ON CONSTRUCTION OF THE FRACTIONALLY REPLICATED $2^m \times 3^n$ EXPERIMENTAL DESIGN

by

WILLIAM K. HSU

B. A., Tunghai University, 1961

A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Statistics

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1968

Approved by:

Young C. Koh
Major Professor

TABLE OF CONTENTS

I. INTRODUCTION

The use of a factorial set of treatment combinations has now become a widely accepted means of investigation. The principle of factorial experimental designs has been elaborated primarily for agricultural field experiments, but has been found valuable in many other types of experimentation. Yates (1937) has described many designs for factors at two and three levels. By these methods, many factors may be introduced into one experiment, but the total number of plots or other experimental units required for a comprehensive test is large. For example, eight factors each used at two levels require 256 plots for one replication, and six factors each at three levels require 729 plots. It is seldom practicable to carry out experiments involving so many plots. A further device for reducing the number of plots is that of fractional replication, whereby certain treatment interactions are assumed negligible and only a selection of all possible treatment combinations is used in the experiment.

In this report, an attempt has been made to formulate the general principles of construction of fractional asymmetrical designs of the type 2^m and 3^n factorials which have been used to obtain the factorial replicated designs of the mixed type $2^m \times 3^n$. The idea behind this work is to estimate the main effects and the two-factor interactions (assuming higher-factor interactions to be absent) with as few assemblies as possible and without too much computation.

II. GENERAL OUTLINE OF CONSTRUCTION OF THE FRACTIONALLY REPLICATED EXPERIMENTAL DESIGNS

2.1 Treatment Combinations and Their Response

Let there be m factors $A_1, A_2, \ldots, A_m$ each at two levels and n factors $B_1, B_2, \ldots, B_n$ each at three levels. Then there are $2^m \times 3^n$ treatment combinations.

The symbol

$$a_1^{x_1}, a_2^{x_2}, \ldots, a_m^{x_m}, b_1^{z_1}, b_2^{z_2}, \ldots, b_n^{z_n} \qquad (2.1.1)$$

denotes the treatment in which the factors $A_1, A_2, \ldots, A_m$ occur at the levels $x_1, x_2, \ldots, x_m$; $x_i = 0, 1$ $(i = 1, 2, \ldots, m)$ and the factors $B_1, B_2, \ldots, B_n$ occur at the levels $z_1, z_2, \ldots, z_n$; $z_j = 0, 1, 2$ $(j = 1, 2, \ldots, n)$.

For simplicity the treatment combination (2.1.1) will be called an assembly and denoted by

$$(x_1, x_2, \ldots, x_m, z_1, z_2, \ldots, z_n) \qquad (2.1.2)$$

In accordance with standard convention, denote an assembly and the mean response to an assembly by the same symbol. Thus if $Y(x_1, x_2, \ldots, x_m, z_1, z_2, \ldots, z_n)$ is an observed response corresponding to an assembly (2.1.2), then

$$E\left[Y(x_1, x_2, \ldots, x_m, z_1, z_2, \ldots, z_n)\right]$$

$$= (x_1, x_2, \ldots, x_m, z_1, z_2, \ldots, z_n)$$

where E stands for 'expectation'.

2.2 Fractionally Replicated Asymmetrical Factorial Experimental Designs

Let S be a column vector of u assemblies, say,

$$S = \begin{bmatrix} (x_{11}, x_{12}, \ldots, x_{1m}) \\ (x_{21}, x_{22}, \ldots, x_{2m}) \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ (x_{u1}, x_{u2}, \ldots, x_{um}) \end{bmatrix}$$

which also is an array of some strength in Euclidean Geometry EG(m, 2), by Barnard (1936), i.e., factorial experimental design with m factors each at 2 levels. Let T be a column vector of v assemblies, say

$$T = \begin{bmatrix} (z_{11}, z_{12}, \ldots, z_{1n}) \\ (z_{21}, z_{22}, \ldots, z_{2n}) \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ (z_{v1}, z_{v2}, \ldots, z_{vn}) \end{bmatrix}$$

which is again an array of some strength in EG(n, 3). Then $S \otimes T$ will mean the symbolic direct product of two column vectors of assemblies S and T and is taken as

$$
S \otimes T = \begin{bmatrix}
(x_{11}, x_{12}, \ldots, x_{1m}, z_{11}, z_{12}, \ldots, z_{1n}) \\
\vdots \\
(x_{11}, x_{12}, \ldots, x_{1m}, z_{v1}, z_{v2}, \ldots, z_{vn}) \\
\hdashline
\vdots \\
\hdashline
(x_{u1}, x_{u2}, \ldots, x_{um}, z_{11}, z_{12}, \ldots, z_{1n}) \\
\vdots \\
(x_{u1}, x_{u2}, \ldots, x_{um}, z_{v1}, z_{v2}, \ldots, z_{vn})
\end{bmatrix}
$$

which is a (uv x 1) column vector of mixed assemblies. It can also be looked upon as a fractionally replicated design consisting of uv assemblies of a $2^m \times 3^n$ design. (Ref.: MacDuffee (12))

2.3 Effect of the 2^m Factorial Experiment

Any interaction of a 2^m experiment will be denoted by

$$A_1^{k_1} A_2^{k_2} \ldots A_m^{k_m} \tag{2.3.1}$$

$$(k_i = 0, 1;\ i = 1, 2, \ldots, m)$$

It also includes the main effects and the average response of all assemblies. If $k_1 = 1$ and the rest of the k_i's $(i = 2, 3, \ldots, m)$ are zero, (2.3.1) represents the main effect A_1. If $k_1 = k_2 = 1$ and the rest of the k_i's $(i = 3, 4, \ldots, m)$ are zero, then the (2.3.1) represents the two-factor interaction A_1A_2, and similarly for other factors.

Let

$$H = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & c(0) \\ 1 & c(1) \end{bmatrix} \qquad (2.3.2)$$

then

$$H^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

where the constants $c(x)$, $(x = 0, 1)$, are defined by (2.3.2). Let

$$U^{(m)} = U \otimes U \otimes U \otimes \ . \ . \ . \otimes U$$

denote the product of U taken m times. Then we shall define all the interactions in (2.3.1) by the matrix identity

$$\begin{bmatrix} 1 \\ a_1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ a_2 \end{bmatrix} \otimes \ . \ . \ . \otimes \begin{bmatrix} 1 \\ a_m \end{bmatrix} = H^{(m)} \ . \begin{bmatrix} I \\ A_1 \end{bmatrix} \otimes \begin{bmatrix} I \\ A_2 \end{bmatrix} \otimes \ . \ . \ . \otimes \begin{bmatrix} I \\ A_m \end{bmatrix} \qquad (2.3.3)$$

or

$$\begin{bmatrix} I \\ A_1 \end{bmatrix} \otimes \begin{bmatrix} I \\ A_2 \end{bmatrix} \otimes \ . \ . \ . \otimes \begin{bmatrix} I \\ A_m \end{bmatrix} = H^{-(m)} . \begin{bmatrix} 1 \\ a_1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ a_2 \end{bmatrix} \otimes . \ . \ . \otimes \begin{bmatrix} 1 \\ a_m \end{bmatrix}$$

with the convention

$$I \cdot I = I, \quad I\, A^k = A^k = A^k\, I, \quad A^o = I$$

The expression for the typical effect $A_1^{k_1} A_2^{k_2} \ . \ . \ . \ A_m^{k_m}$ is

$$A_1^{k_1} A_2^{k_2} \ . \ . \ . \ A_m^{k_m} = D \prod_i \ (-1 + a_i)^{x_i} \ (1 + a_i)^{1-x_i}$$

where the divisor $D = (1/2^i)$, $i = 1, 2, . . ., m$.

For example, if $m = 3$,

$$I = 1/8\ (1 + a_1)(1 + a_2)(1 + a_3)$$

$$A_1 = 1/8\ (-1 + a_1)(1 + a_2)(1 + a_3)$$

$$A_1A_2A_3 = 1/8\ (-1 + a_1)(-1 + a_2)(-1 + a_3)$$

where I is the average of the mean responses of all assemblies. (Ref.: Bose and Connor (5).)

