending this subsection, let me use the form of curve given in (4.2.24) to verify that it also gives the correct monodromies, which is expected as it is birational to the previous forms of Seiberg-Witten curves.

First, we prove that the curve can reproduce the correct periods, thus the prepotential function $\mathcal{F}_{1-loop}$ near ∞.

Proposition 4.2.1. *Choosing Seiberg-Witten differential on the curve (4.2.24) to be $\lambda_{SW} = \frac{dx}{y}$ (as it is the unique holomorphic differential in this case, up to a normalizing constant), then the period integrals of the curve give the desired expressions of special Kähler coordinates $(a\ a_D)$ near $u = \infty$ neighborhood.*

Proof. As $u^2 \gg \Lambda^4$, i.e., in semi-classical region, we have that

$$z_+ = u - \sqrt{u^2 - \Lambda^4} = \frac{\Lambda^4}{u + \sqrt{u^2 - \Lambda^4}} \approx \frac{\Lambda^4}{2u}$$

while

$$z_- = u + \sqrt{u^2 + \Lambda^4} \approx 2u$$

Since α cycle can be represented as small circle around the branch points 0 and z_+, and on this small circle α, since x is small compared with $2u$, and thus $x - 2u \approx -2u$. Consequently, the α-period, and hence the "electric coordinate" a, can be computed as

$$a = \oint_\alpha \lambda_{SW} = \oint_\alpha \frac{dx}{y} = \oint_\alpha \frac{dx}{\sqrt{x(x-z_+)(x-z_-)}} =$$

$$= \oint_\alpha \frac{dx}{\sqrt{x(x-\frac{\Lambda^4}{2u})(x-2u)}} \approx \frac{1}{\sqrt{2u}i} \oint_\alpha \frac{dx}{\sqrt{x(x-\frac{\Lambda^4}{2u})}}$$

$$\approx \frac{1}{\sqrt{2u}i} \oint_\alpha \frac{dx}{x} = \frac{1}{\sqrt{2u}i} 2\pi i = \pi\sqrt{\frac{2}{u}}$$

On the other hand, the β cycle is obtained by circling the branch points z_+ and z_-, thus we can compute the β-period a_D, i.e., the "magnetic coordinate" as follows:

$$a_D = \oint_\beta \frac{dx}{y} = 2\int_{z_+}^{z_-} \frac{dx}{\sqrt{x(x-\frac{\Lambda^4}{2u})(x-2u)}} = -2i\int_{z_+}^{z_-} \frac{dx}{\sqrt{x(x-\frac{\Lambda^4}{2u})(2u-x)}}$$

To estimate the above integral, we split it into two parts by choosing a point x_c such that $\frac{\Lambda^4}{2u} \ll x_c \ll 2u$, thus

$$\int_{z_+}^{z_-}(\cdots) = \int_{\frac{\Lambda^4}{2u}}^{2u}(\cdots) = \int_{\frac{\Lambda^4}{2u}}^{x_c}(\cdots) + \int_{x_c}^{2u}(\cdots)$$

Then we notice that over $\left[\frac{\Lambda^4}{2u},\, x_c\right]$, x is very small compared with $2u$, thus the first part of integral in above formula can be approximated by

$$\int_{\frac{\Lambda^4}{2u}}^{x_c} \frac{dx}{\sqrt{x(x-\frac{\Lambda^4}{2u})(2u)}} = \frac{1}{\sqrt{2u}}\int_{\frac{\Lambda^4}{2u}}^{x_c} \frac{dx}{\sqrt{x(x-\frac{\Lambda^4}{2u})}} \approx \frac{1}{\sqrt{2u}}\log\frac{8u\,x_c}{\Lambda^4}$$

while over $[x_c,\, 2u]$, $x - \frac{\Lambda^4}{2u} \approx x$, thus the second piece of integral can be approximated by

$$\int_{x_c}^{2u}\frac{dx}{x\sqrt{2u-x}} \overset{2ut=x}{=\!=\!=} \int_{\frac{x_c}{2u}}^{1}\frac{2u\,dt}{2ut\sqrt{2u-2ut}} = \frac{1}{\sqrt{2u}}\int_{\frac{x_c}{2u}}^{1}\frac{dt}{t\sqrt{1-t}}$$
$$\approx \frac{1}{\sqrt{2u}}\log\frac{8u}{x_c}$$

Combing the above computations, we see that as $u \to \infty$, we have that

$$a_D \approx -2i\left(\frac{1}{\sqrt{2u}}\log\frac{8u\,x_c}{\Lambda^4} + \frac{1}{\sqrt{2u}}\log\frac{8u}{x_c}\right) = -2i\sqrt{\frac{2}{u}}\log\frac{8u}{\Lambda^2}$$

□

Given the above expressions for $(a_D\ a)$, by considering the loop around $u = \infty$, i.e., in the above expressions, by letting $u \mapsto e^{2\pi\theta}u$, we see that the semi-classical monodromy $\mathcal{M}_\infty$ is reproduced, thus the following:

Corollary 4.2.1.1. *The curve (4.2.24) could be used to produce the semi-classical monodromy $\mathcal{M}_\infty$.*

Remark 4.2.5. *The semi-classical monodromy obtained above uses can also be read off geometrically as follows: We know from the above that near $u = \infty$, the four branch points are given respectively by $0, \frac{\Lambda^4}{2u^2}, 2u$ and ∞, we will obtain the monodromy $\mathcal{M}_\infty$ by keeping track with the motion of these branch points when circling around $u = \infty$. To make life easy, let us first re-scale the variables (which does not affect the topological information to be concerned here) so to make $2u$ stays finite during the process. Thus, define $\tilde{x} = \frac{x}{u}$, so that the four branch points get mapped into $0, \frac{\Lambda^4}{2u^2}, 2, \infty$. And the accompanied re-scaling on y variable is given by $\tilde{y} = y/u^{\frac{3}{2}}$, so that the equation for the elliptic curve transforms into the following*

$$\tilde{y}^2 = \tilde{x}^3 - 2\tilde{x}^2 + \frac{\Lambda^4}{u^2}\tilde{x}$$

while the re-scaled Seiberg-Witten differential is given by

$$\lambda_{SW} = \frac{dx}{x} \longmapsto \frac{d\tilde{x}}{\tilde{y}} = u^{-\frac{1}{2}}\frac{dx}{y}$$

Let us trace along the $u = \infty$ by letting $u \mapsto e^{2\pi i\theta}u, \ \ 0 \le \theta \le 1$. It easy to see that the branch point $\frac{\Lambda^4}{2u^2}$ will rotate twice around the branch point 0, while the other two branch points stay unchanged during the process. As α cycle is the one encircling 0 and $\frac{\Lambda^4}{2u^2}$, we see that it remains the same, however, the β-cycle, which encircles $\frac{\Lambda^4}{2u^2}$ and 2, will be get shifted by -4α as it would across the branch cut connecting 0 and $\frac{\Lambda^4}{2u^2}$ twice during the process. Thus the monodromy around $u = \infty$ acts on α and β cycles as

$$\alpha \longmapsto \alpha, \qquad \beta \longmapsto \beta - 4\alpha$$

However, since the factor $u^{-\frac{1}{2}}$ in the re-scaled SW differential as above changes sign when circling around $u = \infty$, we conclude that the monodromy acts on $(a_D \ \ a)$ as follows:

$$a_D \longmapsto 4a - a_D \qquad a \longmapsto -a$$

which is indeed the desired semi-classical monodromy $\mathcal{M}_\infty$ given before.

Remark 4.2.6. *Besides the above proposition and corollary, the monodromies as $u \to \pm\Lambda^2$ can also be checked to coincide with that computed by using the Picard-Lefschtz formula before, we omit the details here.*

From the discussion in the remark 4.2.5 above, we can deduce the following proposition, which tells us about the vanishing cycle associated to $u = \infty$.

Proposition 4.2.2. *The vanishing cycle at ∞, which, in either forms of Seiberg-Witten curve given, correspond to more than two roots colliding with each other, i.e., "non-stable" degeneration case, is given by $\nu_\infty = 2\alpha$.*

Proof. As already noted in remark 4.2.5 above, we see that as we loop around $u = \infty$ once (counter-clock wisely), SW differential λ_{SW} changes its sign, we just need to know how the monodromy acts on the cycles. To this end, let us assume that its effect on the symplectic basis $\{\alpha, \beta\}$ is given by

$$\alpha \longmapsto m\alpha + n\beta \qquad \beta \longmapsto p\alpha + q\beta$$

where the matrix $\begin{pmatrix} m & n \\ p & q \end{pmatrix} \in SL(2, \mathbb{Z})$. This means that its effect on the period matrix $(a_D \ \ a)$ is given by

$$a_D \longmapsto -pa - qa_D \qquad a \longmapsto -ma - na_D$$

Comparing with the known result for $\mathcal{M}_\infty$ computed by using the prepotential $\mathcal{F}$, we find that

$$\begin{pmatrix} m & n \\ p & q \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}$$

Next, suppose that $\nu_\infty = a\alpha + b\beta$, then compute by using the Picard-Lefschtz formula:

$$\alpha \mapsto \alpha + \langle \nu_\infty, \alpha \rangle \, \nu_\infty = (1 - ab)\,\alpha + b^2\beta$$
$$\beta \mapsto \beta + \langle \nu_\infty, \beta \rangle \, \nu_\infty = a^2\alpha + (1 + ab)\,\beta$$

Comparing with the above matrix, we conclude that $b = 0$ and $a^2 = 4$. By choosing proper orientation, a can be chosen to be 2, consequently $\nu_\infty = 2\alpha$. □

Computing ν_∞ analytically

By slightly modifying the argument in the paper [KTL95] in obtaining ν_∞, we have the following analytic way of determining the monodromy at infinity. Again, we shall related the two equations defining the Seiberg-Witten curve, namely

$$\begin{aligned} y^2 = x^3 - 2ux^2 + \Lambda^4 x \quad & c.f.\,\text{equation (4.2.24)} \\ y^2 = (x^2-u)^2 - \Lambda^4 \quad & c.f.\,\text{equation (4.1.21)} \end{aligned}$$

Since they define the same curve, i.e., birational to each other, there exists projective transformation

$$x \mapsto \frac{ax+b}{cx+d}, \quad \text{where} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

such that the four branch points of (4.1.21) are transformed into the four branch points of (4.1.24) in the following fashion:

$$\begin{aligned} e_2^+ = -\sqrt{u-\Lambda^2} &\longmapsto 0 && (1) \\ e_1^- = \sqrt{u+\Lambda^2} &\longmapsto u + \sqrt{u^2-\Lambda^4} && (2) \\ e_2^- = -\sqrt{u+\Lambda^2} &\longmapsto \infty && (3) \end{aligned}$$

We want to know where does the branch point e_1^+ go under the above transformation? We do not need the precise solution since our purpose is to determine the monodromy near ∞. Is is sufficient to exhibit its behavior as $\frac{u}{\Lambda^2} \to \infty$.

The equation (3) above reads: $\dfrac{b - a\sqrt{u+\Lambda^2}}{d - c\sqrt{u+\Lambda^2}} = \infty$, which implies that the relation $d = c\sqrt{u+\Lambda^2}$ holds. Then by using this relation, the equations (1) and (2) become:

$$\begin{aligned} \frac{b - a\sqrt{u-\Lambda^2}}{c\left(\sqrt{u+\Lambda^2} - \sqrt{u-\Lambda^2}\right)} &= 0 && (1)' \\ \frac{b + a\sqrt{u+\Lambda^2}}{2c\sqrt{u+\Lambda^2}} &= u + \sqrt{u^2-\Lambda^4} && (2)' \end{aligned}$$

By dividing both sides by Λ of the above two equations, and after a little bit arrangement, we get the followings

$$\begin{aligned} \frac{\frac{b}{\Lambda} - a\sqrt{\frac{u}{\Lambda^2}-1}}{c\left(\sqrt{\frac{u}{\Lambda^2}+1} - \sqrt{\frac{u}{\Lambda^2}-1}\right)} &= 0 && (1)'' \\ \frac{\frac{b}{\Lambda} + a\sqrt{\frac{u}{\Lambda^2}+1}}{2c\sqrt{\frac{u}{\Lambda^2}+1}} &= \frac{u}{\Lambda} + \sqrt{\left(\frac{u}{\Lambda}\right)^2 - \Lambda^2} && (2)'' \end{aligned}$$

From the above equation $(1)''$, we infer that asymptotically, we should have

$$\frac{b}{\Lambda} = a\sqrt{\frac{u}{\Lambda^2}} \qquad (1)'''$$

inserting this relation into $(2)''$, we see that asymptotically as $\frac{u}{\Lambda^2} \to \infty$, the equation $(2)''$ becomes

$$\frac{a}{2c} \approx \frac{u}{\Lambda} \qquad (2)'''$$

With the above preparation, we see that under the projective transformation, we have that as $\frac{u}{\Lambda^2} \to \infty$:

$$e_1^+ = \sqrt{u-\Lambda^2} \longmapsto \tilde{u} := \frac{a\sqrt{u-\Lambda^2}+b}{c\sqrt{u-\Lambda^2}+d} = \frac{a\sqrt{\frac{u}{\Lambda^2}-1}+\frac{b}{\Lambda}}{c\left(\sqrt{\frac{u}{\Lambda^2}-1}+\sqrt{\frac{u}{\Lambda^2}+1}\right)}$$

$$\approx \frac{a\sqrt{\frac{u}{\Lambda^2}-1}+a\sqrt{\frac{u}{\Lambda^2}}}{c\left(\sqrt{\frac{u}{\Lambda^2}-1}+\sqrt{\frac{u}{\Lambda^2}+1}\right)} \approx \frac{a}{c} \approx \frac{2u}{\Lambda}$$

but the root $u+\sqrt{u^2-\Lambda^4}$ also approaches $\frac{2u}{\Lambda}$ as $\frac{u}{\Lambda^2} \to \infty$. Consequently, we see that in this limit, the branch points $u+\sqrt{u^2-\Lambda^4}$ and $\tilde{u}$ collide with each other, which corresponds to the collapse of e_1^+ and e_1^-. We thus infer that the cycle α vanishes (since after gluing the branch cuts, the cycle enclosing $e_1^\pm$ is get identified with that enclosing $e_2^\pm$ which gives the α-cycle).

To determine exactly the vanishing cycle ν_∞ associated to $u=\infty$, we need to know the asymptotic behavior of $\tilde{u}$ as $\frac{u}{\Lambda^2} \to \infty$.

To this end, defining $t = \frac{1}{s} = \frac{u}{\Lambda^2}$. Let us first estimate the behavior of the quantity $\tilde{u} - \frac{2u}{\Lambda}$ as $\frac{u}{\Lambda^2} \to \infty$. In terms of the variable s, we then write as:

$$\tilde{u} - \frac{2u}{\Lambda} = \frac{a\sqrt{1-s}+\frac{b}{\Lambda}\sqrt{s}}{c\left(\sqrt{1-s}+\sqrt{1+s}\right)} - \frac{2\Lambda}{s}$$

$$= \frac{\left(a\sqrt{1-s}+\frac{b}{\Lambda}\sqrt{s}\right)\left(\sqrt{1+s}-\sqrt{1-s}\right)}{2cs} - \frac{2\Lambda}{s} \qquad (*)$$

By the relations $(1)'''$ and $(2)'''$ above, as well as bi-normal series, we get that the above equation $(*)$ can be approximated when $s \to 0$ as follows

$$(*) \approx \frac{\left(a\sqrt{1-s}+a\right)\left[(1+\frac{s}{2}-\frac{s^2}{8}+\frac{s^3}{16}-\cdots)-(1-\frac{s}{2}-\frac{s^2}{8}-\frac{s^3}{16}+\cdots)\right]}{2cs} - \frac{2\Lambda}{s}$$

$$\approx \frac{a}{2cs}(\sqrt{1-s}+1)\left(s+\frac{s^3}{8}\right) - \frac{2\Lambda}{s} \approx \frac{\Lambda}{s}(1+\sqrt{1-s})\left(1+\frac{s^2}{8}\right) - \frac{2\Lambda}{s}$$

$$\approx \frac{2\Lambda}{s}\left(1+\frac{s^2}{8}\right) - \frac{2\Lambda}{s} = \frac{\Lambda s}{4} = \frac{\Lambda^3}{4u}$$

Consequently, tracing one circle around $u=\infty$, i.e., by letting $u \mapsto e^{2\pi i\theta}u,\ 0 \le \theta \le 1$, we get a whole loop around in $\tilde{u}-$plane, which implies that e_1^+ would rotate around e_1^- by one whole circle. From this we infer that the vanishing cycle associated to $u=\infty$ should by 2α as the effect of the rotation, i.e., α cycle is dragged by rotating itself so that it is get shifted by α.

Remark 4.2.7. *It is curious to note that if we take the monodromy with respect to the parameter* Λ, *i.e., by letting* $\Lambda \mapsto e^{2\pi i\theta}$, $0 \le \theta \le 1$, *the branch points* $e_1^+ = \sqrt{u - \Lambda^2}$ *and* $e_1^- = \sqrt{u + \Lambda^2}$ *will trade place, while at the same time, the other pair of branch points, namely* $e_2^+ = -\sqrt{u - \Lambda^2}$ *and* $e_1^+ = \sqrt{u - \Lambda^2}$ *will be transformed into* e_2^- *and* e_1^- *respectively. As a consequence of this fact, the* α*-cycle remains the same, while the* β*-cycle is being dragged into* $\beta - 2\alpha - 2\alpha = \beta - 4\alpha$. *Thus, the monodromy* $\mathcal{M}_\infty$ *can be seen as the effect of monodromy with respect to the parameter* Λ. *For the moment being, I do not know if there is some intrinsic explanation of this fact though a possible explanation seems to be suggested in the reference [KTL95]. The idea is to view first* Λ^2 *as a complex parameter, and then compactify the moduli space* $\mathcal{B} \cong \mathbb{C}$ *by assigning the homogeneous coordinate on* $\overline{\mathcal{B}} \cong \mathbb{C}P^1$ *as* $[u : \Lambda^2]$. *By this construction, the line of infinity in the moduli space is obtained by letting* $\Lambda^2 = 0$ *(indeed we see in this case multiple branch points get aligned which corresponds to the non-stable degeneration of the elliptic fibers), consequently, it is intuitively clear that we can view the points at* ∞ *as corresponding to the "singularity"* $\{\Lambda = 0\}$. *From this, we expect that the monodromy at* $u = \infty$ *should correspond to the monodromy around the singularity* $\{\Lambda = 0\}$, *which is indeed the case as had been shown at the beginning of this remark.*

Summary of the main results to be cited later

In summary, the quantum monodromy in the $SU(2)$ case consists of $\mathcal{M}_{\pm\Lambda^2}$ corresponding to the two singular points $\pm\Delta^2$. The semi-classical monodromy $\mathcal{M}_\infty$ is given by the monodromy around ∞. The quantum monodromies, together with the semi-classical one, satisfy the relation (4.2.20). The vanishing cycles $\nu_{\pm\Lambda^2}$ associated to $\pm\Lambda^2$ are given by β and $-2\alpha + \beta$ respectively, which correspond to **monodromy invariant** directions of the corresponding $\mathbb{Z}$-affine structure.

In view of the connection to WCS to be discussed later, we conclude that at the (quantum) discriminant locus $\Delta_c = \{\pm\Lambda^2\}$, there are two BPS states with charges given respectively by:

$$\begin{aligned} \gamma_{-\Lambda^2} &= \beta - 2\alpha = (-2, 1) \\ \gamma_{+\Lambda^2} &= \beta = (0, 1) \end{aligned} \tag{4.2.25}$$

where the first entry corresponds to the magnetic charge taking values in the weight lattice Λ_W, while the second entry to the electric charge with values in the root lattice Λ_R. Consequently, the local system of first homology group of hyper-elliptic fibers splits in to the direct sum of local systems of root lattice (spanned by α-cycles, i.e., integral combinations of α) and the weight lattice (spanned by β-cycles, i.e., integral combinations of β). That is:

$$\underline{\Gamma}_u = H_1(\mathcal{C}_u, \mathbb{Z}) = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta = \Lambda_R \oplus \Lambda_W \tag{4.2.26}$$

4.2.3 More on Picard-Lefschetz and Braid Monodromy

In this subsection, we will give a detailed review about the techniques involved in investigating the monodromies associated to the Seiberg-Witten curves that had been scattered in the physics literature (for example, see [AD95][Kle+94][KTL95] [Kle+95][May99][AF][SD12][Seo13]). We will present them coherently so to make them become more accessible (especially for those interested in mathematics other than its physics contents). Also we display the relations between different methods by letting them "interact" with each other.

This subsection serves as a "link" between the last subsection in which we dealt with the $SU(2)$-case quantum monodromies and the next subsection in which the quantum monodromies of the $SU(3)$ case will be discussed. On the one hand, it helps further understanding the methods used in the last subsection, and on the other hand, it provide more general and more effective tools for a proper investigation of the $SU(3)$-case in the last subsection.

The mathematics to be discussed in this subsection are interesting in itself, and service as independent curiosity. For example, the idea that the monodromies associated to the vanishing cycles can be computed either geometrically by employing Picard-Lefschtz formula, or be directly read off from the asymptotic behavior of the prepotential function $\mathcal{F}$ (which is related to the period integrals of the associated hyper-elliptic curves), is seen to be a particular example (toy model) of the famous *Mirror symmetry* scheme. We will compare the two methods by doing detailed computations. And at various occasions, we will give different interpretations on particular result so to make the consistence between various approaches becoming manifest. Furthermore, the general methods exhibited in this subsection could be used in other places.

Although the materials to be discussed in this subsection are detailed and long in length, which may form the impression that they will attract us away from the main focus of this thesis, however, this is not true as these materials are not only indispensable ingredients in studying the geometry of Seiberg-Witten integrable systems, but also shed new lights on some discussions in chapter three. For example, the two remarks (remark 4.2.10 and 4.2.11) given in this subsection provides rationality in the description of the local model near discriminant in sections 3.1.6 and 3.1.7 in the last chapter. Besides, the vanishing cycles and the associated monodromies contain essential information about the initial conditions for the Wall-Crossing Structure that had been discussed in chapter two.

No originality is claimed for the work done in this subsection, though substantial efforts have been poured in arranging and polishing the results existing in the physics literature.

To begin with, let us discuss some ideas that apply to the general $SU(n)$ case. And their effectiveness will be illustrated by specifying the $SU(3)$, which is our main concern in the next subsection.

Recall that in section 4.1.2, we write the Seiberg-Witten curve in the $SU(n)$ case (i.e., the curve associated to the simple singularity of A_{n-1}-type) as follows:

$$\begin{aligned}\mathcal{C}_{\mathbf{u}} : y^2 = P_n(x) &\equiv \big(\mathcal{W}_{A_{n-1}}(x,\mathbf{u})\big)^2 - \Lambda^{2n} \qquad x, y \in \mathbb{C} \\ &= P_{n+} \times P_{n-} \equiv \big(\mathcal{W}_{A_{n-1}}(x,\mathbf{u}) + \Lambda^n\big) \times \big(\mathcal{W}_{A_{n-1}}(x,\mathbf{u}) - \Lambda^n\big) \\ &= \big(\mathcal{W}_{A_{n-1}}(x; u_2, \cdots, u_{n-1}, u_n - \Lambda^n)\big) \times \big(\mathcal{W}_{A_{n-1}}(x; u_2, \cdots, u_{n-1}, u_n + \Lambda^n)\big)\end{aligned}$$

We denote the roots of the polynomials $P_{n\pm}$ by $\{e_i^{\pm}\}$, and their discriminant by $\delta(P_{n\pm})$ respectively. The corresponding vanishing locus of $\delta_{\pm} := \delta(P_{n\pm})$ are denoted by $\Delta_{\pm}$. Thus the (quantum) discriminant δ_{Λ}, defined as the discriminant of the polynomial $P_n(x)$, is given by the product of δ_+ and δ_-. Consequently, the (quantum) discriminant locus, i.e., the locus of zeros of δ_{Λ}, factorizes as

$$\Delta_{\Lambda} = \Delta_+ \sqcup \Delta_-$$

On the other hand, as a genus $g = n-1$ Riemann surface, $\mathcal{C}_{\mathbf{u}}$ is obtained as a double cover of complex x-plane, ramified at the $2n$ branch points $\{e_i^{\pm}\}$. It can be constructed by pasting the n-copies of complex plane along the n-branch cuts given by line segments l_i connecting the pair of branch points $\{e_i^{\pm}\}$. We now construct the canonical symplectic basis for $H_1(\mathcal{C}_{\mathbf{u}}, \mathbb{Z})$ as follows:

- α^i: the cycle encircling the branch cut l_i, i.e., the cycle that encloses the pair of branch points $\{e_i^{\pm}\}$, for $i = 1, \cdots, n-1$.
- β_i: the cycle encircling the branch points e_i^+ and e_n^+, for $i = 1, \cdots, n-1$.

By proper and consistent choice of orientations, the above cycles can be arranged into a symplectic basis in the sense of that the intersections among $\{\alpha^i, \beta_i\}$ satisfy the following conditions:

$$\langle \alpha^i, \alpha^j \rangle = \langle \beta_i, \beta_j \rangle = 0 \qquad \langle \alpha^i, \beta_j \rangle = \delta^i_j. \tag{4.2.27}$$

A particular basis for the space of vanishing cycles

The vanishing cycles $\nu \in H_1(\mathcal{C}_{\mathbf{u}}, \mathbb{Z})$ are caused by the coincidence of two or more roots among e_i^+ or e_i^- as the moduli parameter $\mathbf{u}$ approaches to the various components of the discriminant locus Δ_{Λ}. We denote by $\nu_{i,j}^{\pm}$ the vanishing cycles associated to the colliding of the two roots $e_i^{\pm}$ and $e_j^{\pm}$, for $i < j$. Of course, not all of these $\frac{n(n-1)}{2}$ vanishing cycles are independent, we thus selects from them the following $2(n-1)$ ones as follows, which generate the first (middle) homology group $H_1(\mathcal{C}_{\mathbf{u}}, \mathbb{Z})$.

- ν_i^+: the cycle encircling the branch points e_i^+ and e_{i+1}^+, for $i = 1, \cdots, n-1$.
- ν_i^-: the cycle encircling the branch points e_i^- and e_{i+1}^-, for $i = 1, \cdots, n-1$.

We also denote by ν_n^+ the cycle encircling the branch points e_1^+ and e_n^+, and by ν_n^- the cycle encircling the branch points e_1^- and e_n^-.

In terms of the symplectic basis $\{\alpha^i, \beta_j\}$, the set of vanishing cycles $\{\nu_i^\pm\}$ can be expressed (see: [SD12][Seo13]) as:

$$\begin{cases} \nu_i^+ = \beta_{i+1} - \beta_i - \alpha^i, \ 1 \le i \le n-2 \\ \nu_i^- = \beta_{i+1} - \beta_i + \alpha^{i+1} - 2\alpha^i, \ 1 \le i \le n-2 \\ \nu_{n-1}^+ = -\beta_{n-1} + \sum_{i=1}^{n-2} \alpha^i \\ \nu_{n-1}^- = -\beta_{n-1} - 2\alpha^{n-1} \\ \nu_n^+ = \beta_1 \\ \nu_n^- = \beta_1 + \sum_{i=1}^{n-1} \alpha^i + \alpha^1 \end{cases} \tag{4.2.28}$$

For example, in the $n = 2$ case, we compute by using the above formulas that there are exactly two vanishing cycles, namely $\nu_2^+ = \beta_1$ and $\nu_2^- = \beta_1 + 2\alpha^1$, which are exactly the vanishing cycles associated with singularities $\pm\Lambda^2$ respectively in the $SU(2)$-case. While in the $n = 3$ case, we have the vanishing cycles $\nu_i^\pm$, which corresponds to the coincidence of the branch points $e_i^\pm$ and $e_{i+1}^\pm$, for $i = 1, 2$, as well as the vanishing cycles $\nu_3^\pm$ corresponding to the coincidence of $e_1^\pm$ and $e_3^\pm$. In terms of the formula (4.2.28) above, these vanishing cycles are given in the symplectic basis as follows:

$$\begin{cases} \nu_1^+ = \beta_2 - \beta_1 - \alpha^1 \qquad \nu_1^- = \beta_2 - \beta_1 + \alpha^2 - 2\alpha^1 \\ \nu_2^+ = -\beta_2 + \alpha^1 \qquad \nu_2^- = -\beta_2 - 2\alpha^2 \\ \nu_3^+ = \beta_1 \qquad \nu_3^- = \beta_1 + 2\alpha^1 + \alpha^2 \end{cases} \tag{4.2.29}$$

Remark 4.2.8. *By the construction of the set of vanishing cycles (4.2.28) above, it is clear that $-\nu_n^\pm = \sum_{i=1}^{n-1} \nu_i^\pm$, that is*

$$\sum_{i=1}^{n} \nu_i^\pm = 0 \tag{4.2.30}$$

which is consistent with the formulas above as can be easily verified as

$$\begin{aligned} \sum_{i=1}^{n} \nu_i^+ &= \left(\sum_{i=1}^{n-2} \beta_{i+1} - \beta_i - \alpha^i \right) + \left(-\beta_{n-1} + \sum_{i=1}^{n-2} \alpha^i \right) + \beta_1 \\ &= \left(\sum_{i=1}^{n-2} \beta_{i+1} - \beta_i \right) - \sum_{i=1}^{n-2} \alpha^i - \beta_{n-1} + \sum_{i=1}^{n-1} \alpha^i + \beta_1 \\ &= \beta_{n-1} - \beta_1 - \beta_{n-1} + \beta_1 = 0. \end{aligned}$$

Similarly, we have

$$\sum_{i=1}^{n} \nu_i^- = \left(\sum_{i=1}^{n-2} \beta_{i+1} - \beta_i + \alpha^{i+1} - 2\alpha^i \right) - \beta_{n-1} - 2\alpha^{n-1} + \beta_1 + \sum_{i=1}^{n-1} \alpha^i + \alpha^1$$

$$= \left(\sum_{i=1}^{n-2} \beta_{i+1} - \beta_i \right) - \beta_{n-1} + \beta_1 + \left(\sum_{i=1}^{n-2} \alpha^{i+1} - \alpha^i \right) - \sum_{i=1}^{n-2} \alpha^i - 2\alpha^{n-1} + \sum_{i=1}^{n-1} \alpha^i + \alpha^1$$

$$= \left[\left(\sum_{i=1}^{n-2} \beta_{i+1} - \beta_i\right) - \beta_{n-1} + \beta_1\right] + \left[\left(\sum_{i=1}^{n-2} \alpha^{i+1} - \alpha^i\right) - \alpha^{n-1} + \alpha^1\right] = 0.$$

In connection with the SW-integrable system, this means that we can choose the coordinates $\{a_i\}_{i=1}^{n-1}$ *on* $\mathcal{B} \cong \mathbb{C}^{n-1}$ *as*

$$a_i := \oint_{\nu_i^\pm} \lambda_{SW} \qquad i = 1, \cdots, n-1. \tag{4.2.31}$$

such that: $\sum_{i=1}^{n-1} a_i = 0$*, which follows from the condition:* $\sum_{i=1}^{n-1} \nu_i^\pm = 0$.

As the set of vanishing cycles generate $H_1(\mathcal{C}, \mathbb{Z})$, we could select $2(n-1)$ vanishing cycles such that they are linearly independent.

Proposition 4.2.3. *The vanishing cycles* $\nu_i^\pm$, $1 \le i \le n-1$ *form a basis of the middle homology group* $H_1(\mathcal{C}, \mathbb{Z})$ *of the SW curve* $\mathcal{C}$.

Proof. From the formula (4.2.28), we see that the transition matrix between $\left(\nu_1^+ \cdots \nu_{n-1}^+ \, \nu_1^- \cdots \nu_{n-1}^-\right)^T$ and $\left(\beta_1 \cdots \beta_{n-1} \, \alpha^1 \cdots \alpha^{n-1}\right)^T$ is given by the following matrix:

$$M = \left(\begin{array}{cccccc|cccccc}
-1 & 1 & 0 & \cdots & 0 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\
0 & -1 & 1 & \cdots & 0 & 0 & 0 & -1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -1 & 1 & 0 & 0 & 0 & \cdots & -1 & 0 \\
0 & 0 & 0 & \cdots & 0 & -1 & 1 & 1 & 1 & \cdots & 1 & 0 \\
\hline
-1 & 1 & 0 & \cdots & 0 & 0 & -2 & 1 & 0 & \cdots & 0 & 0 \\
0 & -1 & 1 & \cdots & 0 & 0 & 0 & -2 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -1 & 1 & 0 & 0 & 0 & \cdots & -2 & 1 \\
0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 & 0 & \cdots & 0 & -2
\end{array}\right)$$

$$= \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

From the expression above, we conclude that $det(A) = det(C) = (-1)^{n-1}$, and $det(B) = 0$, $det(D) = (-2)^{n-1}$, consequently

$$det(M) = det(A)det(D) - det(B)det(C) = 2^{n-1} \neq 0$$

$\square$

For example, in the $SU(3)$ case described in (4.2.29), we see that the remaining vanishing cycles $\nu_3^\pm$ can be expressed in terms of the basis $\{\nu_i^\pm\}_{i=1,2}$ as

$$\begin{aligned} \nu_3^+ &= -\nu_1^+ - \nu_2^+ \\ \nu_3^- &= -\nu_1^- - \nu_2^- \end{aligned} \tag{4.2.32}$$

Proposition 4.2.4. *The only non-trivial intersections among various vanishing cycles given in (4.2.28) are given by the followings:*

$$\begin{aligned} \langle \nu_i^+, \nu_{i+1}^+ \rangle &= \langle \nu_i^-, \nu_{i+1}^- \rangle = 1 \\ \langle \nu_n^+, \nu_1^+ \rangle &= \langle \nu_n^-, \nu_1^- \rangle = 1 \\ \langle \nu_i^+, \nu_i^- \rangle &= -2 \;\; \langle \nu_i^+, \nu_{i+1}^- \rangle = 2 \\ \langle \nu_n^+, \nu_n^- \rangle &= -2 \;\; \langle \nu_n^+, \nu_1^- \rangle = 2 \end{aligned} \tag{4.2.33}$$

Proof. First, we consider the case that the intersections are between those with " + " or " − " sign only, i.e., not of the "mixing type" to be considered momentarily. By the form of vanishing cycles $\nu_i^\pm$ for $1 \le i \le n-2$, we see that when $|i-j| > 1$, $\langle \nu_i^\pm, \nu_j^\pm \rangle = 0$, as the appearance of the terms like $\langle \alpha^i, \beta_i \rangle$ would be forbidden in this case. Thus, in this sub-case $(1 \le i \le n-3)$, the only possible non-vanishing intersections would fall into the following two types:

$$\begin{aligned} \langle \nu_i^+, \nu_{i+1}^+ \rangle &= \langle \beta_{i+1} - \beta_i - \alpha^i, \beta_{i+2} - \beta_{i+1} - \alpha^{i+1} \rangle \\ &= \langle \beta_{i+1}, -\alpha^{i+1} \rangle = 1 \end{aligned}$$

Similarly, we can show that $\langle \nu_i^-, \nu_{i+1}^- \rangle = 1$ for $1 \le i \le n-3$, while

$$\langle \nu_{n-2}^+, \nu_{n-1}^+ \rangle = \left\langle \beta_{n-1} - \beta_{n-2} - \alpha^{n-2}, -\beta_{n-1} + \sum_{i=1}^{n-2} \alpha^i \right\rangle = \langle -\beta_{n-2}, \alpha^{n-2} \rangle = 1$$

and

$$\langle \nu_{n-1}^+, \nu_n^+ \rangle = \left\langle -\beta_{n-1} + \sum_{i=1}^{n-2} \alpha^i, \beta_1 \right\rangle = \langle \alpha^1, \beta_1 \rangle = 1$$

Similarly, we have that

$$\begin{aligned} \langle \nu_{n-2}^-, \nu_{n-1}^- \rangle &= \langle \beta_{n-1} - \beta_{n-2} + \alpha^{n-1} - 2\alpha^{n-2}, -\beta_{n-1} - 2\alpha^{n-1} \rangle \\ &= \langle \beta_{n-1}, -2\alpha^{n-1} \rangle + \langle \alpha^{n-1}, -\beta_{n-1} \rangle = 2 - 1 = 1 \end{aligned}$$

and

$$\begin{aligned} \langle \nu_{n-1}^-, \nu_n^- \rangle &= \left\langle -\beta_{n-1} - 2\alpha^{n-1}, \beta_1 + \sum_{i=1}^{n-1} \alpha^i + \alpha^1 \right\rangle \\ &= \langle -\beta_{n-1}, \alpha^{n-1} \rangle = 1 \end{aligned}$$

We still need to check the cases when $\nu_{n_1}^\pm$ and $\nu_n^\pm$ intersect with $\nu_i^\pm$ for $1 \le i \le n-3$.

$$\begin{aligned} \langle \nu_i^+, \nu_{n-1}^+ \rangle &= \left\langle \beta_{i+1} - \beta_i - \alpha^i, -\beta_{n-1} + \sum_{i=1}^{n-2} \alpha^i \right\rangle \\ &= \left\langle \beta_{i+1}, \sum_{i=1}^{n-2} \alpha^i \right\rangle - \left\langle \beta_i, \sum_{i=1}^{n-2} \alpha^i \right\rangle = 1 - 1 = 0 \end{aligned}$$

and

$$\langle \nu_i^-, \nu_{n-1}^- \rangle = \langle \beta_{i+1} - \beta_i + \alpha^{i+1} - 2\alpha^i, -\beta_{n-1} - 2\alpha^{n-1} \rangle = 0$$

Next, we show that $\langle \nu_n^+, \nu_1^+\rangle = \langle \nu_{n^-}, \nu_1^-\rangle = 1$. Indeed, for $1 \le i \le n-3$, we have that

$$\langle \nu_n^+, \nu_i^+\rangle = \langle \beta_1, \beta_{i+1} - \beta_i - \alpha^i\rangle = \langle \beta_1, -\alpha^i\rangle = \delta_1^i.$$

and

$$\langle \nu_n^-, \nu_i^-\rangle = \left\langle \beta_1 + \sum_{i=1}^{n-1} \alpha^i + \alpha^1, \beta_{i+1} - \beta_i + \alpha^{i+1} - 2\alpha^i \right\rangle$$

$$= \langle \beta_1, -2\alpha^i\rangle + \left\langle \sum_{i=1}^{n-1} \alpha^i, \beta_{i+1} \right\rangle - \left\langle \sum_{i=1}^{n-1} \alpha^i, \beta_i \right\rangle - \langle \alpha^1, \beta_i\rangle$$

$$= 2\delta_1^i + 1 - 1 - \delta_i^1 = 2\delta_1^i - \delta_i^1$$

Now we consider the "mixing" of " $\pm$ " sigh types. First, note that for $1 \le i \le n-2$, we compute that

$$\langle \nu_i^+, \nu_j^-\rangle = \langle \beta_{i+1} - \beta_i - \alpha^i, \beta_{j+1} - \beta_j + \alpha^{j+1} - 2\alpha^j\rangle$$

$$= \langle \beta_{i+1}, \alpha^{j+1}\rangle - 2\langle \beta_{i+1}, \alpha^j\rangle - \langle \beta_i, \alpha^{j+1}\rangle - \langle \beta_i, -2\alpha^j\rangle - \langle \alpha^i, \beta_{j+1}\rangle - \langle \alpha^i, -\beta_j\rangle$$

$$= -\delta_{i+1}^{j+1} + 2\delta_{i+1}^j + \delta_i^{j+1} - 2\delta_i^j - \delta_{j+1}^i + \delta_j^i$$

From the above identity, we infer that when $i = j$, the value of the above identity specialize into $-1-2+1 = -2$, while for $j = i+1$, it equals to $2+1-1 = 2$, which finishes the proof for the case $1 \le i \le n-3$. Next, we check the remaining cases:

For $1 \le i \le n-2$, we consider:

$$\langle \nu_{n-1}^-, \nu_i^+\rangle = \left\langle -\beta_{n-1} - 2\alpha^{n-1}, \beta_{i+1} - \beta_i - \alpha^i \right\rangle = \langle -2\alpha^{n-1}, \beta_{i+1}\rangle = -2\delta_{i+1}^{n-1}$$

$$\langle \nu_{n-1}^+, \nu_i^-\rangle = \left\langle -\beta_{n-1} + \sum_{i=1}^{n-2} \alpha^i, \beta_{i+1} - \beta_i + \alpha^{i+1} - 2\alpha^i \right\rangle$$

$$= \langle -\beta_{n-1}, \alpha^{i+1}\rangle + \langle -\beta_{n-1}, -2\alpha^i\rangle + \left\langle \sum_{i=1}^{n-2} \alpha^i, \beta_{i+1} \right\rangle - \left\langle \sum_{i=1}^{n-2} \alpha^i, \beta_i \right\rangle$$

$$= \delta_{n-1}^{i+1} - 2\delta_{n-1}^i + \left\langle \sum_{i=1}^{n-2} \alpha^i, \beta_{i+1} \right\rangle - \left\langle \sum_{i=1}^{n-2} \alpha^i, \beta_i \right\rangle$$

from which we deduce that when $i = n-2$, the above identity becomes $1-2\times 0+0-1 = 0$, while for $i < n-2$, it becomes $0-0+1-1 = 0$.

Similarly, for $1 \le i \le n-2$, we have that

$$\langle \nu_n^-, \nu_i^+\rangle = \left\langle \beta_1 + \sum_{i=1}^{n-1} \alpha^i + \alpha^1, \beta_{i+1} - \beta_i - \alpha^i \right\rangle$$

$$= \langle \beta_1, -\alpha^i\rangle + \left\langle \sum_{i=1}^{n-1} \alpha^i, \beta_{i+1} \right\rangle - \left\langle \sum_{i=1}^{n-1}, \beta_i \right\rangle + \langle \alpha^1, \beta_{i+1}\rangle - \langle \alpha^1, \beta_i\rangle$$

$$= \delta_1^i + \left\langle \sum_{i=1}^{n-1} \alpha^i, \beta_{i+1} \right\rangle - \left\langle \sum_{i=1}^{n-1}, \beta_i \right\rangle - \delta_i^1$$

$$= \left\langle \sum_{i=1}^{n-1} \alpha^i, \beta_{i+1} \right\rangle - \left\langle \sum_{i=1}^{n-1} \alpha^i, \beta_i \right\rangle = 1 - 1 = 0.$$

Next

$$\langle \nu_n^+, \nu_i^- \rangle = \langle \beta_1, \beta_{i+1} - \beta_i + \alpha^{i+1} - 2\alpha^i \rangle = \langle \beta_1, -2\alpha^i \rangle = 2\delta_1^i$$

then

$$\langle \nu_{n-1}^+, \nu_n^- \rangle = \left\langle -\beta_{n-1} + \sum_{i=1}^{n-2} \alpha^i, \beta_1 + \sum_{i=1}^{n-1} \alpha^i + \alpha^1 \right\rangle$$

$$= \left\langle -\beta_{n-1}, \sum_{i=1}^{n-1} \alpha^i \right\rangle + \left\langle \sum_{i=1}^{n-2} \alpha^i, \beta_1 \right\rangle = 1 + 1 = 2.$$

and then

$$\langle \nu_{n-1}^-, \nu_n^+ \rangle = \langle -\beta_{n-1} - 2\alpha^{n-1}, \beta_1 \rangle = 0.$$

and then

$$\langle \nu_{n-1}^+, \nu_{n-1}^- \rangle = \left\langle -\beta_{n-1} + \sum_{i=1}^{n-2} \alpha^i, -\beta_{n-1} - 2\alpha^{n-1} \right\rangle = \langle -\beta_{n-1}, -2\alpha^{n-1} \rangle = 2.$$

and finally

$$\langle \nu_n^+, \nu_n^- \rangle = \left\langle \beta_1, \beta_1 + \sum_{i=1}^{n-1} \alpha^i + \alpha^1 \right\rangle = \langle \beta_1, \alpha^1 \rangle + \langle \beta_1, \alpha^1 \rangle = -2.$$

which completes the proof of the proposition. □

With the above proposition, the monodromy action $\mathcal{M}_{\nu_i^\pm}$ associated to the vanishing cycle $\nu_i^\pm$, in the basis of $H_1(\mathcal{C}, \mathbb{Z})$ given by the vanishing cycles $\{\nu_i^\pm\}_1^{n-1}$ can be determined easily by using the **Picard-Lefschetz formula**:

$$\mathcal{M}_{\nu_i^\pm}(\delta) = \delta + \langle \nu_i^\pm, \delta \rangle \, \nu_i^\pm \tag{4.2.34}$$

More explicitly, we have for $1 \le i, j \le n-2$ that

$$\mathcal{M}_{\nu_i^+}(\nu_j^+) = \begin{cases} \nu_j^+ - \nu_i^+ & j = i-1 \\ \nu_j^+ + \nu_i^+ & j = i+1 \\ \nu_j^+ & j \neq i+1, i-1 \end{cases} \qquad \mathcal{M}_{\nu_i^+}(\nu_j^-) = \begin{cases} \nu_j^- - 2\nu_i^+ & j = i \\ \nu_j^- + 2\nu_i^+ & j = i+1 \\ \nu_j^- & j \neq i, i+1 \end{cases}$$

$$\mathcal{M}_{\nu_i^-}(\nu_j^+) = \begin{cases} \nu_j^+ - 2\nu_i^- & j = i-1 \\ \nu_j^+ + 2\nu_i^- & j = i \\ \nu_j^+ & j \neq i-1, i \end{cases} \qquad \mathcal{M}_{\nu_i^-}(\nu_j^-) = \begin{cases} \nu_j^- + \nu_i^- & j = i+1 \\ \nu_j^- - \nu_i^- & j = i-1 \\ \nu_j^- & j \neq i-1, i+1 \end{cases}$$

Now we list the remaining case:

$$\mathcal{M}_{\nu_i^+}(\nu_{n-1}^+) = \begin{cases} \nu_{n-1}^+ + \nu_i^+ & i = n-2 \\ \nu_{n-1}^+ & i < n-2 \end{cases} \qquad \mathcal{M}_{\nu_i^+}(\nu_{n-1}^-) = \begin{cases} \nu_{n-1}^- + 2\nu_i^+ & i = n-2 \\ \nu_{n-1}^- & i < n-2 \end{cases}$$

$$\mathcal{M}_{\nu_i^+}(\nu_n^+) = \begin{cases} \nu_n^+ - \nu_i^+ & i = 1 \\ \nu_n^+ & i \neq 1 \end{cases} \qquad \mathcal{M}_{\nu_i^+}(\nu_n^-) = \nu_n^- \quad 1 \leq i \leq n-2$$

$$\mathcal{M}_{\nu_i^-}(\nu_{n-1}^+) = \nu_{n-1}^+ \quad 1 \leq i \leq n-2 \qquad \mathcal{M}_{\nu_i^-}(\nu_{n-1}^-) = \begin{cases} \nu_{n-1}^- + \nu_i^- & i = n-1 \\ \nu_{n-1}^- & i \neq n-2 \end{cases}$$

$$\mathcal{M}_{\nu_i^-}(\nu_n^+) = \begin{cases} \nu_n^+ - 2\nu_i^- & i = 1 \\ \nu_n^+ & i \neq 1 \end{cases} \qquad \mathcal{M}_{\nu_i^-}(\nu_n^-) = \begin{cases} \nu_n^- - \nu_i^- & i = 1 \\ \nu_n^- & i \neq 1 \end{cases}$$

$$\mathcal{M}_{\nu_{n-1}^+}(\nu_i^+) = \begin{cases} \nu_i^+ - \nu_{n-1}^+ & i = n-2 \\ \nu_i^+ & i \neq n-2 \end{cases} \qquad \mathcal{M}_{\nu_{n-1}^+}(\nu_i^-) = \nu_i^- \quad 1 \leq i \leq n-2$$

$$\mathcal{M}_{\nu_{n-1}^+}(\nu_{n-1}^-) = \nu_{n-1}^- - 2\nu_{n-1}^+, \qquad \begin{cases} \mathcal{M}_{\nu_{n-1}^+}(\nu_n^+) = \nu_n^+ + \nu_{n-1}^+ \\ \mathcal{M}_{\nu_{n-1}^+}(\nu_n^-) = \nu_n^- + 2\nu_{n-1}^+ \end{cases}$$

$$\begin{cases} \mathcal{M}_{\nu_n^+}(\nu_i^+) = \nu_i^+ \\ \mathcal{M}_{\nu_n^+}(\nu_i^-) = \nu_i^- \end{cases} \qquad \begin{cases} \mathcal{M}_{\nu_n^+}(\nu_{n-1}^+) = \nu_{n-1}^+ - \nu_n^+ \\ \mathcal{M}_{\nu_n^+}(\nu_{n-1}^-) = \nu_{n-1}^- \end{cases}$$

$$\begin{cases} \mathcal{M}_{\nu_n^+}(\nu_n^+) = \nu_n^+ \\ \mathcal{M}_{\nu_n^+}(\nu_n^-) = \nu_n^- - 2\nu_n^- \end{cases} \qquad \begin{cases} \mathcal{M}_{\nu_n^+}(\nu_1^+) = \nu_1^+ + \nu_n^+ \\ \mathcal{M}_{\nu_n^+}(\nu_1^-) = \nu_1^- + 2\nu_n^+ \end{cases}$$

$$\mathcal{M}_{\nu_{n-1}^-}(\nu_i^+) = \begin{cases} \nu_i^+ - 2\nu_{n-1}^- & i = n-2 \\ \nu_i^+ & i \neq n-2 \end{cases} \qquad \mathcal{M}_{\nu_{n-1}^-}(\nu_i^-) = \nu_i^- \quad 1 \leq i \leq n-2$$

$$\mathcal{M}_{\nu_{n-1}^-}(\nu_{n-1}^+) = \nu_{n-1}^+ - 2\nu_{n-1}^-, \qquad \begin{cases} \mathcal{M}_{\nu_{n-1}^-}(\nu_n^+) = \nu_n^+ \\ \mathcal{M}_{\nu_{n-1}^-}(\nu_n^-) = \nu_n^- + \nu_{n-1}^+ \end{cases}$$

$$\mathcal{M}_{\nu_n^-}(\nu_i^+) = \nu_i^+$$

$$\mathcal{M}_{\nu_n^-}(\nu_i^-) = \begin{cases} \nu_i^- + \nu_n^- & i = 1, \\ \nu_i^- & i \neq 1 \end{cases} \qquad \begin{cases} \mathcal{M}_{\nu_n^-}(\nu_{n-1}^+) = \nu_{n-1}^+ - \nu_n^- \\ \mathcal{M}_{\nu_n^-}(\nu_{n-1}^-) = \nu_{n-1}^- - \nu_n^- \end{cases}$$

$$\begin{cases} \mathcal{M}_{\nu_n^-}(\nu_n^+) = \nu_n^+ - 2\nu_n^- \\ \mathcal{M}_{\nu_n^-}(\nu_n^-) = \nu_n^- \end{cases} \qquad \begin{cases} \mathcal{M}_{\nu_n^-}(\nu_1^+) = \nu_1^- \\ \mathcal{M}_{\nu_n^-}(\nu_1^-) = \nu_1^- + \nu_n^- \end{cases}$$

Remark 4.2.9. *Let a, b be two variables that can take value from $\{\pm\}$. ϵ_{ab} is the two dimensional anti-symmetric tensor. Denote by I_{ij} the intersection matrix among the vanishing cycles $\{\nu_i^+\}_{i=1}^{n-1}$, which in our case was chosen to be the standard symplectic metric. Recall that the Cartan matrix for A_{n-1} is given by $(\langle\alpha_i, \alpha_j\rangle)$. then the above computation can be summarized more compactly as*

$$\mathcal{M}_{\nu_i^a}(\nu_j^b) = \nu_j^b + (\epsilon_{ba} + I_{ji})\langle\alpha_i, \alpha_j\rangle\, \nu_i^a$$

Remark 4.2.10. *The intersection matrix of the standard symplectic basis $\{\alpha^i, \beta_i\}$ is the standard* ***symplectic metric*** *given as*

$$\Omega = \left(\begin{array}{c|c} \mathbf{0}_{n-1\times n-1} & \mathbb{I}_{n-1\times n-1} \\ \hline -\mathbb{I}_{n-1\times n-1} & \mathbf{0}_{n-1\times n-1} \end{array}\right) \tag{4.2.35}$$

Then there exists some symplectic transformation $U \in Sp\,(2(n-1), \mathbb{Z})$ on the symplectic basis such that the transformed intersection matrix has the following form

$$\widetilde{\Omega} = \left(\begin{array}{c|c} \mathbf{0}_{n-1\times n-1} & C_{A_{n-1}} \\ \hline -C_{A_{n-1}} & \mathbf{0}_{n-1\times n-1} \end{array}\right) \tag{4.2.36}$$

where the matrix $C_{A_{n-1}}$ is the Cartan matrix of type A_{n-1}, which is given as follows

$$C_{A_{n-1}} = \begin{pmatrix} 2 & -1 & 0 & 0\cdots & 0 & 0 \\ -1 & 2 & -1 & 0\cdots & 0 & 0 \\ 0 & -1 & 2 & -1\cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0\cdots & 2 & -1 \\ 0 & 0 & 0 & 0\cdots & -1 & 2 \end{pmatrix} \tag{4.2.37}$$

Notice that the matrix $\widetilde{\Omega}$ is again symplectic since it satisfies the following defining property of symplectic matrix

$$\Omega^{-1}\widetilde{\Omega}\,\Omega = \widetilde{\Omega}$$

This means that the basis of one cycles $\{\alpha^i, \beta_i\}$ can be transformed into $\{\widetilde{\alpha}^i, \widetilde{\beta}_i\}$ such that

$$\left\langle \widetilde{\alpha}^i, \widetilde{\beta}_i \right\rangle = 2, \quad \left\langle \widetilde{\alpha}^i, \widetilde{\beta}_{i\pm 1} \right\rangle = -1$$

We note that the basis cycles satisfying the above condition can be obtained by "pushing down" the $n-1$ independent two spheres S_i, $1 \le i \le n-1$ (see proposition 4.1.14) in the ALE fibers when lifting of SW geometry to K3 fibration constructed in section 4.1.4. Here, the term "pushing down" means that after degenerating the ALE fiber in the $K3$ fibration, the vanishing two sphere S_i becomes the two closed "strings" encircling $\{e_i^\pm\}$ and $\{e_i^+, e_n^+\}$ respectively.

