AN APPROXIMATE STABILITY ANALYSIS OF A
TANGENTIALLY LOADED COLUMN SUPPORTED
BY MAXWELL-TYPE VISCOELASTIC FOUNDATION

by

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INTRODUCTION

During recent years, the study and analysis of elastic systems with follower forces has become very important. There are many examples of todays' engineering problems in this category; cantilever pipes conveying fluid [1], aerodynamic flutter of panels [2], vertical take-off and landing aircraft [3,4,5], and pod-mounted jet engines [6]. Under certain conditions, chemical or electromagnetic energy can also induce follower type forces into a system [7].

All these systems are subjected to forces which follow the motion of the system in some prescribed manner. The presence of the follower forces causes the system to be nonconservative. The nonconservative system has two modes of instability, static and dynamic. Static instability, which is called buckling or divergence, occurs when the system assumes a new equilibrium position close to the original configuration of the system. Dynamic instability, or flutter, results when a small disturbance about an equilibrium position causes oscillations which increase without bound. The static method produces conditions of instability for conservative systems but does not provide any information about the dynamic instabilities of the nonconservative systems.

There have been many researchers who have worked on the analysis of the nonconservative systems. Beck [9] was the first to obtain the correct solution of the linear elastic cantilevered Euler's column under the action of a concentrated tangential force. This system is referred to as

Beck's problem. Nemat-Nasser [10] used the Timoshenko beam theory to develop the equation of motion. He found the resulting critical load to be less than that based on the Euler-Bernoulli beam theory. Ziegler [11] used a cantilever column made of a viscoelastic material. His results showed that under certain conditions small internal viscous damping affected the stability of stable system. Then Prasad and Herrmann [12] and Nemat-Nasser and Herrmann [13] showed that the flutter load found for an undamped elastic system subjected to follower forces is the upper bound for systems with slight internal damping.

External damping plays an uncertain role in the stability of non-conservative systems as well. This has been shown by Plaut and Infante [14], Anderson [15], and Pedersen [16]. They found that the flutter load increases with external viscous damping but only to an asymptotic value. Therefore, the damping plays an uncertain role on the system of stability.

The problem of stability has also included several boundary conditions on the free end. Rigid or elastic end supports used by Bartz [17] and Sundararajan [18] with no stabilizing effect. Pedersen [16] used a concentrated tip mass, a linear elastic spring, and a partial follower force and described their effects on the flutter load.

Elastic and viscoelastic foundations have been used to design support for nonconservative systems in order to prevent failures. Anderson [20] discussed the effect of an elastic foundation on the stability of canti-lever columns subjected to uniformly and linearly distributed tangential forces. Peterson [21], Smith and Herrmann [22], and Sundararajan [23] added a continuous elastic support to the column and found that the flutter frequency increased but the flutter load remained unchanged. Wahed [24] considered the elastically supported cantilever with external viscous damping and found that the flutter load depends on both the stiffness and the damping coefficients.

Morgan [8] observed that these studies had certain limitations and were limited to small range of system parameters. He performed an exact stability analysis for the Beck's column supported by the Standard Linear viscoelastic foundation which includes the Kelvin-Voigt and the Maxwell models as special cases. He derived the stability constraints for the full range of foundation parameters. He also showed, for the first time, that a Maxwell type viscoelastic foundation does have stabilizing effects on the column. This is very significant because such a foundation has no effect on the critical load if the column is subjected to a conservative load.

However, the exact solution by Morgan is rather involved and time consuming. This definitely suggests the need for exploring the possibility of employing an approximate analysis which can yield meaningful results. Moreover it would also be interesting to see if an approximate method could yield results which are valid for the entire range of system parameters. In the past, such studies [24] Kar [37] have been limited to rather small range of system parameters.

In this study, Galerkin's method is employed to perform an approximate stability analysis of Beck's column supported by Maxwell-type viscoelastic foundation. The results are compared with the exact analysis reported by Morgan [8].

The equation of motion is developed in Chapter 2 and written in terms of the appropriate nondimensional constants. A separable solution is then assumed and the associated eigenvalue problem is formulated. The Galerkin approximation method outlined by Evan-Iwanowski [25] is used to obtain the real and complex eigenvalues of the characteristic equation. In Chapter 3, the effect of the eigenvalues on the time dependent part of the solution is considered. To investigate the stability characteristics, a modified Routh-Hurwitz criteria, developed by Morgan [8], is used. Then the stability constraints are derived for the full range of foundation parameters and compared to the exact analysis. The primary results and recommendations for further study are given in Chapter 4.

Chapter 2

EQUATION OF MOTION AND GENERAL SOLUTION

2.1 Equation of Motion

Consider small transverse motion about the undisturbed equilibrium position of a uniform elastic column, of length ℓ, supported with a Maxwell visco-elastic foundation and subjected to a constant compressive axial load p as shown in Figure 2.1. Let y be the lateral displacement, x the distance along the beam, and q(x,t) the total reaction pressure from the foundation. Note that the supporting foundation is assumed to be continuously connected to the beam so it opposes motion regardless of the direction. The flexural rigidity EI and the density per unit length ρ are assumed to be constant.

Using Euler-Bernoulli beam theory and Hamilton's principle [26], the dynamic equation of motion for the system is

$$EI \frac{\partial^4 y}{\partial x^4} + p\frac{\partial^2 y}{\partial x^2} + \rho\frac{\partial^2 y}{\partial t^2} = -q(x,t). \tag{2.1}$$

To obtain a realistic effect of the foundation on the motion, its "in-phase" mass M* must be included in the inertia term of equation (2.1). Following the suggestion of Veletsos [27], the total foundation reaction q* is

$$q^* = q - M^* \frac{\partial^2 y}{\partial t^2}. \tag{2.2}$$

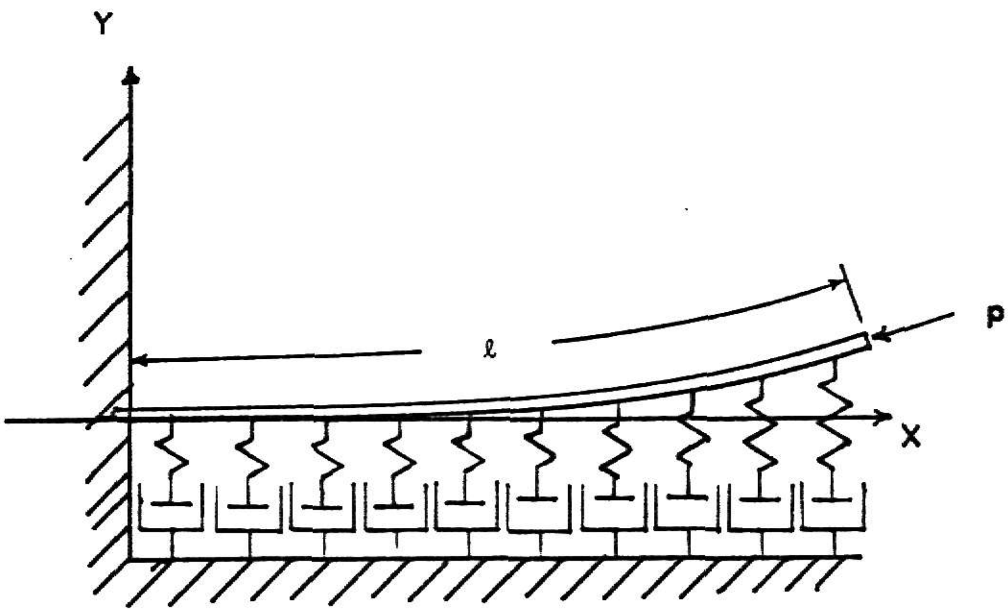


Figure 2.1 Cantilever Column Supported with Maxwell Foundation Subjected to Follower Force.

