THE NORMAL DISTRIBUTION IN LIFE TESTING

by

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TABLE OF CONTENTS

I Introduction

Often it is desirable to estimate the average life of a product on the basis of a random sample. For example, the average life of a production lot of dry cell battries may be estimated by taking a random sample of size N from the lot, putting a "load" across each one, and keeping track of the amount of time required for the output of each dry cell to fall below a specified limit. This is a typical life testing situation. However, the amount of time required for the last battery to go bad may be quite long, especially if N is large. In order to reduce the time required to finish the test, one may stop the test when the Kth battery goes bad, where K is an integer less than N. Notice that the observations become available in an ordered manner,i.e., the smallest value (lifetime) first, the second smallest next, and so on, up to the largest value last. Thus, if the test is terminated when the Kth failure occurs, the data consist of the first K order statistics from a random sample of size N.

Davis (1952) has analyzed data from life tests on different products. Although many products have lives that are exponentially distributed, a few products have lives that are normally distributed. In particular, the "lifetimes in minutes of 100 flashlight cells" were normally distributed with a mean of 746 and a standard deviation of 40. Also, the "lifetimes in hours of 417 forty watt 110-volt internally frosted incandescent lamps" were approximately normally distributed with a mean of about 1070 and a standard deviation of about 200. Finally, the number of miles to the first major breakdown of 191 bus motors were normally distributed with a mean of 97,000 miles and a standard deviation of about 30,000 miles. In this report, we will be concerned only with the case where the lifetimes are normally distributed. For treatment of the exponential case, the reader is referred to a series of articles by B.

Epstein (1960 a,b). The statistical problem, then, is to estimate the mean and the variance of a normal distribution based on the first K order statistics from a random sample of size N. The sample is said to be censored on the right; the largest (N-K) observations are the censored observations. The sample is of size N, and K/N is the proportion of uncensored observations. It will be assumed the sample is taken from an infinite population.

Several estimators have been derived to solve this problem. The most important estimators are the maximum likelihood estimators and the best linear unbiased estimators. First, I will derive the maximum likelihood estimators and then the best linear unbiased estimators, in both cases following the work of Gupta (1952) closely.

II Maximum Likelihood Estimators

Let $X_1, X_2, \ldots, X_K$ be the first K order statistics from a random sample of size N. The joint density of these order statistics is:

$$f_{X_1,X_2,\ldots,X_K}(x_1,x_2,\ldots,x_K) = \frac{N!}{(N-K)!} \prod_{i=1}^{K} f(x_i)\ [1-F(x_K)]^{N-K}$$

For the normal distribution, the likelihood function is:

$$L(\mu,\sigma^2|x_1,x_2,\ldots,x_K) = \frac{N!}{(N-K)!} [\sigma^2 2\pi]^{-K/2} \exp\left\{\frac{-1}{2\sigma^2} \sum_{i=1}^{K} (x_i-\mu)^2\right\}$$

$$\times \left[(\sigma^2\ 2\pi)^{-1/2} \int_{x_K}^{\infty} \exp\left\{\frac{-1}{2\sigma^2} (x-\mu)^2\right\} dx\right]^{N-K} \tag{2.1}$$

The natural logarithm of the likelihood function is:

$$\ln L(\mu,\sigma^2|x_1,x_2,\ldots,x_K) = \ln\left[\frac{N!}{(N-K)!}\right] - K\left[\ln\left\{(2\pi)^{1/2}\right\}\right] - K \ln\sigma$$

$$+ \left[\frac{-1}{2\sigma^2} \sum_{i=1}^{K} (x_i-\mu)^2\right] + (N-K) \ln\left[(\sigma^2\ 2\pi)^{-1/2}\right.$$

$$\left.\times \int_{x_K}^{\infty} \exp\left\{\frac{-1}{2\sigma^2} (x-\mu)^2\right\} dx\right] \tag{2.2}$$

Let $c = \ln\left\{\frac{N!}{(N-K)!}\right\} - K \ln\{(2\pi)^{1/2}\}$

$$t = \frac{x-\mu}{\sigma}$$

$$\eta = \frac{x_K-\mu}{\sigma}$$

$$\Phi(\eta) = (2\pi)^{-1/2} \int_{\eta}^{\infty} \exp\left\{\frac{-1}{2} t^2\right\} dt$$

$$\phi(\eta) = (2\pi)^{-1/2} \exp\left\{\frac{-1}{2}\eta^2\right\}$$

$$A = \phi(\eta) \;/\; \Phi(\eta).$$

Then we can write (2.2) as

$$\ln L(\mu,\sigma^2|x_1, \ldots, x_K) = c - K \ln \sigma - (1/2\sigma^2) \sum_{i=1}^{K} (x_i-\mu)^2 + (N-K) \ln \Phi(\eta) \tag{2.3}$$

The next step is to take the derivative of the logarithm of the likelihood function with respect to μ.

$$\frac{\partial \ln L(\mu,\sigma^2|x_1, \ldots, x_K)}{\partial\mu} = (-1/2\sigma^2) \sum_{i=1}^{K} \left(\frac{\partial(x_i-\mu)^2}{\partial\mu}\right) + (N-K) \frac{\partial \ln \Phi(\eta)}{\partial\mu} \tag{2.4}$$

$$\frac{\partial(x_i-\mu)^2}{\partial\mu} = 2(x_i-\mu)(-1) = -2(x_i-\mu) \tag{2.5}$$

$$\frac{\partial \ln \Phi(\eta)}{\partial\mu} = \frac{\partial \ln \Phi(\eta)}{\partial\eta} \frac{\partial\eta}{\partial\mu} = \frac{\phi(\eta)}{\Phi(\eta)} \frac{1}{\sigma} \tag{2.6}$$

Therefore:

$$\frac{\partial \ln L(\mu,\sigma^2|x_1, \ldots, x_K)}{\partial\mu} = \sigma^{-2} \sum_{i=1}^{K} (x_i-\mu) + (N-K)(A/\sigma) \tag{2.7}$$

