EQUIPMENT REPLACEMENT BY THE CONTINUOUS MAXIMUM PRINCIPLE

by

DAHYABHAI RANCHHODBHAI CHAUDHARI

B.S. (Mech. Engg.), Sardar Vallabhbhai Vidyapeeth,
Vallabh Vidyanagar, Gujarat, India, 1965

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Ching-Lai Hwang

Major Professor

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INTRODUCTION

The objective of this report is to apply the continuous maximum principle to the equipment replacement problems.

One kind of problem faced by manufacturing firms, which is susceptible to total value analysis, is that of making decision concerning investment in capital equipment. Here, management is spending money with the expectation that it will produce revenue in the future. A sum of money is "sunk" in a machine, for example, and the machine is used up over a period of years in the production of goods from which the firm derives revenue.

The "replacement problem" is primarily concerned with how frequently the machine should be replaced. A firm uses a machine which "wears out" after a certain amount of time and is then replaced by a new machine.

A decision model for replacement of equipment should portray the basic economic problem in such terms that the parameters may be evaluated with generally available business data. At the very least, it must be economically feasible to secure the required data. Also, the model should be capable of modification to fit the requirements of as wide a range of situations as possible.

From the efficiency point of view, two general kinds of equipment may be distinguished: the "constant efficiency" and the "diminishing efficiency" types. Under the first category we may classify those items whose efficiency remains fairly constant throughout their service lives and whose service terminates abruptly with their first failure. An electric light bulb is the best example of this type of equipment. To the second classification belong those durable goods whose service life may be extended almost indefinitely if their component parts are replaced or repaired as necessary. This type of equipment is characterized by a decline in productivity and/or increase in maintenance costs as they are used over time.

The economics of replacement associated with these two types of equipment are quite different. For those equipments displaying constant efficiency, a probability distribution for the length of their lives may be obtained from life tests and various replacement policies may be evaluated on the basis of this distribution. Since there is no cost of declining efficiency associated with the problem, the analysis is very often reduced to a comparison of the expected values of the several alternatives.

In the case of "diminishing efficiency" type of equipment (13), for each year of operation the machine produces a certain revenue, each year there is a maintenance cost, and at the end of any year the equipment may be sold for salvage at a certain price. The problem of determining when to replace a piece of equipment depends on the productivity of equipment, the maintenance cost on the equipment, the trade in or salvage value of the equipment as a function of the equipment age, and also the purchase cost of a new equipment. In general, with the age of the equipment (a) the net revenue decreases, (b) the maintenance cost increases and (c) the salvage value decreases. It is this "diminishing efficiency" type of equipment that will be considered in this report for the applicability of the continuous maximum principle.

A problem faced by a manufacturing company when investing in production equipment is that of either maximizing the net present worth of investment or minimizing the present worth of all expenses on the investment.

Case 1 is an extension of the work done by Daccarett (2) for a single machine problem. It deals with the maximization of the net present worth of an investment in production equipment for a chain of machines problem. A basic model for profit maximization treated by Preinreich (9) and others (1, 6, 7, 11) is used to illustrate how the optimum life of the equipment can be determined by means of the continuous maximum principle.

Case 2 deals with the minimization of the present worth of all expenses as sometimes it is difficult to allocate the portion of the revenue of the product to a particular machine when many different operations are carried out on the same product by different machines. This case has also been considered for a chain of machines, as it is the usual situation in practice, rather than a single machine. A basic model treated by Bowman and Fetter (1) for cost minimization is used to illustrate how the optimum life of the equipment in a chain can be determined by means of the continuous maximum principle.

Case 3 deals with a more realistic model than that of Case 2 by taking into account production rate as the second decision variable. It also takes into account variable costs, fixed overhead costs and maintenance costs separately for the minimization of the net present worth of all expenses per unit of production for a single machine. The theoretical solution has been obtained by means of the continuous maximum principle for this model.

A numerical problem has been solved for each of the first two cases in order to show the validity of theoretical results obtained. For third case, as it involves a number of simultaneous non-linear differential equations, a further study by numerical methods is required.

STATEMENT OF THE ALGORITHM OF THE CONTINUOUS MAXIMUM PRINCIPLE

The representation of the continuous simple process is shown schematically in Fig. 1. The performance equations of the process have the form (3, 4, 8, 10)

$$\frac{dx_i}{dt} = f_i(x_1(t), x_2(t), \ldots, x_5(t); \theta_1(t), \ldots, \theta_r(t)),$$

$$t_0 \leq t \leq T,$$

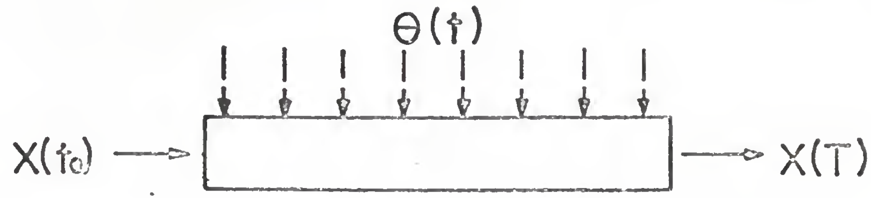

Fig. I Simple process.

$$x_i(t_o) = \alpha_i, \quad i = 1, 2, \ldots, s,$$

or in the vector form

$$\frac{dx}{dt} = f(x(t); \theta(t)), \quad x(t_o) = \alpha, \tag{1}$$

where x(t) is an s-dimensional vector function representing the state of the process at time t and θ(t) is an r-dimensional vector function representing the decision at time t. It may be noted that the variable t may represent the distance in a steady-state continuous process in space.

A typical optimization problem associated with such a process is to find a piecewise continuous decision vector function, θ(t), subject to the constraints

$$\phi_i(\theta_1(t), \theta_2(t), \ldots, \theta_r(t)) \leq 0, \quad i = 1, 2, \ldots, m, \tag{2}$$

which makes a function of the final values of the state

$$S = \sum_{i=1}^{s} c_i x_i(T), \quad c_i = \text{constant}, \tag{3}$$

an extremum when the initial condition $x(t_o) = \alpha$ is given. The function, S, which is to be maximized (or minimized), is termed the objective function of the process. The decision vector function so chosen is called an optimal decision vector function or simply an optimal decision and is denoted by θ(t).

When the time interval is fixed, there are two different types of basic problem: a fixed right-end problem and a free right-end problem, depending on whether the final condition is given or not. In this section we shall consider only the free right-end problem.

The procedure for solving the problem is to introduce an s-dimensional adjoint vector $z(t)$ and a Hamiltonian function H which satisfy the following relations:

$$H(z(t),x(t),\theta(t)) = (z)^T f(x;\theta) = \sum_{i=1}^{s} z_i f_i(x(t);\theta(t)), \quad (4)$$

$$\frac{dz_i}{dt} = -\frac{\partial H}{\partial x_i} = -\sum_{j=1}^{s} z_j \frac{\partial f_j}{\partial x_i}, \quad i = 1, 2, \ldots, s, \quad (5)$$

$$z_i(T) = c_i, \quad i = 1, 2, \ldots, s. \quad (6)$$

The optimal decision vector function $\theta(t)$, which makes S a maximum (or minimum), is the decision vector function, $\theta(t)$, which renders the Hamiltonian function, H, maximum (or minimum) for almost every t, $t_o \leq t \leq T$. If the optimal decision vector function $\theta(t)$ is interior to the set of admissible decisions $\theta(t)$ (the set given by equation (2)), a necessary condition for S to be an extremum with respect to $\theta(t)$ is

$$\frac{\partial H}{\partial \theta} = 0. \quad (7)$$

If $\theta(t)$ is constrained, the optimal decision vector function $\theta(t)$ is determined either by solving equation (7) for $\theta(t)$ or by searching the boundary of the set given by equation (2).

Once the decision vector function $\theta(t)$ is chosen the adjoint vector function $z(t)$ is uniquely determined by equations (5) and (6) and the initial condition at $t = t_o$, $x(t_o) = \alpha$. It may be noted that the performance equation (1) can be written in terms of the Hamiltonian function as

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial z_i}, \quad i = 1, 2, \ldots, s. \tag{8}$$

Pontryagin's maximum principle can be summarized in the following theorem.

<u>THEOREM</u>. Let $\theta(t)$, $t_o \leq t \leq T$ be a piecewise continuous vector function satisfying the constraints given in equation (2). In order that the scalar function S given in equation (3) may be a maximum (or minimum) for a process described by equation (1), with the initial condition at $t = t_o$, $x(t_o) = \alpha$, given, it is necessary that there exists a non-zero continuous vector function $z(t)$ satisfying equations (5) and (6) and that the vector function $\theta(t)$ be so chosen that $H(z(t), x(t), \theta(t))$ is a maximum (or minimum) for every t, $t_o \leq t \leq T$. Furthermore, the maximum (or minimum) value of H is a constant for every t. When T is not fixed, the value of this constant is fixed at zero for every t.

NON-AUTONOMOUS SYSTEMS (4)

A system is called non-autonomous if the right hand side of the performance equation depends explicibly on time t. The performance equation is the form of

$$\frac{dx}{dt} = f(x;t;\theta). \tag{9}$$

This can be transformed to the standard form of equation (1) by introducing a new state variable x_{s+1} to satisfy

$$x_{s+1}(t) = t, \quad t_o \leq t \leq T. \tag{10}$$

Hence the corresponding component of the adjoint vector can also be written in the form

$$\frac{dz_{s+1}}{dt} = -\frac{\partial H}{\partial t} .$$

Then equation (9) becomes

$$\frac{dx}{dt} = f(x;\theta), \tag{11}$$

where x represents an (s+1)-dimensional vector, $(x_1, x_2, \ldots, x_s, x_{s+1})$. The new state variable satisfies the differential equation

$$\frac{dx_{s+1}}{dt} = 1, \quad t_o \leq t \leq T, \tag{12}$$

and the initial condition

$$x_{s+1}(t_o) = t_o . \tag{13}$$

Equation (11), which includes equation (10), is the performance equation of an enlarged system in the form of equation (1) with initial conditions given by $x(t_o) = \alpha$ and equation (13).

