Development of the theory of Kumjian-Pask fibrations, their path groupoids, and their C*-algebras

by

Lydia de Wolf

B.S., Union University, 2015

M.S., Kansas State University, 2019

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

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Abstract

The higher-rank graphs, or k-graphs, of Kumjian and Pask [1], introduced in 2000 to generalize the construction of C*-algebras of directed graphs, are in fact examples of a notion in category theory called a discrete Conduché fibration. In 2017, Brown and Yetter developed a theory of C*-algebras associated to discrete Conduché fibrations, enforced additional conditions to define Kumjian-Pask fibrations, and investigated the generalized infinite path spaces and path groupoids arising from these fibrations [2]. This dissertation continues to explore this generalization, investigating the properties of k-graphs that may or may not hold in the general case, and providing examples of Kumjian-Pask fibrations beyond the motivating k-graph case, to justify and motivate further study. We generalize various constructions, properties, and techniques that are useful in the theory of graph algebras, as well as integrating recent studies to bring our general results up to date. After reformulating or introducing several examples, we cover two classes of Kumjian-Pask fibrations in detail which admit much more development and demonstrate opportunity for continuing research.

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Dedication

To my mother and father, Deb and Jeff, and my siblings Caleb, Hannah, and Chloe.

Introduction

The higher-rank graphs, or k-graphs, of Kumjian and Pask [1], introduced in 2000 to generalize the construction of C*-algebras of directed graphs, are in fact examples of a notion in category theory called a discrete Conduché fibration. In 2017, Brown and Yetter developed a theory of C*-algebras associated to discrete Conduché fibrations, enforced additional conditions to define Kumjian-Pask fibrations, and investigated the generalized infinite path spaces and path groupoids arising from these fibrations [2]. This dissertation continues to explore this generalization, investigating the properties of k-graphs that may or may not hold in the general case, and providing examples of Kumjian-Pask fibrations beyond the motivating k-graph case, to justify and motivate further study. To that end, in the first chapter we provide the fundamental definitions and summarize results established in [2]; original work on the core subalgebra appears in the final subsection. Chapter 2 develops in the general case various types of constructions that are studied and/or useful in the theory of k-graphs. Chapter 3 further investigates the quality of aperiodicity and its relevance to a generalized definition of cycline pairs and Cartan subalgebras. Chapter 4 reformulates or introduces a few examples of KPfs, while Chapters 5 and 6 cover two examples in particular which admit much more development and warrant their own treatment.

Before launching into the definitions and literature review, we establish general assumptions and notational conventions. Throughout, categories are assumed to be small, and composition of morphisms is rendered in anti-diagrammatic order; that is, the composition $C \xrightarrow{\beta} B \xrightarrow{\alpha} A$ is denoted $\alpha\beta$.

The source object of a morphism α is denoted by $s(\alpha)$, and its range or target object is denoted by $r(\alpha)$. We denote the set of objects in a category $\mathcal{C}$ by $\mathrm{Obj}(\mathcal{C})$, and often identify the category with its set of morphisms, so that if λ is a morphism of $\mathcal{C}$, we simply write $\lambda \in \mathcal{C}$.

In this dissertation, $\mathbb{N}$ will denote the natural numbers including zero, and strictly positive integers will be denoted $\mathbb{Z}_{>0}$ explicitly. When examples reference other topics with previously-established notation, such as directed graphs in [3] or higher-rank graphs in [1], we will generally only translate into our own notation style where strictly expedient.

Chapter 1

Discrete Conduché Fibrations, Kumjian-Pask Fibrations, and Their C*-algebras

This chapter begins by summarizing the definitions and major results of [2], which is the initial work emulating the k-graph analysis of [1] in the case of a more general functor than the traditional degree map. Conduché functors or fibrations are a topic of study in category theory and game semantics (see [4], [5], [6], [7]), and this application of them uses the "strict" or "discrete" definition; although true to form for category theory, there also exists a "weak" version defined up to isomorphism.

1.1 Discrete Conduché Fibrations and C*-algebras

Definition 1.1.1. A *discrete Conduché fibration (dCf)* is a functor $F : \mathcal{E} \to \mathcal{B}$ with the *unique factorization property* or *Conduché criterion*: for every morphism $\phi : Y \to X$ in $\mathcal{E}$, every factorization of $F(\phi)$ in $\mathcal{B}$

$$F(Y) \xrightarrow{\lambda} B \xrightarrow{\rho} F(X)$$

lifts uniquely to a factorization of ϕ

$$Y \xrightarrow{\tilde{\lambda}} Z \xrightarrow{\tilde{\rho}} X$$

with $F(\tilde{\lambda}) = \lambda$, $F(\tilde{\rho}) = \rho$, and $F(Z) = B$.

Note that by [2] Lemma 2.2 the unique lifting property extends to finitely many factors.

Definition 1.1.2. A functor $F : \mathcal{E} \to \mathcal{B}$ is *row-finite* if for every object $X \in \mathcal{E}$ and every morphism $\beta : B \to F(X)$ in $\mathcal{B}$, the class of morphisms with target X whose image under F is β is a finite set.

The following surjectivity condition replaces the no-sources or locally-convex conditions commonly used in the study of k-graphs.

Definition 1.1.3. A functor $F : \mathcal{E} \to \mathcal{B}$ between small categories is *strongly surjective* if (i) it is surjective on objects, and (ii) given any object X in $\mathcal{E}$ the map induced from the set of morphisms targeted at X to the set of morphisms targeted at $F(X)$ in $\mathcal{B}$ is surjective.

Definition 1.1.4. Given a strongly surjective, row-finite dCf $F : \mathcal{E} \to \mathcal{B}$, a *Cuntz-Krieger system associated to* F is the following data in a C*-algebra D:

- A projection P_X for each object X of $\mathcal{E}$,
- A partial isometry S_α for each morphism $\alpha : Y \to X$ of $\mathcal{E}$,

satisfying the relations

1. For $X \neq Y$, $P_X \perp P_Y$,

2. If α and β are composable, then $S_{\alpha\beta} = S_\alpha S_\beta$,

3. For all X, $P_X = S_{\mathrm{Id}_X} = S^*_{\mathrm{Id}_X}$,

4. For all $\alpha : Y \to X$, $S^*_\alpha S_\alpha = P_{s(\alpha)} = P_Y$ (source projections),

5. If $F(\alpha) = F(\beta)$ and $\alpha \neq \beta$ then $S^*_\beta S_\alpha = 0$,

6. For all X, and for all morphisms $b : B \to F(X)$ in $\mathcal{B}$,

$$\sum_{\{\alpha \in X\mathcal{E} : F(\alpha) = b\}} S_\alpha S^*_\alpha = P_X.$$

We denote the smallest C*-subalgebra of D containing P, S by $C^*(P, S)$.

Note that Item 5 above follows from Item 6, but it is referenced often enough to justify including it in the list of axioms. The following theorem established the existence of a universal C*-algebra generated by these relations.

Theorem 1.1.5 ([2], Proposition 2.9)**.** *Let $F : \mathcal{E} \to \mathcal{B}$ be a discrete Conduche Fibration (dCf). Then there exists a C*-algebra, $C^*(F)$, generated by a Cuntz-Krieger system $\{p_X, s_\alpha\}$ such that for any Cuntz-Krieger system $\{Q_X, T_\alpha\}$ associated to F in a C*-algebra B, there is a unique $*$-homomorphism from $C^*(F)$ to B extending the map $s_\alpha \mapsto T_\alpha$.*

1.2 Kumjian-Pask Fibrations and Infinite Paths

Definition 1.2.1. A category satisfies the *right Ore condition* / is a *right Ore category* if every cospan $A \xrightarrow{m} B \xleftarrow{n} C$ can be completed to a commutative square.

A category is *strongly right Ore* if it is right Ore and moreover, for every cospan $m, n \in \mathcal{B}$ with $p_1, p_2, q_1, q_2 \in \mathcal{B}$ with $mp_i = nq_i$ for $i = 1, 2$, there exist $a, b \in \mathcal{B}$ with $p_1 a = p_2 b$ and $q_1 a = q_2 b$.

Any category with pullbacks is automatically strongly right Ore, and any left-cancellative right Ore category is strongly right Ore ([2], Proposition 3.3). For example, all lattices, groups, groupoids, and the positive cones of lattice-ordered groups are strongly right Ore.

The following definition of a slice category is a special case of a more general construction known as a comma category.

Definition 1.2.2. Given a category $\mathcal{C}$ and an object X in $\mathcal{C}$, the *slice category* $\mathcal{C}/X$ is the category whose objects are morphisms of $\mathcal{C}$ with target X, and morphisms are given by commutative triangles: that is, for $\alpha, \beta \in X\mathcal{C} = \mathrm{Obj}(\mathcal{C}/X)$, a morphism from β to α is given by a $\gamma \in \mathcal{C}$ with $\alpha\gamma = \beta$. We often see morphisms in $\mathcal{C}/X$ as ordered pairs $(\alpha, \gamma) \in X\mathcal{C} \times \mathcal{C}$ with $r(\gamma) = s(\alpha)$. The range of (α, γ) is α and its source is $\alpha\gamma$. Composition $(\alpha, \gamma)(\beta, \delta)$ can only occur if $\beta = \alpha\gamma$ and in this case we have $(\alpha, \gamma)(\alpha\gamma, \delta) = (\alpha, \gamma\delta)$.

For any object $X \in \mathcal{C}$, a functor $F : \mathcal{C} \to \mathcal{D}$ will induce a natural functor from $\mathcal{C}/X$ to $\mathcal{D}/F(X)$ which by abuse of notation we also denote F.

Definition 1.2.3. A functor $F : \mathcal{E} \to \mathcal{B}$ is *locally split* if for every object $X \in \mathcal{E}$, the induced functor $F : \mathcal{E}/X \to \mathcal{B}/F(X)$ admits a splitting (or section) $x : \mathcal{B}/F(X) \to \mathcal{E}/X$; that is, a functor such that $F \circ x = \mathrm{Id}_{\mathcal{B}/F(X)}$.

Definition 1.2.4. A functor $F : \mathcal{E} \to \mathcal{B}$ is a *Kumjian-Pask fibration (KPf)* if $\mathcal{B}$ is a strongly right Ore category and F is a locally split discrete Conduché fibration.

When the commutative monoid $\mathbb{N}^k$ is considered as a cancellative one-object category $\mathcal{B}$, the degree map $d : \Lambda \to \mathbb{N}^k$ of a row-finite k-graph with no sources does indeed satisfy the definition of a KPf. The unique lifting property is exactly satisfied by the k-graph's unique factorization; $\mathbb{N}^k$ is strongly right Ore; and being both surjective on the singular object of $\mathbb{N}^k$ and source-free, d is locally split.

The splittings or sections described above provide the general analogue to the notion of infinite paths for k-graphs. Thus we adopt the notation and language conventions below from [2], which also details the consistency between these definitions and the notation used by [1] in particular. For a detailed breakdown of the agreement between our setting and the infinite path notation of k-graphs, see [2] Example 3.15. Indeed, the convention adopted in our definition of slice categories by seeing morphisms as pairs emulates the k-graph definition of infinite paths as morphisms defined from $\Omega_k := \{(m,n) \in \mathbb{N}^k \times \mathbb{N}^k : m \leq n\}$ to the small category Λ.

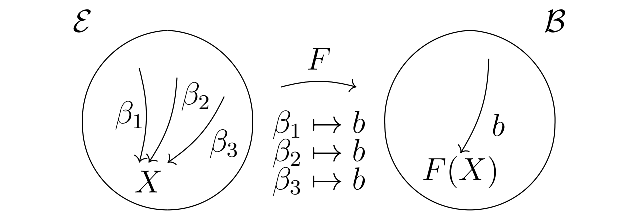
Definition 1.2.5. Let $F : \mathcal{E} \to \mathcal{B}$ be a row-finite KPf.

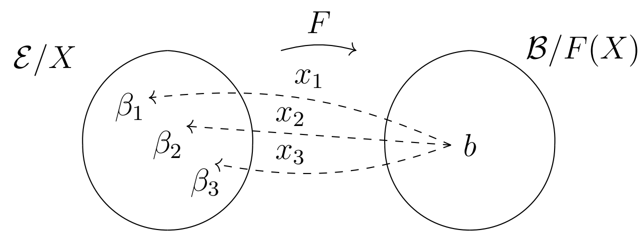
1. For an object X of $\mathcal{E}$, an *infinite path to* X is a section $x : \mathcal{B}/F(X) \to \mathcal{E}/X$ of the functor $F : \mathcal{E}/X \to \mathcal{B}/F(X)$. The set of all infinite paths to X is denoted by $Z(X)$, and the set of all infinite paths (that is $\bigcup_{X \in \mathrm{Obj}(\mathcal{E})} Z(X)$) is denoted by F^∞.

2. For a morphism $\mu : Y \to X$ in $\mathcal{E}$, an *infinite path ending in* μ is an infinite path x to X for which $x(F(\mu)) = \mu$. The set of all infinite paths ending in μ is denoted by $Z(\mu)$.

3. When an infinite path x is introduced without explicitly stating that it is a functor from $\mathcal{B}/F(X)$ to $\mathcal{E}/X$ for a specified object X, we denote the object over which its target category is a slice category by $r(x)$.

The following figure is a visual representation of infinite paths seen as splittings using slice categories. $F : \mathcal{E} \to \mathcal{B}$ is not injective, taking three morphisms $\beta_1, \beta_2, \beta_3$ to $b \in \mathcal{B}$. The morphisms targeting X become the objects in the slice category $\mathcal{E}/X$. Three infinite paths x_1, x_2, x_3 in $Z(X)$ are shown, distinguished by their endings $\beta_1, \beta_2, \beta_3$, such that $x_1(F(b)) = \beta_1$, $x_2(F(b)) = \beta_2$, and $x_3(F(b)) = \beta_3$.





Note that if $X \in \mathrm{Obj}(\mathcal{E})$ and $b \in \mathcal{B}$ targets $F(X)$, then the cylinder set $Z(X)$ may be decomposed into the disjoint union

$$Z(X) = \bigcup_{\beta \in X\mathcal{E}: F(\beta)=b} Z(\beta).$$

In the k-graph case, a map $\sigma^p : \Lambda^\infty \to \Lambda^\infty$ is frequently used, which can be thought of as stripping off the degree-p ending of an infinite path. Additionally, there is a map characterized by $x \mapsto \lambda x$ when $s(\lambda) = r(x)$, which yields the unique infinite path formed by adding on λ to the end of x. These are generalized as the vital maps ind and res in ([2], Proposition 3.17), so we reproduce their introduction and briefly summarize their definitions.

Proposition 1.2.1. *If $F : \mathcal{E} \to \mathcal{B}$ is a KPf, for any morphism $\mu : X \to Y \in \mathcal{E}$, there exist bijections*

$$\mathrm{ind}_\mu : Z(X) \to Z(\mu),$$

$$\mathrm{res}_\mu : Z(\mu) \to Z(X)$$

satisfying the following properties:

1. $\mathrm{ind}_\mu(x)(F(\mu)a) = \mu x(a)$ *for any* $a \in F(s(\mu))\mathcal{B}$*;*

2. $\mathrm{ind}_\mu \circ \mathrm{res}_\mu = \mathrm{Id}_{Z(\mu)}$ *and* $\mathrm{res}_\mu \circ \mathrm{ind}_\mu = \mathrm{Id}_{Z(X)}$*;*

3. *for X an object in $\mathcal{E}$,* $\operatorname{ind}_{\mathrm{Id}_X} = \operatorname{res}_{\mathrm{Id}_X} = \mathrm{Id}_{Z(X)}$*;*

4. *for $\mu, \nu \in \mathcal{E}$ with $s(\mu) = r(\nu)$,* $\operatorname{res}_\mu(Z(\mu\nu)) = Z(\nu)$ *and* $\operatorname{ind}_\mu(Z(\nu)) = Z(\mu\nu)$*;*

5. *for $\mu, \nu \in \mathcal{E}$ with $s(\mu) = r(\nu)$, the domain of* $\operatorname{res}_\nu \circ \operatorname{res}_\mu$ *is $Z(\mu\nu)$ and* $\operatorname{res}_\nu \circ \operatorname{res}_\mu = \operatorname{res}_{\mu\nu}$*;*

6. *for $\mu, \nu \in \mathcal{E}$ with $s(\mu) = r(\nu)$, the domain of* $\operatorname{ind}_\mu \circ \operatorname{ind}_\nu$ *is $Z(s(\nu))$ and* $\operatorname{ind}_\mu \circ \operatorname{ind}_\nu = \operatorname{ind}_{\mu\nu}$.

If $x \in Z(\mu)$, then it is an infinite path to Y, that is a splitting defined on pairs $(c, d) \in \mathcal{B}/F(Y)$, denoted

$$x(c, d) = (x_1(c, d), x_2(c, d)),$$

and $x(cd)$ is a morphism in $\mathcal{E}$ whose factorization with respect to cd is $x_1(c, d)x_2(c, d)$. In a sense, res_μ is thought of as stripping off the ending μ from x, generalizing the behavior of the σ^p operation studied under k-graphs. Thus the result $\operatorname{res}_\mu(x)$ is an infinite path to $s(\mu)$ and is a splitting defined on pairs $(a, b) \in \mathcal{B}/F(X)$. Its values are determined by reference to x as follows:

$$\operatorname{res}_\mu(x)(a, b) := (x_2(F(\mu), a), x_2(F(\mu)a, b)).$$

An infinite path x for a k-graph may be concatenated with a compatible finite path λ whose source is $r(x)$ to form a unique infinite path λx, but the existence of multiple objects in $\mathcal{B}$ complicates the matter here. The ind_μ map takes the place of this process, acting on an infinite path $x \in Z(s(\mu))$ to produce $\operatorname{ind}_\mu(x)$, an infinite path to $r(\mu)$. As such, it must be defined on pairs $(a, b) \in \mathcal{B}/F(Y)$. Since $F(\mu), ab$ form a cospan in $\mathcal{B}$, there exist $c, d \in \mathcal{B}$ such that $F(\mu)c = abd$. By the unique lifting property, there exist unique morphisms $\alpha^{c,d}_{a,b}, \beta^{c,d}_{a,b}, \delta^{c,d}_{a,b} \in \mathcal{E}$ such that

$$\mu x(c) = \alpha^{c,d}_{a,b}\beta^{c,d}_{a,b}\delta^{c,d}_{a,b}$$

and $F(\alpha^{c,d}_{a,b}) = a$, $F(\beta^{c,d}_{a,b}) = b$, and $F(\delta^{c,d}_{a,b}) = d$. Then define

$$\operatorname{ind}_\mu(x)(a, b) := (\alpha^{c,d}_{a,b}, \beta^{c,d}_{a,b}).$$

The proof of ([2], Proposition 3.17) shows that the above definitions are well-defined and satisfy the desired properties.

