MATHEMATICAL ANALYSIS OF THE CHARACTERISTICS OF AXIALLY SYMMETRIC DOWNWARD JET

by

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NOMENCLATURE

a Subscript, indicates values of surrounding air.

b Shape factor to be used in defining the jet boundary of zero downward velocity.

c Subscript, indicates values on axis of jet.

C Constant of integration.

e Natural logarithm.

g Gravitational acceleration; 32.2 feet per second.

h Vertical coordinate from the imaginary point source

k Thermal conductivity of air.

k_1 Shape factor to be used in recirculating type velocity distribution.

k_2 Shape factor to be used in error function type velocity distribution.

k_3 Shape factor to be used in error function type temperature distribution.

k_b Shape factor to be used in defining the jet boundary of the zero downward velocity.

L Vertical coordinates from the nozzle outlet.

m Subscript, indicates values at the end of the jet.

o Subscript, indicates values at the orifice.

p Subscript, indicates values at the beginning of the principal zone.

r Radial distance from axis of jet to point of jet under consideration.

r_f Radial distance from axis of jet to point of outside jet boundary established as surface of zero downward velocity.

T Temperature.

V Velocity.

' Denotes value based on temperature.

INTRODUCTION

From the thermodynamics viewpoint, the air jet can be classified into isothermal jet and non-isothermal jet, according to the thermal state of the fully developed jets. If the mixing of the jet with the surrounding stationary air takes place at an approximately constant pressure, the significant difference in the internal mechanism between these two jets is mainly due to the temperature variations that occur in the non-isothermal jet but not in the isothermal jet.

For the non-isothermal jet, the temperature variations produce the variations in both air density and turbulent kinematic viscosity. The variations of the air density will produce buoyant forces which will act on the fluid elements as external forces. The variations in the turbulent kinematic viscosity will produce a complicated variation in the pattern of viscous force which will also act on the fluid elements as external forces. With this complicated internal mechanism, together with the phenomenon of heat transmission in the non-isothermal jet, the theoretical analysis is much more difficult even with some plausible assumptions. However, the non-isothermal heated jet has been studied from its microscopic viewpoint, and correspondingly the internal mechanism of the air was not considered in the overall treatment.

The isothermal jet, due to its much more simplified internal mechanism than that of the non-isothermal jet, has been studied by solving the equation of continuity. The hyperbolic distribution of center-line velocity of the isothermal jet, which has been obtained experimentally at the Experiment Station at Kansas State College, was obtained by solving an equation of continuity which employed spherical coordinates.

CHARACTERISTICS OF THE HEATED JETS

The Velocity Distribution of Heated Jets

For heated jets projected downward from a nozzle into a stationary air space, there will be viscous forces and buoyant forces acting on a fluid element. These forces will balance the rate of change of momentum of the fluid element, and the dynamically steady state can be obtained.

Due to the predominant buoyant force acting on a fluid element, the downward directed component of velocity decreases with travel and near the bottom of the jet the air leaves the jet boundary and turns upward. The buoyant forces are considered to be greatest at the center line of the jet, and to decrease as the distance from the center line increases. Within the region of the boundary of zero downward velocity, the air flow is directed downward across the jet section, and in the region outside the jet boundary the air which has been turned upward at the bottom of the jet continuously flows upward.

This dynamical picture suggests a velocity profile which has the maximum downward velocity at the center line of the jet, then decreases to zero velocity at the jet boundary and has an upward velocity distribution in the region outside the jet boundary. Consequently, the velocity distribution was assumed to have the form

$$\frac{u}{u_c} = \left[1 - \left(\frac{r}{r_f}\right)^2\right] e^{-k_1(r/r_f)^2} \qquad (1)$$

in which k_1 is a shape factor, and r_f is the radial distance of the jet boundary measured from the jet axis. The value of the shape factor, k_1, can be determined in such a way that the net mass flow at each level is zero.

The total mass flow at a section can be integrated as

$$M = 2\pi \int_0^\infty \rho urdr \qquad (2)$$

Since
$$\frac{u}{u_c} = \left[1 - \left(\frac{r}{r_f}\right)^2\right] e^{-k_1(r/r_f)^2}$$

$$M = 2\pi \int \rho u_c \left[1 - \left(\frac{r}{r_f}\right)^2\right] e^{-k_1(r/r_f)^2} rdr \qquad (3)$$

If the air density is treated as a constant,

$$M = \frac{\pi \rho u_c^2 r_f^2}{k_1} \left[1 - \frac{1}{k_1}\right] \qquad (4)$$

For the net mass flow at each level to be zero, it is obvious that $k_1 = 1$.

The maximum upward velocity can be obtained by differentiating equation 1 with respect to $\frac{r}{r_f}$. This will result in the simple relation

$$\frac{r}{r_f} = \sqrt{\frac{k_1 + 1}{k_1}} \qquad (5)$$

For $k_1 = 1$,

$$\frac{r}{r_f} = 1.414$$

Correspondingly, $$\left(\frac{u}{u_c}\right)_{max.up} = -\frac{1}{k_1} e^{-(k_1+1)} \qquad (6)$$

For $k_1 = 1$,

$$\left(\frac{u}{u_c}\right)_{max.up} = -e^{-2} = -0.135$$

The velocity profile for shape factor k equal to 1 was plotted (Plate I). The velocity profile has its maximum downward velocity at the axis of the jet and reaches zero at the boundary of the jet (i e $r = r_f$). For $r > r_f$, the velocity becomes negative; in other words, it is directed upward. It first increased until $r = r_{max.up}$, which was found to be 1.414 r_f for maximum upward velocity. The upward velocity decreases for r greater than this value, and becomes zero when r is infinity.

The Boundary of the Heated Jets

The location of the boundary of zero downward velocity--inside of which the velocity is downward, while on the outside the velocity is upward--was determined by the following method.

The empirical equation for the boundary of jet velocity was assumed to be

$$r_f^2 + h^2 = h_m^2 \cos b\,\theta \qquad (7)$$

$$\frac{r_f}{h} = \tan\theta \qquad (8)$$

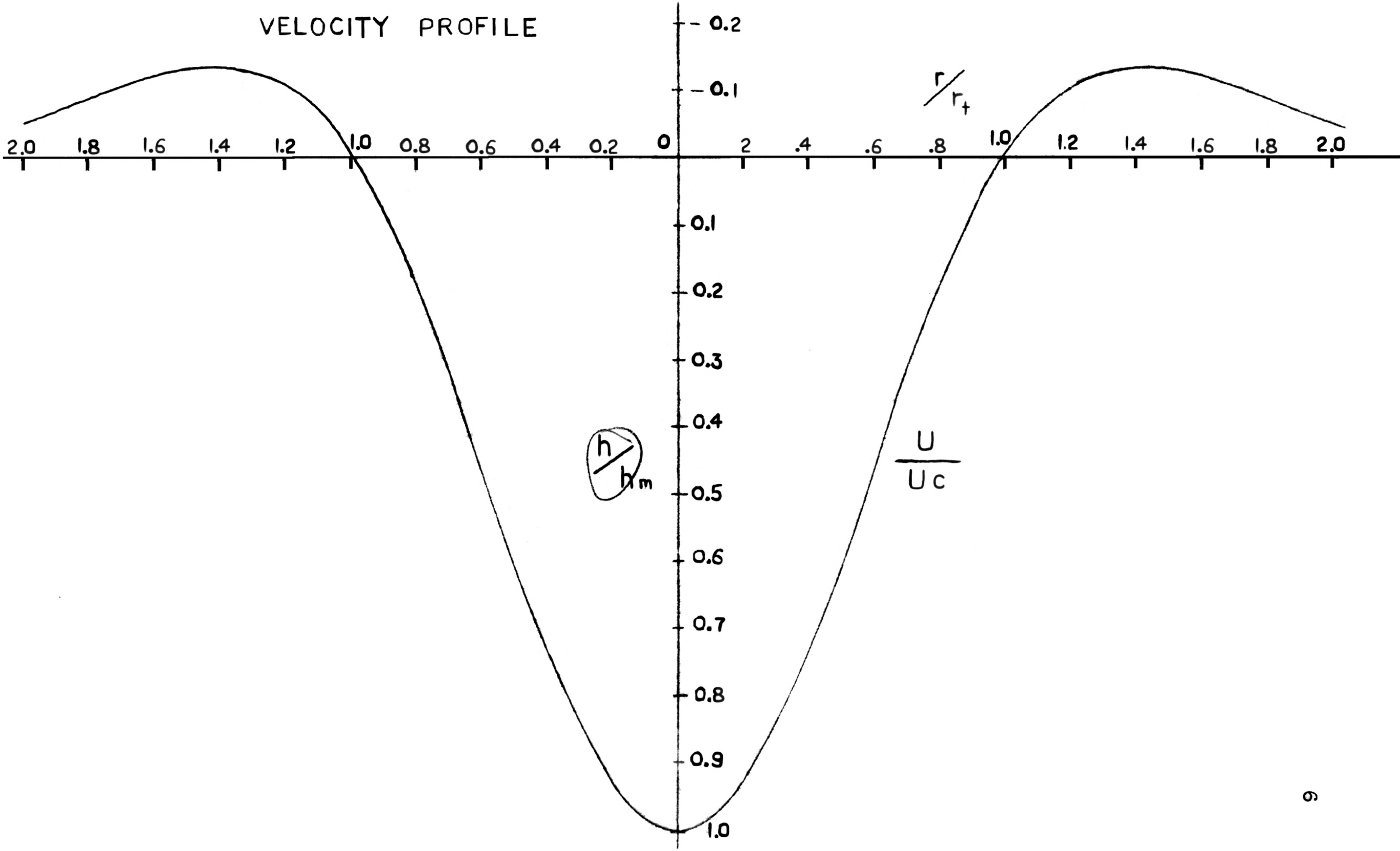
EXPLANATION OF PLATE I

Downward velocity profile of recirculating flow

$$\frac{u}{u_c} = \left[1 - \left(\frac{r}{r_f}\right)^2\right] e^{-k_1 (r/r_f)^2}$$