Problems involving non-autonomous systems can also be solved without introducing an additional state variable. The basic theorem, with the exception of the condition that the maximum (or minimum) value of H is a constant for every t, is valid.

CASE 1. A CLASSICAL MODEL FOR PROFIT MAXIMIZATION--CHAIN OF MACHINES REPLACEMENT PROBLEM

For a single machine of the diminishing efficiency type, the net present value of the investment to the firm is given by (9)

$$V_1 = \int_0^T \left[R(t) - U(t)\right] e^{-it}dt + D(T)e^{-iT} - B \tag{14}$$

where

V_1 = net present worth of the investment,

B = installation cost of the equipment,

T = economic life of the equipment,

$D(T)$ = Salvage value of the equipment at time T,

i = annual rate of interest,

$R(t)$ = revenue function at time t,

$U(t)$ = maintenance and operating expenses function at time t.

Note that the expense function, U(t), excludes depreciation costs and interest on investment in order to avoid double counting of these items in equation (14).

The model of equation (14) assumes that the firm uses a machine for some kind of production until the end of its life, T, and then sells it for D(T), and never again engages in production of this kind. The more usual situation is that the firm intends to continue the given kind of production over an indefinite future period and will consider the acquisition of a chain of equipment to do this. When the chain is infinite, the capitalized value of all future income will be constant and the lifetimes of the machines in the chain will be the same. In this case, the formula for the net present value given by equation (14) becomes (9)

$$V_{\infty} = \left\{ \int_0^T \left[R(t) - U(t) \right] e^{-it} dt + D(T) e^{-iT} - B \right\} (1 + e^{-iT} + e^{-2iT} + \ldots)$$

$$= \left\{ \int_0^T \left[R(t) - U(t) \right] e^{-it} dt + D(T) e^{-iT} - B \right\} \frac{1}{1 - e^{-iT}} \; . \qquad (15)$$

Equations (14) and (15) are very often of the discrete character in which a summation of the discrete revenue and expenditure discounted to the present replaces the integrals of equations (14) and (15).

We shall consider only the continuous case for a chain of machines. The objective function for the case under consideration can be written

$$S = V_{\infty} \; . \qquad (16)$$

The problem, therefore, becomes that of determining the optimum life of each equipment, $\bar{T}$, so that the net present value as given by equation (15) attains its maximum.

OPTIMIZATION BASED ON THE SIMPLE MODEL

Before we proceed to solve the optimization problem stated above, let us briefly discuss the applicability of the maximum principle to the problem.

Figure 2 is a graphical representation of the optimal trajectory concept used in such variational technique as the maximum principle and the classical calculus of variations. The problem usually treated by these techniques is that of selecting a decision function, $\bar{\theta}(t)$, to obtain an optimum trajectory, $\bar{x}(t)$, which renders the objective function, S(t), an extremum in the closed interval, $t_o \leq t \leq T$. Very often the boundaries of the interval are also to be chosen. These techniques are also applicable when the initial and/or final conditions are specified.

For the optimization under consideration, the determination of the optimum upper bound, $\bar{T}$, alone will extremize the objective function. That is, the problem belongs to the "zero control" category in which no decision function is involved and, consequently, there are no trajectories involved. This problem, therefore, does not belong to a class of problem in which the application of variational techniques is advantageous. This type of problem is amenable to solution by the classical calculus.

Taking the derivative of equation (15) with respect to T and applying the condition

$$\frac{dV_\infty}{dT} = 0 \qquad (17)$$

given by the classical calculus, we obtain

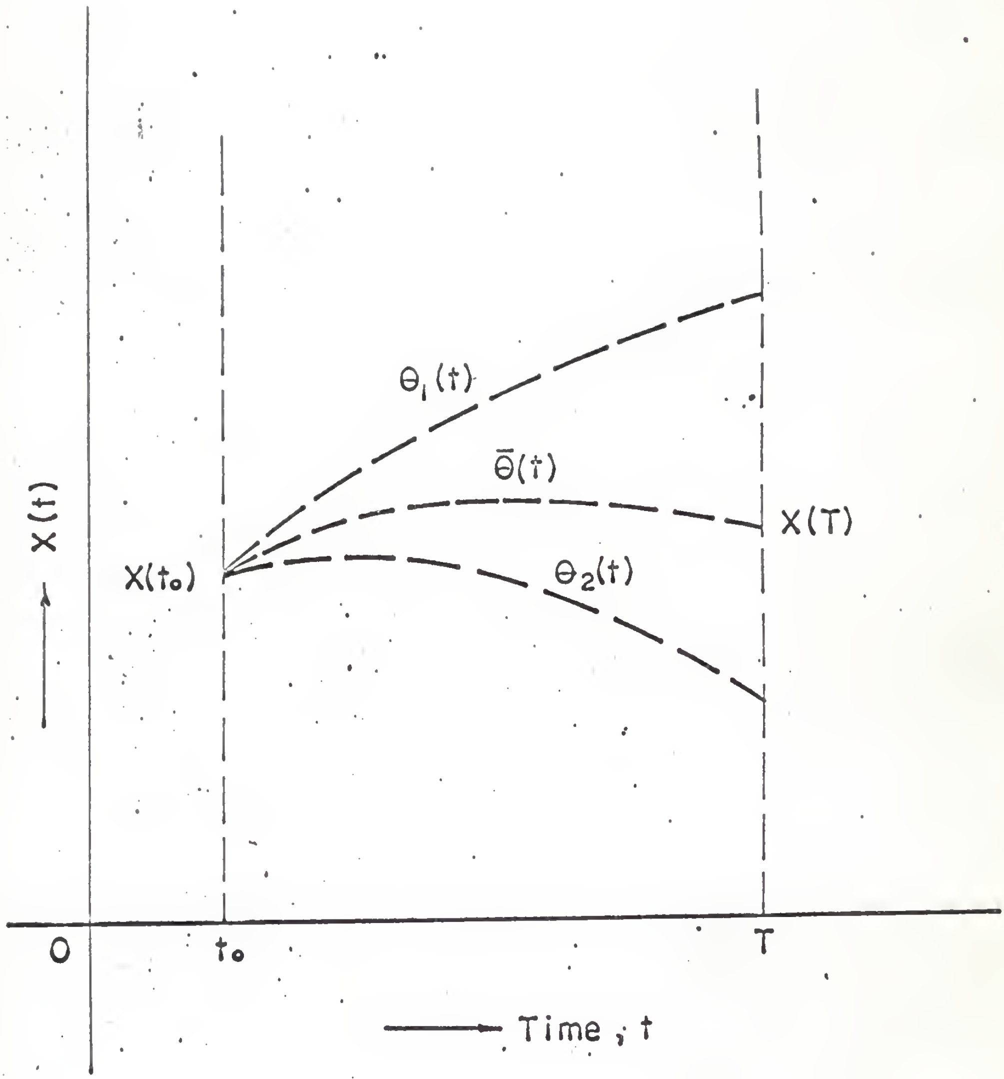

Fig. 2 Optimum Trajectory with the decision vector, $\theta(t)$ as the parameter.

$$\frac{dV_\infty}{dT} = \left\{ \left[R(T)-U(T)\right] e^{-iT} + D'(T)e^{-iT} - iD(T)e^{-iT} \right\} \frac{1}{1-e^{-iT}}$$

$$- \frac{ie^{-iT}}{(1-e^{-iT})^2} \left\{ \int_0^T \left[R(t)-U(t)\right] e^{-it}dt + D(T)e^{-iT} - B \right\}$$

$$= 0,$$

or

$$\left[R(T)-U(T)\right] = iD(T)-D'(T) +$$

$$\frac{i}{1-e^{-iT}} \left\{ \int_0^T \left[R(t)-U(t)\right] e^{-it}dt + D(T)e^{-iT} - B \right\}. \quad (18)$$

Equation (18) indicates that each machine in the chain will be kept until that time when the earnings of the machine (left-hand side) just cover interest on its salvage value plus decline in salvage value plus interest on the present worth of all future earnings of machines in the chain. If the functions for revenue, expenditure, depreciation and interest rate are known, the optimum service life, $\bar{T}$, can be obtained from equation (18) by means of a numerical analysis.

SOLUTION BY THE MAXIMUM PRINCIPLE OF THE SIMPLE MODEL

In order to apply the maximum principle, let us define

$$x_1(t) = \frac{\int_0^t \left[R(t)-U(t)\right] e^{-it}dt}{1-e^{-it}}, \quad x_1(0) = 0, \tag{19}$$

$$\frac{dx_1(t)}{dt} = \frac{\left[R(t)-U(t)\right] e^{-it}}{1-e^{-it}} - \left\{\int_0^t \left[R(t)-U(t)\right] e^{-it}\right\} \frac{ie^{-it}}{(1-e^{-it})^2} \tag{20}$$

$$x_2(t) = \frac{D(t)e^{-it}-B}{1-e^{-it}}, \quad x_2(0) = 0, \tag{21}$$

$$\frac{dx_2}{dt} = \frac{D'(t)e^{-it}-D(t)e^{-it}}{1-e^{-it}} - \frac{\left[D(t)e^{-it}-B\right]}{(1-e^{-it})^2} ie^{-it} \tag{22}$$

where $D'(t) = \frac{dD}{dt}$.

Since the system defined by equations (20) and (22) is non-autonomous (the right-hand side of equation (20) and (22) depend explicitly on time), we shall introduce a new state variable, x_3, defined by

$$\frac{dx_3}{dt} = 1, \quad x_3(0) = t_o = 0, \tag{23}$$

It is obvious that $x_3(t) = t$.