1.3 The Path Groupoid

In this section, we describe the construction of the path groupoid of a KPf as the generalization of the same construction for k-graph path groupoids and summarize the conclusions given in Sections 4 and 5 of [2].

Let $F : \mathcal{E} \to \mathcal{B}$ be a KPf. Then the set $\{Z(\alpha) : \alpha \in \mathcal{E}\}$ is a basis for a topology on F^∞. Furthermore, if F is row-finite, then F^∞ is Hausdorff and $\mathrm{ind}_\mu : Z(\mu) \to Z(s(\mu))$ and $\mathrm{res}_\mu : Z(s(\mu)) \to Z(\mu)$ are continuous. If in addition F is strongly surjective, and moreover every slice category $\mathcal{B}/B$ is countable, then for each $\beta \in \mathcal{E}$, $Z(\beta)$ is compact and F^∞ is totally disconnected.

To construct the path groupoid, we begin with triples

$$\mathcal{G}_F := \{(\mu, \nu, x) \in \mathcal{E} \times \mathcal{E} \times F^\infty : x \in Z(\nu) \text{ and } s(\mu) = s(\nu)\}$$

and define an equivalence relation by $(\mu, \nu, x) \sim (\mu', \nu', x')$ if

1. $x = x'$
2. $\exists\, \lambda \in \mathcal{E}$ with $x \in Z(\lambda) \subset Z(\nu) \cap Z(\nu')$ and $\mathrm{ind}_\mu \circ \mathrm{res}_\nu |_{Z(\lambda)} = \mathrm{ind}_{\mu'} \circ \mathrm{res}_{\nu'} |_{Z(\lambda)}$,
3. there exist $a, b \in \mathcal{B}$ with $F(\mu)a = F(\mu')b$ and $F(\nu)a = F(\nu')b$.

The elements of the path groupoid for F, denoted G_F, are the equivalence classes under this relation, denoted by $[\mu, \nu, x]$, and seen as arrows from x to $\mathrm{ind}_\mu \circ \mathrm{res}_\nu(x)$. The objects of the groupoid G_F may be identified with F^∞ by $x \mapsto [r(x), r(x), x]$. Elements $g = [\mu_g, \nu_g, x]$ and $h = [\mu_h, \nu_h, z]$ of G_F are composable when $x = \mathrm{ind}_{\mu_h} \circ \mathrm{res}_{\nu_h}(z)$. In that case, the composition results in

$$gh = [\mu_g \xi, \nu_h \eta, z] = [\mu_{gh}, \nu_{gh}, z],$$

where $\xi, \eta \in \mathcal{E}$ such that $\nu_g \xi = \mu_h \eta$ The inverse of $[\mu, \nu, x]$ is $[\nu, \mu, \mathrm{ind}_\mu \circ \mathrm{res}_\nu(x)]$. With multiplication provided by the composition and with the existence of inverses, G_F is indeed a groupoid.

A basis for a topology on G_F is given by the cylinder sets

$$Z(\mu, \nu) := \{[\alpha, \beta, x] \in G_F : [\alpha, \beta, x] = [\mu, \nu, x]\}.$$

With respect to this topology, G_F is a topological groupoid; that is, composition is continuous and inversion is a homeomorphism. As in the k-graph case, G_F is étale. If moreover $F : \mathcal{E} \to \mathcal{B}$ is a row-finite KPf with every slice category $\mathcal{B}/B$ countable, then G_F is totally disconnected locally compact Hausdorff.

G_F is also a topologically principal groupoid under an aperiodicity condition generalized from the k-graph case, given below.

Definition 1.3.1. An infinite path $x \in F^\infty$ is *aperiodic* if for all $\alpha, \beta \in \mathcal{E}$, $\mathrm{res}_\alpha(x) = \mathrm{res}_\beta(x)$ implies $\alpha = \beta$. A KPf $F : \mathcal{E} \to \mathcal{B}$ is aperiodic if $Z(X)$ contains an aperiodic path for every $X \in \mathrm{Obj}(\mathcal{E})$.

G_F is topologically principal if and only if F is aperiodic. One direction of the proof is given by [2], Theorem 5.6, and the other will be shown in Chapter 3 following more exploration of aperiodicity for KPfs.

Brown and Yetter noted that while the path groupoid G_Λ of a k-graph (Λ, d) is amenable, this is not guaranteed for G_F in the general case. For example, if $F = \mathrm{Id}_H : H \to H$ when H is a nonamenable group such as $\mathbb{F}_2$, then $G_F \cong H$ (see [2] Example 4.18 for justification of the isomorphism for groupoids). Distinguishing the cases and conditions under which G_F is amenable is one of our avenues of investigation, as the version of the Cuntz-Krieger uniqueness theorem in [2] Corollary 5.7 relies upon the amenability of the path groupoid.

Finally, recall that a C*-algebra $C^*(G)$ can be constructed from a groupoid G in the manner of [8]. For a k-graph (Λ, d), $C^*(\Lambda) \cong C^*(G_\Lambda)$ and the two are often tacitly identified.

Section 5 of [2] proves that for a general row-finite KPf $F : \mathcal{E} \to \mathcal{B}$ with the added condition that $\mathcal{B}$ is left-cancellative, the same isomorphism $C^*(F) \cong C^*(G_F)$ holds. Said isomorphism is given by $s_\alpha \mapsto 1_{Z(\alpha,s(\alpha))}$ for $\alpha \in \mathcal{E}$.

This concludes the review of the background work completed on discrete Conduche fibrations and Kumjian-Pask fibrations as generalizations of k-graphs. We proceed to further results on properties of dCfs and KPfs.

1.4 The Core of $C^*(F)$

Since it was not explicitly established in [2], we prove here that, as for k-graphs, $C^*(F) = \overline{\operatorname{span}}\{s_\alpha s_\beta^* : s(\alpha) = s(\beta)\}$ when F is a KPf.

Proposition 1.4.1. *Let $F : \mathcal{E} \to \mathcal{B}$ be a row-finite, strongly surjective dCf targeting a strongly right Ore category. For $\lambda, \mu \in \mathcal{E}$, every nonzero word in s_λ, s_μ^* may be written as a finite sum of partial isometries of the form $s_\alpha s_\beta^*$ where $s(\alpha) = s(\beta)$. Their linear span forms a dense $*$-subalgebra of $C^*(F)$.*

Proof. Observe that since $\mathrm{Id}_{r(\mu)}\, \mu = \mu = \mu\, \mathrm{Id}_{s(\mu)}$ in a category, Conditions 2 and 3 of Definition 1.1.4 yield $s_\mu = s_\mu p_{s(\mu)} = p_{r(\mu)} s_\mu$. If $a \in \mathcal{B}$ with $r(a) = F(s(\mu)))$, we will also make use of the expansion

$$s_\mu s_\mu^* = s_\mu p_{s(\mu)} s_\mu^* = s_\mu \left(\sum_{r(\eta)=s(\mu), F(\eta)=a} s_\eta s_\eta^* \right) s_\mu^* = \sum_{r(\eta)=s(\mu), F(\eta)=a} s_{\mu\eta} s_{\mu\eta}^*$$

deduced from Condition 6 applied to $s(\mu)$.

It suffices to consider the possible outcomes of the product $s_\lambda s_\mu^*$. Since the object projections are mutually orthogonal, we have $s_\lambda^* s_\mu = s_\lambda^* p_{r(\lambda)} p_{r(\mu)} s_\mu = 0$ unless $r(\lambda) = r(\mu)$.

Now assuming $r(\lambda) = r(\mu)$, we have two cases to consider. First, if $F(\lambda) = F(\mu)$, then $s_\lambda^* s_\mu = p_{s(\lambda)}$ if $\mu = \lambda$ and $s_\lambda^* s_\mu = 0$ if $\mu \neq \lambda$.

If $F(\lambda) \neq F(\mu)$, then since $r(\lambda) = r(\mu)$, $F(\lambda)$ and $F(\mu)$ form a cospan in $\mathcal{B}$, so there exist $a, b \in \mathcal{B}$ such that $F(\lambda)a = F(\mu)b$. Then

$$\begin{aligned} s_\lambda^* s_\mu &= s_\lambda^*(s_\lambda s_\lambda^*)(s_\mu s_\mu^*)s_\mu \\ &= s_\lambda^* \left(\sum_{r(\alpha)=s(\lambda),F(\alpha)=a} s_{\lambda\alpha}s_{\lambda\alpha}^* \right) \left(\sum_{r(\beta)=s(\mu),F(\beta)=b} s_{\mu\beta}s_{\mu\beta}^* \right) s_\mu, \end{aligned}$$

and when expanded, we have terms of the form $s_\lambda^* s_{\lambda\alpha}s_{\lambda\alpha}^* s_{\mu\beta}s_{\mu\beta}^* s_\mu$ which are zero unless α and β are such that $\lambda\alpha = \mu\beta$. Denoting for convenience $I_{a,b} := \{(\alpha, \beta) : \lambda\alpha = \mu\beta, F(\alpha) = a, F(\beta) = b\}$, the sum reduces to

$$\begin{aligned} \sum_{(\alpha,\beta)\in I_{a,b}} s_\lambda^* s_{\lambda\alpha}s_{\lambda\alpha}^* s_{\lambda\alpha}s_{\mu\beta}^* s_\mu &= \sum_{(\alpha,\beta)\in I_{a,b}} s_\lambda^* s_{\lambda\alpha}s_{\mu\beta}^* s_\mu \\ &= \sum_{(\alpha,\beta)\in I_{a,b}} s_\lambda^* s_\lambda s_\alpha s_\beta^* s_\mu^* s_\mu \\ &= \sum_{(\alpha,\beta)\in I_{a,b}} p_{s(\lambda)}s_\alpha s_\beta^* p_{s(\mu)}. \end{aligned}$$

In these terms, since $\lambda\alpha = \mu\beta$, we have $p_{s(\lambda)}s_\alpha = p_{r(\alpha)}s_\alpha = s_\alpha$, $s_\beta^* p_{s(\mu)} = s_\beta^*$, and $s(\alpha) = s(\beta)$. To summarize,

$$s_\lambda^* s_\mu = \begin{cases} p_{s(\lambda)} & \text{if } \mu = \lambda \\ \sum_{(\alpha,\beta)\in I_{a,b}} s_\alpha s_\beta^* & \text{if } r(\lambda) = r(\mu) \text{ and } F(\lambda) \neq F(\mu) \\ 0 & \text{otherwise} \end{cases}$$

where $I_{a,b} = \{(\alpha, \beta) : F(\alpha) = a, F(\beta) = b, \lambda\alpha = \mu\beta\}$. Thus $C^*(F)$ has standard generators of the form $s_\alpha s_\beta^*$ where $s(\alpha) = s(\beta)$. $\square$

Definition 1.4.1. Let $F : \mathcal{E} \to \mathcal{B}$ be a row-finite, strongly surjective dCf. The *core* of $C^*(F)$ is defined as $\overline{\operatorname{span}}\{s_\alpha s_\beta^* : s(\alpha) = s(\beta) \text{ and } F(\alpha) = F(\beta)\}$.

Proposition 1.4.2. *The core as defined above is a C^*-subalgebra of $C^*(F)$.*

Proof. We use a similar method to show that, if $s_\lambda s_\mu^*$ and $s_\alpha s_\beta^*$ are such that $s(\lambda) = s(\mu), F(\lambda) = F(\mu)$, and $s(\alpha) = s(\beta), F(\alpha) = F(\beta)$, then $(s_\lambda s_\mu^*)(s_\alpha s_\beta^*)$ is also in the core. First note that the product is zero unless $r(\alpha) = r(\mu)$. If in fact $\mu = \alpha$, then

$$(s_\lambda s_\mu^*)(s_\alpha s_\beta^*) = s_\lambda p_{s(\mu)} s_\beta^* = s_\lambda s_\beta^*,$$

which satisfies $s(\lambda) = s(\beta)$ and $F(\lambda) = F(\beta)$.

Now assume $\mu \neq \alpha$. If we still have $F(\alpha) = F(\mu)$, the product is again zero. If $F(\alpha) \neq F(\mu)$, they form a cospan in $\mathcal{B}$ and there exist $a, b \in \mathcal{B}$ such that $F(\mu)a = F(\alpha)b$. Then using the same argument with extensions, we find

$$s_\lambda s_\mu^* s_\alpha s_\beta^* = \sum_{(\eta,\xi)\in I_{a,b}} s_{\lambda\eta} s_{\beta\xi}^*,$$

where $I_{a,b} = \{(\eta, \xi) : r(\eta) = s(\mu), r(\xi) = s(\alpha), F(\eta) = a, F(\xi) = b, \mu\eta = \alpha\xi\}$. Each of these terms has $s(\lambda\eta) = s(\eta) = s(\xi) = s(\beta\xi)$ and

$$F(\lambda\eta) = F(\lambda)a = F(\mu)a = F(\alpha)b = F(\beta)b = F(\beta\xi).$$

Thus the core is a $*$-subalgebra of $C^*(F)$, and since it is the closed span, it is a C^*-subalgebra, and written $C^*(\{s_\alpha s_\beta^* : s(\alpha) = s(\beta), F(\alpha) = F(\beta)\})$. □

Let $b \in \mathcal{B}$ be a morphism and denote $\mathscr{F}_b := C^*(\{s_\alpha s_\beta^* : s(\alpha) = s(\beta), F(\alpha) = F(\beta) = b\})$. For $X \in \mathrm{Obj}(\mathcal{E})$, define $\mathscr{F}_b(X) := C^*(\{s_\alpha s_\beta^* : s(\alpha) = s(\beta) = X, F(\alpha) = F(\beta) = b\})$, which will be empty if $F(X) \neq s(b)$.

Recall that a C*-algebra is called *AF* if it is the direct limit of an increasing sequence of finite-dimensional C*-subalgebras. Note that elsewhere in the literature (see [9]) the definition may be an inductive limit over any directed set, which coincides with the usual definition in the separable case.

Theorem 1.4.2. *Let $F : \mathcal{E} \to \mathcal{B}$ be a row-finite, strongly surjective KPf. For $b \in \mathcal{B}$ and $X \in \mathcal{E}$, we have*

$$\mathscr{F}_b(X) \cong \mathcal{K}\left(\ell^2(\{\alpha \in \mathcal{E}X : F(\alpha) = b\})\right)$$

and $\mathscr{F}_b \cong \bigoplus_{X\in\mathcal{E}} \mathscr{F}_b(X)$. Moreover, if $\mathcal{B}$ is countable and can be seen as a directed set equipped with a preorder $\leq$ such that $b_1 \leq b_2$ for $b_1, b_2 \in \mathcal{B}$ implies the existence of $a \in \mathcal{B}$ such that $b_1 a = b_2$, then the C^*-algebras $\mathscr{F}_b, b \in \mathcal{B}$, form a directed system and

$$C^*(\{s_\alpha s_\beta^* : s(\alpha) = s(\beta), F(\alpha) = F(\beta)\}) \cong \overline{\bigcup_{b\in\mathcal{B}} \mathscr{F}_b}$$

is AF.

Proof. By noticing that if $F(\alpha) = F(\beta) = F(\mu) = F(\lambda) = b$,

$$s_\lambda s_\mu^* s_\alpha s_\beta^* = \delta_{\mu,\alpha} s_\lambda s_\beta^*,$$

we can fix $X \in \mathcal{E}$ and map $s_\alpha s_\beta^* \in \mathscr{F}_b(X)$ to the rank-one matrix units $E_{\alpha,\beta}$ in $\mathcal{K}\left(\ell^2(\{\alpha \in \mathcal{E}X : F(\alpha) = b\})\right)$. This map extends to an isomorphism on $\mathscr{F}_b(X)$. If $\mathscr{F}_b(X)$ is empty, so is the set $(\{\alpha \in \mathcal{E}X : F(\alpha) = b\}$.

Then from this follows the isomorphism $\mathscr{F}_b = \bigoplus_{X\in\mathcal{E}} \mathscr{F}_b(X)$, and these C*-algebras are AF since the $\mathcal{K}(\ell^2)$ is always AF.

Now, if we have a preorder $\leq$ on a countable $\mathcal{B}$ such that $b_1 \leq b_2$ implies the existence of $a \in \mathcal{B}$ such that $b_1 a = b_2$, we show that $\mathscr{F}_{b_1}$ is contained in $\mathscr{F}_{b_2}$ whenever $b_1 \leq b_2$. We use the same expansion argument to say when $s_\lambda s_\mu^* \in \mathscr{F}_{b_1}$,

$$s_\lambda s_\mu^* = s_\lambda p_{s(\lambda)} s_\mu^* = s_\lambda \left(\sum_{\alpha\in s(\lambda)\mathcal{E}:F(\alpha)=a} s_\alpha s_\alpha^* \right) s_\mu^* = \sum_{\alpha\in s(\lambda)\mathcal{E}:F(\alpha)=a} s_{\lambda\alpha} s_{\mu\alpha}^*.$$

Each term in the sum has $s(\lambda\alpha) = s(\mu\alpha)$ and $F(\lambda\alpha) = F(\lambda)a = F(\mu)a = b_1 a = b_2$, and thus we can write the generators of $\mathscr{F}_{b_1}$ as sums of terms from $\mathscr{F}_{b_2}$, and $\mathscr{F}_{b_1} \subset \mathscr{F}_{b_2}$. Then the core subalgebra is the direct limit of the $\mathscr{F}_b$ C*-algebras over $\mathcal{B}$ seen as a directed set. As a direct limit of an increasing sequence of AF algebras, the core is AF under these conditions. $\square$

The final conditions hold in the case of many examples considered in this dissertation, such as when $\mathcal{B}$ is chosen to be infinite sequences of natural numbers with finitely many nonzero entries (see Chapter 5), or non-negative rational numbers (see Chapter 6). When $\mathcal{B}$ is the lattice of open sets on a topological space (see Section 4.1), we have the factorability condition, though $\mathcal{B}$ may or may not be countable and care should be used in the definition of AF.

Chapter 2

Constructions for dCfs and KPfs

This chapter covers basic theory on dCfs and KPfs not previously published. The chapter begins with the consequences of composing of dCfs and KPfs, including the special cases involving quotient categories. We then generalize several constructions for k-graphs that were given in [1], as far as possible. These include a generalization and expansion of "pullback graphs" and a generalization of skew products, which in some examples prove useful for analyzing the amenability of the path groupoid. Finally, we consider group actions and a KPf induced by a group action on the skew product.

2.1 Compositions of dCfs and KPfs

Proposition 2.1.1. *Let $F : \mathcal{E} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be functors, and let their composition be denoted $GF : \mathcal{E} \to \mathcal{C}$.*

(i) If F and G are dCfs, then GF is a dCf. Also, if GF is a dCf, then F is a dCf.