Assuming three or more factor interactions as negligible, (2.3.3) reduces to

$$(a_1^{x_1}\ a_2^{x_2}\ .\ .\ .\ a_m^{x_m}) = I + \sum_{i=1}^{m} c(x_i)A_i + \sum_{i<i'} c(x_i)\ c(x_{i'})\ A_iA_{i'}$$

where $c(x_i)$, $x_i = 0, 1$, $i = 1, 2, . . ., m$ are as given in (2.3.3) and $\sum_{i<i'}$ means the summation over all pairs of indices i, i' for $i < i'$ $(i, i' = 1, 2, . . ., m)$.

2.4 Effects of the 3^n Factorial Experiment

Any interaction of a 3^n experiment will be denoted by

$$B_1^{w_1}\ B_2^{w_2}\ .\ .\ .\ B_n^{w_n}. \qquad (2.4.1)$$

$(w_j = 0, 1, 2;\ j = 1, 2, . . ., n)$.

When only one w is different from zero, (2.4.1) represents a main effect. It is a linear effect if $w = 1$ and a quadratic effect if $w = 2$. When two w's are different from zero, (2.4.1) represents a two-factor interaction. It is a linear x linear, linear x quadratic effect according as the couplet of non-zero w's is (1,1), (1,2) or (2,1) and (2,2).

Let

$$K = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & d_1(0) & d_2(0) \\ 1 & d_1(1) & d_2(1) \\ 1 & d_1(2) & d_2(2) \end{bmatrix} \tag{2.4.2}$$

Then

$$K^{-1} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ -1/2 & 0 & 1/2 \\ 1/6 & -1/3 & 1/6 \end{bmatrix}$$

where the constants $d_w(z)$; $z = 0, 1, 2$; $w = 1, 2$ are defined by (2.4.2). Then we shall define all interactions in (2.4.1) by the matrix identity

$$\begin{bmatrix} 1 \\ b_1 \\ b_1^2 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ b_2 \\ b_2^2 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 1 \\ b_n \\ b_n^2 \end{bmatrix} = K^{(n)} \cdot \begin{bmatrix} I \\ B_1 \\ B_1^2 \end{bmatrix} \otimes \begin{bmatrix} I \\ B_2 \\ B_2^2 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} I \\ B_n \\ B_n^2 \end{bmatrix} \tag{2.4.3}$$

or

$$\begin{bmatrix} I \\ B_1 \\ B_1^2 \end{bmatrix} \otimes \begin{bmatrix} I \\ B_2 \\ B_2^2 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} I \\ B_n \\ B_n^2 \end{bmatrix} = K^{(-n)} \cdot \begin{bmatrix} 1 \\ b_1 \\ b_1^2 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ b_2 \\ b_2^2 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 1 \\ b_n \\ b_n^2 \end{bmatrix}$$

with the convention

$$I \cdot I = I, \quad IB^w = B^w = B^w I, \quad B^0 = I .$$

The expression for the typical effect $B_1^{w_1} B_2^{w_2} \ldots B_n^{w_n}$ is

$$B_1^{w_1} B_2^{w_2} \ldots B_n^{w_n}$$

$$= D \prod_j (1+b_j+b_j^2)^{f_0(z_j)}(-1+b_j^2)^{f_1(z_j)}(1-2b_j+b_j^2)^{f_2(z_j)},$$

where $D = (2^{n_1} \times 3^{n_0} \times 6^{n_2})^{-1}$, n = the number of B factors with exponent w_j, $n_0 + n_1 + n_2 = n$, $(j = 1, 2, \ldots, n)$;

$$f_0(z_j) = 1/2(z_j - 1)(z_j - 2),\ f_1(z_j) = z_j(-z_j + 2), \text{ and}$$

$$f_2(z_j) = 1/2\ z_j(z_j - 1).$$

For example, if n = 2,

$$I = 1/9(1 + b_1 + b_1^2)(1 + b_2 + b_2^2)$$

$$B_1B_2^2 = 1/12(-1 + b_1^2)(1 - 2b_2 + b_2^2).$$

As before, I is the average mean response, and the effects defined here do not obey the convention that the sum of the positive coefficients is unity. $B_1B_2^2$ is the interaction of the linear component of B_1 and the quadratic component of B_2. (Ref.: Bose and Conner (5).)

Assuming three or more factor interactions as negligible, (2.4.3) reduces to

$$b_1^{z_1}b_2^{z_2} \ldots b_n^{z_n} = I + \sum_{w=1}^{2} \sum_{j=1}^{n} d_w(z_j)\ B_j^w$$

$$+ \sum_{w,w'} \sum_{j<j'} d_w(z_j)\ d_{w'}(z_{j'})\ B_j^w B_{j'}^{w'}$$

where $d_w(z_j)$, $w = 1, 2$, $z_j = 0, 1, 2$, $j = 1, 2, \ldots, n$, are given in (2.4.2), and $\sum_{j<j'}$ means the summation over all pairs of indices j, j' for $j < j'$ $(j, j' = 1, 2, \ldots, n)$, and $\sum_{w,w'}$ means the summation over all pairs of indices w, w' $(w, w' = 1, 2)$.

2.5 Effects of the $2^m \times 3^n$ Factorial Experiment

Any interaction in this experiment will be denoted by

$$A_1^{k_1} A_2^{k_2} \ldots A_m^{k_m} B_1^{w_1} B_2^{w_2} \ldots B_n^{w_n} \tag{2.5.1}$$

which may be regarded as the symbolic product of $A_1^{k_1} A_2^{k_2} \ldots A_m^{k_m}$ and $B_1^{w_1} B_2^{w_2} \ldots B_n^{w_n}$. If $k_i = 1$ and $w_j = 1$ and the rest of (k, w)'s are zero, (2.5.1) represents the linear effect $A_i B_j$ of the i^{th} factor of the 2^m factorial and the j^{th} factor of the 3^n factorial. If $k_i = 1$ and $w_j = 2$ and the rest of (k, w)'s are zero, then (2.5.1) represents the quadratic effect $A_i B_j^2$ of the same factors $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$.

All interactions in (2.5.1) will be defined by the matrix identity

$$\begin{bmatrix} 1 \\ a_1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ a_2 \end{bmatrix} \otimes \ldots \otimes \begin{bmatrix} 1 \\ a_m \end{bmatrix} \otimes \begin{bmatrix} 1 \\ b_1 \\ b_1^2 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ b_2 \\ b_2^2 \end{bmatrix} \otimes \ldots \otimes \begin{bmatrix} 1 \\ b_n \\ b_n^2 \end{bmatrix}$$

$$= H^{(m)} \otimes K^{(n)} \cdot \begin{bmatrix} I \\ A_1 \end{bmatrix} \otimes \begin{bmatrix} I \\ A_2 \end{bmatrix} \otimes \ldots \otimes \begin{bmatrix} I \\ A_m \end{bmatrix} \otimes \begin{bmatrix} I \\ B_1 \\ B_1^2 \end{bmatrix} \otimes \begin{bmatrix} I \\ B_2 \\ B_2^2 \end{bmatrix} \otimes \ldots \otimes \begin{bmatrix} I \\ B_n \\ B_n^2 \end{bmatrix}$$

$$\begin{bmatrix} I \\ A_1 \end{bmatrix} \otimes \begin{bmatrix} I \\ A_2 \end{bmatrix} \otimes \ . \ . \ . \otimes \begin{bmatrix} I \\ A_m \end{bmatrix} \otimes \begin{bmatrix} I \\ B_1 \\ B_1^2 \end{bmatrix} \otimes \begin{bmatrix} I \\ B_2 \\ B_2^2 \end{bmatrix} \otimes \ . \ . \ . \otimes \begin{bmatrix} I \\ B_n \\ B_n^2 \end{bmatrix}$$

$$= H^{-(m)} \cdot K^{-(n)} \cdot \begin{bmatrix} 1 \\ a_1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ a_2 \end{bmatrix} \otimes \ldots \otimes \begin{bmatrix} 1 \\ a_m \end{bmatrix} \otimes \begin{bmatrix} 1 \\ b_1 \\ b_1^2 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ b_2 \\ b_2^2 \end{bmatrix} \otimes \ldots \otimes \begin{bmatrix} 1 \\ b_n \\ b_n^2 \end{bmatrix}$$

(Ref.: Bose and Connor (5).)