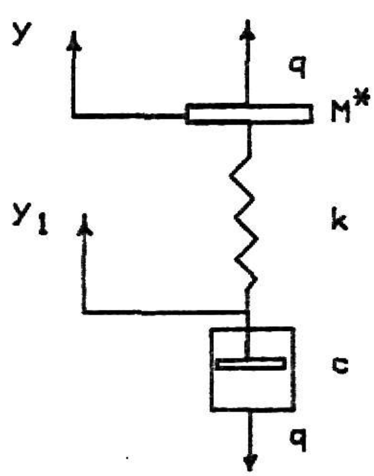


Figure 2.2 Maxwell Viscoelastic Foundation Model

Consequently, equation (2.1) becomes

$$EI \frac{\partial^4 y}{\partial x^4} + p\frac{\partial^2 y}{\partial x^2} + (\rho+M^*)\frac{\partial^2 y}{\partial t^2} = -q^*. \qquad (2.3)$$

The right hand side of equation (2.3) depends on the type of visco-elastic foundation used.

2.2 Maxwell Foundation

Freudenthal and Lorsch [27] reported that a visco-elastic medium reproduces actual foundation behavior better than an elastic medium since its force-deformation relations are time dependent. For analysis, the continuum is represented by independent visco-elastic elements, each supporting a differential beam element. The Maxwell model shown in Figure 2.2, consists of an elastic spring and viscous dashpot combined in series, and represents the characteristics of creep and relaxation.

Using the procedure by Lin [28], let q^* be the pressure acting on the spring. Referring to Figure 2.2, let y_1 be the displacement of the node connecting k and c. The force q is transmitted through both the spring and the dashpot so the following relationships hold. First, note that

$$q^* = k(y-y_1). \qquad (2.4)$$

Dividing by k and Δt, an increment of time, yields

$$\frac{1}{\Delta t}\left[\frac{q^*}{k}\right] = \frac{1}{\Delta t}(y-y_1). \qquad (2.5)$$

In the limit Δt approaches zero and equation (2.5) takes the form

$$\frac{1}{k}\left[\frac{\partial q^*}{\partial t}\right] = \frac{\partial y}{\partial t} - \frac{\partial y_1}{\partial t}. \qquad (2.6)$$

From the force in the dashpot,

$$\frac{1}{c}\, q^* = \frac{\partial y_1}{\partial t}. \tag{2.7}$$

Using equations (2.6) and (2.7)

$$\frac{\partial y}{\partial t} = \frac{1}{k}\frac{\partial q^*}{\partial t} + \frac{1}{c}\, q^*. \tag{2.8}$$

Differentiating with respect to t, equation (2.3) becomes

$$\frac{\partial q^*}{\partial t} = -\left[EI\frac{\partial^5 y}{\partial t\partial x^4} + P\frac{\partial^3 y}{\partial t\partial x^2} + (\rho+M^*)\frac{\partial^3 y}{\partial t^3}\right]. \tag{2.9}$$

Substituting equations (2.3) and (2.9) into equation (2.8) yields the following partial differential equation of motion for the Maxwell foundation.

$$\frac{EI}{k}\frac{\partial^5 y}{\partial t\partial x^4} + \frac{EI}{c}\frac{\partial^4 y}{\partial x^4} + \frac{P}{k}\frac{\partial^3 y}{\partial t\partial x^2} + \left[\frac{\rho+M^*}{k}\right]\frac{\partial^3 y}{\partial t^3} +$$

$$+ \frac{P}{c}\frac{\partial^2 y}{\partial x^2} + \left[\frac{\rho+M^*}{c}\right]\frac{\partial^2 y}{\partial t^2} + \frac{\partial y}{\partial t} = 0. \tag{2.10}$$

Introducing nondimensional parameters

$$w = \frac{y}{\ell}\,,\quad \xi = \frac{x}{\ell}\,,\quad \tau = t\left[\frac{(\rho+M^*)\ell^4}{EI}\right]^{-1/2}$$

$$k = \frac{k\ell^4}{EI}\,,\quad P = \frac{p\ell^2}{EI}\,,\quad C = c\left[\frac{\ell^4}{(\rho+M^*)EI}\right]^{1/2}, \tag{2.11}$$

equation (2.10) can be rewritten in the nondimensional form

$$\frac{C}{K}\frac{\partial^5 w}{\partial\tau\partial\xi^4} + \frac{\partial^4 w}{\partial\xi^4} + \frac{C}{K}P\frac{\partial^3 w}{\partial\tau\partial\xi^2} + \frac{C}{K}\frac{\partial^3 w}{\partial\tau^3} + P\frac{\partial^2 w}{\partial\xi^2} + \frac{\partial^2 w}{\partial\tau^2} + C\frac{\partial w}{\partial\tau} = 0. \tag{2.12}$$

2.3 Separable Solution

As shown in Figure 2.1 the column is fixed at one end and subjected to a constant tangential follower force p on the free end. The entire length is supported by the visco-elastic foundation. The complete boundary-value problem is given by the equation of motion (2.10) about the undisturbed equilibrium position and the following boundary conditions,

$$y(o,t) = 0,\; y'(o,t) = 0,\; y''(\ell,t) = 0,\; Y'''(\ell,t) = 0. \tag{2.13}$$

These can be rewritten in terms of the nondimensional variables as

$$w(o,\tau) = 0,\; w'(o,\tau) = 0,\; w''(1,\tau) = 0,\; w'''(1,\tau) = 0 \tag{2.14}$$

Consider a solution of the form

$$w(\xi,\tau) = \phi(\xi)\; T\,(\tau). \tag{2.15}$$

Substituting this into equation (2.12) yields

$$(C\dot{T} + KT)\phi'''' + P(C\dot{T} + KT)\phi'' + (C\dddot{T} + K\ddot{T} + CK\dot{T})\phi = 0\;. \tag{2.16}$$

Dividing by $\phi \cdot (C\dot{T} + KT)$, equation (2.16) can be rearranged as

$$\frac{\phi'''' + P\phi''}{\phi} = \frac{-(C\dddot{T} + K\ddot{T} + CK\dot{T})}{C\dot{T} + KT}. \tag{2.17}$$

If equation (2.17) is to hold for all values of ξ and τ, both sides must equal a constant. Let this constant be denoted by λ^2. Equation (2.17) now separates into the two ordinary differential equations.

$$\phi'''' + P\phi'' - \lambda^2\,\phi = 0 \tag{2.18}$$

$$C\dddot{T} + K\ddot{T} + C(\lambda^2 + K)\dot{T} + \lambda^2 KT = 0 \tag{2.19}$$

The boundary conditions in terms of ϕ become

$$\phi(0) = 0,\ \phi'(0) = 0,\ \phi''(1) = 0,\ \phi'''(1) = 0. \tag{2.20}$$

It is observed that the boundary value problem defined by equations (2.18) and (2.20) must be solved for λ^2 in order to define equation (2.19), which determines the temporal part of the solution. The linear differential operator in equation (2.18) is not self-adjoint due to the boundary conditions given by equation (2.20) and therefore the eigenvalue, λ^2, is not always real. As a result, the constant λ^2 is complex and can be defined as

$$\lambda^2 = \alpha \pm i\beta,\ i = (-1)^{1/2}\,. \tag{2.21}$$

Let the solution of equation (2.19) be of the form

$$T(\tau) = A\ e^{z\tau} \tag{2.22}$$

where

$$z = \sigma + iw. \tag{2.23}$$

Substituting equation (2.22) into equation (2.19) yields

$$C\ z^3 + K\ z^2 + C(\lambda^2 + K)\ z + \bar{\lambda}^2\ K = 0 \tag{2.24}$$

Since, in general, λ^2 is complex, the above polynomial has complex coefficients. The roots of the complex polynomial from equation (2.22) determine the stability criteria for the system. From equation (2.23), the necessary and sufficient condition for the solution to remain bounded is $\sigma \leq 0$. If $\sigma > 0$, the solution is unbounded and instability prevails. Therefore, the stability conditions for the system are functions of K, C, and λ^2 which depend on the loading parameter P.