(ii) If F and G are row-finite, then so is GF. Also, if GF is row-finite, then so is F.

(iii) GF is surjective on objects if and only if F and G are both surjective on objects.

(iv) If F and G are KPfs, then GF is a KPf.

Proof. (i) Assuming F and G are dCfs, we wish to show the unique lifting property for GF. Let $\phi \in \mathcal{E}$ and suppose $GF(\phi) = ab$ is a factorization of the image in $\mathcal{C}$ under the composition of the dCfs. Consider $F(\phi) \in \mathcal{B}$ so that ab is a factorization of $G(F(\phi))$. Since G is a dCf, there is a unique lift to a factorization $F(\phi) = a'b'$ in $\mathcal{B}$ where $G(a') = a, G(b') = b$. Then $a'b'$ is a factorization of $F(\phi)$ in $\mathcal{B}$ and since F is also a dCf, there is a unique lift $\phi = \alpha\beta$ in $\mathcal{E}$ with $F(\alpha) = a', F(\beta) = b'$. Thus the α, β are the desired unique lift from a, b and GF has the unique lifting property of a dCf.

Next, assume GF is a dCf and suppose F fails the unique lifting property. That is, let $\phi \in \mathcal{E}$ be such that $F(\phi) = ab$ is a factorization in $\mathcal{B}$ with distinct lifts $\phi = \alpha\beta = \alpha'\beta'$ with $F(\alpha) = F(\alpha') = a, F(\beta) = F(\beta') = b$. The same ϕ carried through to $GF(\phi) = G(ab) = G(a)G(b)$ forms a factorization for which $\alpha\beta$ and $\alpha'\beta'$ are both lifts while $\alpha \neq \alpha'$ and $\beta \neq \beta'$, yielding a contradiction.

(ii) Assuming F and G are row-finite, take $GF(X)$ for an object $X \in \mathcal{E}$ and consider a morphism $b \in \mathcal{C}$ targeting $GF(X)$. We need that the set of morphisms $B := \{\beta \in \mathcal{E} : GF(\beta) = b\}$ is finite. Since G is row-finite, the set $B' := \{b' \in \mathcal{B} : G(b') = b\}$ is a finite set. Next, since F is also row-finite,

$$B = \bigcup_{b' \in B'} \{\beta \in \mathcal{E} : F(\beta) = b'\}$$

is a finite union of finite sets, and thus B is finite.

Next, assume GF is row-finite and suppose F is not row-finite. That is, there exists $X \in \mathrm{Obj}(\mathcal{E})$ and $b \in F(X)\mathcal{B}$ such that the set $\{\alpha \in X\mathcal{E} : F(\alpha) = b\}$ is not finite. Then $G(b) \in C$ targets $GF(X) \in \mathrm{Obj}(\mathcal{C})$ with the same set of preimages, and we have a contradiction.

(iii) Assuming F and G are both surjective on objects, we wish to show that for any $y \in \mathcal{C}$, there exists a $Y \in \mathcal{E}$ such that $GF(Y) = y$. Since G is surjective on objects, there exists $y' \in \mathcal{B}$ such that $G(y') = y$, and since F is surjective on objects, there exists $Y \in \mathcal{E}$

such that $F(Y) = y'$. This $Y \in \mathcal{E}$ satisfies $GF(Y) = y$.

Conversely, suppose that F is not surjective on objects, so that there exists $Y \in \mathcal{B}$ such that no $X \in \mathrm{Obj}(\mathcal{E})$ has $F(X) = Y$. Then $G(Y) \in \mathrm{Obj}(\mathcal{C})$ has no preimage in $\mathcal{E}$. On the other hand, if G is not surjective on objects, then GF certainly is not.

(iv) If F and G are assumed to be KPfs, that is, row-finite locally split dCfs targeting strongly right Ore categories $\mathcal{B}$ and $\mathcal{C}$, respectively, then $GF : \mathcal{E} \to \mathcal{C}$ is also a KPf. To show that GF is locally split, consider the induced functor GF between slice categories $\mathcal{E}/Y \to \mathcal{C}/GF(Y)$. This functor admits a splitting $x_{GF} : \mathcal{C}/GF(Y) \to \mathcal{E}/Y$ given by $x_F \circ x_G$, where x_G is the splitting from $\mathcal{C}/GF(Y)$ to $\mathcal{B}/F(Y)$ guaranteed by G being locally split, and similarly $x_F : \mathcal{B}/F(Y) \to \mathcal{E}/Y$ is a splitting for $F : \mathcal{E}/Y \to \mathcal{B}/F(Y)$. Indeed, for $c \in \mathcal{C}/GF(Y)$,

$$GF \circ x_{GF}(c) = G \circ F \circ x_F \circ x_G(c) = G \circ \mathrm{Id}_{\mathcal{B}/F(Y)} \circ x_G(c) = c,$$

showing that $GF \circ x_{GF} = \mathrm{Id}_{\mathcal{C}/GF(Y)}$.

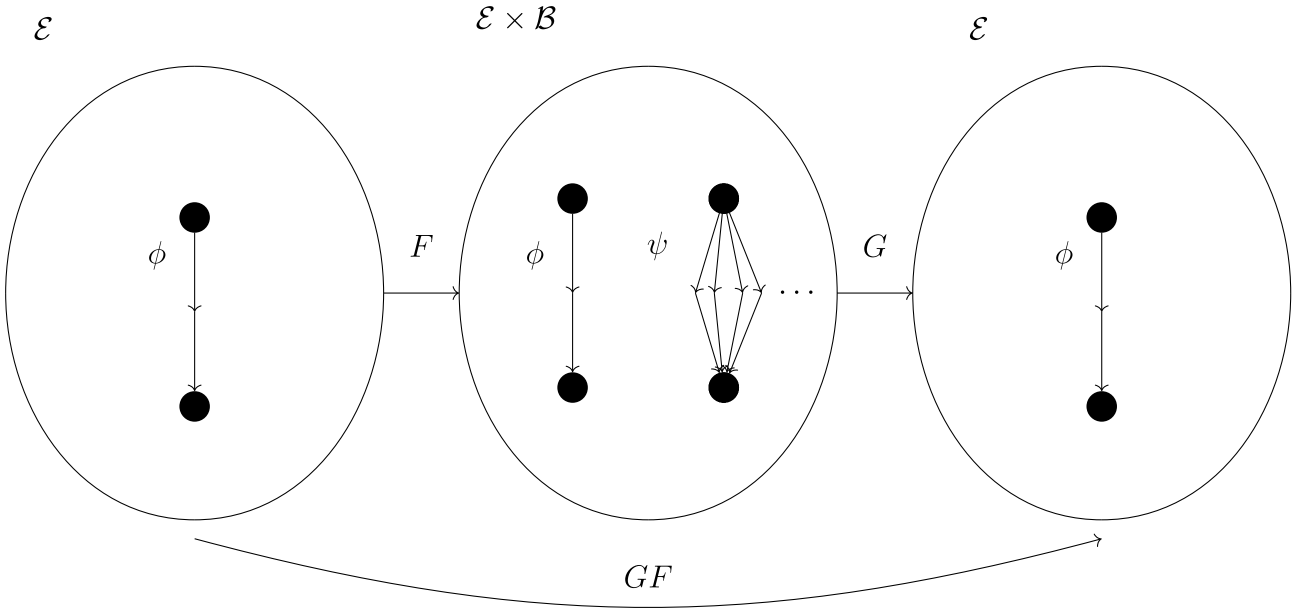
□

Note that as long as F and G are locally split, they need not be surjective on objects, as any $y \in \mathcal{C}$ not in the image of G is also not in the image of GF. If they are surjective on objects in addition to being locally split, they are strongly surjective and we may consider the C*-algebras $C^*(F), C^*(G)$, and $C^*(GF)$.

Example 2.1.1. In particular, if $F : \mathcal{E} \to \Lambda$ is a KPf targeting a k-graph (Λ, d) regarded as a category with degree functor $d : \Lambda \to \mathbb{N}^k$, then $\mathcal{E}$ can be seen as a k-graph with degree map $d \circ F : \mathcal{E} \to \mathbb{N}^k$.

Note the lack of biconditionals in Proposition 2.1.1. It is possible for GF to be a dCf while G is not. There may be instances in which for some $\phi \in \mathcal{B}$, there is a factorization $G(\phi) = ab$ in $\mathcal{C}$ with two lifts α, β and α', β' in $\mathcal{B}$ such that $F(\alpha) = F(\alpha') = a$ and $F(\beta) = F(\beta') = b$ and $\alpha\beta = \phi = \alpha'\beta'$ while $\alpha \neq \alpha'$ and $\beta \neq \beta'$. But if F happens to be a non-surjective dCf and ϕ is not in the image of F, this does not cause a breakdown in the unique lifting property on GF.

See below a simplistic example of this happening. Suppose F maps from $\mathcal{E}$ into a copy of $\mathcal{E}$ in a new category which also contains a pathological ψ consisting of commutative squares $\psi = \alpha_1\beta_1 = \alpha_2\beta_2 = \cdots$ in which $\alpha_i \neq \alpha_j$ and $\beta_i \neq \beta_j$ for all $i \neq j$. Although not surjective, $F : \mathcal{E} \to (\mathcal{E} \times \mathcal{B})$ is a KPf. Now suppose G takes $G(\phi) = G(\psi) = \phi$ back to $\mathcal{E}$. Now $G : \mathcal{E} \times \mathcal{B} \to \mathcal{E}$ is neither row-finite nor a dCf, and yet $GF = \mathrm{Id}_{\mathcal{E}}$, which is a KPf.



The following discussion shows that if F is a row-finite, strongly surjective dCf and $\{s_\alpha, p_X\}$ is a Cuntz-Krieger system associated to F, then $\{s_\alpha, p_X\}$ is not generally a Cuntz-Kreiger system associated to GF. Explicitly in this case, we have the following definition of such a system.

Definition 2.1.2. Given strongly surjective, row-finite functors $F : \mathcal{E} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$, a Cuntz-Krieger system associated to $GF : \mathcal{E} \to \mathcal{C}$ is the following data in a C*-algebra D:

- A projection P_X for each object X of $\mathcal{E}$,
- A partial isometry S_α for each morphism $\alpha : Y \to X$ of $\mathcal{E}$,

satisfying the relations

1. For $X \neq Y$, $P_X \perp P_Y$,

2. If α and β are composable, then $S_{\alpha\beta} = S_\alpha S_\beta$,

3. For all X, $P_X = S_{\mathrm{Id}_X} = S^*_{\mathrm{Id}_X}$,

4. For all $\alpha : Y \to X$, $S^*_\alpha(S_\alpha) = P_Y$,

5. If $GF(\alpha) = GF(\beta)$ and $\alpha \neq \beta$ then $S^*_\beta(S_\alpha) = 0$, which may arise from non-injective F or non-injective G.

6. For all X, and for all morphisms $b : C \to GF(X)$ in $\mathcal{C}$,

$$\sum_{\{\alpha \in X\mathcal{E} : GF(\alpha) = b\}} S_\alpha S^*_\alpha = P_X.$$

But in relation (6), the set $\{\alpha \in X\mathcal{E} : GF(\alpha) = b\}$ is finite, and can be written

$$\bigcup_{b' \in F(X)\mathcal{B} : G(b') = b} \{\alpha \in X\mathcal{E} : F(\alpha) = b'\}$$

as discussed in the proof of the proposition. By re-organizing the sum,

$$P_X = \sum_{b' \in F(X)\mathcal{B} : G(b') = b} \left(\sum_{\alpha \in X\mathcal{E} : F(\alpha) = b'} S_\alpha S^*_\alpha \right).$$

But if $S_\alpha S^*_\alpha = s_\alpha s^*_\alpha$ where $\{s_\alpha, p_X\}$ is a Cuntz-Krieger system for F, each summand is a copy of P_X, yielding

$$P_X = \sum_{b' \in F(X)\mathcal{B} : G(b') = b} P_X.$$

So $\{s_\alpha, p_X\}$ cannot be a Cuntz-Krieger system for GF unless G is injective.

2.2 Induced Quotient Categories

We recall the definition of a quotient from category theory.

Definition 2.2.1. For a category $\mathcal{C}$, a *congruence relation* R on $\mathcal{C}$ is given by, for each pair of objects $X, Y \in \mathcal{C}$, an equivalence relation $R_{X,Y}$ on $\mathrm{Hom}(X, Y)$ such that the equivalence relations respect composition. That is, if $f_1 R_{X,Y} f_2$ for $f_1, f_2 \in \mathrm{Hom}(X, Y)$ and $g_1 R_{Y,Z} g_2$ for $g_1, g_2 \in \mathrm{Hom}(Y, Z)$, then $g_1 f_1 R_{X,Z} g_2 f_2$. The *quotient category* $\mathcal{C}/R$ has objects given by $\mathrm{Obj}(\mathcal{C})$ and morphisms given by equivalence classes of morphisms in $\mathcal{C}$. That is, $\mathrm{Hom}_{\mathcal{C}/R}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)/R_{X,Y}$. We may refer to the natural quotient map $q : \mathcal{C} \to \mathcal{C}/R$ defined by $q(\alpha) = [\alpha]$, the equivalence class under equivalence relation $R_{X,Y}$.

A functor $F : \mathcal{E} \to \mathcal{B}$ induces a congruence relation R_F on $\mathcal{E}$, which produces a special quotient category, $\mathcal{E}/R_F$. Namely, R_F is the congruence relation with equivalence classes defined by $\alpha' \in [\alpha]$ when $F(\alpha') = F(\alpha)$ in $\mathcal{B}$. Denote by q_F the quotient map $\alpha \mapsto [\alpha]$. We can induce a functor $F_q : \mathcal{E}/R_F \to \mathcal{B}$ defined by $F_q([\alpha]) = F(\alpha)$ (which is well defined since $F(\alpha) = F(\alpha')$ for any $\alpha' \in [\alpha]$). We claim this functor F_q is a dCf.

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "F"] \arrow[d, "q_F"'] & \mathcal{B} \\
\mathcal{E}/R_F \arrow[ur, dashed, "F_q"'] &
\end{tikzcd}$$

Proposition 2.2.1. *Let F, q_F, and F_q be defined as above using the congruence relation R_F. If F is a dCf, then F_q is a dCf. Also, F is a dCf if and only if q_F is a dCf. Finally, if F is a KPf, then F_q is a KPf.*

Proof. We first show that F_q satisfies the unique lifting property, assuming that F was a dCf. Suppose $[\phi]$ is a morphism in $\mathcal{E}/R_F$ and let $F_q([\phi]) = F(\phi) = ab$ be a factorization in $\mathcal{B}$. By unique lifting of F, there exist unique $\alpha, \beta \in \mathcal{E}$ such that $\phi = \alpha\beta$, $F(\alpha) = a$, and $F(\beta) = b$. We then take $[\alpha], [\beta]$ as the lift in $\mathcal{E}/R_F$ such that $[\phi] = [\alpha][\beta]$ and $F_q([\alpha]) =$

$F(\alpha) = a$ and $F_q([\beta]) = F(\beta) = b$. That lift is indeed unique, since if there were to be distinct morphisms $[\alpha'], [\beta']$ in $\mathcal{E}/R_F$ that still satisfied the requirements, we would have $F_q([\alpha']) = F_q([\alpha]) = F(\alpha') = F(\alpha)$ showing that in fact $\alpha' \in [\alpha]$ and the equivalence classes were not distinct.

For the second statement, the ($\implies$) direction is clear from the previous section's Proposition 2.1.1(i) as the q_F is the first step in the composition $F = F_q \circ q_F$. We show the ($\impliedby$) direction by contrapositive. Suppose F is not a dCf, so there is some $\phi \in \mathcal{E}$ with $F(\phi) = ab$ a factorization in $\mathcal{B}$ which lifts back to $\phi = \alpha\beta = \alpha'\beta'$ in $\mathcal{E}$ such that $\alpha \neq \alpha', \beta \neq \beta'$, but $F(\alpha) = F(\alpha') = a, F(\beta) = F(\beta') = b$. (In other words, ϕ is a commuting square whose composands are nonequal.) Then $q_F(\phi) = [\phi]$, the equivalence class of morphisms $\phi' \in \mathcal{E}$ such that $F(\phi) = F(\phi')$. By the definition of a congruence relation, we must have $[\alpha][\beta] = [\alpha\beta] = [\phi]$, showing that $[\alpha][\beta]$ is a factorization of $[\phi] = q_F(\phi)$ in $\mathcal{E}/R_F$. Now there is a lift in $\mathcal{E}$ using α, β since $\alpha\beta = \phi$ and $q_F(\alpha) = [\alpha], q_F(\beta) = [\beta]$. But there is also a lift using α', β' because $\alpha'\beta' = \phi$ and $F(\alpha) = F(\alpha')$ shows that $q_F(\alpha') = [\alpha]$ (similarly $q_F(\beta') = [\beta]$). Therefore, q_F is not a dCf. Equivalently, if q_F satisfies the unique lifting condition, then F does.

For the final statement, it suffices to show that F_q is locally split. For $(\mathcal{E}/R_F)/[X]$, the slice category over an object $[X] \in \mathcal{E}/R_F$, we have $F_q([X]) = F(X)$ in $\mathcal{B}$, and we are guaranteed a splitting from $\mathcal{B}/F(X)$ to $\mathcal{E}/X$, say $x_F \in F^\infty$. Then we define a desired splitting $x : \mathcal{B}/F_q([X]) \to (\mathcal{E}/R_F)/[X]$ by

$$x(a, b) = (q_F(x_F(a)), q_F(x_F(b)))$$

for $(a, b) \in \mathcal{B}/F(X)$. $\square$

2.3 G^*F Induced by a KPf and a Functor

Kumjian and Pask define a construction using a k-graph and a monoid morphism, which we reproduce here from [1].

Definition 2.3.1 ([1] Definition 1.9)**.** Let $f : \mathbb{N}^\ell \to \mathbb{N}^k$ be a monoid morphism, then if (Λ, d) is a k-graph we may form the ℓ-graph $f^*(\Lambda)$ as follows: (the objects of $f^*(\Lambda)$ may be identified with those of Λ and) $f^*(\Lambda) = \{(\lambda, n) : d(\lambda) = f(n)\}$ with $d(\lambda, n) = n$, $s(\lambda, n) = s(\lambda)$, and $r(\lambda, n) = r(\lambda)$.

We create a generalization of this construction, culminating in a version of the definition for the KPf case.