The next step is to find the derivative of the likelihood function with respect to σ.

$$\frac{\partial \ln L(\mu,\sigma^2|x_1, \ldots, x_K)}{\partial\sigma} = - K/\sigma - (1/2) \sum_{i=1}^{K} (x_i-\mu)^2$$

$$\times\ (-2)\ \sigma^{-3} + (N-K) \frac{\partial \ln \Phi(\eta)}{\partial\sigma} \tag{2.8}$$

$$\frac{\partial \ln \Phi(\eta)}{\partial\sigma} = \frac{\partial \ln \Phi(\eta)}{\partial\eta} \cdot \frac{\partial\eta}{\partial\sigma} = (-A) \frac{-\eta}{\sigma} \tag{2.9}$$

Therefore:

$$\frac{\partial \ln L\ (\mu,\sigma^2|x_1,\ \ldots,\ x_K)}{\partial\sigma} = -\ K/\sigma + \sum_{i=1}^{K} (x_i-\mu)^2\ \sigma^{-3} + (N-K)\ A(\eta/\sigma). \tag{2.10}$$

Next we set equations (2.7) and (2.10) both equal to zero, then solve the first for A, and substitute this value for A into (2.10). We obtain:

$$A = \left\{\frac{-1}{\hat{\sigma}^2} \sum_{i=1}^{K} (x_i-\hat{\mu})\right\} \frac{\hat{\sigma}}{N-K} \tag{2.11}$$

$$(-\ K/\hat{\sigma}) + \left[\sum_{i=1}^{K} (x_i-\hat{\mu})^2\ \hat{\sigma}^{-3}\right] - (\eta/\hat{\sigma}^2) \sum_{i=1}^{K} (x_i-\mu) = 0. \tag{2.12}$$

Now we multiply equation (2.12) by $\hat{\sigma}^3$ and expand the second term by adding and subtracting $\bar{x}$ inside the square, where $\bar{x} = \sum_{i=1}^{K} x_i/K$. We obtain

$$-\ K\hat{\sigma}^2 + \sum_{i=1}^{K} (x_i-\bar{x})^2 + \sum_{i=1}^{K} (\bar{x}-\hat{\mu})^2 - \eta\hat{\sigma} \sum_{i=1}^{K} (x_i-\hat{\mu}) = 0. \tag{2.13}$$

Now divide equation (2.13) by K, and substitute $(x_K-\hat{\mu})/\hat{\sigma}$ for η and we obtain

$$-\ \hat{\sigma}^2 + s^2 + (\bar{x}-\hat{\mu})^2 - (x_K-\hat{\mu})(\bar{x}-\hat{\mu}) = 0 \tag{2.14}$$

where $s^2 = \sum_{i=1}^{K} (x_i-\bar{x})^2/K.$

Rewrite equation (2.14) as:

$$(\bar{x}-\hat{\mu})^2 - (x_K-\hat{\mu})(\bar{x}-\hat{\mu}) = \hat{\sigma}^2-s^2 \tag{2.15}$$

By expanding the square of the first term, multiplying the factors of the second term, and combining terms, we obtain:

$$\bar{x}^2 - \bar{x}\hat{\mu} - \bar{x}x_K + \hat{\mu}x_K = \hat{\sigma}^2-s^2$$

which can be rewritten as:

$$\hat{\mu}(x_K-\bar{x}) - \bar{x}(x_K-\bar{x}) = \hat{\sigma}^2-s^2 \tag{2.16}$$

Now letting $d = (x_K-\bar{x})$, equation (2.16) can be rewritten as

$$\hat{\mu} = \bar{x} + (\hat{\sigma}^2 - s^2)/d \; . \tag{2.17}$$

The maximum likelihood equations contain only the two unknown parameters, the known constants N, K, and the observations. However, algebra will not solve the two equations in two unknowns because they contain the function A. Hence a table will be needed to solve the maximum likelihood equations. The purpose of rewriting the maximum likelihood equations is to obtain a convenient form for the table. Cohen (1961) has called the quanity that is looked up in the table an auxiliary estimating function. Gupta (1952) and Cohen (1961) use different auxiliary estimating functions. The estimates obtained will be the same except for rounding error.

Set equation (2.7) equal to zero.

$$(1/\hat{\sigma}^2) \sum_{i=1}^{K} (x_i - \hat{\mu}) + (N-K)\, A/\hat{\sigma} = 0 \tag{2.18}$$

Multiply equation (2.18) by $(\hat{\sigma}^2/K)$ and we obtain:

$$(\bar{x} - \hat{\mu}) + \frac{N-K}{K} A\hat{\sigma} = 0 \tag{2.19}$$

The first term of equation (2.19) can be rewritten:

$$(\bar{x} - \hat{\mu}) = (\bar{x} - x_K) + (x_K - \hat{\mu}) = -d + \hat{\sigma}\hat{\eta}$$

Therefore equation (2.19) becomes

$$\hat{\sigma}[\hat{\eta} + (N-K)\, A/K] = d \tag{2.20}$$

Let $z = \hat{\eta} + (N-K)\, A/K$ and equation (2.20) becomes:

$$\hat{\sigma} z = d \quad \text{or} \quad \hat{\sigma} = d/z \tag{2.21}$$

Equation (2.21) is used to obtain the maximum likelihood estimate of σ. Also, z is the auxiliary estimating function. It is necessary to obtain expressions for determining the value of z. Let us first write equation (2.14) as

$$-\hat{\sigma}^2 + s^2 + (\bar{x} - \hat{\mu})^2 - (\bar{x} - \hat{\mu})(x_K - \hat{\mu}) = 0. \tag{2.22}$$

By expanding the square in the third term and using the definitions of d and $\hat{\eta}$, we obtain:

$$-\hat{\sigma}^2 + s^2 + d^2 + \hat{\sigma}^2\hat{\eta}^2 + 2(-d)\,\hat{\sigma}\hat{\eta} - (\bar{x}-\hat{\mu})(x_K-\hat{\mu}) = 0. \tag{2.23}$$