The objective function as given by equation (16) can now be written

$$S = \sum_{i=1}^{3} c_i x_i(T)$$

$$= x_1(T) + x_2(T) \tag{24}$$

therefore, $c_1 = c_2 = 1, \quad c_3 = 0.$

The Hamiltonian and the adjoint variables are

$$H = \sum_{i=1}^{3} z_i \frac{dx_i}{dt}$$

$$= z_1 \frac{dx_1}{dt} + z_2 \frac{dx_2}{dt} + z_3 \frac{dx_3}{dt}$$

$$= z_1 \left\{ \frac{\left[R(t)-U(t)\right] e^{-ix_3}}{1-e^{-ix_3}} - \frac{\left\{\int_0^t \left[R(t)-U(t)\right] e^{-ix_3}\right\} ie^{-ix_3}}{(1-e^{-ix_3})^2} \right\}$$

$$+ z_2 \left\{ \frac{D^1(t)e^{-ix_3}-iD(t)e^{-ix_3}}{1-e^{-ix_3}} - \frac{D(t)e^{-ix_3}-B}{(1-e^{i-x_3})^2} ie^{-ix_3} \right\}$$

$$+ z_3\ (1)\ , \tag{25}$$

$$\frac{dz_1}{dt} = -\frac{\partial H}{\partial x_1} = 0\ , \tag{26}$$

$$z_1(T) = c_1 = 1 \;, \tag{27}$$

$$\frac{dz_2}{dt} = -\frac{\partial H}{\partial x_2} = 0 \;, \tag{28}$$

$$z_2(T) = c_2 = 1 \;, \tag{29}$$

$$\frac{dz_3}{dt} = -\frac{\partial H}{\partial x_3} = -z_1 \left\{ \frac{-i\,\left[R(t)-U(t)\right]\, e^{-ix_3}}{1-e^{-ix_3}} \right.$$

$$\left. - \frac{\left\{\left[R(t)-U(t)\right]\, e^{-ix_3}\right\}\, ie^{-ix_3}}{(1-e^{-ix_3})^2} \right\}$$

$$+z_1 \left[\frac{(1-e^{-ix_3})^2 \left\{ \int_0^t \left[R(t)-U(t)\right] dt \right\}}{(1-e^{-ix_3})^4} \right.$$

$$\times \;\frac{\left[-ie^{-ix_3} ie^{-ix_3} + e^{-ix_3}\right](-i^2 e^{-ix_3})}{}$$

$$\left. - \frac{\left\{ \int_0^t \left[R(t)-U(t)\right] e^{-ix_3} \right\} ie^{-ix_3} \left\{2ie^{-ix_3}\right\}}{(1-e^{-ix_3})^3} \right]$$

$$-z_2 \left[\frac{(1-e^{-ix_3})\left\{D'(t)-iD(t)\right\}(-ie^{-ix_3})-e^{-ix_3}}{(1-e^{-ix_3})^2} \right.$$

$$\times \;\frac{\left\{D'(t)-iD(t)\right\}(ie^{-ix_3})}{}$$

$$-\alpha\left\{\frac{(1-e^{-ix_3})^2\left[\alpha\left\{-iD(t)e^{-ix_3}-0\right\}ie^{-ix_3}+\alpha\left\{D(t)e^{-ix_3}-B\right\}\right.}{}\right.$$

$$\frac{\left.\left\{-i^2e^{-ix_3}\right\}\right]}{(1-e^{-ix_3})^4}$$

$$\left.\left.-\frac{\left[\alpha\left\{D(t)e^{-ix_3}-B\right\}ie^{-ix_3}\right]\ 2ie^{-ix_3}}{(1-e^{-ix_3})^3}\right\}\right] \tag{30}$$

$$z_3(T) = c_3 = 0. \tag{31}$$

Solving equations (26) through (29), we obtain

$$z_1(t) = 1, \qquad 0 \leq t \leq T. \tag{32}$$

$$z_2(t) = 1, \qquad 0 \leq t \leq T. \tag{33}$$

Equations (30) and (31) can now be solved for $z_3(t)$ to yield

$$z_3(t) = \int_t^T \alpha\left\{z_1\alpha\left\{\frac{ie^{-it}\left[R(t)-U(t)\right]}{1-e^{-it}} + \frac{ie^{-2it}\left[R(t)-U(t)\right]}{(1-e^{-it})^2}\right.\right.$$

$$-z_1\alpha\left\{\left[\frac{\left\{\int_0^t\left[R(t)-U(t)\right]e^{-it}dt\right\}\ 2i^2e^{-it}}{(1-e^{-it})^2}\right]\right.$$

$$+\left[\frac{\left\{\int_0^t \left[R(t)-U(t)\right] e^{-it}dt\right\} 2i^2e^{-2it}}{(1-e^{-it})^3}\right]\Bigg\}$$

$$+z_2\left[\frac{(i-e^{-it})\left\{D'(t)-iD(t)\right\}ie^{-it}+e^{-it}\left\{D'(t)-iD(t)\right\}ie^{-it}}{(1-e^{-it})^2}\right.$$

$$-\frac{(1-e^{-it})^2\left[\left\{iD(t)e^{-it}\right\}ie^{-it}+\left\{D(t)e^{-it}-B\right\}i^2e^{-it}\right]}{(1-e^{-it})^4}$$

$$+\frac{\left[\left\{D(t)e^{-it}-B\right\}ie^{-it}\right]2ie^{-it}}{(1-e^{-it})^3} \qquad (34)$$

Substituting equations (32), (33), and (34) back into equation (25), the Hamiltonian function becomes,

$$H=\frac{\left[R(t)-U(t)\right]e^{-it}}{1-e^{-it}}-\frac{\left\{\int_0^t\left[R(t)-U(t)\right]e^{-it}\right\}ie^{-it}}{(1-e^{-it})^2}$$

$$+\frac{D'(t)e^{-it}-iD(t)e^{-it}}{1-e^{-it}}-\frac{\left[D(t)e^{-it}-B\right]ie^{-it}}{(1-e^{-it})^2}$$

$$+\int_t^T\left[\frac{ie^{-it}\left[R(t)-U(t)\right]}{1-e^{-it}}+\frac{ie^{-2it}\left[R(t)-U(t)\right]}{(1-e^{-it})^2}\right.$$

$$-\left[\frac{\left\{\int_0^t \left[R(t)-U(t)\right] e^{-it}dt\right\} 2i^2e^{-it}}{(1-e^{-it})^2}\right.$$

$$\left.+\frac{\left\{\int_0^t \left[R(t)-U(t)\right] e^{-it}dt\right\} 2i^2e^{-2it}}{(1-e^{-it})^3}\right]$$

$$+\left\{\frac{(i-e^{-it})\left[D'(t)-iD(t)\right]ie^{-it}+e^{-it}\left[D'(t)-iD(t)\right]ie^{-it}}{(1-e^{-it})^2}\right.$$

$$-\frac{(1-e^{-it})^2\left\{\left[iD(t)e^{-it}\right]ie^{-it}+\left[D(t)e^{-it}-B\right]\ i^2e^{-it}\right\}}{(1-e^{-it})^4}$$

$$\left.\left.+\frac{\left\{\left[D(t)e^{-it}-B\right]\ ie^{-it}\right\}2ie^{-it}}{(1-e^{-it})^3}\right]\right\}\right] \quad . \tag{35}$$

According to the maximum principle, the optimal decision function, $\theta(t)$, which makes S maximum, makes H maximum for every t, $t_o \leq t \leq T$. Furthermore, the maximum value of H is constant for every t. When T is not fixed, the value of this constant is fixed at zero for every t.

$$\max H = 0 \ , \quad t_o \leq t \leq T.$$

Using this optimality condition and substituting t = T into equation (35), we obtain

$$\frac{\left[R(T)-U(T)\right]e^{-iT}}{1-e^{-iT}} - \frac{\left\{\int_0^T \left[R(t)-U(t)\right] e^{-it}dt\right\} ie^{-iT}}{(1-e^{-iT})^2}$$

$$+ \frac{D'(T)e^{-iT}-iD(T)e^{-iT}}{1-e^{-iT}} - \frac{\left[D(T)e^{-iT}-B\right] ie^{-iT}}{(1-e^{-iT})^2}$$

$$= 0 \ ,$$

or

$$\frac{\left[R(T)-U(T)\right] e^{-iT}}{1-e^{-iT}} = - \frac{D'(T)e^{-iT}-iD(T)e^{-iT}}{1-e^{-iT}}$$

$$+ \frac{\left\{\int_0^T \left[R(t)-U(t)\right]e^{-it}dt\right\} ie^{-iT} + \left[D(T)e^{-iT}-B\right]ie^{-iT}}{(1-e^{-iT})^2} \ ,$$

i.e.

$$R(T)-U(T) = iD(T)-D'(T)$$

$$+ \frac{i}{1-e^{-iT}} \left[\left\{\int_0^T \left[R(t)-U(t)\right] e^{-it}dt\right\} + D(T)e^{-iT}-B\right] \quad (36)$$

Equation (36) is the same solution given by the classical differential calculus. It can be seen that the calculus solution requires only one differentiation while the maximum principle requires considerably more manipulation than that required in the use of calculus.

A NUMERICAL EXAMPLE (1)

Assume the following information for the given chain of machines:

B = Installment cost of each equipment
= $5,000.

$R(t)$ = Revenue function at time t
= 3000 (1-.01t), where $3000 is the beginning annual rate of revenue.

$U(t)$ = 1000 (1+0.14t), where $1000 is the beginning annual rate of expenses.

i = annual rate of interest
= 0.10

An estimated schedule for end-of-year salvage values is as follows:

T	1	2	3	4	5	6	7	8	9	10
D(T)	3500	2800	2400	2000	1600	1300	1000	800	600	500

T	11
D(T)	500

These data may be approximated by

$$D(T) = 5000\ e^{-T/4}\ .$$

SOLUTION

In order to solve equation (36) for T, it can be written in the simplified form as

$$R(T)-U(T)+D'(T)-iD(T)$$

$$= \frac{1}{1-e^{-iT}}\Big\{-e^{-iT}\Big[R(T)-U(T)\Big] - \frac{e^{-iT}}{i}\Big[R'(T)-U'(T)\Big]$$

$$+ \Big[R(0)-U(0)\Big] + \frac{1}{i}\Big[R'(0)-U'(0)\Big] + iD(T)e^{-iT} - iB\Big\}, \quad (37)$$

where

$$R'(0) = \frac{dR}{dt}\Big|\ t = 0, \qquad \text{and } U'(0) = \frac{dU}{dt}\Big|\ t = 0\ .$$

Substituting the numerical values in equation (37), we get the simplified form as

$$\Big[2000 - 170\ T\Big] - 1750\ e^{-T/4}$$

$$= \frac{1}{e^{0.1T}-1}\Big[-300 + 170T - 200e^{0.1T} + \frac{500}{e^{0.25T}}\Big]\ . \quad (38)$$

Equation (38) has only one unknown, T, which can be solved as stated in Table 1.