2.4 Galerkin's Approximation of the Eigenvalue Problem

First, the eigenvalue problem given by equations (2.18) and (2.20) is considered. It is known that the general solution of equation (2.18) can be expressed as

$$\phi(\xi) = C_1 \sin a\,\xi + C_2 \cos a\,\xi + C_3 \sinh b\,\xi + C_4 \cosh b\xi. \qquad (2.25)$$

Upon substitution of boundary conditions (2.20), this yields a transcendental equation which can be solved numerically to obtain the real as well as the complex eigenvalues. However, this approach is very complicated and time consuming as shown by Morgan [8]. In this study, Galerkin's method has been utilized to obtain approximate expressions for the eigenvalues. Following Ivan-Iwanowski [25], a two term approximation for $\phi(\xi)$ is assumed as

$$\phi(\xi) = C_1 S_1(\xi) + C_2 S_2(\xi);\quad C_1,\ C_2 \text{ constants} \qquad (2.26)$$

where $S_1(\xi)$ and $S_2(\xi)$ are the two eigenfunctions of the free vibration of a uniform cantilever beam given by

$$s_1(\xi) = 4.148(\cos 1.875\,\xi - \cosh 1.875\xi) - 3.037(\sin 1.875\,\xi - \sinh 1.875\xi) \qquad (2.27)$$

$$s_2(\xi) = 53.640(\cos 4.694\,\xi - \cosh 4.694\xi) - 54.631(\sin 4.694\,\xi - \sinh 4.694\xi). \qquad (2.28)$$

Then according to Galerkin's method

$$\int_0^1 R(\xi)\, s_1(\xi)\, d\xi = 0 \qquad (2.29)$$

$$\int_0^1 R(\xi)\, s_2(\xi)\, d\xi = 0, \qquad (2.30)$$

where the residual error $R(\xi)$ is defined as,

$$R(\xi) = c_1[s_1''''(\xi) + Ps_1''(\xi) - \lambda^2 s_1(\xi)] + c_2[s_2''''(\xi) + Ps_2''(\xi) - \lambda^2 s_2(\xi)]. \tag{2.31}$$

Carrying out the integrations of equations (2.29) and (2.30) yields

$$\begin{bmatrix} 12.3596 + 0.8713\ P - \lambda^2 & -151.5497 \\ 0.1460 & 485.4811 - 13.5287\ P - \lambda^2 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = 0 \tag{2.32}$$

Setting the determinant of the matrix equal to zero results in

$$\lambda^4 - (497.8 - 12.66\ P)\ \lambda^2 + 10.34\ P^2 + 255.8\ P + 6{,}000. = 0 \tag{2.33}$$

Using equations (2.21) and (2.33), the real and the complex parts of the eigenvalues can be found in terms of the load P as

$$\alpha = 248.9 - 6.329\ P \tag{2.34}$$

$$\beta = (-55{,}961. + 3{,}406.\ P - 29.71\ P^2)^{1/2}. \tag{2.35}$$

Since the eigenvalue problem is the same as the original Beck's column, it is of interest to compare the results of this approximate analysis with the results of the exact solutions obtained by Beck [9] and Morgan [8].

Beck [9] found that the two lowest eigenvalues become equal at the critical load of $P_f = 20.05$. His study showed that this critical load occurs when λ^2 becomes complex since higher loads would cause one z root of equation (2.23) to have a positive real part, inducing flutter instability. This would be true because $z = \pm i\ \lambda^2$. From equations (2.34) and (2.35) we find that the approximate critical load is 19.87. This load is hereafter by P_{fA}.

Exact eigenvalues of this problem up to a load value of P = 100 have been reported by Morgan [8].

The approximate Eigenvalues obtained from equation (2.35) are compared with the values obtained by Morgan in Figure 2.3. Note that α and β are the real and the imaginary parts of λ^2, respectively. Points p_0 through p_4 mark the complex roots corresponding to loads of special interest. The first two real natural frequencies corresponding to P = 0 are shown at points p_0. The roots move toward each other on the real axis, with increasing P, until the load becomes P_f at point p_1. Complex roots exist for loads higher than P_f. At point P_2 the real parts become zero, which corresponds to a load P = 37.7 for the actual solution and P = 39.3 by the approximation method. This is the allowable load for the system in the presence of infinite external viscous damping, as shown by many researchers. [14,15,16] Higher loads yield eigenvalues with negative real parts. This is demonstrated with points p_3 and p_4 which represent loads of 50 and 75, respectively. It is observed that there is a considerable difference between the exact and the approximate eigenvalues for loads greater than p_2. This difference and its effect on system stability will be discussed in more detail in the next chapter.

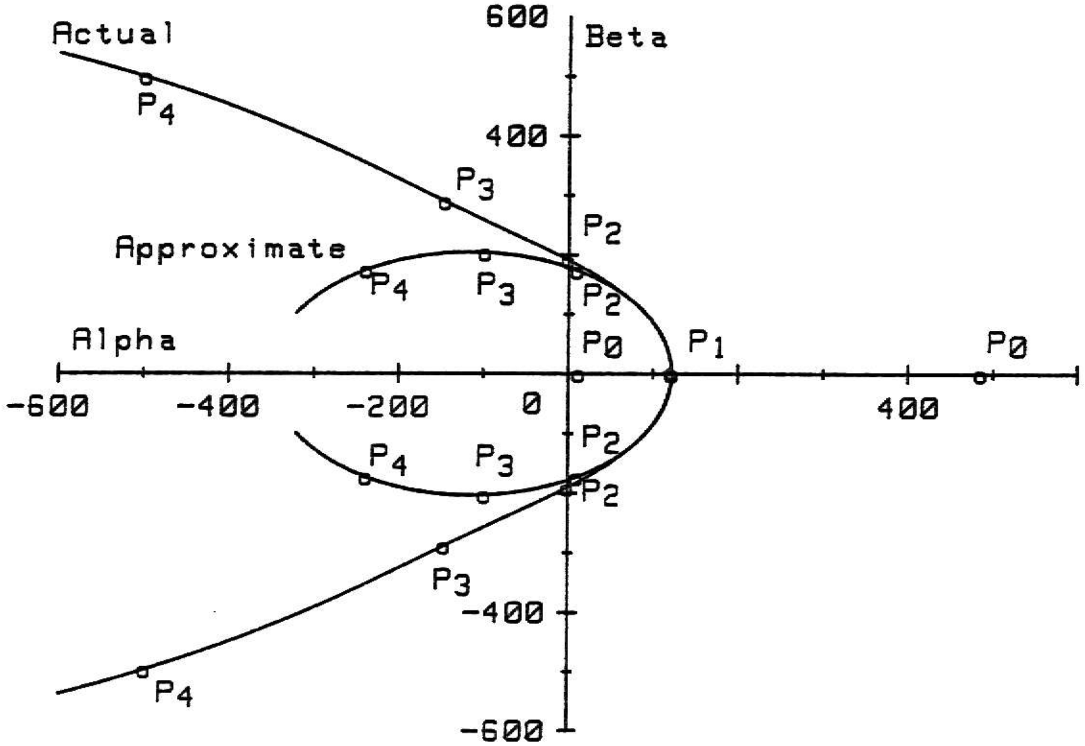


Figure 2.3 Complex Eigenvalues of Transcendental Equation

Chapter 3

STABILITY ANALYSIS AND RESULTS

The time dependent part of the assumed solution governed by equation (2.20) determines the stability criteria of the system. If the motion about the equilibrium configuration after a disturbance is bounded then the system is considered stable; otherwise it is unstable. The system is considered to be asymptotically stable if the steady state motion returns to the original equilibrium configuration.

Assuming a solution of the form of equation (2.23), the temporal equation becomes a third order characteristic equation in z with complex coefficients, as seen from equation (2.25). If the real parts of the roots of this polynomial are negative, then asymptotic stability is insured. The problem thus reduces to the discussion of a characteristic equation with complex coefficients.