$k_1 = 1$

PLATE I



VELOCITY PROFILE
r/r_t
U/Uc
h/h_m
2.0 1.8 1.6 1.4 1.2 1.0 0.8 0.6 0.4 0.2 0 2 .4 .6 .8 1.0 1.2 1.4 1.6 1.8 2.0
0.2 0.1
0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0

in which r_f is referred to as the radial distance of the boundary measured from the jet axis.

Eliminating h in equations 7 and 8, there follows

$$\frac{r_f}{h_m} = \sqrt{\cos b\,\theta}\,\sin\theta \qquad (9)$$

$\frac{r_f}{h_m}$ was calculated by equation 9 for given $\frac{h}{h_m}$, and the corresponding boundary was plotted in Plate II for the case of $b = 9$. The other experimental equation for the jet boundary was assumed to have the form

$$\frac{h}{h_m} = e^{-k_b \frac{r_f^2}{h^2}} \qquad (10)$$

in which k_b is a shape factor of the jet.

The shape of the jet was determined by the shape factor of k_b and for various k_b the shapes of the jet change as shown in Plate III.

If equation 10 is differentiated with respect to h, the maximum r_f is obtained, and also the level, h (distance downward from the point source), of the maximum r_f is determined.

If one takes logarithmic differential on both sides of equation 10,

$$\log \frac{h}{h_m} = -k_b\left(\frac{r_f}{h}\right)^2$$

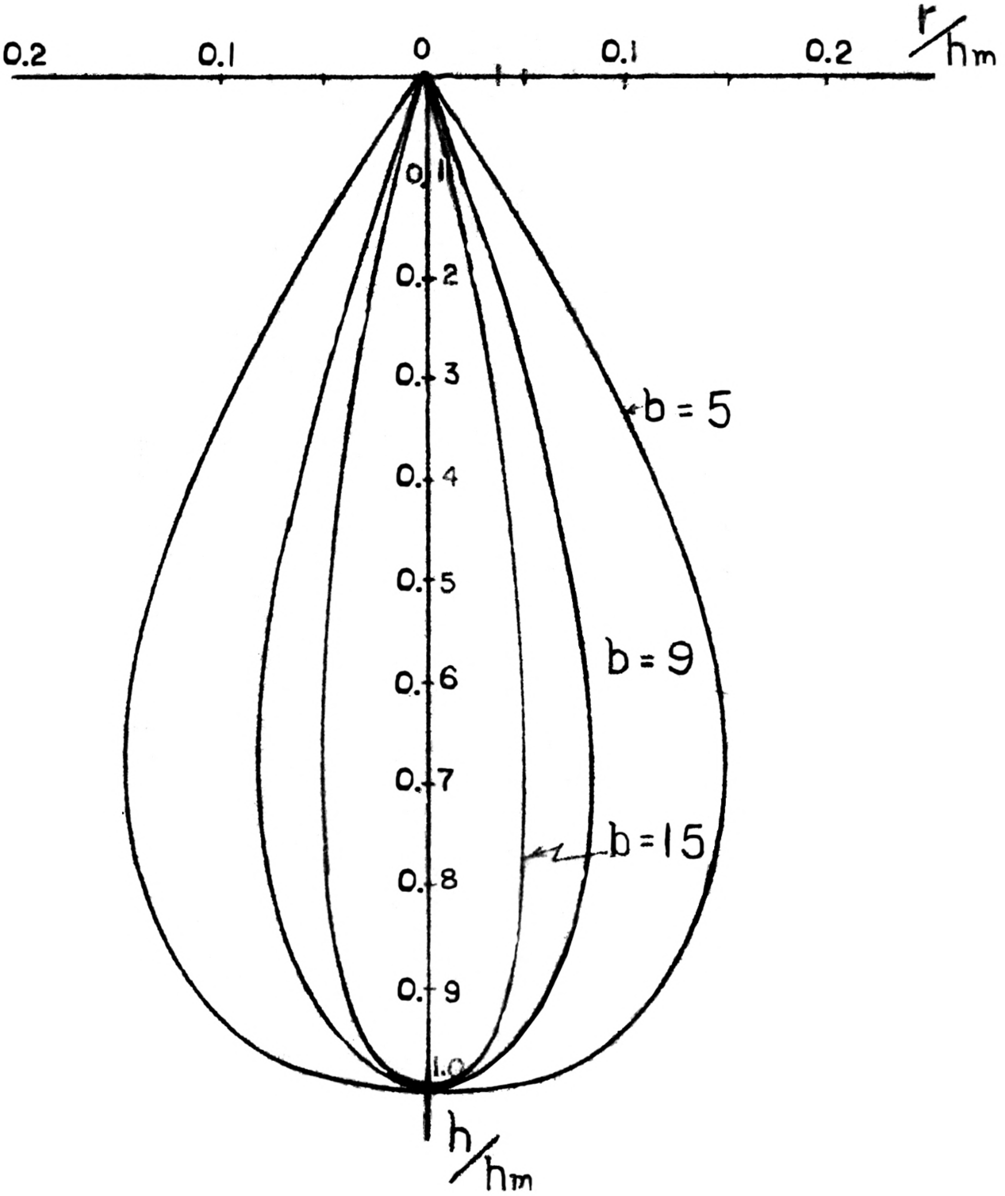
$$\frac{dh}{h} = -k_b\,\frac{2\,r_f\,dr_f}{h} + 2h\,\frac{r_f^2\,dh}{h^2}$$