Table 1. NUMERICAL VALUES OF EQUATION (38)

T	L.H.S. of equation (38)	R.H.S. of equ. (38)
1	468	362
2	600	447
3	663	504
4	676	540
5	649	559
6	590	558
7	505	565

Remark:

L.H.S. = Left Hand Side

R.H.S. = Right Hand Side

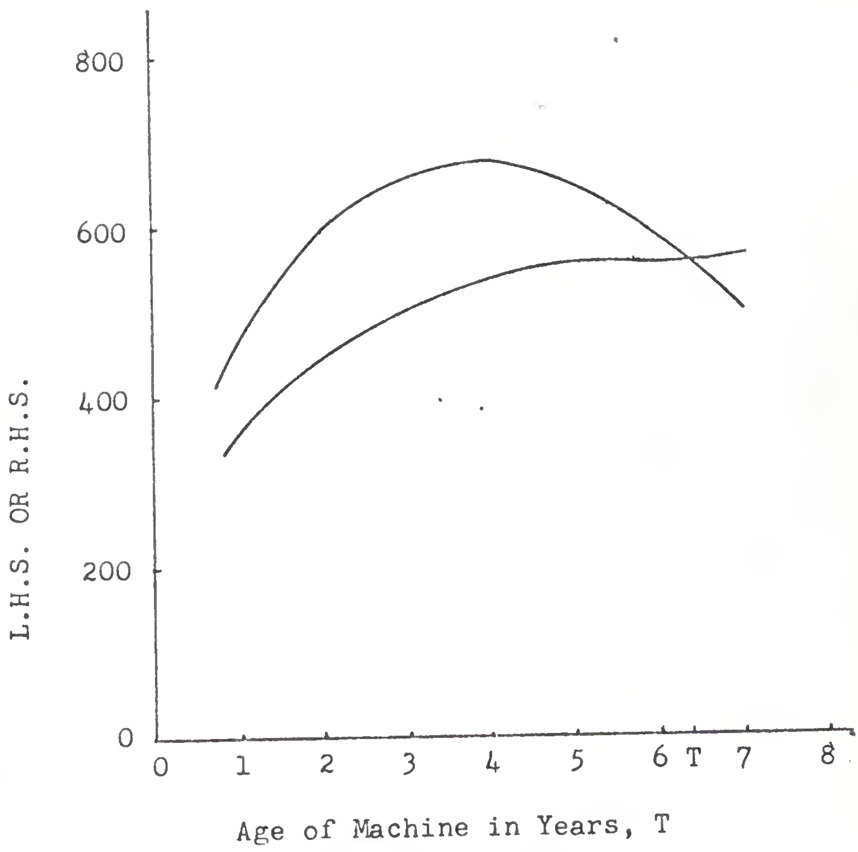

Fig. 3 SOLUTION OF EQUATION (38).

Plotting the graph of each side of equation (38), vs. Time, we can find out the value of T which satisfies the equation (38). This will be the optimum service life of each machine in a chain of machines.

From the graph we find that T = 6.4 years satisfies the equation (38).

Hence optimum service life of each machine in the chain is 6.4 years.

This service life will give the maximum value of net present worth.

Hence from equation (15), we get

$$V_{\infty} = \left\{ \int_0^T \left[R(t) - U(t) \right] e^{-it} dt + D(T) e^{-iT} - B \right\} \frac{1}{1 - e^{-iT}} \quad ,$$

or

$$V_{\infty} = \left\{ \int_0^{6.4} \left[2000 - 170t \right] e^{-it} dt + 500 e^{-iT} - 5000 \right\} \frac{1}{1 - e^{-iT}}$$

$$= \left\{ \left[\frac{e^{-iT}}{i^2} \left\{ -i \left[2000 - 170t \right] + \left[-170 \right] \right\} \right]_0^{6.4} + 500 e^{-iT} - 5000 \right\}$$

$$\times \frac{1}{1 - e^{-iT}}$$

$$= \$5{,}700.$$

CASE 2 A COST MINIMIZATION MODEL (1)

The decision models which make use of a net present worth formula, such as those presented in the previous section, require that the following be known:

(1) A revenue function and

(2) A corresponding expense function (or a profit function in place of (1) and (2)).

(3) The salvage value for all conditions of age

(4) The proper rate of interest.

The revenue function may be easily specified for a firm which uses only a single piece of equipment in its production process. Here, all sales revenue may be attributed to the single machine. However, if the firm's production process consists of a series of operations, each requiring a different piece of equipment, then, in order to use capitalization models, the sales revenue must be distributed back to each piece of equipment. The problem of distributing such joint revenue in order to determine that portion of revenue due to each piece of equipment is very complex and has perplexed economists and accountants for a long time (6). Before general use can be made of this kind of model, ways will have to be found to determine revenue functions for machines.

The expense function is, in most cases, much easier to determine than the revenue function. The operating and maintenance cost of a piece of equipment in current use are often available from accounting records. There are, of course, many difficulities connected with the allocation of various fixed and semivariable expenses to the production of a single machine. However, the problem of estimating an appropriate expense function is not nearly so difficult as that of estimating revenue. Information is usually secured from the equipment manufacturer and/or firms which have had experience using that kind of equipment.

Determining expected salvage values for various periods of use of a piece of equipment is a problem in prediction which is certainly hazardous. However, some knowledge of the market for used equipment of the type in question can often provide clues to these figures. Furthermore, in many cases the timing of the decision is not very sensitive to errors in the magnitude of estimation of D(T).

The selection of an appropriate rate of interest for use in any investment decision model presents problems which neither businessmen nor economists have bet completely resolved. Various concepts have been presented but no satisfactory theory exists. In general, market rate of interest rate is used.

In view of the difficulties just described, it is necessary to make some modifications in the basic model in order to cover the majority of situations actually encountered. Since the objective is one of comparing alternatives, if the revenue which the various alternatives are to produce over time is the same, then it may be assumed a constant in the comparison, and the problem is turned into one of cost minimization. If there are significant differences in revenue function for the alternatives, and these differences can be estimated, they may be treated as additions to, or substractions from, expenses and included in the expense function. Under these special conditions, the cost is given by (1)

$$C_1 = \left[B - D(T)e^{-iT}\right] + \int_0^T U(t)e^{-it}dt \qquad (39)$$

for a firm whose future is limited to the acquisition and use of a single machine. For a chain of machines, the above cost formula can be written in the form (1)

$$C_\infty = \left[(B-D(T)e^{-iT}) + \int_0^T U(t)e^{-it}dt\right] \frac{1}{1-e^{-iT}} \qquad (40)$$

where

B = Installation cost of the equipment,

T = Optimum service life of the equipment,

D(T) = Salvage value at time, T,

i = Annual rate of interest.

The objective function for the case under consideration can be written

$$S = C_\infty . \qquad (41)$$

The problem, therefore, becomes that of determining the optimum life of the equipment, T, so that the net present value of the total cost as given by equation (40) attains its minimum.

OPTIMIZATION BASED ON THE CLASSICAL CALCULUS METHOD

For the optimization under consideration, the determination of the optimum upper bound, $\bar{T}$, alone will extremize the objective function. That is, the problem belongs to the "zero control" category in which no decision function is involved and, consequently, there are no trajectories involved. This type of problem is amenable to solution by the classical calculus.

Taking the derivative of equation (40) with respect to T, and equating it to zero, we get

$$\frac{dC_\infty}{dt} = \frac{1}{1-e^{-iT}}\left[-D'(T)e^{-iT}+iD(T)e^{-iT}+U(T)e^{-iT}\right]$$

$$+\left[B-D(T)e^{-iT}+\int_0^T U(t)e^{-it}dt\right]\frac{\left\{-1(ie^{-iT})\right\}}{(1-e^{-iT})^2}$$

$$= 0,$$

or

$$U(T) + iD(T) - D'(T)$$

$$= \frac{1}{1-e^{-iT}}\left[B-D(T)e^{-iT}+\int_0^T U(t)e^{-it}dt\right] . \tag{42}$$

If the functions for expenses and depreciation, and values of B and i are known, the optimum service life, $\bar{T}$, can be obtained from equation (42) by means of a numerical analysis.

SOLUTION BY THE MAXIMUM PRINCIPLE

In order to apply the maximum principle, let us define

$$x_1(t) = \frac{\int_0^t U(t)e^{-it}dt}{1-e^{-it}}, \qquad x_1(0) = 0, \tag{43}$$

$$\frac{dx_1(t)}{dt} = \frac{U(t)e^{-it}}{1-e^{-it}} + \frac{\int_0^t U(t)e^{-it}dt \left\{-(ie^{-it})\right\}}{(1-e^{-it})^2}$$

$$= \frac{U(t)e^{-it}}{1-e^{-it}} - \frac{ie^{-it}\left[\int_0^t U(t)e^{-it}dt\right]}{(1-e^{-it})^2}, \tag{44}$$

$$x_2(t) = \frac{B-D(t)e^{-it}}{1-e^{-it}}, \qquad x_2(0) = 0, \tag{45}$$

$$\frac{dx_2(t)}{dt} = \frac{iD(t)e^{-it}-D'(t)e^{-it}}{1-e^{-it}} + \frac{\left[B-D(t)e^{-it}\right]\left\{-1(ie^{-it})\right\}}{(1-e^{-it})^2}$$

$$= \frac{iD(t)e^{-it}-D'(t)e^{-it}}{(1-e^{-it})} - \frac{ie^{-it}\left[B-D(t)e^{-it}\right]}{(1-e^{-it})^2}, \tag{46}$$

where $$D'(t) = \frac{dD}{dt}.$$

Since the system defined by equations (44) and (46) is non-autonomous (the right hand side of equations (44) and (46) depend explicitly on time), we shall introduce a new state variable, x_3, defined by

$$\frac{dx_3(t)}{dt} = 1 \ , \quad x_3(0) = t_o = 0, \tag{47}$$

It is obvious that $x_3(t) = t$.