The effect of $A_1^{k_1} A_2^{k_2} \ . \ . \ . \ A_m^{k_m} B_1^{w_1} B_2^{w_2} \ . \ . \ . \ B_n^{w_n}$ is

$$D \prod_{ij} \Big[(-1+a_i)^{x_i} (1+a_i)^{1-x_i} (1+b_j+b_j^2)^{f_0(z_j)} (-1+b_j^2)^{f_1(z_j)} (1-2b_j+b_j^2)^{f_2(z_j)} \Big] ,$$

where $D = (2^{n_1+m} \times 3^{n_o} \times 6^{n_2})^{-1}$, n_w = the number of B factors with exponent w_j, $n_o + n_1 + n_2 = n$, $(j = 1, 2, \ . \ . \ ., n)$, and $f_o(z_j)$, $f_1(z_j)$, $f_2(z_j)$ are defined below (2.4.3). (Ref.: Connor, W. S. (7).) If $m = 3$, $n = 2$

$$A_1 B_1 B_2^2 = 1/96 \ (-1+a_1)(1+a_2)(1+a_2)(-1+b_1^2)(1-2b_2+b_2^2)$$

where I denotes the average of the mean responses of all mixed assemblies.

Assuming three or more factor interactions as negligible,

$$(a_1^{x_1} a_2^{x_2} \ldots a_m^{x_m} \cdot b_1^{z_1} b_2^{z_2} \ldots b_n^{z_n})$$

$$= I + \sum_{i=1}^{m} c(x_i)A_i + \sum_{i<i'} c(x_i)c(x_{i'})A_iA_{i'} + \sum_{w=1}^{2}\sum_{j=1}^{n} d_w(x_j)B_j$$

$$+ \sum_{w,w'} \sum_{j<j'} d_w(z_j)d_{w'}(z_{j'})B_j^wB_{j'}^{w'} + \sum_{w=1}^{2}\sum_{i,j} c(x_i)d_w(z_j)A_iB_j^w \quad (2.5.2)$$

where $c(x_i)$, $d_w(z_j)$, $\sum_{i<i'}$, $\sum_{j<j'}$, are as already defined earlier, and $\sum_{i,j}$ means the summation over all pairs of indices i, j ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$).

2.6 Fractional Replicate of the 2^m Factorial Experimental Design

The treatment combination of the 2^m experiment may be represented as points of an n-dimensional lattice with axes $x_1, x_2, \ldots, x_m$ (as given in section 2.1). Each axis of the lattice will have two points, these points being the elements of the Galois Field GF(2). Thus, for example, the treatment combinations of the 2^2 factorial experiment can be represented by the points (0, 0), (1, 0), and (1, 1). The two factors may be denoted by A_1 and A_2. The points (0, 0) and (1, 1) both satisfy the equation $x_1 + x_2 = 0$, and comprise a 1/2 replicate of the 2^2 experiment defined by the identity relationship $I = A_1A_2$. The symbol $A_1^0A_2^0$ can be used to denote the set of treatment combinations for which $x_1 + x_2 = 0$. Similarly, the symbol $A_1^1A_2^1$ denotes the set of treatment combinations for which $x_1 + x_2 = 1$.

Consider m factors, $A_1, A_2, \ldots, A_m$ each with two levels. Then $A_1^{k_1} A_2^{k_2} A_3^{k_3} \ldots A_i^{k_i}$ denotes the set of treatment combinations for which $k_1x_1 + k_2x_2 + \ldots + k_mx_m = i$, where i is an element of the Galois Field, GF(2). A subscript i can be associated with every effect or interaction of the identity relationship. For example, the treatment combinations of the four possible 1/4 replicates of the 2^5 experiment defined by the identity relationship,

$$I = A_1^{k_1} A_2^{k_2} A_3^{k_3} = A_1^{k_1} A_4^{k_4} A_5^{k_5} = A_2^{k_2} A_3^{k_3} A_4^{k_4} A_5^{k_5}$$

are those treatment combinations which satisfy

$$\begin{aligned} (1)\quad & k_1x_1 + k_2x_2 + k_3x_3 = 0 \\ & k_1x_1 + k_4x_4 + k_5x_5 = 0 \\ & k_2x_2 + k_3x_3 + k_4x_4 + k_5x_5 = 0 \end{aligned}$$

$$\begin{aligned} (2)\quad & k_1x_1 + k_2x_2 + k_3x_3 = 0 \\ & k_1x_1 + k_4x_4 + k_5x_5 = 1 \\ & k_2x_2 + k_3x_3 + k_4x_4 + k_5x_5 = 1 \end{aligned}$$

$$\begin{aligned} (3)\quad & k_1x_1 + k_2x_2 + k_3x_3 = 1 \\ & k_1x_1 + k_4x_4 + k_5x_5 = 0 \\ & k_2x_2 + k_3x_3 + k_4x_4 + k_5x_5 = 1 \end{aligned}$$

$$\begin{aligned} (4)\quad & k_1x_1 + k_2x_2 + k_3x_3 = 1 \\ & k_1x_1 + k_4x_4 + k_5x_5 = 1 \\ & k_2x_2 + k_3x_3 + k_4x_4 + k_5x_5 = 0 , \end{aligned}$$

respectively.

Since the interaction $A_2A_3A_4A_5$ is the generalized interaction of $A_1A_2A_3$ and $A_1A_4A_5$, the subscripts associated with $A_2A_3A_4A_5$ in the four fractional replicates may be obtained as the sum of the corresponding subscripts of $A_1A_2A_3$ and $A_1A_4A_5$.

Thus a $1/2^r$ fraction of the 2^m factorial consists of the assemblies $(x_1, x_2, \ldots, x_m)$ satisfying the r linearly independent equations.

$$L_s = a_{s1}x_1 + a_{s2}x_2 + \ldots + a_{sm}x_m = d_s \tag{2.6.1}$$

$(d_s = 0, 1)$, $(s = 1, 2, \ldots, r)$,
in Galois Field (2), where

$$(a_{s1}, a_{s2}, \ldots, a_{sm}) \neq (0, 0, \ldots, 0).$$

Because each of the d_s can assume one of two values, 0 or 1, there are 2^r different fractions which can be produced in this way. Each fraction contains 2^{m-r} treatment combinations, and because the different fractions do not have any treatment combinations in common, the fractions collectively contain all 2^m treatment combinations.

We shall say that the interaction,

$$A_1^{k_1} A_2^{k_2} \ldots A_m^{k_m} \tag{2.6.2}$$

corresponds to the linear form $k_1x_1 + k_2x_2 + \ldots + k_mx_m$ in GF(2).

The interaction (2.6.2) is estimated by the contrast such that

$$1/2^m \left[(k_1x_1 + k_2x_2 + \ldots + k_mx_m = 1) - (k_1x_1 + k_2x_2 + \ldots + k_mx_m = 0)\right] \tag{2.6.3}$$

where the parenthesis ( ) means the sum of the observed responses of assemblies satisfying the equation within it. If only one k differs from zero, (2.6.2) represents a main effect. If two k's differ from zero, it represents a two-factor interaction, and so on.

Consider any linear form L which is not of the form,

$$k_1L_1 + k_2L_2 + \ldots + k_rL_r \tag{2.6.4}$$

$$k_s = 0,\ 1;\ s = 1,\ 2,\ \ldots\ r$$

$$(k_1,\ k_2,\ \ldots\ k_r) \neq (0,\ 0,\ \ldots,\ 0)$$

the interactions corresponding to the linear form,

$$L + (k_1L_1 + k_2L_2 + \ldots + k_rL_r)$$

are said to be aliases of the interaction corresponding to L.

It is known from the theory of fractional replication that each interaction not corresponding to any linear form $k_1L_1 + k_2L_2 + \ldots + k_rL_r$ is a member of one and only one alias set.

It is clear that the assemblies of the fraction defined by (2.6.1) which satisfy L = 0 also satisfy $L + (k_1L_1 + k_2L_2 + \ldots + k_rL_r) = k_1d_1 + k_2d_2 + \ldots + k_rd_r$ and those satisfying L = 1, satisfy $L + (k_1L_1 + k_2L_2 + \ldots + k_rL_r) = 1 + k_1d_1 + k_2d_2 + \ldots + k_rd_r$, where $\sum_{s=1}^{r} k_sd_s = 0$ or 1. The rank of the matrix formed by the column vectors of coefficients belonging to effects which correspond to $L + (k_1L_1 + k_2L_2 + \ldots + k_rL_r)$ is 1.