3.1 Characteristic Equations with Complex Coefficients

Consider an n^{th} order characteristic equation represented by

$$P(z) = c_0 z^n + c_1 z^{n-1} + \ldots + c_n = 0 \tag{3.1}$$

where the coefficients $c_0, c_1, \ldots, c_n$ are, in general, complex. Let the polynomial $R(z)$ be defined by

$$R(z) = \bar{c}_0\, z^n + \bar{c}_1\, z^{n-1} + \ldots + \bar{c}_n = 0, \tag{3.2}$$

where the bars denote the complex conjugates of the quantities. Since both P(z) and R(z) are independently zero their product will also be zero. Multiplying equations (3.1) and (3.2) results in a polynomial

$$Q(z) = P(z)\ R(z) = a_o z^{2n} + a_1\ z^{2n-1} + \ldots + a_n = 0, \tag{3.3}$$

where the real coefficients $a_0, a_1, \ldots, a_n$ are given by

$$\begin{aligned}
a_0 &= c_0\bar{c}_0, \\
a_1 &= c_o\bar{c}_1 + \bar{c}_0 c_1, \\
a_2 &= c_0\bar{c}_2 + c_1\bar{c}_1 + c_2\bar{c}_0, \\
&\cdot \\
&\cdot \\
&\cdot \\
a_n &= c_n\bar{c}_n
\end{aligned} \tag{3.4}$$

The 2n roots of the equation (3.3) are the n roots of polynomial P(z) and the n roots of polynomial R(z). If the roots of P(z) are real or complex conjugate pairs then the 2n roots will occur as n roots of multiplicity two. [30]

This procedure is now applied to equation (2.25). Equations corresponding to P(z) and R(z) become

$$P(z) = C\ z^3 + K\ z^2 + C[\alpha+K+i\beta]z + K[\alpha+i\beta] = 0, \tag{3.5}$$

$$R(z) = C\ z^3 + K\ z^2 + C[\alpha+K-i\beta]z + K[\alpha-i\beta] = 0, \tag{3.6}$$

Multiplying P(z) and R(z) yields a sixth order equation with real coefficients

$$C^2z^6 + 2KCz^5 + [K^2+2C^2(\alpha+K]z^4 + C[4\alpha K+2K^2]z^3$$

$$+ [2K^2\alpha + C^2(\alpha+K)^2 + C^2\beta^2]z^2 + 2KC[\alpha(\alpha+K)+\beta^2]z$$

$$+ K^2[\alpha^2+\beta^2] = 0. \tag{3.7}$$

The Routh-Hurwitz-Mikhailov Criteria is used in the next section to analyze the roots of equation (3.7).

3.2 Routh-Hurwitz-Mikhailov (RHM) Criteria [31,32,33]

Routh and Hurwitz developed an algebraic criteria to determine the stability of systems described by characteristic equation (3.3) with real coefficients. Hurwitz showed that the necessary and sufficient condition for a system to be asymptotically stable is that the principle minors of the square matrix

$$\begin{bmatrix} a_1 & a_3 & a_5 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ a_0 & a_2 & a_4 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 0 & a_1 & a_3 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & a_{n-3} & a_{n-1} & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & a_{n-4} & a_{n-2} & a_A \end{bmatrix} \tag{3.8}$$

be positive. The conditions can be written as

$$\Delta_1 = a_1 > 0 , \tag{3.9}$$

$$\Delta_2 = \begin{vmatrix} a_1 & a_3 \\ a_o & a_2 \end{vmatrix} > 0, \tag{3.10}$$

$$\Delta_3 \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} > 0 \tag{3.11}$$

and so on up to Δ_n. The Δ_n can also be expressed as

$$\Delta_n = a_n \Delta_{n-1} , \tag{3.12}$$

which reduces the positive definiteness condition on Δ_n to

$$a_n > 0. \tag{3.13}$$

The "stability limit" is determined by

$$\Delta_{n-1} = 0, \tag{3.14}$$

with a pair of imaginary roots, or by

$$a_n = 0, \tag{3.15}$$

with a zero root, provided all other determinants are positive. However, an equivalent approach called the Routh-Hurwitz-Mikhailov (RHM) criteria is used here also to derive the necessary and sufficient stability conditions for the sixth order equation (3.7). The details of the method are covered by Morgan [8] and Popov [34].

The sixth order characteristic equation (3.7) has the following real coefficients

$$a_0 = C^2 ,$$
$$a_1 = 2KC ,$$
$$a_2 = K^2 + 2C^2 (\alpha+K) ,$$
$$a_3 = 2CK (2\alpha+K) ,$$
$$a_4 = 2\alpha K^2 + C^2[(\alpha+K)^2 + \beta^2] , \quad (3.16)$$
$$a_5 = 2KC[(\alpha+K)\alpha + \beta^2] ,$$
$$a_6 = K^2 (\alpha^2+\beta^2) .$$

According to the RHM criteria [8,34], for asymptotic stability, the coefficients a_0, a_1, ... , a_6 should remain positive and the following inequalities should be satisfied.

$$a_3(a_1a_2 - a_0a_3) - a_1(a_1a_4 - a_0a_5) > 0 , \quad (3.21)$$

and

$$[2a_1^3 a_6 + (a_3^2 - 2a_1 a_5) (a_1a_2 - a_0a_3) - a_1a_3 (a_1a_4 - a_0a_5)]$$
$$< [a_3(a_1a_2 - a_0a_3) - a_1(a_1a_4 - a_0a_5)] (a_3^2 - 4a_1a_5)^{1/2} . \quad (3.22)$$

Equation (3.22) can be rewritten as

$$(a_1a_2 - a_0a_3) [a_5(a_4a_3 - a_2a_5) + a_6(2a_1a_5 - a_3^2)] +$$
$$+ (a_1a_4 - a_0a_5) [a_1a_3a_6 - a_5 (a_1a_4 - a_0a_5)] - a_1^3 a_6^2 > 0 . \quad (3.23)$$

At the stability boundary equation (3.23) becomes an equality. Using (3.16) and substituting into the equations above, the following expressions are found.

$$(a_1a_2 - a_0a_3) = 2K^2\, C(K + C^2)\ ,$$

$$(a_1a_4 - a_0a_5) = 2K^2\, C(2\alpha K + \alpha C^2 + K\, C^2)\ ,$$

$$a_5(a_4a_3 - a_2a_5) + a_6\,(2a_1a_5 - a_3^2) = 4C^2\, K^3\, [(K + C^2)\alpha^4 +$$

$$+ (3C^2\, K + 2K^2)\alpha^3 + (3K^2\, C^2 + 2K\beta^2)\alpha^2 + (KC^2\beta^2 + K^3\, C^2 -$$

$$- 2K^2\beta^2)\alpha + (K - C^2)\beta^4 + K^2\,(C^2 - K)\beta^2]\ , \tag{3.24}$$

$$a_1a_3a_6 - a_5(a_1a_4 - a_0a_5) = - 4C^2K^3\, [C^2\alpha^3 + K(K +$$

$$+ 2C^2)\alpha^2 + C^2(K^2+\beta^2)\alpha - K\,(K-C^2)\beta^2]\ ,$$

$$a_1^3a_6^2 = 8C^3\, K^7(\alpha^2 + \beta^2)^2\ .$$

From (3.23) the stability boundary is given by

$$d_1C^4 + d_2C^2 + d_3 = 0, \tag{3.25}$$

where

$$d_1 = \alpha\beta^2\,(\alpha + K) + \beta^4\ ,$$

$$d_2 = \alpha K^2\,(2\beta^2 - K^2) - K^3\beta^2\ , \tag{3.26}$$

$$d_3 = K^4\,\beta^2\ .$$

Equation (3.25) is quadratic in C^2 with the roots

$$C_{1,2}^2 = \frac{d_2}{2d_1} \pm \frac{1}{2d_1}\,(d_2^2 - 4d_1d_3)^{1/2}\ . \tag{3.27}$$

The right hand side of the above equation must be non-negative for real damping. In addition, the expression

$$(d_2^2 - 4d_1d_3)^{1/2} \tag{3.28}$$

must also be non-negative. Combining (3.26) and (3.28), the following inequality is obtained.

$$[\alpha + \frac{\beta^2}{K}]\ (K^2 - 4\beta^2)^{1/2} \geq 0\ . \tag{3.29}$$

From equation (3.27), if the inequality

$$4d_1d_3 \geq 0 \tag{3.30}$$

is satisfied, then the right hand side is non-negative, as long as (3.29) remains valid. Then from (3.30), the system parameters must satisfy the inequality

$$\alpha^2 + \alpha K + \beta^2 \geq 0\ . \tag{3.31}$$

Therefore, to insure real damping, inequalities (3.29) and (3.31) place bounds on K as

$$2\ \beta \leq K \leq \frac{\beta^2}{-\alpha} \tag{3.32}$$

and

$$K \leq \frac{\alpha^2+\beta^2}{-\alpha} \tag{3.33}$$

respectively.

A simple computer program in BASIC language was written to find the roots of equation (3.25) for a given set of α and β, corresponding to a unique load P, as computed by Galerkin's approximation method. The program takes into consideration the stability boundary and insures that the coefficients defined in equation (3.16) remain positive and the inequality of (3.21) holds true. The two roots of equation (3.25) were obtained for sets of α and β. This computation was performed to generate the stability boundaries for various values of the spring parameter K. An HP 9845 desktop computer and an HP 9872 plotter system was used to produce the graphical results in this chapter.