The last term can be rewritten:

$$(\bar{x}-\hat{\mu})(x_K-\hat{\mu}) = (\bar{x} - x_K + x_K - \hat{\mu})(x_K-\hat{\mu})$$

$$= (-d + \hat{\sigma}\hat{\eta})(\hat{\sigma}\hat{\eta})$$

Then multiplying equation (2.23) by minus one and collecting terms, we obtain:

$$\hat{\sigma}^2 - (s^2+d^2) + \hat{\eta}\hat{\sigma}d = 0. \tag{2.24}$$

Now we substitute $\hat{\sigma} = d/z$ into equation (2.24) and obtain:

$$d^2/z^2 + \hat{\eta}d^2/z - (s^2+d^2) = 0. \tag{2.25}$$

$$\frac{d^2 + \hat{\eta}d^2 z}{z^2} = s^2 + d^2$$

$$\frac{1 + \hat{\eta}z}{z^2} = \frac{s^2+d^2}{d^2}$$

Now subtracting one from each side:

$$\frac{1 + \hat{\eta}z - z^2}{z^2} = \frac{s^2}{d^2}$$

$$\frac{z^2}{1 + \hat{\eta}z + z^2} + 1 = \frac{d^2}{s^2} + 1$$

$$\frac{z^2 + (1 + \hat{\eta}z + z^2)}{z^2} = \frac{d^2+s^2}{s^2}$$

$$\frac{1 + \hat{\eta}z - z^2}{1 + \hat{\eta}z} = \Psi \quad \text{where } \Psi = \frac{s^2}{s^2+d^2} \tag{2.26}$$

Note the Ψ can be calculated from the data. Gupta (1952) has provided a table (table 1, p. 262) for looking up the value of z from Ψ (row headings) and p (column headings), where p is the proportion of uncensored observations.

Once z is determined, the maximum likelihood estimate of σ is obtained from equation (2.21). The maximum likelihood estimate of μ is obtained by substituting the estimate of σ^2 into equation (2.17).

Gupta also derived the asymptotic variances and covariance of the maximum likelihood estimators of the mean and standard deviation. He gives these as coefficients of the variance divided by N, the sample size. The coefficients are presented in table 2 (Gupta, 1952,p.263) indexed by p, the proportion of uncensored observations in the sample. In a given problem, multiply the table entry by the estimate of the variance divided by N to obtain an estimate of the asymptotic variance or covariance. One may use the asymptotic variances of the maximum likelihood estimators to calculate large sample confidence intervals for the parameters, based on Halperin's (1952) results and generalizations: the maximum likelihood estimators from a censored sample are asymptotically normally distributed. Cohen (1961) also gives in table 3, p.539, the coefficients for the asymptotic variances and covariance, and includes the correlation coefficient already calculated for the reader.

Worked Example: Suppose the City Transit Co. purchased 50 new buses with improved motors. It is desired to estimate the average number of miles to the first major motor breakdown of these buses after the first 25 buses have had their first major motor breakdown. The data for the first 25 are (assume the remaining 25 buses have milages greater than 101770):

42850	55030	63040	82540	93850
43390	55720	66700	83830	96850
51970	56620	71410	87730	98230
54400	56980	76300	88390	98800
55030	61420	81610	92470	101770

We have $N = 50$, $K = 25$, $p = K/N = .50$, $\bar{x} = 72677.2$,

$s^2 = 343038836.2$, $s = 18521.30763$;

$\Psi = s^2/(s^2 + d^2) = 0.2884061105$; $d = x_K - \bar{x} = 29092.8$; $d^2 = 846391011.8$;

z is found by linear interpolation in the table provided by Gupta (1952, table 1,p.262). The value of z is .8563896094, and the estimates of the parameters are:

$\hat{\sigma} = d/z = 33971.45374$, $\hat{\sigma}^2 = 1154059669$, $\hat{\mu} = \bar{x} + (\hat{\sigma}^2 - s^2)/d = 100554.2291$.

Using Gupta's table 2 (1952, p.263) to find the asymptotic variances and standard errors of the estimates of the mean and standard deviation, we have:

$\hat{\sigma}^2/N = 23081193.39$; estimated variance of $\hat{\mu} = 35016247.68$; estimated standard error of $\hat{\mu} = 5917.452803$; estimated variance of $\hat{\sigma} = 28654147.53$; estimated standard error of $\hat{\sigma} = 5352.956896$.

Using the standard errors and the asymptotic distribution of the estimators, we can find confidence intervals for the mean and standard deviation. The 95% confidence interval for the mean is

$$\hat{\mu} - Z_{.975}\,SE(\hat{\mu}) < \mu < \hat{\mu} + Z_{.975}\,SE(\hat{\mu}),$$

or (88956.02, 112152.43), where $Z_{.975} = 1.96$.

The 95% confidence interval for the standard deviation is

$$\hat{\sigma} - Z_{.975}\,SE(\hat{\sigma}) < \sigma < \hat{\sigma} + Z_{.975}\,SE(\hat{\sigma}),$$

or (23479.66, 44463.25).

III. Best Linear Unbiased Estimators

Let $\underline{X}' = (X_1, X_2, \ldots, X_K)$ be the first K order statistics from a random sample of size N from a distribution which depends only on a location parameter, μ, and a scale parameter, σ. Let μ_i be the expected value of the ith order statistic from a random sample of size N from the standardized distribution. Then

$$E(X_i) = \mu + \sigma\mu_i \tag{3.1}$$

Let V be the covariance matrix of the first K order statistics from a random sample of size N from the standardized distribution. Then $\sigma^2 V$ is the is the covariance matrix for $\underline{X}$. We have:

$$E(\underline{X}) = B\underline{\theta} \quad \text{where} \quad \underline{\theta} = \begin{bmatrix} \mu \\ \sigma \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & \mu_1 \\ 1 & \mu_2 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & \mu_K \end{bmatrix} \tag{3.2}$$