The identity relationships for the fraction defined by (2.6.1) are

$$I = G_1 = G_2 = \ldots = G_r = G_1G_2 = \ldots = G_{r-1}G_r$$
$$= \ldots = G_1G_2 \ldots G_r \qquad (2.6.5)$$

where the G's are the interactions corresponding to the linear form (2.6.4).

The relationship (2.6.5) is useful in determining the set of effects aliased with a given effects.

2.7 Fractional Replicate of the 3^n Factorial Experimental Design

In a 3^n design, suppose each treatment combination to be represented in the conventional fashion by a code containing n digits. The K^{th} digit represents the level of the K^{th} factor, and take the values 0, 1, 2 representing the three levels, as given in section 2.4. Then each such code may be thought of as the coordinates of a point in an n-dimensional factor space, and the whole design consists of 3^n such points arranged in a hypercubic lattice. This geometrical way of considering the fractional replicate of the 3^n factorial design agrees with the method given in section 2.6.

Thus a $1/3^q$ fractional replicate of a 3^n factorial consists of the assemblies $(z_1, z_2, \ldots, z_n)$ satisfying the q linearly independent equations

$$M_t = b_{t1}z_1 + b_{t2}z_2 + \ldots + b_{tn}z_n = e_t \qquad (2.7.1)$$
$$(e_t = 0, 1, 2), \ (t = 1, 2, \ldots, q)$$

in GF(3).

We shall say that the interaction

$$B_1^{w_1}\ B_2^{w_2}\ .\ .\ .\ B_n^{w_n}$$

corresponds to the linear form $w_1z_1 + w_2z_2 + \ldots + w_nz_n$ which we may assign to the two orthogonal components, linear and quadratic, denoted by $L(B_1^{w_1}\ B_2^{w_2}\ \ldots\ B_n^{w_n})$ and $Q(B_1^{w_1}\ B_2^{w_2}\ \ldots\ B_n^{w_n})$, respectively. The linear effect $L(B_1^{w_1}\ B_2^{w_2}\ \ldots\ B_n^{w_n})$ is estimated by the contrast

$$\frac{1}{2 \times 3^{n-1}} \Big[(w_1z_1 + w_2z_2 + \ldots + w_nz_n = 2) - (w_1z_1 + w_2z_2 + \ldots + w_nz_n = 0)\Big]$$

and the quadratic effect $Q(B_1^{w_1}\ B_2^{w_2}\ \ldots\ B_n^{w_n})$, by the contrast

$$\frac{1}{2 \times 3^{n}} \Big[(w_1z_1 + w_2z_2 + \ldots + w_nz_n = 0) - 2(w_1z_1 + w_2z_2 + \ldots + w_nz_n = 1) + (w_1z_1 + \ldots + w_nz_n = 2)\Big]$$

where ( ) has the same meaning as in (2.6.3).

Let M be any linear form which is not of the form,

$$w_1M_1 + w_2M_2 + \ldots + w_qM_q \tag{2.7.2}$$

$$w_t = 0,\ 1,\ 2;\ t = 1,\ 2,\ \ldots,\ q;$$

$$(w_1,\ w_2,\ \ldots,\ w_q) \neq (0,\ 0,\ \ldots,\ 0).$$

The interactions corresponding to the linear form $M + (w_1M_1 + w_2M_2 + \ldots + w_qM_q)$ are said to be aliases of the interaction corresponding to the linear form M.

It is clear that the assemblies of the fraction defined by (2.7.1) satisfying M = 0, 1, 2 also satisfy

$$M + (w_1M_1 + w_2M_2 + \ldots + w_qM_q)$$
$$= \sum_{t=1}^{q} w_te_t,\ \sum_{t=1}^{q} w_te_t + 1,\ \sum_{t=1}^{q} w_te_t + 2$$

respectively, where $e_t = 0, 1, 2$.

The identity relationships for fraction defined by (2.7.1) are

$$I' = G_1' = G_2' = \ldots = G_q' = G_1'G_2' = \ldots = G_{q-1}'G_q'$$
$$\ldots = \ldots = G_1'G_2'G_3' \ldots G_q'^2 \qquad (2.7.3)$$

where the (G')'s are the interactions corresponding to the linear form (2.7.2).

The relationship (2.7.3) is useful in determining the set of effects aliased with a given effect.

2.8 Fractional Replicate of the $2^m \times 3^n$ Factorial Experimental Design

W. S. Connor (7) has published an example of his general method of fractionating $2^m \times 3^n$ designs. The example is a half-replicate of a $2^3 \times 3^2$ design. Briefly, the method by which it is constructed is as follows. The 2^3 part of the design is split into two conventional half-replicates S_1 and S_2. The 3^2 part of the design is split into three conventional one-third replicates T_1, T_2, and T_3. The final design can then be represented symbolically by

$$S_1T_1 + S_2T_2 + S_2T_3$$

where S_1T_1, for example, represents all possible combinations of treatment combinations in S_1 with treatment combinations on T_1. This gives a design of 36 out of a total of 72 trials. For constructing plans for the $2^m \times 3^n$ arrangements are adjoining fractions of the 3^n factorial to fractions of the 2^m factorial in such a way as to preserve two-factor interaction estimates. To illustrate the method of construction, consider the $2^5 \times 3^2$ arrangement. A 1/4 replicate of the $2^5 \times 3^2$ can be constructed by adjoining the combinations of a 1/3 replicate of the 3^2 factorial to a 3/4 replicate of the 2^5 factorial.

An interaction $A_1^{k1} A_2^{k2} \ldots A_m^{km} B_1^{w1} B_2^{w2} \ldots B_n^{wn}$ may be said to belong to two linear forms $k_1x_1 + k_2x_2 + \ldots + k_mx_m$ in GF(2) and $w_1z_1 + w_2z_2 + \ldots + w_nz_n$ in GF(3).

$$L(A_1^{k1} A_2^{k2} \ldots A_m^{km} B_1^{w1} B_2^{w2} \ldots B_n^{wn})$$

and $$Q(A_1^{k1} A_2^{k2} \ldots A_m^{km} B_1^{w1} B_2^{w2} \ldots B_n^{wn}).$$

Let the equations

$$k_1x_1 + k_2x_2 + \ldots + k_mx_m = d$$
$$w_1z_1 + w_2z_2 + \ldots + w_nz_n = e$$

be satisfied by points $(x_1, x_2, \ldots, x_m, z_1, z_2, \ldots, z_n)$ such that $(x_1, x_2, \ldots, x_m)$ satisfy $k_1x_1 + k_2x_2 + \ldots + k_mx_m = d$ in GF(2), and $(z_1, z_2, \ldots, z_n)$ satisfy $w_1z_1 + w_2z_2 + \ldots + w_nz_n = e$ in GF(3), then $(2 \times 3^{n-1} \times 2^m)$

· $L(A_1^{k_1} A_2^{k_2} \dots A_m^{k_m} B_1^{w_1} B_2^{w_2} \dots B_n^{w_n})$ is estimated by tne contrast

$$\left[\begin{pmatrix} k_1x_1 + \dots + k_mx_m = 1 \\ w_1z_1 + \dots + w_nz_n = 2 \end{pmatrix} - \begin{pmatrix} k_1x_1 + \dots + k_mx_m = 1 \\ w_1z_1 + \dots + w_nz_n = 0 \end{pmatrix}\right]$$

$$- \left[\begin{pmatrix} k_1x_1 + \dots + k_mx_m = 0 \\ w_1z_1 + \dots + w_nz_n = 2 \end{pmatrix} - \begin{pmatrix} k_1x_1 + \dots + k_mx_m = 0 \\ w_1z_1 + \dots + w_nz_n = 0 \end{pmatrix}\right],$$

and $(2 \times 3^n \times 2^m) \times Q(A_1^{k_1} A_2^{k_2} \dots A_m^{k_m} B_1^{w_1} B_2^{w_2} \dots B_n^{w_n})$

is estimated by the contrast

$$\left[\begin{pmatrix} k_1x_1 + \dots + k_mx_m = 1 \\ w_1z_1 + \dots + w_nz_n = 0 \end{pmatrix} - 2\begin{pmatrix} k_1x_1 + \dots + k_mx_m = 1 \\ w_1z_1 + \dots + w_nz_n = 1 \end{pmatrix}\right.$$

$$\left. + \begin{pmatrix} k_1x_1 + \dots + k_mx_m = 1 \\ w_1z_1 + \dots + w_nz_n = 2 \end{pmatrix}\right] - \left[\begin{pmatrix} k_1x_1 + \dots + k_mx_m = 0 \\ w_1z_1 + \dots + w_nz_n = 0 \end{pmatrix}\right.$$

$$\left. - 2\begin{pmatrix} k_1x_1 + \dots + k_mz_m = 0 \\ w_1z_1 + \dots + w_nz_n = 1 \end{pmatrix} + \begin{pmatrix} k_1x_1 + \dots + k_mx_m = 0 \\ w_1z_1 + \dots + w_nz_n = 2 \end{pmatrix}\right]$$

where ( ) has the same meaning as in (2.6.3).