Let $F : \mathcal{E} \to \mathcal{B}$ be a KPf and let $G : \mathcal{C} \to \mathcal{B}$ be a functor from a strongly right Ore category $\mathcal{C}$.

$$\begin{tikzcd} \mathcal{E} \arrow[r, "F"] & \mathcal{B} & \mathcal{C} \arrow[l, "G"'] \\ & \mathcal{D} \arrow[ur, dashed, bend right, "G^*F"'] & \end{tikzcd}$$

We induce a new functor G^*F from a new category $\mathcal{D} = \{(\alpha, a) \in \mathcal{E} \times \mathcal{C} : F(\alpha) = G(a)\}$ to target $\mathcal{C}$, defined by $G^*F(\alpha, a) = a$. The source and range maps for $\mathcal{D}$ will be $s(\alpha, a) = (s(\alpha), s(a))$ and $r(\alpha, a) = (r(\alpha), r(a))$, since

$$F(s(\alpha)) = s(F(\alpha)) = s(G(a)) = G(s(a))$$

shows that such pairs do appear as objects in $\mathcal{D}$. Composition in $\mathcal{D}$ is possible when α, β are composable in $\mathcal{E}$ and a, b are composable in $\mathcal{C}$, and then $(\alpha, a)(\beta, b) = (\alpha\beta, ab)$ since $F(\alpha\beta) = F(\alpha)F(\beta) = G(a)G(b) = G(ab)$.

To see that G^*F is a dCf, we show the unique lifting property. Suppose $(\phi, p) \in \mathcal{D}$ so we know $F(\phi) = G(p) \in \mathcal{B}$. Then $G^*F(\phi, p) = p$ in $\mathcal{C}$. Suppose there is a factorization $p = ab$ in $\mathcal{C}$. Since G is a functor, $G(p) = G(ab) = G(a)G(b)$ is a factorization of the morphism

$G(p) = F(\phi)$ in $\mathcal{B}$. Since F is a KPf, the factorization lifts uniquely to $\phi = \alpha\beta$ for some $\alpha, \beta \in \mathcal{E}$ such that $F(\alpha) = G(a)$ and $F(\beta) = G(b)$. Then (α, a) and (β, b) appear in $\mathcal{D}$ and can be taken as the unique lifts for G^*F. Clearly $(\alpha, a)(\beta, b) = (\alpha\beta, ab) = (\phi, p)$ and $G^*F(\alpha, a) = a$ and $G^*F(\beta, b) = b$.

We now wish to show that G^*F is a KPf; that is, a row-finite locally split dCf targeting a strongly right Ore category.

G^*F is row-finite: Let $(X, x) \in \mathcal{D}$ and $(G^*F)(X, x) = x$ in $\mathcal{C}$. Let b be a morphism targeting x in $\mathcal{C}$. We wish to show the set of morphisms $(\alpha, a) \in (X, x)\mathcal{D}$ which are taken to b by G^*F is a finite set. Since $G^*F(\alpha, a) = a$, instead write these morphisms as (β, b). In these pairs, $F(\beta) = G(b)$, and the set of morphisms $\beta \in X\mathcal{E}$ taken to $G(b)$ by F is finite by the fact that F is row-finite. Thus G^*F is row-finite when F is.

G^*F being locally split again follows from F being a KPf. To construct the necessary splittings, let $(Y, y) \in \mathrm{Obj}(\mathcal{D})$ and consider the induced G^*F from $\mathcal{D}/(Y, y)$ to $\mathcal{C}/G^*F(Y, y) = \mathcal{C}/y$. We want an $x : \mathcal{C}/y \to \mathcal{D}/(Y, y)$ such that $F \circ x = \mathrm{Id}$. For $a \in \mathcal{C}/y$, we want its image under x to be some pair $(\alpha, a) \in \mathcal{D}/(Y, y)$ where $F(\alpha) = G(a)$, so $G^*F(\alpha, a) = a$. Since $G(y) = F(Y)$, the slice category $\mathcal{B}/G(y)$ is $\mathcal{B}/F(Y)$, so take a splitting $x_F : \mathcal{B}/F(Y) \to \mathcal{E}/Y$ and consider $\alpha = x_F(G(a))$. Since $F(\alpha) = F(x_F(G(a))) = G(a)$, this provides our pair (α, a). So define $x(a) = (x_F(G(a)), a)$ to be a splitting for the induced G^*F on the slice categories.

We are ready to give the generalized definition of this construction.

Definition 2.3.2. Let $F : \mathcal{E} \to \mathcal{B}$ be a KPf and let $G : \mathcal{C} \to \mathcal{B}$ be a functor from a strongly Right Ore category $\mathcal{C}$. We construct a new KPf $G^*F : \mathcal{D} \to \mathcal{C}$, where

$$\mathcal{D} = \{(\alpha, a) \in \mathcal{E} \times \mathcal{C} : F(\alpha) = G(a)\}$$

is a category with range and source maps $r(\alpha, a) = (r(\alpha), r(a)), s(\alpha, a) = (s(\alpha), s(a))$, and specifically $G^*F(\alpha, a) = a$.

A notable difference between the general case and the k-graph case is that in the latter, the objects of $f^*(\Lambda)$ could be identified with the objects of Λ, because $\mathbb{N}^k$ is a one-object category. However, in the general case, the objects of $\mathcal{D}$ are pairs (X, x) where $X \in \mathrm{Obj}(\mathcal{E}), x \in \mathrm{Obj}(\mathcal{C})$, and $F(X) = G(x)$ in $\mathrm{Obj}(\mathcal{B})$. If G is not injective, there may exist multiple objects (X, x_i) for each $X \in \mathrm{Obj}(\mathcal{E})$ where $G(x_i) = F(X)$.

In the case where G is also a KPf and $\mathcal{E}$ is strongly right Ore, F^*G exists as a KPf from $\mathcal{D}'$ to $\mathcal{E}$, where

$$\mathcal{D}' = \{(a, \alpha) \in \mathcal{C} \times \mathcal{E} : F(\alpha) = G(a)\}$$

has range and source maps $r(a, \alpha) = (r(a), r(\alpha)), s(a, \alpha) = (s(a), s(\alpha))$, and $F^*G(a, \alpha) = \alpha$. Since $\mathcal{D} \cong \mathcal{D}'$ by $(\alpha, a) \mapsto (a, \alpha)$, we have the diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "F"] & \mathcal{B} & \mathcal{C} \arrow[l, "G"'] \\
& \mathcal{D} \cong \mathcal{D}' \arrow[ul, "F^*G"] \arrow[u, dashed, "H"] \arrow[ur, "G^*F"'] &
\end{tikzcd}$$

in which $H = G \circ G^*F = F \circ F^*G$. Indeed, $G \circ G^*F(\alpha, a) = G(a) = F(\alpha) = F \circ F^*G(a, \alpha)$.

The following proposition shows that [1] Proposition 1.11 still holds in the general case.

Proposition 2.3.1. *Let $F : \mathcal{E} \to \mathcal{B}$ be a (strongly surjective) KPf and $G : \mathcal{C} \to \mathcal{B}$ a functor from a strongly right Ore category $\mathcal{C}$. Then there is a $*$-homomorphism $\pi_G : C^*(G^*F) \to C^*(F)$ such that $s_{(\alpha,a)} \mapsto s_\alpha$. Moreover, if G is surjective onto $\mathcal{B}$, then π_G is surjective.*

Proof. We prove this by showing that a Cuntz-Krieger system associated to F, $\{s_\alpha : \alpha \in \mathcal{E}\}$, is also a Cuntz-Krieger system in $C^*(G^*F)$ and then using the universal property. We check below the conditions in the definition of a Cuntz-Krieger system.

1. For $(X, x) \neq (Y, y)$, $p_X \perp p_Y$ is clear.
2. If $(\alpha, a), (\beta, b)$ composable, then α, β were composable and so $s_{\alpha\beta} = s_\alpha s_\beta$ still holds.
3. For all (X, x), $p_X = s_{\mathrm{Id}_X} = s^*_{\mathrm{Id}_X}$ still holds.
4. For all $(\alpha, a) : (Y, y) \to (X, x)$, $s_\alpha s^*_\alpha = p_{s(\alpha)} = p_Y$ still holds.

5. If $G^*F(\alpha, a) = G^*F(\beta, b)$ and $(\alpha, a) \neq (\beta, b)$, that is $a = b$ but $\alpha \neq \beta$, then $F(\alpha) = G(a) = F(\beta)$, so $s_\beta^* s_\alpha = 0$.

6. For all (X, x), and for all morphisms b targeting $G^*F(X, x) = x \in \mathcal{C}$,

$$\sum_{(\alpha,a)\in(X,x)\mathcal{D}:G^*F(\alpha,a)=b} s_\alpha s_\alpha^* = \sum_{(\alpha,b)\in(X,x)\mathcal{D}:G^*F(\alpha,b)=b} s_\alpha s_\alpha^* = \sum_{\alpha\in X\mathcal{E}:F(\alpha)=G(b)} s_\alpha s_\alpha^* = p_X.$$

Then the universal property of $C^*(G^*F) = C^*(\{s_{(\alpha,a)}\})$ implies the existence of a $*$-homomorphism π_G such that $s_{(\alpha,a)} \mapsto s_\alpha$.

Now if G is surjective onto $\mathcal{B}$, there must exist $a \in \mathcal{C}$ such that $G(a) = F(\alpha)$ in $\mathcal{B}$, so every generator s_α of $C^*(F)$ is in the image of π_G as the value of $\pi_G(s_{(\alpha,a)})$. Thus π_G is surjective. $\square$

We conclude this section with a discussion of the infinite path space arising from a KPf constructed in this manner.

Recall the construction of at least one splitting of G^*F for any slice category: Let $(Y, y) \in \mathrm{Obj}(\mathcal{D})$ and denote also by G^*F the induced functor between slice categories $\mathcal{D}/(Y, y)$ and $\mathcal{C}/y$. Then G^*F admits a splitting $x : \mathcal{C}/y \to \mathcal{D}/(Y, y)$ defined by $x(a) = (x_F(G(a)), a)$, where $a \in \mathcal{C}/y$ and $x_F : \mathcal{B}/F(Y) \to \mathcal{E}/Y$ is an appropriate splitting of F.

Remark 2.3.3. On the relationship between F^∞ and $(G^*F)^\infty$:

(i) If G is not surjective, there can exist infinite paths $x_F \in F^\infty$ that do not give rise to any $x \in (G^*F)^\infty$. Namely, if $F(X)$ is not in the image of G, any $x_F \in Z(X)$ will never be used.

(ii) If G is surjective, then any $x_F \in F^\infty$ does give rise to an $x \in (G^*F)^\infty$, since if $x_F \in Z(Y)$, we are sure to have a $G(y) = F(Y)$, and sure to have a $(Y, y) \in \mathcal{D}$, and the resulting x will be in $Z(Y, y)$. However, this is not any kind of unique/injective relation.

(iii) If G is not injective, e.g. if there exist objects $(Y, y), (Y, y')$ in $\mathcal{D}$ such that

$$G(y) = G(y') = F(Y),$$

we can have different infinite paths in $Z(Y, y)$ and $Z(Y, y')$ in $(G^*F)^\infty$ all arising from the same x_F.

2.4 Skew Products and Group Actions

We conclude this chapter by showing that the skew product construction from [1] Section 5 is reproducible for the dCf and KPf case, but the application to the path groupoid reaches a roadblock in the general case. The skew product construction is useful in the k-graph case because it leads to information about the structure of the original path groupoid G_Λ for a k-graph Λ. However, we will see that the same strategy cannot be executed in the general case. The section concludes by discussing a group action on the skew product.

Let $F : \mathcal{E} \to \mathcal{B}$ be a dCf and let $c : \mathcal{E} \to \Gamma$ be a functor where Γ is a discrete group. The *skew product* $(\Gamma \times_c \mathcal{E})$ is a category with objects $(g, X) \in \Gamma \times \mathrm{Obj}(\mathcal{E})$ and with morphisms consisting of pairs $(g, \alpha) \in \Gamma \times \mathcal{E}$ with range and source given by

$$s(g, \alpha) = (gc(\alpha), s(\alpha)), \quad r(g, \alpha) = (g, r(\alpha)).$$

If $s(\alpha) = r(\beta)$, then (g, α) and $(gc(\alpha), \beta)$ are composable, and the composition is

$$(g, \alpha)(gc(\alpha), \beta) = (g, \alpha\beta).$$

Now c and F induce a functor $F_c : (\Gamma \times_c \mathcal{E}) \to \mathcal{B}$ defined by $F_c(g, \alpha) = F(\alpha)$. We claim F_c is also a dCf. To show the unique lifting property, suppose (g, μ) is a morphism in $(\Gamma \times_c \mathcal{E})$ whose image under F_c is $F_c(g, \mu) = F(\mu)$. If $F(\mu) = ab$ is a factorization in $\mathcal{B}$, then there exist unique $\alpha, \beta \in \mathcal{E}$ such that $\mu = \alpha\beta$ with $F(\alpha) = a, F(\beta) = b$. Then the lift of the factorization in $(\Gamma \times_c \mathcal{E})$ is given by $(g, \alpha), (gc(\alpha), \beta)$ so that

$$(g, \alpha)(gc(\alpha), \beta) = (g, \alpha\beta) = (g, \mu).$$

Next, assume moreover that $F : \mathcal{E} \to \mathcal{B}$ is a KPf. We claim $F_c : (\Gamma \times_c \mathcal{E}) \to \mathcal{B}$ is also a KPf. We identify $\Gamma \times F^\infty$ with F_c^∞: for $(g, x) \in \Gamma \times F^\infty$, where $x \in Z(X)$, define the splitting $(g, x) : \mathcal{B}/F(X) \to (\Gamma \times_c \mathcal{E})/(g, X)$ when $a, b \in \mathcal{B}/F(X)$ with $r(b) = s(a)$ by

$$(g, x)(a, b) = \big((g, x)_1(a, b), (g, x)_2(a, b)\big) = \big((g, x(a)), (gc(x(a)), x_2(a, b))\big),$$

or, when worried about conservation of parentheses in the wild, as the morphism and factorization

$$(g, x)(ab) = \big(g, x(a)\big)\big(gc(x(a)), x_2(a, b)\big) = (g, x_1(a, b)x_2(a, b)) = (g, x(ab)).$$

Indeed,

$$F_c \circ (g, x)(a, b) = F_c\big((g, x(a)), (gc(x(a)), x_2(a, b))\big) = (F(x(a)), F(x_2(a, b))) = (a, b)$$

since x is a splitting of F. The range object of infinite path (g, x) is $(g, r(x))$, and the endings of (g, x) are morphisms of the form (g, μ), where $x \in Z(\mu)$, since

$$(g, x)(F_c(h, \mu)) = (g, x)(F(\mu)) = (g, x(F(\mu))) = (g, \mu) = (h, \mu)$$

when $h = g$.

If $(g,x) \in F_c^\infty$ ends in (g,μ), then the infinite path $\mathrm{res}_{(g,\mu)}(g,x)$ is defined by

$$\begin{aligned}
\mathrm{res}_{(g,\mu)}(g,x)(a,b) &= \big((g,x)_2(F_c(g,\mu),a),(g,x)_2(F_c(g,\mu)a,b)\big)\\
&= \big((g,x)_2(F(\mu),a),(g,x)_2(F(\mu)a,b)\big)\\
&= \big((gc(x(F(\mu))),x_2(F(\mu),a)),(gc(x(F(\mu)a)),x_2(F(\mu)a,b))\big)\\
&= \big((gc(\mu),x_2(F(\mu),a)),(gc(\mu x(a)),x_2(F(\mu)a,b))\big),
\end{aligned}$$

or, more succinctly, the morphism

$$\begin{aligned}
\mathrm{res}_{(g,\mu)}(g,x)(ab) &= \big(gc(\mu),x_2(F(\mu),a)\big)\big(gc(\mu)c(x(a)),x_2(F(\mu)a,b)\big)\\
&= (gc(\mu),x_2(F(\mu),a)x_2(F(\mu)a,b))\\
&= (gc(\mu),\mathrm{res}_\mu(x)(ab))
\end{aligned}$$

using the composition rule.

Since $s(h,\mu) = (hc(\mu),s(\mu)) = r(g,x) = (g,r(x))$ when $hc(\mu) = g$ and $s(\mu) = r(x)$, we can see $(gc(\mu)^{-1},\mu)$ has as source the same object as the range of (g,x), and then the infinite path $\mathrm{ind}_{(gc(\mu)^{-1},\mu)}(g,x)$ can be defined. This path has range $(gc(\mu)^{-1},r(\mu))$, so for $(a,b) \in \mathcal{B}/F(r(\mu))$, we have that $F_c(gc(\mu)^{-1},\mu) = F(\mu)$ and ab form a cospan and can be completed to a commuting square $F(\mu)c = abd$ for some $c,d \in \mathcal{B}$. Let the unique lifts in $(\Gamma \times_c \mathcal{E})$ of this morphism for these factorizations be denoted

$$(gc(\mu)^{-1},\mu)(g,x)(c) = (h_1,\alpha_1)(h_2,\alpha_2)(h_3,\alpha_3),$$

where $F_c(h_1,\alpha_1) = F(\alpha_1) = a$, and so on. But we must have $\alpha_1\alpha_2\alpha_3$ composable, and we must have $h_2 = h_1c(\alpha_1)$ and $h_3 = h_1c(\alpha_1)c(\alpha_2) = h_1c(\alpha_1\alpha_2)$. Also, to be equal we must have $r(gc(\mu)^{-1},\mu) = (gc(\mu)^{-1}, r(\mu)) = r(h_1,\alpha_1) = (h_1, r(\alpha_1))$. Thus in fact,

$$\left(gc(\mu)^{-1},\mu\right)(g,x)(c) = \left(gc(\mu)^{-1},\alpha_1\right)\left(gc(\mu)^{-1}c(\alpha_1),\alpha_2\right)\left(gc(\mu)^{-1}c(\alpha_1\alpha_2),\alpha_3\right).$$

Then

$$\mathrm{ind}_{(gc(\mu)^{-1},\mu)}(g,x)(a,b) = \left((gc(\mu)^{-1},\alpha_1),(gc(\mu)^{-1}c(\alpha_1),\alpha_2)\right),$$

but noting that $\alpha_1,\alpha_2,\alpha_3$ satisfy $\mu x(c) = \alpha_1\alpha_2\alpha_3$, we see that $\mathrm{ind}_\mu(x)(a,b) = (\alpha_1,\alpha_2)$, so we might rewrite this as a morphism and factorization

$$\begin{aligned}\mathrm{ind}_{(gc(\mu)^{-1},\mu)}(g,x)(ab) &= \left(gc(\mu)^{-1},\alpha_1\right)\left(gc(\mu)^{-1}c(\alpha_1),\alpha_2\right)\\ &= (gc(\mu)^{-1},\alpha_1\alpha_2)\\ &= (gc(\mu)^{-1},\mathrm{ind}_\mu(x)(ab)).\end{aligned}$$

Thus we see the relationship between $\mathrm{res}_{(g,\mu)}(g,x) \in F_c^\infty$ and $\mathrm{res}_\mu(x) \in F^\infty$, and similarly for $\mathrm{ind}_{(gc(\mu)^{-1},\mu)}(g,x) \in F_c^\infty$ and $\mathrm{ind}_\mu(x) \in F^\infty$.