Figure 3.1 shows that at lower values of K there is very little difference between the approximate and the exact stability boundaries. As K increases past a value of 300, the boundaries differ (as seen from Figure 3.2) but do show some similarities. Like the exact analysis, the approximate results also show that for a fixed K, there is an optimum load value that can be associated with the damping constant C. These results verify that the approximate solution does provide an insight into the behavior of the original problem.

For a fixed K, the two roots of the damping constant approach each other as the load is increased to a maximum value where the two roots become equal. From equation (3.27), the roots become equal when the expression (3.28) is zero. For the values of K shown in Figure 3.1 and Figure 3.2, the expression is zero when

$$K = 2\beta , \tag{3.34}$$

i.e., the maximum load occurs when β equals K/2. Then equation (3.27) reduces to

$$C^3 = -\frac{d_2}{2d_I} , \tag{3.35}$$

which, in terms of the system parameters defines the damping value at the peak load as

$$C^2 = \frac{4\beta^2}{\alpha+\beta} . \tag{3.36}$$

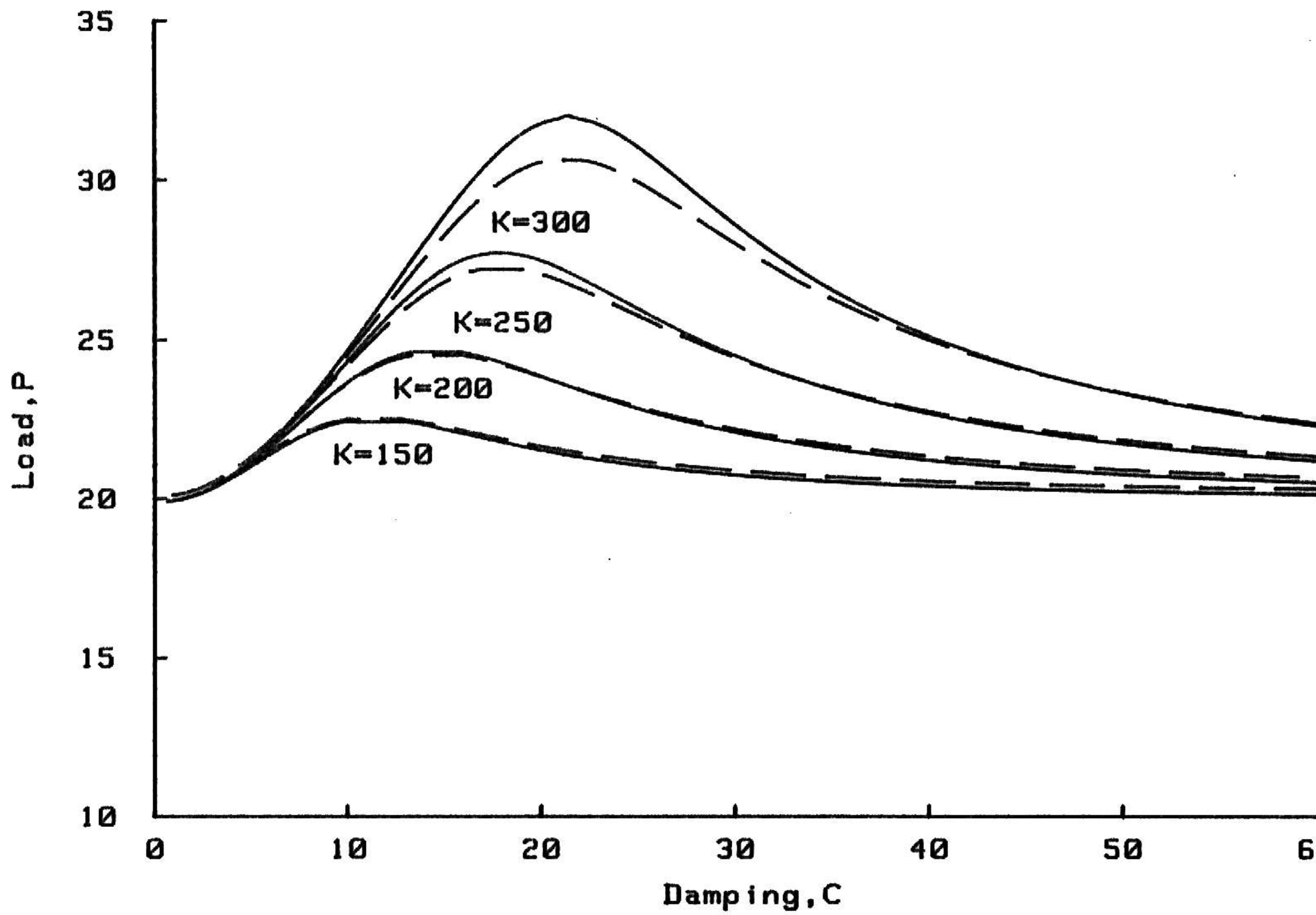
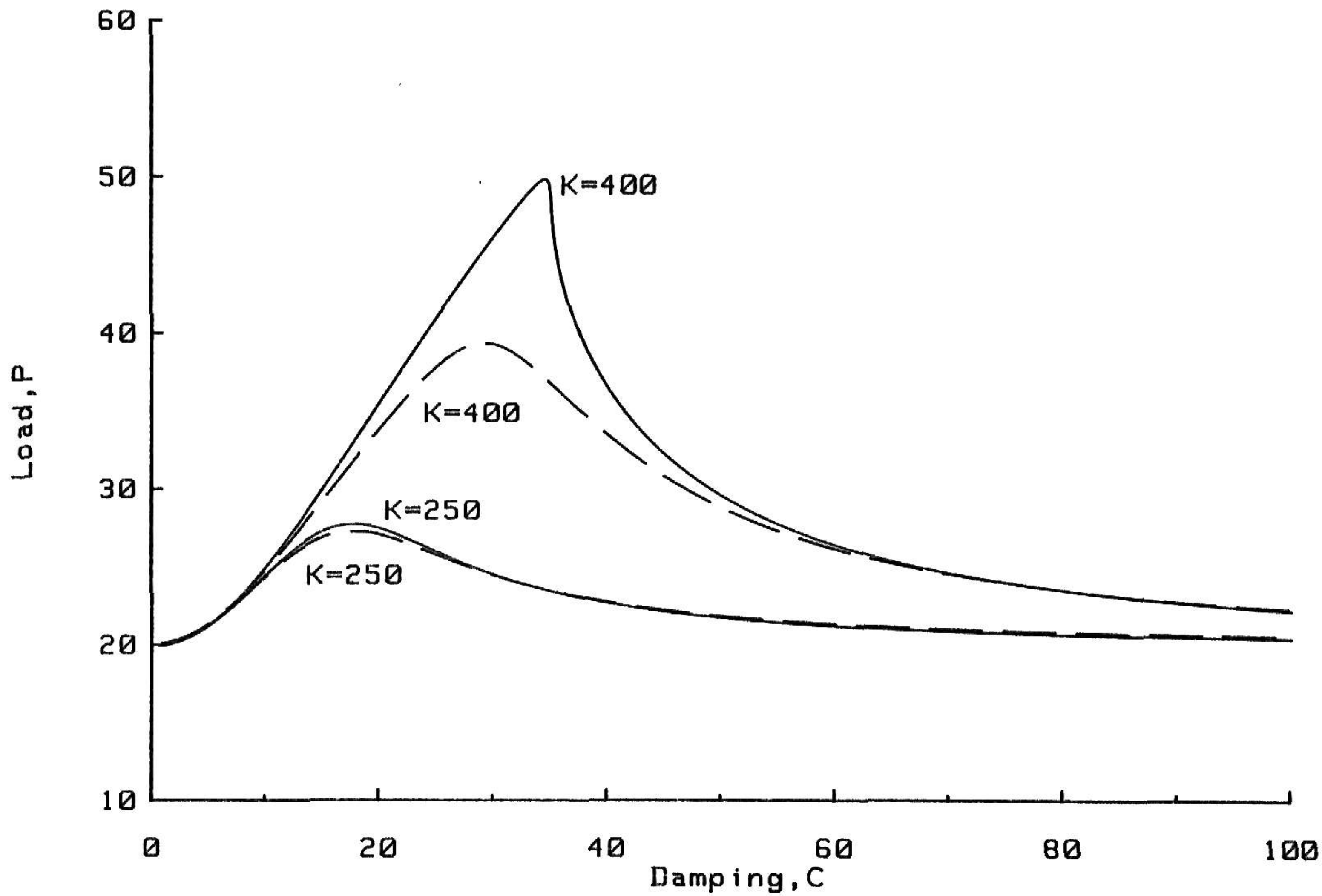


Figure 3.1 Effect of Maxwell Foundation on Flutter Load (0 < K < 300) —— Approximate Solution; --- Exact Solution.