The identity relationships for the fractional replicate of $2^m \times 3^n$ design can be obtained from those of 2^m and 3^n design. If the fraction is given by the mixed assemblies obtained by combining the assemblies of the fraction defined by (2.6.1) and the assemblies of the fraction defined by (2.7.1) as in section 2.2, then the identity relationships are

$$I = G_1 = G_2 = G_1G_2 \; . \; . \; . \; G_r = G_1' = G_2' = \; . \; . \; . \; = G_1'G_2'$$
$$. \; . \; . \; . \; G_q' = \; . \; . \; . \; = G_1G_1' = G_2G_2' = \; . \; . \; . \; . = G_rG_q'$$
$$. \; . \; . = G_1G_2 \; . \; . \; . \; G_rG_1'G_2' \; . \; . \; . \; G_q'$$

where the interactions are obtained by taking all products of interactions from the identity relationships for the fractions defined by (2.6.1) and (2.7.1).

To make these points clear, consider a fraction of a $2^2 \times 3^2$ design consisting of the assemblies (x_1, x_2, z_1, z_2) obtained by taking the symbolic direct product of assemblies of an array given by $x_1 + x_2 = 0$ in EG(m, 2) and those of an array given by $z_1 + z_2 = 0$ in EG(n, 3).

Writing

$$S_1 = \begin{bmatrix} (0, 0) \\ (1, 1) \end{bmatrix} \qquad \text{and } T_1 = \begin{bmatrix} (0, 0) \\ (1, 2) \\ (2, 1) \end{bmatrix}$$

the required assemblies are given by the definition in section (2.2), thus

$$S_1 \otimes T_1 = \begin{bmatrix} (0, 0, 0, 0) \\ (0, 0, 1, 2) \\ (0, 0, 2, 1) \\ (1, 1, 0, 0) \\ (1, 1, 1, 2) \\ (1, 1, 2, 1) \end{bmatrix} \tag{2.8.1}$$

Let

$$P' = \big[\mu,\ A_1,\ A_2,\ A_1A_2\ ,\ L(B_1),\ Q(B_1),\ L(B_2),\ Q(B_2),\ L(B_1B_2),\ Q(B_1B_2),\ L(B_1B_2^2),\ Q(B_1B_2^2),\ L(A_1B_1),\ Q(A_1B_1),\ L(A_1B_2),\ Q(A_1B_2),\ L(A_2B_1),\ Q(A_2B_1),\ L(A_2B_2),\ Q(A_2B_2)\big]\ .$$

Then

$$E = \begin{bmatrix} y(0,\ 0,\ 0,\ 0) \\ y(0,\ 0,\ 1,\ 2) \\ y(0,\ 0,\ 2,\ 1) \\ y(1,\ 1,\ 0,\ 0) \\ y(1,\ 1,\ 1,\ 2) \\ y(1,\ 1,\ 2,\ 1) \end{bmatrix} \cdot$$

$$\begin{bmatrix}
1 & -1 & -1 & +1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -1 & -1 & +1 & 0 & -2 & 1 & 1 & -1 & 1 & 1 & 1 & 0 & 2 & -1 & -1 & 0 & 2 & -1 & -1 \\
1 & -1 & -1 & +1 & 1 & 1 & 0 & -2 & -1 & 1 & 0 & -2 & -1 & -1 & 0 & 2 & -1 & -1 & 0 & 2 \\
1 & 1 & 1 & +1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & 1 & 1 & +1 & 0 & -2 & 1 & 1 & -1 & 1 & 1 & 1 & 0 & -2 & 1 & 1 & 0 & -2 & 1 & 1 \\
1 & 1 & 1 & +1 & 1 & 1 & 0 & -2 & -1 & 1 & 0 & -2 & 1 & 1 & 0 & -2 & 1 & 1 & 0 & -2
\end{bmatrix} \cdot P \qquad (2.8.2)$$

The identity relationship for the fraction (2.8.1) and the other aliased sets of effects are

$$I = A_1A_2 = B_1B_2$$

$$\begin{aligned}
&[A_1,\ A_2] \\
&[B_1,\ B_2,\ B_1B_2^2] \qquad (2.8.3) \\
&[A_1B_1,\ A_2B_1,\ A_1B_2,\ A_2B_2]
\end{aligned}$$

assuming higher factor interaction to be absent.

Thus the six estimable functions of effects are one each for the first two and two each for the remaining two sets of effects in (2.8.3).

From (2.8.2), the normal equations estimating them are

$$\begin{bmatrix} 6 & -6 & -6 & 6 \\ -6 & 6 & 6 & -6 \\ -6 & 6 & 6 & -6 \\ 6 & -6 & -6 & 6 \end{bmatrix} \cdot \begin{bmatrix} \hat{I} \\ \widehat{A_1A_2} \\ \hat{L}(B_1B_2) \\ \hat{Q}(B_1B_2) \end{bmatrix} \doteq \begin{bmatrix} Y(I) \\ Y(A_1A_2) \\ Y \cdot L(B_1B_2) \\ Y \cdot Q(B_1B_2) \end{bmatrix} \quad \ldots\ldots (1)$$

$$\begin{bmatrix} 6 & 6 \\ 6 & 6 \end{bmatrix} \cdot \begin{bmatrix} \hat{A}_1 \\ \hat{A}_2 \end{bmatrix} = \begin{bmatrix} Y(A_1) \\ Y(A_2) \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots (2)$$

$$\begin{bmatrix} 4 & 0 & 2 & -6 & 2 & -6 \\ 0 & 12 & -6 & -6 & -6 & -6 \\ 2 & -6 & 4 & 0 & 4 & 0 \\ -6 & -6 & 0 & 12 & 0 & 12 \\ 2 & -6 & 4 & 0 & 4 & 0 \\ -6 & -6 & 0 & 12 & 0 & 12 \end{bmatrix} \cdot \begin{bmatrix} \hat{L}(B_1) \\ \hat{Q}(B_1) \\ \hat{L}(B_2) \\ \hat{Q}(B_2) \\ \hat{L}(B_1B_2^2) \\ \hat{Q}(B_1B_2^2) \end{bmatrix} = \begin{bmatrix} Y \cdot L(B_1) \\ Y \cdot Q(B_1) \\ Y \cdot L(B_2) \\ Y \cdot Q(B_2) \\ Y \cdot L(B_1B_2^2) \\ Y \cdot Q(B_1B_2^2) \end{bmatrix} \quad \ldots (3)$$

$$\begin{bmatrix} 4 & 0 & 2 & -6 & 4 & 0 & 2 & -6 \\ 0 & 12 & -6 & -6 & 0 & 12 & -6 & -6 \\ 2 & -6 & 4 & 0 & 2 & -6 & 4 & 0 \\ -6 & -6 & 0 & 12 & -6 & -6 & 0 & 12 \\ 4 & 0 & 2 & -6 & 4 & 0 & 2 & -6 \\ 0 & 12 & -6 & -6 & 0 & 12 & -6 & -6 \\ 2 & -6 & 4 & 0 & 2 & -6 & 4 & 0 \\ -6 & -6 & 0 & 12 & -6 & -6 & 0 & 12 \end{bmatrix} \begin{bmatrix} \widehat{L(A_1B_1)} \\ \widehat{Q(A_1B_1)} \\ \widehat{L(A_1B_2)} \\ \widehat{Q(A_1B_2)} \\ \widehat{L(A_2B_1)} \\ \widehat{Q(A_2B_1)} \\ \widehat{L(A_2B_2)} \\ \widehat{Q(A_2B_2)} \end{bmatrix} = \begin{bmatrix} YL(A_1B_1) \\ YQ(A_1B_1) \\ YL(A_1B_2) \\ YQ(A_1B_2) \\ YL(A_2B_1) \\ YQ(A_2B_1) \\ YL(A_2B_2) \\ YQ(A_2B_2) \end{bmatrix} \quad (4) \qquad (2.8.4)$$