Since res and ind behave as inverses, we confirm that

$$\begin{aligned}\mathrm{res}_{(gc(\mu)^{-1},\mu)} \circ \mathrm{ind}_{(gc(\mu)^{-1},\mu)}(g,x)(ab) &= \mathrm{res}_{(gc(\mu)^{-1},\mu)}(gc(\mu)^{-1},\mathrm{ind}_\mu(x)(ab))\\ &= (gc(\mu)^{-1}c(\mu),\mathrm{res}_\mu \circ \mathrm{ind}_\mu(x)(ab))\\ &= (g,x)(ab).\end{aligned}$$

Elements of the path groupoid G_{F_c} of $F_c : (\Gamma \times_c \mathcal{E}) \to \mathcal{B}$ consist of equivalence classes $[(h,\mu),(g,\nu),(g,x)]$ where $(g,x) \in F_c^\infty$ so $g \in \Gamma, x \in F^\infty$, ending in (g,ν) with ν an ending of x, and (h,μ) having the same source as (g,ν). But in particular, $s(h,\mu) = (hc(\mu), s(\mu))$ and $s(g,\nu) = (gc(\nu), s(\nu))$, so we must have $s(\mu) = s(\nu)$ and $h = gc(\nu)c(\mu)^{-1}$. Thus G_{F_c}

consists of

$$[(gc(\nu)c(\mu)^{-1},\mu),(g,\nu),(g,x)],\quad x\in Z(\nu)\subset F^{\infty}, g\in\Gamma, s(\mu)=s(\nu).$$

Note that this appears to arise from $[\mu,\nu,x]\in G_F$ and choice of $g\in G$, but that is not the whole picture. Consider which triples $((g'c(\nu')c(\mu')^{-1},\mu'),(g',\nu'),(g',x'))$ in $\mathcal{G}_{F_c}$ lie in the equivalence class in G_{F_c} above, under the equivalence relation $\sim$ from Section 1.3.

By Condition 1, we must have $(g',x')=(g,x)$ in F_c^{∞}. If Condition 3 is satisfied, there exist $a,b\in\mathcal{B}$ such that $F_c(gc(\nu')c(\mu')^{-1},\mu')a=F_c(gc(\nu)c(\mu)^{-1},\mu)b$ and $F_c(g,\nu')a=F_c(g,\nu)b$, which is true whenever $F(\mu')a=F(\mu)b$ and $F(\nu')a=F(\nu)b$. However, for Condition 2 to be satisfied, there must be $(g,\lambda)\in(\Gamma\times_c\mathcal{E})$ with $(g,x)\in Z(g,\lambda)\subset Z(g,\nu)\cap Z(g,\nu')$ such that

$$\mathrm{ind}_{(gc(\nu')c(\mu')^{-1},\mu')}\circ\mathrm{res}_{(g,\nu')}|_{Z(g,\lambda)}=\mathrm{ind}_{(gc(\nu)c(\mu)^{-1},\mu)}\circ\mathrm{res}_{(g,\nu)}|_{Z(g,\lambda)}.$$

But by the definitions of ind and res for G_{F_c}, for $(g,y)\in Z(g,\lambda)$,

$$\begin{aligned}\mathrm{ind}_{(gc(\nu')c(\mu')^{-1},\mu')}\circ\mathrm{res}_{(g,\nu')}(g,y)(ab)&=\mathrm{ind}_{(gc(\nu')c(\mu')^{-1},\mu')}(gc(\nu'),\mathrm{res}_{\nu'}(y)(ab))\\&=(gc(\nu')c(\mu')^{-1},\mathrm{ind}_{\mu'}\circ\mathrm{res}_{\nu'}(y)(ab))\end{aligned}$$

and similarly

$$\mathrm{ind}_{(gc(\nu)c(\mu)^{-1},\mu)}\circ\mathrm{res}_{(g,\nu)}(g,y)(ab)=(gc(\nu)c(\mu)^{-1},\mathrm{ind}_{\mu}\circ\mathrm{res}_{\nu}(y)(ab)).$$

These are equal when $\mathrm{ind}_{\mu}\circ\mathrm{res}_{\nu}|_{Z(\lambda)}=\mathrm{ind}_{\mu'}\circ\mathrm{res}_{\nu'}|_{Z(\lambda)}$ and $c(\nu')c(\mu')^{-1}=c(\nu)c(\mu)^{-1}$, which is more restrictive than simply requiring $(\mu',\nu',x)\in[\mu,\nu,x]$. In other words, there may be $[\mu',\nu',x]=[\mu,\nu,x]\in Z(\mu,\nu)$ in G_F while $[(gc(\nu')c(\mu')^{-1},\mu'),(g,\nu'),(g,x)]\neq[(gc(\nu)c(\mu)^{-1},\mu),(g,\nu),(g,x)]$ in G_{F_c}. The source of $[(gc(\nu)c(\mu)^{-1},\mu),(g,\nu),(g,x)]$ is (g,x)

and the range is

$$\mathrm{ind}_{(gc(\nu)c(\mu)^{-1},\mu)} \circ \mathrm{res}_{(g,\nu)}(g,x) = (gc(\nu)c(\mu)^{-1}, \mathrm{ind}_\mu \circ \mathrm{res}_\nu(x)).$$

Thus two elements $[(gc(\nu)c(\mu)^{-1}, \mu), (g, \nu), (g, x)], [(hc(\tau)c(\sigma)^{-1}, \sigma), (h, \tau), (h, y)]$ of G_{F_c} are composable when

$$(g, x) = (hc(\tau)c(\sigma)^{-1}, \mathrm{ind}_\sigma \circ \mathrm{res}_\tau(y)),$$

that is, when not only $[\mu, \nu, x], [\sigma, \tau, y]$ are composable in G_F but also $gc(\sigma) = hc(\tau)$.

At this point, we encounter a breakdown in the general case that precludes our following the k-graph case. Namely, the map $\tilde{c} : G_\Lambda \to \Gamma$ defined in Section 5 of [1] as

$$\tilde{c}(\alpha z, d(\alpha) - d(\beta), \beta z) = c(\alpha)c(\beta)^{-1}$$

is well-defined for Λ a k-graph (or an $\mathbb{N}$-graph; see Chapter 5), but the analogous map from G_F to Γ given by $[\mu, \nu, x] \mapsto c(\mu)c(\nu)^{-1}$ is not at all well-defined. Explicitly, suppose $(\mu', \nu', x) \in [\mu, \nu, x]$. Then there exist $a, b \in \mathcal{B}$ such that $F(\mu')a = F(\mu)b$ and $F(\nu')a = F(\nu)b$. Since $x \in Z(\nu') \cap Z(\nu)$, we can use it to find the lift of the second equation in $\mathcal{E}$,

$$\nu' x_2(F(\nu'), a) = \nu x_2(F(\nu), b),$$

but in general there is no reason to have $x_2(F(\nu'), a) = x_2(F(\mu'), a)$ and $x_2(F(\nu), b) = x_2(F(\mu), b)$, and thus despite having

$$c(\nu') = c(\nu)c(x_2(F(\nu), b))c(x_2(F(\nu'), a))^{-1}$$

and

$$c(\mu') = c(\mu)c(x_2(F(\mu), b))c(x_2(F(\mu'), a))^{-1},$$

we cannot assert $c(\mu')c(\nu')^{-1} = c(\mu)c(\nu)^{-1}$. Notice, however, that this equality is equivalent to $c(\nu)c(\mu)^{-1} = c(\nu')c(\mu')^{-1}$, the very one required to have $[(gc(\nu')c(\mu')^{-1}, \mu'), (g, \nu'), (g, x)] = [(gc(\nu)c(\mu)^{-1}, \mu), (g, \nu), (g, x)]$ in G_{F_c}. Thus a map $\hat{c} : G_{F_c} \to \Gamma$ given by

$$\hat{c}([(gc(\nu)c(\mu)^{-1}, \mu), (g, \nu), (g, x)]) = c(\mu)c(\nu)^{-1}$$

is well-defined. To see that it is a homomorphism, note that for a composable pair

$$[(gc(\nu)c(\mu)^{-1}, \mu), (g, \nu), (g, \mathrm{ind}_\sigma \circ \mathrm{res}_\tau(y))], \quad [(g, \sigma), (h, \tau), (h, y)]$$

in G_{F_c}, the composition when found as in Section 4 of [2] is

$$\begin{aligned}
&[(gc(\nu)c(\mu)^{-1}, \mu)(gc(\nu)c(\mu)^{-1}c(\mu), \gamma), (h, \tau)(hc(\tau), \nu), (h, y)] \\
&\qquad = [(gc(\nu)c(\mu)^{-1}, \mu)(gc(\nu), \gamma), (h, \tau)(gc(\sigma), \nu), (h, y)] \\
&\qquad = [(gc(\nu)c(\mu)^{-1}, \mu\gamma), (h, \tau\eta), (h, y)],
\end{aligned}$$

where $\nu\gamma = \sigma\eta$. Then

$$\hat{c}([(gc(\nu)c(\mu)^{-1}, \mu), (g, \nu), (g, \mathrm{ind}_\sigma \circ \mathrm{res}_\tau(y))]) = c(\mu)c(\nu)^{-1},$$

$$\hat{c}([(g, \sigma), (h, \tau), (h, y)]) = c(\sigma)c(\tau)^{-1},$$

and

$$\hat{c}([(gc(\nu)c(\mu)^{-1}, \mu\gamma), (h, \tau\eta), (h, y)]) = c(\mu\gamma)c(\tau\eta)^{-1} = c(\mu)c(\gamma)c(\eta)^{-1}c(\tau)^{-1},$$

but $c(\nu)c(\gamma) = c(\sigma)c(\eta)$ implies $c(\gamma)c(\eta)^{-1} = c(\nu)^{-1}c(\sigma)$, so

$$(c(\mu)c(\nu)^{-1})(c(\sigma)c(\tau)^{-1}) = c(\mu\gamma)c(\tau\eta)^{-1}$$

and $\hat{c} : G_{F_c} \to \Gamma$ is a homomorphism. Being constant on basis opens $Z(\mu, \nu)$ of G_{F_c}, it is a continuous homomorphism.

Remark 2.4.1. If Γ is countable and abelian, note that G_{F_c} and $\hat{c} : G_{F_c} \to \Gamma$ satisfy the premises of [10] Proposition 9.3, yielding an amenability result. Namely, if the subgroupoid $\hat{c}^{-1}(0)$ is amenable, so is G_{F_c}. Furthermore, under a later and more general result, [11] Corollary 4.5, Γ need not be abelian as long as it is amenable.

Finally, we discuss group actions generalized from [12] Section 3 in this context and conclude with an example of an induced KPf arising from a KPf $F : \mathcal{E} \to \mathcal{B}$, a countable group Γ, and its action on the skew product $(\Gamma \times_c \mathcal{E})$.

Adjusting the discussion from [1] Section 5, say that a countable group Γ acts on a KPf $F : \mathcal{E} \to \mathcal{B}$ if there is a (strict) action of Γ on $\mathcal{E}$ compatible with F. That is, the strict group action of Γ on $\mathcal{E}$ is a group homomorphism from Γ to $\mathrm{Aut}(\mathcal{E})$, denoting the automorphism for g by writing the image $g \cdot \alpha$ for $\alpha \in \mathcal{E}$, and moreover requiring $F(g \cdot \alpha) = F(\alpha)$. (We emphasize the strict, though naive, definition of a group action on a category as it seems most appropriate for the KPf setting, which depends on equalities.)

Proposition 2.4.1. *The action of Γ on $F : \mathcal{E} \to \mathcal{B}$ induces an action β on $C^*(F)$ such that $\beta_g s_\alpha = s_{g\cdot\alpha}$.*

Proof. Let the universal C*-algebra $C^*(F)$ be generated by Cuntz-Krieger system $\{p_X, s_\alpha\}$ and let $q_X = p_{g\cdot X}, t_\alpha = s_{g\cdot\alpha}$. Note that

1. For $X \neq Y$ in $\mathrm{Obj}(\mathcal{E})$, q_X and q_Y are orthogonal since $g \cdot X \neq g \cdot Y$.
2. For $\alpha, \beta \in \mathcal{E}$ composable, $t_\alpha t_\beta = s_{g\cdot\alpha} s_{g\cdot\beta} = s_{g\cdot\alpha\beta} = t_{\alpha\beta}$ holds.
3. For $X \in \mathrm{Obj}(\mathcal{E})$, $q_X = p_{g\cdot X} = s_{\mathrm{Id}_{g\cdot X}} = s_{g\cdot \mathrm{Id}_X} = t_{\mathrm{Id}_X}$.
4. If $\alpha : X \to Y$ in $\mathcal{E}$, then since $g \cdot \alpha : g \cdot X \to g \cdot Y$, we have $t_\alpha^* t_\alpha = s_{g\cdot\alpha}^* s_{g\cdot\alpha} = p_{g\cdot X} = q_X$.

5. For all $X \in \mathcal{E}$ and for all $b \in F(X)\mathcal{B}$, since automorphisms are isomorphisms and $F(g \cdot \alpha) = F(\alpha)$,

$$\begin{aligned}
\sum_{\alpha \in X\mathcal{E}, F(\alpha)=b} t_\alpha t_\alpha^* &= \sum_{\alpha \in X\mathcal{E}, F(\alpha)=b} s_{g\cdot\alpha} s_{g\cdot\alpha}^* \\
&= \sum_{g\cdot\alpha \in g\cdot X\mathcal{E}, F(g\cdot\alpha)=b} s_{g\cdot\alpha} s_{g\cdot\alpha}^* \\
&= p_{r(g\cdot\alpha)} \\
&= p_{g\cdot r(\alpha)} = q_{r(\alpha)}
\end{aligned}$$

holds.

Thus $\{q_X, t_\alpha\}$ is a Cuntz-Krieger system for F, so as in the k-graph case, the action of Γ induces an action β on $C^*(F)$ such that $\beta_g s_\alpha = s_{g\cdot\alpha}$.

□

The group action of Γ on $F_c : (\Gamma \times_c \mathcal{E}) \to \mathcal{B}$ given by $g \cdot (h, \alpha) = (gh, \alpha)$ is free. That is, if for all $(h, \alpha) \in (\Gamma \times_c \mathcal{E})$ we have $g \cdot (h, \alpha) = (h, \alpha)$, then $gh = h$ for all h and thus g is the identity element of Γ. We form the quotient $(\Gamma \times_c \mathcal{E})/\Gamma$ by the equivalence relation $(h', \alpha') \sim (h, \alpha)$ if $(h', \alpha') = g \cdot (h, \alpha)$ for some $g \in \Gamma$. That is, the quotient consists of equivalence classes

$$[h, \alpha]_\Gamma := \{(h', \alpha) \in (\Gamma \times_c \mathcal{E}) : h' = gh \text{ for some } g \in \Gamma\}.$$

First, note that the map $q : (\Gamma \times_c \mathcal{E})/\Gamma \to \mathcal{E}$ given by $q([h, \alpha]_\Gamma) = \alpha$ is injective. Indeed, if

$\alpha = \alpha'$ then

$$\begin{aligned}[h, \alpha]_\Gamma &= \{(h', \alpha) : h' = gh \text{ for some } g \in \Gamma\} \\ &= \{(h', \alpha') : h' = gh \text{ for some } g \in \Gamma\} \\ &= [h, \alpha']_\Gamma\end{aligned}$$

and if $[h, \alpha]_\Gamma = [h', \alpha']_\Gamma$, then $(h', \alpha') = g \cdot (h, \alpha)$ for some $g \in \Gamma$; that is, $h' = gh$ and $\alpha' = \alpha$. In other words, elements of the form $(h', \alpha) \in (\Gamma \times_c \mathcal{E})$ for any $h' \in \Gamma$ must all fall into the same equivalence class by $h' = gh$ for $g = h'h^{-1}$ in particular. Thus it seems natural to define a multiplication $[h, \alpha]_\Gamma [hc(\alpha), \beta]_\Gamma$ for $\alpha, \beta \in \mathcal{E}$ composable by

$$\begin{aligned}[(h, \alpha)(hc(\alpha), \beta)]_\Gamma &= [h, \alpha\beta]_\Gamma \\ &= \{(h', \alpha\beta) : h' = gh \text{ for some } g \in \Gamma\} \\ &= \{(h', \alpha)(h'c(\alpha), \beta) : h' = gh \text{ for some } g \in \Gamma\} \\ &= [h, \alpha]_\Gamma [hc(\alpha), \beta]_\Gamma.\end{aligned}$$

Theorem 2.4.2. *Let $F : \mathcal{E} \to \mathcal{B}$ be a KPf, and let Γ be a countable group acting freely on the skew product $(\Gamma \times_c \mathcal{E})$ as above. Then $(\Gamma \times_c \mathcal{E})/\Gamma$ is isomorphic to $\mathcal{E}$, and yields a KPf which we denote $F_q : (\Gamma \times_c \mathcal{E})/\Gamma \to \mathcal{B}$, and namely $F_q([h, \alpha]_\Gamma) = F(\alpha)$.*

Proof. The first statement follows from the injectivity of the map q discussed above. For the second statement, we show that F_q as defined is a dCf and a KPf. To show the unique lifting property, suppose the image of $[h, \alpha]_\Gamma$ under F_q, that is $F_q([h, \alpha]_\Gamma) = F(\alpha)$ has a factorization $F(\alpha) = ab$ in $\mathcal{B}$. Since F is a KPf, there exist unique $\mu, \nu \in \mathcal{E}$ such that $F(\mu) = a, F(\nu) = b$, and $\alpha = \mu\nu$. Take as lifts in $(\Gamma \times_c \mathcal{E})/\Gamma$ the orbits $[h, \mu]_\Gamma, [hc(\mu), \nu]_\Gamma$, so that

$$[h, \mu]_\Gamma [hc(\mu), \nu]_\Gamma = [h, \mu\nu]_\Gamma = [h, \alpha]_\Gamma.$$

Uniqueness again follows from injectivity of q.