The best linear unbiased estimator of $\underline{\theta}$ is the value, say $\underline{\theta}^*$, that minimizes

$$\Omega = (\underline{X}-B\underline{\theta})' \; V^{-1} \; (\underline{X}-B\underline{\theta}) \tag{3.3}$$

We will find the minimum by taking the derivative of (3.3) with respect to $\underline{\theta}$ and setting it equal to zero.

$$\frac{\partial\Omega}{\partial\underline{\theta}} = \frac{\partial\Omega}{\partial(\underline{X}-B\underline{\theta})} \cdot \frac{\partial(\underline{X}-B\underline{\theta})}{\partial\underline{\theta}} = 2V^{-1}(\underline{X}-B\underline{\theta})(-B) = 2(-B)'V^{-1}(\underline{X}-B\underline{\theta})$$

$$= -2B'V^{-1}\underline{X} + 2B'V^{-1}B\underline{\theta} \tag{3.4}$$

Setting (3.4) equal to zero, we obtain:

$$-2B'V^{-1}\underline{X} + 2B'V^{-1}B\underline{\theta}^* = 0$$

$$\underline{\theta}^* = (B'V^{-1}B)^{-1}B'V^{-1}\underline{X} \tag{3.5}$$

Also, the covariance of the estimators of μ and σ is:

$$\text{cov}(\underline{\theta}^*) = \sigma^2(B'V^{-1}B)^{-1} \tag{3.6}$$

The best linear unbiased estimators, μ^* and σ^*, are just linear combinations of the observations:

$$\mu^* = \sum_{i=1}^{K} \beta_i X_i \quad \text{and} \quad \sigma^* = \sum_{i=1}^{K} \gamma_i X_i$$

The coefficients, β_i and γ_i, have been calculated and tabled. At the time Gupta (1952) computed the coefficients, the variances and covariance of the order statistics from the standard normal distribution were available only for samples of size ten or less. Later tables, which also had more significant digits, gave the variances and covariances for samples of size twenty or less. For the coefficients of the best linear unbiased estimators computed from these later tables, see Sarhan and Greenberg (1956,1958a,1958b, 1962).

Worked Example: Suppose eight light bulbs are put on test and the first six lives (X_i) are as given below, along with the coefficients, β_i and γ_i, for calculating the best linear unbiased estimates of μ and σ. The coefficients are taken from tables 3 and 4 in Gupta (1952), pp.267-268.

X_i	β_i	$\beta_i X_i$	γ_i	$\gamma_i X_i$
832.0	.05692	47.357	-.36376	-302.65
877.4	.09621	84.415	-.17876	-156.84
943.8	.11532	108.839	-.08808	- 83.13
1102.2	.13090	144.278	-.01320	- 14.55
1152.4	.14512	167.236	.05698	65.66
1219.6	.45552	555.552	.58682	715.69
		1107.677		224.18

We have $\mu^*= 1107.677$, $\sigma^*= 224.18$, and $\sigma^{2*}= 50256.67$. Using coefficients (.13988 and .11707) given by Gupta (1952) in his tables 5 and 6, p.269, we obtain the estimated variances of μ^* and σ^*:

$$var(\mu^*)= \sigma^{2}*(.13988)= 7029.90$$

$$var(\sigma^*)= \sigma^{2}*(.11707)= 5883.55$$

The coefficients used here (.13988 and .11707) apply only to the case N=8, K=6. Gupta (1952) does not give the coefficient of $\sigma^{2}*$ for estimating the covariance of μ^* and σ^*, but Sarhan and Greenberg (1956,1962) do. The estimated covariance of μ^* and σ^* is

$$cov(\mu^*,\sigma^*)= \sigma^{2}*(.0250)= 1256.42$$

Because Gupta could only compute the coefficients for the best linear unbiased estimators for samples of size ten or less, and because he thought that censored samples of sizes slightly larger than ten would be not sufficiently large for the asymptotic properties of the maximum likelihood estimators to hold reasonable well, he proposed alternative linear estimators.

The alternative linear estimators are found by assuming the covariance matrix of the order statistics is the identity matrix, that is, V=I. Gupta (1952) gives formulae (equations 33,34,35, p.269) for computing the coefficients for the alternative linear estimators directly from the expected values of the order statistics. At that time the expected values of the order statistics from the standard normal distribution were available for samples of size fifty or less.

It may be that the alternative linear estimators are no longer needed, due to the extension of the coefficients for the best linear unbiased estimators for samples up to size twenty. Also, a Monte Carlo simulation study of the maximum likelihood estimators shows that the maximum likelihood estimators are quite good, even for sample sizes as small as ten (Harter and Moore, 1966).

IV. Monte Caro Simulation of the Maximum Likelihood Estimators

Harter and Moore (1966) did a computer simulation study of the maximum likelihood estimators of μ and σ based on censored samples. They obtained the means, variances, covariance, and mean square errors of the estimates of μ and σ in 1000 samples of sizes ten and twenty from the standard normal distribution. The 1000 samples of each size were subjected to varying degrees of censoring. Hence the estimates of μ and σ for different degrees of censoring are not independent, as they are based on the same 1000 samples. The proportion of censored observations in the samples of size ten went from 0 to .8 by increments of .1, and for samples of size twenty, the proportion of censored observations ranged from 0 to .9, again by increments of one-tenth. My major criticism of their work was their failure to construct a histogram of the 1000 estimates of μ and σ for any combination of sample size and percent censoring. Hence the sampling distribution of the maximum likelihood estimators remains unknown for small samples, and the problem of setting confidence intervals on μ and σ has no adequate solution.

Parts of their tables 2 and 3 (p. 210) are reproduced in table 1. We may conclude that the maximum likelihood estimators of μ and σ are negatively biased in a life testing situation, where the largest (N-K) observations are censored. The bias of the estimator of σ is approximately equal to $-1/K$, that is, the average of the estimates of σ is about $1-(1/K)$ or $(K-1)/K$ of the true value of σ.