Compatibility of the Picard-Lefschetz formula with the monodromies computed in terms of the prepotential

Suppose that the cycle ν vanishes at the component $\{\mathbf{u} : Z_{\mathbf{u}}(\nu) = 0\} \subset \Delta_\Lambda$ of the (quantum) discriminant locus. We know that the monodromy acting on the middle-dimensional homology group $\underline{\Gamma}_{\mathbf{u}} = H_1(\mathcal{C}_{\mathbf{u}}, \mathbb{Z})$ of the torus fiber $\mathcal{C}_{\mathbf{u}}$ is given in terms of the Picard-Lefschetz formula, that is, for $\delta \in \underline{\Gamma}_{\mathbf{u}}$, the (quantum) monodromy $\mathcal{M}_\nu$ associated to ν acting on δ as

$$\mathcal{M}_\nu : \delta \longmapsto \delta + \langle \nu, \delta \rangle \, \nu \tag{4.2.38}$$

We now show that the above formula can be obtained analytically from the prepotential $\mathcal{F}$ computed from the periods of the torus fibers.

Recall that in terms of the period integrals, the prepotential function is given by the following

$$\mathcal{F}(\mathbf{a}) = \int \mathbf{a}_D \, d\mathbf{a}$$

where

$$\Xi := \begin{pmatrix} \mathbf{a}_D & \mathbf{a} \end{pmatrix} = \left(\oint_{\beta_i} \lambda_{SW} \cdots \oint_{\alpha^i} \lambda_{SW} \right)$$

Denote by $\Phi^+ = \{\alpha_i\}$ the set of positive roots. We have computed in great details in the $SU(2)$ case in the last subsection (see the lines of the proof of the proposition 4.2.1) the analytic formula for $\mathbf{a}_D$ (and consequently that for $\mathcal{F}(\mathbf{a})$). For general $SU(n)$ case, the expression of $\mathbf{a}_D$ in the limit where $|\mathbf{a} \cdot \alpha_i| \gg \Lambda$ was given in the physics literature (for example in [FH97] and[Hol97][Kuc08]) as follows

$$\begin{aligned} \mathbf{a}_D &= \frac{1}{2\pi i} \sum_{\alpha \in \Phi^+} \alpha(\mathbf{a} \cdot \alpha) \left[\ln \left(\frac{\mathbf{a} \cdot \alpha}{\Lambda} \right)^2 + 1 \right] \approx \frac{1}{2\pi i} \sum_{\alpha \in \Phi^+} \alpha(\mathbf{a} \cdot \alpha) \, \ln \frac{\mathbf{a}^2}{\Lambda^2} \\ &= \frac{1}{2\pi i} \sum_{i,j} (\alpha_i \cdot \alpha_j) \, \mathbf{a} \, \ln \frac{\mathbf{a}^2}{\Lambda^2} = \frac{1}{\pi i} h^\vee \, \mathbf{a} \, \ln \frac{\mathbf{a}}{\Lambda} \end{aligned} \tag{4.2.39}$$

where $h^\vee$ denotes the dual coxter number (see for example [Kac90]) of the gauge group $SU(n)$, which equals n in this case.

Now, suppose that the cycle (charge)

$$\nu = (\mathbf{g} \quad \mathbf{q}) = \mathbf{a}_D \cdot \mathbf{g} + \mathbf{a} \cdot \mathbf{q} = \sum_i g^i \beta_i + q_i \alpha^i \in \underline{\Gamma}_{\mathbf{u}} = H_1(\mathcal{C}_{\mathbf{u}}, \mathbb{Z})$$

vanishes at some place in the moduli space, then there exists some transformation $U \in Sp\,(2n-2, \mathbb{Z})$, such that

$$\nu \cdot U = (\mathbf{g} \quad \mathbf{q}) \; U = (\mathbf{0}; \tilde{q}_1, 0, \cdots, 0)$$

where $\tilde{q}_1 = gcd\,(g^1, \cdots, g^{n-1}; q_1, \cdots, q_{n-1})$, the greatest common divisor of the numbers inside the bracket.

Thus, the singularity in the moduli space corresponds to $\{\tilde{q}_1 a^1 = 0\}$. By using $\{a_1\}$ as local coordinate near the singular locus, it follows from (4.2.39) that $\mathbf{a}_D$ is expressed near $\{\tilde{q}_1 a^1 = 0\}$ as

$$a_{D,j} \approx \frac{1}{2\pi i}\, \delta_1^j\, (\tilde{q}_1)^2\, a^1 \ln \frac{\tilde{q}_1 a^1}{\Lambda} \tag{4.2.40}$$

Remark 4.2.11. *Assume for simplicity that $\tilde{q}_1 = 1$, then by integrating the above formula for $a_{D,1}$ with respect to a^1 gives that near the singularity, we have the following expression*

$$\mathcal{F}(a^1) = \frac{1}{2\pi i}\frac{(a^1)^2}{2} \ln \frac{a^1}{\Lambda} + \textit{holomorphic part}$$

By setting the parameter Λ to be 1, we see that this is exactly the from of the ***local model*** *of the prepotential function near the discriminant locus. (compare with the formula (3.1.78) in subsection 3.1.6).*

Using the formula (4.2.40), we can compute the monodromy around the origin as follows:

Consider a loop around $\{a^1 = 0\}$, i.e., $a^1 \mapsto a^1 e^{2\pi i\theta}$, $0 \leq \theta \leq 1$. Inserting in the formula (4.2.40) above, we see that

$$\begin{aligned} a_{D,1} = \frac{1}{2\pi i}\, (\tilde{q}_1)^2\, a^1 \ln \frac{\tilde{q}_1 a^1}{\Lambda^2} &\mapsto \frac{1}{2\pi i}\, (\tilde{q}_1)^2\, a^1\, e^{2\pi i\theta} \ln \frac{\tilde{q}_1 a^1\, e^{2\pi i\theta}}{\Lambda} \\ &= \frac{1}{2\pi i}\, (\tilde{q}_1)^2\, a^1\, e^{2\pi i\theta} \ln \frac{\tilde{q}_1 a^1}{\Lambda} + \frac{1}{2\pi i}\, (\tilde{q}_1)^2\, a^1\, e^{2\pi i\theta} \ln \left(e^{2\pi i\theta}\right) \\ &= \frac{1}{2\pi i}\, (\tilde{q}_1)^2\, a^1\, e^{2\pi i\theta} \ln \frac{\tilde{q}_1 a^1}{\Lambda} + \theta\, (\tilde{q}_1)^2\, a^1\, e^{2\pi i\theta} \end{aligned}$$

By specializing $\theta = 1$, we get that $a_{D,1} \mapsto a_{D,1} + (\tilde{q}_1)^2\, a^1$, consequently, we have that the following proposition:

Proposition 4.2.5. *The monodromy $\mathcal{M}_{(\mathbf{0},q^1,\cdots,0)}$ acts on $\begin{pmatrix} \mathbf{a}_D & \mathbf{a} \end{pmatrix}$ by the following matrix:*

$$\left(\begin{array}{c|c} \mathbb{I} & \mathbf{0} \\ \hline (\tilde{q}_1)^2\, \mathbb{E}_{11} & \mathbb{I} \end{array}\right) \tag{4.2.41}$$

where $\mathbb{I}$ is the $(n-1)\times(n-1)$ identity matrix, while the $(n-1)\times(n-1)$ matrix $\mathbb{E}_{11}$ is defined as

$$(\mathbb{E}_{11})^{ij} = \delta_1^i\, \delta_1^j.$$

Remark 4.2.12. *In comparison with the remark 4.2.10 above, we see that when $\tilde{q}_1 = 1$, the monodromy (4.2.41) above acts on $(a_1\, a_D^1)$ by the following 2×2 matrix:*

$$\mathcal{M}_{\textit{focus-focus}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

which is easily seen to be the ***focus-focus*** *type singularity (see formula (3.1.73) in section 3.1.6), i.e., the monodromy near the local model of the discriminant locus.*

This provides further rationality in the structure of the local model near the discriminant in section 3.1.6

Let us remark that if $\mathcal{M}_\nu$ acts on the row vector $\Xi := (\mathbf{a}_D \quad \mathbf{a})$ by the right multiplication of the matrix $\mathcal{M}_\nu$, then it acts on the cycle $\gamma = (\mathbf{g} \quad \mathbf{q})$ by the right multiplication of the matrix $\left(\mathcal{M}_\nu^{-1}\right)^t$. This follows from the requirement that the central charge of ν (and consequently its "mass") should keep invariant under the monodromy action. Indeed, suppose that the action on γ is given by the matrix $\widetilde{\mathcal{M}}_\nu$, then we have that

$$
\begin{aligned}
Z(\nu) = \mathbf{g}\cdot\mathbf{a}_D + \mathbf{q}\cdot\mathbf{a} = \gamma\cdot\Xi^t &\longmapsto \left(\gamma\cdot\widetilde{\mathcal{M}}_\nu\right)\cdot\left(\Xi\cdot\mathcal{M}_\nu\right)^t \\
&= \gamma\cdot\left(\widetilde{M}_\nu\cdot\mathcal{M}_\nu^t\right)\cdot\Xi \equiv Z(\nu) \\
\Longrightarrow \widetilde{M}_\nu\cdot\mathcal{M}_\nu^t \equiv \mathbb{I} &\Longleftrightarrow \widetilde{M}_\nu = \left(\mathcal{M}_\nu^{-1}\right)^t
\end{aligned}
\tag{4.2.42}
$$

Thus, by (4.2.41), the monodromy $\mathcal{M}_{(\mathbf{0},\tilde{q}_1,\cdots,0)}$ acts on the charge lattice $\underline{\Gamma}_{\mathbf{u}}$ by right multiplication of the following matrix

$$
\mathcal{M}_{\nu\cdot U} = \left(\begin{array}{c|c} \mathbb{I} & (\tilde{q}_1)^2\,\mathbb{E}_{11} \\ \hline \mathbf{0} & \mathbb{I} \end{array}\right)
\tag{4.2.43}
$$

Besides, since our basis $\{\alpha^i, \beta_i\}$ is chosen to have the intersection matrix given by the standard symplectic metric Ω (see (4.2.35)), we see that the intersection number between the two cycles $\gamma_1 = (\mathbf{g}_1 \quad \mathbf{q}_1)$ and $\gamma_2 = (\mathbf{g}_2 \quad \mathbf{q}_2)$ can be computed as

$$
\begin{aligned}
\langle\gamma_1,\gamma_2\rangle = \langle(\mathbf{g}_1\,\mathbf{q}_1),(\mathbf{g}_2\,\mathbf{q}_2)\rangle &= \left\langle \sum_i (g_1^i\,\beta_i + q_{1i}\,\alpha^i), \sum_i (g_2^i\,\beta_i + q_{2i}\,\alpha^i)\right\rangle \\
&= \sum_i g_2^i\,q_{1i} - \sum_i g_1^i\,q_{2i} = \mathbf{g}_2\cdot\mathbf{q}_1 - \mathbf{g}_1\cdot\mathbf{q}_2 \\
&= (\mathbf{g}_2\,\mathbf{q}_2)\cdot\begin{pmatrix} \mathbf{0} & \mathbb{I} \\ -\mathbb{I} & \mathbf{0}\end{pmatrix}\cdot\begin{pmatrix}\mathbf{g}_1 \\ \mathbf{q}_1\end{pmatrix} = \gamma_2\cdot\Omega\cdot\gamma_1^t = \gamma_1\cdot\Omega^t\cdot\gamma_2^t
\end{aligned}
\tag{4.2.44}
$$

Remark 4.2.13. *In physics literature, the above intersection formula between two "charges" (cycles), which gives the polarization of the complex integrable system, is called the* ***Dirac-Schwinger-Zwanziger*** *pair (DSZ-pair for short). It is an integer-valued bilinear form which gives quantization condition in physics.*

Definition 4.2.1. *We say that two charge vectors γ_1 and γ_2 to be* ***local*** *at some place $\mathbf{u}\in\mathcal{B}$ if and only if $\langle\gamma_1,\gamma_2\rangle$=0 at this place; and* ***non local*** *otherwise.*

Proposition 4.2.6. *For $\nu \in \underline{\Gamma}_{\mathbf{u}} = H_1(\mathcal{C}_{\mathbf{u}}, \mathbf{Z})$, a vanishing cycle situated at the point $\mathbf{u}\in\Delta_\Lambda$, and $U \in Sp\,(2(n-1),\mathbb{Z})$, we have the following relation*

$$
\mathcal{M}_{\nu\cdot U} = U^{-1}\cdot\mathcal{M}_\nu\cdot U
\tag{4.2.45}
$$

Proof. Suppose that the monodromy $\mathcal{M}_\nu$ is implemented by taking a small contractible loop around the singularity $\mathbf{u}$ (or irreducible component of the discriminant locus), and the monodromy $\mathcal{M}_{\nu\cdot U}$ by a small contractible loop around the singularity $\mathbf{u}'$. Connect the two singularity component by a line l, which is so chosen to avoid any other singular component. Thus, to go (which means by parallel transportation) from the place $\mathbf{u}$ to the place $\mathbf{u}'$ along the line l has the effect to transform an arbitrary cycle γ situated near $\mathbf{u}$ into the cycle $\gamma \cdot U$ near $\mathbf{u}'$. In order to compute the action associated with $\nu \cdot U$, i.e., $\mathcal{M}_{\nu \cdot U}(\gamma)$, we need to place the cycle γ near the singular component $\mathbf{u}'$, and then go around a small loop around it to see to where it carries the cycle in question. However, this can also be obtained by first parallel transporting the cycle γ near to $\mathbf{u}$ along the line l by using the Gauss-Manin connection. This has the effect that the cycle γ will be transformed into the cycle $\gamma \cdot U^{-1}$. Then, we go around a small loop around $\mathbf{u}$, which resulted in the new cycle $\gamma \cdot U^{-1} \cdot \mathcal{M}_\nu$. Finally, we take the cycle back to the starting point near $\mathbf{u}'$ along the line l. The total effect therefore is to transform γ into the cycle $\gamma \cdot U^{-1} \cdot \mathcal{M}_\nu \cdot U$. This finishes the proof of the proposition. □

Before stating the next proposition, we introduce the notion of Kronecker product of two matrices (c.f., [HJ91]), which would be very useful in writing the monodromy matrices compactly.

Definition 4.2.2. *Let A be a $m \times n$ matrix and B a $p \times q$ matrix, then the **Kronecker product** of A and B, denoted by $A \otimes B$, is defined as*

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix} \tag{4.2.46}$$

The following identities hold whenever the operations make sense:

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1} \qquad (A \otimes B)^t = A^t \otimes B^t$$
$$(A \otimes B) \otimes C = A \otimes (B \otimes C) \qquad (A \otimes B) \cdot (C \otimes D) = (A \cdot C) \otimes (B \cdot D)$$

Proposition 4.2.7. *Associated to the vanishing cycle $\nu = (\mathbf{g} \quad \mathbf{q})$, the monodromy $\mathcal{M}_\nu$ is given by the following matrix*

$$\mathcal{M}_\nu = \mathbb{I} + (\Omega \cdot \nu^t) \otimes \nu \tag{4.2.47}$$

Proof. Defining a matrix $\widetilde{\mathbf{q}_1} := (\tilde{q}_1\, 0 \cdots 0)$, then by using the definition of Kronecker product, it follows that

$$\widetilde{\mathbf{q}_1}^t \otimes \widetilde{\mathbf{q}_1} = \begin{pmatrix} \tilde{q}_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \otimes (\tilde{q}_1\, 0 \cdots 0) = \begin{pmatrix} (\tilde{q}_1)^2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

which is seen to be exactly the block $(\tilde{q}_1)^2\, \mathbb{E}_{11}$ in the matrix (4.2.43). Thus the monodromy matrix acting on the one cycles can be rewritten as

$$\mathcal{M}_{\nu \cdot U} = \begin{pmatrix} \mathbb{I} & \widetilde{\mathbf{q_1}}^t \otimes \widetilde{\mathbf{q_1}} \\ \mathbf{0} & \mathbb{I} \end{pmatrix} = \mathbb{I}_{2(n-1)\times 2(n-1)} + \begin{pmatrix} \mathbf{0} & \widetilde{\mathbf{q_1}}^t \otimes \widetilde{\mathbf{q_1}} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

By using the standard symplectic metric Ω, the above matrix can be further rewritten in the following form

$$\mathcal{M}_{\nu\cdot U} = \mathbb{I} + \begin{pmatrix} \widetilde{\mathbf{q}}_1{}^t \\ 0 \end{pmatrix} \otimes (\mathbf{0} \quad \widetilde{\mathbf{q}}_1) = \mathbb{I} + \Omega \cdot \begin{pmatrix} 0 \\ \widetilde{\mathbf{q}}_1{}^t \end{pmatrix} \otimes (\mathbf{0} \quad \widetilde{\mathbf{q}}_1)$$

$$= \mathbb{I} + \Omega \cdot \left(\mathbf{0} \quad \widetilde{\mathbf{q}_1}^t\right) \otimes (\mathbf{0} \quad \widetilde{\mathbf{q}}_1) = \mathbb{I} + \Omega \cdot (\nu \cdot U)^t \otimes (\nu \cdot U)$$

Then, by using (4.2.45), we see that

$$\begin{aligned} \mathcal{M}_\nu &= U \cdot \mathcal{M}_{\nu\cdot U} \cdot U^{-1} \\ &= U \cdot \left[\mathbb{I} + \Omega \cdot (\nu \cdot U)^t \otimes (\nu \cdot U)\right] \cdot U^{-1} \\ &= \mathbb{I} + U \cdot \Omega \cdot (\nu \cdot U)^t \otimes (\nu \cdot U) \cdot U^{-1} \\ &= \mathbb{I} + (U \cdot \Omega \cdot U^t) \cdot \nu^t \otimes \nu \cdot (U \cdot U^{-1}) \end{aligned}$$

As the transition matrix U belongs to $Sp(2n-2, \mathbb{Z})$, it follows that $U \cdot \Omega \cdot U^t \equiv \Omega$. Inserting it into the above formula, we finally get that

$$\mathcal{M}_\nu = \mathbb{I} + \Omega \cdot \nu^t \otimes \nu$$

This finishes the proof of the proposition. $\square$

Now suppose that $\nu = (\mathbf{g} \quad \mathbf{q}) = (g^i; q_i) = \sum_i g^i \beta_i + q_i \alpha^i$, then the formula (4.2.47) can be rewritten as

$$\begin{aligned} \mathcal{M}_\nu &= \mathbb{I} + \left(\Omega \cdot (\mathbf{g} \quad \mathbf{q})^t\right) \otimes (\mathbf{g} \quad \mathbf{q}) \\ &= \mathbb{I} + \begin{pmatrix} \mathbf{0} & \mathbb{I} \\ -\mathbb{I} & \mathbf{0} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{g}^t \\ \mathbf{q}^t \end{pmatrix} \otimes (\mathbf{g} \quad \mathbf{q}) \\ &= \mathbb{I} + \begin{pmatrix} \mathbf{q}^t \\ -\mathbf{g}^t \end{pmatrix} \otimes (\mathbf{g} \quad \mathbf{q}) = \mathbb{I} + \begin{pmatrix} \mathbf{q}^t \otimes \mathbf{g} & \mathbf{q}^t \otimes \mathbf{q} \\ -\mathbf{g}^t \otimes \mathbf{g} & -\mathbf{g}^t \otimes \mathbf{q} \end{pmatrix} \end{aligned}$$

that is

$$\begin{aligned} \mathcal{M}_\nu = \mathcal{M}_{(\mathbf{g}\,\mathbf{q})} &= \begin{pmatrix} \mathbb{I} + \mathbf{q}^t \otimes \mathbf{g} & \mathbf{q}^t \otimes \mathbf{q} \\ -\mathbf{g}^t \otimes \mathbf{g} & \mathbb{I} - \mathbf{g}^t \otimes \mathbf{q} \end{pmatrix} \\ &= \begin{pmatrix} \delta_i^j + q_j g^i & q_i q_j \\ -g^i g^j & \delta_i^j - g^j q_i \end{pmatrix} \end{aligned} \tag{4.2.48}$$

Proposition 4.2.8. *The monodromy action $\mathcal{M}_\nu$ computed as above (4.2.47) is given exactly by the Picard-Lefschetz formula (4.2.38).*

Proof. Given $\delta \in H_1(\mathcal{C}_{\mathbf{u}}, \mathbb{Z})$, by the formula (4.2.47), the monodromy action on it is given by

$$\begin{aligned} \mathcal{M}_\nu(\delta) &= \delta \cdot \mathcal{M}_\nu = \delta \cdot \left(\mathbb{I} + \Omega \cdot \nu^t \otimes \nu\right) \\ &= \delta + (\delta \cdot \Omega \cdot \nu^t) \otimes \nu \overset{\text{using (4.2.44)}}{=\!=\!=\!=} \delta + \langle \nu, \delta \rangle \, \nu \end{aligned}$$

$\square$

Proposition 4.2.9. *Given two local vanishing cycles ν_1 and ν_2, we have that*

$$[\mathcal{M}_{\nu_1}, \mathcal{M}_{\nu_2}] = 0 \tag{4.2.49}$$

Proof. Using Picard-Lefschetz formula, we can compute that

$$\mathcal{M}_{\nu_2} \cdot \mathcal{M}_{\nu_1}(\delta) = \delta + \langle \nu_1, \delta \rangle \, \nu_1 + \langle \nu_2, \delta \rangle \, \nu_2 + \langle \nu_2, \langle \nu_1, \delta \rangle \, \nu_1 \rangle \, \nu_2$$

$$= \delta + \langle \nu_1, \delta \rangle \, \nu_1 + \langle \nu_2, \delta \rangle \, \nu_2 + \langle \nu_1, \delta \rangle \langle \nu_2, \nu_1 \rangle \, \nu_2 \overset{\langle \nu_1, \nu_2 \rangle = 0}{=\!=\!=\!=} \delta + \langle \nu_1, \delta \rangle \, \nu_1 + \langle \nu_2, \delta \rangle \, \nu_2$$

which is invariant under the exchange: $\nu_1 \longleftrightarrow \nu_2$, consequently,

$$\mathcal{M}_{\nu_2} \cdot \mathcal{M}_{\nu_1} = \mathcal{M}_{\nu_1} \cdot \mathcal{M}_{\nu_2} \Longleftrightarrow [\mathcal{M}_{\nu_1}, \mathcal{M}_{\nu_2}] = 0.$$

□

Another proof of proposition 4.2.9

A direct proof by using the matrix representation (4.2.48) of $\mathcal{M}_\nu$ is also interesting. Suppose $\mathcal{M}_{\nu_2} \cdot \mathcal{M}_{\nu_1} = A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, then by the formula (4.2.48), the entries are given respectively by

$$a_{11} = \mathbb{I} + \mathbf{q}_1^t \otimes \mathbf{g}_1 + \mathbf{q}_2^t \otimes \mathbf{g}_2 + (\mathbf{q}_2^t \otimes \mathbf{g}_2) \cdot (\mathbf{q}_1^t \otimes \mathbf{g}_1)$$

$$\overset{\mathbf{q}_2^t \otimes \mathbf{g}_2 \equiv \mathbf{q}_2 \otimes \mathbf{g}_2^t}{=\!=\!=\!=\!=} \mathbb{I} + \mathbf{q}_1^t \otimes \mathbf{g}_1 + \mathbf{q}_2^t \otimes \mathbf{g}_2 + (\mathbf{q}_2 \otimes \mathbf{g}_2^t) \cdot (\mathbf{q}_1^t \otimes \mathbf{g}_1)$$

$$\overset{(A \otimes B) \cdot (C \otimes D) = (A \cdot B) \otimes (C \cdot D)}{=\!=\!=\!=\!=\!=\!=} \mathbb{I} + \mathbf{q}_1^t \otimes \mathbf{g}_1 + \mathbf{q}_2^t \otimes \mathbf{g}_2 + (\mathbf{q}_2 \cdot \mathbf{g}_1^t) \otimes (\mathbf{q}_2^t \cdot \mathbf{g}_1)$$

which is clearly invariant under the exchange of indices $1 \longleftrightarrow 2$. Similarly:

$$a_{12} = \mathbf{q}_1^t \otimes \mathbf{q}_1 + \mathbf{q}_2^t \otimes \mathbf{q}_2 + (\mathbf{q}_2^t \otimes \mathbf{g}_2) \cdot (\mathbf{q}_1^t \otimes \mathbf{g}_1) - (\mathbf{q}_2^t \otimes \mathbf{q}_2) \cdot (\mathbf{g}_1^t \otimes \mathbf{q}_1)$$

$$\underset{\mathbf{q}_2^t \otimes \mathbf{q}_2 \equiv \mathbf{q}_2 \otimes \mathbf{q}_2^t}{\overset{\mathbf{q}_2^t \otimes \mathbf{g}_2 \equiv \mathbf{q}_2 \otimes \mathbf{g}_2^t}{=\!=\!=\!=\!=}} \mathbf{q}_1^t \otimes \mathbf{q}_1 + \mathbf{q}_2^t \otimes \mathbf{q}_2 + (\mathbf{q}_2 \otimes \mathbf{g}_2^t) \cdot (\mathbf{q}_1^t \otimes \mathbf{g}_1) - (\mathbf{q}_2 \otimes \mathbf{q}_2^t) \cdot (\mathbf{g}_1^t \otimes \mathbf{q}_1)$$

$$\overset{(A \otimes B) \cdot (C \otimes D) = (A \cdot B) \otimes (C \cdot D)}{=\!=\!=\!=\!=\!=\!=} \mathbf{q}_1^t \otimes \mathbf{q}_1 + \mathbf{q}_2^t \otimes \mathbf{q}_2 + (\mathbf{q}_2 \cdot \mathbf{q}_1^t) \otimes (\mathbf{g}_2^t \cdot \mathbf{g}_1) - (\mathbf{q}_2 \cdot \mathbf{g}_1^t) \otimes (\mathbf{q}_2^t \cdot \mathbf{q}_1)$$

As $\langle \nu_1, \nu_2 \rangle = 0$, we have that $\mathbf{g_2} \cdot \mathbf{q_1^t} = \mathbf{g}_1 \cdot \mathbf{q}_2^t$. Inserting it in the above identity, we get that

$$a_{12} = \mathbf{q}_1^t \otimes \mathbf{q}_1 + \mathbf{q}_2^t \otimes \mathbf{q}_2 + (\mathbf{q}_2 \cdot \mathbf{q}_1^t) \otimes (\mathbf{g}_2^t \cdot \mathbf{g}_1) - (\mathbf{g}_2 \cdot \mathbf{q}_1^t) \otimes (\mathbf{q}_2^t \cdot \mathbf{q}_1)$$

By exchanging the indices $1 \longleftrightarrow 2$, the last two terms in the above identities become

$$(\mathbf{q}_1 \cdot \mathbf{q}_2^t) \otimes (\mathbf{g}_1^t \cdot \mathbf{g}_2) - (\mathbf{g}_1 \cdot \mathbf{q}_2^t) \otimes (\mathbf{q}_1^t \cdot \mathbf{q}_2)$$

$$\overset{\langle \nu_1, \nu_2 \rangle = 0 \Longrightarrow \mathbf{g}_1 \cdot \mathbf{q}_2^t = \mathbf{g}_2 \cdot \mathbf{q}_1^t}{=\!=\!=\!=\!=\!=\!=} (\mathbf{q}_2 \cdot \mathbf{q}_1^t) \otimes (\mathbf{g}_2^t \cdot \mathbf{g}_1) - (\mathbf{g}_2 \cdot \mathbf{q}_1^t) \otimes (\mathbf{q}_2^t \cdot \mathbf{q}_1)$$

from which it follows that a_{12} is again invariant under exchange of indices.

By exactly the same reasoning, the invariance of the other two entries: a_{21} and a_{22} under $1 \longleftrightarrow 2$ can also be obtained. As a consequence, we see that the two matrices $\mathcal{M}_{\nu_1}$ commutes with $\mathcal{M}_{\nu_2}$. □.

In terms of SW curve as weight diagram fibration

Now, we give the description of the vanishing cycles in terms of the SW curve written as the weight diagram fibration (see section 4.1.3). The curve is given by the following

$$\mathcal{C}_{\mathbf{u}} : z + \frac{\Lambda^{2n}}{z} + 2\,\mathcal{W}_{A_{n-1}}(x, \mathbf{u}) = 0$$

with the associated Seiberg Witten differential given by (formula (4.1.47))

$$\lambda_{SW} = -x\frac{dz}{z}$$

The advantage of this form of equation makes the relations between the vanishing cycles and the roots of $SU(n)$ more transparent. The idea presented here is due to [Tac15] and [Hol97].

Recall that for given z and $\mathbf{u}$, the solutions to the equation defining the curve $\mathcal{C}_{\mathbf{u}}$ above are given by

$$\{e_1(z, \mathbf{u}), \cdots, e_n(z, \mathbf{u}))\}$$

which correspond to the n weights of the fundamental representation of $SU(n)$, and could be used to serve as local coordinates on the corresponding sheets of the fibration. Thus, on the i-th sheet, the coordinate is given by the Seiberg-Witten differential restricted on it as

$$\lambda_i = -e_i(z, \mathbf{u})\frac{dz}{z}$$

Define the α cycles to be the lift of the unit circle $S^1 := \{z : |z| = 1\} \subset \mathbb{C}P^1$ of the base of fibration, i.e.,

$$\alpha^i = \text{lift of } S^1 \text{ to the i-th sheet of the fibration.} \tag{4.2.50}$$

Remark 4.2.14. *Notice that in the proposition 4.1.10, we used the notation γ_i for the cycle α^i defined here. As in the A_{n-1}-case, we have that $\sum_i e_i(z, \mathbf{u}) \equiv 0$, which implies that the cycles α^i's are not independent such that*

$$\alpha^n = -\alpha^1 - \cdots - \alpha^{n-1}$$

By defining the $\mathbf{a}$ periods as $a^i := \oint_{\alpha^i} \lambda_{SW}$, the above identity becomes the desired

$$\sum_{i=1}^{n} a^i = \sum_{i=1}^{n} \oint_{\alpha^i} \lambda_{SW} \equiv 0$$

As had been discussed before, there are $n-1$ pairs of branch points $z_i^{\pm}$, $i = 1, \cdots, n-1$, each of which related by the symmetry relation (c.f., (4.1.55))

$$z_i^+ z_i^- = \Lambda^{2n}$$

plus the other two branch points, which are situated at 0 and ∞. The pair $z_i^{\pm}$ corresponds to the colliding of the roots e_i and e_{i+1}, i.e., the colliding of i-th branch and $(i+1)$-th branch of the fibration, which is associated to the simple root α of the Lie algebra. Consequently, by proposition 4.1.11, the monodromy around this pair of branch points is given by the fundamental Weyl reflection M_{α_i} (see (4.1.16)) associate to the simple root α_i.

Remark 4.2.15. *Each of the branch point* $z_i^{\pm}$ *has degree* 2*, while that for* 0 *and* ∞ *being* $n-1$*. Applying the Riemann-Hurwitz formula to the cover*

$$\mathcal{C}_{\mathbf{u}} \longrightarrow \mathbb{C}P^1 \cong S^2$$

we get that the relation between the Euler characteristics as

$$\chi(\mathcal{C}_{\mathbf{u}}) = (\textit{number of sheets})\, \chi(S^2) - \sum_{\textit{branch points}\, p_i} deg(p_i) - 1$$

$$= 2n - 2(n-1) - (n-1)2 = 4 - 2n$$

Using the relation between Euler characteristic and the genus of the curve $\mathcal{C}_{\mathbf{u}}$

$$\chi(\mathcal{C}_{\mathbf{u}}) = 2 - 2g\,(\mathcal{C}_{\mathbf{u}})$$

we get that

$$g\,(\mathcal{C}_{\mathbf{u}}) = n - 1 \quad \textit{as desired.}$$

The β-cycles, which are dual to the α-cycles are defined as

$$\beta_i = \widetilde{\overrightarrow{0\, z_i^+}}\ , \textit{i.e.}, \text{the lift of path up to i-th sheet connecting } z_i^+ \text{and } 0. \tag{4.2.51}$$

Proposition 4.2.10. *The cycles* $\{\alpha^i, \beta_i\}$ *constructed above satisfy the following intersection properties*

$$\langle \alpha^i, \alpha^j \rangle = \langle \beta_i, \beta_j \rangle = 0;$$
$$\langle \alpha^i, \beta_j \rangle = \delta^i_j.$$

Proof. The identities in the first line are clear since the two entries inside the intersection bracket come from different sheets, thus are void of intersection. For the non-trivial ones in the second line, we notice that the path $\mathscr{B}_i$ intersects with S^1 on the base of the fibration in exactly one point positively (by proper choice of orientation), and by lifting the configuration up to the i-th sheet, the intersection pattern remains unchanged. □

Next we show that in this setting, the large $\mathbf{u}$ (semi-classical) behaviour of $\mathbf{a}_D$ (see (4.2.40)) can be reproduced, which generalizes similar computation we had performed in the $SU(2)$ between the remark 4.2.3 and remark 4.2.4.

Proposition 4.2.11. *For* $|\mathbf{u}| \gg \Lambda^n$*, we have that (compare with the formula (4.2.39))*

$$a_{D,k} \approx \frac{na^k}{2\pi i} \ln \frac{a^k}{\Lambda}$$

Lemma 4.2.12. *On the* k*-th sheet, the Seiberg-Witten differential can be approximated by*

$$\lambda_k \approx \frac{a^k}{2\pi i} \frac{dz}{z}$$

Proof. By definition, we have

$$a^k = \oint_{\alpha^k} -x\frac{dz}{z} = \oint_{|z|=1} -e_k\frac{dz}{z} \approx -e_k \oint_{|z|=1} \frac{dz}{z} = -2\pi i e_k$$

Using this, we get that

$$\lambda_k = -e_j\frac{dz}{z} \approx \frac{a^k}{2\pi i}\frac{dz}{z}$$

□

Lemma 4.2.13. *For large* $|\mathbf{u}|$, $z_k^\pm$ *can be approximated by*

$$z_k^+ \approx (a^k)^n \qquad z_k^- \approx \frac{\Lambda^{2n}}{(a^k)^n}$$

Proof. Write the left hand side of $z + \frac{\Lambda^{2n}}{z} + 2\,\mathcal{W}_{A_{n-1}}(x, \mathbf{u}) = 0$ as $2\prod_i(x - e_i)$. When $|\mathbf{u}|$ becomes very large, i.e., Λ is relatively very small, we see that the pair of branch points $z_k^\pm$ approaches to the pair $\{0, \infty\}$ (for examples, see the formula (4.1.54) and (4.1.55)). We assume in the following that z_k^+ is the larger one. Inserting $z = z_k^+$ in the equation defining $\mathcal{C}_\mathbf{u}$ above, as z_k^+ is very large, we see that $\frac{\lambda^{2n}}{z_k^+} \approx 0$. Also, some roots collide with e_k, without loss of generality, we assume that $e_i \approx e_k$, $\forall i \neq k$. Thus, we get the relation $z_k^+ - u_n = e_k^n$, but as $e_k \approx a^k$, we get that $z_k^+ \approx (a^k)^n$. Since $z_k^+ z_k^- = \Lambda^{2n}$, we get the remaining approximation. □

Proof. (**of the proposition 4.2.11**) As had been noted in the proof of the lemma above, we see that in the semi-classical regime, $z_k^- \to 0$, thus by lemma 4.2.12, we have follows:

$$\begin{aligned} a_{D,k} &= \oint_{\beta_k} \lambda_{SW} = \int_0^{z_k^+} \lambda_k \approx \int_{z_k^-}^{z_k^+} \frac{a^k}{2\pi i}\frac{dz}{z} \approx \frac{a^k}{2\pi i}\int_{\frac{\Lambda^{2n}}{(a^k)^n}}^{(a^k)^n} \frac{dz}{z} \\ &= \frac{a^k}{2\pi i}\ln\frac{(a^k)^n}{\Lambda^{2n}/(a^k)^n} = \frac{a^k}{2\pi i}\ln\left(\frac{a^k}{\Lambda}\right)^{2n} = \frac{na^k}{\pi i}\ln\frac{a^k}{\Lambda}. \end{aligned}$$

□

Vanishing cycles and the associated charges

First, let $\mathscr{P}_i^+ = \widetilde{z_i^+ z_i^-}$ denote the path that connects the pair of branch points $z_i^\pm$ By lifting $\mathscr{P}_i^+$ to the i-th and $(i+1)$-th sheet, we get the cycles denoted by $\widetilde{\mathscr{P}}_i^+$ and $\widetilde{\mathscr{P}}_{i+1}^+$ respectively. Since the Weyl reflection r_i associated with the root α_i acts on the sheet by the permutation simple transposition $(i, i+1)$, i.e., by permutating the i-th sheet with the $i+1$-th sheet, we see that the difference cycle $\widetilde{\mathscr{P}}_i^+ - \widetilde{\mathscr{P}}_{i+1}^+$ is closed as $\widetilde{\mathscr{P}}_{i+1}^+$ represents the reverse cycle of $\widetilde{\mathscr{P}}_i^+$ which intersects with each other over the points $z_i^\pm$ where the i-th and $(i+1)$-th sheets collide. We then define the following closed cycles

$$\widetilde{\nu}_i^+ = \widetilde{\mathscr{P}}_i^+ - \widetilde{\mathscr{P}}_{i+1}^+ \tag{4.2.52}$$

When the moduli $\mathbf{u}$ approaches to some components of the discriminant locus $\Delta_\Lambda \subset \mathcal{B}$, the two branch points $z_i^\pm$ come together, which causes the path $\mathscr{P}_i^+$, and consequently the cycle $\widetilde{\nu}_i^+$ to shrink into zero. This implies that $\widetilde{\nu}_i^+$ is the vanishing cycle associated to the pair of branch points $\{z_i^\pm\}$. However, as the pair of branch points $z_i^\pm$ satisfies the relations $z_i^+ z_i^- = \Lambda^{2n}$, we see that when the two branch points hit each other, their common value equals $\pm\Lambda^n$. This implies that there are two ways for the two branch points to get together corresponding to the $\pm$ signs. Thus, besides the $(n-1)$ vanishing cycles $\widetilde{\nu_i}^+$ associated to the $(n-1)$ pairs of branch points, there should be another $(n-1)$ vanishing cycles which are to be constructed in the following. Together, they will form the basis of the first homology cycles of the SW curve. Denote then by the path $\mathscr{P}_i^-$ with homology class represented by $\mathscr{P}_i^+$ followed by winding around the origin once counterclockwise, i.e., as homology one cycle on the base $\mathbb{C}P^1$, we have that

$$[\mathscr{P}_i^-] = [\mathscr{P}_i^+] + [S^1] \tag{4.2.53}$$

We need to lift these paths to the sheets of fibration to get the required vanishing cycles, we will employ an method that generalizes the construction of $\widetilde{\nu}_i^+$. Denote by $\widetilde{\mathscr{P}}_i^{\pm}(w)$ the lift of the path $\mathscr{P}_i^{\pm}$ to the sheet labeled by the weight w of the fundamental representation of $SU(n)$. As the pair of branch points $\{z_i^{\pm}\}$ is associated with the simple roots $\alpha_i = \alpha_{i,i+1}$, with the monodromy around the branch points given by the Weyl reflection $r_i = r_{\alpha_i}$,we see that the path $\widetilde{\mathscr{P}}_i^{\pm}(w)$ on the sheet labeled by w is followed by the returning path $\widetilde{\mathscr{P}}_i^{\pm}(r_i(w))$ on the sheet labeled by the weight $r_i(w)$. Motivated by this, we define the following one cycles

$$\widetilde{\nu}_i^{\pm} := \frac{1}{N} \sum_w \langle w, \alpha_i \rangle \, \widetilde{\mathscr{P}}_i^{\pm}(w) \tag{4.2.54}$$

where N is some normalizing constant so to make the vanishing cylces above independent of the weights of the representation (thus independent of the representation itself).

Claim: To make the cycles constructed above independent of the representation, the normalizing factor N can be chosen in such a way that [Hol97]:

$$\frac{N}{2\pi i} \cdot \mathbb{I} = \sum_w w \otimes w$$

Proof of the claim: We consider the integral of the SW differential over these cycles. First note that on the sheet labeled by the weight w, the differential λ_{SW} becomes:

$$\lambda_{SW} = -x\frac{dz}{z} = -e_i(z, \mathbf{u})\frac{dz}{z} = -\langle w, \phi \rangle \, \frac{dz}{z}$$

where $\phi = \mathbf{a} \cdot \mathbf{H} = \sum_{i=1}^{n-1} a^i \, \mathbf{H}_i \in \mathfrak{h}_n$, the element in the Cartan algebra defined in section 4.1.1, then we compute as follows:

$$\begin{aligned}
\oint_{\widetilde{\nu}_i^{\pm}} \lambda_{SW} &= \frac{1}{N} \sum_w \langle w, \alpha_i \rangle \oint_{\widetilde{\mathscr{P}}_i^{\pm}(w)} \lambda_{SW} = \frac{2}{N} \sum_w \langle w, \alpha_i \rangle \oint_{\widetilde{\mathscr{P}}_i^{\pm}(w) - \widetilde{\mathscr{P}}_i^{\pm}(r_i(w))} \lambda_{SW} \\
&= \frac{2}{N} \sum_w \langle w, \alpha_i \rangle \left(\oint_{\widetilde{\mathscr{P}}_i^{\pm}(w)} \lambda_{SW} - \oint_{\widetilde{\mathscr{P}}_i^{\pm}(r_i(w))} \lambda_{SW} \right) \\
&= \frac{2}{N} \sum_w \langle w, \alpha_i \rangle \oint_{\widetilde{\mathscr{P}}_i^{\pm}(w)} \left(\langle r_i(w), \phi \rangle - \langle w, \phi \rangle \right) \frac{dz}{z} \\
&= -\frac{2}{N} \sum_w \langle w, \alpha_i \rangle \oint_{\widetilde{\mathscr{P}}_i^{\pm}(w)} \frac{\langle w, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \langle \alpha_i, \phi(z) \rangle \frac{dz}{z} \\
&= -\frac{2}{N} \sum_w \langle w, \alpha_i \rangle \frac{\langle w, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \oint_{\widetilde{\mathscr{P}}_i^{\pm}(w)} \langle \alpha_i, \phi(z) \rangle \frac{dz}{z} \\
&= -\frac{2}{N} \sum_w \frac{\langle w, \alpha_i \rangle^2}{\alpha_i^2} \oint_{\widetilde{\mathscr{P}}_i^{\pm}(w)} \langle \alpha_i, \phi(z) \rangle \frac{dz}{z} \\
&\xlongequal[\text{simply laced group}]{\alpha_i^2 = 2 \text{ for}} -\frac{1}{N} \sum_w \langle w, \alpha_i \rangle^2 \oint_{\widetilde{\mathscr{P}}_i^{\pm}(w)} \langle \alpha_i, \phi(z) \rangle \frac{dz}{z}
\end{aligned}$$

Consequently, by choosing N such that $\frac{N}{2\pi i} = \sum_w \langle w, \alpha_i\rangle\langle w, \alpha_i\rangle$, we see that the integral is independent of the representation. $\square$.

Thus, the overall effect is that as homology class of one cycles, i.e., when the integration over them is concerned, we have that

$$\left[\widetilde{\nu}_i^{\pm}\right] = -\left[\widetilde{\mathscr{P}}_i^{\pm}\right]$$

Besides the $(n-1)$ pair of branch points $\{z_i^{\pm}\}$, we also have two more branch points $\{0, \infty\}$. Let us connect them by a path $\mathscr{P}_{0\infty}$, which connects all the sheets with weights belonging to one orbit of the cyclic group generated by the Coexter elements. The Coexter element specifies how the sheets are connected above ∞.

By lifting the path $\mathscr{P}_{0\infty}$ by using formula (4.2.54), we get the closed one cycle denoted by $\widetilde{\nu}_n^{\pm}$, together with $\{\widetilde{\nu}_i^{\pm}\}_{i=1}^{n-1}$, we claim that their intersections are the same as that being satisfied by the vanishing cycles $\{\nu_i^{\pm}\}_{i=1}^{n}$ constructed before.

Proposition 4.2.14. *The intersections numbers among $\{\widetilde{\nu}_i^{\pm}\}_{i=1}^{n}$ satisfy the relations (4.2.33) given in proposition 4.2.4.*

Proof. Instead of direct computation, we establish the one to one correspondence between the two sets of vanishing cycles, namely $\{\widetilde{\nu}_i^{\pm}\}_{i=1}^{n}$ and $\{\nu_i^{\pm}\}_{i=1}^{n}$, thus the proposition follows from the proposition 4.2.4. Before establishing the correspondence, let us first note that their are two ways that the pair of branch points $\{z_i^{\pm}\}$ can come close to each other, which correspond to two ways that the roots of $y^2 = (\mathcal{W}_{A_{n-1}}(x, \mathbf{u}))^2 - \Lambda^{2n}$ collapse with each other, say, $z_i^+ = z_i^- = \Lambda^n$ corresponds to the roots e_i^+ colliding with e_{i+1}^+, while $z_i^+ = z_i^- = -\Lambda^n$ corresponds to the coincidence of the roots e_i^- with e_{i+1}^-. Similarly, by approximating 0 by $\epsilon\Lambda^n$ and ∞ by $\frac{\Lambda^n}{\epsilon}$ as $\epsilon \to 0$, we deem that $\epsilon \to 1$ corresponds to the root e_1^+ colliding with e_n^+, while $\epsilon \to -1$ corresponds to the colliding of e_1^- and e_n^-. In this way, we associate the vanishing paths $\{\widetilde{\mathscr{P}}_i^{\pm}\}_{i=1}^{n-1}$ (and consequently the vanishing cycles $\{\widetilde{\nu}_i^{\pm}\}_{i=1}^{n-1}$) to $\{\nu_i^{\pm}\}_{i=1}^{n-1}$; as well as $\widetilde{\mathscr{P}_{0\infty}}^{\pm}$ (and thus the vanishing cycles $\widetilde{\nu}_n^{\pm}$) to $\nu_n^{\pm}$. $\square$

Proposition 4.2.15. *Let a, b be two variables that can take value from $\{\pm\}$, then the monodromy actions associated with the vanishing cycles can be summarized as*

$$\mathcal{M}_{\widetilde{\nu}_i^a}(\widetilde{\nu}_j^b) = \widetilde{\nu}_j^b + (\epsilon_{ab} + I_{ij})\langle\alpha_j, \alpha_i\rangle\, \widetilde{\nu}_i^a \tag{4.2.55}$$

where ϵ_{ab} is the two dimensional anti-symmetric tensor, and I_{ij} the intersection matrix among the vanishing cycles $\{\widetilde{\nu}_i^{+}\}_{i=1}^{n-1}$, which in our case is chosen to be the standard symplectic metric. And the Cartan matrix for A_{n-1} is given by $\left((C_{A_{n-1}})_{ij}\right) = (\langle\alpha_i, \alpha_j\rangle)$.

Proof. This follows from the above proposition and the remark 4.2.9. $\square$

By the above proposition, we will not distinguish between $\widetilde{\nu}_i^{\pm}$ and $\nu_i^{\pm}$, we simply use notation $\nu_i^{\pm}$ to denote both of them. Using the same notation $\widetilde{\nu}_i^{\pm} := (\mathbf{g} \quad \mathbf{q})$ for the charge vectors associated to the vanishing cycles $\widetilde{\nu}_i^{\pm}$. Namely, we have

$$\widetilde{\nu}_i^{\pm} = \sum_{i=1}^{n-1} g^i\beta_i + q_i\alpha^i = (g^i \ q_i)\cdot(\beta_i \ \alpha^i)$$

Next, we will use the information we have obtained so far to give the expressions of $\widetilde{\nu}_i^{\pm}$ in terms of the roots of the Lie group $SU(n)$ (c.f., [Hol97]).