which implies the estimability of

$$I - A_1A_2 - L(B_1B_2) + Q(B_1B_2) \ , \qquad (2.8.5)$$

$$A_1 + A_2 \ ,$$

$$4L(B_1) + 2L(B_2) - 6Q(B_2) + 2L(B_1B_2^2) - 6Q(B_1B_2^2),$$

$$12Q(B_1) - 6L(B_2) - 6Q(B_2) - 6L(B_1B_2^2) - 6Q(B_1B_2^2) \ ,$$

$$4L(A_1B_1) + 2L(A_1B_2) - 6Q(A_1B_2) + 4L(A_2B_1) + 2L(A_2B_2) - 6Q(A_2B_2),$$

$$12Q(A_1B_1) - 6L(A_1B_2) - 6Q(A_1B_2) + 12Q(A_2B_1) - 6L(A_2B_2) - 6Q(A_2B_2) \ .$$

Thus the six assemblies of $2^2 \times 3^2$ factorial design, taken as above, estimate the six linear functions of effects given in (2.8.5).

The Y functions on the right-hand side of the equations in (2.8.4) are the usual $C'\underline{y}$ components in the normal equations

$$C' \, C \, \hat{\underline{P}} = C'\underline{y}$$

where

$$E(\underline{y}) = C \, \underline{P} \ .$$

III. UNIVARIATE LINEAR MODEL WITH FIXED EFFECTS

Let the fractional design consist of N assemblies ($N < 2^m \times 3^n$) with $y_1, y_2, \ldots, y_N$ as their responses. These N responses are observable quantities with

$$y_r = \theta_r + e_r \qquad (r = 1, 2, \ldots, N) \tag{3.1}$$

where θ_r is the true value of y_r and e_r is an error in observation. The true values are assumed to be linear functions of main effects and two-factor interactions (including the grand average $I = \mu$) $\mu, \mu_1, \ldots, \mu_{(p-1)}$ in some order, which are unknown and fixed with known non-stochastic coefficients 1, $C_{r1}, C_{r2}, \ldots, C_{r(p-1)}$ as given in (2.5.2). For instance,

$$\theta_r = \mu + C_{r1}\mu_1 + C_{r2}\mu_2 + \cdots + C_{r(p-1)}\mu_{(p-1)} \tag{3.2}$$
$$(r = 1, 2, \ldots, N).$$

Written out in full, the coefficients C_{rv} $\left[r = 1, 2, \ldots, N;\ v = 1, 2, \ldots, (p-1)\right]$ are as indicated below.

1. If μ_v corresponds to the main effect A_i $(i = 1, 2, \ldots, m)$ of the 2^m factorial design, (3.3)

$$C_{rv} = \begin{cases} -1 \text{ if } y_r \text{ has level 0 of the } i^{th} \text{ factor} \\ 1 \text{ if } y_r \text{ has level 1 of the } i^{th} \text{ factor.} \end{cases}$$

2. If μ_v corresponds to the two main effects B_j, B_j^2 $(j = 1, 2, \ldots, n)$ of the 3^n factorial design,

$$C_{rv} = \begin{cases} -1 \quad 1 & \text{if } y_r \text{ has level 0 of the } j^{th} \text{ factor} \\ 0 \quad -2 & \text{if } y_r \text{ has level 1 of the } j^{th} \text{ factor} \\ 1 \quad 1 & \text{if } y_r \text{ has level 2 of the } j^{th} \text{ factor.} \end{cases}$$

3. If μ_v corresponds to the interaction $A_{i_1}A_{i_2}$ ($i_1 \neq i_2 = 1, 2, \ldots, m$) of the 2^m factorial design,

$$C_{rv} = \begin{cases} 1 & \text{if } y_r \text{ has levels 0, 0 of } i_1 \text{ and } i_2 \text{ factors} \\ -1 & \text{if } y_r \text{ has levels 0, 1 of } i_1 \text{ and } i_2 \text{ factors} \\ -1 & \text{if } y_r \text{ has levels 1, 0 of } i_1 \text{ and } i_2 \text{ factors} \\ 1 & \text{if } y_r \text{ has levels 1, 1 of } i_1 \text{ and } i_2 \text{ factors.} \end{cases}$$

4. If μ_v corresponds to the interactions $B_{j_1}B_{j_2}$, $B_{j_1}B_{j_2}^2$, $B_{j_1}^2B_{j_2}$, $B_{j_1}^2B_{j_2}^2$ ($j_1 \neq j_2 = 1, 2, \ldots, n$ of the 3^n factorial design,

$$C_{rv} = \begin{cases} 1 \quad -1 \quad -1 \quad 1 & \text{if } y_r \text{ has levels 0, 0 of } j_1 \text{ and } j_2 \text{ factors} \\ 0 \quad 0 \quad 2 \quad -2 & \text{if } y_r \text{ has levels 1, 0 of } j_1 \text{ and } j_2 \text{ factors} \\ -1 \quad 1 \quad -1 \quad 1 & \text{if } y_r \text{ has levels 2, 0 of } j_1 \text{ and } j_2 \text{ factors} \\ 0 \quad 2 \quad 0 \quad -2 & \text{if } y_r \text{ has levels 0, 1 of } j_1 \text{ and } j_2 \text{ factors} \\ 0 \quad 0 \quad 0 \quad 4 & \text{if } y_r \text{ has levels 1, 1 of } j_1 \text{ and } j_2 \text{ factors} \\ 0 \quad -2 \quad 0 \quad -2 & \text{if } y_r \text{ has levels 2, 1 of } j_1 \text{ and } j_2 \text{ factors} \\ -1 \quad -1 \quad 1 \quad 1 & \text{if } y_r \text{ has levels 0, 2 of } j_1 \text{ and } j_2 \text{ factors} \\ 0 \quad 0 \quad -2 \quad -2 & \text{if } y_r \text{ has levels 1, 2 of } j_1 \text{ and } j_2 \text{ factors} \\ 1 \quad 1 \quad 1 \quad 1 & \text{if } y_r \text{ has levels 2, 2 of } j_1 \text{ and } j_2 \text{ factors.} \end{cases}$$

5. If μ_v corresponds to the interactions A_iB_j, $A_iB_j^2$ ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$) of the $2^m \times 3^n$ design,

$$C_{rv} = \begin{cases} 1 \quad -1 \text{ if } y_r \text{ has levels } 0,\ 0 \text{ of } i \text{ and } j \text{ factors} \\ 0 \quad\ \ 2 \text{ if } y_r \text{ has levels } 0,\ 1 \text{ of } i \text{ and } j \text{ factors} \\ -1 \quad -1 \text{ if } y_r \text{ has levels } 0,\ 2 \text{ of } i \text{ and } j \text{ factors} \\ -1 \quad\ \ 1 \text{ if } y_r \text{ has levels } 1,\ 0 \text{ of } i \text{ and } j \text{ factors} \\ 0 \quad -2 \text{ if } y_r \text{ has levels } 1,\ 1 \text{ of } i \text{ and } j \text{ factors} \\ 1 \quad\ \ 1 \text{ if } y_r \text{ has levels } 1,\ 2 \text{ of } i \text{ and } j \text{ factors .} \end{cases}$$

Thus the model (2.5.2) can be written as

$$Y = \underline{y}_{(N\times 1)} = C_{(N\times p)} \cdot \underline{\mu}_{(p\times 1)} + \underline{e}_{(N\times 1)}$$

where $C = ((1,\ C_{rv}))$ as defined in (3.3),

$$\underline{\mu}' = \left[\mu,\ \mu_1,\ \mu_2,\ .\ .\ .\ .,\ \mu_{(p-1)}\right],$$

$$E(\underline{e}) = \underline{0}$$

and $V(\underline{e}) = \sigma^2 I_N$ (σ^2 unknown).