Harter and Moore (1966) also compare the exact variance of the best linear unbiased estimators of μ and σ with their empirical estimates of the mean squared errors of the maximum likelihood estimators of μ and σ. This comparison is the correct one to make because the maximum likelihood estimators are biased while the best linear unbiased estimators are unbiased. Even for a sample of size ten, the maximum likelihood estimators are more precise than the best linear unbiased estimators, that is, they have smaller mean squared errors. Parts of their tables 4 and 5 (p. 211) are reproduced in table 2.

V. Censorship and Truncation

Up to now we have discussed only the case where N items are placed on test and we wait until the Kth item fails, where K is a preassigned number. This is referred to as index termination (K is the index) or type II censoring. Also possible is termination of the test at a predetermined time, say x_o=1000 hours. This is called time termination or type I censoring. Note that in type I censoring, K is a random variable. Cohen (1961) gives formulae for the maximum likelihood estimators of μ and σ for both type I and type II censoring. There are other generalizations of censored samples. A discussion of them follows.

A sample censored such that the largest values are not available (as in life testing situations) is said to be censored on the right. A sample censored such that the smallest observations are not available is said to be censored on the left. A sample censored such that some observations at both ends are not available is said to be doubly censored, as opposed to singly censored. The work of Sarhan and Greenberg (1956, 1958a, 1958b, 1962) on the best linear unbiased estimators takes into consideration all possible cases of single or double censoring at the extremes for samples of size twenty or less. (It does not cover censoring of the middle values of a sample.) Cohen's work (1961) covers single censoring on either the right or the left, but not double censoring.

It is also desirable to distinguish between censored samples and truncated samples. In censored samples, the values of some observations are unknown, but the observations are known to exist, and in fact are counted. In a truncated sample, no observations exist beyond (either to the right or to the left) some point, x_o. The values of all observations are known. It may be that no observations beyond x_o occur due to the limitation of some measuring device. In this case it should be clear that the sample is truncated, but the population itself extends beyond x_o. The other possibility is that the population is truncated, i.e., no values beyond x_o occur in the population. Cohen (1961) gives the maximum likelihood estimators of μ and σ based on truncated samples from a normal population.

The above generalizations are obviously valuable extensions of the problem of estimating the parameters of the normal distribution on the basis of the first K order statistics from a sample of size N. Many of these generalizations are not particularly relevant to life testing situations. Type I censoring clearly is, and a worked example will be given later. A truncated population model may be relevant to life testing because obviously no values below zero can occur. Therefore I will briefly cover the case of truncated populations.

Cohen (1961) says that the method of estimation is the same for truncated populations as for truncated samples. This may be, but only if you realize that you are estimating the mean and standard deviation of the (imaginary) non-truncated population, not of the actual population which has no elements beyond the truncation point. Consider a population that is distributed normally, with mean 1, and standard deviation 1, then truncated at zero (so all elements of the population below zero are lost).

First of all, note that the mean of the population is no longer 1, but some value greater than 1. Also note that the new (truncated) population is skewed to the right. The p.d.f. of the new population is:

$$f(x) = \frac{1.18863664}{2(\pi)^{\frac{1}{2}}} \exp[-\tfrac{1}{2}(x-1)^2] \qquad \text{for } x \geq 0$$

$$= 0 \qquad \text{for } x < 0$$

In some sense $\mu = 1$ and $\sigma^2 = 1$ may be considered the parameters of this distribution, but they are not the mean and variance. In general the p.d.f. of a truncated normal population in a life testing situation is:

$$f(x) = \{c/(2\pi)^{\frac{1}{2}}\sigma\} \exp\{(-1/2\sigma^2)(x-\mu)^2\} \qquad \text{for } x \geq 0$$

$$= 0 \qquad \text{for } x < 0$$

We are assuming $X_o = 0$ is the trucation point; the value of c depends on the amount of area that is truncated below zero, i.e., on the values of μ and σ.

Worked example: The following 20 observations are a random sample from a population that is believed to be approximately normal, except that a sizeable portion of the population is truncated below zero.

1.01	4.46	4.87	5.12	5.28	6.43	6.46	7.39	7.70	8.50
11.37	12.25	12.80	13.58	15.35	15.98	17.56	21.80	26.78	28.27

Using Cohen's (1961) formulae (4) and (1), page 536:

$\bar{x} = 11.648$ $\quad s^2 = 53.648276$ $\quad x_o = 0$

$\hat{\mu} = \bar{x} - \hat{\theta}(\bar{x} - x_o) = 8.653$

$\hat{\sigma}^2 = S^2 + \hat{\theta}(\bar{x}-x_o)^2 = 88.53$ $\quad \hat{\sigma} = 9.409$

$\hat{\gamma} = S^2/(\bar{x}-x_o)^2 = .3954$ $\quad \hat{\theta} = .25712$ (interpolated from Cohen's (1961) table 1, p. 537)

If we had a truncated sample, we could obtain the asymptotic variances and covariance of the estimators by using Cohen's (1961) table 3, page 539. However, we have a truncated population, and Cohen's table 3 does not apply.

Worked Example: Time termination or type I censoring. Suppose fifty buses with new improved motors are purchased by City Transit Co. We wish to estimate the average number of miles until the first major motor breakdown of these "new improved" buses. It is convenient to terminate the test when each bus has gone 100,000 miles. We have N=50 x_o=100,000. The data are:

42880	66820	75640	85150	93310
46210	67450	77470	85360	95260
52870	67600	77860	86380	98590
56380	73420	77890	88510	99190
58570	73450	81520	91180	
61120	74050	82090	91330	

$\bar{x} = 75983.92857$ $\quad S^2 = 227535523.9$

$\hat{\mu} = \bar{x} - \hat{\lambda}(\bar{x}-x_o) = 94994.552$

$\hat{\sigma}^2 = S^2 + \hat{\lambda}(\bar{x}-x_o)^2 = 684096014.1$

$(\bar{x}-x_o) = -24016.07143$

$(\bar{x}-x_o)^2 = 576771686.9$

$\hat{\gamma} = S^2/(\bar{x}-x_o)^2 = .3944984282 \qquad h = \frac{N-K}{N} = .44$