Proposition 4.2.16. *The charge vectors associated to the vanishing cycles can be expressed as*

$$\nu_i^+ = (\alpha_i,\, (p_i+1)\,\alpha_i) \qquad \nu_i^- = (\alpha_i,\, p_i\,\alpha_i) \tag{4.2.56}$$

where for i, j such that $\langle \alpha_i, \alpha_j\rangle \neq 0$, we have that the integers p_i are restrained by

$$p_i - p_j = I_{ji} \tag{4.2.57}$$

Proof. First, by (4.2.54) and the proof of the claim, we have that

$$\begin{aligned}
Z(\nu_i^+ - \nu_i^-) &= \oint_{\nu_i^+} \lambda_{SW} - \oint_{\nu_i^-} \lambda_{SW} = \oint_{\widetilde{\mathscr{P}_i^+}\,\lambda_{SW}} - \oint_{\widetilde{\mathscr{P}_i^-}} \lambda_{SW} \\
&\xlongequal{\text{by (4.2.53)}: \mathscr{P}_i^- - \mathscr{P}_i^+ = S^1} - \oint_{\text{lift of } S^1} \lambda_{SW} = \oint_{\text{lift of } S^1} \langle \alpha_i, \phi(z)\rangle \frac{dz}{z} \\
&\xlongequal{\phi = \mathbf{a}\cdot\mathbf{H}} \oint_{\text{lift of } S^1} \langle \alpha_i, \mathbf{a}\cdot\mathbf{H}\rangle \frac{dz}{z} = \oint_{\text{lift of } S^1} \langle \alpha_i, \mathbf{H}\rangle\, \mathbf{a}\, \frac{dz}{z} = \alpha_i \cdot \mathbf{a}
\end{aligned}$$

But

$$Z(\nu_i^+ - \nu_i^-) = \Xi \cdot (\mathbf{g} \quad \mathbf{q}) = (\mathbf{a_D} \quad \mathbf{a}) \cdot (\nu_i^+ - \nu_i^-)$$

Comparing, we get that

$$\nu_i^+ - \nu_i^- = (0, \alpha_i) \tag{4.2.58}$$

Next, by Picard-Lefschtz formula, we have that

$$\mathcal{M}_{\nu_i^a}(\nu_j^b) = \nu_j^b + \left\langle \nu_i^a, \nu_j^b \right\rangle\, \nu_i^a$$

Comparing it with the formula (4.2.55), we have the following relation

$$\left\langle \nu_i^a, \nu_j^b \right\rangle = (\epsilon_{ab} + I_{ij}) \langle \alpha_j, \alpha_i\rangle \tag{4.2.59}$$

Now suppose that $\nu_i^- = (\mathbf{g}_i, \mathbf{q}_i)$, then by (4.2.58), $\nu_i^+ = (\mathbf{g}_i, \mathbf{q}_i) + (0, \alpha_i)$. Then, by (4.2.59), we get the following relation:

$$\langle \nu_i^+, \nu_j^+\rangle = I_{ij}\langle \alpha_j, \alpha_i\rangle \qquad \langle \nu_i^-, \nu_j^-\rangle = I_{ij}\langle \alpha_j, \alpha_i\rangle$$
$$\langle \nu_i^+, \nu_j^-\rangle = (I_{ij} - 1)\langle \alpha_j, \alpha_i\rangle$$

Using these identities, we get that

$$\langle \nu_i^- + (0, \alpha_i), \nu_j^-\rangle = \langle \nu_i^-, \nu_j^-\rangle + \langle (0, \alpha_i), (\mathbf{g}_j, \mathbf{q}_j)\rangle$$
$$= I_{ij}\langle \alpha_j, \alpha_i\rangle - \langle \alpha_i, \mathbf{g}_j\rangle = (I_{ij} - 1)\langle \alpha_j, \alpha_i\rangle \quad \Rightarrow \mathbf{g}_j = \alpha_j$$

By using the third identity again, we have that

$$\langle \nu_i^+, \nu_i^-\rangle = \langle (\alpha_i, \alpha_i + \mathbf{q}_i), (\alpha_i, \mathbf{q}_i)\rangle = \langle \alpha_i, \mathbf{q}_j\rangle - \langle \alpha_j, \mathbf{q}_i\rangle - \langle \alpha_i, \alpha_j\rangle$$
$$= I_{ij}\langle \alpha_i, \alpha_j\rangle - \langle \alpha_i, \alpha_j\rangle \qquad \Rightarrow \langle \alpha_i, \mathbf{q}_j\rangle - \langle \alpha_j, \mathbf{q}_i\rangle = I_{ij}\langle \alpha_i, \alpha_j\rangle$$

It is clear then that the above linear equation is solved by

$$\mathbf{q}_i = p_i\, \alpha_i$$

for some integer $p_i \in \mathbb{Z}$, and the integers $\{p_i\}$ satisfy the condition (4.2.57). □

Remark 4.2.16. *There are certain ambiguities in determining the charge vectors of the associated vanishing cycles. Namely, the integer p_i in the formulae (4.2.56) can be shifted by arbitrary $n \in \mathbb{Z}$ by looping the paths $\mathscr{P}_i^+$ n-times around the origin (orientation depends on whether n is positive or negative). We can thus resolve the ambiguity by fixing $p_i \equiv n$ for all i. Thus, the formula (4.2.57) can be specified as*

$$\nu_i^+ = (\alpha_i, (n+1)\,\alpha_i) \qquad \nu_i^- = (\alpha_i, n\,\alpha_i) \tag{4.2.60}$$

Note that the charge vector $(\alpha_i, n\,\alpha_i)$ corresponds to the vanishing cycle that is the lift of the path $\mathscr{P}_i^+ + (n - p_i)\,S^1$. In physics literature, this phenomenon is called the manifestation of ***the democracy of dyons*** *(see [Hol97] for more information).*

Braid group and the Monodromy

We now introduce another approach to the computation of the vanishing cycles and the associated monodromies. This method is based on the concept of braid group and its representations. For its history and development, please see the book [Han89], the paper [Lön06], as well as the references contained there in. Its application in the physics content related to the Seiberg-Witten integrable system are discussed in [Cer95] and [Bil+96].

We give some preliminary discussions. Recall that our purpose is to compute the monodromy acting on the local system $\underline{\Gamma} := H_1(\mathcal{C}_{\mathbf{u}}, \mathbb{Z})$ (thus on the period integrals $(\mathbf{a}_D\,\mathbf{a})$) when the moduli $\mathbf{u}$ circles the discriminant locus (singularities) Δ_Λ. This amounts to studying the representation of the fundamental group of the complement of the discriminant Δ_Λ in $\mathcal{B}$ on the local system of first homology lattices of the hyper-elliptic curves fibration, i.e., we study the following homomorphism (by choosing a fixed point $b_0 \in \mathcal{B}\backslash\Delta_\Lambda$)

$$\widetilde{\rho} : \pi_1(\mathcal{B}\backslash\Delta_\Lambda, b_0) \longmapsto GL(\underline{\Gamma}) = GL(H_1(\mathcal{C}_{\mathbf{u}}, \mathbb{Z})) \tag{4.2.61}$$

Since we require that the intersection paring $\langle\cdot,\cdot\rangle$ is invariant under the monodromy, we see that the monodromy should factorize into the symplectic representation, namely, the image of $\widetilde{\rho}$ should lie in the subgroup

$$Sp\,(H_1(\mathcal{C}_{\mathbf{u}}, \mathbb{Z})) \cong Sp(2n-2, \mathbb{Z})$$

Consequently, we have the following **monodromy representation**

$$\rho : \pi_1(\mathcal{B}\backslash\Delta_\Lambda, b_0) \longrightarrow Sp\,(2n-2, \mathbb{Z}) \tag{4.2.62}$$

From this, we see that to solve the monodromy problem, we first need to understand the structure of the fundamental group of the complement of the discriminant locus, and then we should understand how its generators acts on the first homology cycles.

We will display in the following how the **braid group action** is related to the study of the above representation.

Given the local system of lattice $\pi : \underline{\Gamma} \to \mathcal{B}^0 := \mathcal{B}\backslash\Delta_\Lambda$, the action of the fundamental group $\pi_1(\mathcal{B}^0, b_0)$ on the fiber $\underline{\Gamma}_{\mathbf{u}} = H_1(\mathcal{C}_{\mathbf{u}}, \mathbb{Z})$ can be constructed as follows:

Given a small loop $\gamma : [0,1] \to \mathcal{B}^0$ based at b_0, consider the pull back of the fibration π over $[0,1]$, which gives us a trivial fibration $\pi^* : \gamma^*(\underline{\Gamma}) \to [0,1]$, with fiber $(\pi^*)^{-1}(t)$ over $t \in [0,1]$ being $\pi^{-1}(\gamma(t))$, i.e., we have the following commutative diagram:

$$\begin{array}{ccc} \gamma^*(\underline{\Gamma}) & \longrightarrow & \underline{\Gamma} \\ \Big\downarrow{\scriptstyle \pi^*} & & \Big\downarrow{\scriptstyle \pi} \\ [0,1] & \xrightarrow{\gamma} & \mathcal{B}^0 \end{array}$$

Note that π^* is trivial with the following trivilization map

$$\phi : [0,1] \times \underline{\Gamma} \to \gamma^*(\underline{\Gamma})$$

Then a section s of the π can be pulled back to a section $\widetilde{s}$ of π^* via

$$\widetilde{s} := \phi^{-1} \circ s \circ \gamma.$$

Now given an one cycle $\alpha \in \underline{\Gamma}_{\mathbf{u}}$, we want to describe the monodromy action of γ on α, namely $\rho(\gamma)(\alpha) =: \alpha^\gamma$.

To this end, let us first view the cycle α as a path $\alpha : [0,1] \to \{0\} \times \underline{\Gamma}_{\mathbf{u}}$, which is a lifting of the zero constant path $\mathbf{0} : [0,1] \to [0,1]$. Considering the homotopy $h : [0,1] \times [0,1] \to [0,1]$ given by $h(s,t) = t$ that takes the constant zero path $\mathbf{0}$ to the constant one path $\mathbf{1}$. By the *homotopy lifting property*, the homotopy can be lifted to the fiber level as the following map

$$\widetilde{h} : [0,1] \times [0,1] \to [0,1] \times \underline{\Gamma}_{\mathbf{u}}$$

such that $\widetilde{h}(s,0) = \alpha(s)$ and $\widetilde{h}(0,t) = \widetilde{h}(1,t) = \widetilde{s}(t)$.

Finally, the monodromy action of γ on α is given by the following

$$\alpha^\gamma(s) := \phi \circ \widetilde{h}(s,1) = s(\gamma)^{-1} \circ \alpha \circ s(\gamma) \in \underline{\Gamma}. \tag{4.2.63}$$

That is, roughly speaking, we push the α cycle fiberwisely along the loop γ while keeping the base point along the section s. Note that by the homotopy lifting property, the α^γ thus constructed is independent of the choice of the trivilization map ϕ.

After discussing the monodromy action, we show next that it is given by the braid group action. We note that the fundamental group of $\mathcal{B}^0$ can be identified with the braid group B_n. Denote by

$$Conf_n := \{(z_1, \cdots, z_n) \in \mathbb{C}^n : z_i \neq z_j,\ i \neq j\}$$

the configuration space of ordered n-distinct points on the complex plane $\mathbb{C}$.

The symmetry group S_n acts on it by permutating the n-points. Then, the configuration space of non-ordered sets of distinct n-points on $\mathbb{C}$ is defined by the following quotient

$$\widetilde{Conf_n} := Conf_n / S_n$$

The **braid group on n-strands** is defined as the fundamental group of the above configuration space based on a point $S \in \widetilde{Conf_n}$, namely

$$B_n := \pi_1\left(\widetilde{Conf_n}, S\right)$$

Denote by $\mathbb{D} \subset \mathbb{C} \cong \mathbb{R}^2$ the unit disk in $\mathbb{C} \cong \mathbb{R}^2$. Consider

$$\widetilde{\mathbb{D}^n} := (\mathbb{D}^n \backslash (\text{big diagonal}))\,/S_n$$

Then it is easy to see that $\widetilde{\mathbb{D}^n}$ is homotopy equivalent to $\widetilde{Conf_n}$ as the inclusion $\mathbb{D} \hookrightarrow \mathbb{C}$ is a homotopy equivalent. As a consequence, it follows that

$$B_n = \pi_1\left(\widetilde{\mathbb{D}^n}, p\right)$$

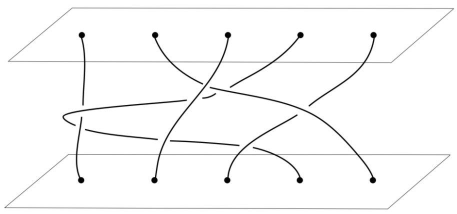

Figure 4.15: A braid

where the base point $p \in \widetilde{\mathbb{D}^n}$ can be chosen to be the point $S \in \widetilde{Conf_n}$ which consists of n distinct points situated on the line segment $[-1, 1] \subset \mathbb{D} \subset \mathbb{C}$.

Given a point $(z_1, \cdots, z_n) \in \widetilde{Conf_n}$, we associate it with the following polynomial that has exactly n-roots $\{z_i\}$

$$\prod_{i=1}^{n}(z - z_i) = (z - z_1)\cdots(z - z_n) = z^n + a_1\, z^{n-1} + \cdots + a_{n-1}\, z + z_n$$

The above polynomial determines uniquely (as the polynomial associated is monic) the $n-1$-coefficients $\{a_i\}_{i=1}^n$ of the above polynomial.

The totality of these coefficients, which is seen to be $\mathbb{C}^{n-1}$, parametrizing the monic polynomials of degree n. Inside $\mathbb{C}^{n-1}$, we have the discriminant locus Δ, which is the location where the monic polynomials associated have two or more roots colliding with each other. Thus, by associating the point $(z_1, \cdots, z_n) \in \widetilde{Conf_n}$ to the $n-1$ coefficients $(a_1, \cdots, a_{n-1})$, we have established an one to one correspondence

$$\widetilde{Conf_n} \longleftrightarrow \mathbb{C}^{n-1}\backslash\Delta$$

which implies that

$$B_n = \pi_1\left(\widetilde{Conf_n}, S\right) = \pi_1\left(\mathbb{C}^{n-1}\backslash\Delta, b\right) \tag{4.2.64}$$

By a theorem of Artin, the braid group B_n is generated by $n-1$ elements $\sigma_1, \cdots, \sigma_{n-1}$, which are subject the following relations:

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i-j| > 1; \tag{4.2.65}$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for } i = 1, \cdots, n-1. \tag{4.2.66}$$

Geometrically, the generator σ_i corresponds to the exchange of the i-th and $(i+1)$-th strand of the braid (see the below figure for illustration).

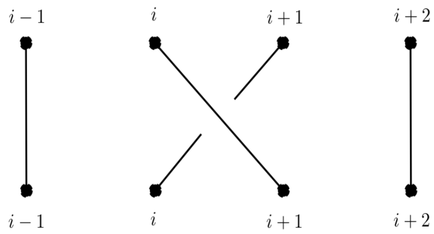

Figure 4.16: Geometric representation of the generator σ_i.

The nontrivial braid relation (4.2.66) can be illustrated as follows

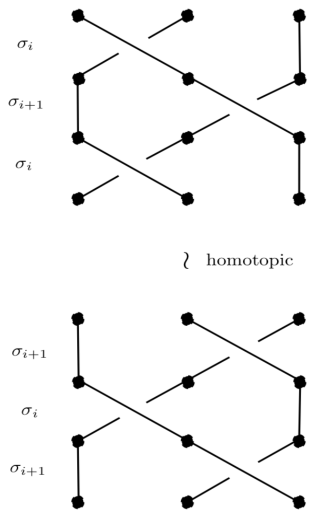

Figure 4.17: Illustration of $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

Consider the free group F_n generated by n letters $\{f_1, \cdots, f_n\}$, it is shown by E.Artin in [Art47] that the braid group acts on the free group F_n in the following way

$$f_j^{\sigma_i} = \begin{cases} f_j & j \neq i, i+1 \\ f_i f_{i+1} f_i^{-1} & j = i \\ f_i & j = i+1 \end{cases} \tag{4.2.67}$$

Next, we give a geometric realization of the above *Artin representation* (which is faithful).

Consider the universal family of the complements $\mathscr{C}$ on $\widetilde{\mathbb{D}^n}$, defined as

$$\mathscr{C} := \left\{(S, y) \in \widetilde{\mathbb{D}^n} \times \mathbb{C} : y \notin S\right\}$$

where the point S is viewed as a subset of n-points in $\mathbb{C}$. Then the projection to the first factor defines a locally trivial fibration $p : \mathscr{C} \to \widetilde{\mathbb{D}^n}$ with fiber over the point S being the complement $\mathbb{C}\backslash S$ to the subset S. A particular section of p is given by $S \mapsto (S, 2i)$ as $2i \notin \mathbb{D}$. Then it is easy to see that

$$\pi_1 \left(p^{-1}(S), 2i\right) = F_n$$

with n-generators given by the loops γ_i that starts at the base point $2i$, circling around the n-distinct points z_i contained in S.

Proposition 4.2.17. *The action of* $\pi_1 \left(\widetilde{\mathbb{D}^n}, S\right) = B_n$ *on* $\pi_1 \left(p^{-1}(S), 2i\right)$ *is given by the Artin representation (4.2.67).*

Proof. Note that the generator σ_i of the braid group B_n corresponds to the exchange of two points z_i and z_{i+1} in S, which can be realized by a rotation of z_i and z_{i+1} about their center. This causes the loops γ_i and γ_{i+1} to be dragged while the rest of the loops kept intact. The effect is that the loop γ_{i+1} is being dragged into γ_i, while the loop γ_i being dragged into $\gamma_i\gamma_{i+1}\gamma_i^{-1}$, which corresponds exactly to the relations given in (4.2.67). $\square$

With these preparation, we can study the the relation between the vanishing cycles and the braid group action. We want to see how the action of $\pi_1(\mathcal{B}^0, b_0)$ on the first homology group $H_1(\mathcal{C}_{\mathbf{u}}, \mathbb{Z})$ could be induced from the braid group action. The generator σ_i corresponds to the exchange (braiding) of the two roots of the polynomials

$$P_n(x, \mathbf{u}) = \left(\mathcal{W}_{A_{n-1}}(x, \mathbf{u})\right)^2 - \Lambda^{2n}$$
$$= \left(\mathcal{W}_{A_{n-1}}(x, \mathbf{u}) + \Lambda^n\right) \cdot \left(\mathcal{W}_{A_{n-1}}(x, \mathbf{u}) - \Lambda^n\right) = P_n^+ \cdot P_n^-$$

defining the curve $\mathcal{C}_{\mathbf{u}}$. We have pointed out before that any two roots of the polynomial $P_n^{\pm}$ can be exchanged by forming a suitable words in the generators σ_i. The roots of the polynomial splits into two categories, namely the roots of P_n^+ and P_n^- respectively, denoted respectively by $\{\nu_i^+\}_{i=1}^n$ and $\{\nu_i^-\}_{i=1}^n$.

Thus, the fundamental group $\pi_1(\mathcal{B}^0, b_0)$ should be generated by those elements $\sigma \in B_{2n}$ that respects the above splitting of the roots of the polynomial. And to each such element in the braid group, we can associate it to the corresponding vanishing cycles $\nu_\sigma = (\mathbf{g}, \mathbf{q}) = (g^i, q_i)$ in the homology basis $\{\alpha^i, \beta_i\}$.

Since we know that the monodromy associated to the vanishing cycle ν_σ is given by the Picard-Lefschtz formula, which in matrix form, is given by the formula (4.2.48) as

$$\mathcal{M}_{\nu_\sigma} = \begin{pmatrix} \mathbb{I} + \mathbf{q}^t \otimes \mathbf{g} & \mathbf{q}^t \otimes \mathbf{q} \\ -\mathbf{g}^t \otimes \mathbf{g} & \mathbb{I} - \mathbf{g}^t \otimes \mathbf{q} \end{pmatrix} = \begin{pmatrix} \delta_i^j + q_j g^i & q_i q_j \\ -g^i g^j & \delta_i^j - g^j q_i \end{pmatrix}$$

Or in more compact form, it reads as

$$\mathcal{M}_{\nu_\sigma} = \mathbb{I} + (\Omega \cdot \nu_\sigma^t) \otimes \nu_\sigma$$

Proposition 4.2.18. *(c.f.[Bil+96]) The subgroup of B_{2n} that respects the splitting of the roots $\{\nu_i^+\}_{i=1}^n \bigcup \{\nu_i^-\}_{i=1}^n$ is generated by the following set of $2n$-generators*

$$\left\{\sigma_1, \cdots, \sigma_{n-1}, \sigma_n^2, \sigma_{n+1}, \cdots, \sigma_{2n-1}, T = (\sigma_1\sigma_2 \cdots \sigma_{2n-1})^n\right\}$$

Before proving the above proposition, let us first illustrate it with the example of the simplest $SU(2)$-case.

In this case, the four roots split into two groups, namely $\{\nu_1^+, \nu_2^+\}$ and $\{\nu_1^-, \nu_2^-\}$. The element σ_1 corresponds to the exchanges of the two roots ν_1^+ and ν_2^+, while σ_2^2 acts as identity. Now σ_3 corresponds to the exchange of ν_1^- with ν_2^-, while the braiding T corresponds to the exchange of the two groups of the roots. These indeed generate all possible bradings among the two groups of roots, thus the proposition holds in this situation.

Generalizing a little bit, we can prove the proposition as follows

Proof. First, it is easy to see that the elements belong to the set $\{\sigma_1, \cdots, \sigma_{n-1}\}$ generates the braidings among the roots in the group $\{\nu_i^+\}_{i=1}^n$; while the elements in the group $\{\sigma_{n+1}, \cdots, \sigma_{2n-1}\}$ generate the braidings among the roots in the second group $\{\nu_i^-\}_{i=1}^n$. Clearly, the element σ_n^2 corresponds to the identity element. Finally, since the product $\sigma_1\sigma_2 \cdots \sigma_{2n-1}$ induces the cyclic permutation of the set (ordered as indicated) $\{\nu_1^+, \cdots, \nu_n^+, \nu_1^- \cdots, \nu_n^-\}$, consequently, the element T corresponds to the exchange of the two groups of the roots as a whole. This completes the proof of the proposition. □

Remark 4.2.17. *By the discussion in the remark 4.2.7, the above proposition implies that the brading T corresponds to the monodromy around ∞ in the moduli space.*

The vanishing cycles ν_{σ_i} associated to the generators σ_i should satisfy the following constraints which are imposed by the braid group relations (4.2.65) and (4.2.66). Indeed, the relation $\sigma_i\sigma_j = \sigma_j\sigma_i$ implies that

$$\left[\mathcal{M}_{\nu_{\sigma_i}}, \mathcal{M}_{\nu_{\sigma_j}}\right] = 0$$

which by proposition 4.2.9, translates into the following relation

$$\langle \nu_{\sigma_i}, \nu_{\sigma_j} \rangle = \nu_{\sigma_i} \cdot \Omega^t \cdot \nu_{\sigma_j}^t = 0 \qquad |i-j| > 1 \tag{4.2.68}$$

Similarly, the relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ becomes

$$\mathcal{M}_{\nu_{\sigma_i}} \cdot \mathcal{M}_{\nu_{\sigma_{i+1}}} \cdot \mathcal{M}_{\nu_{\sigma_i}} = \mathcal{M}_{\nu_{\sigma_{i+1}}} \cdot \mathcal{M}_{\nu_{\sigma_i}} \cdot \mathcal{M}_{\nu_{\sigma_{i+1}}}$$

which translates into the following relation

$$\langle \nu_{\sigma_i}, \nu_{\sigma_{i+1}} \rangle = \nu_{\sigma_i} \cdot \Omega^t \cdot \nu_{\sigma_{i+1}}^t = 1 \qquad i = 1, \cdots, n-1 \tag{4.2.69}$$

In terms of coordinates, the above equations (4.2.68) and (4.2.69) becomes

$$\begin{cases} \sum_k g_j^k\, q_{ik} - g_i^k\, q_{jk} = 0 & |i-j| > 1 \\ \sum_k g_{i+1}^k\, q_{ik} - g_i^k\, q_{i+1\,k} = 1 & i = 1, \cdots, n-1 \end{cases} \tag{4.2.70}$$

By the proposition 4.2.4 and the remark 4.2.16, the solutions to the above equations can be given in terms of the simple roots as follows

$$\begin{cases} \nu_{\sigma_i} = \nu_i^+ = (\alpha_i, (n+1)\,\alpha_i) & i = 1, \cdots, n \\ \nu_{\sigma_{i+n}} = \nu_i^- = (\alpha_i, n\,\alpha_i) & i = 1, \cdots, n \end{cases} \tag{4.2.71}$$

Remark 4.2.18. *In [Bil+96], another solution to the equations (4.2.68) and (4.2.69) is given by the following*

$$\begin{cases} \nu_{\sigma_{2i-1}} = (\mathbf{0}, \mathbf{e}_i - \mathbf{e}_{i-1}) & i = 1, \cdots, n-1 \\ \nu_{\sigma_{2n}} = (\mathbf{0}, -\mathbf{e}_{n-1}) \\ \nu_{\sigma_{2i}} = (-\mathbf{e}_i, \mathbf{0}) & i = 1, \cdots, n-1 \\ \nu_{\sigma_{2n}} = (\mathbf{e}_1 + \cdots + \mathbf{e}_{n-1}) \end{cases} \tag{4.2.72}$$

where $\mathbf{e}_i$ denotes an orthonormal basis in $\mathbb{R}^{n-1}$ (viewing $\mathbf{e}_0 = 0$). The merit of this form of solution is that the electric charges of the odd-numbered ν and the magnetic charges of the even numbered ν are given explicitly in terms of the roots and the fundamental weights of $SU(n)$. Actually, it can be shown that the two solutions differ by a duality transformation, i.e., a $Sp(2n, \mathbb{Z})$ transformation. Thus they can be viewed as equivalent solutions.

Zariski-Van Kampen Theorem and the Fundamental Group of the Complement

The fundamental group $\pi_1(\mathcal{B}^0, b_0) = \pi_1(\mathbb{C}^{n-1}\backslash\Delta_\Lambda, b_0)$ can be computed by using method initiated by Zariski and Van Kampen [Zar29][Van33]. Our exposition follows the paper [Cog11]. Note that by *Zariski-Lefschetz hyperplane section theorem*, the computation of the above fundamental group can be reduced in the complex two dimensional case as follows

$$\pi_1(\mathbb{C}^{n-1}\backslash\Delta_\Lambda) = \pi_1(H\backslash H \cap \Delta_\Lambda) \tag{4.2.73}$$

where $H \cong \mathbb{C}^2 \subset \mathbb{C}^{n-1}$ is a hyperplane, and $H \cap \Delta_\Lambda$ is the corresponding hyperplane section. Thus, in the following, we consider the hypersurface $S \subset \mathbb{C}^2$ defined by a polynomial $f(x, y) = 0$, and find the method to compute the fundamental group $\pi_1(\mathbb{C}^2\backslash S)$. To this end, take a point $P \in \mathbb{C}^2\backslash S$, and consider the projection from P, which defines a map $Pr : \mathbb{C}^2 \to \mathbb{C}^1$. The restrict of it on the curve S will be denoted by the same symbol Pr. Thus, let us consider the map

$$Pr : S \longrightarrow \mathbb{C} \tag{4.2.74}$$

The fiber over $z \in \mathbb{C}$ is the intersection points of the line l_z connecting between z and the graph of the curve C, which generically consist of $deg(f)$ points. The number of intersection points becomes less that $deg(f)$ over the discriminant locus $\Delta = \{q_1, \cdots, q_m\} \subset \mathbb{C}$.

Denote by $\mathcal{L} := L_1 \cup \cdots \cup L_m = Pr^{-1}(\Delta)$, i.e., the union of non-generic vertical lines.

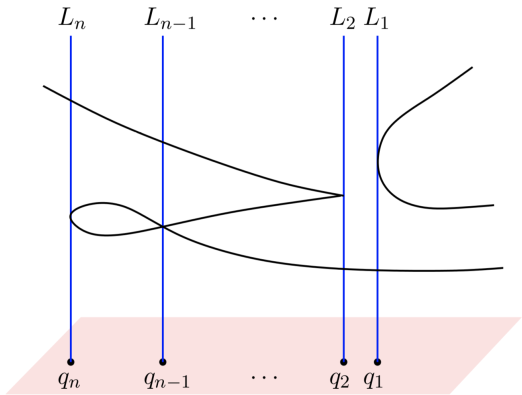

Figure 4.18: The picture is taken from [Cog11]

The fibration Pr is not "good" in the sense that the fibers come close to each other near the projection point P. In order to make a good locally trivial fibration out of Pr, we need to separate the directions based at the point P by replacing it with a copy of $\mathbb{C}P^1$ which parameterizes all directions issuing from P, that is, we need to consider the *blow up* $\mathcal{BL}_P(\mathbb{C}^2)$ of $\mathbb{C}^2$ at P. Making a coordinate transformation, we can assume that $P = (0, 0)$, the explicit construction of the blow up goes as follows:

$$\mathcal{BL}_P(\mathbb{C}^2) := \big\{((z, w); [u, v]) \in \mathbb{C}^2 \times \mathbb{C}P^1 | \, uw = vz\big\} \tag{4.2.75}$$

Denote by $\epsilon : \mathcal{BL}_P(\mathbb{C}^2) \to \mathbb{C}^2$ the projection map onto the first factor. Note that the fiber of ϵ over the point $P = (0, 0)$ is a copy of $\mathbb{C}P^1$, which we call it the exceptional fiber, and denote it by E, while the rest fiber is given by $((z, w); [z, w])$, i.e., the point (z, w) together with its "direction" specified by the element $[z, w] \in \mathbb{C}P^1$. Now consider the following composition map

$$\widetilde{Pr} = Pr \circ \epsilon : \mathcal{BL}_P(\mathbb{C}^2) \longrightarrow \mathbb{C} \tag{4.2.76}$$

Then it is easy to see that the fiber of $\widetilde{Pr}$ over a point $z \in \mathbb{C}$ is given by $\mathbb{C}$. It can be seen from the above construction that there is a birational isomporphism

$$\mathcal{BL}_P(\mathbb{C}^2)\backslash\{E\} \cong \mathbb{C}^2\backslash\{P\} \tag{4.2.77}$$

as well as the following identity

$$\widetilde{Pr}|_{\mathcal{BL}_P(\mathbb{C}^2)\backslash\{E\}} = Pr|_{\mathbb{C}^2\backslash\{P\}} \tag{4.2.78}$$

Now let us lift the curve S and the union of lines $\mathcal{L}$ from $\mathbb{C}^2$ to $\mathcal{BL}_P(\mathbb{C}P^2)$ through the blowing up map ϵ, namely

$$\widetilde{S} := \epsilon^{-1}(S) \qquad \widetilde{\mathcal{L}} := \epsilon^{-1}(\mathcal{L})$$

Note that we have that $\widetilde{S} \cong S$ as $P \notin S$. As the blowing up is birational, $\widetilde{Pr}$ is a proper submersion. Consequently, by *Ehresmann's fibration theorem*, we conclude that $\widetilde{Pr}$ is a locally trivial fibration with fiber being isomorphic to the projective line $\mathbb{C}$, which induces the following locally trivial fibration

$$\widetilde{Pr} : \mathcal{BL}_P(\mathbb{C}^2)\backslash(\widetilde{S} \cup \widetilde{\mathcal{L}}) \longrightarrow \mathbb{C}\backslash\Delta \tag{4.2.79}$$

with the generic fiber being $F := \mathbb{C}\backslash Z_d$, where Z_d is the union of $d = deg(f)$ distinct points. It can be shown (for example [Cog11]) that the blow up map ϵ will induce the following isomorphism on the fundamental group lever, namely

$$\epsilon_* : \pi_1\left(\mathcal{BL}_P(\mathbb{C}^2)\backslash(\widetilde{S} \cup \widetilde{\mathcal{L}})\right) \cong \pi_1(\mathbb{C}^2\backslash(S \cup \mathcal{L})) \tag{4.2.80}$$

The fundamental group of the fiber F is a free abelian group generated by d-meridians $\gamma_i,\ i = 1, \cdots, d$ with γ_i circling around the i-th distinct point, that is

$$\pi_1(F) = \langle \gamma_1, \cdots, \gamma_d \rangle \tag{4.2.81}$$

Apparently, the fundamental group of the base of the fibration $\mathbb{C}\backslash\Delta$ is a free abelian group generated the meridians $\alpha_i,\ i = 1, \cdots, m$ with the meridians α_i being associated with $q_i \in \Delta$, that is

$$\pi_1(\mathbb{C}\backslash\Delta) = \langle \alpha_1, \cdots, \alpha_m \rangle \tag{4.2.82}$$

Next we show that $\pi_1(\mathbb{C}^2\backslash S)$ can be computed from the action of $\pi_1(\mathbb{C}\backslash\Delta)$ on $\pi_1(F)$.

Let us begin by first eliminating the effect $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ in the isomorphism ϵ_* in (4.2.80). Note that the inclusion of $\widetilde{\mathcal{L}}$ in $\mathcal{BL}_P(\mathbb{C}^2)\backslash(\widetilde{S} \cup \widetilde{\mathcal{L}})$ induces the following surjection on the fundamental group level

$$i_* : \pi_1\left(\mathcal{BL}_P(\mathbb{C}^2)\backslash(\widetilde{S} \cup \widetilde{\mathcal{L}})\right) \twoheadrightarrow \pi_1\left(\mathcal{BL}_P(\mathbb{C}^2)\backslash\widetilde{S}\right) \tag{4.2.83}$$

We claim that the kernel of the above surjection is generated exactly by the meridians of $\widetilde{\mathcal{L}}$, which is the same as the meridians of $\mathbb{C}\backslash\Delta$. Indeed, suppose that γ represents a class $[\gamma] \in \pi_1(\mathcal{BL}_P(\mathbb{C}^2)\backslash\widetilde{S})$ that maps to zero under the above surjection. It follows that there exists a disc $\mathbb{D}$ such that $\gamma = \partial\,\mathbb{D}$. Then by putting the disk in general position if necessary, we assume that $\mathbb{D}$ intersects transversely with $\widetilde{\mathcal{L}}$ at the points $\{s_1, \cdots, s_m\}$.

Choosing small disks $\mathbb{D}_i \subset \mathbb{D}$ such that $\mathbb{D}_i \cap \widetilde{\mathcal{L}} = \{s_i\}$. Then by denoting $\alpha_i = \partial\,\mathbb{D}_i$, i.e., the meridian associated to $L_i \subset \widetilde{\mathcal{L}}$, it is easy to see that γ is homotopic (with orientations being properly chosen) to $\prod_{i=1}^{s} \alpha_i$, which is seen to be the meridian associated to $\widetilde{\mathcal{L}}$. The claim is thus proved, and we get the following

$$\begin{aligned} \pi_1(\mathbb{C}^2\backslash S) = \pi_1(\mathcal{BL}_P(\mathbb{C}^2)\backslash\widetilde{S}) &\cong \pi_1\left(\frac{\mathcal{BL}_P(\mathbb{C}^2)\backslash(\widetilde{S} \cup \widetilde{\mathcal{L}})}{ker(i_*)}\right) \\ &= \pi_1\left(\mathcal{BL}_P(\mathbb{C}^2)\backslash(\widetilde{S} \cup \widetilde{\mathcal{L}})\right) / \langle \alpha_1, \cdots, \alpha_m \rangle \end{aligned} \tag{4.2.84}$$

To continue, let us proof the following proposition

Proposition 4.2.19. *Given a locally trivial fibration $\pi : E \to B$ with fiber F and a section $s : B \to E$. Then we have that*

$$\pi_1(E) = \pi_1(F) \rtimes \pi_1(B) \tag{4.2.85}$$

where $\pi_1(B)$ acts on $\pi_1(F)$ by the monodromy representation (4.2.62).

Proof. The existence of the section s implies the splitting of the following exact sequence

$$1 \longrightarrow \pi_1(F) \xrightarrow{i_*} \pi_1(E) \underset{s_*}{\overset{\pi_*}{\rightleftarrows}} \pi_1(B) \longrightarrow 1$$

Thus, as a set we have that

$$\pi_1(E) = \{(\alpha, \beta) = s_*(\alpha)i_*(\beta) : \alpha \in \pi_1(B),\ \beta \in \pi_1(F)\}$$

while the product structure is given as follows

$$(\alpha_1, \beta_1) \cdot (\alpha_2, \beta_2) = s_*(\alpha_1)i_*(\beta_1)s_*(\alpha_2)i_*(\beta_2)$$
$$= s_*(\alpha_1)s_*(\alpha_2)s_*(\alpha_2)^{-1}i_*(\beta_1)s_*(\alpha_2)i_*(\beta_2)$$
$$= (\alpha_1\alpha_2,\ s_*(\alpha_2)^{-1}\beta_1 s_*(\alpha_2)\beta_2) = (\alpha_1\alpha_2,\ \beta_1^{\alpha_2}\,\beta_2)$$

Thus

$$\pi_1(E) = \pi_1(F) \rtimes \pi_1(B)$$

□

Note that the fibration $\widetilde{Pr}$ (see (4.2.79)) is endowed with a section given by

$$s : z \mapsto ((z, z); [z, z])$$

By applying the above proposition to $\widetilde{Pr}$ and using the isomorphism (4.2.84), we get

$$\pi_1(\mathbb{C}^2\backslash S) = (\pi_1(F) \rtimes \pi_1(\mathbb{C}\backslash\Delta))\,/\langle\alpha_1, \cdots, \alpha_m\rangle \tag{4.2.86}$$

Also note that when we trace along a generator α_i of $\pi_1(\mathbb{C}\backslash\Delta)$ based at the point $z_0 \in \mathbb{C}\backslash\Delta$, which causes the motion of the roots of $f(z_0, w)$. This induces a braid among the roots. Consequently, we get the following *braid monodromy map*

$$\theta : \pi_1(\mathbb{C}\backslash\Delta, z_0) \longrightarrow B_d \tag{4.2.87}$$

Thus $\pi_1(\mathbb{C}\backslash\Delta)$ acts on $\pi_1(F)$ through the braid monodromy map θ, which is given by Artin representation, i.e., the α_i action is given by the action of $\theta(\alpha_i) \in B_d$ on $\pi_1(F) = F_d$ through (4.2.67).

Now we can state the following *Zariski-Van Kampen theorem*

Proposition 4.2.20. *Given a curve $S \subset \mathbb{C}^2$ that gives rise to a d-sheeted cover of $\mathbb{C}$ which ramifies at the discriminant locus $\Delta \subset \mathbb{C}$ that consists of n points. Then the fundamental group $\pi_1(\mathbb{C}^2\backslash S)$ has the following finite presentation*

$$\pi_1(\mathbb{C}^2\backslash S) = \langle\gamma_1, \cdots, \gamma_d : \theta(\alpha_i)\,\gamma_j = \gamma_j,\ j = 1, \cdots, d\rangle \tag{4.2.88}$$

with the meridians γ_i and α_i constructed as before.

Proof. By (4.2.86) we have that

$$\pi_1(\mathbb{C}^2\backslash S) = (\pi_1(F) \rtimes_\theta \pi_1(\mathbb{C}\backslash\Delta))\,/\langle\alpha_1, \cdots, \alpha_m\rangle$$
$$= \langle\gamma_1, \cdots, \gamma_d, \alpha_1, \cdots, \alpha_m : \theta(\alpha_i)\,\gamma_j = \alpha_i^{-1}\,\gamma_j\,\alpha_i\rangle/\langle\alpha_1, \cdots, \alpha_m\rangle$$
$$= \langle\gamma_1, \cdots, \gamma_d : \theta(\alpha_i)\,\gamma_j = \gamma_j,\ j = 1, \cdots, d\rangle$$

□

Examples: Define $\widetilde{Conf_n}^0$ to be the configuration space of unordered n-distinct points $\{z_1, \cdots, z_n\}$ on $\mathbb{C}$ with center of mass $\frac{1}{n}\sum_{i=1}^{n} z_i$ being situated at the origin. Clearly $\widetilde{Conf_n}^0$ is homotopic to $\widetilde{Conf_n}$ with the homotopy given by

$$H(t)(z_1, \cdots, z_n) = \left(z_1 - \frac{t}{n}\sum_{i=1}^{n} z_i, \cdots, z_n - \frac{t}{n}\sum_{i=1}^{n} z_i\right) \quad 0 \le t \le 1$$

However, the associated polynomial for a point $(z_1, \cdots, z_n) \in \widetilde{Conf_n}^0$ is given by $\prod_{i=1}^{n}(z - z_i)$ with the coefficient of z being zero, which is exactly our A_{n-1} polynomial given before. For the "trivial" $n = 2$ case, we get that

$$\widetilde{Conf_2}^0 \cong \left\{z^2 + bz + c : \ b^2 - 4c \neq 0, b = 0\right\} \cong \mathbb{C}^*$$

Thus

$$B_2 = \pi_1\left(\widetilde{Conf_2}^0, \text{base point}\right) \cong \pi_1(\mathbb{C}^*, \text{base point}) = \mathbb{Z} = \langle\gamma\rangle$$

where γ represents the class of loop that circling around the origin based at the "base point", which under the monodromy representation corresponds to the exchange of two roots of the quadratic equation.

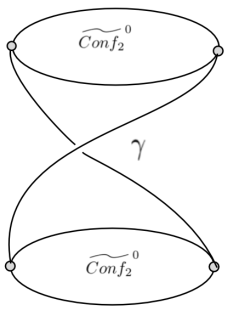

Figure 4.19: Fundamental group of $\widetilde{Conf_2}^0$

Next we compute the fundamental group of $\widetilde{Conf_n}^0$ in the case of $n = 3$. In this (less trivial) case, the configuration space can be represented as

$$\widetilde{Conf_3}^0 = \left\{z^3 + uz + v : \ v^2 - \frac{4}{27}u^3 \neq 0\right\} \cong \mathbb{C}^2 \backslash \left\{4u^3 - 27v^2 = 0\right\} \tag{4.2.89}$$

where the curve $C := \{(u, v) \in \mathbb{C}^2 : \ 4u^3 - 27v^2 = 0\}$ is singular at $(0, 0)$ with "cusp" type singularity.

Now we apply the Zariski- Van Kampen theorem to compute the fundamental group of $\widetilde{Conf_3}^0$. To this end, let us choose the projection

$$\pi : \mathbb{C}^2 \longrightarrow \mathbb{C} \qquad (u, v) \longmapsto u$$

Restricting it to the curve C, we get a double branch covering $\pi|_C$ of $\mathbb{C}$ ramified at $\Delta = \{u = 0\} \subset \mathbb{C}$. The fiber over u consists of the solution to the equation $4u^3 - 27v^2 = 0$ in v. The fundamental group $\pi_1(\mathbb{C}\backslash\Delta, \text{base point})$ is generated by a single loop α around the origin that start and end at the "base point". By tracing along the loop α, the induced braid monodromy $\theta(\alpha) \in B_2 = \langle\sigma_1\rangle$ can be computed as follows.

Let α be represented by a circle about the origin of radius δ, then replace u by $\delta\, e^{2\pi i\theta}\, 0 \leq \theta \leq 1$ in the above equation, we get the solutions

$$v = \pm\sqrt{\frac{4}{27}}\,\delta^{\frac{3}{2}}\, e^{3\pi i\theta} \quad 0 \leq \theta \leq 1.$$

Thus, we see that when θ goes continuously from 0 to 1, the two roots first change the sign when $\theta = \frac{1}{3}$, and change it again when θ hits $\frac{2}{3}$ and 1. As a consequence of it, the induced monodromy $\theta(\alpha)$ reads in this case as

$$\theta(\alpha) = \sigma_1^3 \tag{4.2.90}$$

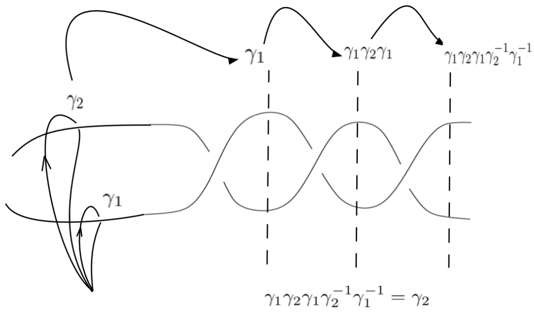

Figure 4.20: The monodromy $\theta(\alpha) = \sigma_1^3$ and the generator of the fundamental group

As the covering π is two-folded, we see that the fundamental group of the fiber being the free group F_2 generated by two meridians $\langle\gamma_1, \gamma_2\rangle$ associated two two separated branches of the curve. Then we have the following

Proposition 4.2.21. *The fundamental group of the complement of the cusp curve C in $\mathbb{C}^2$ has the following presentation*

$$\pi_1(\mathbb{C}^2\backslash C) = \langle\gamma_1, \gamma_2 : \gamma_1\gamma_2\gamma_1 = \gamma_2\gamma_1\gamma_2\rangle \tag{4.2.91}$$

Proof. By the Zariski-Van Kampen theorem, we get that

$$\pi_1(\mathbb{C}^2\backslash C) = \langle\gamma_1, \gamma_2 : \theta(\alpha)\,\gamma_i = \gamma_i,\, i = 1, 2\rangle$$

By the Artin presentation (4.2.67), we compute the monodromy action of $\theta(\alpha)$ as follows

$$\begin{aligned}
\theta(\alpha)\,\gamma_1 = \sigma_1^3(\gamma_1) &= \sigma_1^2(\gamma_1\gamma_2\gamma_1^{-1}) = \sigma_1\left(\sigma_1(\gamma_1)\sigma_1(\gamma_2)\sigma_1(\gamma_1)^{-1}\right) \\
&= \sigma_1\left(\gamma_1\gamma_2\gamma_1^{-1}\gamma_1\gamma_1\gamma_2^{-1}\gamma_1^{-1}\right) = \sigma_1\left(\gamma_1\gamma_2\gamma_1\gamma_2^{-1}\gamma_1^{-1}\right) \\
&= \sigma_1(\gamma_1)\sigma_1(\gamma_2)\sigma_1(\gamma_1)\sigma_1(\gamma_2)^{-1}\sigma_1(\gamma_1)^{-1} \\
&= \underline{\gamma_1\gamma_2\gamma_1^{-1}}\gamma_1\underline{\gamma_1\gamma_2\gamma_1^{-1}}\gamma_1^{-1}\underline{\gamma_1\gamma_2^{-1}\gamma_1^{-1}} = \gamma_1\gamma_2\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}\gamma_1^{-1}
\end{aligned}$$

Similarly, we have that

$$\theta(\alpha)\,\gamma_2 = \sigma_1^3(\gamma_2) = \sigma_1^2\,(\gamma_1) = \sigma_1(\gamma_1\gamma_2\gamma_1^{-1}) = \sigma_1(\gamma_1)\sigma_1(\gamma_2)\sigma_1(\gamma_1)^{-1}$$
$$= \underline{\gamma_1\gamma_2\gamma_1^{-1}}\gamma_1\underline{\gamma_1\gamma_2^{-1}\gamma_1^{-1}} = \gamma_1\gamma_2\gamma_1\gamma_2^{-1}\gamma_1^{-1}$$

Consequently, the two relations $\theta(\alpha)\,\gamma_1 = \gamma_1$ and $\theta(\alpha)\,\gamma_2 = \gamma_2$ becomes the following

$$\gamma_1\gamma_2\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}\gamma_1^{-1} = \gamma_1 \text{ and } \gamma_1\gamma_2\gamma_1\gamma_2^{-1}\gamma_1^{-1} = \gamma_2$$

The second relation simplifies immediately into $\gamma_1\gamma_2\gamma_1 = \gamma_2\gamma_1\gamma_2$, while the first is easily seen to be equivalent to $\gamma_1\gamma_2\gamma_1\gamma_2 = \gamma_1\underline{\gamma_1\gamma_2\gamma_1}$, which, by using the braid relation (4.2.66) to the underlined part, simplifies further into $\gamma_1\gamma_2\gamma_1\gamma_2 = \gamma_1\gamma_2\gamma_1\gamma_2$, which is trivial. This completes the proof of the proposition. □

Remark 4.2.19. *A faithful representation of B_3 can be realized by letting the two generators γ_1 and γ_2 acting on $\mathbb{C}^2$ through the following $SL(2,\mathbb{Z})$ matrices as follows*

$$\mathcal{M}_{\gamma_1} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \quad \mathcal{M}_{\gamma_2} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

The geometry of the cusp curve can be studied by surrounding the cusp curve C by a 3-sphere $S^3_R := \{|u|^2 + |v|^2 = R^2\} \subset \mathbb{C}^2$ with varying radius R centered at the origin. Considering the homeomorphism given by

$$u \mapsto 2u^{\frac{3}{2}} \qquad v \mapsto 3\sqrt{3}v,$$

we see that the three sphere is diffeomorphic to $\{4|u|^3 + 27|v|^2 = R^2\}$, which intersects the cusp curve $C : \{(u,v) :\ 4u^3 = 27v^2\}$ at the locus

$$\left\{ 4|u|^3 = 27|v|^2 = \frac{1}{2}R^2 \right\}$$

Writing u as $u = |u|\,e^{i\theta_u}$, and v as $v = |v|\,e^{i\theta_v}$, we see that the above locus can be described in the following way

$$|u|^3 = \frac{R^2}{8}; \quad |v|^2 = \frac{R^2}{54} \qquad 3\theta_u \equiv 2\theta_v \mod 2\pi$$

which describes a curve that wraps the a torus in one direction (corresponding to the angle θ_u) three times, while wrapping in another direction (corresponding to the angle θ_v) two times. That is, we get a (2,3)-torus knot, which is of course the simplest non-trivial knot—the *trefoil knot*.

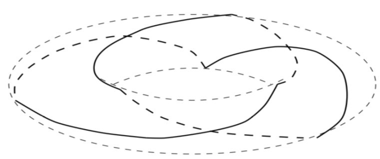

Figure 4.21: Trefoil knot as torus knot, the picture is taken from [Tay01]

Denote by U the complement of the $\mathbb{C}^2$ to the three sphere S^3_R, we have that

$$\mathbb{C}^2\backslash C \approx \left(\mathbb{C}^2 \cap U\right) \setminus (C \cap U) = S^3\backslash\{\text{trefoil knot}\}$$

where "≈" denotes homeomorphism. Thus, by above proposition 4.2.21, we get the knot group for trefoil knot as follows

$$\pi_1\left(S^3\backslash\{\text{trefoil knot}\}\right) = \pi_1(\mathbb{C}^2\backslash C) = \langle \gamma_1, \gamma_2 : \gamma_1\gamma_2\gamma_1 = \gamma_2\gamma_1\gamma_2 \rangle \tag{4.2.92}$$

Note that in the computation in the proof of the proposition above, we have used the projection to the u coordinate, which gives a two-fold branched covering. Similarly, by projecting to the v coordinate, we will get a three-fold covering. Using this cover to do the computation should give us the same result. Let us now verify this.

Consider the projection

$$\pi : \mathbb{C}^2 \longrightarrow \mathbb{C} \quad (u, v) \longmapsto v$$

Restricting it to the curve C, we get a three-fold branch covering $\pi|_S$ of the complex plane $\mathbb{C}$ ramified at $\Delta = \{v = 0\} \subset \mathbb{C}$. The fiber over v consists of the solution to the equation $4u^3 - 27v^2 = 0$ in u. The fundamental group $\pi_1(\mathbb{C}\backslash\Delta, \text{base point})$ is generated by a single loop α around the origin that start and end at the "base point". By tracing along the loop α, the induced braid monodromy $\theta(\alpha) \in B_3 = \langle \sigma_1, \sigma_2 \rangle$ can be computed as follows.