The objective function as given by equation (41) can now be written

$$S = \sum_{i=1}^{3} c_i x_i(T)$$

$$= x_1(T) + x_2(T) \tag{48}$$

therefore, $c_1 = c_2 = 1, \quad c_3 = 0.$

The Hamiltonian and adjoint variables are

$$H = \sum_{i=1}^{3} z_i \frac{dx_i}{dt}$$

$$= z_1 \frac{dx_1}{dt} + z_2 \frac{dx_2}{dt} + z_3 \frac{dx_3}{dt}$$

$$= z_1 \left\{ \frac{U(t)e^{-ix_3}}{1-e^{-ix_3}} - \frac{ie^{-ix_3}\left[\int_0^t U(t)e^{-ix_3}dt\right]}{(1-e^{-ix_3})^2} \right\}$$

$$+ z_2\left\{\frac{iD(t)e^{-ix_3}-D'(t)e^{-ix_3}}{(1-e^{-ix_3})} - \frac{ie^{-ix_3}\left[B-D(t)e^{-ix_3}\right]}{(1-e^{-ix_3})^2}\right\}$$

$$+ z_3(1) \tag{49}$$

$$\frac{dz_1}{dt} = -\frac{\partial H}{\partial x_1} = 0 \tag{50}$$

$$z_1(T) = c_1 = 1 \tag{51}$$

$$\frac{dz_2}{dt} = -\frac{\partial H}{\partial x_2} = 0 \tag{52}$$

$$z_2(T) = c_2 = 1. \tag{53}$$

$$\frac{dz_3}{dt} = -\frac{\partial H}{\partial x_3} = -\left[z_1\left\{\frac{U(t)\left[-ie^{-ix_3}\right]}{(1-e^{-ix_3})} + \frac{\left[U(t)e^{-ix_3}\right]\left[-1(ie^{-ix_3})\right]}{(1-e^{-x_3})}\right\}\right.$$

$$z_1\left\{\left[\frac{\int_0^t U(t)dt\left\{ie^{-ix_3}\left[-ie^{-ix_3}\right]+e^{-ix_3}\left[-i^2e^{-ix_3}\right]\right\}}{(1-e^{-ix_3})^2}\right]\right.$$

$$\left.+\frac{ie^{-ix_3}\left[\int_0^t U(t)e^{-ix_3}\,dt\right]\left[-2(+ie^{-ix_3})\right]}{(1-e^{-ix_3})^3}\right\}$$

$$+ z_2\left\{\frac{\left[iD(t)-D'(t)\right]\left[-ie^{-ix_3}\right]}{(1-e^{-ix_3})} + \frac{\left[iD(t)-D'(t)\right]e^{-ix_3}\left[-1(ie^{-ix_3})\right]}{(1-e^{-ix_3})^2}\right\}$$

$$- z_2\left\{\frac{\left[-i^2e^{-ix_3}\left\{B-D(t)e^{-ix_3}\right\}\right] + ie^{-ix_3}\left[iD(t)e^{-ix_3}\right]}{(1-e^{-ix_3})^2}\right.$$

$$\left.+ \frac{ie^{-ix_3}\left[B-D(t)e^{-ix_3}\right]\left[-2(+ie^{-ix_3})\right]}{(1-e^{-ix_3})^3}\right\}\Bigg] ,$$

or

$$\frac{dz_3}{dt} = Z_1\left\{\frac{U(t)e^{-it_3}i}{1-e^{-ix_3}} + \frac{iU(t)e^{-2ix_3}}{(1-e^{-ix_3})^2}\right.$$

$$- \frac{2i^2e^{-ix_3}\left[\int_0^t U(t)e^{-ix_3}dt\right]}{(1-e^{-ix_3})^2} - \left.\frac{2i^2e^{-2ix_3}\left[\int_0^t U(t)e^{-ix_3}dt\right]}{(1-e^{-ix_3})^2}\right\}$$

$$+ z_2\left\{\frac{ie^{-ix_3}\left[iD(t)-D'(t)\right]}{(1-e^{-ix_3})} + \frac{ie^{-2ix_3}\left[iD(t)-D'(t)\right]}{(i-e^{-ix_3})^2}\right.$$

$$- \frac{i^2e^{-ix_3}\left[B-D(t)e^{-ix_3}\right] - i^2e^{-2ix_3}D(t)}{(1-e^{-ix_3})^2}$$

$$\left.- \frac{2i^2e^{-2ix_3}\left[B-D(t)e^{-ix_3}\right]}{(1-e^{-ix_3})^3}\right\} , \qquad (54)$$

and

$$z_3(T) = c_3 = 0. \qquad (55)$$

Solving equations (50) through (53) we obtain

$$z_1(t) = 1, \quad 0 \leq t \leq T. \tag{56}$$

$$z_2(t) = 1, \quad 0 \leq t \leq T. \tag{57}$$

Equations (54) and (55) can now be solved for $z_3(t)$ to yield

$$z_3(t) = \int_t^T \Bigg[z_1 \Bigg\{ \frac{U(t)e^{-it} \cdot i}{1-e^{-it}} + \frac{iU(t)e^{-2it}}{(1-e^{-it})^2}$$

$$- \frac{2i^2 e^{-it}\left[\int_0^t U(t)e^{-it}dt\right]}{(1-e^{-it})^2} - \frac{2i^2 e^{-2it}\left[\int_0^t U(t)e^{-it}dt\right]}{(1-e^{-it})^3} \Bigg\}$$

$$+ z_2 \Bigg\{ \frac{ie^{-it}\left[iD(t)-D'(t)\right]}{(1-e^{-it})} + \frac{ie^{-2it}\left[iD(t)-D'(t)\right]}{(1-e^{-it})^2}$$

$$- \frac{i^2 e^{-it}\left[B-D(t)e^{-it}\right] - i^2 e^{-2it} D(t)}{(1-e^{-it})^2}$$

$$- \frac{2i^2 e^{-2it}\left[B-D(t)e^{-it}\right]}{(1-e^{-it})^3} \Bigg\} \Bigg] . \tag{58}$$

Substituting equations (56), (57) and (58) back into equation (49), the Hamiltonian function becomes

$$H = \left\{ \frac{U(t)e^{-it}}{1-e^{-it}} - \frac{ie^{-it}\left[\int_0^t U(t)e^{-it}\,dt\right]}{(1-e^{-it})^2} \right\}$$

$$+ \left\{ \frac{iD(t)e^{-it} - D'(t)e^{-it}}{1-e^{-it}} - \frac{ie^{-it}\left[B-D(t)e^{-it}\right]}{(1-e^{-it})^2} \right\}$$

$$+ \int_t^T \left[\left\{ \frac{U(t)e^{-it}i}{1-e^{-it}} + \frac{iU(t)e^{-2it}}{(1-e^{-it})^2} - \frac{2i^2e^{-it}\left[\int_0^t U(t)e^{-it}dt\right]}{(1-e^{-it})^2} \right. \right.$$

$$\left. - \frac{2i^2e^{-2it}\left[\int_0^t U(t)e^{-it}dt\right]}{(1-e^{-it})^3} \right\} + 1 \left\{ \frac{ie^{-it}\left[iD(t)-D'(t)\right]}{(1-e^{-it})} \right.$$

$$+ \frac{ie^{-2it}\left[iD(t)-D'(t)\right]}{(1-e^{-it})^2} - \frac{i^2e^{-it}\left[B-D(t)e^{-it}\right] - i^2e^{-2it}D(t)}{(1-e^{-it})^2}$$

$$\left. \left. - \frac{2i^2e^{-2it}\left[B-D(t)e^{-it}\right]}{(1-e^{-it})^3} \right\} \right] . \qquad (59)$$

According to the maximum principle, the optimal decision vector, $\theta(t)$, which makes S minimum makes H minimum for every t, $t_o \leq t \leq T$. Furthermore, the minimum value of H is constant for every t. When T is not fixed, the value of this constant is fixed at zero for every t. That is

$\min H = 0, \quad 0 \leq t \leq T.$

Using this optimality condition and substituting $t = T$ into equation (59) we obtain

$$H = \frac{U(T)e^{-iT}}{1-e^{-iT}} - \frac{ie^{-it}\left[\int_0^T U(t)e^{-it}dt\right]}{(1-e^{-iT})^2}$$

$$+ \frac{iD(T)e^{-iT}-D'(T)e^{-iT}}{(1-e^{-iT})} - \frac{ie^{-iT}\left[B-D(T)e^{-iT}\right]}{(1-e^{-iT})^2}$$

$$= 0,$$

or

$$U(T) + i\,D(T) - D'(T)$$

$$= \frac{i}{1-e^{-iT}}\left[\int_0^T U(t)e^{-it}dt + B - D(T)e^{-iT}\right] . \qquad (60)$$

This result is the same as that obtained by the classical calculus method.

SOLUTION OF THE COST MINIMIZATION PROBLEM (1)

We will use the same data as in the previous example (of present net worth maximization). i.e.

B = Installation cost of equipment

= \$5,000

$$i = \text{Annual interest rate}$$
$$= 0.10$$
$$D(T) = \text{Salvage value at time } T$$
$$= 5{,}000\ e^{-T/4}.$$

Now, the expense function must be modified to account for the given revenue loss over time. This may be done by

$$U(t) = 1000\ (1 + 0.17\ t)$$

where \$1,000 is the yearly expense rate when the machine is new.

From equation (60), we get the final relation as

$$U(T) + i\ D(T) - D'(T)$$
$$= \frac{i}{1-e^{-iT}} \left[\int_0^T U(t)e^{-it}dt + B-D(T)\ e^{-it} \right].$$

By substituting the given values, the above expression should be solved for T, i.e.,

$$(1000 + 170\ T) + 500\ e^{-T/4} + 1250\ e^{-T/4}$$
$$= \frac{i}{1-e^{-iT}} \left[\int_0^T (1000+170t)\ e^{-it}dt + 5000 - 5000\ e^{-.35T} \right]$$

Simplifying the above expression, we obtain

$$1000 + 170\,T + 1750\,e^{-.25T}$$

$$= \frac{1}{e^{0.1T}-1}\left[-\left\{2700 + 170T\right\} + 3200\,e^{0.1T} - 500\,e^{-.25T}\right] \tag{61}$$

In order to solve this expression, for each value of T, L.H.S. and R.H.S. are calculated and then they are plotted against time, T. The intersection of the two curves gives the optimal value of T, i.e., T. This optimal value, T, will minimize the total costs.