<u>Figure 3.2</u> Effect of Maxwell Foundation on Flutter Load (0 < K < 400) —— Approximate Solution; --- Exact Solution.

Beyond this value of load, the roots of equation (3.25) become complex.

For stiffness values greater than K = 407.73, the expression again becomes zero if

$$\alpha + \frac{\beta^2}{K} = 0 \ . \tag{3.37}$$

For these values of K, as the load increases, the two roots of C approach each other and become equal when

$$K = \frac{\beta^2}{-\alpha} \ . \tag{3.38}$$

Substituting equation (3.38) in equation (3.27) yields the value of the damping constant for this load as

$$C^2 = - \frac{\beta^4}{\alpha^3} \ . \tag{3.39}$$

As the load is increased from $P_{fA} = 19.9$, the roots of C approach each other and become equal when equation (3.38) is satisfied. This results in the maximum load. Beyond this load, the roots diverge and inequality (3.21) is violated.

Figure 3.3 shows a comparison between the exact and the approximate results for relatively larger values of K. It is seen that large values of K result in lower values of peak loads. The agreement between the two results is good because of the fact that at lower values of P, the approximate values of α and β are close to the exact ones (c.f. Figure 2.3).

The stability characteristics for $0 < K < \infty$ can be summarized as follows: For $K < K_A^*$ the peak load occurs when equation (3.34) is satisfied, while for $K > K_A^*$ the peak load is attained when equation (3.38) is satisfied. At K_A^* both constraints on the stiffness parameter K are met simulataneously, i.e.,

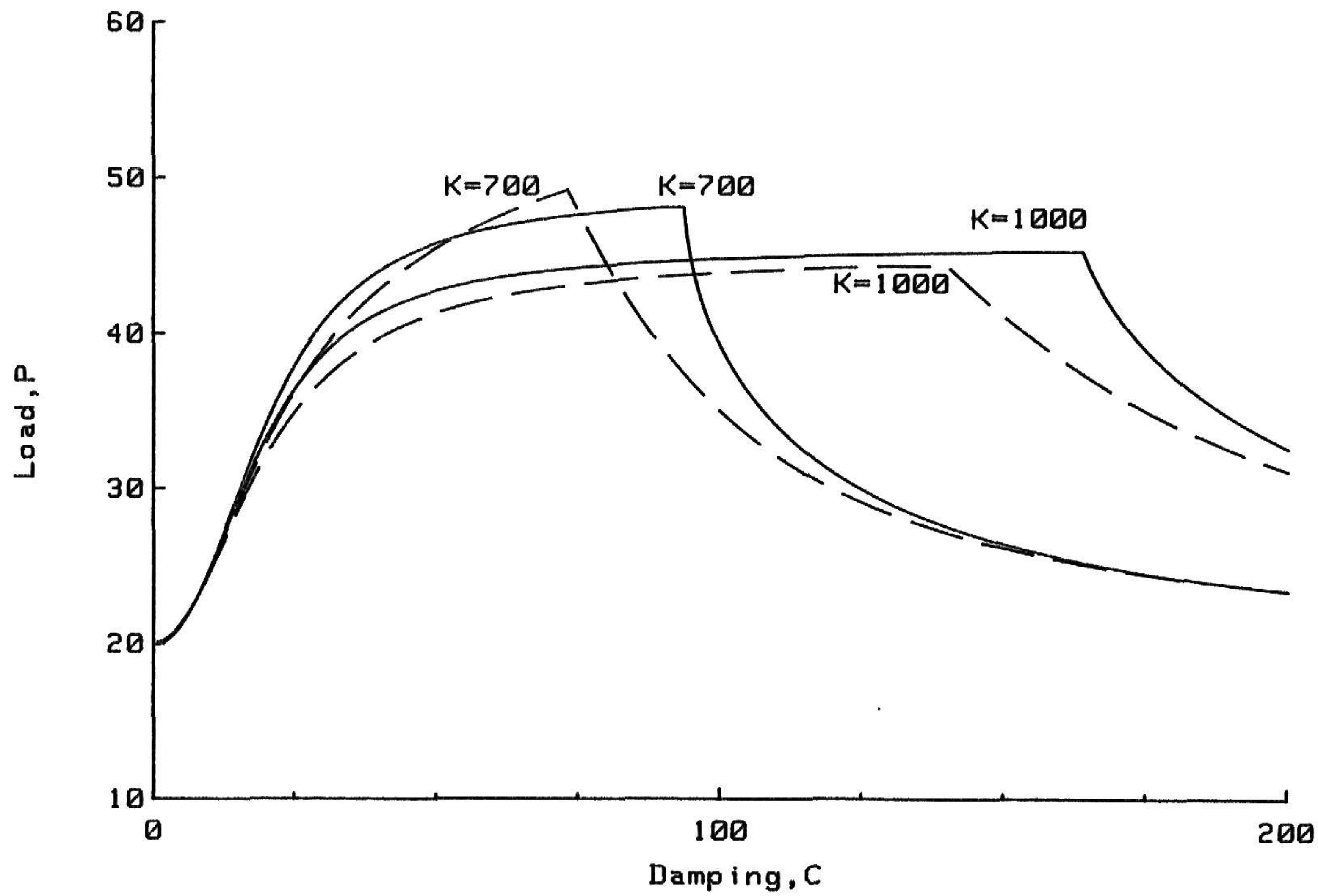


Figure 3.3 Stability region of Maxwell Foundation for Large Values of K; —— Approximate Solution; --- Exact Solution.

$$2\ \beta = \frac{\beta^2}{-\alpha} \tag{3.40}$$

or

$$\beta = -2\alpha \tag{3.41}$$

This relationship is satisfied at a unique load value of approximately 55.4 (α = -101.68, β = 203.86). The value of the stiffness parameter K for this load is

$$K_A^* = 2\ \beta = 407.7318, \tag{3.42}$$

and the required damping constant is given by

$$C_A^* = 2\ \beta(\alpha + \beta)^{-1/2} = K_A^*\ (\alpha + \beta)^{-1/2} = 40.34. \tag{3.43}$$

A comparison is made in Figure 3.4 between the stability boundaries corresponding to K_A^* and the exact value K* as reported by Morgan [8]. It is seen that at this point the approximation breaks down because there is a significant difference between the exact and the approximate complex eigenvalues as seen from Figure 2.3. In fact the approximate values of α and β yield results that are physically impossible.

From Figure 3.4, it is seen that for a value of damping constant C close to C_A^*, there exist two values of load P which lie on the stability boundary. This anamoly is attributed to the incorrect values of α and β for larger loads. Surprisingly, the approximate maximum load, 55.4 designated as P_A^*, compares very favorably to P* = 54.9 [8].