Let α be represented by a circle about the origin of radius δ, then replace v by $\delta\, e^{2\pi i\theta}\ 0 \le \theta \le 1$ in the above equation, we get the solutions

$$u_k = \sqrt[3]{\frac{27}{4}}\,\delta^{\frac{2}{3}}\, e^{\frac{4\pi i\theta}{3}}\omega^{k-1} = \sqrt[3]{\frac{27}{4}}\,\delta^{\frac{2}{3}}\, e^{\frac{2\pi i(2\theta+k-1)}{3}} \qquad 0 \le \theta \le 1, \quad k = 1, 2, 3.$$

from which we see that when θ first hit $\theta = \frac{1}{2}$, the three roots u_k undergo a cyclically permutation represented by the brading $\sigma_2\sigma_1$, and the same happens when θ further hit $\theta = 1$. Consequently the braid monodromy in this case reads as

$$\theta(\alpha) = \sigma_2\sigma_1\sigma_2\sigma_1 = (\sigma_2\sigma_1)^2 = \sigma_2\sigma_2\sigma_1\sigma_2 = \sigma_2^2\sigma_1\sigma_2$$

Thus, by Zariski-Van Kampen theorem, we get that

$$\pi_1(\mathbb{C}^2\backslash C) = \pi_1(S^3\backslash\{\text{trefoil knot}\}) = \langle \gamma_1, \gamma_2, \gamma_3 : \theta(\alpha)\,\gamma_i = \gamma_i,\ i = 1, 2, 3 \rangle$$

The results of the computations read as follows

$$\begin{cases} \theta(\alpha)(\gamma_1) = \gamma_1^{-1}\gamma_2^{-1}\gamma_3^{-1}\gamma_2\gamma_3\gamma_2\gamma_1 \\ \theta(\alpha)(\gamma_2) = \gamma_1^{-1}\gamma_2^{-1}\gamma_3\gamma_2\gamma_1 \qquad \theta(\alpha)(\gamma_3) = \gamma_1 \end{cases}$$

From this we see that the relations $\theta(\alpha)(\gamma_i) = \gamma_i,\ i = 1, 2, 3$ specify into the following

$$\begin{cases} \gamma_3\gamma_2\gamma_1 = \gamma_2\gamma_1\gamma_2 \\ \gamma_2\gamma_3\gamma_2 = \gamma_3\gamma_2\gamma_1 \qquad \gamma_1 = \gamma_3 \end{cases}$$

Thus only two among the γ_i's are independent. By eliminating γ_3, for example, we get the relation between γ_1 and γ_2 as: $\gamma_1\gamma_2\gamma_1 = \gamma_2\gamma_1\gamma_2$, exactly the one given in (4.2.92), which gives another proof of the proposition 4.2.21.

Remark 4.2.20. *The equivalent of the two projections in computing the fundamental group lies in the fact that as a torus knot, the trefoil knot can be either represented as a* $(2,3)$*-knot or as* $(3,2)$*-knot, which corresponds to projecting to different projections.*

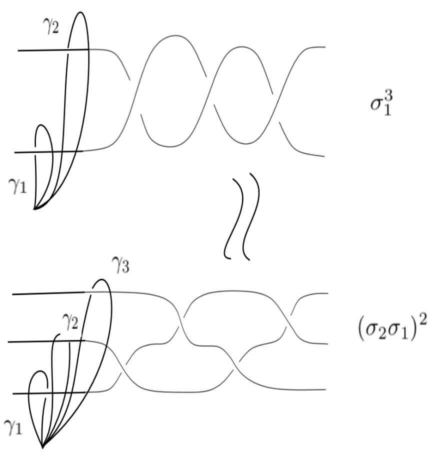

Figure 4.22: The trefoil knot as $(2,3)$-knot and $(3,2)$-knot.

4.2.4 Quantum Monodromy: $SU(3)$ case

In this subsection, we study the monodromies associated to the vanishing cycles in $SU(3)$ case. As the moduli space in this case is given by $\mathcal{B} \cong \mathbb{C}^2$, and the (quantum) discriminant locus Δ_Λ is of codimension one. The situation is much more complicated than the $SU(2)$ case, and more involved tools (as had been reviewed substantially in the last subsection) are needed. Let us begin by computing

$$\pi_1(\mathbb{C}^2 \backslash \Delta_\Lambda, \text{ base point}) = \pi_1(\mathcal{B}^0) = \pi_1(\mathbb{C}^{n-1} \backslash \Delta_\Lambda)$$

for $n = 3$, *i.e.*, $SU(3)$ case. That is, we need to compute the fundamental group

$$\pi_1\left(\mathbb{C}^2 \backslash \{4u^3 = 27(v \pm \Lambda^3)^2\}, \text{ base point}\right)$$

The discriminant curve is given in this case by

$$\Delta_\Lambda := \Delta_\Lambda^+ \cup \Delta_\Lambda^- = \left\{(u,v) \in \mathbb{C}^2 : \, 4u^3 = 27(v \pm \Lambda^3)^2\right\}$$

where $\Delta_\Lambda^\pm$ is defined by

$$\Delta_\Lambda^\pm := \left\{(u,v) \in \mathbb{C}^2 : \, 4u^3 = 27(v \mp \Lambda^3)^2\right\}$$

This tells us that the curve Δ_Λ is composed of two cusp curves $\Delta_\Lambda^\pm$ linked together. Or equivalently, by intersecting the curve with the 3-sphere and by employing the isomorphism (4.2.92) established in last subsection, we see that $\mathbb{C}^2 \backslash \Delta_\Lambda$ is homeomorphic to $S^3 \backslash \{\mathscr{L}\}$, where $\mathscr{L}$ is the link made by tangling the two trefoil knots together. The two trefoil knots $K_\pm$ are given by the intersection $S^3 \cap \Delta_\Lambda^\pm$ and

$$\mathscr{L} = K_+ \cup K_- = \left(S^3 \cap \Delta_\Lambda^+\right) \cup \left(S^3 \cap \Delta_\Lambda^-\right) \tag{4.2.93}$$

Thus

$$\pi_1(\mathbb{C}^2 \backslash \Delta_\Lambda) = \pi_1(S^3 \backslash \mathscr{L}) = \pi_1(S^3 \backslash K_+ \cup K_-) \tag{4.2.94}$$

To determine how the knots $K_\pm$ get linked together, we observe that since the equations defining the two curves $\Delta_\Lambda^\pm$ differ from the standard cusp curves $4u^3 = 27v^2$ by a shift in the v-direction by the amount $\pm\Lambda^3$. Thus by shifting the standard trefoil knots defined by intersection the curve $4u^3 = 27v^2$ with a three sphere S^3 in the v-direction by the amount $\pm\Lambda^3$, we will get the knots $K_\pm$ respectively. Thus, the knot K_+ can be obtained by shifting the knot K_- in the direction corresponding to $-v$ by $2\Lambda^3$ unit.

As a consequence of this, and in viewing of the computation laid down at the end of the last subsection, we see that the fundamental group $\pi_1(\mathbb{C}^2 \backslash \Delta_\Lambda)$ is generated by six generators: $\left\{\gamma_i^\pm\right\}_{i=1}^3$, where $\left\{\gamma_i^+\right\}_{i=1}^3$ are the three generators corresponding to the knot K_+, while the generators $\left\{\gamma_i^-\right\}_{i=1}^3$ are that for the knot K_-. Each set of three generators satisfy the relations computed by using Zariski-Van Kampen theorem.

The three generators, in terms of the Trefoil knot, can be illustrated in Figure 4.23 taken from [AF95] (Fig.2 there). When in the quantum situation, the trefoil knot in figure 4.23 "splits" into two, and the three generators become six generators which are illustrated in Figure 4.24 taken again from [AF95] (Fig.3 there).

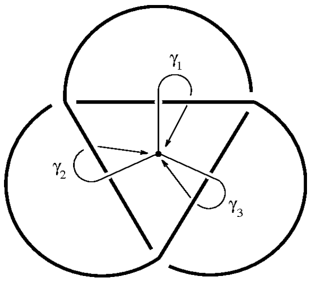

Figure 4.23: Three generators of the fundamental group being illustrated, figure taken from [AD95]

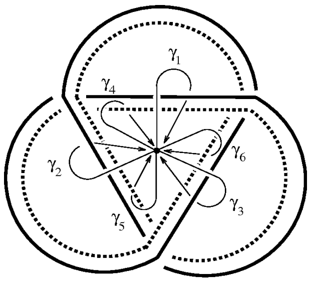

Figure 4.24: Six generators of the fundamental group illustrated, figure taken from [AD95]

Previously, we have displaced that the six generators are not independent homotopically, there are some relations among them. For example (proposition 4.2.21), as homotopic classes, we have that

$$[\gamma_1^\pm] = [\gamma_3^\pm] \qquad [\gamma_1^\pm][\gamma_2^\pm][\gamma_1^\pm] = [\gamma_2^\pm][\gamma_1^\pm][\gamma_2^\pm]$$

or simply, we write directly as follows (with homotopic classes implicitly understood)

$$\gamma_1^\pm = \gamma_3^\pm \qquad \gamma_1^\pm\gamma_2^\pm\gamma_1^\pm = \gamma_2^\pm\gamma_1^\pm\gamma_2^\pm \tag{4.2.95}$$

For example, the relation $\gamma_1 = \gamma_3$ origins from the geometric fact that as you slide the loop γ_1 in figure 4.23 along the Trefoil knot to becomes the loop γ_3, the loop γ_1 will come cross the loop γ_2 during the process, thus geometrically, we have that $\gamma_1 \sim \gamma_2 \cdot \gamma_3 \cdot \gamma_2^{-1}$, thus as homotopical classes, we have the identity $\gamma_1 \equiv \gamma_3$ in the fundamental group $\pi_1\left(\mathbb{C}^2\backslash\Delta_\Lambda\right)$. Now consider the monodromy representation (4.2.62) in this case:

$$\rho : \pi_1(\mathcal{B}\backslash\Delta_\Lambda) \longrightarrow Sp(4,\mathbb{Z})$$

For an element $\gamma \in \pi_1(\mathcal{B}\backslash\Delta_\Lambda)$, denote the matrix that represents the monodromy associated to $\rho(\gamma)$ by $\mathcal{M}_\gamma$. Then since ρ is a group homeomorphism, they should respect the relations (4.2.95), namely

$$\mathcal{M}_{\gamma_1^\pm} = \mathcal{M}_{\gamma_2^\pm} \cdot \mathcal{M}_{\gamma_3^\pm} \cdot \mathcal{M}_{\gamma_2^\pm}^{-1} \quad \text{plus cyclic permutations of } \{1,2,3.\} \tag{4.2.96}$$

$$\mathcal{M}_{\gamma_1^\pm} \cdot \mathcal{M}_{\gamma_2^\pm} \cdot \mathcal{M}_{\gamma_1^\pm} = \mathcal{M}_{\gamma_2^\pm} \cdot \mathcal{M}_{\gamma_1^\pm} \cdot \mathcal{M}_{\gamma_2^\pm} \tag{4.2.97}$$

Detailed Analysis of the Geometry of Moduli Space

First, the two curves intersect with each other at the following three points, we call them $\mathbb{Z}_2$-points in accordance with the convention in physics literature

$$\Delta_\Lambda^+ \cap \Delta_\Lambda^- = \left\{ \left(\sqrt[3]{\frac{27}{4}}\Lambda^2, 0 \right), \left(\omega\sqrt[3]{\frac{27}{4}}\Lambda^2, 0 \right), \left(\omega^2 \sqrt[3]{\frac{27}{4}}\Lambda^2, 0 \right) \right\} \tag{4.2.98}$$

where $\omega := e^{\frac{2\pi i}{3}}$ is the primitive cubic root of unit one. We denote the above three $\mathbb{Z}_2$-points by p_1, p_2, p_3 respectively, that is

$$p_k = \left(\omega^{k-1}\sqrt[3]{\frac{27}{4}}\Lambda^2, 0 \right) \qquad k = 1, 2, 3. \tag{4.2.99}$$

Denote by $\delta_\Lambda^\pm(u, v) = 4u^3 - 27(v \mp \Lambda^3)^2$ the discriminant polynomial. Then the discriminant locus $\Delta_{\mathbf{u}}\Delta_\Lambda$ (where we denote by $\mathbf{u} := (u, v)$) of the discriminant curve Δ_Λ itself is given by the solutions to the following set of equations

$$\delta_\Lambda^\pm = \frac{\partial \delta_\Lambda^\pm}{\partial u} = \frac{\partial \delta_\Lambda^\pm}{\partial v} = 0 \iff \begin{cases} 4u^3 = 27(v \pm \Lambda^3)^2 \\ 12u^2 = 0 \\ -54(v \pm \Lambda^3) = 0 \end{cases}$$

The solutions are given by the two points set

$$\Delta_{\mathbf{u}}\Delta_\Lambda := \{q_+, q_-\} = \big\{(0, \Lambda^3), (0, -\Lambda^3)\big\} \tag{4.2.100}$$

which we call them $\mathbb{Z}_3$-points, again in accordance to the conventions in physics literature. Note that $q_\pm$ corresponds to the cusp singularities of the two curves $\Delta_\pm$ respectively. Next, we find the roots of the polynomial

$$P_3(x, \mathbf{u}) = (x^3 - ux - v)^2 - \Lambda^6$$
$$= (x^3 - ux - v + \Lambda^3)(x^3 - ux - v - \Lambda^3) = P_3^+ \cdot P_3^-$$

which defines the Seiberg-Witten curve in the $SU(3)$ case.

Recall that for a general cubic equation $t^3 + pt + q = 0$, its solutions are given by the Cardano's formula as follows

$$t_k = \omega^{k-1}\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \omega^{2(k-1)}\sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \quad k = 1, 2, 3.$$

We apply it to the case of $P_3(x, \mathbf{u}) = 0$. To this end, let us denote by $e_k^\pm,\ k = 1, 2, 3.$ the three roots of the polynomials $P_3^\pm$ respectively. Also recall that the discriminant of the polynomials $P_3^\pm$ is given by $\delta_\pm = 4u^3 - 27(v \mp \Lambda^3)^2$ respectively. Then by Cardano's formula above, we have the following explicit expressions of these roots:

$$e_k^\pm = \omega^{k-1}\sqrt[3]{\frac{v \mp \Lambda^3}{2} + \frac{\sqrt{-\delta_\pm}}{6\sqrt{3}}} + \omega^{2(k-1)}\sqrt[3]{\frac{v \mp \Lambda^3}{2} - \frac{\sqrt{-\delta_\pm}}{6\sqrt{3}}} \tag{4.2.101}$$

We see that when the moduli $\mathbf{u} = (u, v)$ lies in the locus Δ_Λ, then roots degenerate. For example, when $\delta_+ = 0$, we see easily that $e_2^+ = e_3^+$, and at the node point, i.e., $\mathbb{Z}_3$-point where $v = \Lambda^3$, all three roots $e_k^+,\ k = 1, 2, 3$ degenerate. Further, at the intersection points p_k, i.e., $\mathbb{Z}_2$-point where $v = 0$, again we have the degenerate situation: $e_2^+ = e_3^+$.

Since Δ_Λ is a complex surface sitting inside $\mathbb{C}^2$, in order to study the monodromy around the various components (branches) of it, we cut the surface by a family of real planes defined by

$$H_t := \{Im\, v = t,\, t \in \mathbb{R}\} \subset \mathbb{C}^2 \tag{4.2.102}$$

which will give us a family of real curves $\mathscr{L}_t := H_t \cap \Delta_\Lambda,\ t \in \mathbb{C}^2$. Thus, we reduce the study of monodromies associated to various branches of Δ_Λ into the studying of the monodromies associated to various branches of $\mathcal{L}_t$ for different t. Near the two nodes $q_\pm$, we cut the surface by the three sphere S_3, which give us the trefoil knots. We can study the associated monodromies by using the braid monodromy as before. We first prove that it is sufficient to study the situation in the case of $t = 0$.

Proposition 4.2.22. *The monodromy group associated to the locus Δ_Λ equals that associated to the locus $\mathscr{L}_0 = H_0 \cap \Delta_\Lambda$, i.e., we have that*

$$\pi_1(\Delta_\Lambda) = \pi_1(H_0 \cap \Delta_\Lambda) = \pi_1(\mathscr{L}_0) \tag{4.2.103}$$

Proof. We note that the singularities of the surfaces, that is: the $\mathbb{Z}_3$-points $\{q_\pm\}$ lies on the hyperplane section $\mathscr{L}_0$ by (4.2.100), thus, the complement set $\Delta_\Lambda \backslash \mathscr{L}_0$ is smooth. Then by applying the *Lefschetz hyperplane theorem* (for example, see [Mil63]), we get the desired isomorphism. □

Consequently, we focus on the hyperplane section $\mathscr{L}_0 = H_0 \cap \Delta_\Lambda$ in the following discussion. We simply write $\mathscr{L}_0$ and H_0 as $\mathscr{L}$ and H respectively. We write $u = x + iy$, $v = z + iw$, where $(x, y, z, w) \in \mathbb{C}^2 \cong \mathbb{R}^4$. Then $\Delta_\Lambda \cap H$ is described by the equation:

$$4(x + iy)^3 = 27(z \pm \Lambda^3)^2$$

Equating the real and imaginary part, we get the equations describing $\mathscr{L}$:

$$\mathscr{L} : \begin{cases} 4(x^3 - 3xy^2) = 27(z \pm \Lambda^3)^2 \\ 3x^2 y = y^3 \end{cases} \tag{4.2.104}$$

which "splits" into three different cases:

$$\mathscr{L}_1 : \begin{cases} 4x^3 = 27(z \pm \Lambda^3)^2 \\ y \equiv 0 \end{cases} \qquad \mathscr{L}_2 : \begin{cases} -32x^3 = 27(z \pm \Lambda^3)^2 \\ y \equiv \sqrt{3}x \end{cases}$$

$$\mathscr{L}_3 : \begin{cases} -32x^3 = 27(z \pm \Lambda^3)^2 \\ y \equiv -\sqrt{3}x \end{cases} \tag{4.2.105}$$

which further "split" into the following three pairs of lines:

$$\mathscr{L}_1 = \Sigma_+^1 \cup \Sigma_-^1 \quad \mathscr{L}_2 = \Sigma_+^2 \cup \Sigma_-^2 \quad \mathscr{L}_3 = \Sigma_+^3 \cup \Sigma_-^3 \tag{4.2.106}$$

where $\Sigma_\pm^k,\ k = 1, 2, 3$ is described by the equations in (4.2.105) with $\pm$ signs respectively.

Apparently, the pair of lines $\mathscr{L}_1$ lie on the hyperplane $H_1 := \{y \equiv 0\}$, while the pairs $\mathscr{L}_{2,3}$ lie on the hypreplanes $H_{2,3} := \{y \equiv \pm\sqrt{3}\}$ respectively with 2 corresponding to $+$ and 3 to $-$. The three $\mathbb{Z}_2$-points $p_k,\ k = 1, 2, 3,$ (see equation:(4.2.99)) are given by the following

$$p_k = \Sigma_+^k \cap \Sigma_-^k, \quad k = 1, 2, 3. \tag{4.2.107}$$

All three points lie on the plane $\{v = 0\}$, and is rotated into each other under the $\mathbb{Z}_3$ permutations. Besides, the three "positive" lines Σ_+^k intersect at the $\mathbb{Z}_3$-point $q_- = (0, -\Lambda^3)$, while the three "negative" lines Σ_-^k intersect at the other $\mathbb{Z}_3$-point $q_+ = (0, \Lambda^3)$. That is

$$\Sigma_\pm^1 \cap \Sigma_\pm^2 \cap \Sigma_\pm^3 = q_\mp = (0, \mp\Lambda^3) \tag{4.2.108}$$

The situation described above can be depicted in the following picture (Figure 4.25), this picture is taken from [KTL95] (Fig.4 there).

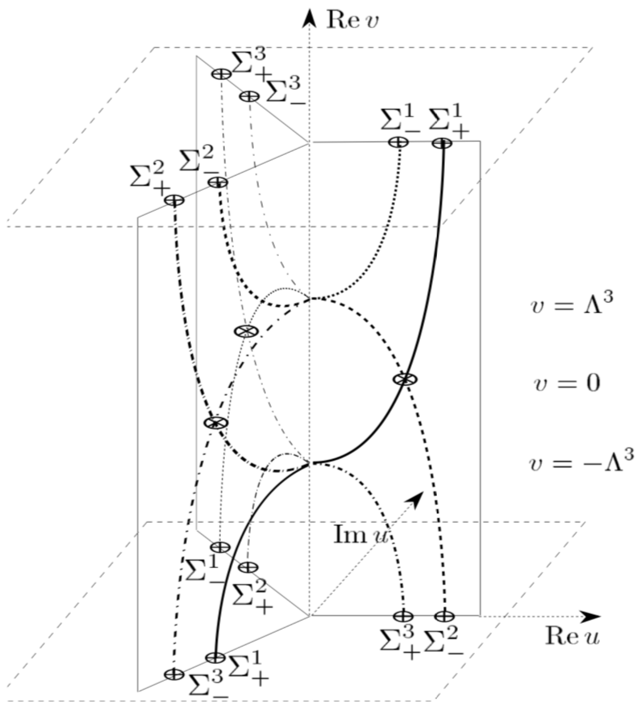

Figure 4.25: Geometry of moduli space for $SU(3)$ when $Im\, v = 0$, figure taken from [KTL95]

Now we can study the monodromies around each six singular lines $\Sigma_\pm^i$ for $i = 1, 2, 3$. Thus we fix a base point (see figure 4.25) u_0 that belongs the the plane $H_{\text{cut}} := \{z = Re(v) \equiv constant > \Lambda^3\}$, we then consider the six loops based at u_0 which encircling the six singular points cut out by the plane above. Denote by $\gamma_i^\pm$ the loops circling around the singular lines $\Sigma_\pm^i$ for $i = 1, 2, 3$. In other words, the loops $\gamma_i^\pm$ circles around the points $P_i^\pm$ on the plane H_{cut} if $P_i^\pm$ denote the intersection points of the base plane with the six singular lines, that is: $P_i^\pm := \Sigma_\pm^i \cap H_{\text{cut}}$.Denote the monodromy matrices associated to the loops $\gamma_i^\pm$ by $\mathcal{M}_i^\pm$ for $i = 1, 2, 3$. Roughly speaking, when looping around the points $P_i^\pm$, certain roots among $\left\{e_k^\pm\right\}_{i=1}^3$ coincide, which causes certain one cycles become the vanishing cycles. And this induces the corresponding monodromies given by Picard-Lefschtz formula.

Remark 4.2.21. *We notice that as Λ approaches to zero, i.e., when the quantum moduli space "degenerate" into the classical moduli space. In particularly, the roots e_i^+ coincides with e_i^-. Consequently, by denoting r_i the loops circling around the pairs of the points $P_i^\pm$ for $i = 1, 2, 3$, we see that in this degenerating situation, the loops r_i vanishes, which produce exactly the classical monodromies $\mathcal{M}_{r_i}^c$ (see (4.2.15)) given before. Besides, by proper choice of the orientations of loops, we see that as homotopical classes, we have that $r_i = \gamma_i^+ \cdot \gamma_i^-$. In terms of the monodromies associated, this infers that*

$$\mathcal{M}_{r_i}^c = \mathcal{M}_{\gamma_i^+} \cdot \mathcal{M}_{\gamma_i^-}, \quad i = 1, 2, 3. \tag{4.2.109}$$

We note that the classical monodromy $\mathcal{M}_{r_i}^c$ corresponds to the braid monodromy B_{γ_i} associated to the braid generator γ_i for the $\pi_1(\mathbb{C}^2 \backslash \{\text{trefoil knot}\})$, thus they should obey the braid relations given in (4.2.97), namely

$$\mathcal{M}_{r_1}^c = \mathcal{M}_{r_2}^c \cdot \mathcal{M}_{r_3}^c \cdot \mathcal{M}_{r_2}^{c-1} \tag{4.2.110}$$

as well as the following

$$\mathcal{M}_{r_1}^c \cdot \mathcal{M}_{r_2}^c \cdot \mathcal{M}_{r_1}^c = \mathcal{M}_{r_2}^c \cdot \mathcal{M}_{r_1}^c \cdot \mathcal{M}_{r_2}^c \tag{4.2.111}$$

Remark 4.2.22. *Notice that since $\delta_\pm = 4u^3 - 27(v \mp \Lambda^3)^2$, we see that $\Lambda \to 0$ amounts to the limit $v \to \infty$. Thus, geometrically, choosing the cut plane H_{cut} far $Re(v) \to \infty$, then the monodromies computed with reference to the plane H_{cut} are reduced to the classical monodromies.*

Remark 4.2.23. *Intuitively, each pair $\mathscr{L}_i = \Sigma_+^i \cup \Sigma_-^i$ of the singular lines correspond to a situation similar to that $SU(2)$ case discussed above. For example, r_i, in this interpretation, becomes the vanishing cycle that induces the classical monodromy at "∞" of the $SU(2)$-quantum moduli space $\mathbb{C}$ (since we know that for $SU(2)$ case, $\mathcal{M}_\infty = \mathcal{M}_{+\Lambda^2} \cdot \mathcal{M}_{-\Lambda^2}$, c.f., see formula (4.2.20)). And the vanishing cycles associated with the two singular lines $\Sigma_\pm^i$ are interpreted as the two vanishing cycles for this copy of $SU(2)$ model. Thus the three pairs of singular lines corresponds to the three ways of the embeddings of the Lie groups $SU(2) \hookrightarrow SU(3)$ (c.f., [KTL95]), which further induces the embedding of the moduli spaces $\mathcal{B}_{SU(2)} \hookrightarrow \mathcal{B}_{SU(3)}$. Roughly speaking, the moduli space $\mathcal{B}_{SU(2)}$ "cuts out" a subspace $\mathbb{C}$ inside $\mathcal{B}_{SU(3)}$, with the two singular points $\{-\Lambda^2, +\Lambda^2\}$ corresponding to one pair of intersections of this complex plane with the discriminant locus Δ_Λ for $SU(3)$ case.*

We see that the discriminant δ_Λ, when restricted to the tree pair of lines $\mathscr{L}_i$, $i = 1, 2, 3$, becomes

$$\begin{aligned} \delta_\pm &= 4u^3 - 27(v \mp \Lambda^3)^2 = 4(x + iy)^3 - 27(z \mp \Lambda^3)^2 \\ &= 4(x^3 - 3xy^2) - 27(z \mp \Lambda^3)^2 + 4i\,(3x^2y - y^3) = 4(x^3 - 3xy^2) - 27(z \mp \Lambda^3)^2 \end{aligned}$$

Consequently, by (4.1.101), the roots $e_k^\pm$, $k = 1, 2, 3$, when restricted to pairs of lines $\mathscr{L}_i$, $i = 1, 2, 3$, becomes

$$\begin{aligned} e_k^\pm &= \omega^{k-1} \sqrt[3]{\frac{v \mp \Lambda^3}{2} + \frac{\sqrt{-\delta_\pm}}{6\sqrt{3}}} + \omega^{2(k-1)} \sqrt[3]{\frac{v \mp \Lambda^3}{2} - \frac{\sqrt{-\delta_\pm}}{6\sqrt{3}}} \\ &= \omega^{k-1} \sqrt[3]{\frac{z \mp \Lambda^3}{2} + \frac{\sqrt{27(z \mp \Lambda^3)^2 - 4(x^3 - 3xy^2)}}{6\sqrt{3}}} \end{aligned}$$

$$+\omega^{2(k-1)}\sqrt[3]{\frac{z\mp\Lambda^3}{2}-\frac{\sqrt{27(z\mp\Lambda^3)^2-4(x^3-3xy^2)}}{6\sqrt{3}}}$$

More specifically, by (4.2.104) and (4.2.105), we have the followings:

On $\Sigma^1_\pm$, we have that $4x^3 = 27(z\mp\Lambda^3)^2$, and $y=0$, thus by circling around $\Sigma^1_\pm$, we see that $e_1^\pm \leftrightarrow e_2^\pm$, with the corresponding vanishing cycle $\nu_1^\pm$;

On $\Sigma^2_\pm$, we have that $-32x^3 = 27(z\mp\Lambda^3)^2$, and $y=\sqrt{3}x$, thus by circling around $\Sigma^2_\pm$, we see that $e_2^\pm \leftrightarrow e_3^\pm$, with the corresponding vanishing cycle $\nu_2^\pm$;

On $\Sigma^3_\pm$, we have that $-32x^3 = 27(z\mp\Lambda^3)^2$, and $y=-\sqrt{3}x$ thus by circling around $\Sigma^3_\pm$, we see that $e_1^\pm \leftrightarrow e_3^\pm$, with the corresponding vanishing cycle $\nu_3^\pm$;

Following [KTL95] and [Seo13], after the proper choice of the homology basis (c.f., (4.2.27) in section 4.2.3)

$$\{\beta_1,\beta_2;\alpha^1,\alpha^2\}\in\Lambda_W\oplus\Lambda_R = H_1(\mathcal{C},\mathbb{Z})$$

the above six vanishing cycles $\nu_i^\pm = (\mathbf{g}_i^\pm,\mathbf{q}_i^\pm)$ associated to the six singular lines $\Sigma^i_\pm$ in the moduli space can be presented as follows (see formula (4.2.29)):

$$\begin{cases}\nu_1^+ = \beta_2-\beta_1-\alpha^1, & \nu_1^- = \beta_2-\beta_1+\alpha^2-2\alpha^1\\ \nu_2^+ = -\beta_2+\alpha^1, & \nu_2^- = -\beta_2-2\alpha^2\\ \nu_3^+ = \beta_1, & \nu_3^- = \beta_1+2\alpha^1+\alpha^2\end{cases}\tag{4.2.112}$$

In coordinates form, they read as follows

$$\begin{cases}\nu_1^+ = (-1,1;-1,0), & \nu_1^- = (-1,1;-2,1);\\ \nu_2^+ = (0,-1;1,0), & \nu_2^- = (0,-1;0,-2);\\ \nu_3^+ = (1,0;0,0), & \nu_3^- = (1,0,2,1);\end{cases}\tag{4.2.113}$$

Remark 4.2.24. *We noted before (see formula (4.2.32)), that the six vanishing cycles are not independent, for example the vanishing cycles $\nu_3^\pm$ can be expressed as*

$$\nu_3^\pm = -\nu_1^\pm - \nu_2^\pm$$

This echos the fact that only two of the three generators $\{\gamma_i\}_{i=1}^3$ for the fundamental group $\pi_1(\mathbb{C}^2\backslash\{\text{trefoil knot}\})$ are independent.

Recall that for the vanishing cycle ν, the associated monodromy matrix $\mathcal{M}_\nu$ is given by

$$\mathcal{M}_\nu = \mathcal{M}_{(\mathbf{g}\,\mathbf{q})} = \begin{pmatrix}\mathbb{I}+\mathbf{q}^t\otimes\mathbf{g} & \mathbf{q}^t\otimes\mathbf{q}\\ -\mathbf{g}^t\otimes\mathbf{g} & \mathbb{I}-\mathbf{g}^t\otimes\mathbf{q}\end{pmatrix}$$

Thus, for $\nu_1^+ = (\mathbf{g}_1^+,\mathbf{q}_1^+) = (-1,1;-1,0)$ above, we compute that

$$\mathbb{I}+(\mathbf{q}_1^+)^t\otimes\mathbf{g}_1^+ = \begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}-1\\0\end{pmatrix}\otimes\begin{pmatrix}-1&1\end{pmatrix}$$

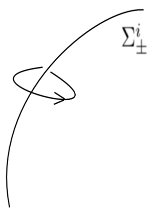

Figure 4.26: Monodromy around the singular lines

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$$

similarly

$$(\mathbf{q}_1^+)^t \otimes \mathbf{q}_1^+ = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix};$$

and

$$(\mathbf{g}_1^+)^t \otimes \mathbf{g}_1^+ = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix};$$

as well as

$$\mathbb{I} - (\mathbf{g}_\mathbf{1}^+)^t \otimes \mathbf{q}_1^+ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} -1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

Putting then above computations together, we finally get

$$\mathcal{M}_{\nu_1^+} = \left(\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \hline -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 1 \end{array}\right)$$

We can do the same for all other vanishing cycles, we summarize the computations as follows

$$\mathcal{M}_{\nu_1^+} = \begin{pmatrix} 2 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 1 \end{pmatrix} \quad \mathcal{M}_{\nu_1^-} = \begin{pmatrix} 3 & -2 & 4 & -2 \\ -1 & 2 & -2 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 2 & 0 \end{pmatrix} \tag{4.2.114}$$

$$\mathcal{M}_{\nu_2^+} = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 \end{pmatrix} \quad \mathcal{M}_{\nu_2^-} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & -4 & 0 & -1 \end{pmatrix} \tag{4.2.115}$$

$$\mathcal{M}_{\nu_3^+} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \mathcal{M}_{\nu_3^-} = \begin{pmatrix} 3 & 0 & 4 & 2 \\ 1 & 1 & 2 & 1 \\ -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.2.116}$$

Remark 4.2.25. *Notice that the quantum monodromy matrices in $SU(2)$ case, namely $\mathcal{M}_{+\Lambda^2}$ and $\mathcal{M}_{-\Lambda_2}$ given in (4.2.18) and (4.2.19) respectively are embedded as sub-matrices in the above $\mathcal{M}_{\nu_i^\pm}s$, which reflects the fact the embedding of the Lie groups $SU(2) \hookrightarrow SU(3)$.*

Remark 4.2.26. *The above results tell us that when restricted to the hyperplane $H = \{Im(v) = 0\}$ (generic situation) inside $\mathbb{C}^2$, the discriminant locus Δ_Λ was cut into an union of six singular lines $\Sigma^i_\pm$, with each point on the singular line $\Sigma^i_\pm$ being associated with the vanishing cycle $\nu_i^\pm$, which in its coordinate form (4.2.112), can be realized as the charge vector of a BPS state $\nu_i^\pm$ (by abusing the notation) with the corresponding mass function vanishes. These "states" are important in the sense that they will later (section 5.2) serve as the initial data for the wall crossing structure (WCS) associated to the Seiberg-Witten integrable system in the $SU(3)$ case.*

Moreover, the $\mathbb{Z}_2$-points and $\mathbb{Z}_3$-points are worth special mentioning here. We notice that at the three $\mathbb{Z}_2$ points $p_k = \Sigma^k_+ \cap \Sigma^k_-$, $k = 1, 2, 3$, there are vanishing cycles with corresponding charge vectors that are mutually local (see definition 4.2.1) to each other, which means that at these three points, two BPS states can become massless simultaneously. In our case, they read as: for p_k, the pair being $\{\nu_k^+, \nu_{k+2}^-\}$ for $k = 1, 2, 3$, and $k + 2$ being taken $\bmod 3$.

The two $\mathbb{Z}_3$-points $\{q_\pm\}$, which by 4.2.108, are the common cusp points of the six lines, which means that there are three massless BPS states situated there. At $q_+ = (0, \Lambda^3)$, the three BPS states are given by $\{\nu_k^+\}, k = 1, 2, 3$, while at $q_- = (0, -\Lambda^3)$, they are given by $\{\nu_k^-\}, k = 1, 2, 3$. respectively. It can be verified directly (or by using the results given in 4.2.33) that the charge vectors in each group are mutually non-local, i.e., there mutual intersections being non-trivial.

Remark 4.2.27. *Note that the vanishing cycles and the associated monodromy matrices we computed differ from the one given in for example [KTL95] and [AD95], however, they differ each other by $SL(4, \mathbb{Z})$ duality transformations. Indeed, in [KTL95], the authors choose the homology cycles to be the ones such that α_1 encircles e_2^+ and e_2^-; α_2 encircles e_3^+ and e_3^-; β_1 encircles e_1^- and e_2^-, while β_2 encircles e_1^+ and e_3^-. The particularity about this choice is that the β-cycles vanish,and α_i corresponds to the fundamental weights, while α_i to the simple roots of $SU(3)$, see the following figure 4.27 for an illustration:*

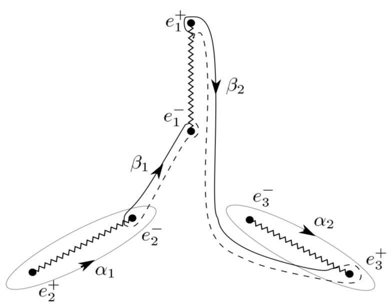

Figure 4.27: One choice of homology basis, figure taken from [KTL95]

In this basis, it was shown in [KTL95] that the vanishing cycles and the associated monodromies are given by

$$\begin{cases} \widetilde{\nu}_1^+ = (1,0;-2,1) \quad \widetilde{\nu}_1^- = (1,0;0,0); \\ \widetilde{\nu}_2^+ = (0,1;0,0) \quad \widetilde{\nu}_2^- = (0,1;-1,2); \\ \widetilde{\nu}_3^+ = (-1,-1;2,-1) \quad \widetilde{\nu}_3^- = (-1,-1;1,-2); \end{cases} \tag{4.2.117}$$

$$\widetilde{\mathcal{M}}_{\nu_1^+} = \begin{pmatrix} -1 & 0 & 4 & -2 \\ 1 & 1 & -2 & 1 \\ -1 & 0 & 3 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \widetilde{\mathcal{M}}_{\nu_1^-} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.2.118}$$

$$\widetilde{\mathcal{M}}_{\nu_2^+} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \quad \widetilde{\mathcal{M}}_{\nu_2^-} = \begin{pmatrix} 1 & -1 & 1 & -2 \\ 0 & 3 & -2 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 \end{pmatrix} \tag{4.2.119}$$

$$\widetilde{\mathcal{M}}_{\nu_3^+} = \begin{pmatrix} -1 & -2 & 4 & -2 \\ 1 & 2 & -2 & 1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & 2 & 0 \end{pmatrix} \quad \widetilde{\mathcal{M}}_{\nu_3^-} = \begin{pmatrix} 0 & -1 & 1 & -2 \\ 2 & 3 & -2 & 4 \\ -1 & -1 & 2 & -2 \\ -1 & -1 & 1 & -1 \end{pmatrix} \tag{4.2.120}$$

Chapter 5

WCS for Seiberg-Witten Integrable systems

In this final chapter, we will use the wall-crossing structure (WCS) formalism that had been laid in chapter two to study the Seiberg-Witten integrable systems in the $SU(2)$ and $SU(3)$ cases.

The ideal is that the WCS associated to the corresponding integrable systems can be used to generate the so-called BPS-states and study the wall-crossing phenomena of the associated BPS-invariants (or DT-invariants) by employing the algorithm given in terms of the split attractor flow.

The success of using WCS in these two cases shows the properness of the WCS formalism, which gives us the confidence that this formalism could be applied to more general settings. For example, it is expected (c.f., [KS14]) that the WCS formalism can be applied to the Hitchin integrable systems, as well as to the integrable system arsing from the studying of $N = 2$ supersymmetric black holes in super-string theory.

The investigation of the BPS states and the corresponding BPS invariants in these situations belong to the current pursuits of theoretical physics.

In section , the WCS for Seiberg-Witten integrable system in $SU(2)$ case will be discussed in details. The main ingredients of the WCS for $SU(2)$ case are the results of physics (see for example [Fay97][BF98]). We just reformulate in terms of WCS.

We will generalize it to the $SU(3)$ case in section . The construction roughly goes as follows:

We know from [Kle+94][KTL95][Kle+95] the vanishing cycles associated to the discriminant locus, which help producing the so-called *initial condition* of the WCS.

The WCS is then studied by cutting the real four dimensional base $\mathcal{B}$ with two complementary hyperplanes, so that the situation is reduced to the real two dimensional case.

On the resulting plane after cutting, the discriminant locus was cut into six singular points that fall into three pairs, each of which determines a situation similar to the $SU(2)$ case. This corresponds to three embeddings of the Lie algebra $\mathfrak{su}(2) \hookrightarrow \mathfrak{su}(3)$. Thus, by applying the WCS formalism to this situation, we get three families of BPS states with non-trivial DT-invariants (or BPS invariants).

We apply again the WCS formalism to another wall (its existence is discussed for example in [Tay01][Hol97]) in order to derive the results about the interaction of the above three families. This gives us new BPS states with DT- invariants equal to 2.

5.1 WCS for SW Integrable System: $SU(2)$ Case

5.1.1 Description of the problem

Recall that the base of the SW integrable system in the $SU(2)$ case is given by $\mathcal{B} \cong \mathbb{C}$, and we have a local system $\underline{\Gamma}$ of rank 2 lattice over the smooth part $\mathcal{B}^0 = \mathcal{B} \backslash \Delta_\Lambda$, where the discriminant locus Δ_Λ (corresponding to the singularities of the $\mathbb{Z}$-affine structure on $\mathcal{B}$) consists of just two points $\{\pm\Lambda^2\}$.

Endowing $\mathcal{B}$ with coordinate u, then the elliptic curve $\mathcal{C}_u$ sitting over $u \in \mathcal{B}^0$ is given by (c.f., (4.1.21))

$$\mathcal{C}_u : y^2 = \mathcal{W}_{A_1}^2 - \Lambda^4 = (x^2 - u)^2 - \Lambda^4 \tag{5.1.1}$$

Then the stalk of the local system $\underline{\Gamma}$ of charge lattices at u is described by

$$\underline{\Gamma}_u = H_1(\mathcal{C}_u, \mathbb{Z}) = \underline{\Gamma}_u^e \oplus \underline{\Gamma}_u^m = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta = \Lambda_R \oplus \Lambda_W \tag{5.1.2}$$

which splits the lattices (4.2.26) into the direct sum of two sub-lattices spanned by the α and β cycles, denote respectively by $\underline{\Gamma}_u^e$ and $\underline{\Gamma}_u^m$ as above.

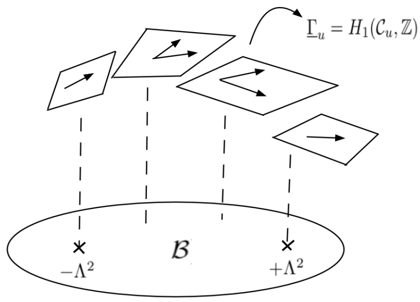

Figure 5.1: Local system $\underline{\Gamma}$ over $\mathcal{B}$

When u approaches to $\pm\Delta_\Lambda^2$, the cycles $\gamma_{\pm\Lambda^2}$ shrinks to size zero, thus the vanishing cycles associated are given by (c.f., (4.2.27))

$$\begin{cases} \gamma_{-\Lambda^2} & = \beta - 2\alpha = (-2, 1) \\ \gamma_{+\Lambda^2} & = \beta = (0, 1) \end{cases} \tag{5.1.3}$$

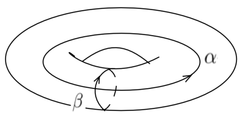

Figure 5.2: Basis cycles on an elliptic curve

Thus, any charge $\gamma \in \Gamma$ can be decomposed uniquely as

$$\gamma = q\,\alpha + g\,\beta$$

where g and q are integers, which are called the **"electric"** and **"magnetic" coordinate** of the **charge vector** γ.

We need a **central function** $Z_u \in Hom_{\mathbb{Z}}(\Gamma_u, \mathbb{C})$. To describe it, we denote the central charges of α and β respectively by

$$a(u) := Z_u(\alpha) \qquad a_D(u) := Z_u(\beta)$$

Consequently, for the general charge vector $\gamma \in \underline{\Gamma}_u$ given as above, its **central charge** is given by

$$Z_u(\gamma) = q\, a(u) + g\, a_D(u) \tag{5.1.4}$$

It had been shown (see the relevant discussion in section 4.1.2) that the central charge function, as a holomorphic function, can be expressed as the integral over the charge vectors with respect to a holomorphic differential one form

$$\lambda_{SW} \in H^1(\mathcal{C}_u, \mathbb{C})$$

called the Seiberg-Witten differential (see formula (4.1.41)). The central charge function is then expressed explicitly as

$$Z_u(\gamma) = \oint_\gamma \lambda_{SW} = q \oint_\alpha \lambda_{SW} + g \oint_\beta \lambda_{SW} \tag{5.1.5}$$

where λ_{SW} in this $SU(2)$ case is given by (see formula (4.1.42)) :

$$\begin{aligned} \lambda_{SW} &= \left(\frac{\partial}{\partial x}\mathcal{W}_{A_1}\right)\frac{x\,dx}{y} = \frac{\partial}{\partial x}(x^2-u)\,\frac{x\,dx}{y} \\ &= \frac{2x^2\,dx}{y} = \frac{2x^2\,dx}{\sqrt{(x^2-u)^2-\Lambda^4}} \end{aligned} \tag{5.1.6}$$

In particular, we see that the special Kähler coordinates $a(u)$ and $a_D(u)$ can be expressed in terms of λ_{SW} as

$$a(u) = \oint_\alpha \lambda_{SW} \qquad a_D(u) = \oint_\beta \lambda_{SW} \tag{5.1.7}$$

Recall (see definition (2.2.2) in section 2.2) that the wall of the first kind (which is called the wall of marginal stability by physicists, see appendix A for more information on it from the physicists' perspective) is defined to be the locus in the moduli space where two or more charge vectors with equal phase of the corresponding central charge functions, namely

$$\mathcal{W}^1 := \bigcup_\gamma \mathcal{W}^1_\gamma \tag{5.1.8}$$

where

$$\mathcal{W}^1_\gamma := \bigcup_{\gamma=\gamma_1+\gamma_2} \mathcal{W}_{\gamma_1,\gamma_2} \tag{5.1.9}$$

and

$$\mathcal{W}_{\gamma_1,\gamma_2} := \{u \in \mathcal{B} : \mathbb{R}_{>0}\cdot Z_u(\gamma_1) = \mathbb{R}_{>0}\cdot Z_u(\gamma_2)\}$$

$$= \left\{ u \in \mathcal{B} : Im\left(\frac{Z_u(\gamma_1)}{Z_u(\gamma_2}\right) = 0 \right\}$$

We claim that the wall of the first kind $\mathcal{W}^1$ in $SU(2)$ case is described by

$$\mathcal{W}^1 = \left\{ u \in \mathcal{B} : Im\left(\frac{a_D(u)}{a(u)}\right) = 0 \right\} \tag{5.1.10}$$

Lemma 5.1.1. *Given two complex numbers z_1 and z_2, we have that*

$$Im\left(\frac{z_1}{z_2}\right) = \frac{Im(z_1\overline{z}_2)}{|z_2|^2}$$

The lemma follows from

$$\frac{z_1}{z_2} = \frac{z_1\overline{z}_2}{z_2\overline{z}_2} = \frac{z_1\overline{z}_2}{|z_2|^2}$$

Proof. (of the claim) Given two charge vectors $\gamma_i = q_i\alpha + g_i\beta$, for $i = 1, 2$. Applying the lemma above, we compute as follows

$$Im\left(\frac{Z_u(\gamma_1)}{Z_u(\gamma_2}\right) = Im\left(\frac{q_1 a(u) + g_1 a_D(u)}{q_2 a(u) + g_2 a_D(u)}\right) = Im\left(\frac{q_1 + g_1\frac{a_D(u)}{a(u)}}{q_2 + g_2\frac{a_D(u)}{a(u)}}\right)$$

$$\xlongequal{z:=\frac{a_D(u)}{a(u)}} Im\left(\frac{q_1 + g_1 z}{q_2 + g_2 z}\right) = Im\left(\frac{(q_1 + g_1 z)(q_2 + g_2\overline{z})}{(q_2 + g_2 z)(q_2 + g_2\overline{z})}\right)$$

$$= Im\left(\frac{q_1q_2 + g_1g_2|z|^2 + q_1g_2\overline{z} + g_1q_2 z}{q_2^2 + g_2^2|z|^2 + q_2g_2Re(z)}\right) = Im(q_1g_2\overline{z} + g_1q_2 z)$$

$$= (g_1q_2 - q_1g_2)\, Im(z) = (g_1q_2 - q_1g_2)\, Im\left(\frac{a_D(u)}{a(u)}\right)$$

Since we consider all possible γ_i, so the integers q_i and g_i are arbitrary, but the above identity holds for all possibilities, thus follows the proposition. □

It can be shown (for example [BF96]) by using the analytic expressions for $a(u)$ and $a_D(u)$ that the curve $\mathcal{W}^1$ (see the picture below) is a closed curve that passes through two singularities $\pm\Lambda^2$ and looks like an ellipse, but not exactly.

The wall $\mathcal{W}^1$ divides the moduli space $\mathcal{B} = \mathbb{C}$ into two connected components, the inside of the curve is called the *strong coupling region*, and is denoted by $\mathcal{B}_{strong}$, while the outside of the curve is called the *weak coupling region*, and is denoted by $\mathcal{B}_{weak}$.

It had been shown (c.f.,[BF96],[BF96]) that the BPS states for the underlying physics theory associated to the $SU(2)$ Seiberg Witten integrable system were given as follows:

Within $\mathcal{B}_{strong}$, we have

- the "magnetic monopoles" with charge vectors given by $(q, g) = \pm(0, 1)$;
- a "dyon" with charge vectors given by $(q, g) = \pm(-2, 1)$ in the lower u plane, and $\pm(2, 1)$ in the upper u plane.

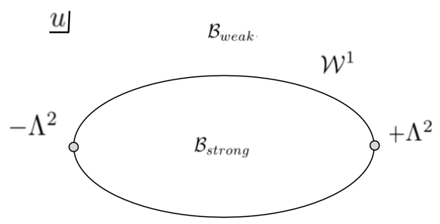

Figure 5.3: Curve of marginal stability in $SU(2)$ case

Within $\mathcal{B}_{weak}$, we have

- the "magnetic monopole" with charge vectors given by $(q, g) = \pm(0, 1)$
- "dyons" with charge vectors given by $(q, g) = \pm(2k, 1)$, with $k \in \mathbb{Z}$.
- $W^{\pm}$ "gauge bosons" with charge vectors given by $(q, g) = (\pm 2, 0)$

And the BPS invariants $\Omega(\gamma)$ of these states all equal to one except for the W-bosons, which are equal to -2. Now we can state the main problem to be answered in this section.

Problem: To encode these charges and the corresponding BPS invariants using the WCS formalism for complex integrable system that had been laid down in chapter 3 (see section 3.2.3 for more details).

5.1.2 Attractor flows

Recall that in section 3.1.4, we have displayed that the base $\mathcal{B}^0$, as a complex manifold, can be endowed with the Kähler metric $g_{\mathcal{B}^0}$ (see equation (3.1.43)):

$$g_{\mathcal{B}^0} = Im\left(da_D d\overline{a}\right) \tag{5.1.11}$$

with the corresponding Kähler potential given by

$$K = Im\left(a_D \overline{a}\right) \tag{5.1.12}$$

Given $\gamma \in \underline{\Gamma}_{u_0}$, let $\theta = Arg\, Z_{u_0}(\gamma)$, we can consider the attractor flow to be given by the gradient flow of the following function

$$F_\gamma(u) := Re\left(e^{-i\theta} Z_u(\gamma)\right) \tag{5.1.13}$$

where the gradient is taken with respect to the Kähler metric $g_{\mathcal{B}^0}$ given above. Thus the flow equation is given by (see formula (3.2.5))

$$\dot{u} + grad\, F_\gamma(u) = 0 \tag{5.1.14}$$

Denote by $\mathcal{L}_\gamma$ the flow line that satisfies the flow equation above, then by the proposition 3.2.2 in section 3.2, we see that along the flow line $\mathcal{L}_\gamma$, the quantity $Im(e^{-i\theta} Z_u(\gamma))$ stays constant. However since at the point $u_0 \in \mathcal{B}^0$, we have that $\theta = Arg\, Z_{u_0}(\gamma)$, we deduce that $Im(e^{-i\theta} Z_u(\gamma)) \equiv 0$. Consequently, as a set, the flow line is described by

$$\mathcal{L}_\gamma := \left\{u \in \mathcal{B} : Im\left(\left(e^{-i\theta} Z_u(\gamma)\right) = 0\right\}\right. \tag{5.1.15}$$

We endow the base $\mathcal{B}^0$ with the $\mathbb{Z}$-affine coordinates (see remark 3.1.17 in chapter 2):

$$\begin{cases} x := Im\left(e^{-i\theta} Z_u(\alpha)\right) = Im\left(e^{-i\theta}\, a(u)\right) \\ y := Im\left(e^{-i\theta} Z_u(\beta)\right) = Im\left(e^{-i\theta}\, a_D(u)\right) \end{cases}$$

By writing the charge vector $\gamma = q\,\alpha + g\,\beta$, we see from above that the attractor flows correspond to the $\mathbb{Z}$-affine lines on the $\mathbb{Z}$-affine manifold $\mathcal{B}^0$ given by the following equation

$$\mathcal{L}_\gamma :\ q\,x(t) + g\,y(t) = 0 \tag{5.1.16}$$

where t is the "time" parameter for the flow line $\mathcal{L}_\gamma$.