IV. MULTIVARIATE LINEAR MODEL WITH FIXED EFFECTS

In a fractional design with N assemblies out of $2^m \times 3^n$ ($N < 2^m \times 3^n$), there are N experimental units to be observed. The problem is univariate or multivariate according as we make one observation or more than one observation on each experimental unit.

Suppose we make "q" observations ($q > 1$) on each experimental unit, then the model analogous to that in (3.1) is

$$Y_r = 1 \cdot \underline{\mu} + C_{r1}\underline{\mu}_1 + C_{r2}\underline{\mu}_2 + \cdots + C_{r(p-1)}\underline{\mu}_{(p-1)} + \underline{e}_r \quad (4.1)$$

where each $\underline{y}_r$, $\underline{\mu}$, $\underline{\mu}_1$, . . ., $\underline{\mu}_{(p-1)}$, $\underline{e}_r$ is a ($q \times 1$) column vector, $\underline{\mu}$, $\underline{\mu}_v$ ($v = 1, 2, \ldots, p-1$) are assumed to be fixed and unknown, and $\underline{e}$ is NID ($\underline{0}(q \times 1)$, $(q \times q)$). The matrix of constants $C = ((1, C_{rv}))$, $r = 1, 2, \ldots, N$; $v = 1, 2, \ldots, (p - 1)$ is the same as in the univariate case.

The model (4.1) can be compactly written in the form

$$Y'(N \times q) = C(N \times p) \cdot \mu_{(pxq)} + e'(Nxq), \qquad (4.2)$$

where

$$Y'_{(Nxq)} = \begin{array}{cc} \begin{bmatrix} \underline{y}_1' \\ \underline{y}_2' \\ \cdot \\ \cdot \\ \cdot \\ \underline{y}_N' \end{bmatrix} & \begin{array}{c} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{array} \\ q & \end{array} \qquad e'_{(Nxq)} = \begin{array}{cc} \begin{bmatrix} \underline{e}_1' \\ \underline{e}_2' \\ \cdot \\ \cdot \\ \cdot \\ \underline{e}_N' \end{bmatrix} & \begin{array}{c} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{array} \\ q & \end{array}$$

$$\mu_{(p\times q)} = \begin{array}{c} \begin{bmatrix} \underline{\mu}' \\ \underline{\mu}_1' \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \underline{\mu}_{(p-1)}' \end{bmatrix} \\ q \end{array} \begin{array}{c} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ \end{array} = \begin{bmatrix} \mu^{(1)} & \mu^{(2)} & \cdot & \mu^{(q)} \\ \mu_1^{(1)} & \mu_1^{(2)} & \cdot & \mu_1^{(q)} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \mu_{(p-1)}^{(1)} & \mu_{(p-1)}^{(2)} & \cdot & \mu_{(p-1)}^{(q)} \end{bmatrix}$$

We shall assume (i) $\Sigma(q \times q)$ to be an unknown positive definite matrix, (ii) $N > p + q$ where p is the rank of C.

V. ESTIMATION OF EFFECTS (LEAST-SQUARE METHOD)

From the model (3.4), we know the sum of squares due to error is

$$\underline{e}'\underline{e} = (Y - C\underline{\mu})' \; (Y - C\underline{\mu}) \; .$$

The value of $\underline{\mu}$ that minimizes $\underline{e}'\underline{e}$ is given by the solution to

$$\frac{\partial}{\partial \underline{\mu}} (\underline{e}'\underline{e}) = 0.$$

We get

$$\frac{\partial}{\partial \underline{\mu}} (\underline{e}'\underline{e}) = 2C'Y - 2\, C'C\hat{\underline{\mu}} = 0 \; .$$

The least-square estimate of $\underline{\mu}$,

$$\hat{\underline{\mu}} = (C'C)^{-1} \, C'Y \; ,$$

is the best linear unbiased estimate of $\underline{\mu}$, and we get var. $(\hat{\underline{\mu}}) = (C'C)^{-1}\sigma^2$.

The unbiased estimate of σ^2 based on the least-square estimate of $\underline{\mu}$ is given by

$$\hat{\sigma}^2 = \frac{Y'(I_N - C(C'C)^{-1} \, C')Y}{N - p}$$

All the p effects represented by the effect vector $\underline{\mu}$ are estimable only if rank C = rank C'C = p; otherwise some of them are non-estimable. (Ref.: Graybill (10).)

VI. TESTING OF SIGNIFICANCE AND CONFIDENCE BOUNDS

We again assumed that $\underline{e}_{(N\times 1)}$ is NID $(\underline{0}_{(N\times 1)}, \sigma^2 I_N)$ and on the basis of information supplied by the N responses y_1, y_2, . . ., y_N certain test procedures about the nature of effects are given.

6.1 Single Hypothesis (Student's t-test)

Consider
$$H_o: \underline{\mu}_o = 0$$
$$H_a: \underline{\mu}_o \neq 0$$

where μ_o denotes a certain specified effect. Let $\hat{\mu}_o$ be the least-square estimate of μ_o with var. $(\hat{\mu}_o) = d\sigma^2$, where d is some constant and

$$E\left[\frac{\underline{y}'(I_N - C(C'C)^{-1} C')\underline{y}}{N - p} = \sigma^2\right] = \sigma^2$$

(Ref.: Graybill (10).)

Then an α level test of the above hypothesis is given by

$$\phi(y_1, y_2, \ldots, y_N) = \begin{cases} 1 \text{ (i.e., reject } H_o) \text{ if } t = \left|\dfrac{\hat{\mu}_o}{\sqrt{d\sigma^2}}\right| > t_\alpha \\ 0 \text{ (i.e., accept } H_o) \text{ otherwise} \end{cases}$$

where t_α is such that

$$E\left[\phi(y_1, y_2, \ldots, y_N)/H_o\right] = \alpha \qquad (0 < \alpha < 1).$$

6.2 Multiple Hypothesis (F-test)

Let $(\underline{\mu}^*)' = (\mu_1, \mu_2, \ldots, \mu_r)$

be the row vector of r effects $(r < p)$ with

$$(\hat{\underline{\mu}}^*)' = (\hat{\mu}_1, \hat{\mu}_2, \ldots, \hat{\mu}_r) .$$

Then an α level test for the hypothesis

$$
\begin{array}{cc}
r & \begin{bmatrix}
0 & 1 & 0 & 0 & \cdots\cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots\cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots\cdots & 0 & 0 & \cdots & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdots\cdots & \cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots\cdots & \cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots\cdots & \cdot & \cdot & \cdots & \cdot \\
0 & 0 & 0 & 0 & \cdots 0 & 1 & 0 & \cdots & 0
\end{bmatrix} \cdot \underline{\mu}_{(p\mathrm{x}1)} \\
 & 1 \qquad\qquad r \qquad\qquad\qquad p - (r+1)
\end{array}
$$

$$= \underline{\mu}^*_{(r\mathrm{x}1} = \underline{0}_{(r\mathrm{x}1)} ,$$

or $$H_o = (G\underline{\mu} = \underline{\mu}^*) = \underline{0}$$
$$H_a = (G\underline{\mu} = \underline{\mu}^*) \neq \underline{0}$$

is given by

$$\phi(y_1, y_2, \ldots, y_N) = \begin{cases} 1 \text{ (i.e., reject } H_o) \text{ if } F > F_\alpha \\ 0 \text{ (i.e., accept } H_a) \text{ elsewhere} \end{cases} ,$$

where

$$F = \frac{(N - p)\, \underline{y}' \left[C(C'C)^{-1}C' - G(G'G)^{-1}G'\right] \underline{y}}{r\, \underline{y}'\left[I_N - C(C'C)^{-1}C'\right]\underline{y}}$$

and F_α is such that

$$E\left[\phi(y_1, y_2, \ldots, y_N)/H_o\right] = \alpha \qquad (0 < \alpha < 1)$$

(Ref.: Graybill (10).)