EXPLANATION OF PLATE II

Jet boundary of zero downward velocity defined by

$$\frac{r_f}{h_m} = \sqrt{\cos b\,\theta}\,\sin\theta$$

PLATE II

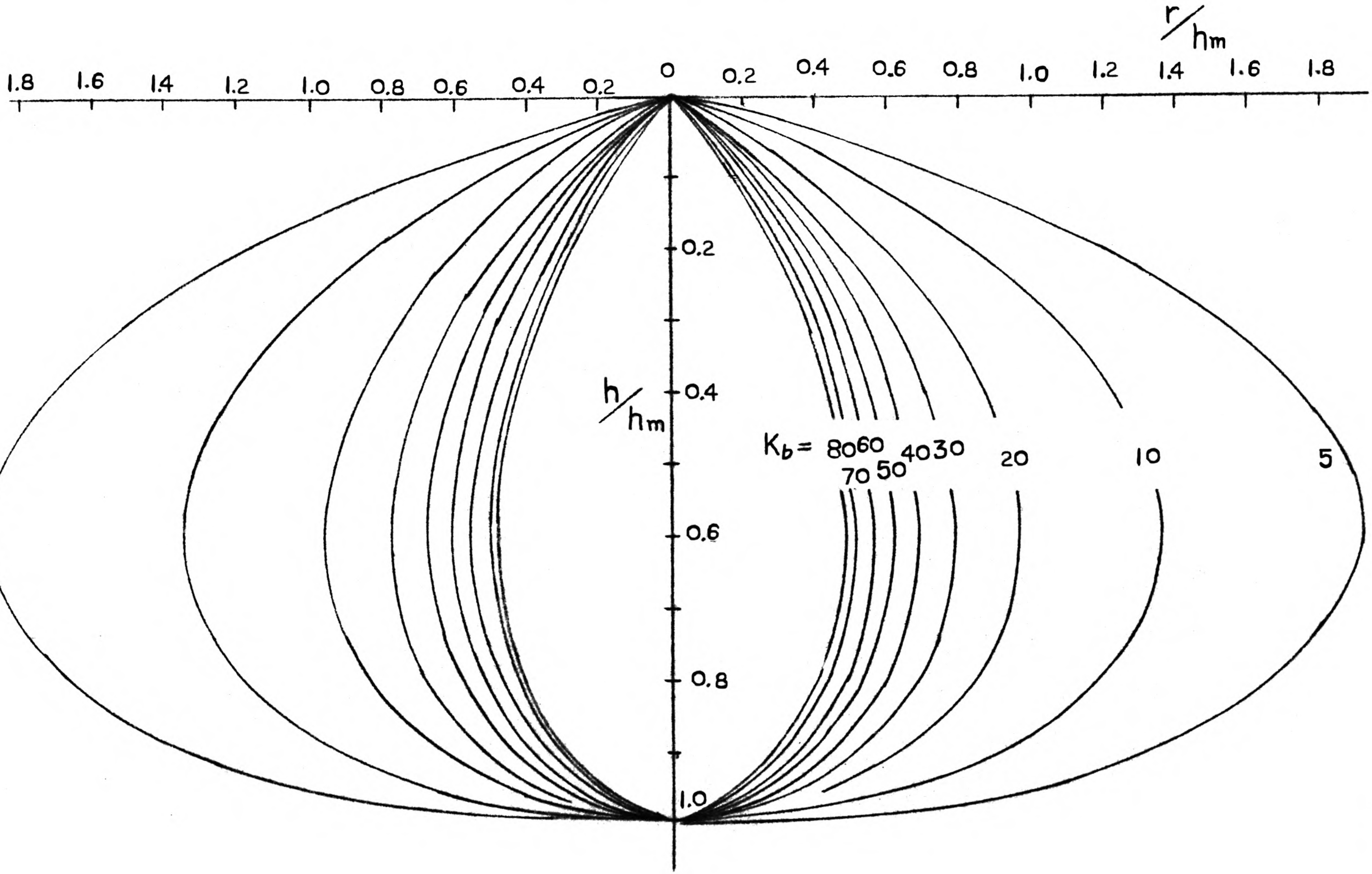


EXPLANATION OF PLATE III

Jet boundary of zero downward velocity defined by

$$\frac{h}{h_m} \quad e^{-k_b\left(\frac{rf}{h}\right)^2}$$

PLATE III



r/hm
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
h/hm
0.2
0.4
0.6
0.8
1.0
Kb = 80 60 40 30
70 50
20
10
5

Therefore
$$\frac{dr_f}{dh} = \frac{2\,k_b\left(\frac{r_f}{h}\right)^2 - 1}{2\,k_b\left(\frac{r_f}{h}\right)} \tag{11}$$

By letting equation 11 be zero,

$$2\,k_b\,\left(\frac{r_f}{h}\right)^2 - 1 = 0$$

$$\left(\frac{r_f}{h}\right)^2 = \frac{1}{2\,k_b}$$

$$\frac{r_f}{h} = \frac{1}{\sqrt{2\,k_b}} = \frac{r_{f\ max}}{h} \tag{12}$$

If we substitute equation 12 back in equation 10,

$$\left(\frac{h}{h_m}\right)_{\text{for } r_f max} = e^{-k_b \frac{1}{2k_b}} = e^{-\frac{1}{2}} = .606$$

and $r_{f\ max}$ can be calculated correspondingly as

$$(r_f)_{max} = \frac{1}{\sqrt{2\,k_b}}\,e^{-\frac{1}{2}} = \frac{.428}{\sqrt{k_b}}\,h_{max}$$

For $k_b = 2$,

$$r_{f\ max} - \frac{.428}{1.414}\,h_{max} = 0.302\,h_{max}$$

The jet will spread more widely as the value of k decreases finally for $k_b = 0$. r_f tends to infinity.

The Recirculating Stream Lines

In the flow of a jet, it is known that the stream line can be maintained only near the exit of the nozzle. At the downstream the flow is turbulent so that the stream line will not exist in reality. But it is of interest to represent equation 1 by means of plotting the stream function, ψ, and correspondingly get the flow pattern for the particular type of recirculating flow. The stream function was introduced thus:

$$\frac{\partial \psi}{\partial r} = \rho\, u\, r \qquad (13)$$

The stream function ψ can be solved for this particular case.

$$\psi = \int_0^x \rho\, u\, r\, dr + f(h) \qquad (14)$$

If one substitutes the expression of u, equation 14 can be integrated as:

$$\psi = \frac{\rho r_f^2 u_c}{2 k_1} e^{-k_1 (r/r_f)^2} \left[\left(\frac{r}{r_f}\right)^2 + \frac{k_1 - 1}{k_1} \right] + f(h \cdot c) \qquad (15)$$

in which $f(h \cdot c)$ is a function of h and can be determined by the initial condition, namely, $\psi = 0$ for $r = 0$. Thus

$$0 = \frac{\rho r_f^2 u_c}{2 k_1} \left[\frac{k_1 - 1}{k_1} \right] + f(h \cdot c)$$

1 - K1

Therefore $$f(h \cdot c) = - \frac{\rho r_f^2 u_c}{2 k_1} \; \frac{k_1 - 1}{k_1} \qquad (16)$$

Thus the stream function was found as

1 - K1

$$\psi = \frac{\rho r_f^2 u_c}{2 k_1} \left[\left(\frac{r}{r_f}\right)^2 e^{-k_1 (r/r_f)^2} + \frac{k_1 - 1}{k_1} \left(e^{-k_1 (r/r_f)^2} - 1\right) \right] \qquad (17)$$

For the particular case of $k_1 = 1$, the equation can be reduced to a simple form:

$$\psi = \frac{\rho r_f^2 u_c}{2} \left[\left(\frac{r}{r_f}\right)^2 e^{-1 (r/r_f)^2} \right] \qquad (18)$$

Since
$$\frac{u_c}{u_o} = C_1 \frac{h_p}{h}\left[1 - \left(\frac{h}{h_m}\right)^2\right]^n \Big/ \left[1 - \left(\frac{h_p}{h_m}\right)^2\right]^n$$

$$C_2 \Psi = \frac{2 h_m}{h}\left[1 - \left(\frac{h}{h_m}\right)^2\right]^n \left(\frac{r}{\cancel{r_f}}\right)^2 e^{-(r/r_f)^2} \qquad (19)$$

h/a $\quad \frac{a^2}{h^2} \quad \frac{h}{h^2} = \frac{r^2}{h}$

in which
$$C_2 = \frac{2 h_m}{h_p}\left[1 - \left(\frac{h_p}{h_m}\right)^2\right]^n \frac{u_o}{\rho\, r_f^2}$$

The stream lines were constructed and are shown in Plate IV. It is obvious that for a given value of Ψ_1 and h, there will exist two roots of corresponding r/r_f, single root, and none, for $\dfrac{C_2 \Psi_1}{\frac{h_m}{h}\left[1 - \left(\frac{h}{h_m}\right)^2\right]^n}$ greater than e^{-1}, equal to e^{-1}, and less less than e^{-1}, respectively.[1]

If two roots of r/r_f exist, one of which will be less than one, which is nothing more than the downward path, the other is greater than one, which is for the upward path.

Single roots, then, correspond to a point of turning when the upward flow will turn downward, or the downward flow will turn upward at the point, namely, $r = r_f$. Thus the stream lines form individual recirculating flow paths.

The stream line of $\Psi = 0$ is constructed by a line of jet axis, a horizontal infinite line at $h = h_m$, and also the boundary of top ceiling, and the vertical line at r is finity.