Note that the flow line equation (5.1.15) implies that the phase of the central charge function $Z_u(\gamma)$ is preserved along the flow line $\mathcal{L}_\gamma$, namely

$$Arg\,(Z_u(\gamma)) \equiv \theta \tag{5.1.17}$$

For this reason, we also call the attractor flow lines the **lines of constant phase**. Besides, the choice of the function $F_\gamma(u)$ in (5.1.13) implies that along the flow line $\mathcal{L}_\gamma$, we have

$$\begin{aligned} F_\gamma(u) &= Re\left(e^{-i\theta} Z_u(\gamma)\right) = Re\left(e^{-i\theta} e^{i\,Arg(Z_u(\gamma))}|Z_u(\gamma)|\right) \\ &= Re\left(e^{-i\theta}\, e^{i\theta}|Z_u(\gamma)|\right) = Re\,(|Z_u(\gamma)|) = |Z_u(\gamma)| \end{aligned}$$

which is exactly the "mass" function introduced before (in remark 3.2.3). Denote it by

$$m_u(\gamma) := |Z_u(\gamma)| \tag{5.1.18}$$

By applying the proposition 3.2.5 in section 3.2, we get the following

Proposition 5.1.2. *Away from the discriminant locus Δ_Λ, the mass function $m_u(\gamma)$ is decreasing along the flow line $\mathcal{L}_\gamma$.*

Remark 5.1.1. *The mass function $m_u(\gamma)$ in physics literature measures exactly the physical masses of the BPS states, for this aspect of the story, please consult appendix A for relevant information.*

From the proposition above, we conclude that away from the singularities and the critical points of the function $F_\gamma(u)$, along the flow line $\mathcal{L}_\gamma$, $F_\gamma(u)$ is always decreasing, converging to a local minimum, the so called *attractor points* (see definition 3.2.2).

Proposition 5.1.3. *The attractor points in $SU(2)$ case considered here coincide with the discriminant locus $\Delta_\Lambda = \{\pm\Lambda^2\}$.*

Proof. Indeed, as the mass function $m_{u(\gamma)} = F_\gamma(u)$ is decreasing along the attractor flow line $\mathcal{L}_\gamma$, and it is non-negative by definition, thus, the terminal points of the flow correspond to the points where the mass function vanishes. However, we know from 5.1.3 that the cycles $\gamma_{\pm\Lambda^2}$ are the associated vanishing one cycles over $\pm\Lambda^2$. Thus the possible attractor points for the attractor flow lines in this case are $\pm\Lambda^2$. □

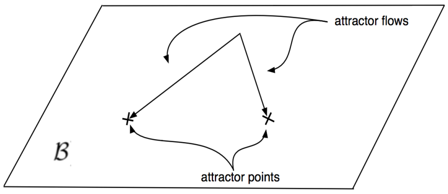

Figure 5.4: Attractor flows and attractor points

Remark 5.1.2. *By the above proof, we see that the attractor flow converges to the minimum of* $|Z(\gamma)|$*, which should be the zeros of the central charge function* $Z(\gamma) = qa(u) + ga_D(u)$*. Consequently,* $\frac{a_{D(u)}}{a(u)} = -\frac{q}{g} \in \mathbb{R}$ *(assuming* $q \neq 0$*), which means that the zeros of it must lie on the curve of marginal stability* $\mathcal{W}^1$*. This is indeed the case as had been noted before that the points* $\pm\Lambda^2$ *lie on* $\mathcal{W}^1$*.*

Remark 5.1.3. *It is possible that the attractor flow lines terminate at a "regular zero" of the central charge function lying on* $\mathcal{W}^1$*, however, this case can be excluded by the following argument. Suppose that* $Z_{u_m}(\gamma)$ *vanishes, then since Seiberg-Witten differential* λ_{SW} *is regular, thus* $Z_{u_m}(\gamma) = \oint_\gamma \lambda_{SW} \equiv 0$ *implies that* γ *must be the vanishing cycles over the point* u_m*. But we know that the only possible vanishing cycles are situated at* $\pm\Lambda^2$*.*

Remark 5.1.4. *The proposition 5.1.3 also follows from the observation (5.1.17) that the phase of the central charge function* $Z_u(\gamma)$ *stays constant along the flow line. For suppose that the attractor flow line does not terminate at the zeros of* $Z_u(\gamma)$*, then the attractor flow line could be continued further and the "constant phase" condition will be violated.*

Proposition 5.1.4. *The attractor flow associated to the charge* γ *can intersect with the wall* $\mathcal{W}^1_{\gamma_1,\gamma_2}$ *at most once. In particular, the attractor flow line intersects with the wall transversely. And since our flow is confined in the complex plane, thus it is atomically planar (see definition 3.2.6), thus by using the terminology defined in definition 3.2.9, we conclude that the attractor flow lines form an attractor tree that is **good**.*

Proof. This is the special case of the proposition 3.2.10. □

Thus by proposition 3.2.17, we expect that by considering the split attractor flow lines on the base $\mathcal{B}$ of the SW integrable system in $SU(2)$ case, we will get an local embedding

$$\mathcal{B}^0 \hookrightarrow Stab(\mathfrak{g}_b) \quad \text{for each } b \in \mathcal{B}^0. \tag{5.1.19}$$

from which we can obtain information about the charge vectors of the BPS states and the corresponding BPS invariants (DT-invariants).

We will see that these are exactly the ones that had been obtained by physics considerations.

5.1.3 Algorithm for computing the BPS invariants

To get a WCS on the moduli space $\mathcal{B} \cong \mathbb{C}$ suitable for our purpose, we consider the following map

$$\begin{aligned} Y : \mathcal{B} \times \ S_\theta &\to \Gamma^*_{\mathbb{R}} \\ (u, \theta) &\mapsto Y_\theta(u)(\gamma) := Im(e^{-i\theta} Z_u(\gamma)) \end{aligned} \tag{5.1.20}$$

Remark 5.1.5. *Intuitively speaking, this means that we want to "scan over" all possible charge vectors $\gamma \in \Gamma_u$ over all points $u \in \mathcal{B}$, in all directions, in order to see which charge vectors can be realized as the BPS states at the given place in the moduli space.*

Notice that this essentially amounts to the (local) embedding of the moduli space into $Stab(\mathfrak{g}_u)$, i.e., the space of stability conditions on $\mathfrak{g}$. For this reason, we say that the information about the BPS states are "encoded" by WCS.

Choosing $\theta = Arg\, Z_u(\gamma)$. By (5.1.17), we know that the attractor flow lines lie at the locus where the central charge has constant phase θ.

The attractor flow lines on the base $\mathcal{B}$ can be lifted to (see the discussion in section 3.2.2) the flows on the following two sub-spaces of the total space of the local system of lattice $\underline{\Gamma}$ over $\mathcal{B}$:

$$\begin{aligned} \mathcal{B}^{0\prime} &:= \{(u, v) \in tot\, \underline{\Gamma}_{\mathbb{R}} : Y(u)(v) = 0\} \\ \mathcal{B}^{0\prime}_{\mathbb{Z}} &:= \{(u, \gamma) \in tot\, \underline{\Gamma} : Y(u)(v) = 0\} \end{aligned}$$

Moreover, the lifted flow lines equation are given by the following

$$\begin{cases} \dot{u} = \iota(\gamma) \\ \dot{\gamma} = 0 \end{cases} \tag{5.1.21}$$

Before giving the algorithm for computing the BPS invariants using the WCS, let us first recall that the vanishing cycles associated to $\pm\Lambda^2$ are given by $\gamma_{\pm\Lambda^2}$ respectively (see the formula 5.1.3). They correspond to the monodromy invariant directions of the corresponding monodromy matrices $\mathcal{M}_{\pm\Lambda^2}$, namely

$$\mathcal{M}_{\pm\Lambda^2} \cdot \gamma_{\pm\Lambda^2} = \gamma_{\pm\Lambda^2}$$

From this, we easily see that the **tail set** (see section 3.2.3) $\mathcal{T}_{\mathcal{B}^{0\prime}_{\mathbb{Z}}} \subset \mathcal{B}^{0\prime}_{\mathbb{Z}}$ in this case reads

$$\mathcal{T}_{\mathcal{B}^{0\prime}_{\mathbb{Z}}} = \left(\pm\Lambda^2, \gamma_{\pm\Lambda^2}\right) \subset \mathcal{B}^{0\prime}_{\mathbb{Z}} \tag{5.1.22}$$

Indeed, we can verify that the fibers of $\pm\Lambda^2$ under the projection of $tot\, \underline{\Gamma}_{\mathbb{R}}$ to $\mathcal{B}^0$ are one dimensional vector subspaces spanned by two vanishing charge vectors respectively, which are strict convex cones. Further more, the tail set above is easily seen to be preserved by the "inverse attractor flow".

Consequently, the **initial data** (see definition 3.2.7) of our WCS, which is given by the restriction of the map $a : tot\underline{\Gamma} \to \Gamma^*_{\mathbb{R}}$ to the tail set, reads in this case as

$$a_{\pm\Lambda^2}(\gamma_{\pm\Lambda^2}) = DT(\gamma_{\pm\Lambda^2}) \cdot e_{\gamma_{\pm\Lambda^2}} \in \underline{\mathfrak{g}}_{\pm\Lambda^2,\gamma_{\pm\Lambda^2}} \tag{5.1.23}$$

That is, the local systems attached to the initial data are typically trivial of rank one. And by the remark 3.2.14, we can assign the BPS invariants for the initial data by

$$\Omega_{\pm\Lambda^2}(\gamma_{\pm\Lambda^2}) = 1. \tag{5.1.24}$$

Remark 5.1.6. *We see that the **Tail assumption** and **Mass function assumption** are easily verified to be true in $SU(2)$ case. Indeed, as the "initial points" $\pm\Lambda^2$ are the attractor points for all attractor flows, thus given open subset $U \subset \mathcal{B}^{0\prime}$, the subset of points $(u,\gamma) \in U$ such that their maximal positive trajectories intersect the tail set $\mathcal{T}_{\mathcal{B}^{0\prime}_{\mathbb{Z}}}$ is dense in U. The mass function that satisfies the mass function assumption is simply $F_\gamma(u)$ since its restriction on the attractor flow line $\mathcal{L}_\gamma$ is simply the "mass" function $m_u(\gamma)$, which had been shown to be decreasing and strictly positive.*

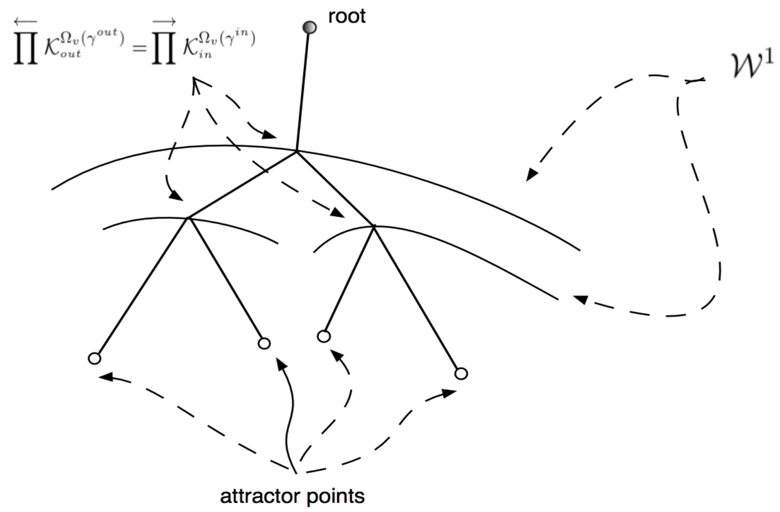

Figure 5.5: Attractor flows on the moduli space

Algorithm for computing BPS invariants in $SU(2)$ case

- Initial data: at the attractor points $\pm\Lambda^2$, we have BPS states with charges $(0,1)$ and $(2,-1)$ respectively. We assign the corresponding BPS invariants to be $\Omega_{\pm\Lambda^2}(\gamma_{\gamma_{\pm\Lambda^2}}) = 1$. These are the only non-trivial invariants at the attractor points.

- Constructing attractor trees: Given $(u,\gamma) \in \underline{\Gamma}_u$, we consider all trees, called the *attractor trees* on $\mathcal{B}$ with (the only) root at u, and the external vertexes all ending at the attractor points. All edges of the trees are given by the attractor flow associated with (u,γ), and each internal vertex should have valency at least three and lies in the wall of the first kind $\mathcal{W}^1$. Moreover, the following "balance condition" at each such internal vertex v must be satisfied:

  Balance condition: If the incoming edge at v is the attractor flow associated to γ^{in}, and the out coming edges are associated to charges γ_i^{out}, then we have that at each internal vertex v:

$$\gamma^{in} = \sum_i \gamma_i^{out} \tag{5.1.25}$$

- Determining $\Omega_u(\gamma)$: Under the assumption that the number of the attractor trees considered above is finite, we start at the attractor points of the union of above trees, and move backward toward (u,γ), and at each inner vertex, we apply the KSWCF

$$\overleftarrow{\prod} \mathcal{K}_{\gamma^{out}}^{\Omega_v(\gamma^{out})} = \overrightarrow{\prod} \mathcal{K}_{\gamma^{in}}^{\Omega_v(\gamma^{in})} \tag{5.1.26}$$

  so that at each inner vertex, $\Omega_v(\gamma^{in})$ can be computed from $\Omega_v(\gamma^{out})$.

  By induction, we will eventually arrive at the value of the $\Omega_u(\gamma)$.

Before applying the above algorithm, let me first remark that the strong and weak coupling region are characterized as

$$\mathcal{B}_{strong} = \left\{ u \in \mathcal{B} : Im\,\frac{a_D}{a} < 0 \right\}$$
$$\mathcal{B}_{weak} = \left\{ u \in \mathcal{B} : Im\,\frac{a_D}{a} > 0 \right\}$$

5.1.4 WCS in the strong coupling region

We first work in the strong coupling region $\mathcal{B}_{strong}$, i.e., the region inside $\mathcal{W}^1$.

Given $(u_0, \gamma) \in \underline{\Gamma}_{u_0}$, where $u_0 \in \mathcal{B}_{strong}$, we define $\theta_0 = arg\, Z_{u_0}(\gamma)$, and consider the attractor flow equation issuing from u_0

$$\dot{u} + grad\, F_\gamma(u) = 0$$

which by (5.1.15), is solved by

$$\mathcal{L}_{u_0,\gamma} := \{u \in \mathcal{B} : arg\, Z_u(\gamma) \equiv \theta_0\} \tag{5.1.27}$$

From proposition 5.1.2 and proposition 5.1.3, we see that in order for the line $\mathcal{L}_{u_0,\gamma}$ to end at the attractor points, the only possibilities for the choice of γ are the vanishing cycles $\gamma_{\pm\Lambda^2}$ situated at the attractor points $\pm\Lambda^2$ respectively.

We prove in the following that $\mathcal{L}_{u_0,\gamma}$ is the unique line that connect u_0 with either $\pm\Lambda^2$ depending on if $\gamma = \gamma_{\pm\Lambda^2}$.

Proposition 5.1.5. *The attractor flow can cross the wall $\mathcal{W}^1$ only at its end point.*

Proof. Let $\gamma = (q, g)$, then $Z_u(\gamma) = qa(u) + ga_D(u)$. By proposition 3.2.8, the flow line $\mathcal{L}_{u_0,\gamma}$ can be rewritten as

$$qa(u) + ga_D(u) = c(1-t), \quad t \in [0, 1] \tag{5.1.28}$$

where t is the real time parameter (for the flow) and the constant c is given by

$$c = qa(u_0) + ga_D(u_0) \ \text{ such that } \ Arg\, c = \theta_0$$

Assuming $q > 0$, we consider the Kähler potential $K(t) = Im\, a_D\bar{a}$ of the metric

$$ds^2 = Im\tau\, da_D d\bar{a}$$

which is easily seen to have the same sigh as that of $Im(\frac{a_D}{a})$. We compute its restriction on the flow line $\mathcal{L}_{u_0,\gamma}$ as

$$\begin{aligned} K(u(t)) = Im\, a_D\bar{a} &= Im\left(\frac{c(1-t) - ga}{q}\bar{a}\right) \\ &= Im\left(\frac{ca(1-t)}{q}\right) = \frac{1-t}{q} Im(c\bar{a}) \end{aligned} \tag{5.1.29}$$

from which we see that if $\mathcal{L}_{u_0,\gamma}$ intersects $\mathcal{W}^1$, i.e., $Im(\frac{a_D}{a}) = 0$, then it either intersects at the end point where $t = 1$ or at points where $Im(c\bar{a}) = 0$. But the following lemma shows that this quantity cannot be zero. The proposition thus follows. $\square$

Lemma 5.1.6. *Along the flow line $\mathcal{L}_{u_0,\gamma}$, the quantity $Im(c\bar{a})$ never vanishes.*

Proof. Since $u_0 \in \mathcal{B}_{strong}$, thus when $t = 0$,

$$K(u_0) = \frac{1}{q} Im(c\bar{a}) < 0$$

by our assumption that $q > 0$, we conclude that initially $Im(c\bar{a}) < 0$. We then prove that it can not be zero by showing that along the flow line, it is a decreasing function. This can be done by showing instead that $Im(\bar{c}a)$ is increasing along $\mathcal{L}_{u_0,\gamma}$. Indeed, by using the definition $\tau = \frac{da_D}{da}$ and taking the derivatives on both sides of the flow line equation (5.1.28) with respect to t , we see that

$$\frac{da}{dt} = \frac{-c}{p + q\tau}$$

then we compute as follows

$$\frac{d}{dt} Im(\bar{c}a) = Im(\bar{c}\dot{a}) = Im\left(\frac{-c\bar{c}}{p + q\tau}\right)$$

$$= \frac{-|c|^2}{|p + q\tau|^2} Im(p + q\bar{\tau}) = \frac{q|c|^2}{|p + q\tau|^2} Im\tau > 0$$

This completes the proof of the lemma. $\square$

In summary, we conclude that in the strong coupling region $\mathcal{B}_{strong}$, the only attractor flow issuing from a point is the straight line (in the affine coordinate) that connects it to the corresponding attractor points $\pm\Lambda^2$ depending on in which monodromy invariant direction is our attractor flow pointing to.

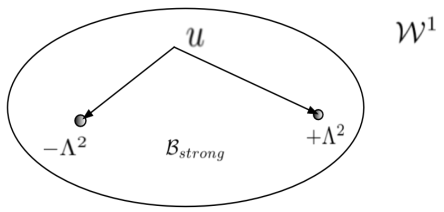

Figure 5.6: Attractor flows in strong coupling region

Consequently, within $\mathcal{B}_{strong}$, the only BPS charges existing are those corresponding to the two vanishing cycles $\gamma_{\pm\Lambda^2}$ at the two singularities $\pm\Lambda^2$, namely, the magnetic monopole with charge vector $(0, 1)$ and the dyon with charge vector $(2, -1)$.

Remark 5.1.7. *The proofs of the above two propositions are based on the treatment in [Fay97].*

5.1.5 WCS in the weak coupling region

Given $(u_0, \gamma) \in \underline{\Gamma}_{u_0}$, where $u_0 \in \mathcal{B}_{weak}$, i.e., outside of the wall $\mathcal{W}^1$. In this weak coupling region, we have $Im\,(\frac{a_D}{a}) > 0$. Define: $\theta_0 = arg\, Z_{u_0}(\gamma)$, and consider the attractor flow associated to the function

$$F_\gamma(u) = Re(e^{-i\theta_0} Z_\gamma(u))$$

as before, the flow line $\mathcal{L}_{u_0,\gamma}$ is given as before, see equation (5.1.15).

From the algorithm described above, we need to consider rooted attractor trees with root at u_0 that terminates at the attractor points $\pm\Lambda^2$. We expect the following scenery:

- For $\gamma = \gamma_{\pm\Lambda^2}$, the flow line $\mathcal{L}_{u_0,\gamma}$ will directly flow into the attractor points $\{\pm\Lambda^2\}$ without splitting, which means the two states with charges $\gamma_{\pm\Lambda^2}$ persists in the weak coupling region.
- For charges γ other than the ones represented by the vanishing cycles, the flow line $\mathcal{L}_{u_0,\gamma}$ will intersect the wall $\mathcal{W}^1$ at the point u_* where the flow line splits into two flows $\mathcal{L}_{u_*,m\gamma_{+\Lambda^2}}$ and $\mathcal{L}_{u_*,n\gamma_{-\Lambda^2}}$ that start at u_* and terminate at the attractor points $\pm\Lambda^2$ respectively, i.e.,

  $$\mathcal{L}_{u_0,\gamma} = \mathcal{L}_{u_*,m\gamma_{+\Lambda^2}} + \mathcal{L}_{u_*,n\gamma_{-\Lambda^2}} \tag{5.1.30}$$

  corresponding to the split of the charge vector

  $$\gamma = m\gamma_{+\Lambda^2} + n\gamma_{-\Lambda^2}$$

  We see that if such a *split attractor flow* exists, then the corresponding BPS state with charge γ (with non-vanishing BPS invariant) exits in $\mathcal{B}_{weak}$, and its BPS invariant $\Omega_{u_0}(\gamma)$ can be computed by applying the KSWCF at the splitting points.

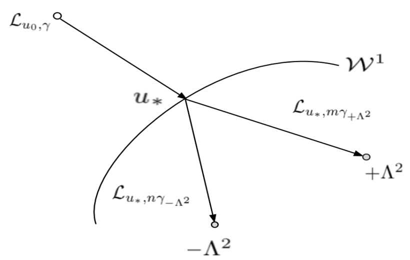

Figure 5.7: Split attractor flow

We need to further investigate the second case above. The first question need to be addressed is when shall a split attractor flow exist?

Apparently, we should make sure that each of the three flow lines in (5.1.30) exits. This is indeed guaranteed by the following two lemmas.

Lemma 5.1.7. *The two attractor flow lines $\mathcal{L}_{u_*,m\gamma_{+\Lambda^2}}$ and $\mathcal{L}_{u_*,n\gamma_{-\Lambda^2}}$ issuing from $\pm\Lambda^2$ respectively can only intersect at the wall of marginal stability $\mathcal{W}^1$.*

Proof. We have that

$$\mathcal{L}_{u_*,m\gamma_{+\Lambda^2}} = \{u \in \mathcal{B} : Arg\, Z_u(\gamma_{+\Lambda^2}) = \theta_0\}$$

$$\mathcal{L}_{u_*,n\gamma_{-\Lambda^2}} = \{u \in \mathcal{B} : Arg\, Z_u(\gamma_{-\Lambda^2}) = \theta_0\}$$

from which we see that $\mathcal{L}_{u_*,m\gamma_{+\Lambda^2}}$ intersects with $\mathcal{L}_{u_*,n\gamma_{-\Lambda^2}}$ at

$$\mathcal{L}_{u_*,m\gamma_{+\Lambda^2}} \cap \mathcal{L}_{u_*,n\gamma_{-\Lambda^2}} = \{u^* \in \mathcal{B} : Arg\, Z_{u^*}(\gamma_{-\Lambda^2}) = Arg\, Z_{u^*}(\gamma_{+\Lambda^2})\}$$

which belongs to the following set

$$\left\{u \in \mathcal{B} : Im\left(\frac{Z_u(\gamma_{+\Lambda^2})}{Z_u(\gamma_{-\Lambda^2})}\right) = 0\right\} = \left\{u \in \mathcal{B} : Im\, \frac{a_D(u)}{2a(u) - a_D(u)} = 0\right\}$$
$$= \left\{u \in \mathcal{B} : Im\, \frac{a_D(u)}{a(u)} = 0\right\} =: \mathcal{W}^1$$

Therefore, the intersection point u^* lies on the wall $\mathcal{W}^1$. ∎

Lemma 5.1.8. *The intersection points u^* coincides with the splitting point u_*.*

Proof. At the splitting point u_* is the point where the central charge function becomes

$$Z_u(\gamma) = mZ_u(\gamma_{+\Lambda^2}) + nZ_u(\gamma_{-\Lambda^2})$$

as we have at the splitting point that

$$\gamma = m\gamma_{+\Lambda^2} + n\gamma_{-\Lambda^2}$$

Suppose that $u^* \neq u_*$, then the phase of $Z_{u_*}(\gamma)$ would be different from that of $Z_{u^*}(\gamma_{\pm\Lambda^2})$, contradicting to

$$Arg\, Z_{u_*}(\gamma) = Arg\, Z_{u^*}(\gamma_{\pm\Lambda^2}) = \theta_0$$

Thus $u^* = u_*$, and the lemma is proved. ∎

Now we can answer the question: When will a split attractor flow tree like (5.1.30) exist? Putting it in another way, we can ask: given a point $u_0 \in \mathcal{B}_{weak}$, for which particular charge vectors $\gamma \in \underline{\Gamma}_{u_0}$, should the attractor flow tree associated to it exist?

To answer this question, we need to reverse the logical above, and consider instead the two attractor flow lines $\mathcal{L}_{+\Lambda^2}$ and $\mathcal{L}_{-\Lambda^2}$ issuing from the two attractor points (singularities) $\pm\Lambda^2$, which are given respectively by

$$\mathcal{L}_{+\Lambda^2} = \{u \in \mathcal{B} : Arg\, Z_u(\gamma_{+\Lambda^2}) = \theta\}$$

$$\mathcal{L}_{-\Lambda^2} = \{u \in \mathcal{B} : Arg\, Z_u(\gamma_{-\Lambda^2}) = \theta\}$$

The two lines correspond to the same θ-value, though the θ here can be arbitrary. In terms of the framework of WCS, we say that we need to consider the **initial data** of the WCS. As the lemma 5.1.8 above shows, these two "incoming rays" $\mathcal{L}_{\pm\Lambda^2}$ will intersect exactly at one point $u_* \in \mathcal{W}^1$.

By KSWCF, the two rays will "scatter" at u_* into possibly infinity many rays $\mathcal{L}_{m,n}$ with all possible charges $m\gamma_{+\Lambda^2} + n\gamma_{-\Lambda^2}$. Together with the incoming rays, they form the so-called *scattering diagram*.

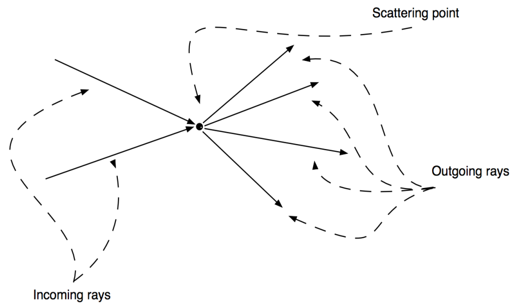

Figure 5.8: Scattering of the incoming two rays

Then by considering a small loop around the scattering point u_*, i.e., the splitting point, the KSWCF is then equivalent to the triviality of the following monodromy (see proposition 2.2.1 in chapter 2)

$$\prod_{t_i}^{\circlearrowleft} \mathbb{S}(l_{\gamma_i}) = id$$

where the product is taken in the increasing order of elements t_i. Written in terms of the KS transformations, this read as:

$$\boxed{\prod_{m,n\geq 0;\, m/n\nearrow} \mathcal{K}_{(m,n)}^{\Omega_{u_-^*}(m,n)} = \prod_{m,n\geq 0;\, m/n\searrow} \mathcal{K}_{(m,n)}^{\Omega_{u_+^*}(m,n)}} \tag{5.1.31}$$

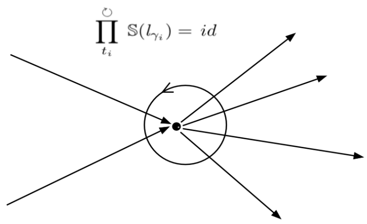

Figure 5.9: Monodromy around the splitting point

In the case in which we are concerned here, we have that

$$k = \langle \gamma_{-\Lambda^2}, \gamma_{+\Lambda^2} \rangle = \langle (2,-1), (0,1) \rangle = 2 \cdot 1 - 0 \cdot (-1) = 2$$

Then the KSWCF above specialize to the following (see formula (2.2.11))

$$\mathcal{K}_{2,-1}\,\mathcal{K}_{0,1} = (\mathcal{K}_{0,1}\,\mathcal{K}_{2,1}\,\mathcal{K}_{4,1}\ \cdots)\,\mathcal{K}_{2,0}^{-2}\,(\cdots\mathcal{K}_{6,-1}\,\mathcal{K}_{4,-1}\,\mathcal{K}_{2,-1}) \tag{5.1.32}$$

from which we infer that after the "scattering", we will get countably many BPS rays associated to new BPS states with the charges indicated by the sub-index on the RHS of the above formula, while the corresponding BPS invariants are indicated by the super-index on the RHS of the formula above.

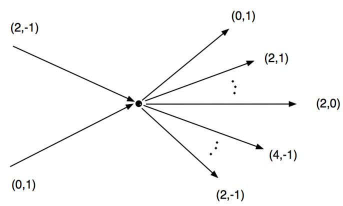

Figure 5.10: Scattering diagram in $k = 2$ case

Finally, we can spell out the WCS that in some sense "foresees" these BPS states by applying the algorithm in section 5.1.3 as follows:

Given $(u_0, \gamma) \in \underline{\Gamma}_{u_0}$, where $u_0 \in \mathcal{B}_{weak}$, our purpose is to test at u_0, if the BPS state with charge γ exists or not. And if it exists, then to determine its BPS invariant $\Omega_{u_0}(\gamma)$. To this end, we follow the procedures (algorithm) as below:

- First, determine $\theta_0 = arg\, Z_{u_0}(\gamma)$, and consider the gradient flow associated to the function $F_\gamma(u) = Re(e^{-i\theta_0} Z_\gamma(u))$.
- Second, the rooted attractor flow tree with root u_0 would be generically a tree that splits only at some point $u_* \in \mathcal{W}^1$, with the two external edges ending at the two attractor points $\{\pm\Lambda^2\}$.
- Third, we start with the attractor points, tracing along the external edges till the scattering point u_*, at this point we use the KSWCF, and from which we can tell if the charge γ coincides one of the charges on the RHS of (5.1.32):

  If the answer is YES, the state with charge vector γ exists at the point u_0 with corresponding BPS invariant $\Omega_{u_0}(\gamma)$ can be read off from the KSWCF (5.1.32); If the answer is NO, then at u_0 there does not exist BPS state with charge vector γ.

Remark 5.1.8. *We see that the possible BPS states that had been "captured" by the procedure above fall into a convex cone on the space*

$$\mathcal{B}^{0\prime}_{\mathbb{Z}} := \{(b, \gamma) \in tot\,\underline{\Gamma} : Y(b)(v) = 0\}$$

which then confirms the claim in proposition 3.2.16, namely that in this case, the WCS is uniquely determined by its initial data.

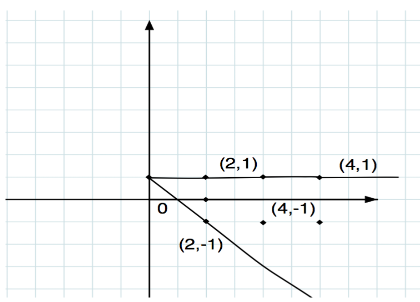

Figure 5.11: Convex cone generated by the BPS charges

5.1.6 Further remarks and miscellaneous discussions

Remark 5.1.9. *As we change θ continuously, i.e., by rotating the affine coordinates on $\mathcal{B}$ (see remark 3.1.17 in chapter 3), we see that any flow line in the scattering diagram described above would sweep the whole complex plane, from which we know that for the BPS states with charge given by KSWCF in the weak coupling region, they actually exist for all point $u \in \mathcal{B}_{weak}$.*

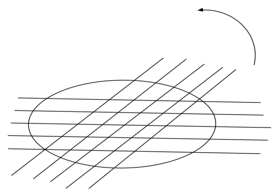

Figure 5.12: Rotating the affine coordinates

Actually, the split attractor flow lines also get rotated after continuously changing the angle θ. Algebraically, this means that the flow lines are still the curves of constant Z-phase, despite that the phases could be arbitrary θ, instead of being 0. Geometrically, this means that if there exits a vanishing cycle in one direction specified by θ_0, then it also exits along all other directions of the $\mathbb{Z}$-affine structure.

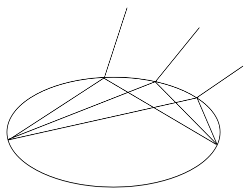

Figure 5.13: Rotating the splitting attractor flows

The discussion above enables us to verify the **compactness assumption** in section 3.2.3, that is: *There exists an open dense subset $\mathcal{B}^{0\prime\prime}_{\mathbb{Z}} \subset \mathcal{B}^{0\prime}_{\mathbb{Z}}$ with the property that for every $(b, \gamma) \in \mathcal{B}^{0\prime\prime}_{\mathbb{Z}}$, there exists a compact subset $K_{(b,\gamma)} \subset \mathcal{B}^{0\prime}_{\mathbb{Z}}$ and an open neighborhood U of (b, γ) such that for every attractor tree T with the root and root edge in U, the corresponding tree T^0 belongs to $K_{(b,\gamma)}$.* Indeed, by the remark above we see that given $b_0 \in \mathcal{B}$ and $\gamma \in \underline{\Gamma}_{b_0}$ and an open neighborhood U_{b_0} around b_0, then for $b \in U_{b_0}$ that is infinitesimally close to b_0, the attractor flow line $\mathcal{L}_{\gamma_b}$ is also infinitesimally close to $\mathcal{L}_{\gamma_{b_0}}$.

Even if in certain regions, the topological type of the attractor trees could "jump", however, as can be seen in the following discussion, they are still bounded by compact sets, thus the compactness assumption holds. The asymptotic expansion for $a(u)$ and $a_D(u)$ are given by (after re-scaling $\Lambda = 1$ (c.f.,[BF98],[BF98]).

$$\text{Near } u = \infty \begin{cases} a(u) & \sim \sqrt{u/2} \\ a_D(u) & \sim \frac{i}{\pi}\sqrt{2u}\,(\ln z + 3\ln 2 - 2) \end{cases} \tag{5.1.33}$$

$$\text{Near } u = -1 \begin{cases} a(u) & \sim \frac{i}{2\pi}\left[\epsilon\frac{u+1}{2}\ln\frac{u+1}{2} + \frac{u+1}{2}\left(-i\pi - \epsilon(1 + 4\ln 2)\right) + 4\epsilon\right] \\ a_D(u) & \sim \frac{i}{\pi}\sqrt{2u}\,(\ln z + 3\ln 2 - 2) \end{cases} \tag{5.1.34}$$

$$\text{Near } u = 1 \quad \begin{cases} a(u) & \sim \frac{2}{\pi} - \frac{1}{2\pi}\frac{u-1}{2}\left[\ln\frac{u-1}{2} + 1 - 4\ln 2\right] \\ a_D(u) & \sim i\frac{u-1}{2} \end{cases} \tag{5.1.35}$$

Certain attractor flow lines can then be depicted by using Mathematica:

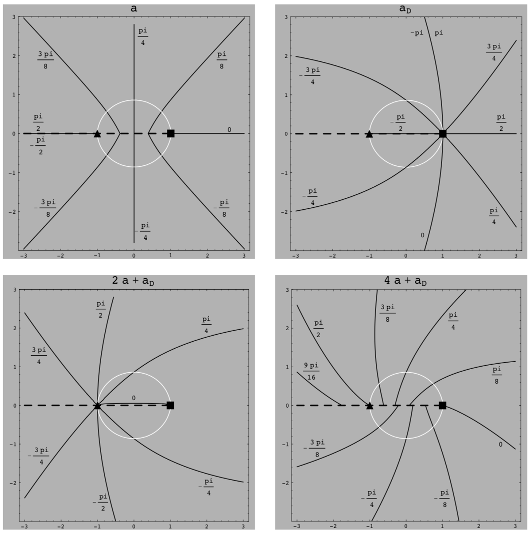

Figure 5.14: Curves of constant $Z(\gamma)$ phase θ for $\gamma = (1,0),(0,1),(2,1),(4,1)$. The graphs are taken from [BF98].

Discussions: From the above graphs, we see that for $\gamma_{-\Lambda^2} = (0,1)$ and $\gamma_{+\Lambda^2} = (2,1)$, there exists attractor flow line that starts at any point $u \in \mathcal{B}$ and ends at the attractor points ($\pm\Lambda^2$ corresponding to $\gamma_{\pm\Lambda^2}$ respectively), which means that the BPS states with charge vectors given by $\gamma_{\pm\Lambda^2}$ exist through the moduli space. However for $\gamma = (1,0)$, i.e., the left upper picture above, we see that there are no flow lines that terminates at the attractor points, which simply means that through out the moduli space, there are no BPS state with charge $(1,0)$. This is indeed the case as had been implied by KSWCF.

The tricky part is about the flow lines associated to the "dyon" $\gamma = (4,1)$. As the right lowest picture shows that for some values of θ (for example $\theta = \frac{\pi}{4}$), the flow lines will first hit the wall $\mathcal{W}^1$, and then proceed to intersect the cut $[-\Lambda^2, \Lambda^2]$. In this case, it can be shown that the flow line could be represented by a flow line that splits at the intersection point into two flow lines that terminate at $\pm\Lambda^2$ respectively, as the charge vector $(4,1)$ can be written as a sum of the primitive charge vectors associated to the monodromy invariant directions at the attractor points $\pm\Lambda^2$, i.e., when above the cut, we have that

$$(4,1) = 2 \times (2,1) - (0,1)$$

and when below the cut, we have that $(4,1) = 2\times(2,-1)+(0,1)$. However, for $\gamma = (1,0)$ above, this decomposition is not possible, thus the flow line associated to γ can not be represented by split attractor flows. For if

$$(1,0) = m(2,1) + n(0,1)$$

then we have $2m = 1$, which is impossible as we are seeking integer solutions.

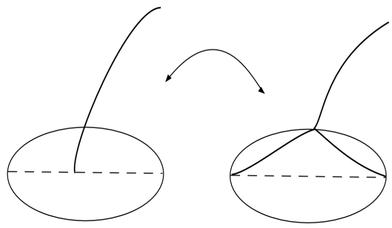

Figure 5.15: Representing the flow line by split attractor flow

More detailed analysis ([BF98]) shows more interesting properties of the rotated attractor flow lines associated to the charge vector $(4,1)$, the same holds for more general $(2n,1)$ and $(2n,-1)$ BPS states, which are called "dyons" in physics literature.

So let $\gamma = (4,1)$, then the attractor flow lines are given by the equation

$$Arg(Z_u(\gamma)) = \theta, \quad \theta \in [0, 2\pi]$$

As θ varies, it can be show that the moduli space are separated by the wall $\mathcal{W}^1$, the flow lines corresponding to $\theta = 0$ and $\theta = \frac{\pi}{2}$ into four regions.

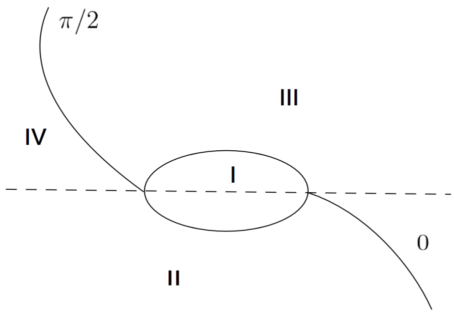

Figure 5.16: Four regions in moduli space

We see from the above graph, that when $\theta = 0, \frac{\pi}{2}$, the attractor flows are given by a single flow lines, however, when crossing the regions divided by the these two lines, the attractor flow would undergone "phase transition" (see figure 5.17) in the sense that it will be represented by splitting attractor flow instead of a single flow line.

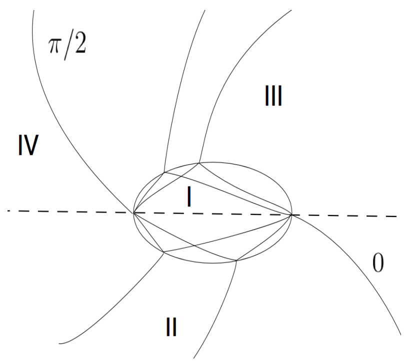

Figure 5.17: "Phase transition" of the attractor flow lines

Before ending this section, let me point out that the split attractor flow trees on the base $\mathcal{B}$ of the integrable system

$$\pi : X \to \mathcal{B}$$

when lifted to X, corresponds to how the torus fibers degenerate.

More precisely, borrowing the terminologies form [Lin17], it corresponds exactly to the pseudo-holomorphic disc with boundaries in the relative class

$$\gamma \in H_2((X^0, \pi^{-1}(b)), \mathbb{Z})$$

Thus when at the splitting point on the wall of the first kind, there would be new holomorphic discs produced, which is called the “bubbling phenomenon” in [Lin14].

With this interpretation, the BPS invariant $\Omega_b(\gamma)$, which is locally constant in b, roughly speaking, corresponds to the counting of the number of pseudo holomorphic discs in the relative class γ.

Near a singularity of focus-focus type, the local model is given by the Ooguri-Vafa space (see section 3.1.6 for its description).

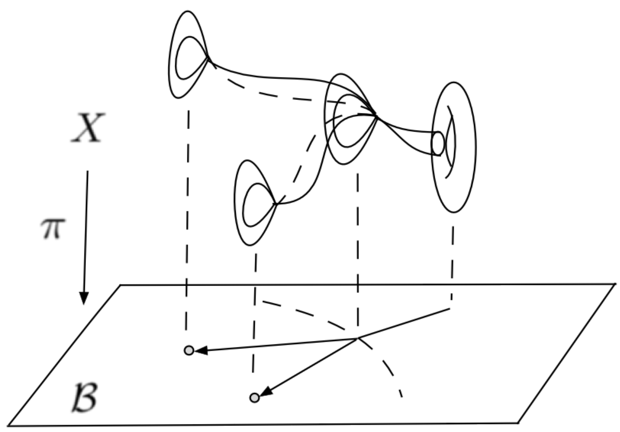

Figure 5.18: WCS and pseudo holomorphic discs

5.2 WCS for SW Integrable System: $SU(3)$ Case

In this section, we will apply the formalism of WCS to the Seiberg-Witten integrable system in the case of $SU(3)$. It turned out that this case is much more involved than the $SU(2)$ case investigated in the last section as the moduli space $\mathcal{B}$ in this case is of complex dimension two, and consequently, the discriminant locus Δ_Λ, as well as the walls of the first kind (walls of marginal stability) $\mathcal{W}^1$, are much more complicated than the $SU(2)$ case. Besides, the combinatory of the split attractor flows in this case is more subtle than the $SU(2)$ case considered before.

However, by employing the methods discussed in section 4.2.4, i.e., by considering a slice of moduli space given by $Im(v) = 0$, the problem can be reduced to the real three dimensional situation, so that the visualization become possible in this case. The proposition 4.2.22 then guarantees that we will not losing essential information after this induction.

Moreover, by remark 4.2.23, remark 4.2.25 and remark 4.2.26 in the last chapter, we see that some aspects of the WCS in $SU(3)$ case can be investigated through the reduction to the $SU(2)$ case which had already been explained in last section. Morally, this corresponds to the embeddings $SU(2) \hookrightarrow SU(3)$.

5.2.1 Description of the problem

First, we collect some facts that had been given in previous chapters regarding the SW integrable system in the $SU(3)$ case. The SW curve in this case is given by (4.1.24) as

$$\mathcal{C}_{\mathbf{u}} : y^2 = \mathcal{W}_{A_2}^2 - \Lambda^6 = (x^3 - ux - v)^2 - \Lambda^6 \tag{5.2.1}$$

where $\mathbf{u} := (u, v)$ denotes the moduli parameter.

Thus the moduli space is given by $\mathcal{B} = \mathbb{C}^2$. And the quantum discriminant is given by (4.1.25) as

$$\delta_\Lambda = 2^6 \Lambda^{18} \left(4u^3 - 27(v + \Lambda^3)^2\right) \left(4u^3 - 27(v - \Lambda^3)^2\right) \tag{5.2.2}$$

So the discriminant curve is given in this case by

$$\Delta_\Lambda := \Delta_\Lambda^+ \cup \Delta_\Lambda^- = \left\{(u, v) \in \mathbb{C}^2 : 4u^3 = 27(v \pm \Lambda^3)^2\right\} \tag{5.2.3}$$

where $\Delta_\Lambda^\pm$ is defined as

$$\Delta_\Lambda^\pm := \left\{(u, v) \in \mathbb{C}^2 : 4u^3 = 27(v \mp \Lambda^3)^2\right\} \tag{5.2.4}$$

And the SW differential is given by (4.1.45) as follows

$$\begin{aligned}\lambda_{SW} &= \left(\frac{\partial}{\partial x}\mathcal{W}_{A_2}\right)\frac{x\,dx}{y} = \frac{\partial}{\partial x}(x^3 - ux - v)\,\frac{x\,dx}{y} \\ &= \frac{(3x^2 - u)\,x\,dx}{y} = \frac{(3x^2 - u)\,x\,dx}{\sqrt{(x^3 - ux - v)^2 - \Lambda^6}}\end{aligned} \tag{5.2.5}$$

The central charge is given by the following

$$Z_{\mathbf{u}}(\gamma) = \oint_{\gamma} \lambda_{SW} \tag{5.2.6}$$

Given standard basis $\{\alpha^1, \alpha^2; \beta_1, \beta_2\}$ of the hyper-elliptic curve $\mathcal{C}_u$ (c.f., 4.2.27 in section 4.2.3), then a charge vector can be written as

$$\gamma = (\mathbf{g} \quad \mathbf{q}) = g^1\beta_1 + g^2\beta_2 + q_1\alpha^1 + q_2\alpha^2 \in \underline{\Gamma}_{\mathbf{u}} = H_1(\mathcal{C}_{\mathbf{u}}, \mathbb{Z})$$

where $\mathbf{g} = \begin{pmatrix} g^1 & g^2 \end{pmatrix}$ denotes the "magnetic charge" of the "BPS state" represented by γ, while $\mathbf{q} = \begin{pmatrix} q^1 & q^2 \end{pmatrix}$ denotes the "electric charge" of the "state" represented sγ.

Defining

$$a^i := \oint_{\alpha^i} \lambda_{SW} \qquad a_{D,i} := \oint_{\beta_i} \lambda_{SW} \quad i = 1, 2.$$

then the central charge of γ can be written as follows

$$Z_{\mathbf{u}}(\gamma) = \oint_{\gamma} \lambda_{SW} = g^1 a_{D,1} + g^2 a_{D,2} + q_1 a^1 + q_2 a^2 \tag{5.2.7}$$

Recall that near the discriminant locus Δ_Λ, the moduli space $\mathcal{B}$ is studied by intersecting the moduli space with the three sphere S^3_R with radius R (see section 4.2.3), while the monodromies near this locus is investigated by intersecting it with the hyperplane given by $\{Im(v) \equiv 0\}$ (see section 4.2.4). Then the region where R becomes very large, or $|Re(v)|$ becomes very large is called the **weak coupling region**. Or equivalently, the weak coupling region is characterized by (see for examples: [FH97][BF96][Pet12])

$$\left| \frac{\alpha_i \cdot \mathbf{a}}{\Lambda} \right| >> 1, \quad \text{for all positive roots } \alpha_i. \tag{5.2.8}$$

Remark 5.2.1. *By remark 4.2.22, we see that the weak coupling region corresponds to the semi-classical limit of the theory, thus the monodromies in this region will be given by semi-classical limit discussed in chapter four.*

Weak coupling spectrum

We list the known BPS states in the weak coupling region ([FH97]). As the above remark indicated, these states should be the ones that are invariant under the semi-classical monodromies. To this end, let us denote by $\{\alpha_1, \alpha_2, \alpha_3\}$ the positive roots of $SU(3)$. In terms of the orthonormal basis $\{e_i\}$ of the plane, the positive roots can be expressed as

$$\alpha_i = \frac{1}{\sqrt{2}}(e_i - e_{i+1}) \quad i = 1, 2.$$

while $\alpha_3 = \alpha_1 + \alpha_2$, namely, we have that

$$\alpha_1 = (1, 0), \ \alpha_2 = (-1/2, \ \sqrt{3}/2), \ \alpha_3 = (1/2, \sqrt{3}/2)$$

Then by proposition 4.2.16 and remark 4.2.16, we see that there exist BPS states that are given by

$$\begin{cases} \nu_i^+ = (\alpha_i, (n+1)\,\alpha_i) \\ \nu_i^- = (\alpha_i, n\,\alpha_i) \end{cases} \tag{5.2.9}$$

for $i = 1, 2$ and $n \in \mathbb{Z}$.

We introduce the following notations for these states in accordance with [Tay01] and [Hol97]:

$$\begin{cases} Q_1^n : & = (\alpha_1, (n-1)\alpha_1) \\ Q_2^n : & = (\alpha_2, n\alpha_2) \end{cases} \tag{5.2.10}$$

By what we had said above, the states obtained from these by applying the semi-classical monodromies should also exist in the weak coupling region. Recall that the semi-classical monodromies (see (4.2.15)) reads:

$$\mathcal{M}_i \equiv \mathcal{M}^c_{r_i} = r_i^{class}\, T^{-1}, \; i = 1, 2, 3.$$

which acts as follows ([FH97])

$$\begin{cases} Q_1^1 \cdot \mathcal{M}_1^{\pm n} & = \pm Q_1^n \\ Q_2^1 \cdot \mathcal{M}_2^{\pm n} & = \pm Q_2^n \end{cases} \tag{5.2.11}$$

Since we have the braid relation (4.2.110)

$$\mathcal{M}_3 = \mathcal{M}_2 \cdot \mathcal{M}_1 \cdot \mathcal{M}_2^{-1}$$

we see that the remaining possible BPS states are given by

$$Q_1^n \cdot \mathcal{M}_2^{\pm} = Q_2^n \cdot \mathcal{M}_1^{\pm} \tag{5.2.12}$$

That is, besides the tower of "dyonic BPS states" Q_1^n and Q_2^n as above, we should also have the following two towers of BPS states:

$$Q_{3\pm}^n := Q_1^n \cdot \mathcal{M}_2^{\pm} = (\alpha_1, (n-1)\alpha_1) \cdot \mathcal{M}_2^{\pm}$$

$$= (\alpha_1 + \alpha_2, (n-1)\alpha_1 \pm \alpha_2) = (\alpha_3, \pm\alpha_1 + n\alpha_3) \tag{5.2.13}$$

Besides these states, there exist also the following family of BPS states in the weak coupling region:

$$W_k := (0, \alpha_k), \quad k = 1, 2. \tag{5.2.14}$$

which are called the "W-bosons" in physics literature.

Remark 5.2.2. *It should be understood that all BPS states above are accompanied by their anti-BPS states, i.e., the BPS states with opposite charge vectors. For this reason, we omit the discussion of them through out.*

Strong coupling spectrum

Since there does not exist a single wall (of the first kind) as in the $SU(2)$ case, we cannot separate the moduli space $\mathcal{B}$ by a single wall as the weak coupling region and strong coupling region. However, inspired by the $SU(2)$ case discussed in previous section, we expect that the *strong coupling region* in $SU(3)$ case corresponds to the region that is near the discriminant locus Δ_Λ. That is the region where the $|\mathbf{u}|$ becomes relatively small.