Table 2. NUMERICAL VALUES OF EQUATION (61)

T, in Years	L.H.S. OF EQU. (61)	R.H.S. OF EQU. (61)
1	2532	2635
2	2400	2535
3	2337	2500
4	2324	2460
5	2351	2440
6	2410	2430
7	2495	2440

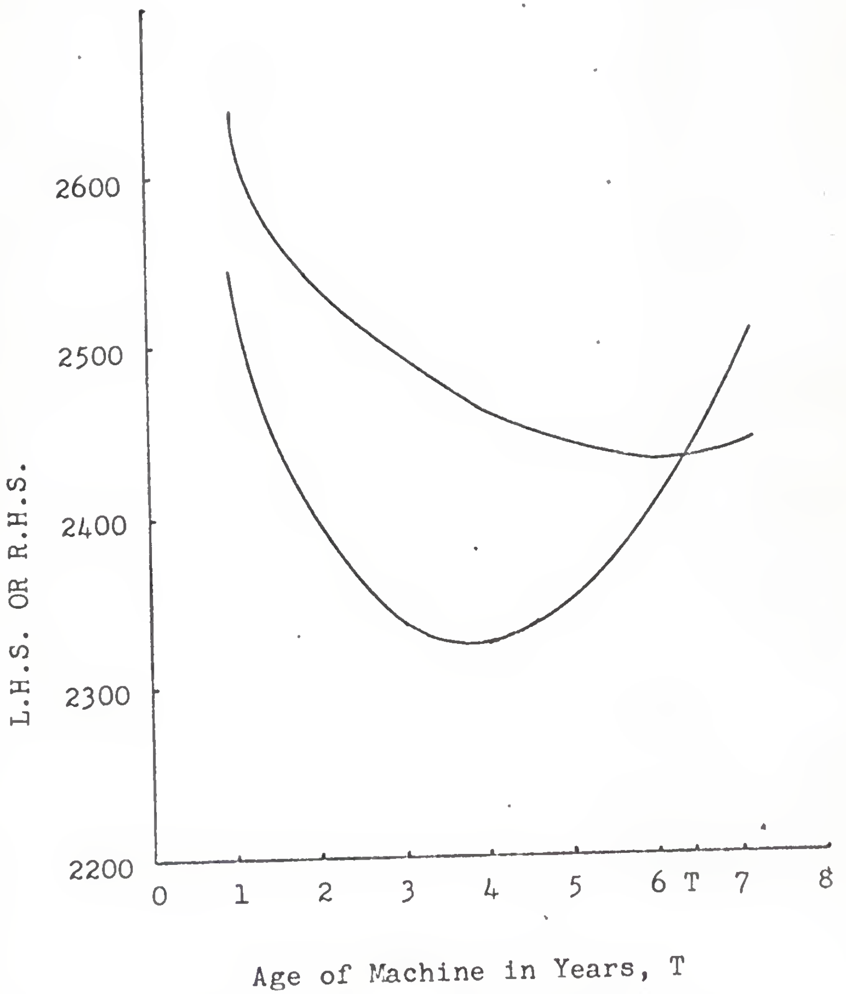

Fig. 4. SOLUTION OF EQUATION (61).

From the curve plotted in Fig. 4, we find that

T = 6.4 years (same as previous answer).

The net present value of the total cost, C_{∞} for this value of T is

$$C = \left[\int_0^{6.4} (1000 + 170t)\, e^{-it} dt + 5000 - 5000\, e^{-.35T} \right]$$

$$\times \frac{1}{1-e^{-.64}}$$

$$= \$24,280.$$

In general, Case 1 and Case 2 are different problems. However, for this special numerical example the problems are equivalent. In Case 1 the revenue function is given by

$$R(t) = 3000\ (1-0.01t)$$

and the expense function is given by

$$U(t) = 1000\ (1+0.14t).$$

As we are not considering revenue function in Case 2 the expense function has been modified to

$$U(t) = 1000\ (1+0.14t) - 3000\ (-0.01t)$$
$$= 1000\ (1+0.17t)$$

in order to account for the loss of revenue function. The expense function of Case 2 is equivalent to the revenue function and the expense function of Case 1 and this is why we get the optimum life of the machine the same in both cases.

CASE 3 A MORE REALISTIC MODEL

We assume in the models discussed previously that the investment time, T, is solely responsible for the maximization of profits or minimization of costs, as the case may be. It is easy to visualize, however, that under actual conditions there are other factors which are equally or more significant than the investment time and which should, therefore, be brought into the analysis. One such factor is the production rate at which the equipment is operated. In the analysis that follows, the production rate is introduced as the second decision variable which is dependent on time.

The manner in which the production affects the operation of the system varies with the market conditions, the manufacturing process (expense function) and the type of equipment used (depreciation function). These factors are not completely independent of each other but for computational purposes they may be considered so without lessening the efficiency of the model.

A mathematical model which accounts for all possible forms of variation in the system is obviously unattainable and therefore, simplifying assumptions are made here.

(1) The company's share of the market, M_s, remains constant throughout the investment time, T.

(2) The cost of any shortage is negligible and no inventory is carried. Consequently, we can write

$$0 \leq P(t) \leq M_s, \quad 0 \leq t \leq T, \tag{62}$$

where P(t) is the production rate.

(3) The amount of maintenance and servicing required per unit of production, M(P,t), can be considered as a function of the commulative service (production) obtained from the machine and can be approximated by the relationship (12)

$$M(P,t) = a\ (1-e^{-bP^1(t)}) \tag{63}$$

where a and b are constants and $P^1(t)$ is the cummulative production at time t, i.e.,

$$P^1(t) = \int_0^t P(t)dt. \tag{64}$$

The constants a and b can be determined from the company record (or manufacturer's data) on similar machines in the past. The unit of 'a' can be written as $ per unit time per unit of production whereas that of b can be written as per unit of production.

The graphical presentation of equation (63) is shown in Fig. 5. It is interesting and enlightening to provide some interpretation (12) of the result contained in Fig. 5 which shows that maintenance cost per unit of production begins at zero when a machine is placed in service and rises at a uniformly declining rate, reaching a horizontal plateau after a certain level of cummulative production is reached. Of course, the nature of this function is going to depend partly on the

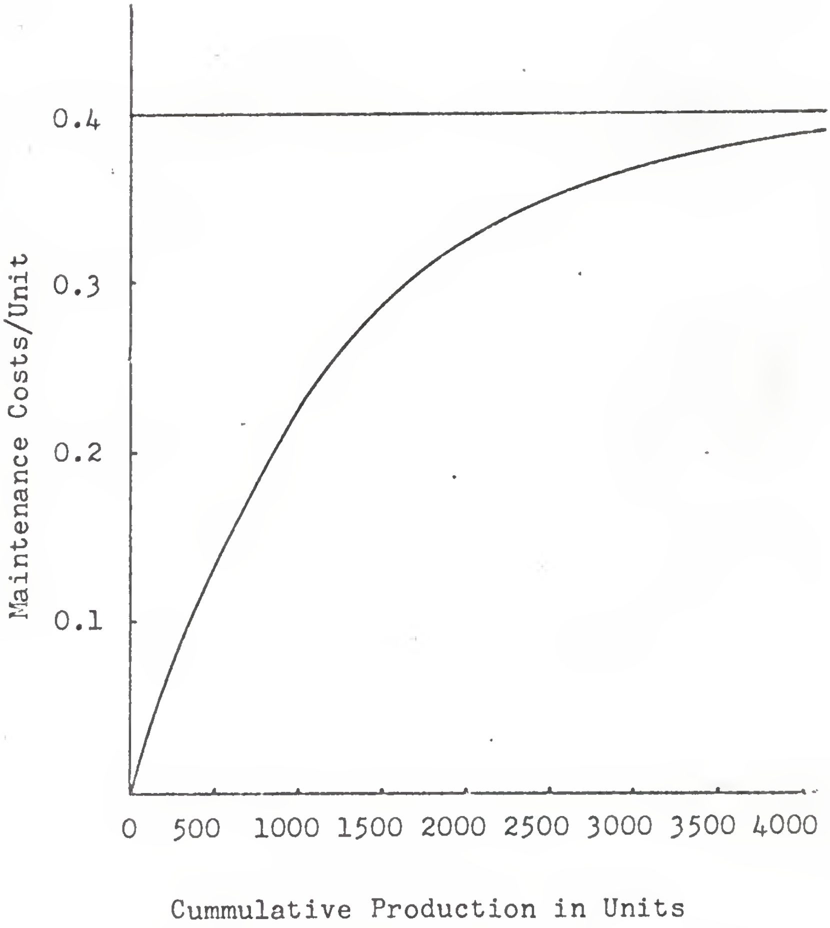

Fig. 5. MAINTENANCE COST VS. CUMMULATIVE PRODUCTION

firm's maintenance policies; it is not a technical relation in the strict sense, as it reflects suboptimal behavior. Assuming, however, that the general characteristics of this function hold for typical machine maintenance policies, we will try to give explanation for this type of behavior.

Reflection will show that a relatively simple stochastic failure model will generate this observed maintenance cost fundtion. When a machine is new, each of its component parts is new, and each of these parts is subject to a probability failure density, with age (cummulative production) as an independent variate. These probability densities will, in general, be different and have different expected values. As a machine renders production service these parts begin to fail, with the parts of lowest expected life tending to fail first and so forth. Very few parts fail early in service, while more and more fail in later service. Hence, the maintenance cost curve tends to rise. Eventually, however, the past replacement of parts creates a machine with a more even age distribution of component parts, rather than a distribution heavily biased by relatively new parts as is the case in the early periods of service. Once the ages of the many component parts begin to fall into a wider distribution, the failure of individual parts tends to become random with the average maintenance cost rate approaching a constant level.

It is believed that this explanation will describe adequately the behavior of any complex mechanical system composed of a large number of component parts, such as automotive equipment, pumping equipment, various kinds of automatic conveying and mechanical fabricating equipment and so on. Therefore, one might expect the maintenance cost function of Fig. 5 to be typical of many complex mechanical systems.