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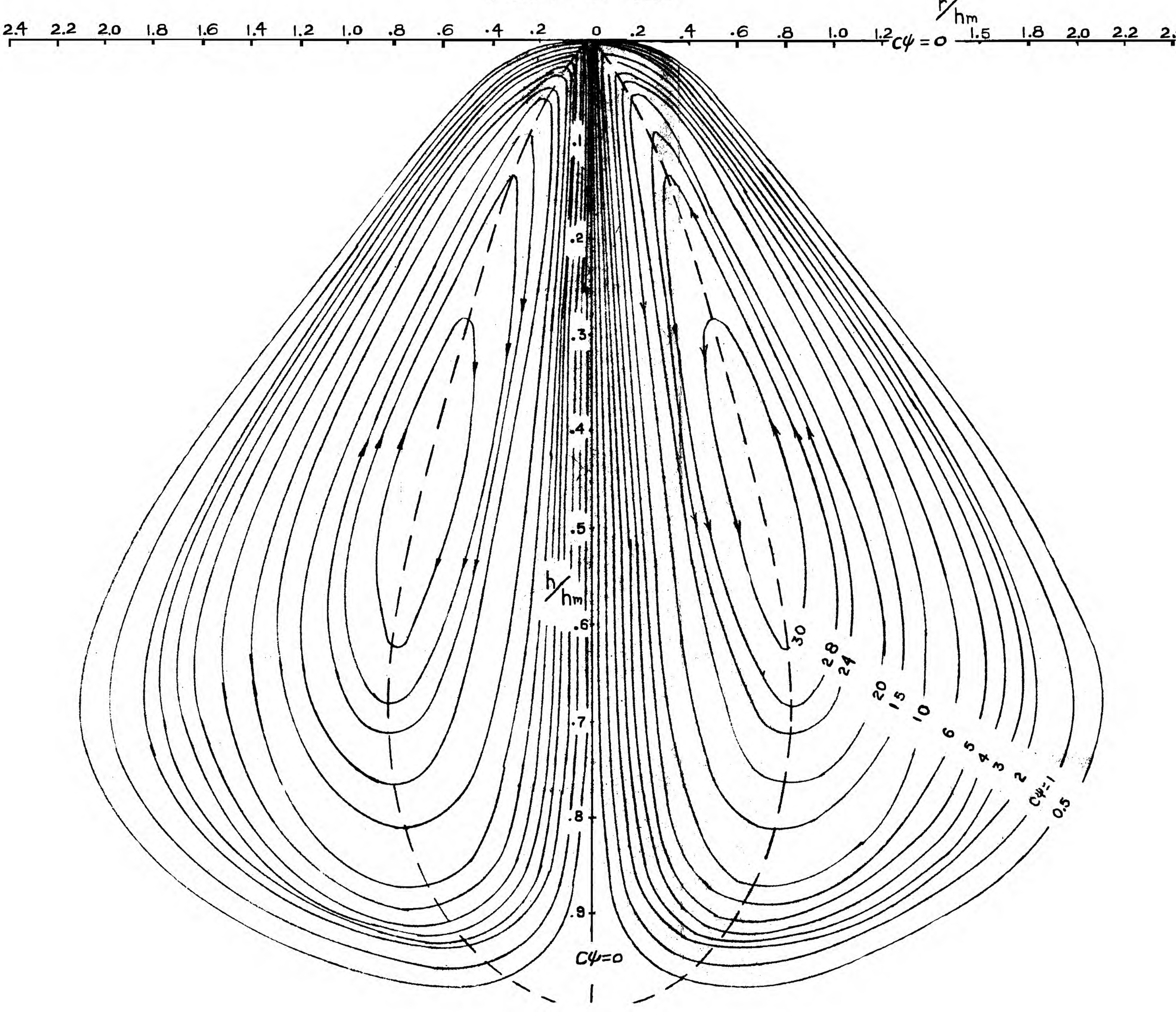
[1] The maximum value of $(r/r_f)^2 e^{-(r/r_f)^2}$ occurs where $r/r_f = 1$. For this case, r/r_f has only one value, i.e., unity. For other values of $(r/r_f)^2 e^{-(r/r_f)^2}$, (r/r_f) will have two values, one less than unity, the other greater than unity.

EXPLANATION OF PLATE IV

Recirculating flow stream lines.

$$c_2 \psi = \frac{h_m}{h} \left[1 - \left(\frac{h}{h_m}\right)^2 \right]^n \left(\frac{r}{r_f}\right)^2 e^{-(r/r_f)^2}$$

PLATE IV


r/hm
2.4 2.2 2.0 1.8 1.6 1.4 1.2 1.0 .8 .6 .4 .2 0 .2 .4 .6 .8 1.0 1.2 1.5 1.8 2.0 2.2 2.4
Cψ = 0
.1 .2 .3 .4 .5 .6 .7 .8 .9
h/hm
30 28 24 20 15 10 6 5 4 3 2
Cψ=1
0.5
Cψ=0

Momentum Analysis

From the foregoing velocity distribution equation, the momentum of the jet at a given cross section can be evaluated.

$$\frac{u}{u_c} = \left[1 - \left(\frac{r}{r_f}\right)^2\right] e^{-k_1 (r/r_f)^2} \qquad (20)$$

$$= \left[1 - \left(\frac{r}{r_f}\right)^2\right] e^{-(r/r_f)^2} \quad \text{for } k_1 = 1$$

Let $\frac{r}{r_f} = x \qquad r = r_f\, x \qquad u = u_c\,(1 - x^2)e^{-x^2}$

For given h, r_f is fixed.

$$dr = r_f\, dx$$

$$dI = \rho u^2\, 2\pi\, r\, dr$$

$$= \rho u^2\, 2\pi\, r_f^2\, x\, dx$$

$$I = 2\pi \rho r_f^2\; u^2\, x\, dx \qquad (21)$$

$$= 2\pi \rho r_f^2\, u_c^2 \int (1 - x^2)^2\, e^{-2x^2}\, x\, dx$$

Let $x^2 = y$

$$I = \pi \rho r_f^2\, u_c^2 \int (1 - y)^2\, e^{-2y}\, dy$$

$$\frac{I(+)}{\pi \rho r_f^2\, u_c^2} = \int_0^1 (1 - y^2)^2\, e^{-2y}\, dy \qquad (22)$$

$$\frac{I(-)}{\pi \rho r_f^2\, u_c^2} = \int_1^\infty (1 - y)^2\, e^{-2y}\, dy \qquad (23)$$

$$\frac{I(+)}{\pi \rho r_f^2\, u_c^2} = \left[-\frac{x^4}{2} e^{-2x^2} + \frac{x^2}{2} e^{-2x^2} - \frac{1}{4} e^{-2x^2}\right]_0^1$$

$$= \frac{1}{4}(1 - e^{-2}) \qquad (24)$$

$$\frac{I(-)}{\pi \rho r_f^2 u_c^2} = \left[-\frac{x^4}{2} e^{-2x^2} + \frac{x^2}{2} e^{-2x^2} - \frac{1}{4} e^{-2x^2} \right]_1^{\infty}$$

$$= \frac{1}{4} e^{-2} \tag{25}$$

$$\frac{I_{net}}{\pi \rho r_f^2 u_c^2} = \frac{I(+) + I(-)}{\pi \rho r_f^2 u_c^2} = \frac{1}{4} (1 - e^{-2}) + \frac{1}{4} e^{-2} = \frac{1}{4} \tag{26}$$

Therefore $$I_{net} = \frac{1}{4} \pi \rho r_f^2 u_c^2 \tag{27}$$

Hence the net momentum at each section is a quarter of the momentum which the jet would have if it had a uniform velocity equal to the center-line velocity over the circular area of radius r_f.

Asymptotic Incoming and Outgoing Stream Lines

Many experimental studies on the downward projection of air jets have shown that velocities along the vertical axis of the jet will take a form as

$$\frac{u_c}{u_o} = \frac{h_p}{h} \left[\frac{1 - \left(\frac{h}{h_m}\right)^2}{1 - \left(\frac{h_p}{h_m}\right)^2} \right]^n \tag{28}$$

in which n is an exponent which has been assumed of the order of one-half and one-third for the case of heated jet[1] and 0 for an isothermal jet. Also many experiments show that the normal

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[1] "Characteristics of Downward Jets from a Vertical Unit Heater," by Linn Helander, S. M. Yen, and L. B. Knee.

probability type curve is a satisfactory representation of the cross-sectional velocity profile of a free jet. Thus

$$\frac{u}{u_c} = e^{-k_2(r/h)^2} \tag{29}$$

If we combine equation 28 and equation 29, we get

$$\frac{u}{u_o} = \frac{h_p}{h}\left[\frac{1 - \left(\frac{h}{h_m}\right)^2}{1 - \left(\frac{h_p}{h}\right)^2}\right]^n e^{-k_2(r/h)^2} \tag{30}$$

It is instructive to investigate the stream function which will be derived from equation 30, and thus we can compare the pattern of stream path with the previously described recirculating stream lines.

If we introduce the stream function

$$\frac{\partial \Psi}{\partial r} = \rho u r \tag{31}$$

Substitute equation 30 in equation 31, and we get the stream function as a function of h and r. Thus

$$\Psi = \int_0^r \rho \frac{u_o h_p}{h}\left[\frac{1 - \left(\frac{h}{h_m}\right)^2}{1 - \left(\frac{h_p}{h_m}\right)^2}\right]^n e^{-k_2(r/h)^2} r\, dr + f(h \cdot c)$$

$$= \frac{u_o h\, h_p}{2k}\left[\frac{1 - \left(\frac{h}{h_m}\right)^2}{1 - \left(\frac{h_p}{h_m}\right)^2}\right]^n \left[1 - e^{-k_2(r/h)^2}\right] + f(h \cdot c)$$

when $r = 0$, $\Psi = 0$, and $f(h \cdot c) = 0$.

After rearrangement, we get

$$C_3\,\psi = \frac{h}{h_m}\left[1 - \left(\frac{h}{h_m}\right)^2\right]^n \left[1 - e^{-k_2 (r/h)^2}\right] \tag{32}$$

in which $$C_3 = \frac{2\,k_2}{u_o\,h_p\,h_m}\left[1 - \left(\frac{h_p}{h_m}\right)^2\right]^{\frac{1}{2}}$$

The stream lines on Plate V were constructed according to equation 32 for the case of $n = \frac{1}{2}$ and $k_2 = 70$.

The shapes of the stream lines are quite different from those of the recirculating stream lines.

The individual stream line has upper asymptote and lower asymptote.