We know that in this region, there are at least six BPS states with vanishing "mass" function (defined as the absolute value of the central charge function). These states are represented by the six vanishing cycles $\nu_i^\pm$, $i = 1, 2, 3$ as had been exhibited in section 4.2.4. These cycles are the invariant cycles under the (Picard-Lefschetz) monodromies around the six singular lines $\sum_\pm^i$, $i = 1, 2, 3$. They are given explicitly by (4.2.113):

$$\begin{cases} \nu_1^+ = (-1, 1; -1, 0) \qquad \nu_1^- = (-1, 1; -2, 1) \\ \nu_2^+ = (0, -1; 1, 0) \qquad \nu_2^- = (0, -1; 0, -2) \\ \nu_3^+ = (1, 0; 0, 0) \qquad \nu_3^- = (1, 0, 2, 1) \end{cases} \tag{5.2.15}$$

In the following, we choose the basis of homology one cycles as in remark 4.2.27. In this basis, the six vanishing cycles become the following (we use the same notation $\nu_i^\pm$)

$$\begin{cases} \nu_1^+ = (1, 0; -2, 1) \qquad \nu_1^- = (1, 0; 0, 0) \\ \nu_2^+ = (0, 1; 0, 0) \qquad \nu_2^- = (0, 1; -1, 2) \\ \nu_3^+ = (-1, -1; 2, -1) \qquad \nu_3^- = (-1, -1; 1, -2) \end{cases} \tag{5.2.16}$$

Like the vanishing cycles given in (5.2.15), these basis are not linearly dependent. In fact, it can be verified (see formulas (4.2.32)) that

$$\begin{cases} \nu_3^+ = -\nu_1^+ - \nu_2^+ \\ \nu_3^- = -\nu_1^- - \nu_2^- \end{cases}$$

From proposition 4.2.4., we know that the intersection numbers among these vanishing cycles are given by

$$\begin{cases} \langle \nu_i^+, \nu_{i+1}^+ \rangle = \langle \nu_i^-, \nu_{i+1}^- \rangle = 1 \quad i = 1, 2. \\ \langle \nu_3^+, \nu_1^+ \rangle = \langle \nu_3^-, \nu_1^- \rangle = 1 \\ \langle \nu_i^+, \nu_i^- \rangle = -2 \qquad \langle \nu_i^+, \nu_{i+1}^- \rangle = 2 \quad i = 1, 2. \\ \langle \nu_3^+, \nu_3^- \rangle = -2 \qquad \langle \nu_3^+, \nu_1^- \rangle = 2 \end{cases} \tag{5.2.17}$$

We can choose the simple roots of $SU(3)$ in Dynkin basis to be

$$\begin{cases} \alpha_1 = (2, -1) \\ \alpha_2 = (-1, 2) \end{cases} \tag{5.2.18}$$

which are the column vectors of the Cartan matrix A_2 for $SU(3)$, that is

$$A_2 := \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

The fundamental weights are given by

$$\begin{cases} w_1 = (1,0) \\ w_2 = (0,1) \end{cases} \tag{5.2.19}$$

Since we know that the "magnetic charge vectors" sit in the weight lattice Λ_w, which have elements expressed in terms of the co-roots, i.e., vector of the form

$$\mathbf{g} = n_1\alpha_1^\vee + n_2\alpha_2^\vee \in \Lambda_w$$

where the co-root is defined through

$$\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$$

In our simply-laced case, the roots are self dual, namely, $\alpha^\vee = \alpha$. In these notations, we can re-express the six BPS states (see 5.2.17) associated to six singular lines $\Sigma_\pm^k$ for $k = 1, 2, 3$ as follows

$$\begin{cases} \nu_1^+ = (\alpha_1, -\alpha_1) \qquad \nu_1^- = (\alpha_1, 0) \\ \nu_2^+ = (\alpha_2, 0) \qquad \nu_2^- = (\alpha_2, \alpha_2) \\ \nu_3^+ = (-\alpha_1 - \alpha_2, \alpha_1) \qquad \nu_3^- = (-\alpha_1 - \alpha_2, -\alpha_2) \end{cases} \tag{5.2.20}$$

In terms of the notation Q_k^n, $k = 1, 2$ and $Q_{3\pm}^n$, the above BPS states can be expressed as

$$\begin{cases} \nu_1^+ = Q_1^2 \qquad \nu_1^- = Q_1^0 \\ \nu_2^+ = Q_2^0 \qquad \nu_2^- = Q_2^1 \\ \nu_3^+ = -Q_{3+}^0 \qquad \nu_3^- = -Q_{3-}^{-1} \end{cases} \tag{5.2.21}$$

Now we can state the main problem to be solved in this section.

Problem: *Starting with the known BPS spectrum (5.2.21) at the strong coupling region (as the initial data for the WCS), to enumerate the weak coupling BPS spectrum (5.2.10), (5.2.13) and (5.2.14) by using WCS. Then to extend the known spectrum if possible.*

5.2.2 Attractor flows and walls of the first kind

Again, in out $SU(3)$ case, the attractor flow (see (3.2.5)) stating at $(\mathbf{u}, \gamma) \in \underline{\Gamma}_{\mathbf{u}}$ is defined to be the gradient flow of the function

$$F_\gamma(\mathbf{u}) = Re(e^{-i\theta} Z_{\mathbf{u}}(\gamma))$$

where $\theta = Arg(Z_{\mathbf{u}}(\gamma))$, that is

$$\dot{\mathbf{u}} + \nabla F_\gamma(\mathbf{u}) = 0 \tag{5.2.22}$$

which, by proposition 3.2.22 and proposition 3.2.23, is equivalent to the Hesse flow associated to the Hesse potential $\widehat{\mathscr{F}^x}$, i.e., the attractor flow lines are the flow lines on which the gradient of the Hesse potential $\widehat{\mathscr{F}^x}$ vanishes identically.

The gradient above is taken with respect to the following Kähler metric on $\mathcal{B}^0$

$$g_{\mathcal{B}^0} = Im\left(da_{D,1}d\bar{a}^1 + da_{D,2}d\bar{a}^2\right)$$

Then by proposition 3.2.3, the above attractor flow equation is solved by

$$Im\left(e^{-i\theta}Z_{\mathbf{u}}(\gamma)\right) = 0 \tag{5.2.23}$$

In $\mathbb{Z}$-affine coordinates (y_i, y^i) on $\mathcal{B}^0_\theta$, the flow lines are straight affine lines

$$g^1\, y_1(t) + g^2\, y_2(t) + q_1\, y^1(t) + q_2\, y^2(t) = 0 \tag{5.2.24}$$

here the charge vector γ is given by

$$\gamma = g^1\beta_1 + g^2\beta_2 + q_1\alpha^1 + q_2\alpha^2 \in H_1(\mathcal{C}_{\mathbf{u}}, \mathbb{Z})$$

Further more, the central charge function is given by

$$Z_{\mathbf{u}}(\gamma) = g^1 a_{D,1}(\mathbf{u}) + g^2 a_{D,2}(\mathbf{u}) + q_1 a^1(\mathbf{u}) + q_2 a^2(\mathbf{u}) \tag{5.2.25}$$

Since we know that the "mass function" $m_{\mathbf{u}}(\gamma) := |Z_{\mathbf{u}}(\gamma)|$ decrease along the flow lines, and the central charge function is free of critical points by proposition 3.2.3, we see that the flow lines terminate (possibly after some splittings, to be discussed later) at the locus where the central charge function vanishes. In other words, as in the $SU(2)$ case, if the BPS state with the charge γ exits at $\mathbf{u} \in \mathcal{B}$, then the attractor flow associated to $(\mathbf{u}, \gamma)$ would eventually terminate at some places situated at the discriminant locus Δ_Λ.

Walls of the first kind: Let us investigate the possible wallS of first kind (wall of marginal stability in physics) $\mathcal{W}^1$. we expect that on certain components of the wall $\mathcal{W}^1$, some BPS states decay into $\nu_i^\pm$s. As each pair $\nu_i^\pm$ corresponds to a $SU(2)$ copy of the situation away from singular locus (see remark 4.2.23), we have the following three *walls of marginal stability*:

$$\mathcal{W}^1(\nu_k^+, \nu_k^-) := \left\{\mathbf{u} = (u, v) \in \mathbb{C}^2 \, : \, Im\left(\frac{Z_{\mathbf{u}}(\nu_k^+)}{Z_{\mathbf{u}}(\nu_k^-)}\right) \equiv 0\right\} \tag{5.2.26}$$

Denote by $\mathcal{W}^1_k$ the wall $\mathcal{W}^1(\nu_k^+, \nu_k^-)$ defined as above, then by the explicate expression of these vanishing cycles in (5.2.16), these walls are described as

$$\begin{aligned}
\mathcal{W}^1_1 &= \left\{\mathbf{u} = (u, v) \in \mathbb{C}^2 \, : \, Im\left(\frac{a_{D,1}}{2a^1 - a^2}\right) \equiv 0\right\} \\
\mathcal{W}^1_2 &= \left\{\mathbf{u} = (u, v) \in \mathbb{C}^2 \, : \, Im\left(\frac{a_{D,2}}{2a^2 - a^1}\right) \equiv 0\right\} \\
\mathcal{W}^1_3 &= \left\{\mathbf{u} = (u, v) \in \mathbb{C}^2 \, : \, Im\left(\frac{a_{D,1} + a_{D,2} + a^2 - 2a^1}{a_{D,1} + a_{D,2} + 2a^2 - a^1}\right) \equiv 0\right\}
\end{aligned} \tag{5.2.27}$$

The above are possible walls near the discriminant locus Δ_Λ, they are of real dimension three and has topology type $S^1 \times \mathbb{C}$ (as had been pointed out in [AD95]) for large moduli $\mathbf{u}$. We will prove this fact later (see proposition 5.2.2).

When restricted to the hyperplane $H_0 = \{Im(v) = 0\}$ inside $\mathcal{B} \cong \mathbb{C}^2$, the above walls have the cylinder topology, i.e., $S^1 \times \mathbb{R}$. Numerical test about their shapes are given in [CAM10], we list some of them as below (see figure 5.19 and figure 5.20):

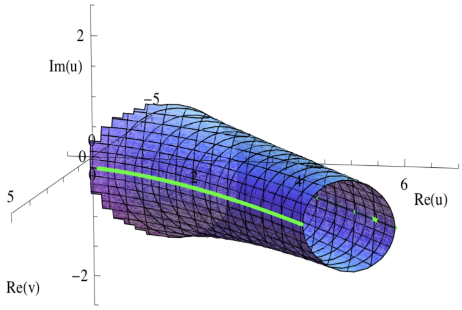

Figure 5.19: $\mathcal{W}_2^1$ away from singular locus when $Im(v) = 0$, $Re(v) > 0$

The change of the value of $Im(v)$ (i.e., foliating the moduli space by $Im(v) \equiv constants$.) has the effect that the above wall are being shifted. We have the illustration for this below taken from [CAM10].

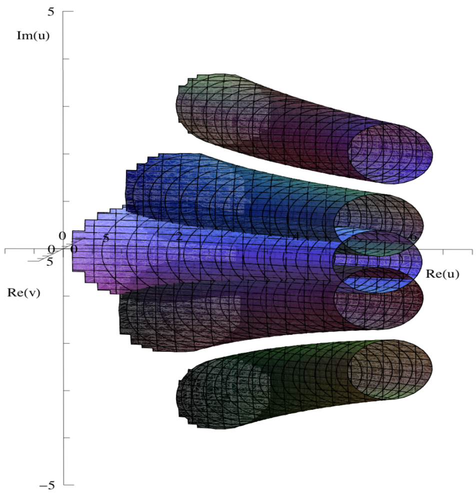

Figure 5.20: Wall $\mathcal{W}_2^1$ away from singular locus when $Im(v) = 3, 1, 0, -1, -3$, $Re(v) > 0$

Indeed, the moduli variable v is "frozen" in this situation. The plane $\mathcal{H}$ is endowed with the coordinate u. Notice that cutting the moduli space further with the plane H_{cut} in figure 5.19 corresponds to taking a slice of cylinder (wall of the first kind in $Im(v) = 0$ hyperplane), which is of course of topological type S^1. Also notice that the green lines in the figure are nothing but the singular lines. Thus the two intersection points $P_k^{\pm}$ plays exactly same role as $\pm\Lambda^2$ played in the $SU(2)$ case.

Away from the discriminant locus Δ_Λ, we are entering into the weak coupling region. By the form of *weak coupling spectrum* given in the last subsection, and since the charges are written in terms of the simple roots α_1 and α_2, we infer that the wall (of the first kind) in the weak coupling spectrum region consists of a single wall, namely

$$\mathcal{W}^1(\alpha_1,\alpha_2) = \left\{(u,v)\in\mathbb{C}^2 : Im\left(\frac{\alpha_1\cdot\mathbf{a}}{\alpha_2\cdot\mathbf{a}}\right) = Im\left(\frac{2a^1-a^2}{2a^2-a^1}\right)\equiv 0\right\} \tag{5.2.28}$$

A mathematical argument supporting this fact will be given later in proposition 5.2.5. This wall then separates the moduli space $\mathcal{B}^0$ within the weak coupling region into two parts in which the towers of BPS states $Q^n_{3\pm}$ sit respectively (see figure 5.21 below).

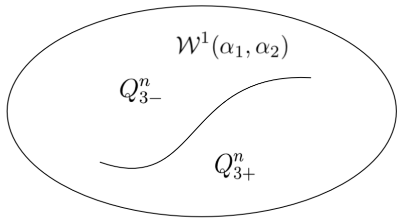

Figure 5.21: Wall in the weak coupling region that separates $Q^n_{3\pm}$

5.2.3 Algorithm for computing the BPS invariants

Now we put our discussion in the framework of WCS. To get a WCS on the moduli space $\mathcal{B}\cong\mathbb{C}^2$ that is suitable for our purpose, we consider the following map

$$\begin{aligned} Y : \mathcal{B}\times\ S_\theta &\to \Gamma^*_{\mathbb{R}} \\ (\mathbf{u},\theta) &\mapsto Y_\theta(\mathbf{u})(\gamma) := Im(e^{-i\theta}Z_{\mathbf{u}}(\gamma)) \end{aligned} \tag{5.2.29}$$

Choosing $\theta = Arg\, Z_u(\gamma)$, we see that the attractor flow lines lie at the locus where the central charge has constant phase θ. And the attractor flow lines on the base $\mathcal{B}$ can be lifted to the flows on the following two sub-spaces of the total space of the local system of lattice $\underline{\Gamma}$ over $\mathcal{B}$:

$$\begin{cases} \mathcal{B}^{0\prime} := \{(u,v)\in tot\,\underline{\Gamma}_{\mathbb{R}} : Y(u)(v) = 0\} \\ \mathcal{B}^{0\prime}_{\mathbb{Z}} := \{(u,\gamma)\in tot\,\underline{\Gamma} : Y(u)(\gamma) = 0\} \end{cases}$$

From previous subsection (see formulas (5.2.16), (5.2.20), (5.2.21)), we know that the **tail set** $\mathcal{T}_{\mathcal{B}^{0\prime}_{\mathbb{Z}}}\subset\mathcal{B}^{0\prime}_{\mathbb{Z}}$ in this case reads

$$\mathcal{T}_{\mathcal{B}^{0\prime}_{\mathbb{Z}}} = \left(\Delta_\Lambda, \nu^{\pm}_{k=1,2,3}\right)\subset\mathcal{B}^{0\prime}_{\mathbb{Z}} \tag{5.2.30}$$

where $\nu^{\pm}_{k=1,2,3}$ denotes the six vanishing cycles associated to the discriminant.

Remark 5.2.3. *The tail set in $SU(3)$ case differs considerably from the $SU(2)$ case in that its projection to the base $\mathcal{B}^0$ is high dimensional manifold instead of just two points $\pm\Lambda^2$ in $SU(2)$ case. Despite this, the associated BPS states, nevertheless, are finite in number, as in $SU(2)$ case. Besides, the set of attractor points where the split attractor flows terminate, belongs to the discriminant locus. Thus the* ***initial data*** *of the WCS, which is given by the restriction of the map $a : tot\underline{\Gamma}\to\Gamma^*_{\mathbb{R}}$ to the tail set, is given in $SU(3)$ case by*

$$a_{\mathbf{u}}(\nu^{\pm}_k) = DT(\nu^{\pm}_k)\cdot e_{\nu^{\pm}_k}\in\underline{\mathfrak{g}}_{\mathbf{u},\nu^{\pm}_k}\quad \mathbf{u}\in\Sigma^k_{\pm}. \tag{5.2.31}$$

That is, the local systems attached to the initial data are typically trivial of rank one. And by the remark 3.2.14, we can assign the invariants

$$\Omega_{\mathbf{u}}(\nu_k^{\pm}) = 1 \quad \mathbf{u} \in \Sigma_{\pm}^k. \tag{5.2.32}$$

The tail assumption and mass function are easily seen to be true in this case.

Given the initial data above, the BPS invariants in $SU(3)$ case can be similarly computed as in the $SU(2)$ case. That is, we need to construct all the rooted attractor trees with root $(\mathbf{u}, \gamma) \in \underline{\Gamma}_u$ such that their edges are given by the attractor flow lines associated to $(\mathbf{u}, \gamma)$. Each internal vertex should have valency at least three and lies in the wall of the first kind $\mathcal{W}^1$.

Moreover, the "**balance condition**" at each such internal vertex v must be satisfied. That is, at each such internal vertex v, if the incoming edge at v is the attractor flow associated to γ^{in}, and the out coming edges are associated to charges γ_i^{out}, then we have that at each internal vertex v

$$\gamma^{in} = \sum_i \gamma_i^{out} \tag{5.2.33}$$

Finally, under the assumption that the number of the attractor trees considered above is finite, we start at the attractor points of the union of above trees, moving backward toward (u, γ), and at each inner vertex, we apply the KSWCF

$$\overleftarrow{\prod} \mathcal{K}_{\gamma^{out}}^{\Omega_v(\gamma^{out})} = \overrightarrow{\prod} \mathcal{K}_{\gamma^{in}}^{\Omega_v(\gamma^{in})} \tag{5.2.34}$$

so that at each inner vertex, $\Omega_v(\gamma^{in})$ can be computed from $\Omega_v(\gamma^{out})$. we will arrive at the desired $\Omega_u(\gamma)$ by induction.

In order to make this algorithm works, we first need to verify the **finiteness assumption**, i.e., given $(\mathbf{u}, \gamma) \in \underline{\Gamma}_{\mathbf{u}}$, we need to show that the number of attractor trees rooted at $\mathbf{u}$ and with the root edge given by the attractor flow line in the direction of γ should be finite. Let us denote the set of such attractor trees by $Att(\mathbf{u}, \gamma)$, now we state the following proposition.

Proposition 5.2.1. *For the $SU(3)$ theory being considered in this chapter, the number of elements in the set $Att(\mathbf{u}, \gamma)$ is finite.*

Proof. First, we notice that for those charge vectors $\nu_k^{\pm}$ associated to the vanishing cycles, the associated attractor trees $Att(\mathbf{u}, \nu_k^{\pm})$ consists of a single flow line that starts at $\mathbf{u}$, and ends at where the vanishing cycle $\nu_k^{\pm}$ is situated. The terminal point is uniquely determined by the initial point $\mathbf{u}$, as well as the direction vector at $\mathbf{u}$ given by $\imath(\nu_k^{\pm})$. Now consider the case for general $\gamma \in \underline{\Gamma}_{\mathbf{u}}$. Since the "mass"function $m_{\mathbf{u}}(\gamma) = |Z_{\mathbf{u}}(\gamma)|$ is decreasing along the flow line $\mathcal{L}_\gamma$, thus, if $\mathcal{L}_\gamma$ terminates at discriminant locus Δ_Λ without spitting along the way, then $Att(\mathbf{u}, \gamma)$ consists of a single flow $\mathcal{L}_\gamma$. In this case, γ must be of the form $\nu_k^{\pm}$.

However, in general, we expect the scenery that the attractor flow will split several times when it finally terminates at Δ_Λ. This corresponds to the splitting of the charge vector γ at various components of walls of the first kind $\mathcal{W}^1$. We argue that there are only finitely many possibilities, and therefore only finitely many attractor trees possible. Indeed, an attractor tree in $Att(\mathbf{u}, \gamma)$ corresponds to a split of the following type:

$$\gamma \to \gamma_1 + \gamma_2 \to (\gamma_{11} + \gamma_{12}) + (\gamma_{21} + \gamma_{22}) \to$$

$$\to (\gamma_{111} + \gamma_{112}) + (\gamma_{121} + \gamma_{122}) + (\gamma_{211} + \gamma_{212}) + (\gamma_{221} + \gamma_{222}) \to \cdots$$

where for each split indicated by " $\to$ ", the balancing condition should be satisfied, for example $\gamma = \gamma_1 + \gamma_2$, $\gamma_2 = \gamma_{21} + \gamma_{22}$, ect.

The final split should give those charge vectors among the ones associated with vanishing cycles $\nu_k^{\pm}$, so that the terminal edges of the attractor flow trees would land on the attractor points as required. It is then clear that there are only finitely many possible splits of the charge vector γ as above, thus only finitely many possible attractor tress corresponding to them.

This shows that there are only finitely many combinatorial types of the attractor trees associated to $(\mathbf{u}, \gamma)$. To finish the proof, we still need to show that for each attractor tree, there are only finitely many edges of it, this can be seen from the following argument.

Suppose that to the contrary, that there are infinitely many possible tree edges, i.e., infinitely many splitting points, then the set of splitting points has an accumulation point which we denote by $\mathbf{u}^*$, and it hits the discriminant locus Δ_Λ.

Then by the proof in proposition 3.2.12, we know that the length of the attractor flow line connecting point on the wall of the first kind and on the point outside the wall must be positive as the "mass"function $m_{\mathbf{u}}(\gamma) = |Z_{\mathbf{u}}(\gamma)|$ is decreasing along the flow line. However, in the situation being considered here, this length can be arbitrarily close to zero when approaching the accumulating point. The contradiction shows the proposition. □

Remark 5.2.4. *As the moduli space is of real four dimensional, it is very hard to visualize the situation. However, as we have noted before that the essential information regarding the WCS retains if we restrict to generic hyperplane section of the moduli space. By doing this, the problem is reduced to the three dimensional scenery, and the situation becomes much more manageable. Moreover the discriminant locus Δ_Λ, after intersecting with the hyperplane, becomes a bunch of one-dimensional curves (see figure 4.26), and each branch is associated with the (quantum) monodromies that are essentially all that could be captured by the (quantum) discriminant locus Δ_Λ.*

Moreover, each pair of singular lines $\Sigma_\pm^k$ corresponds to one way of embedding $SU(2) \hookrightarrow SU(3)$. Consequently, away from the singularities of the discriminant locus, by intersecting with another plane that are complement to the original hyperplane, we will get a local situation that resembles the one in $SU(2)$ case, so we can apply the WCS in $SU(2)$ case for this local situation. And by "patching" together these local situation, we will get the global WCS for the $SU(3)$ SW integrable system.

5.2.4 WCS in the strong coupling region

Strong coupling spectrum: The *strong coupling region* in $SU(3)$ case is characterized by small values of u and v (c.f.,[Gal+13]). It is known that in this region ([Kle+94][KTL95][Kle+95]) there exists only six BPS states with vanishing "mass" function.

These states correspond to the six vanishing cycles $\nu_i^{\pm}$, $i = 1, 2, 3$ that are the invariant cycles under the Picard-Lefschetz monodromies around the six singular lines $\Sigma_{\pm}^i$, $i = 1, 2, 3$ below.

This can be seen by cutting the base by hyperplane $H = \{Imv = 0\}$, in which Δ_Λ becomes three pairs of lines $\mathscr{L}_k = \Sigma_+^k \cup \Sigma_-^k$, $k = 1, 2, 3$ (see [KTL95], from which the figure 4.25 is taken).

There exists a symplectic basis such that in this basis ([Hol97]):

$$\begin{cases} \nu_1^+ = (1, 0; -2, 1) & \nu_1^- = (1, 0; 0, 0) \\ \nu_2^+ = (0, 1; 0, 0) & \nu_2^- = (0, 1; -1, 2) \\ \nu_3^+ = (1, 1; -2, 1) & \nu_3^- = (1, 1; -1, 2) \end{cases}$$

In the strong coupling region $\mathcal{B}_{\text{strong}}$ in $SU(3)$ case, where the moduli $\mathbf{u} := (u, v)$ becomes small, we shall show that the WCS will give us exactly six BPS states $\nu_k^{\pm}$ for $k = 1, 2, 3$, corresponding to the six vanishing cycles in the theory.

To facilitate our investigation, we will not cut the moduli space $\mathcal{B}$ by two complementary hyper-planes as in the investigation in the weak coupling region case, instead, as what had been done in section 4.2.4, we intersect the moduli space by a three sphere S_R^3 centered at the origin, and with radius R large that Λ^3, then, the (quantum) discriminant locus Δ_Λ, after the cut, will become two trefoil knots linked together. The weak coupling region corresponds to large Λ^3, while the strong coupling region to be investigated below corresponding to the radius being close to Λ^3.

Then the walls of the first kind $\mathcal{W}_k^1$, $k = 1, 2, 3$, as defined in (5.2.27), when restricted on the three sphere S_R^3, become the family of closed curves that each of them touch the two trefoil knots in exactly one point, that is, it looks like a tubular neighborhood of the trefoil knot. For illustration, please see the following picture taken from *wikipedia*, the trefoil knot entry.

Figure 5.22: The wall of first kind in strong coupling region

It is clear then that the region bounded by this wall would be the strong coupling region where there exists only six BPS states corresponding to the six vanishing cycles. Because inside this region, there are no possible walls for the attractor flows to split, thus, only for this states with charge vector $\nu_k^{\pm}$s could the associated split attractor flows exist that land on the location where $\nu_k^{\pm}$ vanishes.

If, as before, we cut our moduli space $\mathcal{B}$ by two complementary hyper-planes, and concentrate the WCS on the resulting two plane

$$\mathcal{H} := \mathcal{B} \cap H \cap H_{\text{cut}}$$

where $H = \{Im(v) \equiv 0\}$ and $H_{\text{cut}} := \{z = Re(v) \equiv \mathit{constant}\}$. The constant in this case is allowed to be close to Λ^3.

For $z \neq 0, \pm\Lambda^3$, and given $\mathbf{u} \in \mathcal{H}$, and $\gamma \in \underline{\Gamma}_{\mathbf{u}}$, then the associated attractor flow $\mathcal{L}_\gamma$ exists only for $\gamma = \nu_k^{\pm}$, $k = 1, 2, 3$, which consists a single flow line starting at $\mathbf{u}$ and terminates at the place where γ vanishes. And the direction of the flow line corresponds to the monodromy invariant direction, where the monodromy is given by the Picard-Lefschtz formula associated to the vanishing cycle.

For $z = 0$, then in this case $v = 0$, thus we have a single moduli u as in $SU(2)$ case. This corresponds to the three $\mathbb{Z}_2$ points p_k, $k = 1, 2, 3$, where two mutually local BPS states have vanishing mass simultaneously. And the three walls $\mathcal{W}_k^1$, $k = 1, 2, 3$ all shrink to a point. Thus there are no wall crossing phenomenon near $\mathbb{Z}_2$ points.

Finally, for $z = \pm\Lambda^3$, we are at the two $\mathbf{Z}_3$ points where three mutually non local BPS states have vanishing mass simultaneously. At these two points, the three walls $\mathcal{W}_k^1, k = 1, 2, 3$ merge together to touch each other in exactly one point, which is exactly the $\mathbb{Z}_3$ point where the three cycles vanishing simultaneously.

More precisely, at the $\mathbb{Z}_3$ point $q_+ = (0, \Lambda^3)$, the three walls touch together such that the singular points P_k^+, $k = 1, 2, 3$ coincide, which means that the cycles ν_k^+, $k = 1, 2, 3$ vanish simultaneously. Similarly, at the $\mathbb{Z}_3$ point $q_- = (0, -\Lambda^3)$, the three walls touch together such that the singular points P_k^-, $k = 1, 2, 3$ coincide, which means that the cycles ν_k^-, $k = 1, 2, 3$ vanish simultaneously.

5.2.5 WCS in the weak coupling region

Recall that in section 4.2.4, we cut the moduli space by the hyperplane

$$H = \{Imv = 0\}$$

And Δ_Λ becomes three pairs of lines (4.2.106)

$$\mathscr{L}_1 = \Sigma^1_+ \cup \Sigma^1_- \quad \mathscr{L}_2 = \Sigma^2_+ \cup \Sigma^2_- \quad \mathscr{L}_3 = \Sigma^3_+ \cup \Sigma^3_- \tag{5.2.35}$$

I copy the figure 4.25 to here for convenience.

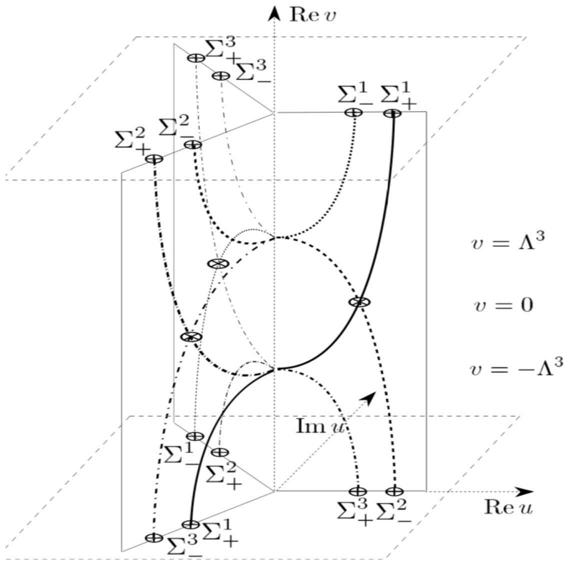

Figure 5.23: Geometry of moduli space for $SU(3)$ when $Im\, v = 0$, figure taken from [KTL95]

The **weakly coupled region** is the region on $\mathcal{B}$ where $|\alpha \cdot \mathbf{a}| \gg \Lambda$ for all positive roots α (c.f.,[FH97]).

In order to study the WCS in the weak coupling region, by remark 5.2.4 at the end of the last subsection, we cut the moduli space by the horizontal plane given by

$$H_{\text{cut}} := \{z = Re(v) \equiv constant >> \Lambda^3\}$$

which intersects with the six singular lines at $P_k^\pm := \Sigma^i_\pm \cap H_{\text{cut}}$.

With these preparation, let us investigate the WCS on the plane

$$\mathcal{H} := \mathcal{B} \cap H \cap H_{\text{cut}}$$

Then the walls of the first kind $\mathcal{W}^1_k$ (see (5.2.27)), when restricted to the plane $\mathcal{H}$, becomes a closed curve that passes through the points $P_k^\pm$.

Proposition 5.2.2. *The walls of the first kind $\mathcal{W}^1_k$, when restricted to the plane $\mathcal{H}$, is topologically homeomorphic to the circle S^1, and plays the same role as the wall of stability in $SU(2)$ case.*

Proof. The variable v is "frozen". $\mathcal{H}$ is endowed with coordinate u. We know from [KTL95] and [CAM10] that the period integrals $\mathbf{a}$ and $\mathbf{a_D}$ can be expressed in terms of Appell function

$$F_4(a,b,c,c',u,v) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_n(c')_m(1)_n(1)_m} u^m v^n$$

where $(a)_n$ is the Pochhammer symbol $(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)}$. However, if one of the variables u, v is set to be constant, the Appell function reduces to the hypergeometric function used in expressing the period integrals in $SU(2)$ case. Thus, the situation will be reduced to the $SU(2)$ case, to which the results about the shape of the $SU(2)$-wall applies. □

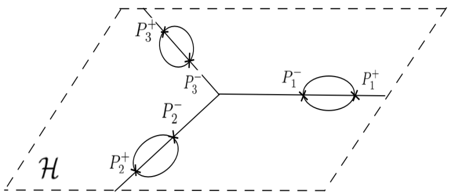

Figure 5.24: when restricted to $\mathcal{H}$, walls reduced to that in $SU(2)$ case.

Proposition 5.2.3. *The WCS formalism, when applied to $\mathcal{H}$, enables us to produce the following BPS states:*

$$\begin{cases} n\nu_k^+ + (n-1)\nu_k^-, & k=1,2,3;\ n=1,2,\cdots \\ \nu_k^+ + \nu_k^-, & k=1,2,3; \\ (n-1)\nu_k^+ + n\nu_k^-, & k=1,2,3;\ n=\cdots-3,-2,-1,1. \end{cases} \tag{5.2.36}$$

All states with DT-invariants $\Omega = 1$, except for the middle row states, which have DT-invariants $\Omega = -2$.

Proof. We know from (5.2.17) that $\langle \nu_k^+, \nu_k^- \rangle = 2$, for $k = 1, 2, 3$, thus by (2.2.12), we can apply the following KSWCF for $k = \langle \gamma_1, \gamma_2 \rangle = 2$ to each pair $\nu_k^{\pm}$

$$\mathcal{K}_{\gamma_2}\mathcal{K}_{\gamma_1} = \left(\prod_{n=1}^{\infty} \mathcal{K}_{n\gamma_1+(n-1)\gamma_2}\right) \mathcal{K}_{\gamma_1+\gamma_2}^{-2} \left(\prod_{n=-\infty}^{1} \mathcal{K}_{(n-1)\gamma_1+n\gamma_2}\right) \tag{5.2.37}$$

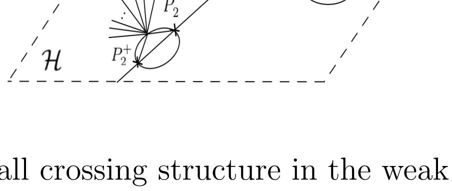

Figure 5.25: Wall crossing structure in the weak coupling region

□

More precisely, the above proposition says that in the weak coupling region $\mathcal{B}_{\text{weak}}$, i.e., where v becomes large compared with Λ^3, we have the following: Given $\mathbf{u} \in \mathcal{B}_{\text{weak}}$ and $\gamma \in \underline{\Gamma}_{\mathbf{u}}$, we consider the attractor flow associated to $(\mathbf{u}, \gamma)$. Then, for γ among those in (5.2.36), the flow line $\mathcal{L}_\gamma$ exists, which are in general (except for $\gamma = \nu_k^{\pm}$) splits according to the splitting of the charge vector as in (5.2.36), where the splitting points occur at $\mathcal{W}_k^1$, $k = 1, 2, 3$.

As an example, let us consider $\gamma = n\nu_k^+ + (n-1)\nu_k^- \in \underline{\Gamma}_{\mathbf{u}}$, the associated attractor flow tree then looks like the following

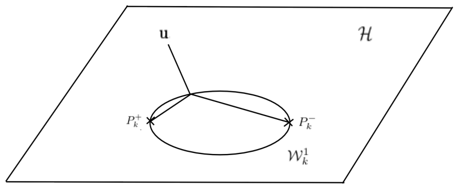

Figure 5.26: Attractor flow in $SU(3)$ WCS

where the two splitting flows that terminate at $P_k^{\pm}$ respectively are given by the flow lines associated to the charges $n\nu_k^+$ and $(n-1)\nu_k^-$ respectively.

Remark 5.2.5. *For those points inside the wall $\mathcal{W}_k^1$ in the weak coupling region, to show the above BPS states exist at this point, we can change the value of $Im(v)$, i.e., by choosing a different vertical intersection, we can move this point outside the wall, and then consider the associated attractor flow, we will show the existence of these BPS states. However, for the region in the moduli space where we restrain the value of v to be small, then this kind of "perturbation" of the point will not be possible, this will restrain the possible BPS states for those points in side the wall, and we are in the strong coupling region to be discussed in the previous subsection.*

Because in the weak coupling region, the three classical monodromies $\mathcal{M}_k^c$, $k = 1, 2, 3$ act on the charge vectors, thus by proposition 4.2.15 and the remark 4.2.16, we know that the charge vectors associated to the vanishing cycles, in the weak coupling region, are represented (see (4.2.60)) by the following

$$\nu_k^+ = (\alpha_k, (n+1)\,\alpha_k) \qquad \nu_k^- = (\alpha_k, n\,\alpha_k) \tag{5.2.38}$$

for $k = 1, 2$ that corresponds to the two simple roots α_1 and α_2, while for the root $\alpha_3 = \alpha_1 + \alpha_2$, the monodromy action on $\nu_3^{\pm}$ gives the following

$$\begin{cases} \nu_3^+ = (\alpha_3, n\alpha_3 - \alpha_1) \\ \nu_3^- = (\alpha_3, n\alpha_3 + \alpha_2) \end{cases} \tag{5.2.39}$$

Consequently, we expect that the attractor flow associated to these charge vectors also exist in the weak coupling region $\mathcal{B}_{\text{weak}}$. We show that the corresponding DT invariants associated to these charge vectors are all equal to one. To this end, we note that since

$$\langle (\alpha_1, (n-1)\alpha_1), (\alpha_2, n\alpha_2) \rangle = n\alpha_1 \cdot \alpha_2 - (n-1)\alpha_1 \cdot \alpha_2 = \alpha_1 \cdot \alpha_2 = 1$$

the KSWCF gives us the following **pentagon identity**

$$\mathcal{K}_{(\alpha_1,(n-1)\alpha_1)} \cdot \mathcal{K}_{(\alpha_2,n\alpha_2)} = \mathcal{K}_{(\alpha_2,n\alpha_2)} \cdot \mathcal{K}_{(\alpha_3,(n-1)\alpha_3+\alpha_2)} \cdot \mathcal{K}_{(\alpha_1,(n-1)\alpha_1)} \tag{5.2.40}$$

Similarly, we also have the following

$$\mathcal{K}_{(\alpha_2,(n-1)\alpha_2)} \cdot \mathcal{K}_{(\alpha_1,n\alpha_1)} = \mathcal{K}_{(\alpha_1,n\alpha_1)} \cdot \mathcal{K}_{(\alpha_3,(n-1)\alpha_3+\alpha_1)} \cdot \mathcal{K}_{(\alpha_2,(n-1)\alpha_2)} \tag{5.2.41}$$

These two pentagon identities imply that the DT invariants of the states in (5.2.38) and (5.2.39), being the exponents of the corresponding KS-transformation $\mathcal{K}$, equal to one.

The states $\nu_3^\pm$ in the formula (5.2.39) are separated in the weak coupling region by the wall $\mathcal{W}^1(\alpha_1, \alpha_2)$, whereas the states $\nu_k^\pm$, $k = 1, 2$ in the formula (5.2.38) are present in both sides being separated by the wall above. Indeed, we can build more complicated KSWCF that takes account this fact by using the above two basic pentagon identities.

Lemma 5.2.4.

$$\mathcal{K}_{(\alpha_1,n\alpha_1)} \cdot \mathcal{K}_{(\alpha_2,n\alpha_2)} = \mathcal{K}_{(\alpha_2,n\alpha_2)} \cdot \mathcal{K}_{(\alpha_1,n\alpha_1)} \tag{5.2.42}$$

Proof. This simply follows from

$$\langle(\alpha_1, n\alpha_1),\, (\alpha_2, n\alpha_2)\rangle = 0$$

thus the associated KS-transformations commute with each other. □

Next, we start with the following trivial identity

$$\prod_{n=-\infty}^{\infty} \mathcal{K}_{(\alpha_1,n\alpha_1)} \cdot \mathcal{K}_{(\alpha_1,(n+1)\alpha_1)} \cdot \mathcal{K}_{(\alpha_2,(n+1)\alpha_2)} \cdot \mathcal{K}_{(\alpha_2,(n+2)\alpha_2)}$$

$$= \prod_{n=-\infty}^{\infty} \mathcal{K}_{(\alpha_2,(n-2)\alpha_2)} \cdot \mathcal{K}_{(\alpha_2,(n-1)\alpha_2)} \cdot \mathcal{K}_{(\alpha_1,(n-1)\alpha_1)} \cdot \mathcal{K}_{(\alpha_1,n\alpha_1)} \tag{5.2.43}$$

Applying the above lemma, we can rewrite it as

$$\prod_{n=-\infty}^{\infty} \mathcal{K}_{(\alpha_1,n\alpha_1)} \cdot \mathcal{K}_{(\alpha_2,(n+1)\alpha_2)} = \prod_{n=-\infty}^{\infty} \mathcal{K}_{(\alpha_2,n\alpha_2)} \cdot \mathcal{K}_{(\alpha_1,(n-1)\alpha_1)} \tag{5.2.44}$$

Then we apply the pentagon identities (5.2.40) and (5.2.41), we get the following wall-crossing formula

$$\prod_{n=-\infty}^{\infty} \mathcal{K}_{(\alpha_2,(n+1)\alpha_2)} \cdot \mathcal{K}_{(\alpha_3,n\alpha_3+\alpha_2)} \cdot \mathcal{K}_{(\alpha_1,n\alpha_1)}$$

$$= \prod_{n=-\infty}^{\infty} \mathcal{K}_{(\alpha_1,(n-1)\alpha_1)} \cdot \mathcal{K}_{(\alpha_3,n\alpha_3-\alpha_1)} \cdot \mathcal{K}_{(\alpha_2,n\alpha_2)} \tag{5.2.45}$$

In terms of the attractor flows, the existence of these BPS states can be verified as follows: For BPS states with charge vectors $(\alpha_k, n\alpha_k)$, for $k = 1, 2$, since it can be written uniquely as

$$(\alpha_k, n\alpha_k) = -n\nu_k^+ + (n+1)\nu_k^-$$

Thus, we see that the corresponding attractor flow is a rooted tree with two terminal points landing where $\nu_k^{\pm}$ vanish, and splits at the wall $\mathcal{W}_k^1$. However, for charge vectors $(\alpha_3, n\alpha_3 + \alpha_2)$ and $(\alpha_3, n\alpha_3 - \alpha_1)$, the situation is different, we show that the associated split attractor flows consist of two tress in each case. First, note that we the two charge vectors can be written as

$$\begin{cases}(\alpha_3, n\alpha_3 + \alpha_2) = n\nu_3^+ - (n+1)\nu_3^- \\ (\alpha_3, n\alpha_3 - \alpha_1) = (n-1)\nu_3^+ - n\nu_3^-\end{cases} \tag{5.2.46}$$

Thus, the corresponding attractor flow, for each of the above charge vector in the weak coupling region, consists of a single rooted tree with exactly two terminal points terminating at the locus where $\nu_3^{\pm}$ vanish. The illustration of the attractor flows is similar to that given in figure 5.25. The two charge vectors above can also be written as follows

$$\begin{cases}(\alpha_3, n\alpha_3 + \alpha_2) = (\alpha_1, n\alpha_1) + (\alpha_2, (n+1)\alpha_2) \\ (\alpha_3, n\alpha_3 - \alpha_1) = (\alpha_1, (n-1)\alpha_1) + (\alpha_2, n\alpha_2)\end{cases} \tag{5.2.47}$$

The splitting of the charge vectors is represented as follows

$$(\alpha_3, n\alpha_3 + \alpha_2) \to (\alpha_1, n\alpha_1) + (\alpha_2, (n+1)\alpha_2) \to$$

$$\to (-n\nu_1^+ + (n+1)\nu_1^-) + (-(n+1)\nu_2^+ + (n+2)\nu_2^-)$$

And similarly, for $(\alpha_3, n\alpha_3 - \alpha_1)$, we have that

$$(\alpha_3, n\alpha_3 - \alpha_1) \to (\alpha_1, (n-1)\alpha_1) + (\alpha_2, n\alpha_2) \to$$

$$\to (-(n-1)\nu_1^+ + n\nu_1^-) + (-n\nu_2^+ + (n+1)\nu_2^-)$$

Therefore, the corresponding attractor flow first splits on the wall $\mathcal{W}^1(\alpha_1, \alpha_2)$ at $\mathbf{u}^*$, and then splits at $\mathbf{u}_1^*$ and $\mathbf{u}_2^*$ on $\mathcal{W}_1^1$ and $\mathcal{W}_2^1$ respectively. See the left of the below picture (figure 5.27) for the split attractor tree $\mathcal{L}_{(\alpha_3, n\alpha_3+\alpha_2)}$ in this case, while the right one denote the other possible splitting attractor tree discussed above.

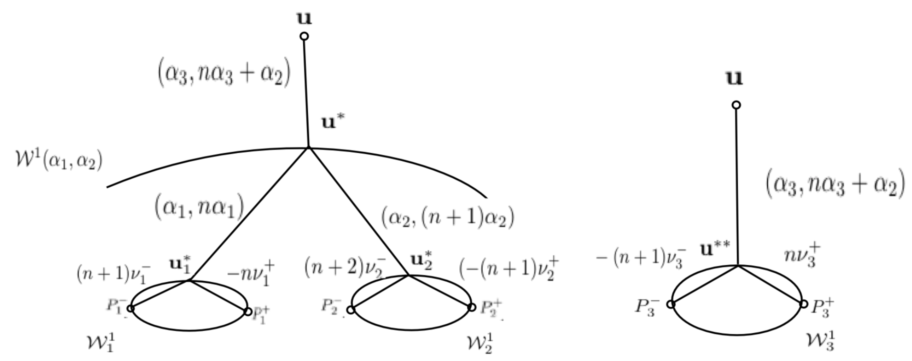

Figure 5.27: Two split attractor trees for $(\alpha_3, n\alpha_3 + \alpha_2)$

To illustrate, we compute the DT-invariant $\Omega_{\mathbf{u}}((\alpha_3, n\alpha_3 + \alpha_2))$ by applying the algorithm to the above splitting attractor tree.

Indeed, for the left picture of the above graph (see figure 5.27), we see that the **initial data** of the WCS consists four "vanishing" charge vectors $\{\nu_1^\pm, \nu_2^\pm\}$ with the DT invariants all equal to one.

Then move toward the splitting points $\mathbf{u}_1^*$ and $\mathbf{u}_2^*$, and applying KSWCF for

$$\langle \nu_k^+, \nu_k^- \rangle = 2, \;\; k = 1, 2$$

we get the the DT invariants for $(\alpha_1, n\alpha_1)$ and $(\alpha_2, (n+1)\alpha_2)$ are both equal to one. Namely, we have that

$$\Omega_{\mathbf{u}_1^*}((\alpha_1, n\alpha_1)) = 1 \qquad \Omega_{\mathbf{u}_2^*}((\alpha_2, (n+1)\alpha_2)) = 1$$

Finally, we move toward the splitting point $\mathbf{u}^*$, and apply the KSWCF there, what we will get is nothing but the Pentagon identity, which further implies that the DT invariant

$$\Omega_{\mathbf{u}}((\alpha_3, n\alpha_3 + \alpha_2)) = 1$$

And for the attractor flow tree in the right of the above figure, by applying the KSWCF once at the only splitting point $\mathbf{u}^{**}$, we will get the same result.

We will have a similar picture for the attractor tree $\mathcal{L}_{(\alpha_3, n\alpha_3 - \alpha_1)}$, so we omit the discussion here.

Interaction among split attractor flows

$SU(3)$ case is interesting in that when $\Lambda \to 0$, there exists wall $\mathcal{W}^1_{\alpha_1,\alpha_2}$ responsible for the "decay" of states like the following

$$(\alpha_3, \alpha_1 + (n-1)\alpha_3) = (\alpha_1, n\alpha_1) + (\alpha_2, (n-1)\alpha_2)$$

which means that the state with charge $(\alpha_3, \alpha_1 + (n-1)\alpha_3)$, besides being created by "scattering" ν_3^+ and ν_3^-, can also be created on one side of $\mathcal{W}^1_{\alpha_1,\alpha_2}$ by "scattering" $(\alpha_1, n\alpha_1)$ and $(\alpha_2, (n-1)\alpha_2)$ (the relevant KSWCF is given by the pentagon identity).

Proposition 5.2.5. *([Tay01][Hol97][Kuc08]) The wall (of the first kind) at weak coupling region is given by*

$$\mathcal{W}^1_{\alpha_1,\alpha_2} = \left\{ (u, v) \in \mathbb{C}^2 : Im\left(\frac{\alpha_1 \cdot \mathbf{a}}{\alpha_2 \cdot \mathbf{a}}\right) = Im\left(\frac{a^1}{a^2}\right) \equiv 0 \right\}$$

Proof. It is known ([Hol97]) that in the weak coupling region, we have the following expression

$$\mathbf{a}_D(\mathbf{a}) = \frac{i}{2\pi} \sum_{\alpha \text{ positive roots}} \alpha(\alpha \cdot \mathbf{a}) \left[\ln\left(\frac{\alpha \cdot \mathbf{a}}{\Lambda}\right)^2 \right]$$

Then, as $\Lambda \to 0$ (or equivalently, $|\alpha \cdot \mathbf{a}| \gg 0$ for all positive roots α), $\mathbf{a}_D \sim (C \cdot \ln \Lambda)\, \mathbf{a}$ for some constant C.Thus, in this limit, $|\mathbf{a}_D| \gg |\mathbf{a}|$, which implies that $Z(\gamma) = \mathbf{a}_D \cdot \mathbf{g} + \mathbf{a} \cdot \mathbf{q} \approx (\ln \Lambda)\, \mathbf{a} \cdot \mathbf{g}$. This means that in this region, the central charge is dominated by the magnetic charge, thus follows the proposition. $\square$

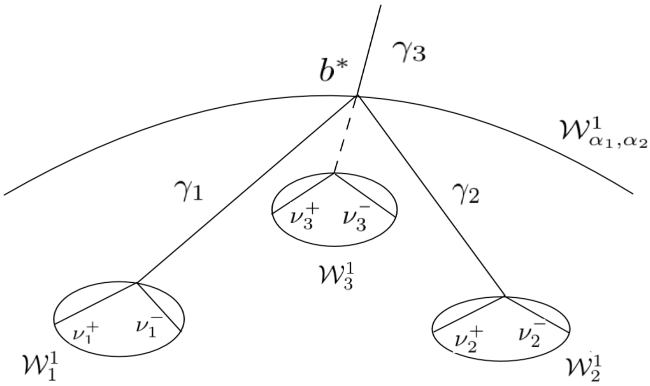

Figure 5.28: Attractor trees for γ_3 outside $\mathcal{W}_{\alpha_1,\alpha_2}$

Therefore, on one side of $\mathcal{W}^1_{\alpha_1,\alpha_2}$ where the walls $\mathcal{W}^1_k$'s are present, the states with charge $(\alpha_3, \alpha_1 + (n-1)\alpha_3)$ exist with DT-invariant equals to 1 (following from $k = 2$ KSWCF), while on the other side, the same states derive their existence in two ways.

Figure 5.28 above illustrates the corresponding split attractor flow. Thus, we expect that the DT-invariant $\Omega((\alpha_3, \alpha_1 + (n-1)\alpha_3))$ to "jump" from 1 to 2 after crossing the wall of the first kind: $\mathcal{W}^1_{\alpha_1,\alpha_2}$.

The KSWCF to be used at the split point b^* is given by the proposition 5.2.6 below.