The approximate stability boundaries for a wider range of K is represented in Figure 3.5. Notice that the combination of K and C allow an optimum load P. As K is increased from zero to K_A^*, peak loads increase from P_{fA} (=19.9) to P_A^* (=55.4). For values of $K > K_A^*$, the peak loads

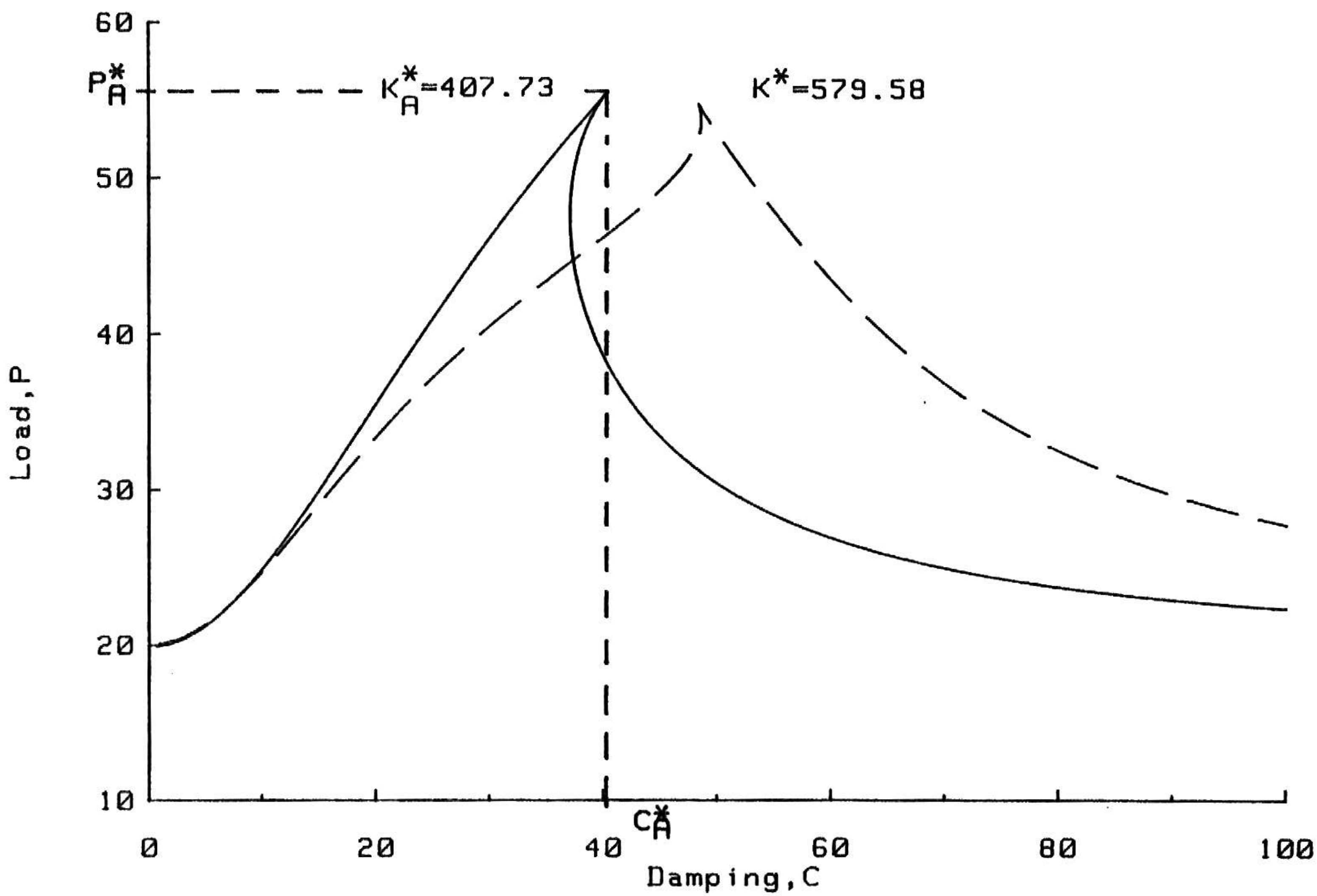


Figure 3.4 Effect of Maxwell Foundation on Flutter Load at the Peak Load for Stability; —— Approximate Solution; --- Exact Solution.

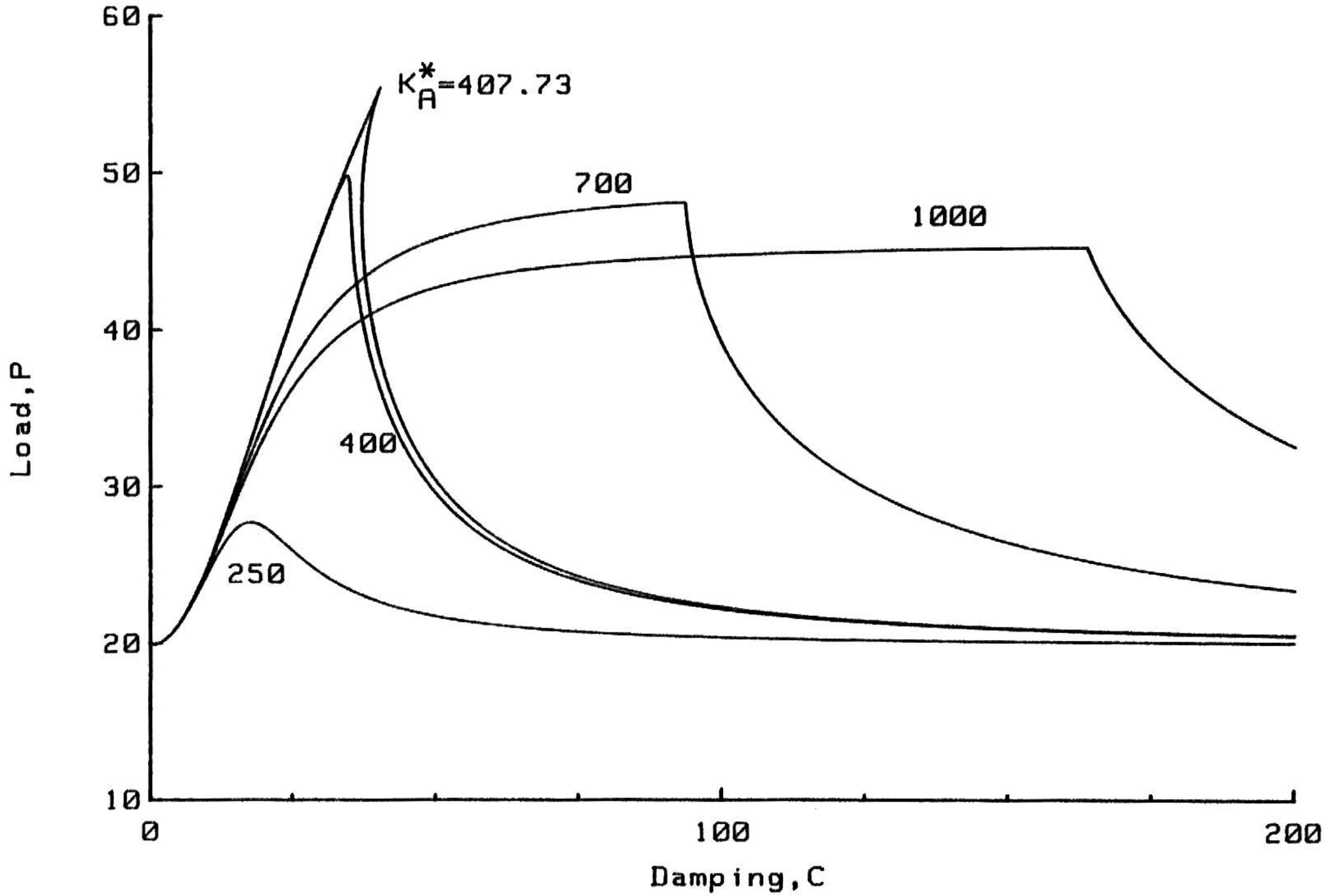


Figure 3.5 Stability Region for Maxwell Foundation ($0 \leq K \leq 1000$)