Finally, to see that F_q is locally split, take an object $[h, X]_\Gamma$, where $h \in \Gamma$ and $X \in \mathrm{Obj}(\mathcal{E})$, with image $F_q([h, X]_\Gamma) = F(X)$ in $\mathcal{B}$. We will define a splitting from $\mathcal{B}/F(X)$ to $((\Gamma \times_c \mathcal{E})/\Gamma)/[h, X]_\Gamma$ on pairs $(a, b) \in \mathcal{B}/F(X)$ using a splitting $x : \mathcal{B}/F(X) \to \mathcal{E}/X$ of F. Let

$$[h, x](a, b) = ([h, x_1(a, b)]_\Gamma, [hc(x_1(a, b)), x_2(a, b)]_\Gamma) \, .$$

Then $F_q \circ [h, x] = \mathrm{Id}_{\mathcal{B}/F(X)}$ and F_q is a KPf. $\square$

Chapter 3

Aperiodicity, the Cycline Subalgebra, and Cartan Subalgebras

Brown et al. in [13] studied a special class of standard generators for the C*-algebra of a k-graph (Λ, d), which produce the cycline subalgebra of $C^*(\Lambda)$. In this section, we show that much of the definition and properties may be generalized with some care to the KPf context. We note that in many places throughout the development of theory around the C*-algebras of KPfs, the difficulty in achieving results comes from either the existence of multiple objects in $\mathcal{B}$, or the inability to factorize $\mathcal{B}$ morphisms into arbitrarily-sized pieces. In the proofs in this section, we will often need to look "upstream" using the strongly right Ore condition on $\mathcal{B}$ to achieve similar results. Later on, in order to apply more recent results from [14], we relate the cycline subalgebra to the interior of the isotropy of the path groupoid. To aid in our discussion, we first flesh out the aperiodicity topics introduced in Section 1.3 along the way.

For this section, let $F : \mathcal{E} \to \mathcal{B}$ be a KPf with $\mathcal{B}$ moreover left-cancellative. Let $C^*(F) = C^*(\{P_X, S_\alpha\})$ be the universal C*-algebra generated by $S_\alpha S_\beta^*$, where $\alpha, \beta \in \mathcal{E}$ and $s(\alpha) = s(\beta)$, by Proposition 1.4.1. Let $\mathscr{D} := C^*(\{P_\alpha : \alpha \in \mathcal{E}\})$, where $P_\alpha := S_\alpha S_\alpha^*$.

By Section 5 of [2], when $\mathcal{B}$ is left-cancellative, $C^*(F) \cong C^*(G_F)$ by the isomorphism

Υ defined by $s_\alpha \mapsto 1_{Z(\alpha,s(\alpha))}$, which gives $S_\alpha S_\alpha^* \mapsto 1_{Z(\alpha)}$. Thus $P_\alpha = P_\beta$ if and only if $Z(\alpha) = Z(\beta)$. Also note that for γ, η such that $r(\gamma) = s(\alpha)$ and $r(\eta) = s(\beta)$,

$$\begin{aligned}
P_{\alpha\gamma} S_\alpha S_\beta^* P_{\beta\eta} &= S_{\alpha\gamma} S_{\alpha\gamma}^* S_\alpha S_\beta^* S_{\beta\eta} S_{\beta\eta}^* \\
&= S_{\alpha\gamma} S_\gamma^* S_\alpha^* S_\alpha S_\beta^* S_\beta S_\eta S_{\beta\eta}^* \\
&= S_{\alpha\gamma} S_\gamma^* P_{s(\alpha)} P_{s(\beta)} S_\eta S_{\beta\eta}^* \\
&= S_{\alpha\gamma} S_\gamma^* S_\eta S_{\beta\eta}^*,
\end{aligned}$$

and if in particular $F(\gamma) = F(\eta)$ then Property 5 of Definition 1.1.4 implies the result will be $S_{\alpha\gamma} S_{\beta\gamma}^*$ if $\gamma = \eta$ and 0 otherwise.

Our aim is to define *cycline pairs* $(\alpha, \beta) \in \mathcal{E} \times \mathcal{E}$ as pairs satisfying a set of equivalent conditions. The k-graph precursors suggest the following generalization.

Proposition 3.0.1. *Assume the premises listed above. For $(\alpha, \beta) \in \mathcal{E} \times \mathcal{E}$ with $s(\alpha) = s(\beta) = X$, the following are equivalent:*

(a) $P_{\alpha\gamma} = P_{\beta\gamma}$ for all $\gamma \in X\mathcal{E}$.

(b) $S_\alpha S_\beta^*$ is normal and commutes with $\mathscr{D} = C^*(\{S_\mu S_\mu^*\})$.

(c) $\operatorname{ind}_\alpha(y) = \operatorname{ind}_\beta(y)$ for all $y \in Z(X)$.

Before proving this proposition, we explore ways of characterizing aperiodicity for KPfs.

3.1 Aperiodicity

We begin by giving an expanded version of Definition 1.3.1, including the introduction of right-aperiodicity suggested by [10] Definition 10.5.

Definition 3.1.1. We say that $x \in F^\infty$ is *aperiodic* if for all $\mu, \nu \in \mathcal{E}$ such that $x \in Z(\mu) \cap Z(\nu)$, $\operatorname{res}_\mu(x) = \operatorname{res}_\nu(x)$ implies $\mu = \nu$.

Thus we say $x \in Z(X) \subset F^\infty$ is *periodic* if there exist $a \neq b \in \mathcal{B}/F(X)$ such that $\mathrm{res}_{x(a)}(x) = \mathrm{res}_{x(b)}(x)$; or equivalently, there exist $\mu \neq \nu \in \mathcal{E}$ with $x \in Z(\mu) \cap Z(\nu)$ (so $F(\mu) \neq F(\nu)$) such that $\mathrm{res}_\mu(x) = \mathrm{res}_\nu(x)$.

We say that x is *right-aperiodic* if for all $\alpha, \beta \in \mathcal{E}$, $\mathrm{ind}_\alpha(x) = \mathrm{ind}_\beta(x)$ implies $\alpha = \beta$.

We say that a KPf $F : \mathcal{E} \to \mathcal{B}$ is *aperiodic* if $Z(X)$ contains an aperiodic path for every $X \in \mathrm{Obj}(\mathcal{E})$.

Proposition 3.1.1. *Let $F : \mathcal{E} \to \mathcal{B}$ be a KPf with $\mathcal{B}$ left-cancellative.*

(i) For $x \in Z(\beta)$, if x is aperiodic, then $\mathrm{res}_\beta(x)$ is aperiodic.

(ii) When $\mathcal{E}$ is right-cancellative, if x is aperiodic, then x is also right-aperiodic.

Proof. To show (i), assume $x \in F^\infty$ is aperiodic and ends in $\beta \in \mathcal{E}$. Suppose $\mathrm{res}_\beta(x)$ ends in both $\mu, \nu \in \mathcal{E}$, recall that $\mathrm{res}_\mu(\mathrm{res}_\beta(x)) = \mathrm{res}_{\beta\mu}(x)$ since $r(\mu) = s(\beta)$, and similarly $\mathrm{res}_\nu(\mathrm{res}_\beta(x)) = \mathrm{res}_{\beta\nu}(x)$. Then if

$$\mathrm{res}_\nu(\mathrm{res}_\beta(x)) = \mathrm{res}_\mu(\mathrm{res}_\beta(x)),$$

the aperiodicity of x implies $\beta\nu = \beta\mu$. Then $F(\beta\nu) = F(\beta)F(\nu) = F(\beta)F(\mu) = F(\beta\mu)$ implies $F(\nu) = F(\mu)$ since $\mathcal{B}$ is left-cancellative. Note of course that by definition, the value of $\mathrm{res}_\beta(x)$ on $F(\mu)$ or $F(\nu)$ agrees with that of x. But then $\mu = x(F(\mu)) = x(F(\nu)) = \nu$, so $\mathrm{res}_\beta(x)$ is an aperiodic path.

To show (ii), assume now that $\mathcal{E}$ is right-cancellative. We wish to show that if x is aperiodic, then x is also right-aperiodic.

By way of contradiction, suppose x is not right-aperiodic; i.e., there exist $\alpha \neq \beta$ with $\mathrm{ind}_\alpha(x) = \mathrm{ind}_\beta(x)$. We must have $s(\alpha) = s(\beta) = r(x)$ and $r(\alpha) = r(\beta)$. Since $\mathcal{B}$ is strongly right Ore, there exist $a, b \in \mathcal{B}$ such that $F(\alpha)a = F(\beta)b$, and since a, b target $r(x)$, we have the existence of $x(a), x(b)$ lifts such that $\alpha x(a) = \beta x(b)$. We cannot have $x(a) = x(b)$, because then $\alpha = \beta$ by right cancellation yields a contradiction. Thus we must have $x(a) \neq x(b)$, but still

$$\mathrm{res}_{x(a)}(x) = \mathrm{res}_{\alpha x(a)} \circ \mathrm{ind}_\alpha(x) = \mathrm{res}_{\beta x(b)} \circ \mathrm{ind}_\beta(x) = \mathrm{res}_{x(b)}(x),$$

which contradicts the assumption that x was aperiodic. So indeed,

$$(\mathrm{res}_\mu(x) = \mathrm{res}_\nu(x) \implies \mu = \nu) \implies (\mathrm{ind}_\alpha(x) = \mathrm{ind}_\beta(x) \implies \alpha = \beta).$$

$\square$

Notice that even if x is aperiodic, in the case that $\mathrm{ind}_\alpha(x) = \mathrm{ind}_\beta(x)$ as described by Condition (c) of Proposition 3.0.1, the infinite path $\mathrm{ind}_\alpha(x)$ is not aperiodic, since

$$\mathrm{res}_\alpha(\mathrm{ind}_\alpha(x)) = x = \mathrm{res}_\beta(\mathrm{ind}_\beta(x)) = \mathrm{res}_\beta(\mathrm{ind}_\alpha(x)),$$

while $\alpha \neq \beta$. This differs from the k-graph case, in which x being aperiodic implies that αx is aperiodic for $\alpha \in \mathcal{E}$ such that $s(\alpha) = r(x)$.

Next, we briefly make explicit equivalences that follow from Proposition 1.2.1. The following statements are equivalent when holding for an $x \in Z(\beta)$:

- $\mathrm{ind}_\alpha \circ \mathrm{res}_\beta(x) = x$
- $\mathrm{res}_\beta(x) = \mathrm{res}_\alpha(x)$
- $\mathrm{ind}_\beta \circ \mathrm{res}_\alpha(x) = x$.

These follow from the fact that $\mathrm{ind}_\mu \circ \mathrm{res}_\mu = \mathrm{Id}_{Z(\mu)}$ and $\mathrm{res}_\nu \circ \mathrm{ind}_\mu = \mathrm{Id}_{Z(X)}$ when $\mu : X \to Y$. Next, we prove a lemma on the level of entire cylinder sets.

Lemma 3.1.2. *The following statements are equivalent:*

(i) $\mathrm{res}_\alpha(x) = \mathrm{res}_\beta(x)$ *for all* $x \in Z(\alpha) = Z(\beta)$.

(ii) $\mathrm{ind}_\alpha(y) = \mathrm{ind}_\beta(y)$ *for all* $y \in Z(s(\alpha)) = Z(s(\beta))$*.*

Proof. We first wish to show that an $x \in Z(\beta)$ can be written uniquely as $\mathrm{ind}_\beta(y)$ for some $y \in Z(s(\beta))$. Let $y = \mathrm{res}_\beta(x)$, then clearly $x = \mathrm{ind}_\beta(y)$. Suppose $x = \mathrm{ind}_\beta(z)$ for a $z \in Z(s(\beta))$. For $c \in \mathcal{B}$ targeting $F(s(\beta))$, $x(c) = z(c)$ as the two infinite paths must agree on all objects in $\mathcal{B}/s(\beta)$. For the same c, $\mathrm{res}_\beta(x)(c) = x(c)$, so $\mathrm{res}_\beta(x)(c) = z(c)$. Thus $z = \mathrm{res}_\beta(x)$ is the y we seek.

(i) $\implies$ (ii) Assume for all $x \in Z(\beta)$, $\mathrm{res}_\alpha(x) = \mathrm{res}_\beta(x)$ and let $y \in Z(s(\beta))$. We may then consider $\mathrm{ind}_\beta(y) \in Z(\beta)$ and apply the assumption to say

$$\mathrm{res}_\alpha \circ \mathrm{ind}_\beta(y) = \mathrm{res}_\beta \circ \mathrm{ind}_\beta(y) = y,$$

and then apply ind_α to both sides to find $\mathrm{ind}_\alpha(y) = \mathrm{ind}_\beta(y)$.

(i) $\impliedby$ (ii) Now assume $\mathrm{ind}_\alpha(y) = \mathrm{ind}_\beta(y)$ for all $y \in Z(s(\beta))$ and let $x \in Z(\beta)$. By the above discussion, we write $x = \mathrm{ind}_\beta(y)$ for a $y \in Z(s(\beta))$, where the $y = \mathrm{res}_\beta(x)$ is known. Then apply the assumption to say

$$\mathrm{res}_\alpha(x) = \mathrm{res}_\alpha \circ \mathrm{ind}_\beta(y) = \mathrm{res}_\alpha \circ \mathrm{ind}_\alpha(y) = y = \mathrm{res}_\beta(x).$$

Thus the statements are equivalent, the key being that they hold for all x or y in the cylinder sets, respectively. □

3.2 Cycline Pairs

The lemma of the previous section allows us to replace Condition (c) of Proposition 3.0.1 with an alternative version, yielding the final version which we will now prove.

Theorem 3.2.1. *Assume* $F : \mathcal{E} \to \mathcal{B}$ *is a KPf with* $\mathcal{B}$ *left-cancellative. For* $(\alpha, \beta) \in \mathcal{E} \times \mathcal{E}$ *with* $s(\alpha) = s(\beta) = X$*, the following are equivalent:*

(a) $P_{\alpha\gamma} = P_{\beta\gamma}$ *for all* $\gamma \in X\mathcal{E}$*.*

(b) $S_\alpha S_\beta^*$ *is normal and commutes with* $\mathscr{D} = C^*(\{S_\mu S_\mu^*\})$*.*

(c) $\mathrm{ind}_\alpha(y) = \mathrm{ind}_\beta(y)$ *for all* $y \in Z(X)$*.*

(c)* $\mathrm{res}_\alpha(x) = \mathrm{res}_\beta(x)$ *for all* $x \in Z(\beta)$*.*

Proof. First note that assuming (a) implies that $Z(\alpha) = Z(\beta)$ and that $r(\alpha) = r(\beta)$. If $F(\alpha) = F(\beta)$, $Z(\alpha) \cap Z(\beta)$ is nonempty if and only if $\alpha = \beta$ ([2], Lemma 4.1), in which case all three items hold automatically, so now assume $F(\alpha) \neq F(\beta)$. Also note that we make heavy use of Item 5 from Definition 1.1.4.

$(c) \iff (c^*)$ by the previous Lemma.

$(a) \implies (b)$: Assume $P_{\alpha\gamma} = P_{\beta\gamma}$ for all $\gamma \in X\mathcal{E}$. Since

$$(S_\alpha S_\beta^*)(S_\alpha S_\beta^*)^* = S_\alpha P_{s(\beta)} S_\alpha^* = S_\alpha S_\alpha^* = P_\alpha$$

and

$$(S_\alpha S_\beta^*)^*(S_\alpha S_\beta^*) = S_\beta P_{s(\alpha)} S_\beta^* = P_\beta,$$

$P_\alpha = P_\beta$ implies $S_\alpha S_\beta^*$ is normal.

Now let $P_\mu = S_\mu S_\mu^* \in \mathscr{D}$. Having assumed (a), we have $r(\alpha) = r(\beta) =: Y$. First consider the case where $r(\mu) \neq Y$, in which

$$S_\mu S_\mu^* S_\alpha S_\beta^* = S_\mu S_\mu^* P_{r(\mu)} P_{r(\alpha)} S_\alpha S_\beta^* = (S_\mu S_\mu^*) 0 (S_\alpha S_\beta^*) = 0,$$

$$S_\alpha S_\beta^* S_\mu S_\mu^* = S_\alpha S_\beta^* P_{r(\beta)} P_{r(\mu)} S_\mu S_\mu^* = 0$$

show that $S_\alpha S_\beta^*$ commutes with $P_\mu \in \mathcal{D}$. We move on to the case where $r(\mu) = Y$. There are three sub-cases.