$\hat{\lambda} = f(h,\hat{\gamma}) = .7915792341$ (interpolated from table 2, p. 538, Cohen, 1961)

To find the asymptotic variance and covariance, we need to find $\hat{\xi} = \frac{(x_o-\hat{\mu})}{\hat{\sigma}} = .1913746564$. The coefficients of $\frac{\hat{\sigma}^2}{N}$ to obtain the estimated asymptotic variances are found by interpolation to Cohen's (1961) table 3, p. 539, using $\eta = -\hat{\xi}$. They are (to seven digits):

$\mu_{11} = 1.319517 \qquad \mu_{12} = .3945325 \qquad \mu_{22} = 1.046810$

$\widehat{var}(\hat{\mu}) = \frac{\hat{\sigma}^2}{N}\mu_{11} = 18053526$

$\widehat{var}(\hat{\sigma}) = \frac{\hat{\sigma}^2}{N}\mu_{22} = 14322370$

$\widehat{covar}(\hat{\mu},\hat{\sigma}) = \frac{\hat{\sigma}^2}{N}\mu_{12} = 5397962.$

Note that we calculated $\hat{\sigma}^2$, but then the estimated variance of $\hat{\sigma}$.

$\hat{\sigma} = \sqrt{\hat{\sigma}^2} = 26155.22919$. The estimated correlation of $\hat{\mu}$ and $\hat{\sigma}$, calculated from figures above, is $\hat{\rho}(\hat{\mu},\hat{\sigma}) = .3358$; from interpolation of Cohen's (1961) table 3, p. 539, $\hat{\rho}(\hat{\mu},\hat{\sigma}) = .3479$. The estimated standard errors of $\hat{\mu}$ and $\hat{\sigma}$ are:

$S.E.(\hat{\mu}) = 4.249 \times 10^3 \qquad S.E.(\hat{\sigma}) = 3.784 \times 10^3$

Using the normal distribution and $Z = 1.96$, large-sample 95% confidence intervals are:

$$8.667 \times 10^4 < \mu < 1.033 \times 10^5$$

$$1.874 \times 10^4 < \sigma < 3.357 \times 10^4$$

VI. Cost, Time Savings, and Efficiency

For the exponential distribution, the cost of a life test based on K out of N items includes the cost of the K items; the (N-K) items that don't fail are as good as new (the "memoryless" property of the exponential distribution). For the normal distribution, the cost of a life test based on K out of N items includes the cost of all N items minus the salvage value of the (N-K) partially used items.

Another application where K out of N items may be used is destructive quality control tests. It may be possible to put N items in a testing device and gradually increase the stress. As the items fail one by one, the failure point for each one is recorded and the test is stopped when the Kth item fails. In this case the items which have not failed may be as good as new. Of course, the question arises, is there any advantage to testing K out of N items instead of just testing K items until they fail? To answer this question, let us compare the mean squared errors of the maximum likelihood estimators of μ and σ based upon a complete sample of size K with the mean squared errors of the maximum likelihood estimators of μ and σ based upon a censored sample of size N, where N is greater than K. The maximum likelihood estimator of μ based upon a complete sample of size K is $\bar{X}$, which is unbiased and has variance (and mean squared error) equal to 1/K for the standard normal distribution. The mean squared error of the maximum likelihood estimator of μ from a censored samples of sizes ten and twenty are available from the Monte Carlo study of Harter and Moore (1966). The variances of $\bar{X}$ and the empirically obtained values of the mean squares errors of $\hat{\mu}$ from the censored samples are presented in table 3. For larger values of N, one could use the asymptotic variance to estimate the mean squared error of $\hat{\mu}$ based upon a censored sample. An example using N=100 is given in table 3. The coefficient σ_{11} is taken from Gupta's table 2 (1952, p. 263), where p=K/N. See table 2 in this report for a comparison of the mean squared error and the asymptotic variance of the estimators of the mean and standard deviation based upon censored samples of sizes ten and twenty.

The maximum likelihood estimator of the standard deviation is biased for both complete and censored samples. Hence I use asymptotic results to compare the estimator based on a complete sample of size K with the estimator based upon K uncensored observations from a sample of size N. The results are given in table 4. Harter and Moore (1966) give the coefficient of σ^2/N for the asymptotic variance of $\hat{\sigma}$ for complete samples as well as censored samples. The value is 0.500000 for complete samples. The values of the coefficient (σ_{22}) for censored samples are taken from Gupta's table 2 (1952, p. 263).

On the basis of the examples in table 3, we may conclude that it may be beneficial to use a censored sample (K out of N) in a destructive quality control test when estimating the mean. Unless the amount of censoring is extreme (70 percent or more), the censored sample will provide a better estimator of the mean than a full sample of size K. Also, based upon the results presented in table 4, it seems that it is never beneficial to use the censored sample when estimating the standard deviation.

In planning a life test, the decision between a censored sample and a complete sample must include the expected length of the test, as well as the cost of the articles and the efficienty of the estimators.

For the exponential case, Epstein (1960a) gives a table showing the ratio of the expected waiting time for a life test based on K out of N to the expected waiting time for a life test based on K out of K. For the normal case, such ratios depend upon the parameters of the distribution as well as K and N. For a normal distribution to be a reasonable approximation to the distribution of lifetimes of a product, the mean must be about three standard deviations above zero. In order to compare normal distribution results with the exponential case, I decided to compute the ratio of expected waiting times, assuming the mean is three times the standard deviation. The use of the mean equal to three standard deviations produces ratios that are small compared to using a mean larger than three standard deviations. To show this let $\mu=c\sigma$ and consider the ratio as a function of c.

$$\frac{E(X_{K,N})}{E(X_{K,K})} = \frac{\mu + \sigma\mu_{K,N}}{\mu + \sigma\mu_{K,K}} = \frac{c\sigma + \sigma\mu_{K,N}}{c\sigma + \sigma\mu_{K,K}} = \frac{c + \mu_{K,N}}{c + \mu_{K,K}} = f(c)$$

where $\mu_{i,j}$ is the expected value of the ith order statistic in a sample of size j from the standard normal distribution.