Using short notations:

$$\gamma_1 := (\alpha_1, n\alpha_1) \quad \gamma_2 := (\alpha_2, (n-1)\alpha_2) \quad \gamma_3 := (\alpha_3, \alpha_1 + (n-1)\alpha_3)$$

we prove the following wall-crossing formula:

Proposition 5.2.6. *The KSWCF at the split point b^* is given by:*

$$\mathcal{K}_{\gamma_1}\mathcal{K}_{\gamma_3}\mathcal{K}_{\gamma_2} = \Xi^+ \, \mathcal{K}^2_{\gamma_3}\mathcal{K}^{-2}_{2\gamma_3} \, \Xi^- \, \mathcal{K}_{\gamma_1} \tag{5.2.48}$$

where

$$\Xi^+ := \prod_{n=1}^{\infty} \mathcal{K}_{n\gamma_2+(n-1)(\gamma_1+\gamma_3)} \, \mathcal{K}_{n\gamma_2+(n-1)(\gamma_1+\gamma_3)+\gamma_3}$$

$$\Xi^- := \prod_{n=\infty}^{1} \mathcal{K}_{(n-1)\gamma_2+n(\gamma_1+\gamma_3)+\gamma_3} \, \mathcal{K}_{(n-1)\gamma_2+n(\gamma_1+\gamma_3)}$$

Proof. Since $\langle\gamma_1, \gamma_2\rangle = \langle\gamma_1, \gamma_3\rangle = 1$, by applying the pentagon identity, we have

$$\mathcal{K}_{\gamma_1}\mathcal{K}_{\gamma_3}\mathcal{K}_{\gamma_2} = \mathcal{K}_{\gamma_3}\mathcal{K}_{\gamma_1+\gamma_3}\mathcal{K}_{\gamma_1}\mathcal{K}_{\gamma_2} = \mathcal{K}_{\gamma_3}\mathcal{K}_{\gamma_1+\gamma_3}\mathcal{K}_{\gamma_2}\mathcal{K}_{\gamma_1+\gamma_2}\mathcal{K}_{\gamma_1} \tag{5.2.49}$$

Note that

$$\langle\gamma_1 + \gamma_3, \gamma_2\rangle = \langle\gamma_1, \gamma_2\rangle + \langle\gamma_3, \gamma_2\rangle = 2$$

so by applying KSWCF for $k=2$, the right hand side of (5.2.49) becomes

$$\mathcal{K}_{\gamma_3}\prod_{n=1}^{\infty}\mathcal{K}_{n\gamma_2+(n-1)(\gamma_1+\gamma_3)}\,\mathcal{K}^{-2}_{\gamma_1+\gamma_2+\gamma_3}\prod_{n=\infty}^{1}\mathcal{K}_{(n-1)\gamma_2+n(\gamma_1+\gamma_3)}\,\mathcal{K}_{\gamma_3}\,\mathcal{K}_{\gamma_1}=P\,\mathcal{K}^{-2}_{2\gamma_3}\,Q\,\mathcal{K}_{\gamma_1}$$

where

$$P:=\mathcal{K}_{\gamma_3}\prod_{n=1}^{\infty}\mathcal{K}_{n\gamma_2+(n-1)(\gamma_1+\gamma_3)}$$

and

$$Q:=\prod_{n=\infty}^{1}\mathcal{K}_{(n-1)\gamma_2+n(\gamma_1+\gamma_3)}\,\mathcal{K}_{\gamma_3}$$

We now simplify P and Q. By repeated use of pentagon identities, we get that

$$\begin{aligned}
P&=\mathcal{K}_{\gamma_3}\left(\mathcal{K}_{\gamma_2}\,\mathcal{K}_{2\gamma_2+(\gamma_1+\gamma_3)}\,\mathcal{K}_{3\gamma_2+2(\gamma_1+\gamma_3)}\cdots\right)\\
&=\mathcal{K}_{\gamma_2}\,\mathcal{K}_{\gamma_2+\gamma_3}\,\mathcal{K}_{\gamma_3}\,\mathcal{K}_{2\gamma_2+(\gamma_1+\gamma_3)}\,\mathcal{K}_{3\gamma_2+2(\gamma_1+\gamma_3)}\,\mathcal{K}_{4\gamma_2+3(\gamma_1+\gamma_3)}\cdots\\
&=\mathcal{K}_{\gamma_2}\,\mathcal{K}_{\gamma_2+\gamma_3}\,\mathcal{K}_{2\gamma_2+(\gamma_1+\gamma_3)}\,\mathcal{K}_{2\gamma_2+(\gamma_1+2\gamma_3)}\,\mathcal{K}_{\gamma_3}\,\mathcal{K}_{3\gamma_2+2(\gamma_1+\gamma_3)}\,\mathcal{K}_{4\gamma_2+3(\gamma_1+\gamma_3)}\cdots\\
&=\mathcal{K}_{\gamma_2}\,\mathcal{K}_{\gamma_2+\gamma_3}\,\mathcal{K}_{2\gamma_2+(\gamma_1+\gamma_3)}\,\mathcal{K}_{2\gamma_2+(\gamma_1+2\gamma_3)}\,\mathcal{K}_{3\gamma_2+2(\gamma_1+\gamma_3)}\,\mathcal{K}_{3\gamma_2+2(\gamma_1+\gamma_3)+\gamma_3}\,\mathcal{K}_{\gamma_3}\,\mathcal{K}_{4\gamma_2+3(\gamma_1+\gamma_3)}\cdots\\
&=\left(\mathcal{K}_{\gamma_2}\,\mathcal{K}_{\gamma_2+\gamma_3}\right)\left(\mathcal{K}_{2\gamma_2+(\gamma_1+\gamma_3)}\,\mathcal{K}_{2\gamma_2+(\gamma_1+\gamma_3)+\gamma_3}\right)\left(\mathcal{K}_{3\gamma_2+2(\gamma_1+\gamma_3)}\,\mathcal{K}_{3\gamma_2+2(\gamma_1+\gamma_3)+\gamma_3}\right)\mathcal{K}_{\gamma_3}\,\mathcal{K}_{4\gamma_2+3(\gamma_1+\gamma_3)}\cdots\\
&=\cdots=\prod_{n=1}^{\infty}\mathcal{K}_{n\gamma_2+(n-1)(\gamma_1+\gamma_3)}\,\mathcal{K}_{n\gamma_2+(n-1)(\gamma_1+\gamma_3)+\gamma_3}\,\mathcal{K}_{\gamma_3}=\Xi^{+}\,\mathcal{K}_{\gamma_3}
\end{aligned}\tag{5.2.50}$$

Similarly, we have that

$$\begin{aligned}
Q&=\left(\cdots\mathcal{K}_{3\gamma_2+4(\gamma_1+\gamma_3)}\,\mathcal{K}_{2\gamma_2+3(\gamma_1+\gamma_3)}\,\mathcal{K}_{\gamma_2+2(\gamma_1+\gamma_3)}\,\mathcal{K}_{\gamma_1+\gamma_3}\right)\mathcal{K}_{\gamma_3}\\
&=\cdots\mathcal{K}_{3\gamma_2+4(\gamma_1+\gamma_3)}\,\mathcal{K}_{2\gamma_2+3(\gamma_1+\gamma_3)}\,\mathcal{K}_{\gamma_2+2(\gamma_1+\gamma_3)}\,\mathcal{K}_{\gamma_3}\,\mathcal{K}_{\gamma_1+2\gamma_3}\,\mathcal{K}_{\gamma_1+\gamma_3}\\
&\cdots\mathcal{K}_{3\gamma_2+4(\gamma_1+\gamma_3)}\,\mathcal{K}_{2\gamma_2+3(\gamma_1+\gamma_3)}\,\mathcal{K}_{\gamma_3}\,\mathcal{K}_{\gamma_2+2(\gamma_1+\gamma_3)+\gamma_3}\,\mathcal{K}_{\gamma_2+2(\gamma_1+\gamma_3)}\\
&\cdots\mathcal{K}_{3\gamma_2+4(\gamma_1+\gamma_3)}\,\mathcal{K}_{\gamma_3}\,\mathcal{K}_{2\gamma_2+3(\gamma_1+\gamma_3)+\gamma_3}\,\mathcal{K}_{2\gamma_2+3(\gamma_1+\gamma_3)}\,\mathcal{K}_{\gamma_2+2(\gamma_1+\gamma_3)}\\
&=\cdots=\mathcal{K}_{\gamma_3}\prod_{n=\infty}^{1}\mathcal{K}_{(n-1)\gamma_2+n(\gamma_1+\gamma_3)+\gamma_3}\,\mathcal{K}_{(n-1)\gamma_2+n(\gamma_1+\gamma_3)}=\mathcal{K}_{\gamma_3}\,\Xi^{-}
\end{aligned}\tag{5.2.51}$$

Putting these together, we get finally that

$$\mathcal{K}_{\gamma_1}\,\mathcal{K}_{\gamma_3}\,\mathcal{K}_{\gamma_2}=P\,\mathcal{K}^{-2}_{2\gamma_3}\,Q\,\mathcal{K}_{\gamma_1}=\Xi^{+}\,\mathcal{K}_{\gamma_3}\,\mathcal{K}^{-2}_{2\gamma_3}\,\mathcal{K}_{\gamma_3}\,\Xi^{-}\,\mathcal{K}_{\gamma_1}=\Xi^{+}\,\mathcal{K}^{2}_{\gamma_3}\mathcal{K}^{-2}_{2\gamma_3}\,\Xi^{-}\,\mathcal{K}_{\gamma_1}$$

□

Remark 5.2.6. *The proposition 5.2.6 above yields the desired "jump" of the BPS invariants:*

$$\Omega(\gamma_3)=\Omega((\alpha_3,\alpha_1+(n-1)\alpha_3))$$

which is compatible with the ***"primitive wall crossing formula"*** *(see for example [MPS11]) that gives the "jump"*

$$\Delta\Omega(\gamma_3\to\gamma_1+\gamma_2)=(-1)^{\langle\gamma_1,\gamma_2\rangle+1}|\langle\gamma_1,\gamma_2\rangle|\,\Omega(\gamma_1)\,\Omega(\gamma_2)=1\tag{5.2.52}$$

Proposition 5.2.7. *In parallel with the above argument, since we have that*

$$(-\alpha_3, \alpha_2 + (n-1)\alpha_3) = (-\alpha_1, (n-1)\alpha_1) + (-\alpha_2, n\alpha_2)$$

we infer that on one side of $\mathcal{W}^1_{\alpha_1,\alpha_2}$, where $\mathcal{W}^1_k$'s are present, the states with charge $(-\alpha_3, \alpha_2 + (n-1)\alpha_3)$ exist with DT-invariant equals to 1, while on the other side, the same states derive their existence in two ways.

Thus, we expect the DT-invariant $\Omega((-\alpha_3, \alpha_2 + (n-1)\alpha_3))$ to "jump" from 1 to 2 after crossing the wall $\mathcal{W}^1_{\alpha_1,\alpha_2}$.

Proof. The same as the proof of the proposition 5.2.6 above. □

Yet, there exist states that are present only on one side of $\mathcal{W}^1_{\alpha_1,\alpha_2}$, which are "scattered" by $(\alpha_1, n\alpha_1)$ and $(\alpha_2, (n+1)\alpha_2)$.

Indeed, by applying pentagon identity, we see that after crossing the wall, new states with charge $(\alpha_3, \alpha_2 + n\alpha_3)$ exist with DT invariants one.

The BPS spectrum for pure $SU(3)$ gauge theory thus obtained by using WCS is consistent with that obtained by physical approaches ([FH97][Hol97][Kuc08]).

Appendices

Appendix A

BPS states in 4d, N=2 theories

In order to motivate the emergence of wall-crossing structures discussed in great detail in the first chapter, we need to review a little bit about the relevant physics background. We will see what a BPS state is and how to associate the invariant (BPS degeneracies) to it. Also, we will use examples to show the phenomena of wall-crossing, i.e., how these invariants "jump" (corresponding to particle creations and decays) when the background parameters are being changed in the moduli space of the theory. Especially, we will give a short account of pure $SU(2)$ supersymmetric Yang-Mills theory.

Recall that for any quantum theory, there is associated Hilbert space describing the "particle states" of the theory. The theories to be discussed in this section describe particle states with charges taking in some charge lattice Γ, thus the Hilbert space $\mathcal{H}$ in these theories are graded by the charge lattice Γ

$$\mathcal{H} = \bigoplus_{\gamma \in \Gamma} \mathcal{H}_\gamma \tag{A.0.1}$$

where the subsector $\mathcal{H}_\gamma$ describes the states with charge γ.

However, the Hilbert space $\mathcal{H}$ sometimes contains too much information, it would be desirable if we could extract some useful information from it. By doing this, we hope that the Hilbert space could be under good control on the one hand, and still provides sufficient information to infer the essential features of the theory in question. Although this would be hard for general quantum field theory, but it turned out that for theories enjoying the *supersymmetry* (SUSY), this could be done.

The results the study of the so-called BPS states which are much more tractable. We will focus here on the theories with $N = 2$ supersymmetric in four dimension because they always posse holomorphic structures so that the powerful tools of complex analysis can be applied to extract useful information. This manifests especially notable in $SU(2)$ Yang-Mills theory, which can be solved exactly by the celebrated Seiberg-Witten solutions [SW94]. Besides, the BPS states can even led us into the non-perturbative realm of physics.

A.1 $4d, N = 2$ supersymmetry algebra

Supersymmetry (SUSY) is a symmetry between bosons with integer spins and fermions with half integer spins. In four dimension, the $N = 2$ SUSY $\mathfrak{s}$ extends the usual Poincaré symmetry $\mathfrak{poin}(1,3)$ in the following way

$$\mathfrak{s} = \mathfrak{s}^0 \oplus \mathfrak{s}^1$$

where the *bosonic algebra* $\mathfrak{s}^0$ is the even part of $\mathfrak{s}$, while the *fermionic algebra* $\mathfrak{s}^1$ is the odd part of $\mathfrak{s}$. More concretely,

$$\mathfrak{s}^0 = \mathfrak{poin}(1,3) \oplus \mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_R \oplus \mathbb{C}$$

here the R-symmetry part $\mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_R$ accounts for the ambiguity in rotating the supercharges into one another and the phase shift of the supercharge. And the $\mathbb{C}$ in the above formula, being the center of the algebra, is represented as the central charge Z of the theory, which is a linear function from the charge lattice to the complex plane, i.e.,

$$Z \in Hom_{\mathbb{Z}}(\Gamma, \mathbb{C})$$

The odd part $\mathfrak{s}^1$, generated by two supercharges Q^A_α and $\bar{Q}^A_{\dot{\alpha}}$, mapping bosons to fermions and vice verse, give a representation of the bosonic algebra. As a representation of $\mathfrak{s}^0$, we have

$$\mathfrak{s}^1 = (2,1;2)_{+1} \oplus (1,2;2)_{-1}$$

By writing $\mathfrak{poin}(1,3)$ as $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$, the first two representations in the parentheses above refers to a singlet or a fundamental of the $\mathfrak{su}(2)$ in the Lorentz group, while the third representation means the fundamental of the R-symmetry $\mathfrak{su}(2)_R$. Besides, the $\pm$ sign refers to the charges associated to $\mathfrak{u}(1)_R$. In particular, under $\mathfrak{u}(1)_R$, Q^A_α has charge $+1$ while $\bar{Q}^A_{\dot{\alpha}}$ has charge -1.

The fermionic part $\mathfrak{s}^1$ consists of the conserved supercharges Q^A_α, and denote the conjugated supercharges as $(Q^A_\alpha)^\dagger := \bar{Q}^A_{\dot{\alpha}}$, where the lower indices $\alpha = 1,2$ and $\dot{\alpha} = 1,2$ denote the spinor indices and $A = 1,2$ the number of supersymmetries. As usual, we can raise and lower indices by using the Levi-civita tensor ϵ_{AB} with $\epsilon^{12} = -\epsilon^{21} = \epsilon_{21} = -\epsilon_{12} = 1$, i.e.,

$$\bar{Q}_{\dot{\alpha}A} = \epsilon_{AB}\bar{Q}^B_{\dot{\alpha}}$$

These supersymmetry (odd) generators satisfy the following anti-commutation relations

$$\begin{aligned}
\{Q^A_\alpha, Q_{\dot{\beta}B}\} &= 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^A_B \\
\{Q^A_\alpha, Q^B_\beta\} &= 2\epsilon_{\alpha\beta}\epsilon^{AB}\bar{Z} \\
\{\bar{Q}_{\dot{\alpha}A}, \bar{Q}_{\dot{\beta}B}\} &= -2\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{AB} Z
\end{aligned} \tag{A.1.1}$$

where $P_\mu = (E, \vec{p})^T$ is the Lorentz four-momentum vector, and $\sigma^\mu_{\alpha\dot{\beta}}$ the Pauli-matrices

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

A.2 Particle representations

The particle states in the Hilbert space $\mathcal{H}$ (A.0.1) should be the representations of the SUSY algebra $\mathfrak{s}$. In this subsection, we concentrate on the *massive representation*, i.e., representations that satisfy $P^2 = M^2$, where P^2 is the Casimir operator and $M > 0$. Again, we will mimic the Wigner's "little group" methods to construct the single-particle representations of the SUSY algebra.

Thus, we first bring the particle with mass $M > 0$ in its rest frame where the only non-trivial component of the four momentum P is $P^0 = E = M$ (in the units that the speed of light c=1), i.e., it represents the one particle state characterized by

$$P^m|\Psi\rangle = M\delta_0^m|\Psi\rangle \tag{A.2.1}$$

then by Wigner's construction, the little superalgebra in this case becomes

$$\mathfrak{s}_l^0 \oplus \mathfrak{s}^1$$

where the Bosonic part $\mathfrak{s}_l^0 = \mathfrak{so}(3) \oplus \mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_R$, with $\mathfrak{so}(3)$ being the little algebra of $\mathfrak{poin}(1,3)$.

The states satisfying (A.2.1) form a finite dimensional representation of the little superalgebra, and the fermionic part $\mathfrak{s}^1$ acts as Clifford algebra. We will construct them in the following.

First notice that a particle in its rest frame in invariant under the operation of spatial parity, denoted by $\mathcal{P}$, which acts on the supercharges as

$$\mathcal{P}(Q_\alpha^A) = \sigma^0_{\alpha\dot{\beta}}\bar{Q}^{\dot{\beta}B}, \quad \mathcal{P}(\bar{Q}^{\dot{\beta}A}) = \sigma^{0\dot{\beta}\alpha}Q_{\alpha A} \tag{A.2.2}$$

Besides, the $\mathfrak{u}(1)_R$ part, being represented by the operator $\mathcal{U}$, can change the phase of the supercharge Q as

$$\mathcal{U}(Q) = \zeta Q, \quad \mathcal{U}(\bar{Q}) = \zeta^{-1}\bar{Q} \tag{A.2.3}$$

Define the composition of the two operations as $\mathcal{I}(\zeta) := \mathcal{U} \circ \mathcal{P}$, which is easily seen to be an involution. Thus, it is desirable to decompose the fermionic algebra $\mathfrak{s}^1$ under this involution, for this, we define the following new generators of the SUSY transformations

$$\mathcal{R}_\alpha^A := \zeta^{1/2}Q_\alpha^A + \zeta^{-1/2}\sigma^0_{\alpha\dot{\beta}}\bar{Q}^{\dot{\beta}A} \tag{A.2.4}$$

$$\mathcal{T}_\alpha^A := \zeta^{1/2}Q_\alpha^A - \zeta^{-1/2}\sigma^0_{\alpha\dot{\beta}}\bar{Q}^{\dot{\beta}A} \tag{A.2.5}$$

with eigenvalues $+1$ and -1 respectively under the involution $\mathcal{I}(\zeta)$, then the fermionic algebra $\mathfrak{s}^1$ canbe splited into two partes generated by $\mathcal{R}_\alpha^A$ and $\mathcal{T}_\alpha^A$ respectiely, i.e.,

$$\mathfrak{s}^1 = \mathfrak{s}^{1,+} \oplus \mathfrak{s}^{1,-}$$

The anti-commutation relations between these generators can be computed by using the relations (A.1.1), for example

$$\{\mathcal{R}_\alpha^A, \mathcal{R}_\beta^B\} = \{\zeta^{1/2}Q_\alpha^A + \zeta^{-1/2}\sigma^0_{\alpha\dot{\gamma}}\bar{Q}^{\dot{\gamma}A}, \zeta^{1/2}Q_\beta^B + \zeta^{-1/2}\sigma^0_{\beta\dot{\delta}}\bar{Q}^{\dot{\delta}B}\}$$

$$
\begin{aligned}
&= \zeta\{Q^A_\alpha, Q^B_\beta\} + \zeta^{-1}\sigma^0_{\alpha\dot\gamma}\sigma^0_{\beta\dot\delta}\,\{\bar Q^{\dot\gamma A}, \bar Q^{\dot\delta} B\} \\
&\quad +\sigma^0_{\alpha\dot\gamma}\,\{Q^B_\beta, \bar Q^{\dot\gamma} A\} + \sigma^0_{\beta\dot\delta}\,\{Q^A_\alpha, \bar Q^{\dot\delta B}\} \\
&= 2\zeta\bar Z\epsilon_{\alpha\beta}\epsilon^{AB} + \zeta^{-1}\sigma^0_{\alpha\dot\gamma}\sigma^0_{\beta\dot\delta}\,\epsilon^{AC}\epsilon^{BD}\epsilon^{\dot\gamma\dot\lambda}\epsilon^{\dot\delta\dot\kappa}\,\{\bar Q_{\lambda C}, \bar Q_{\dot\kappa D}\} \\
&\quad +\sigma^0_{\alpha\dot\gamma}\,\epsilon^{\dot\gamma\dot\lambda}\epsilon^{AC}\,\{Q^B_\beta, \bar Q_{\dot\lambda C}\} + \sigma^0_{\beta\dot\delta}\,\epsilon^{\dot\delta\dot\kappa}\epsilon^{BD}\,\{Q^A_\alpha, \bar Q_{\dot\kappa D}\} \\
&= 2\zeta\bar Z\epsilon_{\alpha\beta}\epsilon^{AB} - 2\zeta^{-1}Z\sigma^0_{\alpha\dot\gamma}\sigma^0_{\beta\dot\delta}\,\epsilon^{AC}\epsilon^{BD}\epsilon^{\dot\gamma\dot\lambda}\epsilon^{\dot\delta\dot\kappa}\epsilon_{\dot\lambda\dot\kappa}\epsilon_{CD} \\
&\quad +2M\sigma^0_{\alpha\dot\gamma}\,\epsilon^{\dot\gamma\dot\lambda}\epsilon^{AC}\,\sigma^0_{\beta\dot\lambda}\delta^B_C + 2M\sigma^0_{\beta\dot\delta}\,\epsilon^{\dot\delta\dot\kappa}\epsilon^{BD}\,\sigma^0_{\alpha\dot\kappa}\delta^A_D \\
&= (2\zeta\bar Z + 2\zeta^{-1}Z + 4M)\,\epsilon^{AB}_{\alpha\beta} = 4(M + Re(Z/\zeta))\epsilon_{\alpha\beta}\epsilon^{AB}
\end{aligned}
$$

Performing similar computations for the other generators, we will get

$$
\{\mathcal{R}^A_\alpha, \mathcal{R}^B_\beta\} = 4(M + Re(Z/\zeta))\,\epsilon_{\alpha\beta}\epsilon^{AB} \tag{A.2.6}
$$

$$
\{\mathcal{T}^A_\alpha, \mathcal{T}^B_\beta\} = 4(-M + Re(Z/\zeta))\,\epsilon_{\alpha\beta}\epsilon^{AB} \tag{A.2.7}
$$

The above relations would yield very important consequences, indeed by the Hermiticity conditions

$$
(\mathcal{R}^A_\alpha)^\dagger = \epsilon^{\alpha\beta}\epsilon_{AB}\mathcal{R}^B_\beta
$$

we have the following

$$
(\mathcal{R}^1_1 + (\mathcal{R}^1_1)^\dagger)^2 = (\mathcal{R}^2_1 + (\mathcal{R}^2_1)^\dagger)^2 = 4(M + Re(Z/\zeta))^2 \tag{A.2.8}
$$

Since the LHS is always non-negative, we obtain the BPS bound as follows:

$$
M + Re(Z/\zeta) \geq 0 \tag{A.2.9}
$$

In particular, if the central charge Z has phase α, i.e., $Z = e^{i\alpha}|Z|$, and choose ζ to be $-e^{i\alpha}$, we will get the so-called *BPS bound*:

$$
M \geq |Z| \tag{A.2.10}
$$

and by this choice of ζ, the anti-commutation relations (A.2.6) and (A.2.7) specialize into

$$
\{\mathcal{R}^A_\alpha, \mathcal{R}^B_\beta\} = 4(M - |Z|)\,\epsilon_{\alpha\beta}\epsilon^{AB} \tag{A.2.11}
$$

$$
\{\mathcal{T}^A_\alpha, \mathcal{T}^B_\beta\} = -4(M + |Z|)\,\epsilon_{\alpha\beta}\epsilon^{AB} \tag{A.2.12}
$$

In particular, in this case we have also the following relation

$$
\{\mathcal{R}^A_\alpha, \mathcal{T}^B_\beta\} = 0 \tag{A.2.13}
$$

Because we have already fixed the gauge ζ in deducing the above relations, thus the $\mathfrak{u}(1)_R$ symmetry is broken, and the above relations define a representation of the new little algebra

$$
\tilde{\mathfrak{s}^0_l} = \mathfrak{so}(3) \oplus \mathfrak{su}(2)_R \tag{A.2.14}
$$

To describe the representation of the SUSY algebra $\mathfrak{s}$, we can now distinguish between two cases, namely, the $M \geq |Z|$ case, which yield the so called “non-BPS” or “long” representations; and the $M = |Z|$ case, which yields the “BPS” or “short” representations.

Long representations

In this case, $M > |Z|$, thus by the relations (A.2.11), (A.2.12) and (A.2.13), we now have Clifford module structures. In order to see this more clearly, let us focus on the $\mathcal{R}$-generators case, since the same conclusions hold for the $\mathcal{T}$-generators. First, the equation (A.2.11), written in its components, gives us

$$\{\mathcal{R}_1^1, \mathcal{R}_1^1\} = \{\mathcal{R}_2^2 = \mathcal{R}_2^2\} = 0 \qquad \{\mathcal{R}_1^1, \mathcal{R}_2^2\} = 4(M - |Z|)$$

as well as

$$\{\mathcal{R}_1^2, \mathcal{R}_1^2\} = \{\mathcal{R}_2^1 = \mathcal{R}_2^1\} = 0 \qquad \{\mathcal{R}_1^2, \mathcal{R}_2^1\} = -4(M - |Z|)$$

We get two copies of commuting Clifford algebras, and $\mathfrak{s}^{1,+}$ is the tensor product of the above two Clifford algebras. By treating $\mathcal{R}_1^A$, $A = 1, 2$ as annihilation operator, we can define the Clifford vacuum $|\Omega\rangle$ as

$$\mathcal{R}_1^A|\Omega\rangle = 0 \quad A = 1, 2$$

thus we get the following basis of the $\mathcal{R}$-Clifford module as

$$\{|\Omega\rangle, \mathcal{R}_2^1|\Omega\rangle, \mathcal{R}_2^2|\Omega\rangle, \mathcal{R}_2^1\mathcal{R}_2^2|\Omega\rangle\}$$

which spans a four dimensional irreducible Clifford representation, which is usually denoted by ρ_{hh}, called the *half-hypermultiplet*.

As noted before, the $\mathcal{R}$-Clifford module should also be a module of the new little algebra $\tilde{\mathfrak{s}_l^0}$ (see (A.2.14). Indeed, we see that

$$T_{(\alpha\beta)} := \frac{1}{2}\mathcal{R}_\alpha^A \mathcal{R}_\beta^B \epsilon_{AB}$$

generates a copy of $\mathfrak{so}(3) \cong \mathfrak{su}(2)$, while

$$T^{(AB)} := \frac{1}{2}\mathcal{R}_\alpha^A \mathcal{R}_\beta^B \epsilon^{\alpha\beta}$$

generates a copy of $\mathfrak{su}(2)$. As the Clifford generators transform in the $(\frac{1}{2}; \frac{1}{2})$ of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$, which is just the vector of $\mathfrak{so}(4)$, thus the half-hypermultiplet ρ_{hh} is the Dirac spinor, i.e.,

$$\rho_{hh} = (0; \frac{1}{2}) \oplus (\frac{1}{2}, 0) \tag{A.2.15}$$

It can be shown that the general representation of $\tilde{\mathfrak{s}_l^0} \oplus \mathfrak{s}^{1,+}$ is of the form

$$\rho_{hh} \oplus \mathfrak{h}$$

where $\mathfrak{h}$ is arbitrary representation of the little algebra $\tilde{\mathfrak{s}_l^0}$.

Now take the $\mathcal{T}$-generators into consideration, we finally get the *long representation* of $\tilde{\mathfrak{s}_l^0} \oplus \mathfrak{s}^1$ as

$$\mathfrak{t}_{long} = \rho_{hh} \otimes \rho_{hh} \otimes \mathfrak{h} \tag{A.2.16}$$

The general representation is then obtained from this rest frame one by a Lorentz boost.

Short (BPS) representations

When the BPS bound (A.2.10) is saturated, i.e., $M = |Z|$, then the $\mathcal{R}$-part (A.2.11) commutation relation vanishes, thus we just have one copy of Clifford module contributing to the full representation. Namely, the short or BPS representation

$$\mathfrak{t}_{BPS} = \rho_{hh} \otimes \mathfrak{h} \tag{A.2.17}$$

There are two special cases to be considered about the BPS representations. First, the *hypermultiplet* is obtained by letting $\mathfrak{h}$ to be the one dimensional trivial representation $(0;0)$. In this case, we are just left with ρ_{hh}. It contains a pair of scalars in a doublet of $\mathfrak{su}(2)_R$ and a Dirac fermion. By taking $\mathfrak{h} = (\frac{1}{2};0)$, we get the vector multiplet

$$\begin{aligned}\rho_{hh} \otimes (\frac{1}{2};0) &= [(\frac{1}{2};0) \otimes (\frac{1}{2};0)] \oplus [(0;\frac{1}{2}) \otimes (\frac{1}{2};0)] \\ &= (\frac{1}{2};\frac{1}{2}) \oplus (1;0) \oplus (0;0)\end{aligned}$$

which consists of a spinor doublet of $\mathfrak{su}(2)_R$, a vector and a complex scalar.

A.3 BPS states and BPS index

BPS states are one particle states that fall into the short (BPS) representations of the SUSY algebra. The Hilbert space of BPS states can thus be defined as

$$\mathcal{H}^{BPS} := \{|\Psi\rangle \in \mathcal{H} : H|\Psi\rangle = |Z|\Psi\rangle\} \tag{A.3.1}$$

where H denotes the Hamiltonian of the corresponding field theory.

Again, the space of BPS states is graded by the charge lattice Γ, in accordance with the decomposition (A.0.1), i.e.,

$$\mathcal{H}^{BPS} = \bigoplus_{\gamma \in \Gamma} \mathcal{H}^{BPS}_{\gamma} \tag{A.3.2}$$

The peculiar nature of the $d = 4, N = 2$ theories to be considered in this thesis is that the vacuum of the theory is in general not unique, but forms a space of vacua, denoted by $\mathcal{B}$, also called the *moduli space* of the theory. Thus, over each point $b \in \mathcal{B}$, there is a copy of BPS states space as above, and by varying b, we get a family of Hilbert spaces parametrized by $\mathcal{B}$. (A.3.1) and (A.3.2) above then get lifted into the following

$$\mathcal{H}^{BPS}_b := \{|\Psi\rangle \in \mathcal{H}_b : H|\Psi\rangle = |Z_b|\Psi\rangle\} \tag{A.3.3}$$

$$\mathcal{H}^{BPS}_b = \bigoplus_{\gamma \in \Gamma} \mathcal{H}^{BPS}_{b,\gamma} \tag{A.3.4}$$

Now, as b varies in the moduli space $\mathcal{B}$, a Non-BPS state that satisfies the BPS bound: $(M_b \geq |Z_b|)$ may becomes a sum of BPS states for some $b \in \mathcal{B}$ where the BPS bound becomes saturated. This kind of states are called *fake BPS* states, to distinguish them from the true BPS states. This can be illustrated by the following

$$\mathfrak{t}_{long} \to \mathfrak{t}_{fake-BPS} = \rho_{hh} \otimes \mathfrak{h}' = \rho_{hh} \otimes \rho_{hh} \otimes \mathfrak{h}$$

Compared with the true BPS states, these fake ones are more difficult to deal with, since they can appear and disappear when other parameters of the theory being varies. Thus, we want to have some means to distinguish the more stable BPS states from the fake ones, it turned out that this could be done by introducing the so-called *BPS index* to be characterized by the property that it vanishes on the long representations. And consequently it also vanishes on the fake BPS states.

Given a representation $\mathfrak{t}$ of the little superalgebra $\mathfrak{s}_l^0 \oplus \mathfrak{s}^1$, we can define its character χ with respect to the Cartan elements J_3 of $\mathfrak{so}(3)$ and I_3 of $\mathfrak{su}(2)_R$ as follows

$$\chi(\mathfrak{t}) := Tr_{\mathfrak{t}}\, q_1^{2J_3} q_2^{2I_3} \tag{A.3.5}$$

It is then easy to see that for $\mathfrak{t} = \mathfrak{t}_{long}$, the character becomes

$$\chi(\mathfrak{t}_{long}) = (q_1 + q_1^{-1} + q_2 + q_2^{-1})^2\, \chi(\mathfrak{h}) \tag{A.3.6}$$

while for short or BPS representations, it reads

$$\chi(\mathfrak{t}_{BPS}) = (q_1 + q_1^{-1} + q_2 + q_2^{-1})\, \chi(\mathfrak{h}) \tag{A.3.7}$$

We now consider the quantity derived from χ as follows

$$\tilde{\chi}(\mathfrak{t}) := q_1 \frac{\partial}{\partial q_1}\bigg|_{q_1=-q_2=y} \left(Tr_{\mathfrak{t}}\, q_1^{2J_3} q_2^{2I_3}\right) \tag{A.3.8}$$

Claim: This quantity satisfies the desired property that it vanishes on the long representations.

Proof: By (A.3.6), we have that

$$q_1 \frac{\partial}{\partial q_1} \chi(\mathfrak{t}_{long}) = 2(q_1 + q_1^{-1} + q_2 + q_2^{-1})(1 - 1/q_1^2)\, \chi(\mathfrak{h})$$

by evaluating at $q_1 = -q_2 = y$, we see it vanishes. $\square$.

we now give a more convenient form of the character.

$$\begin{aligned}
\tilde{\chi}(\mathfrak{t}) := q_1 \frac{\partial}{\partial q_1}\bigg|_{q_1=-q_2=y} \left(Tr_{\mathfrak{t}}\, q_1^{2J_3} q_2^{2I_3}\right) &= \frac{\partial}{\partial \log q_1}\bigg|_{q_1=-q_2=y} \left(Tr_{\mathfrak{t}}\, q_1^{2J_3} q_2^{2I_3}\right) \\
&= \frac{\partial}{\partial \log q_1}\bigg|_{q_1=-q_2=y} \left(Tr_{\mathfrak{t}}\, e^{2I_3 \log q_1} e^{2I_3 \log q_2}\right) \\
&= Tr_{\mathfrak{t}}(2J_3)(y)^{2J_3}(-y)^{2I_3} = Tr_{\mathfrak{t}}(2J_3)(-1)^{2J_3}(-y)^{2(J_3+I_3)} \\
&\overset{\mathcal{J}_3 := J_3 + I_3}{=\!=\!=} Tr_{\mathfrak{t}}(2J_3)(-1)^{2J_3}(-y)^{2\mathcal{J}_3}
\end{aligned} \tag{A.3.9}$$

From it, we easily calculated that

$$\tilde{\chi}(\mathfrak{t}_{BPS}) = (y - y^{-1})\tilde{\chi}(\mathfrak{h}) = (y - y^{-1})\, Tr_{\mathfrak{h}}\, y^{2J_3}(-y)^{2I_3}$$

Provided that each summand on the RHS of (A.3.4) is of finite dimensional, we can restrict the above trace to each summand, and get

$$Tr_{\mathcal{H}_{b,\gamma}}(2J_3)(-1)^{2I_3}(-y)^{2\mathcal{J}_3} \tag{A.3.10}$$

Definition A.3.1. *(Protected spin character): The protected spin character $\Omega_b(\gamma, y)$ associated to charge $\gamma \in \Gamma_b$ is defined through the following relation*

$$(y - y^{-1})\, \Omega_b(\gamma, y) := Tr_{\mathcal{H}_{b,\gamma}}(2J_3)(-1)^{2I_3}(-y)^{2\mathcal{J}_3} \tag{A.3.11}$$

From the definition, we see that for BPS state with charge γ, i.e. belongs to the short representation $\mathfrak{t}_{BPS}$, we have that $\Omega_b(\gamma, y) = 1$. Equivalently, we can write $\Omega_b(\gamma, y) := Tr_{\mathfrak{h}_b} y^{2J_3}\,(-y)^{2I_3}$. By specializing it to $y = -1$, we get the so-called *second helicity supertrace.* We will call it simply the BPS index.

Definition A.3.2. *(BPS index): The BPS index is defined as*

$$\Omega_b(\gamma) := Tr_{\mathfrak{h}_b}\, y^{2J_3}$$

Remark A.3.1. *In the first chapter, the index $\Omega_b(\gamma)$ defined above is also called BPS invariant (as they counts BPS states) since they are stable under generic variation of the parameter $b \in \mathcal{B}$. However, the index will "jump" when b crosses certain codimension one locus in $\mathcal{B}$, a phenomenon called Wall-Crossing, as had been formalized in the first chapter.*

Suppose we have two BPS states with charges γ_1 and γ_2 respectively, for generic $b \in \mathcal{B}$, the BPS index $\Omega_b(\gamma_1)$ and $\Omega_b(\gamma_2)$ stay constant, but for certain b in the moduli space $\mathcal{B}$, it may happen that the two states γ_1 and γ_2 may form a bound state $\gamma_1 + \gamma_2$, thus leave the one particle states space, this will cause the BPS indices would jump at such b, i.e., *Wall-crossing phenomena.*

Since the particles in question are all BPS, so the mass M (= energy E in the units where the speed of light: $c = 1$), consequently, for this process to be possible energetically at least, we must have the following (see the short note [Nei] for explanation):

$$E_{before} = E_{\gamma_1} + E_{\gamma_2} \leq E_{\gamma_1+\gamma_2} = E_{final}$$

which is the same as

$$|Z_b(\gamma_1)| + |Z_b(\gamma_2)| \leq |Z_b(\gamma_1 + \gamma_2)|$$

As Z is additive, by triangle inequality, we also have

$$|Z_b(\gamma_1 + \gamma_2)| = |Z_b(\gamma_1) + Z_b(\gamma_2)| \leq |Z_b(\gamma_1)| + |Z_b(\gamma_2)|$$

The two inequalities are compatible if and only if

$$|Z_b(\gamma_1) + Z_b(\gamma_2)| = |Z_b(\gamma_1)| + |Z_b(\gamma_2)|$$

This is possible when the phase of $Z_b(\gamma_1)$ equals that of $Z_b(\gamma_2)$, i.e.

$$Z_b(\gamma_1)/Z_b(\gamma_2) \in \mathbb{R}_+$$

Motivated by this, we can define the *wall of marginal stability* associated with γ_1 and γ_2 as

$$MS(\gamma_1, \gamma_2) := \{b \in \mathcal{B} : Z_b(\gamma_1)/Z_b(\gamma_2) \in \mathbb{R}_+\} \tag{A.3.12}$$

We may expect that when crossing the wall of stability, some states may form bound states, and some may decay into new states, causing the BPS indices to jump. The jumping is governed by the so-called Kontsevich-Soibelman Wall-Crossing Formula to be explained in chapter 2.

Remark A.3.2. *The wall of marginal stability would be called the wall of first kind in the next section.*

A.4 BPS states in type II superstring theory

In order to have $N = 2$ SUSY in string theoretical framework, we need to compactify the type II string theory on a Calabi-Yau three fold X. Locally, the moduli space of the theory would split into the product of vector multiplet $\mathcal{M}_{vector}$ moduli space and the hyermultiplet moduli space $\mathcal{M}_{hyper}$, i.e.,

$$\mathcal{M} = \mathcal{M}_{vector} \times \mathcal{M}_{hyper}$$

The dimension of the moduli spaces depend on the compactification. In type IIA compactification, the number of hypermultiplets is given by $h^{2,1}(X) + 1$, and the number of vector multiplets in this case is $h^{1,1}$; while in type IIB compactificaion, the number of hypermultiplets is given by $h^{1,1} + 1$, and that of the vector multiplets is $h^{2,1}$, here, $h^{i,j} := dimH^{i,j}(X, \mathbb{C})$ denotes the Hodge number of X.

Type IIA and type IIB compactifications are related by the well-known duality known as Mirror Symmetry.

In order to describe the charged particle state, we need to introduce the concepts of charged Dp-branes.

A Dp-brane is an extended object with $p+1$ dimensional world volume in $10d$ space-time. They are the objects on which open strings can end.

It turned out that the Dp-brane can also carry sorts of electromagnetic charges, in order to define them, let us first review what happens in classical electromagnetism.

Recall that the classical Maxwell equations read

$$\begin{cases} dF \quad = *J_m \\ d*F = *J_e \end{cases} \tag{A.4.1}$$

where J_m is the magenetic four-current, and J_e the electric four current, and " $*$ " denotes the Hodge duality operation in four dimension.

For a particle, the electric charge e and the magnetic charge g is defined through

$$e = \int_{S^2} *F \tag{A.4.2}$$

$$g = \int_{S^2} F \tag{A.4.3}$$

where S^2 is a sphere surrounding the particle source.

Similarly, for a Dp-brane in $D = 10$ dimension, which couples to a $(p+1)$ form $C^{(p+1)}$, and consequently, there is a corresponding $p+2$-form $F_{p+2} = dC^{P+1}$, which is the field strength. We surround the Dp-brane with a $D-(p+2)$-dimensional sphere, then the electric charge is defined as

$$q^p = \int_{S^{D-(p+2)}} *F_{p+2} \tag{A.4.4}$$

The magnetic dual of a Dp-brane is a $D(6-p)$-brane, and its magnetic charge is defined as

$$p_{6-p} = \int_{S^{p+2}} F_{p+2} \tag{A.4.5}$$

In type IIA compactification, the low energy spectrum contains RR 1-form $C^{(1)}$ and a RR 3-form $C^{(3)}$, which are coupled by $D0$-brane and $D4$-brane, thus we could have the following types of Dp-branes and the corresponding charges

•$D0$-branes with electric charge $q^0 = \int_{S^8} * dC^{(1)}$

•$D2$-branes with electric charge $q^2 = \int_{S^6} * dC^{(3)}$

•$D4$-branes with magnetic charge $p_4 = \int_{S^4} dC^{(3)}$

•$D6$-branes with magnetic charge $p_6 = \int_{S^2} dC^{(1)}$

Note that $D0$-branes are dual to $D6$-branes, while $D2$-branes are dual to $D4$-branes.

In type IIB compactification, the low energy spectrum contains a RR 0-form $C^{(0)}$, a RR 2-form $C^{(2)}$, and a RR 4-form $C^{(4)}$, which are coupled by $D(-1)$-brane (D-instanton), $D1$-brane and $D3$-brane respectively. Together with their magnetic duals, we thus have the following types of Dp-branes and the corresponding charges

•$D1$-branes with electric charge $q^1 = \int_{S^7} * dC^{(2)}$

•$D3$-branes with electric charge $q^3 = \int_{S^5} * dC^{(4)}$

•$D3$-branes with magnetic charge $p_3 = \int_{S^5} dC^{(4)}$

•$D5$-branes with magnetic charge $p_5 = \int_{S^3} dC^{(2)}$

Notice that in this case $D3$-branes are self dual.

With these preparations, we can try to describe what are the charged BPS particles in type II string theory.

Roughly speaking, they can be viewed as Dp-branes wrapping on cycles in the Calabi-Yau inner space X, which would yield a point-like objects moving in time from the 4d space-time perspective. In this way, we can "geometrilize" the point particle, or even black holes. Thus, by what had been said above, we conclude that

• In Type IIA, the particle states should be charged in the charge lattice $H^{even}(X, \mathbb{Z})$ with rank $h^{1,1}(X)$.

• In Type IIB, the particle states should be charged in the charge lattice $H^{odd}(X, \mathbb{Z})$ with rank $h^{2,1}(X)$.

We now explain briefly what should be the notion of BPS particles in this situation.

If a particle state is represented by a Dp-brane $\mathcal{D}$, then its mass can be identified (up to normalization) to be the volume of the brane $Vol(\mathcal{D})$.

The central charge function Z in both Type IIA and IIB theory involves an integrate some holomorphic $(3,0)$ form Ω over $\mathcal{D}$, i.e.

$$Z(\mathcal{D}) = \int_{\mathcal{D}} \Omega$$

In both cases, the BPS bound $M \geq |Z|$ easily seen to be hold, and for the states to be BPS, i.e. the BPS bound is saturated, we require the followings

- In Type IIA, the Dp-brane $\mathcal{D}$ needs to be wrapping special Lagrangian 3-cycles.

- In Type IIB, the Dp-brane $\mathcal{D}$ needs to be wrapping holomorphic cycles.

A.5 BPS states in $SU(n)$ SYM theory

A detailed review of the $d = 4, N = 2$ gauge theory would be beyond the scope of this thesis on the one hand, and irrelevant to the mathematical investigations later in later sections on the other hand, thus it is sufficient here to just outline its main features to the extent that the basis for later discussions could be laid down. The interested reader can consult various good expository papers for more details, for example[AH97] and the references contained therein.

Description of the classical moduli space

Given a $d = 4, N = 2$ supersymmetric Yang Mills theory (SYM for short) theory with compact Gauge group G, by gauge invariance, the theory posses a continuum of independent vacua (ground states), which form a manifold $\mathcal{M}_{class}$, called *the moduli space of vacua*, also named as the *classical moduli space* of the theory.

The *classical moduli space* $\mathcal{M}_{class}$ has complex dimension $r = rank(G)$, and it splits into the *Higgs branch*, where the gauge group G is completely broken, and the *Coulomb branch* where G breaks down to its maximal abelian subgroup. We will focus in the following the Coulomb branch, and denote it by $\mathcal{B}_{class}$.

In order to describe the (classical) Coulomb branch $\mathcal{B}_{class}$, we note that the scalar potential of the $N = 2$ SYM takes the following form

$$V(\phi) = \frac{1}{g^2} Tr\left([\phi^\dagger, \phi]^2\right)$$

where g is the coupling constant of the theory, and ϕ is a Higgs field, i.e. a complex field taking values in the adjoint representation of G that transforms as scalar under Poincaré group.

As $[\phi^\dagger, \phi]$ is self-adjoint, the minimum of $V(\phi)$ is obtained when it vanishes, thus ϕ can be diagonalized, i.e. can be rotated into the Cartan subalgebra $\mathfrak{t}$ of the complexified Lie algebra $\mathfrak{g}_\mathbb{C}$ corresponding to G. So the gauge group G generically breaks down to the subgroup H which is generated by elements coming from the Cartan subalgebra $\mathfrak{t}$. In other words, we see that the gauge transformations in G/H acts by changing the choice of this Cartan subalgebra, and consequently they also "permutate" different vacuum. But, there are elements in the Weyl group that leave $\mathfrak{t}$ invariant. So we need to use Weyl invariant coordinates on $\mathcal{B}_{class}$.

Since the Weyl group acts on ϕ by conjugation, thus the following characteristic polynomial of ϕ is manifestly invariant

$$det(\lambda - \phi) = 0 \tag{A.5.1}$$

Specializing to the case when $G = SU(n)$, so that $H = U(1)^{n-1}$ is the maximal torus in G, while the Weyl group in this case becomes $W = S(n)$-the symmetric group on n-letters.

Hence, (A.5.1) becomes in this case the following

$$det(\lambda - \phi) = \sum_{k=0}^{n} (-1)^k c_k(\phi) \lambda^{n-k}$$

$$= \lambda^n - \lambda^{n-1} c_1(\phi) + \lambda^{n-2} c_2(\phi) + \cdots + (-1)^n c_n(\phi) = 0$$

the coefficients $c_k(\phi)$ can be computed by performing the following formal computation

$$det(\lambda - \phi) = \lambda^n det(1 - \phi/\lambda) = \lambda^n e^{Tr \ln(1-\phi/\lambda)} =$$

$$= \lambda^n \exp\left(-\sum_{m=1}^{\infty} \frac{Tr\,(\phi^m)}{m\lambda^m}\right) \overset{Tr(\phi)=0}{=\!=\!=}$$

$$= \lambda^n - \frac{1}{2} Tr(\phi^2)\lambda^{n-2} - \frac{1}{3} Tr(\phi^3)\lambda^{n-3} + \cdots$$

From this we see that the classical moduli space $\mathcal{B}_{class}$ can be parametrized by the coordinates

$$u_k := \lim_{\vec{x} \to \infty} Tr\,(\phi^k(\vec{x})), \text{ for } k = 2, \cdots n. \tag{A.5.2}$$

Let $a_1, \cdots a_n$ denote the roots of the characteristic polynomial (A.5.1), then ϕ can be diagonalized as

$$\phi \sim diag(a_1, \cdots a_n) \quad \text{with} \quad \sum_i a_i = 0$$

then by using the relations $(Tr\,\phi)^k = 0$, we computed that

$$\begin{cases} c_1(\phi) = 0 \\ c_2(\phi) = \sum_{i<j} a_i a_j \\ c_3(\phi) = \sum_{i<j<k} a_i a_j a_k \\ \cdots \\ c_n(\phi) = \prod_{i=1}^n a_i \end{cases}$$

In particularly, when $G = SU(2)$, and suppose $\phi = \frac{1}{2}a\sigma_3$, where σ_3 is the third Pauli matrix, then the moduli space can be parametrized by

$$u = Tr(\phi^2) = \frac{1}{2}a^2 \tag{A.5.3}$$

and in the $SU(3)$ case, the moduli space can be parametrized by

$$u = \frac{1}{2}Tr(\phi^2) = -(a_1a_2 + a_1a_3 + a_2a_3) \;\; v = -\frac{1}{3}Tr(\phi^3) = a_1a_2a_3 \tag{A.5.4}$$

where we have written that $\phi \sim diag\,(a_1, a_2, a_3)$.

The Kähler metric on the moduli space

It is clear from the above description of good coordinates on the moduli space, we see that at low energy, for generic vacuum $u = (u_2, \cdots u_n)$, the Gauge group $SU(n)$ breaks down to its maximal abelian subgroup $U(1)^{n-1}$ by Higgs mechanism, thus we have a family of abelian gauge theories (of the Maxwell type) parametrized by $u \in \mathcal{B}_{class}$, while the full gauge symmetry get restored at the origin of the moduli space. The moduli space is easily seen to be $\mathbb{C}^{n-1}$, which can be compactified by adding points at infinity.

It can be shown that the low energy effective action is fully determined by a prepotential $\mathcal{F}(\mathcal{A})$, which is a holomorphic function in a single $N = 2$ vector multiplet $\mathcal{A}$, in $N = 1$ superspace formalism, the Lagrangian reads

$$\mathcal{L}_{eff} = \frac{1}{4\pi}\left[d^4\int \theta\frac{\partial\mathcal{F}}{\partial A^i}\bar{A}^i + \int d^2\theta\frac{1}{2}\frac{\partial^2\mathcal{F}}{\partial A_i\partial A_j}W^{\alpha i}W^j_\alpha\right] \tag{A.5.5}$$

where the chiral superfield A_i is the $N = 1$ chiral multiplet in the $N = 2$ vector multiplet $\mathcal{A}$, with its scalar components given by a_i. Further more the prepotential induces a Kähler metric on $\mathcal{B}_{class}$ by

$$ds^2 = g_{i\bar{j}}\, da_i d\bar{a}_j = Im\frac{\partial\mathcal{F}}{\partial a_i\partial a_j}da_i d\bar{a}_j \tag{A.5.6}$$

with the corresponding Kähler potential given by

$$K = Im\left(\bar{a}_i\frac{\partial\mathcal{F}}{\partial a_i}\right) \tag{A.5.7}$$

Denote by $\tau_{ij} := \frac{\partial\mathcal{F}}{\partial a_i\partial a_j}$, and call it the *period matrix*, the metric (A.5.6) can be written more compactly as

$$ds^2 = Im(\tau_{ij})\, da_i d\bar{a}_j \tag{A.5.8}$$

It follows that the period matrix should be a symmetric $n \times n$ matrix whose imaginary part is positive definite. By introducing the variables that dual to a_i, i.e.,

$$a^i_D := \frac{\partial\mathcal{F}}{\partial a_i}$$

the above metric becomes

$$ds^2 = Im\,(da^i_D d\bar{a}_i) \tag{A.5.9}$$

with corresponding Kähler potential becomes

$$K(a, \bar{a}) = Im(\bar{a}_i a_D^i) \tag{A.5.10}$$

Besides, the real symplectic structure on the moduli space can be defined as follows

$$\omega = \frac{i}{2}\partial\bar{\partial} K = \frac{i}{2}\partial\bar{\partial}\, Im(\bar{a}_i a_D^i) \tag{A.5.11}$$

Remark A.5.1. *We will discuss later in chpater 4 that this metric actually induces the so called special Kähler structure on the moduli space*

Remark A.5.2. *The positivity of $Im\,\tau_{ij}$ comes from the fact it is part of kinetic action of the bosonic piece*

The above metric posses some singularities, we illustrate this by focusing at the origin $u = 0$ of the moduli space, where the full gauge symmetry $SU(n)$ being restored. At such points, some massive states, which had been integrated out when constructing the effective description at low energy, will become massless, thus renders the effective description meaningless at such points, i.e. they should be viewed as the singularities of the moduli space.