From equation (63), we can see that

$$M(P,t) = 0, \quad \text{when } P^1(t) = 0, \quad \text{at } t = 0$$

and

$M(P,t)$ approaches nearly 'a' when $P^1(t)$ tends to be larger with the increase in service time t of the equipment.

(4) We assume E to be the fixed overhead cost associated with the machine ($ per unit time).

We assume C_v to be the variable cost associated with the machine ($ per unit time per unit of production).

(5) With the total installed cost, B, and a constant rate of depreciation, k, the salvage value of the machine at time t is given by

$$D(t) = Be^{-kt} . \tag{65}$$

Using the cost minimization per unit of production as the criteria for optimality we write

C = Cost per unit of production

$$= \int_0^T \left[C_v + \frac{E}{P(t)} + a(1-e^{-bP^1(t)}) \right] e^{-it} dt + \frac{B}{P^1(T)} - \frac{D(T)e^{-kT}}{P^1(T)} \tag{66}$$

where

$P^1(T)$ = Total production at the end of the optimum life, T, of the machine

$$= \int_0^T P(t)\, dt.$$

The term under the integral sign in equation (66) represents the present worth of all the expenses per unit of production except depreciation. The two terms outside the integral sign may be understood as the net cost per unit of production of buying the equipment and selling it at a price D(T) after T years of use.

Substituting equation (65) into the equation (66) we obtain

$$C = \int_0^T \left[C_v + \frac{E}{P(t)} + a(1-e^{-bP^1(t)}) \right] e^{-it} dt + \frac{B}{P^1(T)} - \frac{Be^{-(k+i)T}}{P^1(T)}$$

$$= \int_0^T \left[C_v + \frac{E}{P(t)} + a(1-e^{-bP^1(t)}) \right] e^{-it} dt + \frac{B(1-e^{-(k+i)T})}{P^1(T)} . \tag{67}$$

Our objective is to minimize the value of C as given in equation (67) by choosing the most profitable rate of production, P(t), during the optimum investment time, T. We shall try to accomplish this through the use of the maximum principle.

OPTIMIZATION BASED ON MORE REALISTIC MODEL

To apply the maximum principle let the production rate be the decision variable, i.e.,

$$\theta(t) = P(t), \quad 0 \leq \theta(t) \leq \theta_{max} \tag{68}$$

The state variables are defined as follows:

$$x_1(t) = a(1-e^{-b\int\theta(t)dt}) \quad , \tag{69}$$

$$\frac{dx_1}{dt} = ab\theta(t)e^{-b\int\theta(t)dt} \quad , \quad x_1(0) = 0, \tag{70}$$

$$x_2(t) = \frac{B\left[1-e^{-(k+i)t}\right]}{\int\theta(t)dt} \quad , \tag{71}$$

$$\frac{dx_2}{dt} = \frac{\left[\int\theta(t)dt\right] B(k+i)e^{-(k+i)t} - B(1-e^{-(k+i)t})\ \theta(t)}{\left[\int\theta(t)dt\right]^2} \quad ,$$

$$x_2(0) = 0, \tag{72}$$

$$x_3(t) = \int_0^t \left[C_v + \frac{E}{\theta(t)} + x_1\right] e^{-it}dt \quad , \tag{73}$$

$$\frac{dx_3}{dt} = \left[C_v + \frac{E}{\theta(t)} + x_1\right] e^{-it} \quad , \quad x_3(0) = 0, \tag{74}$$

$$x_4(t) = \int_0^t \theta(t)dt \ , \tag{75}$$

$$\frac{dx_4}{dt} = \theta(t) \ , \quad x_4(0) = 0. \tag{76}$$

Since the system defined by equations (70), (72), (74) and (76) is non-autonomous (the right hand sides depend explicitly on time), we shall introduce an additional state variable x_5, defined by

$$x_5(t) = t \ , \tag{77}$$

$$\frac{dx_5}{dt} = 1, \qquad x_5(0) = t_o = 0. \tag{78}$$

The objective function to be minimized now becomes

$$S = \sum_{i=1}^{5} c_i x_i(T)$$

$$= x_2(T) + x_3(T). \tag{79}$$

Therefore,

$$c_1 = c_4 = c_5 = 0,$$

$$c_2 = c_3 = 1.$$

The Hamiltonian function and adjoint variables can be written as

$$H = \sum_{i=1}^{5} z_i \frac{dx_i}{dt} \ .$$

Substituting respective values of $\frac{dx_i}{dt}$, $i = 1, 2, \ldots, 5$ from equations (70), (72), (74), (76) and (78) we obtain

$$H = z_1 ab\theta(t)e^{-bx_4}$$

$$+ z_2\left\{\frac{x_4 B(k+i)e^{-(k+i)x_5} - B(1-e^{-(k+i)x_5})\ \theta(t)}{x_4^2}\right.$$

$$+ z_3\left[c_v + \frac{E}{\theta(t)} + x_1\right] e^{-ix_5} + z_4\theta(t) + z_5(1)\ , \qquad (80)$$

$$\frac{dz_1}{dt} = -\frac{\partial H}{\partial x_1} = -\ z_3 e^{-ix_5}\ , \qquad (81)$$

$$z_1(T) = c_1 = 0\ , \qquad (82)$$

$$\frac{dz_2}{dt} = -\frac{\partial H}{\partial x_2} = 0\ , \qquad (83)$$

$$z_2(T) = c_2 = 1\ , \qquad (84)$$

$$\frac{dz_3}{dt} = -\frac{\partial H}{\partial x_3} = 0\ , \qquad (85)$$

$$z_3(T) = c_3 = 1\ , \qquad (86)$$

$$\frac{dz_4}{dt} = -\frac{\partial H}{\partial x_4} = z_1 ab^2\theta(t)e^{-bx_4} + z_2\left\{\frac{B(k+i)e^{-(k+i)x_5}}{x_4^2}\right.$$

$$\left. - \frac{2B(1-e^{-(k+i)x_5})\ \theta(t)}{x_4^3}\right\}, \qquad (87)$$

$$z_4(T) = c_4 = 0 \ , \tag{88}$$

$$\frac{dz_5}{dt} = -\frac{\partial H}{\partial x_5} = z_2 \left\{ \frac{x_4 B(k+i)^2 e^{-(k+i)x_5} + B(k+i)e^{-(k+i)x_5}\theta(t)}{x_4^2} \right.$$

$$\left. + z_3 i \left[c_v + \frac{E}{\theta(t)} + x_1 \right] e^{-ix_5} \right. \ , \tag{89}$$

$$z_5(T) = c_5 = 0. \tag{90}$$

Solving equations (83) through (86) we obtain

$$z_2(t) = 1 \ , \qquad 0 \leq t \leq T \ , \tag{91}$$

$$z_3(t) = 1 \ , \qquad 0 \leq t \leq T \ . \tag{92}$$

Substituting equations (91) and (92) into the equation (80) and separating terms we obtain

H = Variable portion of the Hamiltonian which includes the decision variable, $\theta(t)$ + the portion of the Hamiltonian which does not include $\theta(t)$

= H* + H remainder

or

$$H = H^* + \frac{B(k+i)e^{-(k+i)x_5}}{x_4} + \left[c_v + x_1 \right] e^{-ix_5} + z_5 \ , \tag{93}$$

where

$$H^* = z_1 ab\theta(t)e^{-bx_4} - \frac{B(1-e^{-(k+i)x_5})\ \theta(t)}{x_4^2}$$

$$+ \frac{E}{\theta(t)} e^{-ix_5} + z_4\theta(t)$$

$$= \left[z_1 abe^{-bx_4} - \frac{B(1-e^{-(k+i)x_5})}{x_4^2} + z_4 \right] \theta(t) + \frac{E}{\theta(t)} e^{-ix_5} . \tag{94}$$

The optimum value of $\theta(t)$ can be obtained by taking the partial derivative of equation (94) with respect to $\theta(t)$ and then equating it to zero, i.e.,

$$\frac{\partial H^*}{\partial \theta} = \left[z_1 abe^{-bx_4} - \frac{B(1-e^{-(k+i)x_5})}{x_4^2} + z_4 \right] - \frac{E}{\theta^2} e^{-ix_5}$$

$$= 0 ,$$

or

$$\frac{E}{\theta^2} e^{-ix_5} = \left[z_1 + abe^{-bx_4} - \frac{B(1-e^{-(k+i)x_5})}{x_4^2} + z_4 \right] . \tag{95}$$

Simplifying equation (94), we obtain

$$\theta(t) = \left\{ \sqrt{\frac{Ee^{-ix_5}}{x_4^2 z_1 abe^{-bx_4} - B(1-e^{-(k+i)x_5}) + z_4 x_4^2}} \right\} x_4(t) . \tag{96}$$

Substituting equation (92) into equation (81) and then integrating the resulting equation, we obtain

$$z_1(t) = \frac{e^{-it}}{i} + c \tag{97}$$

where c is a constant of integration. Using the boundary condition $z_1(T) = 0$, we obtain

$$c = - \frac{e^{-iT}}{i}$$

and hence from equation (96), finally

$$z_1(t) = \frac{e^{-it} - e^{-iT}}{i}, \qquad 0 \leq t \leq T. \tag{98}$$

Substituting equation (98) into equation (96) we obtain

$$\theta(t) = \left\{ \sqrt{\frac{E e^{-ix_5}}{x_4^2 \, \frac{e^{-it} - e^{-iT}}{i} \, abe^{-bx_4} - B(1 - e^{-(k+i)x_5}) + z_4 x_4^2}} \right\}^{\rho} x_4(t), \tag{99}$$

Equation (99) gives the optimum value of O(t), as a continuous function of time.

In order to solve $\theta(t)$ from equation (99) explicitly as a function of time, we need to solve equations (70), (76), (78), (87) and (99) simultaneously. An attempt has not been made to solve these simultaneous differential equations as it might involve complex mathematical situations. Instead, a simplifying assumption is made which considers production rate as a constant over the life of the equipment in the numerical problem which follows the theoretical analysis. In order to solve those simultaneous differential equations, a further study by numerical methods is required.