The upper and lower asymptote can be found for a particular stream line. For $n = \frac{1}{2}$, equation 32 becomes

$$C_3\,\psi = \frac{h}{h_m}\left[1 - \left(\frac{h}{h_m}\right)^2\right]^{\frac{1}{2}} \left[1 - e^{-k_2 (r/h)^2}\right]$$

For any given particular stream line, say ψ_1,

$$C_3\,\psi_1 = \frac{h}{h_m}\left[1 - \left(\frac{h}{h_m}\right)^2\right]^{\frac{1}{2}} \left[1 - e^{-k_2 (r/h)^2}\right] \tag{33}$$

For r equal to infinity, it follows:

$$C_3\,\psi_1 = \frac{h}{h_m}\left[1 - \left(\frac{h}{h_m}\right)^2\right]^{\frac{1}{2}}$$

If squared on both sides,

$$\left(\frac{h}{h_m}\right)^2 \left[1 - \left(\frac{h}{h_m}\right)^2\right] = (C_3\,\psi_1)^2 \tag{34}$$

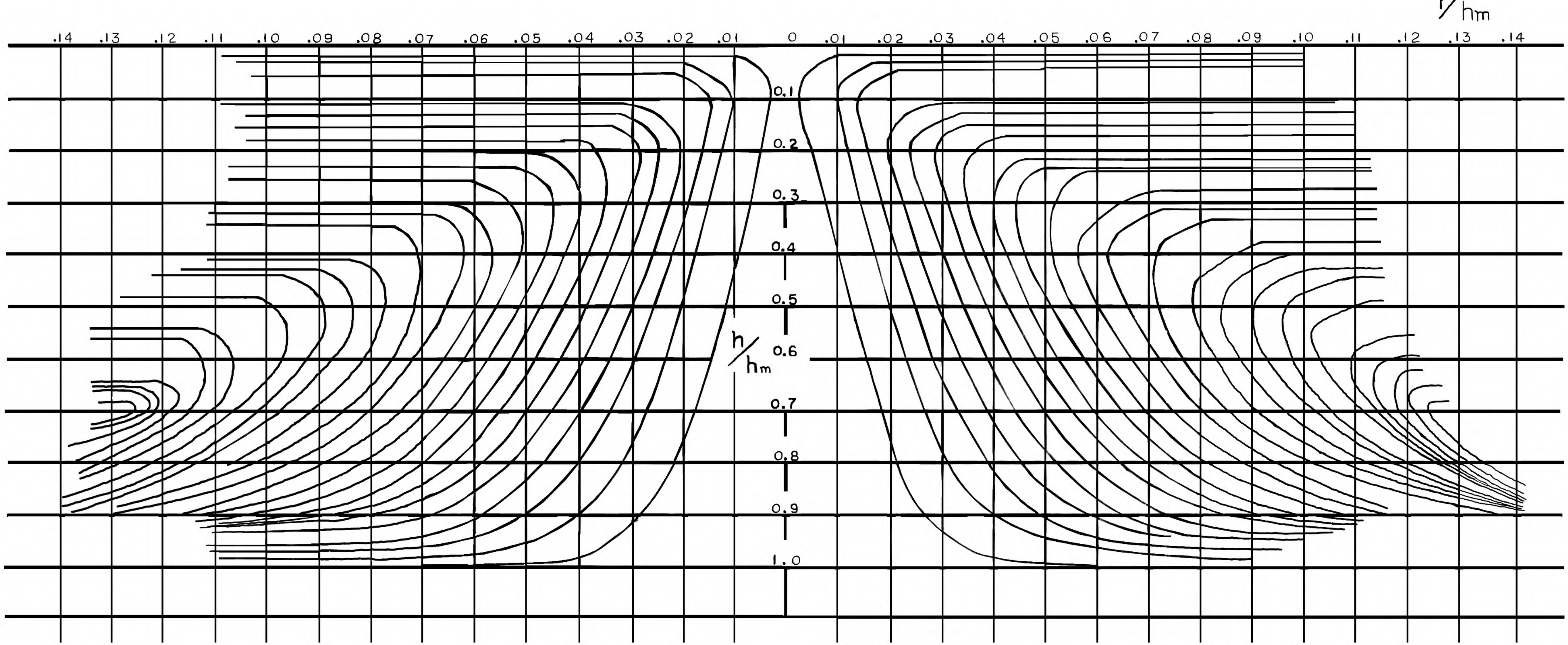
This is a quadratic of $\left(\frac{h}{h_m}\right)^2$.

EXPLANATION OF PLATE V

Asymptotic incoming and outgoing stream lines.

$$C_3 \psi = \frac{h}{h_m}\left[1 - \left(\frac{h}{h_m}\right)^2\right]\left[1 - e^{-k_2(r/h)^2}\right]$$

PLATE V


r/hm
.14
.13
.12
.11
.10
.09
.08
.07
.06
.05
.04
.03
.02
.01
0
.01
.02
.03
.04
.05
.06
.07
.08
.09
.10
.11
.12
.13
.14
h/hm
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0

Obviously, $$\left(\frac{h}{h_m}\right)^2 = \frac{1 \pm \sqrt{1 - 4\,C_3^2\,\psi_1^2}}{2} \qquad (35)$$

Therefore $$\left(\frac{h}{h_m}\right)_{upper} = \sqrt{\frac{1 - \sqrt{1 - 4\,C_3^2\,\psi_1^2}}{2}} \qquad (36)$$

$$\left(\frac{h}{h_m}\right)_{lower} = \sqrt{\frac{1 + \sqrt{1 - 4\,C_3^2\,\psi_1^2}}{2}} \qquad (37)$$

Thus for $\psi_1 = 0$, equation 36 gives $\frac{h}{h_m} = 0$, corresponding to $h = 0$, which represents the upper branch asymptote, and equation 37 gives $\frac{h}{h_m} = 1$, corresponding to $h = h_{max}$, which represents the lower branch asymptote. However, if two asymptotes are correspondents, the equation 34 should have two equal roots which require

$$1 - 4\,C_3^2\,\psi_1^2 = 0$$

Then $$C_3^2\,\psi_1^2 = 1/4, \text{ or } (C_3\,\psi_1) = 1/2$$

Hence $$\left(\frac{h}{h_m}\right)_{uL} = \frac{1}{\sqrt{2}} = 0.707$$

The characteristic difference between the asymptotic type air inflow, outflow, and the recirculating type of air flow is obviously justified by the existence of a boundary of zero velocity at finite distance from the axis of the jet in the latter case, and at infinite distance from the axis of the jet in the former case.

If the surrounding space has no finite boundary, and in extreme cases the jet has no great temperature difference between surrounding space, then the fluid on the axis will travel vertically downward to the level of $h_{max.}$, horizontally to infinity, and then upward.

On the other hand, if the surrounding space is confined to finite walls and the average temperature difference is high, the buoyant force will act on the downcoming jet and push up the air flow which has been decelerated near the bottom of the jet. Due to this upwardly directed flow, the jet, which was originally blowing down, will be decelerated and then turn upward. Thus it will construct the recirculating flow pattern.

The Temperature Distribution of Heated Jets

Several experiments show that the normal probability function is a satisfactory representation of temperature difference ratio for a heated jet. Thus

$$\frac{T - T_a}{T_c - T_a} = e^{-k_3 (r/h')^2} \tag{38}$$

and for the center-line temperature difference ratio is expressed by

$$\frac{T_c - T_a}{T_o - T_a} = \frac{h_p'}{h'} \left[\frac{1 - \left(\frac{h'}{h_m'}\right)^2}{1 - \left(\frac{h_p'}{h_m'}\right)^2} \right]^{(2n-1)} \tag{39}$$

in which n is an exponent which can be found from the experiment.

Finally, if we combine equations 38 and 39, we get

$$\frac{T - T_a}{T_o - T_a} = \frac{h_p'}{h'} \left[\frac{1 - \left(\frac{h'}{h_m'}\right)^2}{1 - \left(\frac{h_p'}{h_m'}\right)^2} \right]^{(2n-1)} e^{-k_3 (r/h')^2} \tag{40}$$

The temperature distribution curve is drawn for the case of $k_3 = 40$, $n = 2/3$, and $1/2$. The constant temperature contour lines are also plotted to show that these lines are spread out radially from the imaginary source. This fact shows the correspondence of the constant temperature contour line and stream lines, which are also spread out radially from the point source. One can easily expect that there would be no heat transmission along the stream line except by convection, and, on the other hand, since no flow would cross the stream line, there would be no heat flow to the tangential direction of the stream line except by conduction due to temperature gradient. However, with this fact it is better to employ spherical coordinates, and thus take advantage of the radial lines as stream lines so that the heat conduction is assumed to occur in a tangential direction, and heat convection is assumed to occur only in a radial direction (along stream line).

The energy balance equation is derived, together with the following assumptions:

1. Ignoring the energy losses which were caused by the viscous effects.

2. Temperature difference ratio $\dfrac{T - T_a}{T_o - T_a}$ is a function of angle θ, which is an angle made by the radial line and the center line of the jet.