Case 1: when $\mu = \alpha\gamma$ or $\mu = \beta\gamma$ for some γ. Then

$$P_\mu S_\alpha S_\beta^* = S_{\alpha\gamma} S_\gamma^* S_\alpha^* S_\alpha S_\beta^* = S_{\alpha\gamma} S_{\beta\gamma}^*$$

and since $P_{\alpha\gamma} = P_{\beta\gamma}$,

$$S_\alpha S_\beta^* P_\mu = S_\alpha S_\beta^* S_{\beta\gamma} S_{\beta\gamma}^* = S_{\alpha\gamma} S_{\beta\gamma}^*.$$

Case 2: when $\alpha = \mu\gamma$ or $\beta = \mu\gamma$ for some γ. Without loss of generality assume the former. Then

$$S_\mu S_\mu^* S_\alpha S_\beta^* = S_\mu S_\mu^* S_{\mu\gamma} S_\beta^* = S_\mu S_\mu^* S_\mu S_\gamma S_\beta^* = S_\mu S_\gamma S_\beta^* = S_\alpha S_\beta^*$$

since these are partial isometries. And since $P_\beta = P_\alpha = P_{\mu\gamma} = S_\mu P_\gamma S_\mu^*$,

$$S_\alpha S_\beta^* S_\mu S_\mu^* = S_\alpha S_\beta^* P_\beta S_\mu S_\mu^* = S_\alpha S_\beta^* (S_\mu P_\gamma S_\mu^*) S_\mu S_\mu^* = S_\alpha S_\beta^* S_\mu P_\gamma S_\mu^* = S_\alpha S_\beta^*.$$

Case 3: When μ neither has α (or β) as a factor, nor is a factor in α (or β), which in particular includes the assumption $\mu \neq \alpha \neq \beta$. If $F(\mu) = F(\alpha)$ or $F(\beta)$ (without loss of generality assume the former) then

$$S_\mu S_\mu^* S_\alpha S_\beta^* = (S_\mu)0(S_\beta^*) = 0,$$

$$S_\alpha S_\beta^* S_\mu S_\mu^* = S_\alpha (S_\beta^* S_\beta S_\beta^*) S_\mu S_\mu^* = S_\alpha S_\beta^* P_\beta S_\mu S_\mu^* = S_\alpha S_\beta^* S_\alpha S_\alpha^* S_\mu S_\mu^* = (S_\alpha S_\beta^* S_\alpha)0(S_\mu^*) = 0.$$

(For the third equality use again $P_\beta = P_\alpha = S_\alpha S_\alpha^*$.) Otherwise if $F(\mu) \neq F(\alpha) \neq F(\beta)$ but they all target $F(Y)$ since $r(\mu) = Y$, we make use of the right Ore condition on $\mathcal{B}$. Since $F(\alpha), F(\beta)$ form a cospan at $F(Y)$, there exist $a, b \in \mathcal{B}$ such that $F(\alpha)a = F(\beta)b =: m$. Then since this m and $F(\mu)$ also form a cospan at $F(Y)$, there exist $p, q \in \mathcal{B}$ such that $mp = nq$. This yields $F(\mu)q = F(\alpha)ap = F(\beta)bp$. We first use $F(\mu)q = F(\alpha)ap$ and the expansion

$$\begin{aligned}
S_\mu S_\mu^* S_\alpha S_\beta^* &= \sum_{\eta\in\mathcal{E},F(\eta)=q} S_{\mu\eta}S_{\mu\eta}^* S_\alpha S_\beta^* && r(\eta)=s(\mu)\\
&= \sum_{\eta\in\mathcal{E},F(\eta)=q} S_{\mu\eta}S_{\mu\eta}^* S_\alpha S_\alpha^* S_\alpha S_\beta^*\\
&= \sum_{\eta\in\mathcal{E},F(\eta)=q} S_{\mu\eta}S_{\mu\eta}^* \left(\sum_{\xi\in\mathcal{E},F(\xi)=ap} S_{\alpha\xi}S_{\alpha\xi}^*\right) S_\alpha S_\beta^* && r(\xi)=s(\alpha)\\
&= \sum_{\eta,\xi\in\mathcal{E},F(\eta)=q,F(\xi)=ap} S_{\mu\eta}S_{\mu\eta}^* S_{\alpha\xi}S_{\alpha\xi}^* S_\alpha S_\beta^*.
\end{aligned}$$

But $F(\mu\eta) = F(\mu)F(\eta) = F(\mu)q = F(\alpha)ap = F(\alpha\xi)$, so the only nonzero terms in the sum are where $\mu\eta = \alpha\xi$. Next, we use $F(\mu)q = F(\beta)bp$ and the expansion

$$\begin{aligned}
S_\alpha S_\beta^* S_\mu S_\mu^* &= \sum_{\eta'\in\mathcal{E},F(\eta')=q} S_\alpha S_\beta^* S_{\mu\eta'}S_{\mu\eta'}^*\\
&= \sum_{\eta',\zeta\in\mathcal{E},F(\eta')=q,F(\zeta)=bp} S_\alpha S_\beta^* S_{\beta\zeta}S_{\beta\zeta}^* S_{\mu\eta'}S_{\mu\eta'}^*
\end{aligned}$$

in which the only nonzero terms are those where $\mu\eta' = \beta\zeta$. The final sum from the first expansion can be rewritten as

$$\sum_{\alpha\xi=\mu\eta,F(\eta)=q,F(\xi)=ap} S_{\alpha\xi}S_{\alpha\xi}^* S_{\alpha\xi}S_{\alpha\xi}^* S_\alpha S_\beta^* = \sum_{\alpha\xi=\mu\eta,F(\eta)=q,F(\xi)=ap} P_{\alpha\xi}S_\alpha S_\beta^*,$$

and we have already seen that $P_{\alpha\xi}$ commutes with $S_\alpha S_\beta^*$ in Case 1. The second expansion's final sum can be written similarly as

$$\sum_{\beta\zeta=\mu\eta',F(\eta')=q,F(\zeta)=bp} S_\alpha S_\beta^* S_{\beta\zeta}S_{\beta\zeta}^* S_{\beta\zeta}S_{\beta\zeta}^* = \sum_{\beta\zeta=\mu\eta',F(\eta')=q,F(\zeta)=bp} S_\alpha S_\beta^* P_{\beta\zeta}.$$

Consider such an ξ that $F(\xi) = ap$; there exists an $x \in Z(\xi)$, i.e. $x(ap) = \xi$. Having assumed (a), $Z(\alpha\xi) = Z(\beta\xi)$ and $\alpha\xi = x(F(\alpha)ap) = x(F(\beta)bp) = \beta x(bp)$. Thus in fact $F(\alpha)ap = F(\beta)bp$ gives us that the sets $\{\alpha\xi : F(\xi) = ap, r(\xi) = s(\alpha)\} = \{\beta\zeta : F(\zeta) = bp, r(\zeta) = s(\beta)\}$ are equal. This means the sums have the same terms.

We pause to summarize the results of all these cases: assuming $F(\alpha) \neq F(\beta)$ and denoting again $r(\alpha) = r(\beta) = Y$,

$$P_\mu S_\alpha S_\beta^* = S_\alpha S_\beta^* P_\mu = \begin{cases} 0 & \text{when } r(\mu) \neq Y \\ S_{\alpha\gamma} S_{\beta\gamma} & \text{when } r(\mu) = Y \text{ and } \mu = \alpha\gamma \text{ or } \beta\gamma \\ S_\alpha S_\beta^* & \text{when } r(\mu) = Y \text{ and } \mu\gamma = \alpha \text{ or } \beta \\ 0 & \text{when } r(\mu) = Y, \text{ no such factoring, } F(\mu) = F(\alpha) \text{ or } F(\beta) \\ P_\mu S_\alpha S_\beta^* & \text{when } r(\mu) = Y \text{ and } F(\mu) \neq F(\alpha) \neq F(\beta). \end{cases}$$

$(b) \implies (a)$: Let $\gamma \in X\mathcal{E}$. By assumption, $S_\alpha S_\beta^*$ is normal and commutes with $P_{\alpha\gamma} = S_{\alpha\gamma} S_{\alpha\gamma}^*$, which means

$$P_{\alpha\gamma} S_\alpha S_\beta^* = S_{\alpha\gamma} S_{\alpha\gamma}^* S_\alpha S_\beta^* = S_{\alpha\gamma} S_{\beta\gamma}^*$$

is also normal. Then

$$P_{\alpha\gamma} = S_{\alpha\gamma} S_{\alpha\gamma}^* = (S_{\alpha\gamma} S_{\beta\gamma}^*)(S_{\alpha\gamma} S_{\beta\gamma}^*)^* = (S_{\alpha\gamma} S_{\beta\gamma}^*)^*(S_{\alpha\gamma} S_{\beta\gamma}^*) = S_{\beta\gamma} S_{\beta\gamma}^* = P_{\beta\gamma}.$$

$(a) \implies (c^*)$ Assume $Z(\alpha\gamma) = Z(\beta\gamma)$ for all $\gamma \in X\mathcal{E}$ and in particular $Z(\alpha) = Z(\beta)$. Then if $x \in Z(\beta)$ it is also in $Z(\alpha)$ and $\mathrm{res}_\alpha(x)$ and $\mathrm{res}_\beta(x)$ are defined. If they agree on all objects in $\mathcal{B}/s(\beta)$, they are equal by [2] Lemma 4.5. So let (a, b) be such that $r(a) = F(s(\beta)) = F(s(\alpha))$ and $r(b) = s(a)$. Now $(F(\beta)a, b) \in \mathcal{B}/F(r(\beta))$ and by definition

$$\mathrm{res}_\beta(x)(a, b) = (x_2(F(\beta), a), x_2(F(\beta)a, b)),$$

while

$$\mathrm{res}_\alpha(x)(a,b) = (x_2(F(\alpha),a), x_2(F(\alpha)a,b)).$$

Now, $F(x_2(F(\beta),a)) = a = F(x_2(F(\alpha),a))$, so $x(a) = x_2(F(\beta),a) = x_2(F(\alpha),a)$, implying that the first positions are equal. Similarly, $F(x_2(F(\beta)a,b)) = b = F(x_2(F(\alpha)a,b))$, implying that $x(b) = x_2(F(\beta)a,b) = x_2(F(\alpha)a,b)$ and the second positions are also equal. Therefore, $\mathrm{res}_\alpha(x) = \mathrm{res}_\beta(x)$.

$(c) \implies (a)$: Let $\gamma \in X\mathcal{E}$. For all $z \in Z(s(\gamma))$, $\mathrm{ind}_{\alpha\gamma}(z) = \mathrm{ind}_\alpha(\mathrm{ind}_\gamma(z)) = \mathrm{ind}_\beta(\mathrm{ind}_\gamma(z)) = \mathrm{ind}_{\beta\gamma}(z)$. If $x \in Z(\alpha\gamma)$ then it can be written as $\mathrm{ind}_{\alpha\gamma}(z) = \mathrm{ind}_{\beta\gamma}(z)$ for some $z \in Z(s(\gamma))$ and so $Z(\alpha\gamma) = Z(\beta\gamma)$, implying that $P_{\alpha\gamma} = P_{\beta\gamma}$. □

Definition 3.2.2. If $\alpha, \beta \in \mathcal{E}$ satisfy one (and therefore all) of the conditions above, then (α, β) is called a *cycline pair*.

As in Section 4 of [13], the relation defined by "$\alpha \sim \beta$ if (α,β) satisfies (c)" is an equivalence relation. It is easily checked that it is reflexive and symmetric, and if $(\alpha,\beta),(\beta,\gamma)$ both satisfy (c), then $s(\alpha) = s(\beta) = s(\gamma) =: X$ gives $\mathrm{ind}_\alpha(y) = \mathrm{ind}_\beta(y) = \mathrm{ind}_\gamma(y)$ for all $y \in Z(X)$.

Observe that (α,β) is cycline if and only if (β,α) is.

We call $S_\alpha S_\beta^*$ a *cycline generator* when (α,β) is a cycline pair, and we generate the *cycline C*-subalgebra*

$$\mathscr{M} = C^*\{S_\alpha S_\beta^* : (\alpha,\beta) \text{ cycline}\} \subset C^*(F).$$

Considering the commutant

$$\mathscr{D}' = \{a \in C^*(F) : aP_\mu = P_\mu a, \forall \mu \in \mathcal{E}\},$$

we see by construction that $\mathscr{D} \subset \mathscr{M} \subset \mathscr{D}'$ by condition (b).

We wish to show that $\mathscr{M}$ is equal to $\overline{\mathrm{span}}\{S_\alpha S_\beta^* : (\alpha,\beta) \text{ cycline}\}$. For $(\alpha,\beta),(\mu,\nu)$

cycline, a product $S_\alpha S_\beta^* S_\mu S_\nu^*$ will be zero unless $r(\alpha) = r(\beta) = r(\mu) = r(\nu)$. Then $F(\mu)$ and $F(\beta)$ are a cospan in $\mathcal{B}$, where by the right Ore condition, we can find $a, b \in \mathcal{B}$ such that $F(\mu)a = F(\beta)b$. Then we use the expansions

$$S_\mu S_\mu^* = \sum_{F(\eta)=a, r(\eta)=s(\mu)} S_{\mu\eta} S_{\mu\eta}^*, \qquad S_\beta S_\beta^* = \sum_{F(\gamma)=b, r(\gamma)=s(\beta)} S_{\beta\gamma} S_{\beta\gamma}^*$$

to show

$$\begin{aligned}
S_\alpha S_\beta^* S_\mu S_\nu^* &= S_\alpha S_\beta^* S_\beta S_\beta^* S_\mu S_\mu^* S_\mu S_\nu^* && \text{partial isometries} \\
&= \sum_{F(\eta)=a, F(\gamma)=b} S_\alpha S_\beta^* S_{\beta\gamma} S_{\beta\gamma}^* S_{\mu\eta} S_{\mu\eta}^* S_\mu S_\nu^* && r(\gamma) = s(\beta), r(\eta) = s(\mu) \\
&= \sum_{F(\eta)=a, F(\gamma)=b, \beta\gamma=\mu\eta} S_\alpha S_\beta^* S_{\beta\gamma} S_{\beta\gamma}^* S_{\beta\gamma} S_{\mu\eta}^* S_\mu S_\nu^* \\
&= \sum_{F(\eta)=a, F(\gamma)=b, \beta\gamma=\mu\eta} S_\alpha S_\beta^* S_{\beta\gamma} S_{\mu\eta}^* S_\mu S_\nu^* \\
&= \sum_{F(\eta)=a, F(\gamma)=b, \beta\gamma=\mu\eta} S_\alpha S_\beta^* S_\beta S_\gamma S_\eta^* S_\mu^* S_\mu S_\nu^* \\
&= \sum_{F(\eta)=a, F(\gamma)=b, \beta\gamma=\mu\eta} S_\alpha S_\gamma S_\eta^* S_\nu^* \\
&= \sum_{F(\eta)=a, F(\gamma)=b, \beta\gamma=\mu\eta} S_{\alpha\gamma} S_{\nu\eta}^*.
\end{aligned}$$

It remains to show that each such $(\alpha\gamma, \nu\eta)$ is a cycline pair. But if (α, β) and (μ, ν) are cycline pairs and $\beta\gamma = \mu\eta$, we have

$$P_{\alpha\gamma\lambda} = P_{\beta\gamma\lambda} = P_{\mu\eta\lambda} = P_{\nu\eta\lambda}$$

so $(\alpha\gamma, \nu\eta)$ satisfies condition (a) and is also cycline.

Corollary 3.2.3. *Let $F : \mathcal{E} \to \mathcal{B}$ be a KPf with $\mathcal{B}$ left-cancellative.*

(i) For α with $r(x) = s(\alpha)$, $\mathrm{ind}_\alpha(x)$ aperiodic implies that x is aperiodic, and that α does

not appear in a cycline pair (α, β) with $\alpha \neq \beta$.

(ii) For any $x \in F^\infty$, if (α, β) is a cycline pair with $\alpha \neq \beta$ and $s(\alpha) = s(\beta) = r(x)$, then $\mathrm{ind}_\alpha(x)$ is periodic.

(iii) If moreover $\mathcal{E}$ is right-cancellative and F is aperiodic, then (α, β) is a cycline pair if and only if $\alpha = \beta$.

Proof. The first part of (i) follows directly from Proposition 3.1.1(i). The second part follows from condition (c); namely, if (α, β) is a cycline pair while $\mathrm{ind}_\alpha(x)$ is aperiodic, we must have $\mathrm{ind}_\alpha(x) = \mathrm{ind}_\beta(x)$, which would imply

$$\mathrm{res}_\alpha(\mathrm{ind}_\alpha(x)) = x = \mathrm{res}_\beta(\mathrm{ind}_\beta(x)) = \mathrm{res}_\beta(\mathrm{ind}_\alpha(x)),$$

and force $\alpha = \beta$.

To see (ii), if (α, β) is a cycline pair with $\alpha \neq \beta$, regardless of the aperiodicity of x, the infinite path $\mathrm{ind}_\alpha(x)$ is not aperiodic since

$$\mathrm{res}_\alpha(\mathrm{ind}_\alpha(x)) = x = \mathrm{res}_\beta(\mathrm{ind}_\beta(x)) = \mathrm{res}_\beta(\mathrm{ind}_\alpha(x)).$$

Finally, to see (iii) assume F is aperiodic. Then for $(\alpha, \beta) \in \mathcal{E} \times \mathcal{E}$ with $s(\alpha) = s(\beta)$, we are guaranteed an aperiodic infinite path $y \in Z(s(\alpha))$. If (α, β) is a cycline pair, then by (c), $\mathrm{ind}_\alpha(y) = \mathrm{ind}_\beta(y)$. But then by Proposition 3.1.1(ii), y is right-aperiodic and thus $\alpha = \beta$. Conversely, if $\alpha = \beta$, they form a cycline pair trivially. $\square$

Note that part (iii) matches [13] Proposition 4.8, but that it requires the further assumption that $\mathcal{E}$ is right-cancellative.

3.3 Cartan Subalgebras of the Path Groupoid

We now move on to considering the relation between cycline pairs and the isotropy of the path groupoid G_F. Recall the path groupoid consists of equivalence classes $[\mu, \nu, x]$ in which $x \in Z(\nu)$, $s(\mu) = s(\nu)$. Since the source of $[\mu, \nu, x]$ is x and the range is $\mathrm{ind}_\mu \circ \mathrm{res}_\nu(x)$, the isotropy consists of classes in which $\mathrm{ind}_\mu \circ \mathrm{res}_\nu(x) = x$, which as mentioned in Section 3.1 is equivalent to $\mathrm{res}_\mu(x) = \mathrm{res}_\nu(x)$.

If (α, β) is a cycline pair, then by Condition (c*), $\mathrm{res}_\alpha(x) = \mathrm{res}_\beta(x)$ for all $x \in Z(\beta)$, and thus $Z(\alpha, \beta) \subset \mathrm{Int}\,\mathrm{Iso}(G_F)$. Conversely, if $Z(\alpha, \beta) \subset \mathrm{Int}\,\mathrm{Iso}(G_F)$, then for all $x \in Z(\beta)$, $\mathrm{ind}_\alpha \circ \mathrm{res}_\beta(x) = x$. Thus

$$(\alpha, \beta) \text{ is cycline} \iff Z(\alpha, \beta) \subset \mathrm{Int}\,\mathrm{Iso}(G_F).$$

Moreover, since $\mathrm{Int}\,\mathrm{Iso}(G_F)$ is a union of open sets and the basis for G_F is given by $\{Z(\mu, \nu)\}$,

$$\mathrm{Int}\,\mathrm{Iso}(G_F) = \bigcup_{(\alpha,\beta) \text{ is a cycline pair}} Z(\alpha, \beta).$$

Thus when $F : \mathcal{E} \to \mathcal{B}$ is a row-finite KPf targeting left-cancellative $\mathcal{B}$ with every slice category $\mathcal{B}/B$ countable, since G_F is a locally compact Hausdorff étale groupoid, the premises of [14] Theorem 3.1 are satisfied. Letting $\iota_r : C_r^*(\mathrm{Int}\,\mathrm{Iso}(G_F)) \to C_r^*(G_F)$ be inclusion, part (b) of that theorem provides a uniqueness theorem; namely, if $\pi : C_r^*(G_F) \to D$ is a C*-homomorphism, then π is injective if and only if $\pi \circ \iota_r$ is an injective homomorphism of $C_r^*(\mathrm{Int}\,\mathrm{Iso}(G_F))$. This is an improvement on [2] Corollary 5.7 as it removes the requirements that F be aperiodic and G_F be amenable. When $\mathcal{B}$ is left-cancellative, the isomorphism $C^*(F) \cong C^*(G_F)$ shows that the generators of $\mathscr{M} = C^*\{S_\alpha S_\beta^*, (\alpha, \beta) \text{ cycline}\}$ correspond to $\{1_{Z(\alpha,\beta)} : (\alpha, \beta) \text{ cycline}\}$, the generators of $C^*(\mathrm{Int}\,\mathrm{Iso}(G_F))$. Thus the Cuntz-Krieger uniqueness can be assessed from the groupoid approach as in the k-graph case.