Note that $\mu_{K,K} \geq 0$ and $\mu_{K,N} < \mu_{K,K}$.

$$f'(c) = \frac{(1)(c+\mu_{K,K}) - (c+\mu_{K,N})(1)}{(c+\mu_{K,K})^2} = \frac{\mu_{K,K} - \mu_{K,N}}{(c+\mu_{K,K})^2}$$

The derivative, $f'(c)$, is greater than zero for all values of c. Hence f(c) is an increasing function of c and the minimum value of f(c) in the interval $(3,\infty)$ is at c=3. Therefore these ratios represent the best (smallest) savings ratios possible for the normal case. The results are presented in table 5. The table entries are $E(X_{K,N})/E(X_{K,K})$. Epstein's figures for the exponential case are given in parentheses. As these ratios are based on the assumption that the mean is three times the standard deviation, it may be more convenient to have a table showing the amount of time saved, which depends only on the standard deviation and not the relative sizes of the mean and standard deviation. The expected value of the time saved is $E(X_{K,K})-E(X_{K,N})=\sigma(\mu_{K,K}-\mu_{K,N})$. Hence the expected time savings can be expressed in standard deviations. The figures are presented in table 6.

Another way of looking at the waiting time problem is to see what combinations of K and N give the same expected waiting time as N' out of N' (N' is less than K). Table 7 contains the expected values of the order statistics from the standard normal distribution for selected values of K and N. Some remarkable comparisons can be found in this table. For example, the value of N=100 and K=90 is 1.250 while the value for N=K=6 is 1.267. This means that the expected waiting time for a life test based on 90 out of 100 items will be less than the expected waiting time for a life test based on 6 out of 6 items.

Also, the phenomenon of diminishing returns can be found from table 7 by finding the reduction in waiting time for each observation censored. For example, for N=10, censoring the largest observation (so that K=9) results in a reduction in expected waiting time of 1.539-1.001=0.538 standard deviations; but censoring the 9th observation, (so that K=8) produces an additional reduction in expected waiting time of only 1.001-0.656=0.345 standard deviations. Likewise, for N=100, censoring the largest 10 percent of the observations results in a time savings of 1.258 standard deviations, but an additional censoring of 10 more observations produces a further time savings of only 0.428 standard deviations. The diminishing returns for additional censoring continues to hold until the sample median is censored, then each additional observation censored will produce a larger time saving than the observation previously censored, i.e., we get increasing returns.

VII. Conclusions

Both maximum likelihood and best linear unbiased estimators have been derived to solve the problem of estimating the parameters of the normal distribution from a censored sample. When the censoring is on the right, as in life tests, the maximum likelihood estimators are negatively biased, i.e., they underestimate the parameters. However, they are asymptotically unbiased and their asymptotic distribution is normal. Therefore, for large samples, the maximum likelihood estimators can be used to set confidence intervals on the parameters. To obtain unbiased estimators when the sample size is small, best linear unbiased estimators have been derived. The coefficients of the observations for the best linear unbiased estimators are available for samples of size twenty or less. The small sample distributions of both the maximum likelihood and best linear unbiased estimators are unknown, and therefore there is no method for obtaining confidence intervals on the parameters for small samples. A computer simulation study of the maximum likelihood estimators indicates that the maximum likelihood estimators are superior to the best linear unbiased estimators for samples of size ten or more. The efficiencies of the maximum likelihood estimator and the best linear unbiased estimator of the mean based on a censored sample relative to a complete sample remain high as long as the central values are not censored. The efficiency in estimating the standard deviation from a censored sample relative to a complete sample is not very high when the censoring is at the extremes. In fact, using asymptotic theory for the maximum likelihood estimator, it was found to be better to use a complete sample of size K, rather than the first K observations from a larger sample. The time savings from censoring the largest values in a life test were found to show the concept of diminishing returns. That is, each additional observation censored produces less time savings than the previous censored observation. This holds true only for censoring the largest half of the sample.

TABLE 1

Monte Carlo Simulation of Maximum Likelihood (ML) Estimators

N	K	$E(\hat{\mu})$	$E(\hat{\sigma})$	$V(\hat{\mu})$	$Cov(\hat{\mu},\hat{\sigma})$	$V(\hat{\sigma})$
10	10	-0.00	0.93	0.100	0.000	0.049
10	9	-.01	.92	.101	.003	.057
10	8	-.02	.90	.104	.009	.067
10	7	-.04	.88	.114	.019	.078
10	6	-.07	.85	.125	.028	.084
10	5	-.11	.81	.146	.047	.101
10	4	-.19	.76	.184	.075	.123
10	3	-.31	.67	.265	.130	.160
10	2	-.57	.51	.412	.210	.204
20	20	0.01	0.97	0.048	0.000	0.023
20	18	.01	.96	.048	.002	.027
20	16	-.00	.95	.050	.005	.032
20	14	-.01	.94	.054	.010	.038
20	12	-.03	.93	.061	.017	.045
20	10	-.04	.91	.073	.029	.057
20	8	-.07	.89	.095	.048	.075
20	6	-.13	.85	.142	.083	.100
20	4	-.27	.76	.257	.151	.141
20	2	-.73	.52	.563	.289	.202

$E(\hat{\mu})$ - average value of ML estimates of μ in 1000 samples

$E(\hat{\sigma})$ - average value of ML estimates of σ in 1000 samples

$V(\hat{\mu})$ - variance of ML estimates of μ in 1000 samples

$Cov(\hat{\mu},\hat{\sigma})$ - covariance of ML estimates of μ and σ in 1000 samples