3. No velocity exists in tangential direction.

The energy equation is expressed as:

$$\rho C_p \bar{V} \cdot \nabla T = k \nabla^2 T$$

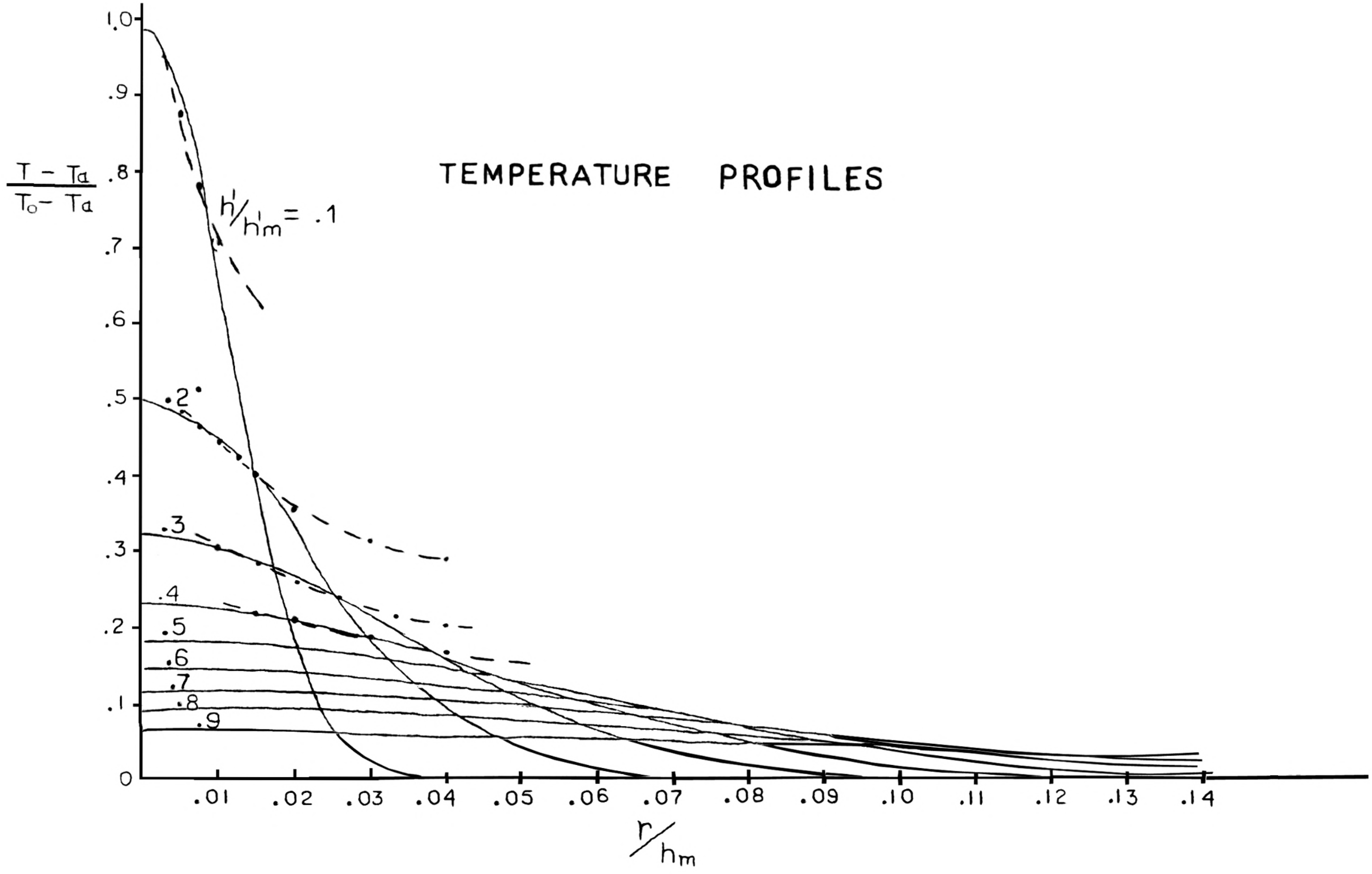
EXPLANATION OF PLATE VI

Temperature profiles.

$$\frac{T - T_a}{T_o - T_a} = \frac{h_p'}{h'} \left[\frac{1 - \left(\frac{h'}{h_m'}\right)^2}{1 - \left(\frac{h_p'}{h_m'}\right)^2} \right]^{2n-1} e^{-k_3 (r/h')^2}$$

in which $n = 2/3$, $k_3 = 40$, $\frac{h_p'}{D_o} = 4.5$, $\frac{h_m'}{D_o} = 21.5$. Dotted lines are temperature distribution, according to equation 46.

PLATE VI



TEMPERATURE PROFILES
$\frac{T - T_a}{T_o - T_a}$
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
$h'/h'_m = .1$
.2
.3
.4
.5
.6
.7
.8
.9
.01 .02 .03 .04 .05 .06 .07 .08 .09 .10 .11 .12 .13 .14
r/h_m

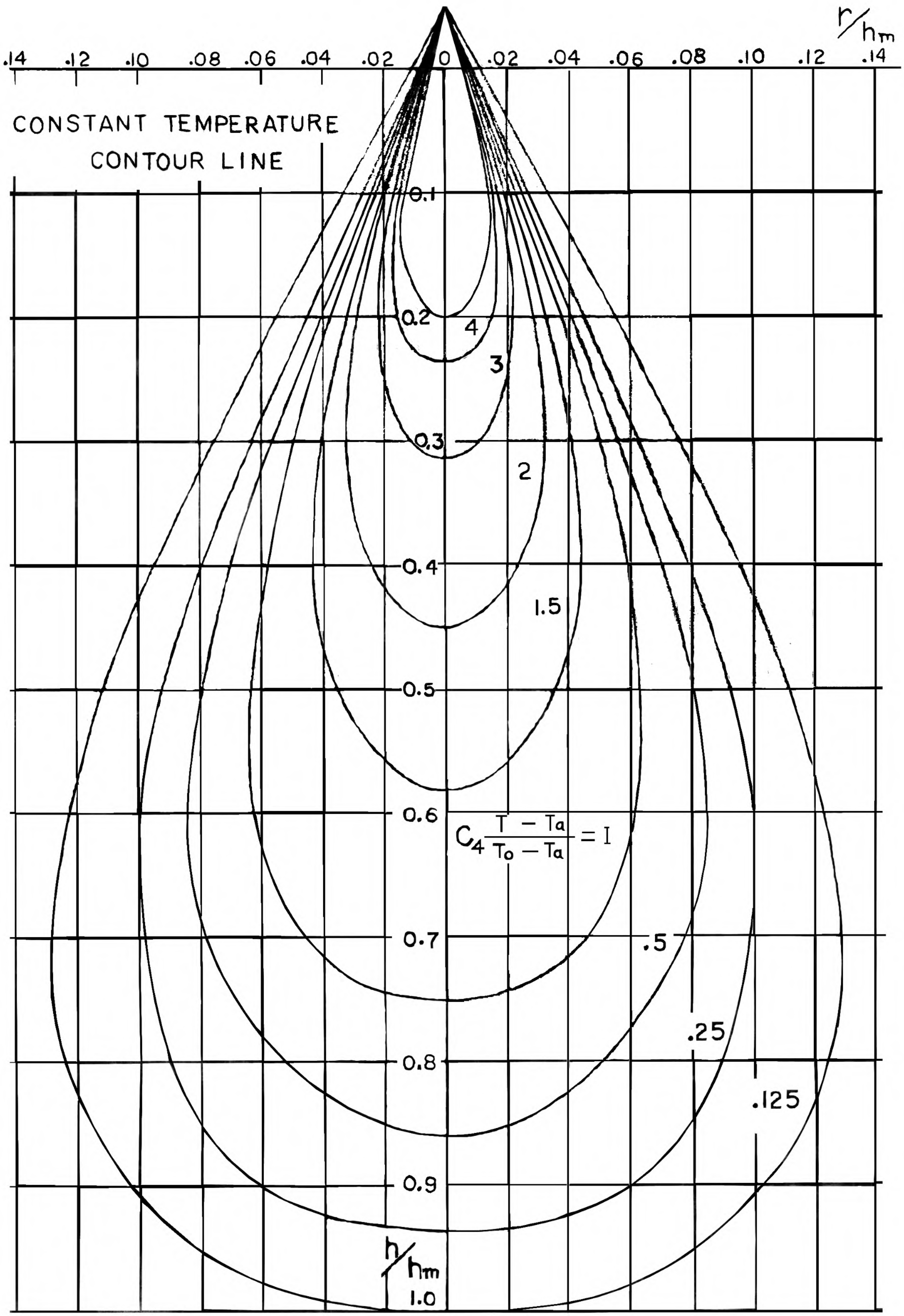
EXPLANATION OF PLATE VII

Constant temperature contour lines according to

$$\frac{T - T_a}{T_o - T_a} = \frac{h_p'}{h'} \left[\frac{1 - \left(\frac{h'}{h_m'}\right)^2}{1 - \left(\frac{h_p'}{h_m'}\right)^2} \right]^{2n-1} e^{-k_3 (r/h')^2}$$

in which $n = 2/3$, $k_3 = 40$, $\frac{h_p'}{D_o} = 4.5$, $\frac{h_m'}{D_o} = 21.5$.