It remains to be determined what the optimum investment time T should be. According to the maximum principle, a condition for optimality is obtained by making use of the fact that min H = 0 for $0 \leq t \leq T$. It should be noted that in order to minimize objective function, the Hamiltonian function should be minimized. Putting t = T in equation (80) we obtain

$$\min H = z_1(T)ab\theta(T)e^{-bx_4(T)}$$

$$+ z_2(T)\left\{\frac{x_4(T)B(k+i)e^{-(k+i)T} - B(1-e^{-(k+i)T})\theta(T)}{\left[x_4(T)\right]^2}\right.$$

$$+ z_3(T)\left[C_v + \frac{E}{\theta(T)} + x_1(T)\right]e^{-it} + z_4(T)\theta(T) + z_5(T)$$

$$= 0 \,. \qquad (100)$$

Substituting equations (82), (84), (86), (88) and (90) into equation (100) we obtain

$$\frac{x_4(T)B(k+i)e^{-(k+i)T} - B(1-e^{-(k+i)T})\theta(T)}{\left[x_4(T)\right]^2}$$

$$+ \left[c_v + \frac{E}{\theta(T)} + x_1(T)\right]e^{-iT} = 0,$$

or

$$\frac{x_4(T)B(k+i)e^{-(k+i)T} - B(1-e^{-(k+i)T})\theta(T)}{\left[x_4(T)\right]^2}$$

$$+ \frac{\left[c_v+\frac{E}{\theta(T)}+x_1(T)\ e^{-iT}\right] x_4(T)^{\ 2}}{} = 0 \ ,$$

i.e.,

$$x_4(T)B(k+i)e^{-(k+i)T}+B\theta(T)e^{-(k+i)T}$$

$$= B\theta(T) - \left[c_v + \frac{E}{\theta(T)} + x_1(T)\right] e^{-iT} \left[x_4(T)\right]^2 \ ,$$

i.e.,

$$Be^{-(k+i)T} \left[x_4(T)(k+i)+\theta(T)\right]$$

$$= B\theta(T) - \left[c_v+\frac{E}{\theta(T)}+x_1(T)\right] e^{-iT} \left[x_4(T)\right]^2$$

i.e.,

$$Be^{-(k+i)T} = \frac{B\theta(T) - \left[c_v+\frac{E}{\theta(T)}+x_1(T)\right] e^{-iT} \left[x_4(T)\right]^2}{x_4(T)(k+i)+\theta(T)}$$

This equation can be used to find the optimum investment life, T, of the equipment when $\theta(T)$, $x_1(T)$, $x_4(T)$ and other constants are known.

Solution of the Numerical Problem

In order to simplify the numerical analysis we assume that production rate is constant throughout the service life of the machine. With this assumption, from equation (75) we obtain

$$x_4(t) = \int_0^t \theta \, dt$$

$$= \theta t, \qquad 0 \leq t \leq T. \qquad (102)$$

Substituting this value in equation (101) we obtain

$$Be^{-(k+i)T} = \frac{B\theta - \left[c_v + \frac{E}{\theta} + a(1-e^{-b\theta T})\right] e^{-iT} \times \theta^2 T^2}{\theta\left[T(k+i) + 1\right]}$$

$$= \frac{B - \left[c_v + \frac{E}{\theta} + a(1-e^{-b\theta T})\right] e^{-iT} \times \theta T^2}{(k+i)T + 1} \qquad (103)$$

DATA FOR THE NUMERICAL PROBLEM

The following data is provided for a particular type of machine.

The installation cost of the machine, B = \$20,000.

Fixed overhead costs, E = \$250 per unit time.

Variable costs, C_v = \$0.60 per unit time per unit of production.

Depreciation rate, $k = 0.30$ (exponential)

Annual interest rate, $i = 10\%$

Market share, M_s = 500 units per unit time

Machine Capacity, $P_m(t)$ = 700 Units per unit time

The maintenance cost function for the machine is given by

$$M(P,t) = a(1-e^{-b\theta t})$$

where

$a = 0.40$, \$ per unit time per unit of production,

$b = 8.4 \times 10^{-4}$ per unit of production,

θ = average rate of production

$$= \min \begin{cases} M_s \\ P_m(t) \end{cases}$$

= 500 units per unit time

Solution:

In order to find optimum investment time, T, we can use the equation

$$Be^{-(k+i)T} = \frac{B - \left[c_v + \frac{E}{\theta} + a(1-e^{-b\theta T})\right] e^{-iT} \times \theta T^2}{(k+i)T+1} \tag{104}$$

Table 3 NUMERICAL VALUES OF EQUATION (104)

T	L.H.S. of equ. (104) $	R.H.S. of equ. (104) $
1	13,400	13,890
2	9,000	9,900
3	6,020	6,980
4	4,040	4,750
5	2,710	3,000
6	1,820	1,618

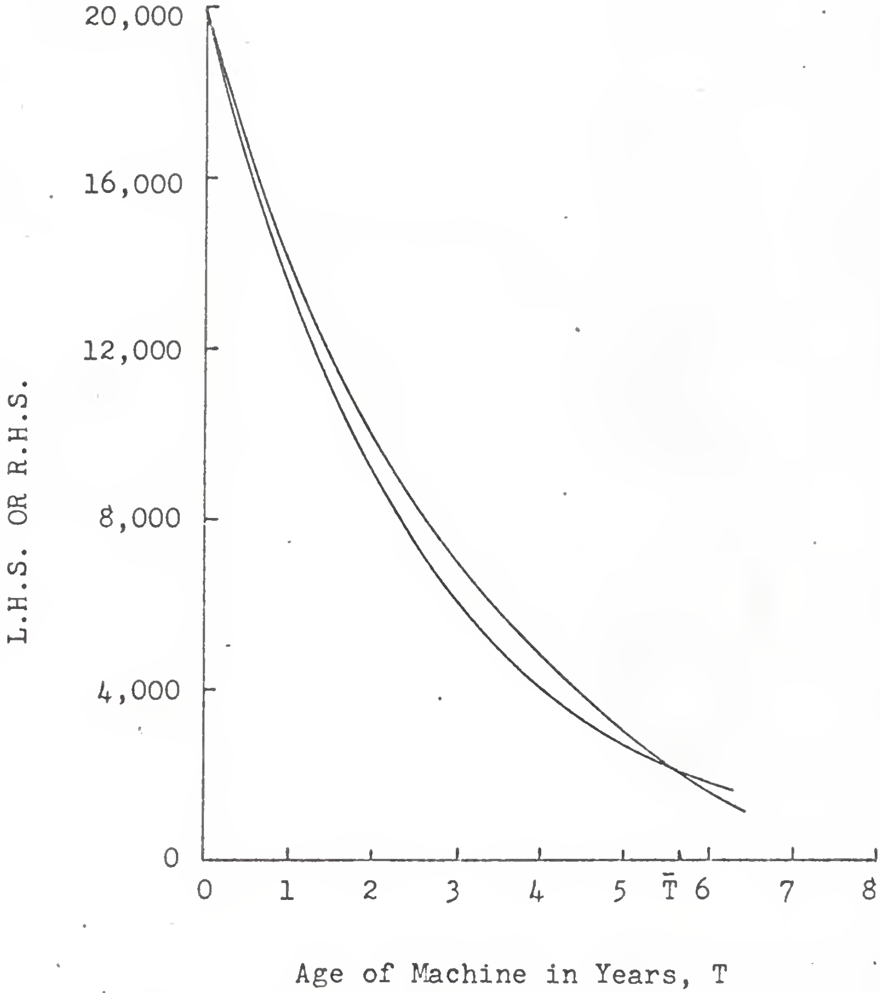

Fig. 6. SOLUTION OF EQUATION (104).

From the graph in Fig. 6, it is seen that the optimum investment life of the machine is

$T = 5.70$ years.

The present worth of total expenses per unit of production can be found from the equation (66) as

$$C = \int_0^T \left[c_v + \frac{E}{\theta} + a(1-e^{-b\theta t})\right] e^{-it} dt + \frac{B(1-e^{-(k+i)T})}{\theta T}$$

$$= \int_0^{5.7} \left[0.60+0.50+0.40(1-e^{-.42t})\right] e^{-it} dt + \frac{20,000(1-e^{-.4T})}{500 \times 5.7}$$

$$= 5.787 + 6.3$$

$$= \$12.087 \text{ per unit of production.}$$

CONCLUSION

This report provides a comparative study of the continuous maximum principle. It can be seen that for first two cases the classical calculus method is comparatively easier than the continuous maximum principle. However, the results obtained by both the methods are same which proves the validity of the continuous maximum principle. Case 3 cannot be solved by the classical calculus method due to the complexities involved in handling such models by this method, the maximum principle definitely shows a method to solve such problems.

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EQUIPMENT REPLACEMENT BY THE CONTINUOUS MAXIMUM PRINCIPLE

by

DAHYABHAI RANCHHODBHAI CHAUDHARI

B.S.(Mech. Engg.), Sardar Vallabhbhai Vidyapeeth, Vallabh Vidyanagar, Gujarat, India, 1965

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The objective of this report is a comparative study of applicability of the continuous maximum principle. The problems treated are those of finding the optimum investment life of the "diminishing efficiency" type of equipment so as to maximize the net present worth of the investment or to minimize the net present worth of all expenses on the equipment.

Three cases have been considered in details in this report. Case 1 deals with the finding of the optimum investment life of the machine so as to maximize the net present worth of all returns on the investment. Sometimes it is difficult to allocate a portion of revenue to a particular machine when many different operations are carried on the same product by different machines. In order to avoid this difficulty Case 2 deals with the minimization of the present worth of all expenses on the machine. These two cases deal with the replacement problem for a chain of machines. Case 3 deals with a more realistic model than that of Case 2 by taking into account production rate as the second decision variable. It also considers variable costs, fixed costs and maintenance costs separately. The problem is to minimize the present worth of the sum of all costs (including depreciation) per unit of production.

A numerical problem has been solved for each of the first two cases in order to show the validity of theoretical results obtained. For third case, as it involves a number of simultaneous non-linear differential equations, a further study by numerical methods is required.

Case 1 and Case 2 are "zero-order" control problems in which the application of variational techniques is not advantageous. Although much more computations are necessary in finding the solution by the maximum principle as compared to the classical calculus method, it certainly gives a correct solution. Case 3 cannot be solved by the classical calculus method due to the complexities involved in handling such models by this method, the maximum principle definitely shows a method to solve such problems.