The sigularities in the exact quantum moduli space $\mathcal{B}$ would be deformed by quantum effects. For example, in the $SU(2)$ case, instead of single singularity at the origin, the exact quantum moduli space would have two singularities, i.e. the original singularity splits into two points.

This is due to that the entries of the imaginary part of period matrix τ_{ij} is harmonic function, so it cannot have a minimum if it is globally defined on the moduli space. But to ensure the unitary of the theory, $Im\,(\tau_{ij}) > 0$ should hole through out $\mathcal{B}_{class}$. Consequently, the period matrix can only be defined locally, so to force the global structure of the *exact quantum moduli space* $\mathcal{B}$ to be differed from that of $\mathcal{B}_{class}$. We will discuss this further in the next subsection for the $SU(2)$ case.

Central charge and BPS states

From the previous discussion, we conclude that there is a singular locus (codimension one) $\mathcal{B}^{sing}$ inside the exact quantum moduli space (or moduli space in short) $\mathcal{B}$. Denote by $\mathcal{B}^0$ its complement, i.e.

$$\mathcal{B}^0 := \mathcal{B}\backslash\mathcal{B}^{sing}$$

Over $\mathcal{B}^0$, the low energy effective theoy is described by a family of abeliean gauge theories, which can be viewed as $r = rk(SU(n)) = n - 1$ copies of SUSY Maxwell theory. Thus, the abelian gauge symmetries $U(1)^r$ would act on the Hilbert space of the theories, which would endow the Hilbert space with a grading coming from the electromagnetic charges.

In the Coulomb branch with which we are concerned here, we can distinguish between the *eclectic charges* q and the *magnetic charges* p. When the rank of gauge group is r, we see that the electric charges and magnetic charges can be combined together to take values in a rank $2r$ charge lattice Γ. Since it depends on the particular point $b \in \mathcal{B}^0$ at which the theory being considered, we actually get a local system of lattices $\underline{\Gamma}$ over $\mathcal{B}^0$.

Schematically, $\Gamma \cong \mathbb{Z}^{\oplus 2r}$, which splits into electric part $\Gamma_e \cong \mathbb{Z}^{\oplus r}$ and the magnetic part $\Gamma_m \cong \mathbb{Z}^{\oplus r}$, i.e., for each point $b \in \mathcal{B}^0$, we have

$$\Gamma_b = \Gamma_{b,e} \oplus \Gamma_{b,m} \tag{A.5.12}$$

and the one particle states Hilbert space $\mathcal{H}_b$ at $b \in \mathcal{B}^0$ can be decomposed as

$$\mathcal{H}_b = \bigoplus_{\gamma\in\Gamma} \mathcal{H}_{b,\gamma} \tag{A.5.13}$$

where $\mathcal{H}_{b,\gamma}$ is the Hilbert space of one particle states with charge γ.

Denote by $\{\gamma_{e,i}\}$ for $i = 1, \cdots, r$ a $\mathbb{Z}$-basis of Γ_e, and $\{\gamma_m^i\}$ for $i = 1, \cdots, r$ a $\mathbb{Z}$-basis of Γ_m. By Dirac quantisation condition, there is a integer-valued symplectic bilinear paring $\langle \cdot, \cdot \rangle$ on Γ such that the above basis can be chosen to satisfy the following

$$\langle \gamma_{e,i}, \gamma_m^j \rangle = \delta_i^j$$

while the pairing restricting to Γ_e and Γ_m vanishes. Given two states γ_1, γ_2 with electromagnetic charges the following

$$\gamma_1 = q_1^i \gamma_{e,i} + p_j^1 \gamma_m^j, \qquad \gamma_2 = q_2^i \gamma_{e,i} + p_j^2 \gamma_m^j$$

then we have the following

$$\langle \gamma_1, \gamma_2 \rangle = q_1^i p_i^2 - p_j^1 q_2^j \tag{A.5.14}$$

From the $N = 2$ SUSY algebra, we see that there should be a central charge function defined on each of the Hilbert space $\mathcal{H}_{b,\gamma}$, which should be linear with respect to γ, i.e.,

$$Z_b(\gamma_1 + \gamma_2) = Z_b(\gamma_1) + Z_b(\gamma_2)$$

For a charge $\gamma = (q^i; p_j) = q^i \gamma_{e,i} + p_j \gamma_m^j \in \Gamma_b$, Z_b acts on it as

$$Z_b(\gamma) = q^i Z_b(\gamma_{e,i}) + p_j Z_b(\gamma_m^j)$$

We claim that $Z_b(\gamma_{e,i})$ can be identified with $a_{b,i}$, while $Z_b(\gamma_m^j)$ should be identified with the dual coordinate $a_{D,b}^j$, thus, we get the following formula

$$Z_b(\gamma) = q^i a_{b,i} + p_j a_{D,b}^j \tag{A.5.15}$$

Since $\mathcal{B}$ has coordinates $u = (u_2, \cdots, u_n)$, we can rewrite the above formula as follows to indicate it is a function of u:

$$Z_u(\gamma) = q^i a_i(u) + p_j a_D^j(u) \tag{A.5.16}$$

or in terms of the period matrix $(\tau_{ij}) = (\frac{\partial a_{D,i}}{\partial a_j})$, this is equivalent to

$$Z_u(\gamma) = a_i(u)(q^i + p_j \tau_{ij}(u)) \tag{A.5.17}$$

Again, the BPS bound $M \geq |Z|$ is satisfied, and the BPS states are defined as those that saturate this bound. We will give the BPS specturm for SU(2) SYM in the next section.

A.6 Details on $SU(2)$ case and the electric-magnetic duality

We now specialize the previous general discussion to the case when gauge group is $G = SU(2)$, the good thing about this theory is that it can be solved exactly. Here, we just review its basic features. Now the classical moduli space $\mathcal{B}_{class} \cong \mathbb{C}$, with good coordinate (A.5.3)

$$u = \frac{1}{2} Tr(\phi^2) = \frac{1}{2} a^2$$

And on it we have the Kähler metric determined as

$$ds^2 = Im(\tau)\, da d\bar{a}, \quad \text{with } \tau(a) = \frac{\partial^2 \mathcal{F}}{\partial a^2} \tag{A.6.1}$$

where $\mathcal{F}$ is the prepotential of the theory. Or in terms of the magnetic dual coordinate $a_D = \frac{\partial \mathcal{F}}{\partial a}$, the metric can be written as

$$ds^2 = Im\left(da_D da\right) = -\frac{i}{2}\left(\frac{da_D}{du}\frac{d\bar{a}}{d\bar{u}} - \frac{da}{du}\frac{d\bar{a}_D}{d\bar{u}}\right) du d\bar{u} \tag{A.6.2}$$

with the corresponding Kähler potential given by

$$K = Im(\bar{a} a_D) \tag{A.6.3}$$

and the Hessian $\tau(u)$ can be expressed as

$$\tau(u) = \frac{da_D}{da} = \frac{\left(\frac{da_D}{du}\right)}{\left(\frac{da}{du}\right)} \tag{A.6.4}$$

Remark A.6.1. $\tau(a)$ *represents the complexified effective gauge coupling of the theory, it is related to the gauge coupling* g *and the theta-angle* θ *in the following way*

$$\tau(a) = \frac{\theta(a)}{\pi} + \frac{4\pi i}{g^2(a)} \tag{A.6.5}$$

We have remarked that the effective action of the theory is fully determined by the prepotential $\mathcal{F}$, which is classically given by

$$\mathcal{F}(a) = \frac{1}{2}\tau_0\, a^2 \tag{A.6.6}$$

where τ_0 is the bare coupling constant.

While in the full quantum theory it receives perturbative (one loop) and non-perturbative corrections, yields the following form

$$\mathcal{F} = \frac{1}{2}\tau_0\, a^2 + \frac{i}{\pi} a^2 \ln\frac{a^2}{\Lambda^2} + a^2 \sum_{k=1}^{\infty} \mathcal{F}_k \left(\frac{\Lambda}{a}\right)^{4k} \tag{A.6.7}$$

where Λ is the dynamically generated scale.

From this, we see that for large a, i.e., we focus on the near ∞ region of the moduli space, the instanton sum above converges. So that by taking derivative twice, we see that near $u = \infty$, the effective constant τ is given by

$$\tau = \text{const} + \frac{4i}{\pi} \ln \frac{u}{\Lambda^2} + \text{single-valued} \tag{A.6.8}$$

Now, the monodromy around $u = \infty$ acts on τ by shifting it to $\tau - 8$.

As had been noted before (after Remark A.5.2) that the strongly coupling singularity $u = 0$ in $\mathcal{B}_{class}$ will be deformed in the exact quantum moduli space $\mathcal{B}$. It is the insights of Seiberg and Witten that the quantum moduli space $\mathcal{B}$ should still be the complex plane $\mathbb{C}$, but with the singularity $u = 0$ being deformed into two singularities located at $u = \pm 1$ respectively. Again, at these singularities, some particle states become massless.

The charge lattice Γ in this case is or rank 2, which form a local system of lattice over $\mathcal{B}^0$. As noted before, Γ splits into magnetic part and electric part. Denote by γ_e the basis of the electric part, and γ_m that of the magnetic part. Then, for any $\gamma = (q, p) = q\gamma_e + p\gamma_m \in \Gamma$, its central charge can be evaluated as

$$Z_u(\gamma) = qa(u) + pa_D(u) \tag{A.6.9}$$

For the state with charge γ to be BPS, the following condition should be satisfied

$$m = |Z_u(\gamma)| = |qa(u) + pa_D(u)| = \left| (p\ q) \begin{pmatrix} a_D \\ a \end{pmatrix} \right| \tag{A.6.10}$$

Electric-Magnetic duality

Electric-Magnetic duality, originally discovered by Montonen and Olive in the context of $N = 4$ SYM, was shown to be still valid in the $N = 2$ context by Seiberg and Witten. It is a duality that exchanges the electric and magnetic charges, as well as the gauge coupling constant g of the theory, i.e.,

$$E \to B, \ \ B \to -E, \qquad g \to \frac{1}{g}$$

where E and B denote the electric and magnetic field respectively.

Seiberg and Witten discovered that starting with the effective Lagrangian $\mathcal{L}_{eff}(a)$, and by applying the super version of Hodge duality operation, they will end up with a new effective Lagrangian $\mathcal{L}^D_{eff}(a_D)$, which is equivalent to the original one, but now written in the magnetic dual variable a_D, i.e.,

$$\mathcal{L}_{eff}(a) = \mathcal{L}^D_{eff}(a_D)$$

By applying this duality once again, we see that a_D would get transformed into $-a$. We now see how this duality acts on the complexified effective coupling constant $\tau(u)$. By definition

$$\tau(u) = \frac{da_D}{da} \tag{A.6.11}$$

and the dual coupling constant τ_D should be defined as

$$\tau_D = \frac{d(-a)}{da_D} \tag{A.6.12}$$

thus we see that under the electric-magnetic duality

$$\tau(a) \to \tau_D(a_D) = -\frac{1}{\tau(a)} \tag{A.6.13}$$

From this, and by (A.6.5), if the theta-angle $\theta = 0$, this specializes into $g \to -\frac{1}{g}$, which is part of the original Montonen-Olive duality.

There is another transformation thus leaves the theory unchanged, namely, the theta angle θ, being of topological origin, is defined only modulo $2\pi\mathbb{Z}$, so that τ is defined up to $\mathbb{Z}$. There for we have the following symmetry $\tau \to \tau + 1$, which is equivalent to $a_D \to a_D + a$. The two duality transformations are represented by the matrices

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{A.6.14}$$

which act on the vector $(a_D, a)^\dagger$ by left matrix multiplication, and on the coupling constant τ by linear fractional transformation. It is well known that S, T generates the group $SL(2,\mathbb{Z})$. which acts on the upper half plane $\mathbb{H}$ by linear fractional transformation. Thus points on the upper half plane that are related by $SL(2,\mathbb{Z})$-action would give rise to equivalent theories. This is called the S duality, or the *electric-magnetic duality*.

Monodromies on the moduli space $\mathcal{B}$

We first study the monodromy of $a(u)$ and $a_D(u)$ around $u = \infty$. At large $|u|$, the theory is asymptotically free, the one loop part of $\mathcal{F}$ (A.6.7) gives a good approximation

$$\mathcal{F}_{1-loop} = \frac{i}{\pi} a^2 \ln \frac{a^2}{\Lambda^2}$$

From this, we computed that

$$a_D = \frac{\partial \mathcal{F}}{\partial a} = \frac{2ia}{\pi} + \frac{4ia}{\pi} \ln \frac{a}{\Lambda}$$

Considering the monodromy around $u = \infty$, i.e. $u \to e^{2\pi i} u$, and from the relation $u = \frac{1}{2}a^2$, we get that

$$a \to -a, \qquad \ln a \to \ln a + i\pi$$

Consequently, we have that $a_D \to -a_D + 4a$. So the monodromy at ∞ can be written in terms of matrix as

$$M_\infty = \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix} \tag{A.6.15}$$

which acts on the vector $(a_D, a)^\dagger$ on the left, it should act on the BPS charge (q, p) at right by its inverse M_∞^{-1} so to make the mass formula (A.6.9) still hold after the monodromy, i.e.,

$$(q, p) \to (-p - 4q, -p)$$

Remark A.6.2. *The monodromy M_∞ can be written as PT^{-4}, where P is the element $-id \in SL(2,\mathbb{Z})$. The Weyl group W in $SU(2)$ case is just $\mathbb{Z}_2$, which acts ons classical level as $a \to -a$, thus we see that the P part of the monodromy already exists at the classical level. Apparently, since the good coordinate is $u = \frac{1}{2}a^2$, we can remove the ambiguity caused by P by working in the u-complex plane instead of the a-plane.*

The nontrivial monodromy M_∞ at weak coupling region implies there are must be other singularities on the exact quantum moduli space $\mathcal{B}$ since as noted before $Im(\tau)$ must be locally defined to ensure its positivity (unitarty requirement). $Im(\tau)$ is defined only semi-classically near $u = \infty$, where $u = \frac{1}{2}a^2$. However, the quantum effects would shift this relation such that the zero $u = 0$ splits into ± 1, related by $\mathbb{Z}_2$ symmetry.

Besides ∞, there must be at least two singularities, which can be relocated at ± 1. This can be seen by requiring that the monodromy should be non-abelian in order to ensure the positivity of $Im(\tau)$ all over the moduli space. Since if otherwise, say, the monodromies around other singularities all commute with $\mathcal{M}_\infty$, then the coordinate $u = \frac{1}{2}a^2$ will become globally defined on $\mathcal{B}$, contradicting to our assumption that it should be locally defined. Seiberg and Witten proposed to consider the simplest situation, namely, there are just two singularities ± 1 (besides ∞) at which some BPS states become massless. In the following we will compute the monodromies $M_{\pm 1}$ around ± 1.

The idea is that besides the natural coordinate $a(u)$ near $u = \infty$, we also consider the magnetic dual $a_D(u)$, and treat it on the equal footing as $a(u)$. At some region in $\mathcal{B}$, the description in terms of $a_D(u)$ should be more preferred. This had been suggested by the electric-magnetic duality discussed above, and can be seen more clearly as follows.

When we move away from the region near $u = \infty$, we see that the instanton sum in (A.6.7) may not be convergent any more, thus to get the new effective Lagrangian $\mathcal{L}_{eff}$, we need new prepotential $\mathcal{F}$, which can be viewed as kind of "analytic continuation" of the holomorphic function $\mathcal{F}$ over $\mathcal{B}$.

Let u_0 be the point where $a_D(u_0) = 0$ (will be transformed into $+1$ later). Near u_0, $a_D \approx c_0(u - u_0)$, and by using renormalization technique, wee see that near u_0, the magnetic dual coupling is given approximately as

$$\tau_D \approx -\frac{i}{2\pi} \ln a_D$$

by the definition of τ_D (see (A.6.12), we computed the following

$$a(u) \approx \int_{u_0}^{u} \tau_D \, da_D \overset{a_0 = a(u_0)}{\approx} a_0 + \frac{i}{2\pi} a_D \ln a_D$$

$$\approx a_0 + \frac{i}{2\pi} c_0(u - u_0) \ln(u - u_0)$$

The monodromy around u_0 can be read off by considering the loop around u_0 as

$$(u - u_0) \to e^{2\pi i} (u - u_0)$$

from which we get that

$$a_D \to a_D \qquad a \to a - a_D$$

Thus by adjusting the parameters, u_0 can be normalized to $+1$, and the corresponding monodromy matrix is given by

$$M_{+1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \tag{A.6.16}$$

which can be written in terms of S and T as $M_{+1} = STS^{-1}$. Note that by the definition of BPS state (A.6.10), the magnetic monopole with charge $(0,1)$ would become massless at the singularity $u = 1$. Near the region where $a_D = 0$, the magnetic dual a_D becomes natrual coordinate, in which the prepotential can be written as

$$\mathcal{F}_D(a_D) = \frac{1}{2}\tau_{0,D}\, a_D^2 - \frac{i}{4\pi} a_D^2 \ln \frac{a_D}{\Lambda} + \Lambda^2 \sum_{k=1}^{\infty} \mathcal{F}_k \left(\frac{a_D}{\Lambda}\right)^k \tag{A.6.17}$$

which is seen to be convergent as $a_D \to 0$.

At the remaining singularity $u = -1$, there is nothing essential new as we can go from $u = 1$ to $u = -1$ simply by using the $\mathbb{Z}_2$ symmetry, i.e. $u \to -u$. Thus, the prepotential near $u = -1$ region is obtained through replacing a_D in (A.6.17) by $2a - a_D$ (corresponding to the state with charge (1,-1) that vanishes at the sigularity $u = -1$), i.e.,

$$\tilde{\mathcal{F}} = \mathcal{F}_D(2a - a_D) \tag{A.6.18}$$

In summary, we have three coordinate patches on the moduli space $\mathcal{B}$, namely, near ∞ patch, and near ± 1 patches, in each patch, we have preferred coordinates, namely $a(u)$, $a_D(u)$ and $a(u) - a_D(u)$ respectively, and the prepotential in each coordinate patches is given in (A.6.7), (A.6.17) and (A.6.18) respectively.

To get a global theory, we need to glue these three coordinate patches together that is consistent globally, i.e., we require the monodromies to satisfy the following relation

$$M_{+1} \cdot M_{-1} = M_\infty \tag{A.6.19}$$

from which the monodromy M_{-1} at the third singularity -1 becomes

$$M_{-1} = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix} \tag{A.6.20}$$

As said above, the particle state that becomes massless at -1 is of electric-magnetic charge $(2, -1)$. From the mathematical point of view, we need to solve the following Riemann-Hilbert problem.

RH problem: To find multi-valued functions $a(u), a_D(u)$ that exhibit the monodromies $M_{\pm 1}$ and M_∞, which are subject to the relation (A.6.19). Further, the coupling τ computed from them should satisfy $Im\tau > 0$.

Sieberg and Witten, in their celebrated paper[SW94], gives a geometric solution to this RH problem involving family of elliptic curves, which in turn gives exact solution to $SU(2)$ SYM theory. Before closing this section, let me comment on the stability of the BPS states in$SU(2)$ case, as well as the spectrum of the theory.

Stability of BPS states and BPS spectrum

Recall that at the end of section A.3, we discussed the necessary condition for BPS states to decay, or conversely, to form bound state when the moduli of the theory being varied. We argued that this is possible only when $b \in \mathcal{B}$ crosses certain one dimensional walls, called the walls of marginal stability. We now describe the wall in $SU(2)$ SYM theory case.

For two BPS states with charges γ_1 and γ_2, the wall of marginal stability associated to them is defined as

$$MS(\gamma_1, \gamma_2) = \left\{ b \in \mathcal{B} : Im\left(\frac{Z_b(\gamma_1)}{Z_b(\gamma_2)}\right) = 0 \right\} \tag{A.6.21}$$

The charge lattice Γ splits into electric part Γ_e and the magnetic part Γ_m, which is generated by γ_e and γ_m respectively. Thus for two states $\gamma_1 = (q^1, p_1)$ and $\gamma_2 = (q^2, p_2)$, the associated central charges evaluated at $u \in \mathcal{B}$ read

$$Z_u(\gamma_1) = q^1 a(u) + p_1 a_D(u)$$
$$Z_u(\gamma_2) = q^2 a(u) + p_2 a_D(u)$$

$$Im\left(\frac{Z_b(\gamma_1)}{Z_b(\gamma_2)}\right) = Im\left(\frac{q^1 a(u) + p_1 a_D(u)}{q^2 a(u) + p_2 a_D(u)}\right) = Im\left(\frac{q^1 + p_1 \frac{a_D(u)}{a(u)}}{q^2 + p_2 \frac{a_D(u)}{a(u)}}\right)$$

from which, we see that

$$Im\left(\frac{Z_b(\gamma_1)}{Z_b(\gamma_2)}\right) = 0 \text{ iff } Im\left(\frac{a_D(u)}{a(u)}\right) = 0$$

Since this holds for arbitrary γ_1 and γ_2, we conclude that the *wall of the first kind* $\mathcal{W}$ in $SU(2)$ SYM can be defined as

$$\mathcal{C} := \left\{ u \in \mathcal{B} : Im\left(\frac{a_D(u)}{a(u)}\right) = 0 \right\} \tag{A.6.22}$$

The shape of this curve is a closed curve passing through the two singularities $\mathcal{B}^{sing}$, which divides the moduli space $\mathcal{B}$ into two connected component. Inside of the curve is called the *strong coupling region*, denoted by $\mathcal{B}_{strong}$, while outside of which is called the *weak coupling region*, denoted by $\mathcal{B}_{weak}$. The BPS spectrum of this theory is then can be described as

Inside $\mathcal{B}_{strong}$, we have

- the magnetic monopole with charge $(q, p) = \pm(0, 1)$
- a dyon with charge $(q, p) = \pm(-2, 1)$ in the lower u plane, and $\pm(2, 1)$ in the upper u plane.

Inside $\mathcal{B}_{weak}$, we have

- the magnetic monopole with charge $(q, p) = \pm(0, 1)$
- dyons with charge $(q, p) = \pm(2k, 1)$, with $k \in \mathbb{Z}$.
- $W^{\pm}$ gauge bosons with charge $(q, p) = (\pm 2, 0)$

Appendix B

Physics motivation of attractor flows

Motivated from the relevant works [Den00] in the study of super-symmetric black holes as well, as the work of Kontsevich and Soibelman on Mirror symmetry [KS06], it seems that the construction of these BPS invariants which is compatible with the formalism of WCS can be obtained by studying certain integer flows on the moduli space of the corresponding theories. Roughly speaking, the flow for a given charge γ is a tree lying on the moduli space, which splits at the points on the walls of first kind (physicists would call them wall of marginal stability). Since this kind of flow was first studied in $4d, N = 2$ supersymmetric gravity under the name of *attractor mechanism*, so these flows would be called the *attractor flows*, or more precisely *split attractor flows*.

The concept of attractor flows and the technique to study them have been proved useful in determining the existence of (multi-centered)BPS black hole solutions as well as their classifications. Indeed, the so called the *attractor flow conjecture* states that a multicentered black hole solution exists if and only if the corresponding flow tree exists. This is very convenient because even if the check that if a supergravity solution exist or if it is well-defined can be very challenging in most cases, yet the corresponding flow tree is simple in nature and to check that if it exists and well-defined is relatively not that hard in general. Thus, under the assumption of this attractor flow conjecture, one can translate the problems of finding certain type of solutions in supergravity into the problems of determining if the corresponding attractor flows exist or not. Although there is no rigorous proof of this conjecture so far, nevertheless, no counterexamples have been found.

Although the technique of attractor flows originated in the supergravity regime, i.e., low energy regime, its power is not limited by this. In fact, in is conjectured that the split attractor flow trees can even be used in the study of existence and classification for BPS states in the full type II string theories.

Even in the context of $N = 2$ field theories decoupled from gravity, it was found out that the same kind of structures still persist and are still useful in describing the BPS states in these theories. This leads to the speculation that the appearance of split attractor flows may be investigated entirely from the microscopic D-brane perspective. This is already reminiscent in the study of mirror symmetry in the above mentioned work of Soibelman and Kontsevich.

In particular, the Seiberg-Witten for $SU(2), N = 4$ super Yang-Mills theory can be obtained from the $N = 2$ super gravity theory describing the low energy physics of type II string theory in a certain rigid, i.e. gravity decoupling limit([Den99]). Hence, we can take the rigid limits of supergravity attractor flows to get the BPS solutions of this effective abelian theory.

As is well known that there are two types of D-branes in type Calabi-Yau compactifications. Namely, the A branes that wrap special Lagrangian (SLG) submanifolds endowed with flat connections, and B branes that wrap holomorphic cycles which are endowed with holomorphic bundles. There have been tremendously amount of rigorous mathematical formalisms focusing on the B branes side even though the current understanding is far more from complete, while the A branes side turned out to be harder to deal with, if not in-attractable at all. This is partly because the explicit constructions of generic SLGs is virtually impossible since there are difficult foundational geometric questions still unsolved, and partly because the quantum corrections on B-side are better in control.

Thus, it would be desirable if the split attractor flows could be used in the studying of these SLGs since the former are much easier to deal with than the later. Indeed, in the work of Frederik Denef [Den01], it is argued that split attractor flows can be used to assemble and disassemble of SLGs by studying their deformations. The process here corresponds to the decay or form of states in the corresponding field theory.

On the A side, the DT-invariant $\Omega_b(\gamma)$ should be thought of as counting the virtual number of certain SLGs with the charge γ, i.e. the ones that are wrapping the three cycles $\gamma \in H_3(X, \mathbb{Z})$ in a CY 3 fold X, where b depends on the complex moduli space. By SYZ torus fibration, these correspond to holomorphic discs in the total space with boundaries on a torus fiber. By the same reasons as mentioned above, the direct counting of these holomorphic discs can be very challenging. Thus it is suggested by Kontsevich and Soibelman that this counting problem maybe tackled by using Non-archimedean geometry, i.e., by tropicalization map, the space X can be retracted to its base B, which is non-archimedean analytic space (c.f.,[KS06]). Therefore, the counting of holomorphic discs in X is translated into the problem of counting of tropical curves in B, which are piecewise linear graphs in B. Certainly, the later problem in much more attractable.

The above construction should be compared with the technique of split attractor flows, and the attractor flow trees should be thought of as analogy of the tropical trees. This provides us with further motivation of the usefulness of this technique.

B.0.1 Attractor flows in supergravity

In the following we will review the works of Denef (see [Den00]) on the correspondence between existence/stability of BPS states in type II string theory compactified on a Calabi-Yau manif old and BPS solutions of four dimensional $N = 2$ supergravity. We will especially focus on the emergence of splitting attractor flows in this story as well as its relation to the stability condition. The main purpose here is to motivate the mathematical formalism of attractor flows proposed by Kontsevich and Soibelman (see section 3 of [KS14]).

It is well known that the investigate of the $4d$ $N = 2$ supergravity can be approached from the super string theoretical techniques by embedding it into either a Type IIA theory (controlled by Kähler moduli) or Type IIB theory (controlled by complex moduli) in ten dimension, and in this way, many supergravities can be realized as low energy descriptions of the corresponding string theories. Even more remarkably, there is well-known mirror symmetry relating the two embeddings, i.e. the four dimensional theory coming from the Type IIA compactified on a Calabi-Yau three fold X should be equivalent to the description of that coming from Type IIB compactified on another Calabi-Yau three fold $\check{X}$, called the mirror dual of X. For certain problem, its manifest in the IIA side maybe more attractable than its counterpart in the IIB side, and vice verse. For this reason, mirror symmetry becomes very powerful think-kit both from computational perspective and from the landscape of theoretical insights in both physics and mathematics.

We are especially interested in how black holes could be realized in such frame work. It turned out that we can model black holes in supergravity with various type of charges by wrapping various types of D-branes around cycles in the extra compactified dimensions in such a way that one could from the three dimensional space point of view, we will obtain a point like object in time (singularities of space-time). Of course, these black hole solutions thus obtained should preserve certain amount of $N = 2$ supersymmetry so that the BPS bound is being saturated. Thus, we normally call these solutions BPS black hole solutions.Further more, the ambiguities for making a BPS black hole by wrapping branes is associted to the entropy of the black hole.

However, it has been known since Denef ([Den00]) that the above single-centered, spherically symmetric solutions can not cover all BPS spectrum. For some BPS state of a given charge, it is often being realized in low energy limit (supergravity regime) as multi-centered stationary bound state of black hole which not only carries the intrinsic angular momentum, but the relative distances between the centers are also subject to certain equilibrium distances constrains.

Besides its metric properties, the multi-centered black hole solution in $N = 2, d = 4$ supergravity theories thus obtained also depends on the values of the gauge fields as well as the scalars in the theories. The scalars fall into either hypermultiplets or vector multiplets. For the discussion of BPS black holes, the solution only depends on the vector multiplets, the values of hypermultiplets scalars can be set to be constants. The most important fact discovered is that the manifold of vector multiplets scalars, as a special Kähler manifold, is highly constrained by the so called *attractor mechanism*.

Let me explain briefly what this mechanism meat to tell us about the geometry of black hole configurations, as well as its relevance to the splitting attractor flows which would be our main concern later.

Roughly speaking, a BPS black hole in this setting is determined by first specifying a "background", i.e. the values of the scalars at the spatial infinity $r = \infty$, where r denotes the radial distance in three dimension, and the values of these scalars at other points are controlled by a first order linear differential equation, called the *attractor flow equation*, which flows these values radially inward to the event horizon of the black hole. The peculiar nature of the attractor flow equation tells us that the value at the event horizon does not change even if the background being perturbed. We will call the fixed value *attractor value* (or the attractor point on the vector multiplet moduli space) for the black hole solution, and for this reason, this mechanism is called the attractor mechanism.

Thus, for a single centered black hole configuration, the scalars associated with it, stating at the values at spatial infinity and ending at the attractor point, would flow along a line on the scalars manifold, this will be called a single attractor flow.

And in the case that the black hole configuration consists of more than one centers, i.e. the so-called multi-centered black hole solution, a similar attractor mechanism still exits, though the flow lines obtained by solving the corresponding attractor flow equation would be different in that the attractor flow in this case would usually splits (could be more than once) along the way when it flows from the spatial infinity down to the even horizon. For each center, there would be one branch of flow terminating on the attractor point associated to this center. These type of flow trees will be called the *split attractor flows*. And the so-called *the split flow conjecture* asserts that the configurations of these flow lines could be used to classify the supergravity solutions. It turned out that the split points of the flow would lie on certain walls in the moduli space of scalars. Physicists call these walls the *walls of marginal stability*, which is analogous to the *walls of the first kind* defined previously.

Bibliography

[Zar29] Oscar Zariski. "On the Problem of Existence of Algebraic Functions of Two Variables Possessing a Given Branch Curve". In: *American Journal of Mathematics* 51.2 (1929), pp. 305–328.

[Van33] Egbert R Van Kampen. "On the Fundamental Group of an Algebraic Curve". In: *American journal of Mathematics* 55.1 (1933), pp. 255–260.

[Art47] Emil Artin. "Theory of Braids". In: *Annals of Mathematics* (1947), pp. 101–126.

[Mil63] John Milnor. "Morse Theory, volume 51 of Annals of Mathematics Studies". In: *Princeton, NJ, USA* (1963).

[Gri70] Phillip A Griffiths. "A Transcendental method in Algebraic Geometry". In: *Actes du Congres International des Mathématiciens (Nice, 1970)*. Vol. 1. 1970, pp. 113–119.

[Bot83] R Bott. "The Yang-Mills Equations over Riemann Surfaces". In: *Phil. Trans. R. Soc. Lond. A* 308.1505 (1983), pp. 523–615.

[GV83] Gerard Gonzalez-Sprinberg and J-L Verdier. "Construction Géométrique de la correspondance de McKay". In: *Annales Scientifiques de l'École Normale Supérieure*. Vol. 16. 3. Elsevier. 1983, pp. 409–449.

[Hit+87a] Nigel Hitchin et al. "Stable Bundles and Integrable Systems". In: *Duke mathematical journal* 54.1 (1987), pp. 91–114.

[Hit+87b] Nigel J Hitchin et al. "Hyperkähler Metrics and Supersymmetry". In: *Communications in Mathematical Physics* 108.4 (1987), pp. 535–589.

[Han89] Vagn Lundsgaard Hansen. *Braids and Coverings: Selected Topics*. Vol. 18. Cambridge University Press, 1989.

[Kro+89] Peter B Kronheimer et al. "The Construction of ALE Spaces as Hyper-Kähler quotients". In: *J. Diff. Geom* 29.3 (1989), pp. 665–683.

[Kac90] VG Kac. *Infinite–Dimensional Lie Algebras, 3d edition, Cambridge U.* 1990.

[Hit91] Nigel Hitchin. "Hyperkähler Manifolds". In: *Astérisque, tome 206 (1992), Séminaire Bourbaki, exp. no 748, p. 137-166* (1991).

[HJ91] RA Horn and CR Johnson. "Topics in Matrix Analysis". In: *Cambridge university press cambridge* (1991).

[Kle+94] Albrecht Klemm et al. "On the Monodromies of $N = 2$ Supersymmetric Yang-Mills Theory". In: *Physics From Planck Scale To Electrweak Scale* (1994), p. 433.

[SW94] Nathan Seiberg and Edward Witten. “Electric-magnetic Duality, Monopole Condensation, and Confinement in N= 2 Supersymmetric Yang-Mills Theory”. In: *Nuclear Physics B* 426.1 (1994), pp. 19–52.

[AD95] Philip C Argyres and Michael R Douglas. “New phenomena in $SU(3)$ Supersymmetric Gauge Theory”. In: *Nuclear Physics B* 448.1-2 (1995), pp. 93–126.

[AF95] Philip C Argyres and Alon E Faraggi. “Vacuum Structure and Spectrum of $N = 2$ Supersymmetric $SU(N)$ Gauge Theory”. In: *Physical Review Letters* 74.20 (1995), p. 3931.

[Cer95] Anna Ceresole. “Duality and Braids in $N = 2$ Yang-Mills Theories”. In: *Prepared for*. 1995, pp. 80–87.

[GR95] Amit Giveon and Martin Roček. “Effective Actions and Gauge Field Stability”. In: *Physics Letters B* 363.3 (1995), pp. 173–179.

[KTL95] AD Klemm, Stefan J Theisen, and Wolfgang Lerche. “Nonperturbative Effective Actions of $N = 2$ Supersymmetric Gauge Theories”. In: *Int. J. Mod. Phys. A* 11.hep-th/9505150 (1995), pp. 1929–1974. URL: https://arxiv.org/abs/hep-th/9505150.

[Kle+95] A Klemm et al. “Simple Singularities and $N = 2$ Supersymmetric Yang-Mills Theory”. In: *Physics Letters. Section B: Nuclear, Elementary Particle and High-Energy Physics* 344.1-4 (1995), pp. 169–175.

[BF96] Adel Bilal and Frank Ferrari. “Curves of Marginal Stability, and Weak-and Strong-Coupling BPS Spectra in $N = 2$ Supersymmetric QCD”. In: *Nuclear Physics B* 480.3 (1996), pp. 589–619.

[Bil+96] M Billó et al. “A Search for Non-perturbative Dualities of Local $N = 2$ Yang-Mills Theories from Calabi-Yau 3-Folds”. In: *Classical and Quantum Gravity* 13.5 (1996), p. 831.

[DW96] Ron Donagi and Edward Witten. “Supersymmetric Yang-Mills Theory and Integrable Systems”. In: *Nuclear Physics B* 460.2 (1996), pp. 299–334.

[Kle+96] Albrecht Klemm et al. “Self-Dual Strings and $N = 2$ Supersymmetric Field Theory”. In: *Nuclear Physics B* 477.3 (1996), pp. 746–764.

[MW96] Emil J Martinec and Nicholas P Warner. “Integrable Systems and Supersymmetric Gauge Theory”. In: *Nuclear Physics B* 459.1-2 (1996), pp. 97–112.

[Mok96] Tahar Mokri. “On Lie Algebroid Actions and Morphisms”. In: *Cahiers de topologie et géométrie différentielle catégoriques* 37.4 (1996), pp. 315–331.

[AH97] Luis Alvarez-Gaumé and SF Hassan. “Introduction to S-Duality in $N = 2$ Supersymmetric Gauge Theories (A Pedagogical Review of the Work of Seiberg and Witten)”. In: *Fortschritte der Physik* 45.3-4 (1997), pp. 159–236.

[BS97] A Brandhuber and S Stieberger. “Self-Dual Strings and Stability of BPS States in $N = 2$, $SU(2)$ Gauge Theories”. In: *Nuclear Physics B* 488.1-2 (1997), pp. 199–222.

[Don97] Ron Y Donagi. “Seiberg-Witten Integrable Systems”. In: *arXiv preprint alg-geom/9705010* (1997). URL: https://arxiv.org/abs/1206.2317.

[Fay97] Ansar Fayyazuddin. “Results in Supersymmetric Field Theory from 3-Brane Probe in F-theory”. In: *Nuclear Physics B* 497.1-2 (1997), pp. 101–109.

[FH97] Christophe Fraser and Timothy J Hollowood. “On the Weak Coupling Spectrum of $N = 2$ Supersymmetric $SU(n)$ Gauge theory”. In: *Nuclear Physics B* 490.1-2 (1997), pp. 217–235.

[Hit97] Nigel J Hitchin. “The Moduli Space of Special Lagrangian Submanifolds”. In: *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze* 25.3-4 (1997), pp. 503–515.

[Hol97] Timothy J Hollowood. “Strong Coupling $N = 2$ Gauge Theory with Arbitrary Gauge Group”. In: *Adv. Theor. Math. Phys.* 2.hep-th/9710073 (1997), pp. 335–355.

[KKV97] Sheldon Katz, Albrecht Klemm, and Cumrun Vafa. “Geometric Engineering of Quantum Field Theories”. In: *Nuclear Physics B* 497.1-2 (1997), pp. 173–195.

[Kle97] A Klemm. “On the Geometry Behind $N = 2$ Supersymmetric Effective Actions in Four Dimensions”. In: *High Energy Physics and Cosmology*. 1997, p. 120.

[BF98] Oren Bergman and Ansar Fayyazuddin. “String Junction Transitions in the Moduli Space of $N = 2$ SYM”. In: *Nuclear Physics B* 535.1-2 (1998), pp. 139–151.

[Bil+98] Marco Billó et al. “The Rigid Limit in Special Kähler Geometry-from-Fibrations to Special Riemann surfaces: a Detailed Case Study”. In: *Classical and quantum gravity* 15.8 (1998), p. 2083.

[Ler98] Wolfgang Lerche. “Introduction to Seiberg-Witten Theory and its Stringy Origin”. In: *AIP Conference Proceedings*. Vol. 419. 1. AIP. 1998, pp. 171–217.

[LW98] Wolfgang Lerche and Nicholas P Warner. “Exceptional SW geometry from ALE fibrations”. In: *Physics Letters B* 423.1-2 (1998), pp. 79–86.

[Moo98] Gregory Moore. “Arithmetic and Attractors”. In: *arXiv preprint hep-th/9807087* (1998). URL: https://arxiv.org/abs/hep-th/9807087.

[Den99] Frederik Denef. “Attractors at Weak Gravity”. In: *Nuclear physics b* 547.1-2 (1999), pp. 201–220.

[Fre99] Daniel S Freed. “Special Kähler Manifolds”. In: *Communications in Mathematical Physics* 203.1 (1999), pp. 31–52.

[Mar99] Andrei Marshakov. *Seiberg-Witten Theory and Integrable Systems*. World Scientific, 1999.

[May99] Peter Mayr. “Geometric Construction of $N = 2$ Gauge Theories”. In: *Fortschritte der Physik: Progress of Physics* 47.1-3 (1999), pp. 39–63.

[Den00] Frederik Denef. “Supergravity Flows and D-brane Stability”. In: *Journal of High Energy Physics* 2000.08 (2000), p. 050.

[Den01] Frederik Denef. “(Dis) assembling Special Lagrangians”. In: *arXiv preprint hep-th/0107152* (2001). URL: https://arxiv.org/abs/hep-th/0107152.

[KS01] Maxim Kontsevich and Yan Soibelman. “Homological Mirror Symmetry and Torus Fibrations”. In: *Symplectic geometry and mirror symmetry*. World Scientific, 2001, pp. 203–263.

[Tay01] Brett J Taylor. “On the Strong-Coupling Spectrum of Pure $SU(3)$ Seiberg-Witten Theory”. In: *Journal of High Energy Physics* 2001.08 (2001), p. 031.

[ACD02] DV Alekseevsky, Vincente Cortés, and Chandrashekar Devchand. “Special Complex Manifolds”. In: *Journal of Geometry and Physics* 42.1 (2002), pp. 85–105.

[KM02] Yvette Kosmann-Schwarzbach and Kirill CH Mackenzie. “Differential Operators and Actions of Lie Algebroids”. In: *Contemporary Mathematics* 315 (2002), pp. 213–234.

[GS03] Mark Gross and Bernd Siebert. “Affine Manifolds, Log Structures, and Mirror Symmetry”. In: *Turkish Journal of Mathematics* 27.1 (2003), pp. 33–60.

[MM03] Ieke Moerdijk and Janez Mrcun. *Introduction to Foliations and Lie Groupoids*. Vol. 91. Cambridge University Press, 2003.

[BM04] Claudio Bartocci and Igor Mencattini. “Hyper-Symplectic Structures on Integrable Systems”. In: *Journal of Geometry and Physics* 50.1 (2004), pp. 339–344.

[BF04] Aleksei Viktorovich Bolsinov and Anatolij Timofeevič Fomenko. *Integrable Hamiltonian Systems: Geometry, Topology, Classification*. CRC Press, 2004.

[Rac04] Sébastien Racaniere. “Lie Algebroids, Lie Groupoids and Poisson Geometry”. In: *Preprint, June* (2004).

[KS06] Maxim Kontsevich and Yan Soibelman. “Affine Structures and Non-Archimedean Analytic Spaces”. In: *The unity of mathematics*. Springer, 2006, pp. 321–385.

[Lön06] Michael Lönne. “Braid Monodromy of Hypersurface Singularities”. In: *arXiv preprint math/0602371* (2006). URL: https://arxiv.org/pdf/math/0602371.pdf.

[KS08] Maxim Kontsevich and Yan Soibelman. “Stability Structures, Motivic Donaldson-Thomas Invariants and Cluster Transformations”. In: *arXiv preprint arXiv:0811.2435* (2008). URL: arXiv:0811.2435.

[Kuc08] Michael Yu Kuchiev. “Charges of Dyons in $N = 2$ Supersymmetric Gauge Theory”. In: *Nuclear physics B* 803.1-2 (2008), pp. 113–134.

[B+09] Ricardo Castano Bernard, Diego Matessi, et al. “Lagrangian 3-torus Fibrations”. In: *Journal of Differential Geometry* 81.3 (2009), pp. 483–573.

[Fer09] Frank Ferrari. “Galois Symmetries in Super Yang-Mills Theories”. In: *Journal of High Energy Physics* 2009.03 (2009), p. 128.

[CAM10] Anna CAMPBELLOVÁ. “Supersymetric Gauge Theory and String Theory”. PhD thesis. Masarykova univerzita, Přırodovědecká fakulta, 2010. URL: https://is.muni.cz/th/dp9rl/thesis_ismuni.pdf.

[Cha10] Kwokwai Chan. “The Ooguri-Vafa Metric, Holomorphic Discs and Wall-crossing”. In: *Mathematical Research Letters* 17.2 (2010), pp. 401–414.

[GMN10] Davide Gaiotto, Gregory W Moore, and Andrew Neitzke. “Four-Dimensional Wall-Crossing via Three-Dimensional Field Theory”. In: *Communications in Mathematical Physics* 299.1 (2010), pp. 163–224.

[Lu10] Wenxuan Lu. “Instanton Correction, Wall-crossing and Mirror Symmetry Of Hitchin’s Moduli Spaces”. In: *arXiv preprint arXiv:1010.3388* (2010). URL: https://arxiv.org/abs/1010.3388.

[ACG11] Enrico Arbarello, Maurizio Cornalba, and Phillip Griffiths. *Geometry of Algebraic curves: volume II with a contribution by Joseph Daniel Harris*. Vol. 268. Springer Science & Business Media, 2011.

[CDP11] Heng-Yu Chen, Nick Dorey, and Kirill Petunin. “Moduli Space and Wall-Crossing Formulae in Higher-Rank Gauge Theories”. In: *Journal of High Energy Physics* 2011.11 (2011), p. 20.

[Cog11] José Ignacio Cogolludo-Agustín. “Braid Monodromy of Algebraic Curves”. In: *Ann. Math. Blaise Pascal* 18 (2011), pp. 141–209.

[CF11] Marius Crainic and Rui Loja Fernandes. “Lectures on Integrability of Lie Brackets”. In: *Lectures on Poisson Geometry, Geom. Topol. Monogr* 17 (2011), pp. 1–107.

[KS11] Maxim Kontsevich and Yan Soibelman. “Lectures on Motivic Donaldson-Thomas Invariants and Wall-crossing Formulas”. In: *preprint* 3.4 (2011), p. 5.

[MPS11] Jan Manschot, Boris Pioline, and Ashoke Sen. “Wall-crossing from Boltzmann Black Hole Halos”. In: *Journal of High Energy Physics* 2011.7 (2011), p. 59.

[Moo12a] G Moore. *Lecture Notes for Felix Klein Lectures*. 2012. URL: http://www.%20physics.%20rutgers.%20edu/%5C~%7B%7D%20gmoore.

[Moo12b] G Moore. *PITP Lectures on BPS States and Wall-crossing in $d = 4, N = 2$ Theories*. 2012. URL: http://www.%20physics.%20rutgers.%20edu/%5C~%7B%7D%20gmoore.

[NP12] Nikita Nekrasov and Vasily Pestun. “Seiberg-Witten Geometry of Four-Dimensional $N = 2$ Quiver Gauge Theories”. In: *arXiv preprint arXiv:1211.2240* (2012). URL: https://arxiv.org/abs/1211.2240.

[Pet12] Kirill Petunin. “Wall-crossing in Supersymmetric Gauge Theories”. In: *arXiv preprint arXiv:1206.2317* (2012). URL: https://arxiv.org/abs/1206.2317.

[SD12] Jihye Seo and Keshav Dasgupta. “Argyres-Douglas Loci, Singularity Structures and Wall-crossings in Pure $N = 2$ Gauge Theories with Classical Gauge Groups”. In: *Journal of High Energy Physics* 2012.5 (2012), p. 72.

[Tod12] Morikazu Toda. *Theory of Nonlinear Lattices*. Vol. 20. Springer Science & Business Media, 2012.

[Vai12] Izu Vaisman. *Lectures on the Geometry of Poisson Manifolds*. Vol. 118. Birkhäuser, 2012.

[Van12] Dieter Van den Bleeken. “BPS Dyons and Hesse Flow”. In: *Journal of High Energy Physics* 2012.2 (2012), p. 67.

[GMN13a] Davide Gaiotto, Gregory W Moore, and Andrew Neitzke. “Spectral Networks”. In: *Annales Henri Poincaré*. Vol. 14. 7. Springer. 2013, pp. 1643–1731.

[GMN13b] Davide Gaiotto, Gregory W Moore, and Andrew Neitzke. “Wall-crossing, Hitchin systems, and the WKB Approximation”. In: *Advances in Mathematics* 234 (2013), pp. 239–403.

[Gal+13] Dmitry Galakhov et al. “Wild Wall-crossing and BPS Giants”. In: *Journal of High Energy Physics* 2013.11 (2013), p. 46.

[Seo13] Jihye Sofia Seo. “Singularity Structure of $N = 2$ Supersymmetric Yang-Mills Theories: A Review”. In: *arXiv preprint arXiv:1307.2691* (2013). URL: https://arxiv.org/abs/1307.2691.

[KS14] Maxim Kontsevich and Yan Soibelman. “Wall-crossing structures in Donaldson-Thomas Invariants, Integrable Systems and Mirror Symmetry”. In: *Homological Mirror Symmetry and Tropical Geometry*. Springer, 2014, pp. 197–308.

[Lin14] Yu-Shen Lin. “Reduced Open Gromov-Witten Invariants on HyperK\” aher Manifolds”. In: *arXiv preprint arXiv:1404.4684* (2014). URL: https://arxiv.org/abs/1404.4684.

[LTY14] Yuan Luo, Meng-Chwan Tan, and Junya Yagi. “$N = 2$ Supersymmetric Gauge Theories and Quantum Integrable Systems”. In: *Journal of High Energy Physics* 2014.3 (2014), p. 90.

[Nei14a] Andrew Neitzke. “Hitchin Systems in N= 2 Field Theory”. In: *arXiv preprint arXiv:1412.7120* (2014). URL: https://arxiv.org/abs/1412.7120.

[Nei14b] Andrew Neitzke. “Notes on a New Construction of Hyperkahler Metrics”. In: *Homological Mirror Symmetry and Tropical Geometry*. Springer, 2014, pp. 351–375.

[Tac15] Yuji Tachikawa. *$N = 2$ Supersymmetric Dynamics for Pedestrians*. Springer, 2015, p. 236.

[Lin17] Yu-Shen Lin. “Open Gromov–Witten Invariants on Elliptic K3 Surfaces and Wall-Crossing”. In: *Communications in Mathematical Physics* 349.1 (2017), pp. 109–164.

[Lon18] Pietro Longhi. “Wall-crossing Invariants from Spectral Networks”. In: *Annales Henri Poincaré*. Vol. 19. 3. Springer. 2018, pp. 775–842.

[KS13] Maxim Kontsevich and Yan Soibelman. “Wall-crossing Structures, Algebracity and Resurgence”. In: (unpublished preprint 2013).

[AF] PC Argyres and AE Faraggi. “The Vacuum Structure and Spectrum of $N = 2$ Supersymmetric $SU(N)$ Gauge theory”. In: *Rev. Lett* 74 (), p. 3931.

[AGV] VI Arnold, SM Gusein-Zade, and AN Varchenko. *Singularities of Differentiable Maps. Volume 2. Monodromy and Asymptotics of Integrals. Translated from the Russian by Hugh Porteous and revised by the authors and James Montaldi. Reprint of the 1988 translation, Mod.*

[Bru] U Bruzzo. *Notes on the Geometry of Lagrangian Torus Fibrations*. URL: http://www.dima.unige.it/~bartocci/gfabstracts/gaavs.pdf.

[Nei] Andrew Neitzke. *What is a BPS State?* URL: https://web.ma.utexas.edu/users/neitzke/expos/bps-expos.pdf.

[Sep] Daniele Sepe. *Notes on Lagrangian Fibrations*. URL: https://www.math.tecnico.ulisboa.%20pt/~jmourao/inves/D_Sepe_Lagrangian_Fibrations.pdf.