PLATE VII


r/hm
.14 .12 .10 .08 .06 .04 .02 0 .02 .04 .06 .08 .10 .12 .14
CONSTANT TEMPERATURE CONTOUR LINE
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
h/hm
4
3
2
1.5
$C_4 \frac{T - T_a}{T_o - T_a} = 1$
.5
.25
.125

or
$$\bar{V} \cdot \nabla T = \frac{k}{C_p \rho} \nabla^2 T \tag{41}$$

If equation 41 is expanded into spherical coordinates, since for axial symmetric jet no circumferential velocity exists, i.e., $V_\phi = 0$, we get:

$$\frac{k}{\rho C_p} \frac{1}{r^2} \left\{ \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial \mu} (1 - \mu^2) \frac{\partial T}{\partial \mu} \right\}$$
$$= V_r \frac{\partial T}{\partial r} + V_\theta \frac{\sqrt{1 - \mu^2}}{r} \frac{\partial T}{\partial \mu} \tag{42}$$

Here $\mu = \cos \theta$.

If radial lines are assumed to be contour temperature lines,

$$\frac{\partial T}{\partial r} = 0$$

(That is, no temperature gradient exists on the constant temperature contour lines.) Then the energy equation reduces to:

$$\frac{k}{C_p \rho} \frac{1}{r^2} \frac{d}{d\mu} (1 - \mu^2) \frac{dT}{d\mu} = +V_\theta \frac{\sqrt{1 - \mu^2}}{r} \frac{\partial T}{\partial \mu} \tag{43}$$

From the third assumption we can neglect the right-hand side of equation 43, which expresses the heat convection in tangential direction. Finally, equation 43 reduces to a simple form:

$$\frac{d}{d\mu} (1 - \mu^2) \frac{dT}{d\mu} = 0 \tag{44}$$

The solution for this is obviously

$$T - C_1 = C_2 \log \frac{1 + \mu}{1 - \mu} \tag{45}$$

or, in more convenient form,

$$\frac{T - T_c}{T_c - T_a} = C_3 \log \frac{1 + \mu}{1 - \mu} \qquad (46)$$

With this expression the temperature ratio becomes infinity at $\mu = 1$, i.e., when $\theta = 0$. Actually, this corresponds to the imaginary point source, because when μ approaches 1 the radial constant temperature lines will squeeze to a point which corresponds to the imaginary point source, where the temperature is theoretically infinity.

The calculated data are shown and compared with the experimental result. However, the temperature ratio curve which is based on equation 46 and fits the experimental equation at θ, is not very large. For a larger value of θ, equation 46 gives a higher temperature ratio than the experimental result. This could be explained by the fact that the tangential velocity was neglected; namely, $V_\theta = 0$.

Since $V_\theta \approx 0$ at the region of small θ, no heat convection can be expected in a tangential direction, and the assumption of $V_\theta = 0$ leads us to equation 44 which neglected the heat convection in a tangential direction. So the solution of equation 44 agrees with the experimental result for small values of θ. However, for a larger value of θ, V_θ is no longer equal to zero and the heat convection will take place in a tangential direction. This will cause a reduction of temperature since heat will be carried away to the surrounding air space by the convection.

Thermal Energy Balance

For a surface of a volume V fixed in the space, the two basic laws, namely, the law of conservation of mass and the conservation of energy, should be satisfied.

For the conservation of mass,

$$\frac{\partial}{\partial t} \int_V \rho \, dV + \int_s \rho u_i n_i \, ds = 0 \tag{47}$$

For the conservation of energy,

$$\int_V \frac{\partial Q}{\partial t} dV + \int_s u_i(\tau_{ij} n_i) ds - \int_s u \rho u_j n_j ds - \int_s \left(-k \frac{\partial T_i}{\partial x_i} n_i\right) ds$$

$$= \frac{\partial}{\partial t} \int u \rho \, dV \tag{48}$$

in which $\frac{\partial Q}{\partial t}$ is the rate of heat production per unit volume in V by external agencies. u is the total energy of the system per unit mass.

$u = \frac{1}{2} u_i u_i + p + E + W_f = \frac{1}{2} u_i u_i + p + H$

H = the enthalpy of the system per unit mass

E = the internal energy of the system per unit mass

p = the potential energy of the system per unit mass

n_i = the i^{th} component of the outer normal

W_f = the flow work per unit mass

For the heated jet operating at a steady state, these two principles require that the net increase of the mass and the energy be zero within a volume V. Then equations 47 and 48 reduce to:

$$\int \rho u_i \, n_i \, ds = 0 \tag{49}$$

$$\int_v \frac{\partial Q}{\partial t} dv + \int u_i \, (T_{ij} n_i) ds - \int u \rho u_j \, n_j \, ds$$

$$- \int (-k \frac{\partial T}{\partial x_i} n_i) \, ds = 0 \tag{50}$$

If one takes a finite volume which is bounded by the constant temperature surface and ignores the potential energy and kinetic energy, the energy equation reduces to:

$$\int_v \frac{\partial Q}{\partial t} dv + \int u_i \, (T_{ij} n_i) ds - \int \rho H \, u_j n_j \, ds$$

$$- \int - (k \frac{\partial T}{\partial x_i} n_i) ds = 0 \tag{51}$$

This energy equation can be further reduced if the energy dissipation due to the viscous effect is ignored.

$$\int_v \frac{\partial Q}{\partial t} dv - \int \rho H \, u_j \, n_j \, ds - \int (-k \frac{\partial T}{\partial x_i} n_i) \, ds = 0 \tag{52}$$

By assuming the air is a perfect gas, the enthalpy H can be expressed as

$$H = C_p T$$

in which C_p is a specific heat at constant pressure and T is an absolute temperature in degree Rankine. Since the temperature is constant on the constant temperature surface, equation 52 becomes

$$\int_v \frac{\partial Q}{\partial t} dv - C_p T \int \rho u_j \, n_j \, ds - \int -(k \frac{\partial T}{\partial x_i} n_i) \, ds = 0 \tag{53}$$

According to equation 49, the first term in the left-hand side vanishes, and finally

$$\int_v \frac{\partial Q}{\partial t} dv + \int_s \left(k \frac{\partial T}{\partial x_i} n_i\right) ds = 0 \qquad (54)$$

Equation 54 states that the net amount of heat transmitted across the constant temperature surface is equal to the rate of heat generated within the volume. The amount of heat conducted to the particular constant temperature surface was evaluated numerically from constant temperature surface profiles plotted on Plate VIII.[1] The results show that the amount of heat conducted across the constant temperature surfaces is a constant, which in turn shows that the heat generated within the hypothetical jet and evaluated as $\int_v \frac{\partial Q}{\partial t} dv$ is generated at the point source.

CHARACTERISTICS OF THE ISOTHERMAL JETS

The law of conservation of mass gives the equation of continuity:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \overline{V} = 0 \qquad (55)$$

in which ρ is the density of the fluid and $\overline{V}$ its velocity. For a steady isothermal jet we assume ρ is constant. Then equation 55 reduces to

$$\nabla \cdot \overline{V} = 0 \qquad (56)$$

If we expand the equation 56 in spherical coordinates ($r \cdot \theta \cdot \phi$), there follows

---

[1] Temperature gradients at constant temperature surfaces were evaluated graphically from Plate VIII, and the heat flow was calculated from these evaluations.