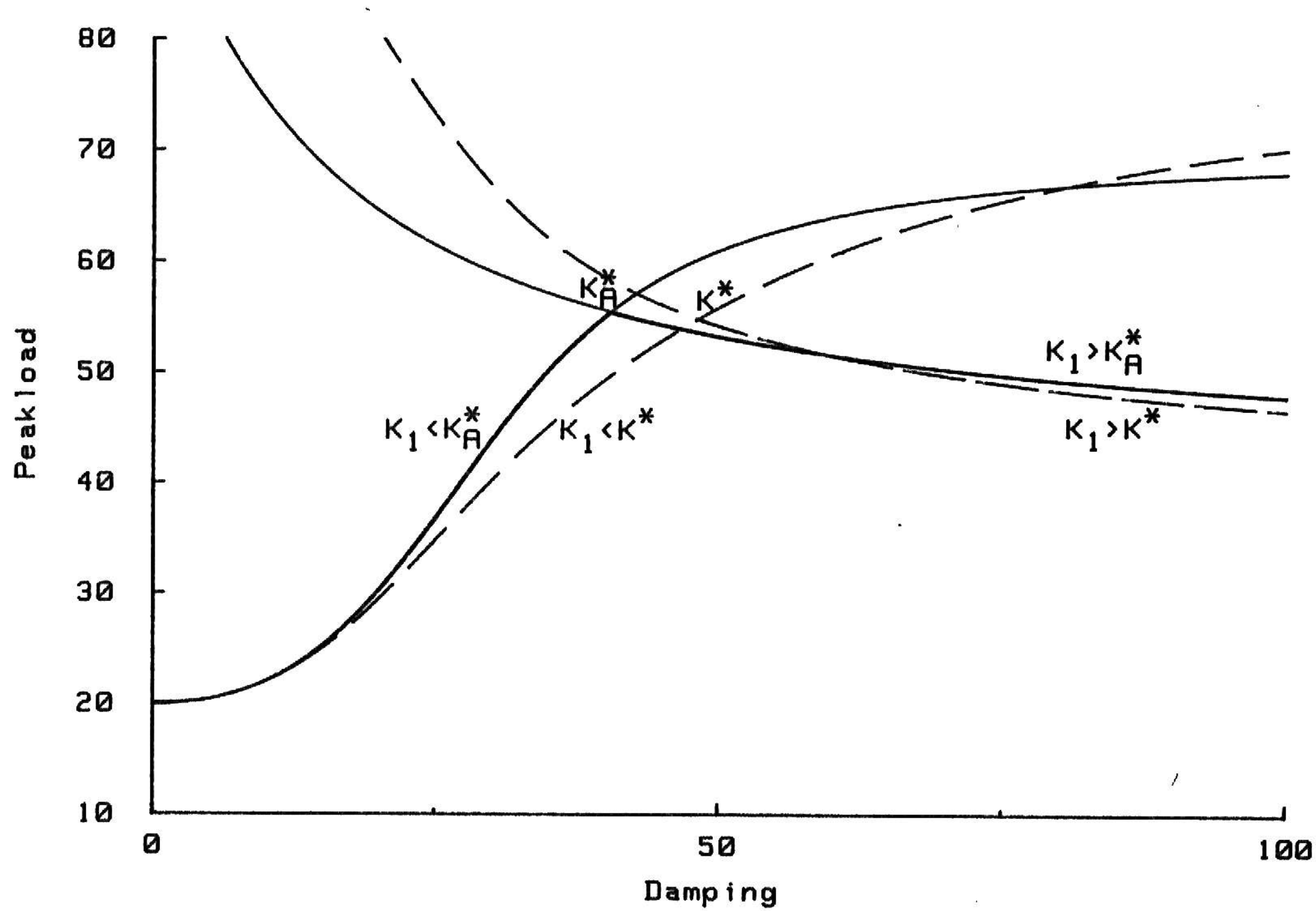


Figure 3.6 Peak Flutter Loads for Maxwell Foundation ($0 \leq K \leq \infty$)

decrease while the corresponding damping values increase. This behavior is expected, since the Maxwell model becomes a viscous dashpot as the value of K approaches infinity. Taking the limit of equation (3.25) as K approaches infinity reduces the expression for the stability limit to

$$-C^2 \alpha + \beta^2 = 0, \tag{3.44}$$

or

$$\alpha = \frac{\beta^2}{C^2} . \tag{3.45}$$

The critical load for Beck's column in the presence of infinite viscous damping is 37.7. The approximate value given by this study is comparable and has the value of 39.3. This verifies the results of the viscous damping case as obtained by Plaut and Infante [14].

A plot of the peak loads for $0 \leq K \leq \infty$ is shown in Figure 3.6. For a stiffness less than K_A^* the peak load is along the curve described by equation (3.36), while for a stiffness greater than K_A^* the peak load is along the curve obtained from equation (3.39). The intersection occurs at $K = K_A^*$. Therefore, the peak loads follow the path represented by the solid line. Morgan's [8] exact results are also plotted on the same graph for a comparison.

The following Table summarizes the comparison between the exact and the approximate values of some important parameters.

Parameters	Exact Values as Obtained by Morgan [8]	Approximate Values From This Analysis
P_f	20.05	19.9
P*	54.9	55.4
P_I	37.7	39.3
C*	48.24	40.34
K*	579.58	407.73

Chapter 4

DISCUSSION AND CONCLUSIONS

This investigation deals with a special nonconservative elastic stability problem which contains a follower force. More specifically the effect of a Maxwell-type viscoelastic foundation on the stability of a tangentially loaded column is studied through an application of the Galerkin's technique.

The equation of motion for a cantilever column subjected to a follower load and continuously supported by the Maxwell Foundation is derived in Chapter 2. A separable solution exists and the resulting boundary value problem can be solved via the Galerkin's procedure. The first two eigenfunctions of the free vibration of a cantilever beam are utilized in the Galerkin's approach to generate approximate eigenvalues. These eigenvalues are, in general, complex because the associated boundary value problem is not self-adjoint. These eigenvalues are then used in the temporal equation to derive the stability conditions for the system. Routh-Hurwitz-Mikhailov (RHM) criteria is applied to yield closed form expressions which reveal the effect of foundation parameters on the stability of the column.

A comparison between the approximate and the exact eigenvalues [8] is presented in Figure 2.3. It is seen that the real and the imaginary parts of the approximate eigenvalues compare well with the exact ones up to the point where the real part becomes zero. Past this point, the exact values of the imaginary part (β) continue to increase whereas the approximate values start decreasing after the load P reaches around 57. Therefore it is expected that the approximate stability boundaries would be somewhat different than the exact stability boundaries as reported by Morgan [8].

The analysis presented in Chapter 3 shows that a Maxwell-type viscoelastic foundation has a stabilizing effect on the column which is subjected to tangential loading. This is in contrast to the case of conservative loading for which such a foundation is known to have no stabilizing effect at all. It is found that for a given value of the stiffness parameter K of the foundation, there is a unique value of the damping parameter C which results in the maximum flutter load. The graphical results presented in Chapter 3 show that there is an excellent agreement between the exact and the approximate stability boundaries for which the flutter load does not cause significant difference between the exact and the approximate eigenvalues. This appears to be the case for $P < 39.3$ which corresponds to the point where the real part of the eigenvalue becomes zero. It is observed that the flutter load increases from P_{fA} to a maximum value as K is increased and then decreases to P_{IA} as K becomes very large, when appropriate damping is present. Although the approximate stability boundary for K_A^* (c.f. Figure 3.4) does not make much sense, the approximate peak load P_A^* (=55.4) is surprisingly very close to the exact P^* (=54.9) obtained by Morgan [8].

In summary, the present study shows that the Galerkin's method can be utilized with some success to investigate the effect of a Maxwell-type viscoelastic foundation on the stability of a column subjected to follower forces. The method is simple and yields results which are quantitatively and qualitatively close to the exact results. The results of the present investigation should be helpful and serve as a guideline for analyzing the behavior of other nonconservative problems.

Many extensions and applications of an approximation method can be proposed from this problem. A few suggestions follow.

1. Study the effect of viscoelastic foundation on the stability of a column loaded by a distributed tangential follower force.
2. Investigate the application of other approximation techniques in the study of dynamic stability problems.
3. Apply such methods to related dynamic stability problems outside the Solid Mechanics area.

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AN APPROXIMATE STABILITY ANALYSIS OF A
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by

Donald R. Pawlowski

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ABSTRACT

The present study investigates the effect of a Maxwell-type visco-elastic foundation on the stability of a cantilever column under the action of a constant tangential follower force at the free end.

It is shown that a separable solution exists for the partial differential equation describing the motion. The resulting boundary value problem is solved approximately through an application of the Galerkin's method.

Routh-Hurwitz-Mikhailov criteria is used to derive stability conditions involving the system parameters. It is found that any combination of foundation parameters increases the flutter load beyond that of the unsupported column. Furthermore, there exists an optimum combination of foundation parameters which yields the maximum flutter load. The approximate results of this study are also compared with the exact results available in the literature. A close agreement is found.