$V(\hat{\sigma})$ - variance of ML estimates of σ in 1000 samples

TABLE 2

Comparison of Maximum Likelihood (ML) and Best Linear Unbiased (BLU) Estimators

N	K	$V(\mu^*)$	$MSE(\hat{\mu})$	$AV(\hat{\mu})$	$V(\sigma^*)$	$MSE(\hat{\sigma})$	$AV(\hat{\sigma})$
10	10	0.100	0.100	0.100	0.058	0.053	0.050
10	9	.102	.101	.102	.068	.064	.059
10	8	.107	.105	.106	.081	.076	.069
10	7	.117	.115	.114	.099	.092	.082
10	6	.134	.130	.127	.124	.106	.099
10	5	.166	.159	.152	.161	.136	.124
10	4	.237	.220	.199	.225	.184	.161
10	3	.417	.359	.302	.354	.269	.225
10	2	1.127	.733	.578	.749	.444	.354
20	20	0.050	0.048	0.050	0.027	0.024	0.025
20	18	.051	.048	.051	.032	.029	.029
20	16	.053	.050	.053	.037	.034	.034
20	14	.058	.054	.057	.045	.042	.041
20	12	.065	.061	.064	.055	.050	.050
20	10	.079	.075	.076	.070	.065	.062
20	8	.108	.100	.099	.094	.087	.081
20	6	.175	.158	.151	.139	.122	.122
20	4	.383	.331	.289	.244	.198	.177
20	2	1.871	1.089	.890	.786	.430	.376

$V(\mu^*)$ - exact variance of BLU estimator of μ

$MSE(\hat{\mu})$ - mean squared error of ML estimator of μ in 1000 samples

$AV(\hat{\mu})$ - asymptotic variance of ML estimator of μ

$V(\sigma^*)$ - exact variance of BLU estimator of σ

$MSE(\hat{\sigma})$ - mean squared error of ML estimator of σ in 1000 samples

$AV(\hat{\sigma})$ - asymptotic variance of ML estimator of σ

TABLE 3

Precision of the Maximum Likelihood Estimators
of μ from Complete and Censored Samples

N=K		N=10
$Var(\bar{X})=1/K$	K	$MSE(\hat{\mu})$
.500	2	.733
.333	3	.359
.250	4	.220
.200	5	.159
.167	6	.130
.143	7	.115
.125	8	.105
.111	9	.101
.100	10	.100

N=K		N=20
$Var(\bar{X})=1/K$	K	$MSE(\hat{\mu})$
.500	2	1.089
.250	4	.331
.167	6	.158
.125	8	.100
.100	10	.075
.083	12	.061
.071	14	.054
.062	16	.050
.056	18	.048
.050	20	.048

N=K		N=100
$Var(\bar{X})=1/K$	K	asymptotic $var(\hat{\mu})=\sigma_{11}/100$
.100	10	.1779
.050	20	.0578
.033	30	.0302
.025	40	.0199
.020	50	.0152
.0167	60	.0127
.0143	70	.0114
.0125	80	.0106
.0111	90	.0102

TABLE 4

Precision of Maximum Likelihood Estimators of σ from Complete and Censored Samples

N=K		N=100
1/2K	K	$\sigma_{22}/100$
.05	10	.0751
.025	20	.0354
.0167	30	.0225
.0125	40	.0162
.0100	50	.0124
.0083	60	.0099
.0071	70	.0082
.00625	80	.0069
.005556	90	.00586
.0050505	99	.0050843

1/2K=asymptotic variance of $\hat{\sigma}$ based upon a complete sample of size K from the standard normal distribution

$\sigma_{22}/100$=asymptotic variance of $\hat{\sigma}$ based upon the first K observations of an ordered sample of size N=100 from the standard normal distribution.

TABLE 5

Ratios of Expected Waiting Time for Life Tests Based on K out of N versus K out of K Observations based on $\mu = 3\sigma$

K \ N	2	3	4	5	10	15	20
1	.81(.5)	.72(.33)	.66(.25)	.61(.2)	.49(.1)	.42(.067)	.38(.05)
2	1	.84(.56)	.76(.39)	.70(.3)	.56(.14)	.49(.092)	.45(.068)
3		1	.86(.59)	.78(.43)	.61(.18)	.53(.12)	.49(.087)
4			1	.87(.62)	.64(.23)	.57(.14)	.52(.104)
5				1	.69(.28)	.60(.18)	.54(.125)
10					1	.73(.35)	.65(.23)

TABLE 6

Savings in Time for Life Tests Based on K out of N instead of K out of K Observations

K \ N	2	3	4	5	10	15	20
1	.56	.85	1.03	1.16	1.54	1.74	1.87
2	0	.56	.86	1.06	1.57	1.81	1.97
3		0	.55	.85	1.50	1.79	1.98
4			0	.53	1.41	1.74	1.95
5				0	1.29	1.68	1.91
10					0	1.20	1.60

TABLE 7

Expected Values of Standard Normal Order Statistics
for Selected Values of K and N

N	K=N	K=.9N	K=.8N	K=.7N	K=.6N
1	0				
2	.564				
3	.846				
4	1.029				
5	1.163				
6	1.267				
7	1.352				
8	1.424				
9	1.485				
10	1.539	1.001	0.656	0.356	.123
20	1.867	1.131	.745	.448	.187
30	2.043	1.179	.777	.473	.209
40	2.161	1.203	.793	.486	.220
50	2.249	1.218	.802	.494	.227
60	2.319	1.229	.809	.499	.231
70	2.377	1.236	.813	.502	.234
80	2.427	1.242	.817	.505	.237
90	2.470	1.246	.820	.507	.238
100	2.508	1.250	.822	.509	.240

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THE NORMAL DISTRIBUTION IN LIFE TESTING

by

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ABSTRACT

In life testing, the problem of estimating the parameters of a distribution from the first K order statistics of a random sample of size N often arises. The case of a normally distributed population is considered, and both maximum likelihood estimators and best linear unbiased estimators are derived. A simulation comparison of the maximum likelihood and best linear unbiased estimators indicates that the maximum likelihood estimators are superior for at least all sample sizes of ten or more. A distinction between censorship and truncation is made, as well as between type I and type II censoring. There is also a discussion of cost, time savings, and efficiency of the estimators based on censored and complete samples. Worked examples are given for maximum likelihood estimation from both type I and type II censored samples, a sample from a truncated population, and for best linear unbiased estimation from a type II censored sample.