EXPLANATION OF PLATE VIII

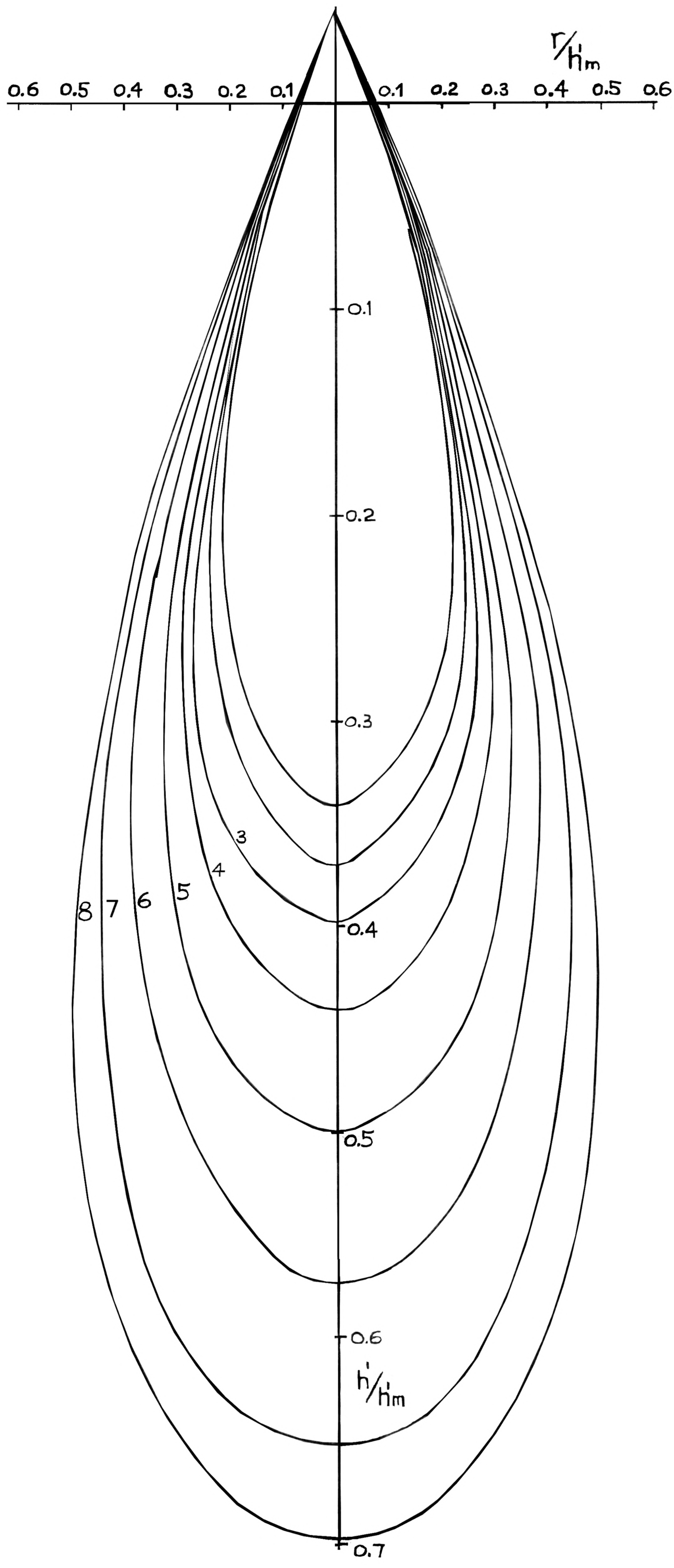
Numerical evaluation of heat conduction across the individual constant temperature surface.

$$Q = \sum k \frac{\Delta T}{\Delta n} \Delta S$$

Q_i = the amount of heat conducted from i^{th} surface to $(i + 1)$ surface.

$C\ Q_1 = 560.5$	$C\ Q_4 = 562.1$
$C\ Q_2 = 549.3$	$C\ Q_5 = 554.6$
$C\ Q_3 = 557.2$	$C\ Q_6 = 553.5$

PLATE VIII


r/h'm
0.6 0.5 0.4 0.3 0.2 0.1 0.1 0.2 0.3 0.4 0.5 0.6
0.1
0.2
0.3
3
4
8 7 6 5
0.4
0.5
0.6
h'/h'm
0.7

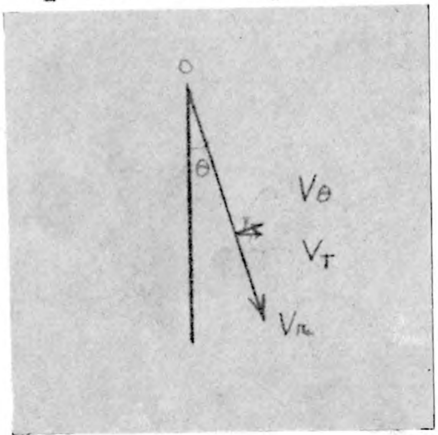
$$\frac{\partial}{\partial r} (r^2 \sin \theta \, V_r) + \frac{\partial}{\partial \theta} (r \sin \theta \, V_\theta) + \frac{\partial}{\partial \phi} (r \, V_\phi) = 0 \qquad (57)$$

For axially symmetric jet $V_\phi = 0$; hence the third term vanishes and equation 57 becomes:

$$\partial r \sin \theta \, V_r + r^2 \sin \theta \frac{\partial V_r}{\partial r} + r \cos \theta \, V_\theta + r \sin \theta \frac{\partial V_\theta}{\partial \theta} = 0 \qquad (58)$$

Since at the vicinity of the jet axis, the velocity component in the direction perpendicular to the jet axis is negligibly small, correspondingly V_r and V_θ have a relation as

$$V_T = V_r \sin \theta + V_\theta \cos \theta = 0 \qquad (59)$$



If we substitute equation 59 into equation 58, we get

$$r^2 \sin \theta \frac{\partial V_r}{\partial r} + r \sin \theta \, V_r + r \sin \theta \frac{\partial V_\theta}{\partial \theta} = 0 \qquad (60)$$

Dividing by $r^2 \sin \theta$,

$$\frac{\partial V_r}{\partial r} + \frac{V_r}{r} + \frac{\partial V_\theta}{r \partial \theta} = 0 \qquad (61)$$

The velocity component in tangential direction is much smaller than the velocity component in radial direction, and the third term in equation 61 which expresses the partial increment in tangential direction is small compared to the first two terms, so by neglecting the third term we have

$$\frac{dV_r}{dr} + \frac{V_r}{r} = 0 \tag{62}$$

The solution of equation 62 is clearly

$$V_r = \frac{C}{r} \tag{63}$$

C can be determined by the initial condition.

For $r = r_p$, $V = V_o$, there follows:

$$\frac{V_r}{V_o} = \frac{r_p}{r} \tag{64}$$

This equation gives a hyperbolic distribution of the velocity at the vicinity of the jet axis which corresponds to the experimental results.

CONCLUSION

1. For heated jets of the recirculating flow, the velocity distribution could be expressed as:

$$\frac{u}{u_c} = \left[1 - \left(\frac{r}{r_f}\right)^2\right] e^{-k_1(r/r_f)^2}$$

The individual stream line follows the recirculating path shown in Plate I.

2. For a heated jet, if its temperature difference is not too high, the velocity distribution might be expressed as:

$$\frac{u}{u_o} = \frac{h_p}{h} \left[\frac{1 - \left(\frac{h}{h_m}\right)^2}{1 - \left(\frac{h_p}{h_m}\right)^2}\right]^n e^{-k_2(r/n)^2}$$

The stream lines come into the jet body asymptotically from the surrounding space and leave the jet body asymptotically to the surrounding space.

3. The jet boundary of zero downward velocity might be defined by the following expressions:

$$\text{(i)} \quad \frac{r_f}{h_m} = (\cos b\theta)^{\frac{1}{2}} \sin \theta$$

$$\text{(ii)} \quad \frac{h}{h_m} = e^{-k_b (r_f/h)^2}$$

4. The total amount of heat crossing a constant temperature surface was constant and this fact proved the consistency of the shapes of the constant temperature contour lines.

5. The hyperbolic distribution of center-line velocity is an accurate flow pattern for isothermal jets.

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MATHEMATICAL ANALYSIS OF THE CHARACTERISTICS
OF AXIALLY SYMMETRIC DOWNWARD JET

by

HUEI-HUANG CHIU

B. S., National Taiwan University,
Taipei, China, 1953

AN ABSTRACT OF
A THESIS

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KANSAS STATE COLLEGE
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1956

The characteristics of mathematical models of non-isothermal and isothermal jets were studied under the guidance of the Department of Mechanical Engineering at Kansas State College.

The objectives of the studies were:

1. To find the flow patterns represented by the models, and, by comparison with the flow pattern of the actual jet, to determine the validity of the equations employed to describe the model.

2. Investigate the thermal characteristics of the models.

3. Investigate empirical equations for the boundary of zero downward velocity.

4. Obtain the pattern of center-line velocity distribution of the unheated jet model for comparison with the corresponding distribution for the actual jet.

5. Obtain guidance for future research.

The conclusions obtained from the study were:

1. For heated jets of the recirculating flow type, the velocity distribution could be expressed as

$$\frac{u}{u_c} = \left[1 - \left(\frac{r}{r_f}\right)^2\right] e^{-k_1 (r/r_f)^2}$$

2. The jet boundary of zero downward velocity might be defined by the following expressions:

$$\text{(i)} \quad \frac{r_f}{h_m} = \sqrt{\cos b\,\theta \sin\theta}$$

$$\text{(ii)} \quad \frac{h}{h_m} = e^{-k_b (r_f/h)^2}$$

3. The constant temperature contour lines, which were

obtained from the experimental equation, converge to the imaginary heat source, and the energy balance on the surface of constant temperature shows that the net amount of heat conducted across constant temperature surfaces is approximately equal to a constant.

4. Both experimental results and the theoretical analysis show that the hyperbolic distribution of center-line velocity in the isothermal jet is an accurate flow pattern.