6.3 Confidence Bounds for Estimate (Single Hypothesis)

Given an estimable effect μ_o (say), the probability statement

$$P_r\left\{\left|\frac{\hat{\mu}_o - \mu_o}{\sqrt{d\sigma^2}}\right| \leq t_\alpha\right\} = 1 - \alpha$$

based on student's "t" distribution can be inverted to read

$$P_r\left\{\hat{\mu}_o - t_\alpha\sqrt{d\hat{\sigma}^2} \leq \mu_o \leq \hat{\mu}_o + t_\alpha\sqrt{d\hat{\sigma}^2}\right\} = 1 - \alpha.$$

6.4 Confidence Bounds for Estimate (Multiple Hypothesis)

The variance ratio statistic, F, in (6.2) can be used to give a probability statement

$$P_r\left\{\frac{N-p}{r} \cdot \frac{(\hat{\underline{\mu}}^* - \underline{\mu}^*)'\left[G(C'C)^{-1}G'\right]^{-1}(\hat{\underline{\mu}}^* - \underline{\mu}^*)}{\underline{y}'\left[I - C(C'C)^{-1}C'\right]\underline{y}} < F_\alpha\right\} = 1 - \alpha,$$

which implies

$$P_r\left\{\left[(\hat{\underline{\mu}}^*)'\left[G(C'C)^{-1}G'\right]^{-1}(\hat{\underline{\mu}}^*)\right]^{1/2} - \left[\frac{r}{N-p}F_\alpha \cdot \left(\underline{y}'\left[I - C(C'C)^{-1}C'\right]\underline{y}\right)\right]^{1/2}\right.$$

$$\leq \left[(\underline{\mu}^*)'\left[G(C'C)^{-1}G'\right]^{-1}(\underline{\mu}^*)\right]^{1/2}$$

$$\leq \left[(\hat{\underline{\mu}}^*)'\left[G(C'C)^{-1}G'\right]^{-1}(\hat{\underline{\mu}}^*)\right]^{1/2}$$

$$\left. + \left[\frac{r}{N-p}F_\alpha \cdot \left(\underline{y}'\left[I - C(C'C)^{-1}C'\right]\underline{y}\right)\right]^{1/2}\right\} \geq 1 - \alpha.$$

(Ref.: Roy, S. N. and Bose, R. C., (14), Roy S. N. (15).)

The inequality in the parenthesis provides a confidence interval for the parametric function

$$\left[(\underline{\mu}^*)' \left[G(C'C)^{-1} G'\right]^{-1} (\underline{\mu}^*)\right]^{1/2}$$

with a confidence coefficient $\geqslant 1 - \alpha$.

VII. SUMMARY

The method of construction is, in brief, as follows. We obtain t fractions of the 2^m complete factorial out of r fractions of the 3^n complete factorial. We then associate each fraction from the 2^m with a fraction from the 3^n. The association is such that every treatment combination in the fraction from the 2^m is adjoined to every treatment combination in the fraction from the 3^n, thereby forming treatment combinations in the fraction from the $2^m \times 3^n$.

Section 2 contains the standard method for obtaining fractional factorial designs from the 2^m, 3^n, and $2^m \times 3^n$ complete factorial. An example is given to describe the method of construction of the mixed type of $2^m \times 3^n$ fractionally replicated factorial design. In sections 3 and 4, there are univariate and multivariate linear model analyses. Section 5 contains a description of least squares estimation. The last section is to test hypotheses that are whether a specified (or each of a set of) effects is significantly different from zero, or the effects within a given set are all equal.

ACKNOWLEDGMENTS

The author wishes to express gratitude to his major professor, Dr. Young O. Koh, for helping in the preparation of this report and giving invaluable advice. He also wants to acknowledge helpful comments of Professor Holly C. Fryer, Professor Arlin M. Feyerherm, and Professor Leonard E. Fuller.

REFERENCES

1. Addelman, S. "Techniques for Constructing Fractional Replicate Plans." <u>Journal of the American Statistical Association</u>, Vol. 58 (1963), pp. 62-69.

2. Banerjee, K. S. "A Note on the Fractional Replication in Factorial Arrangements." <u>Sankhya</u>, <u>10</u> (1950), p. 87.

3. Barnard, M. M. "An Enumeration of the Confounded Arrangements in the 2^n Factorial Designs." <u>Jour. Roy. Stat. Soc. Suppl</u>. 3, pp. 195-202, 1936.

4. Bose, R. C. "Mathematical Theory of the Symmetrical Factorial Design." <u>Sankhya</u>, <u>8</u> (1947), Part 2, pp. 107-166.

5. Bose, R. C. and Connor, W. S. "Analysis of Fractionally Replicated 2^m and 3^n Design." Invited Paper at the 31st Session of the International Statistical Institute, Brussels, 1958.

6. Box, G. E. P. and Hunter, J. S. "The 2^{k-p} Fractional Factorial Design, II." <u>Technometrics</u>, 3, pp. 311-352.

7. Connor, W. S. "Construction of Fractional Factorial Designs of the Mixed 2^m x 3^n Series." Published in <u>Contributions to Probability and Statistics</u>, essays in honor of Harold Hotelling (1960).

8. Connor, W. S. and Shirley, Young. "Fractional Factorial Designs for Experiments with Factors at Two and Three Levels." To be published in <u>Applied Mathematics Series National Bureau of Standards</u>, U. S. Government Printing Office, Washington, D. C., Vol. 58, 1961.

9. Finney, D. J. "The Fractional Replication of Factorial Arrangements." <u>Annals of Eugenics</u>, <u>12</u> (1945), pp. 291-301.

10. Graybill, F. A. "An Introduction to Linear Statistical Models." McGraw-Hill Book Company, Inc. (1961), pp. 113-117.

11. Kempthorne, O. "A Simple Approach to Confounding and Fractional Replication in Factorial Experiments." <u>Biometrika</u>, 34, (1947), pp. 255-272.

12. MacDuffee, C. C. "The Theory of Matrices." Chelsea Publishing Company, New York (1946), pp. 81.

13. Morrison, Milton. "Fractional Replication for Mixed Series." Biometrics, 12 (1956), pp. 1-19.

14. Roy, S. N. and Bose, R. C. "Simultaneous Confidence Interval Estimation." Annals of Mathematical Statistics, 24(1953), pp. 513-536.

15. Roy, S. N. "Some Aspects of Multivariate Analysis." John Wiley & Sons, Inc. (1957).

ON CONSTRUCTION OF THE FRACTIONALLY REPLICATED $2^m \times 3^n$ EXPERIMENTAL DESIGN

by

WILLIAM K. HSU

B. A., Tunghai University, 1961

AN ABSTRACT OF A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Statistics

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1968

The use of a factorial set of treatment combination has now become a widely accepted means of investigation. The principles of factorial experimental design have been elaborated primarily for agricultural field experiments, but have been found valuable in many other types of experimentation. Yates (1937) has described many designs for factors at two and three levels. By these methods, many factors may be introduced into one experiment, but the total number of plots or other experimental units required for a comprehensive test is large. For example, eight factors each used at two levels require 256 plots for one replication, and six factors each at three levels require 729 plots; it is seldom practicable to carry out experiments involving so many plots. A further device for reducing the number of plots is that of fractional replication, whereby certain treatment interactions are assumed negligible and only a selection of all possible treatment combinations is used in the experiment.

In section 2, an attempt has been made to formulate the general principles of construction of fractional asymmetrical designs of the type 2^m and 3^n factorials which have been used to obtain the factorial replicated designs of the mixed type $2^m \times 3^n$. To illustrate the method of construction, consider a half-replicate of a $2^3 \times 3^2$ design. The 2^3 part of the design is split into two conventional half-replicates S_1 and S_2. The 3^2 part of the design is split into three conventional one-third replicates T_1, T_2, and T_3. The final design can then be represented symbolically by $S_1T_1 + S_2T_2 + S_2T_3$. Where S_1T_1, for example, represents all possible combinations of treatment combinations in S_1 with treatment combinations on T_1. This gives a design of 36 out of a total of 72 trials. In section 3 and 4, there are univariate and multivariate linear model analysis. Section 5 contains a description of least square estimation. The last section is to test hypotheses that are whether a specified (or each of a set of) effects is significantly different from zero, or the effects within a given set are all equal. The idea behind this work is to estimate the main effects and the two-factor interactions (assuming higher factor interactions to be absent) with as few assemblies as possible and without too much computation.