By [14] Corollary 4.5, if $\mathrm{Int}\,\mathrm{Iso}(G_F)$ is abelian and closed, then $\iota_r(C^*(\mathrm{Int}\,\mathrm{Iso}(G_F))) \subset$

$C_r^*(G_F)$ is a Cartan subalgebra in the sense of [15] Definition 4.5. Cartan subalgebras have been a subject of recent study in [14, 16] and more. Thus if $\operatorname{Int}\operatorname{Iso}(G_F)$ is abelian and closed, the cycline subalgebra $\mathscr{M}$ provides a Cartan subalgebra in $C^*(F)$ as a whole.

While on the topic of the isotropy of G_F, we complete the proof of the following biconditional.

Theorem 3.3.1. *$F : \mathcal{E} \to \mathcal{B}$ is aperiodic if and only if G_F is topologically principal.*

Proof. The ($\Longrightarrow$) direction is given by [2] Theorem 5.6. We will use the same notation, so identify the objects of G_F with F^∞, and given an object $[r(x), r(x), x] =: x$, denote $xG_Fx = \{g \in G_F : r(g) = s(g) = x\} \subset \operatorname{Iso}(G_F)$. Also denote $T = \{x \in F^\infty : xG_Fx = \{x\}\}$, the set of units with trivial isotropy. G_F is topologically principal if the units with trivial isotropy are dense in the unit space of the groupoid.

We provide the ($\Longleftarrow$) direction by contrapositive. Suppose that F is not aperiodic; that is, there exists an $X \in \operatorname{Obj}(\mathcal{E})$ such that for any $x \in Z(X)$, there exist $\alpha \neq \beta \in \mathcal{E}$ with $\operatorname{res}_\alpha(x) = \operatorname{res}_\beta(x)$, or equivalently, $\operatorname{ind}_\beta \circ \operatorname{res}_\alpha(x) = x$. That is, $r([\beta, \alpha, x]) = s([\beta, \alpha, x])$, so x is not in T. As this is the case for all $x \in Z(X)$, $Z(X) \cap T = \emptyset$. Thus T is not dense in F^∞ and G_F is not topologically principal. $\square$

Using the above result, we will give conditions under which the C*-algebra of a KPf is simple, generalized from [1] Proposition 4.8. First, we define a term below.

Definition 3.3.2. A row-finite, strongly surjective KPf $F : \mathcal{E} \to \mathcal{B}$ is called *cofinal* if for all $x \in F^\infty$ and $X \in \operatorname{Obj}(\mathcal{E})$, there exist $\lambda, \mu \in \mathcal{E}$ such that $x \in Z(\mu), s(\lambda) = s(\mu)$, and $r(\lambda) = X$.

Observe that this is a stronger condition than requiring $\mathcal{E}$ to be a connected category, because we insist on a certain direction for the morphisms, rather than allowing any two objects to be connected by zigzags.

Proposition 3.3.1. *Let $F : \mathcal{E} \to \mathcal{B}$ be a row-finite, strongly surjective KPf satisfying the aperiodicity condition, with $\mathcal{B}$ countable and left-cancellative, and $C^*(G_F) = C^*_r(G_F)$. Then $C^*(F)$ is simple if and only if F is cofinal.*

Proof. Since F is row-finite and $\mathcal{B}$ is countable, we have that F^∞ is second-countable, and G_F is a second-countable, étale, locally compact Hausdorff groupoid. Since $\mathcal{B}$ is left-cancellative, we have $C^*(F) \cong C^*(G_F) = C^*_r(G_F)$. Under these conditions, we may apply [17] Theorem 5.1 to state that $C^*(F)$ is simple if and only if G_F is minimal and topologically principal. By Theorem 3.3.1, we have the latter by the aperiodicity condition on F. Thus we show that F being cofinal is equivalent to G_F being minimal.

Assuming F is cofinal, we wish to show that all orbits are dense in $F^\infty = G_F^{(0)}$. Fix $x \in F^\infty$ and $\lambda \in \mathcal{E}$. At object $s(\lambda)$, we have the existence of $\alpha, \beta \in \mathcal{E}$ such that $x \in Z(\alpha)$, $s(\alpha) = s(\beta)$, and $r(\beta) = s(\lambda)$. That is, there is an element $[\lambda\beta, \alpha, x] \in G_F$ and $\mathrm{ind}_{\lambda\beta} \circ \mathrm{res}_\alpha(x) \in Z(\lambda)$ is in the orbit of x. Thus in any basis cylinder set, we can find an infinite path in the same orbit as x, so G_F is minimal.

Conversely, assume G_F is minimal and let $x \in F^\infty$ and $X \in \mathrm{Obj}(\mathcal{E})$. Then in $Z(X)$ there is some infinite path y from the orbit of x; that is, for some $\alpha, \beta \in \mathcal{E}$, we have $y = \mathrm{ind}_\alpha \circ \mathrm{res}_\beta(x)$. Then $x \in Z(\beta)$, $s(\alpha) = s(\beta)$, and $r(\alpha) = X$, satisfying the desired conditions. $\square$

Chapter 4

Examples

The obvious motivating example of a KPf is the degree functor of a row-finite k-graph with no sources, and Chapters 5 and 6 deal with similarly convenient choices of $\mathcal{B}$. Meanwhile, this chapter collects several other examples of KPfs which admit less development due to the lack of properties like cancellativity. The following example summarizes two given in [2].

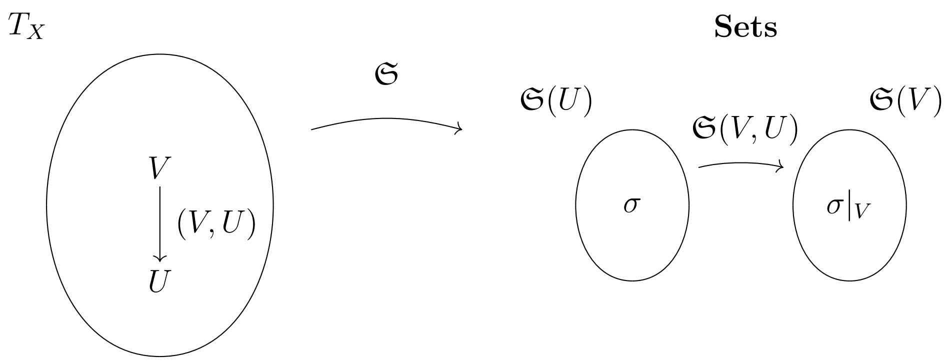
Example 4.0.1 ([2], Examples 3.10 and 3.11)**.** If $\mathcal{B}$ is any strongly right Ore category, $\mathrm{Id}_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$ is a KPf. In particular, if $\mathcal{B} = H$ is a discrete group seen as a one-object category, $C^*(\mathrm{Id}_H) \cong C^*(H)$.

Another example mentioned in that paper was later reformulated in conversation with Yetter, and both are presented with more development in the following section. Meanwhile, we note a previously unmentioned example that has been a subject of previous study by [18].

Example 4.0.2. Let F be a countable discrete abelian group, fix $k \geq 0$, and define $P := \mathbb{N}^k \times F$, viewed as a cancellative abelian monoid. As such, it forms a cancellative one-object category and is strongly right Ore. A P-*graph* consists of a small category Λ equipped with a functor $d : \Lambda \to P$ satisfying the unique factorization property. Therefore, P-graphs are examples of KPfs.

4.1 KPfs Arising From a Presheaf of Sets on a Topological Space

Example 4.1.1 ([2], Example 3.12)**.** Let X be a topological space, and $\mathfrak{S}$ a presheaf of sets on X; that is, a contravariant functor from the lattice of open sets T_X ordered by inclusion (namely, there is an arrow $V \to U$ when $V \subset U$, written (V, U)) to **Sets**, the category of sets and set-functions. Regard the images of elements of $\mathfrak{S}(U)$ under maps $\mathfrak{S}(V, U) : \mathfrak{S}(U) \to \mathfrak{S}(V)$ as restrictions, denoting $\mathfrak{S}(V, U)(\sigma)$ by $\sigma|_V$ for $\sigma \in \mathfrak{S}(U)$.



Let $\mathcal{S}$ be the set of local sections, that is, pairs (U, σ) where U is an open set and $\sigma \in \mathfrak{S}(U)$. Then $\mathcal{S}$ is partially ordered by restriction, with $(V, \tau) \leq (U, \sigma)$ exactly when $V \subset U$ and $\tau = \sigma|_V$. Seeing $\mathcal{S}$ as a category whose objects are pairs and with arrows arising from the partial order, $((V, \sigma|_V), (U, \sigma)) \mapsto (V, U)$ defines a functor $F : \mathcal{S} \to T_X$. If $((V, \sigma|_V), (U, \sigma)) \in \mathcal{S}$ and $V \subset W \subset U$ so that $(V \to W \to U)$ is a factorization of $F((V, \sigma|_V), (U, \sigma)) = (V, U)$, then $((V, \sigma|_V), (W, \sigma|_W), (U, \sigma))$ is the unique factorization of $((V, \sigma|_V), (U, \sigma))$ lifting the factorization of (V, U) as $(V \to W \to U)$. Thus F is a dCf.

Since a set function cannot take an element $\sigma \in \mathfrak{S}(U)$ to $\tau_1 \neq \tau_2$ images in $\mathfrak{S}(V)$, this F is row-finite.

Since T_X has pullbacks by intersections of sets, it is strongly right Ore. And, finally, for any object $(U, \sigma) \in \mathcal{S}$, the induced functor $F : \mathcal{S}/(U, \sigma) \to T_X/U$ admits a splitting given by $(V, U) \mapsto ((V, \sigma|_V), (U, \sigma))$, and thus $F : \mathcal{S} \to T_X$ is a KPf.

A splitting of the KPf defined above targeting T_X is in fact completely determined by its range object, and the cylinder sets $Z(U, \sigma)$ are singletons. Indeed, the endings of $x \in Z(U, \sigma)$ are dictated by the set functions defining the restrictions $\sigma|_V$ for all $V \subset U$. As the sets $Z(U, \sigma)$ forming the basis for the infinite path space are singletons, F^∞ in this case is locally compact Hausdorff and moreover totally disconnected.

To investigate the path groupoid for this KPf, we introduce a notation for morphisms in $\mathcal{S}$ that is shorter: $((V, \sigma|_V), (U, \sigma)) =: (V, U, \sigma)$. The range is (U, σ) and the source is $(V, \sigma|_V)$. If $x_{(U,\sigma)}$ is the infinite path to (U, σ), then $x_{(U,\sigma)}$ ends in the morphism (V, U, σ) for any (V, U) targeting U in T_X. The units $[(U, \sigma), (U, \sigma), x_{(U,\sigma)}]$ may be identified with $x_{(U,\sigma)} \in F^\infty$, and elements in G_F are triples of the form

$$[(V, U', \sigma'), (V, U, \sigma), x_{(U,\sigma)}]$$

because the endings must have the same source, $(V, \sigma|_V)$, but can have different ranges $(U', \sigma'), (U, \sigma)$. In fact, we can show that the choice of V above is irrelevant to the equivalence class.

Proposition 4.1.1. *The elements of G_F are determined by the U, U', σ, and σ' that appear in them. In particular:*

$$[(V, U', \sigma'), (V, U, \sigma), x_{(U,\sigma)}] = [(U \cap U', U', \sigma'), (U \cap U', U, \sigma), x_{(U,\sigma)}].$$

Proof. We refer to the equivalence relation defined in 1.3. Condition 1 is satisfied. Condition 3 is satisfied by trivial a and the b shown in the diagram below.

$$\begin{tikzcd} & & U \\ V \arrow[urr, bend left] \arrow[r, "\exists!\, b"] \arrow[drr, bend right] & U \cap U' \arrow[ur] \arrow[dr] & \\ & & U' \end{tikzcd}$$

Namely, $V \subset U$ and $V \subset U'$ yields a unique $b : V \to (U \cap U')$, which with trivial a satisfies

$$(V, U')a = (V, U') = (U \cap U', U')b, \quad (V, U)a = (V, U) = (U \cap U', U)b.$$

To see that condition 2 is satisfied, take $\lambda = (V, U, \sigma)$. For y ending in (V, U, σ), both $\mathrm{ind}_{(V,U',\sigma')} \circ \mathrm{res}_{(V,U,\sigma)}(y)$ and $\mathrm{ind}_{(U\cap U',U',\sigma')} \circ \mathrm{res}_{(U\cap U',U,\sigma)}(y)$ are infinite paths to the object (U', σ'). Since there is exactly one such $x_{(U',\sigma')}$, these are equal. $\square$

Thus the basis for the path groupoid consists of sets $Z((U \cap U', U', \sigma'), (U \cap U', U, \sigma))$, where $\sigma \in \mathfrak{S}(U), \sigma' \in \mathfrak{S}(U')$, and $\sigma|_{U\cap U'} = \sigma'|_{U\cap U'}$. These sets are also singletons, so G_F has the discrete topology.

This F is aperiodic, since if

$$\mathrm{res}_{(V,U,\sigma)}(x_{(U,\sigma)}) = \mathrm{res}_{(V',U,\sigma)}(x_{(U,\sigma)})$$

then that infinite path $\mathrm{res}_{(V,U,\sigma)}(x_{(U,\sigma)})$ is in the intersection of $Z(V, \sigma|_V) \cap Z(V', \sigma|_{V'})$, but since the sets are singletons, $Z(V, \sigma_V) = Z(V', \sigma|_{V'})$. Then $V = V'$ and $(V, U, \sigma) = (V', U, \sigma)$.

This concludes the discussion of the KPf targeting T_X arising from a presheaf $\mathfrak{S}$. However, another KPf may be defined using the same concepts, and which we present as another example.

Example 4.1.2. Let X be a topological space and let $\mathfrak{S}$ be a presheaf on T_X. We must in this case require that $\mathfrak{S}$ is a flasque presheaf; that is, for all $V \subset U$, the set function $\mathfrak{S}(V, U)$ is surjective. Now let $\mathcal{B} = T_X^{op}$ be the lattice of open sets ordered by $U \to V$ when $V \subset U$. Once again, for morphism $U \to V$ in T_X written as (U, V), we denote the image of $\sigma \in \mathfrak{S}(U)$ under $\mathfrak{S}(V, U)$ by $\sigma|_V$, but it may be more intuitive to think of σ as an extension in $\mathfrak{S}(U)$, a preimage of τ under the set function $\mathfrak{S}(V, U)$. As a set function need not be injective, there may be many such extensions, which ends up providing more variety in this example.

Let $\mathcal{A}$ be a category whose objects are $\mathrm{Obj}(\mathcal{A}) = \{(V,\tau) : \tau \in \mathfrak{S}(V)\}$ and morphisms arise as follows:

$$(U,\sigma) \longrightarrow (V,\tau) \quad \iff \quad V \subset U, \sigma|_V = \tau.$$

Again $\mathcal{A}$ is partially ordered by $(U,\sigma) \leq (V,\tau)$ when $V \subset U$ and $\sigma|_V = \tau$. (Antisymmetric: If $V \subset U$ and $U \subset V$, and $\sigma|_V = \tau$ and $\tau|_U = \sigma$, then $U = V$ and $\sigma = \tau$. Transitive: if $(U,\sigma) \leq (V,\tau)$ and $(V,\tau) \leq (W,\rho)$, then $V \subset U$ and $\sigma|_V = \tau$, and $W \subset V$ and $\tau|_W = \rho$, so we have $W \subset V \subset U$ and $\sigma|_W = (\sigma|_V)|_W = \rho$. Then $(U,\sigma) \leq (W,\rho)$.) Note that $\mathcal{A}$ is the same as $\mathcal{S}^{op}$ when taking the $\mathcal{S}$ from the previous example.

Now we define a functor $F : \mathcal{A} \to T_X^{op}$ by $(U,\sigma) \mapsto U$ and $((U,\sigma),(V,\tau)) \mapsto (U,V)$. Note that in this definition, F is not necessarily row-finite, but it is when $\mathfrak{S} : T_X^{op} \to \mathbf{FinSet}$. To see this, let $(V,\tau) \in \mathrm{Obj}(\mathcal{A})$ so that $F(V,\tau) = V$ and take (U,V) targeting V in T_X^{op}. We need to see that the set of morphisms in $\mathcal{A}$ targeting (V,τ) which are taken by F to (U,V) is a finite set. But these arise from objects (U,σ) where $\sigma \in \mathfrak{S}(U)$ and $\sigma|_V = \tau$, so if $\mathfrak{S}(U)$ is a finite set, so is the set of morphisms in question.

The functor F is a dCf. In this case, given a morphism in $\mathcal{A}$

$$(U,\sigma) \longrightarrow (V,\tau)$$

whose image under F, namely (U,V), has a factorization $(W,V)(U,W)$ in T_X^{op}, we know that $V \subset W \subset U$. Then $\sigma|_W$ also satisfies $(\sigma|_W)|_V = \tau$ since $\mathfrak{S}(U,V) = \mathfrak{S}(W,V) \circ \mathfrak{S}(U,W)$, so the unique lift in $\mathcal{A}$ is

$$(U,\sigma) \longrightarrow (W,\sigma|_W) \longrightarrow (V,\tau).$$

As an extension of τ, $\sigma|_V \in \mathfrak{S}(W)$ is not necessarily the only preimage under $\mathfrak{S}(W,V)$, but as the intermediary between σ and τ, it is determined uniquely. That is, if there were two choices of extension $\rho_1, \rho_2 \in \mathfrak{S}(V)$, $\sigma \in \mathfrak{S}(U)$ could only be an extension of one of them to that U, since $\mathfrak{S}(U,W)$ cannot take σ to two different restrictions.

T_X^{op} is strongly right Ore, with pullbacks given by unions of opens.

Finally, we show that F is locally split, and thus a KPf. Let (V, τ) be an object in $\mathcal{A}$, so $F(V, \tau) = V$ in T_X^{op}, and consider the induced functor $F : \mathcal{A}/(V, \tau) \to T_X^{op}/V$ between slice categories. Objects in the former are morphisms targeting (V, τ); that is, they come from objects (U_i, σ_i), where σ_i is a choice of extension for τ. For a morphism $(U, V) \in T_X^{op}/V$, define a splitting $x : T_X^{op}/V \to \mathcal{A}/(V, \tau)$ by $x(U, V) = ((U, \sigma), (V, \tau))$ with σ an extension of τ. We know that at least one such σ exists by our assumption that $\mathfrak{S}$ was a flasque presheaf.

The set $Z(V, \tau)$ can be written as a disjoint union

$$Z(V, \tau) = \bigcup_{U \in T_X : V \subset U} \left(\bigcup_{\sigma \in \mathfrak{S}(U) : \sigma|_V = \tau} Z((U, \sigma), (V, \tau)) \right).$$

For convenience, we again use a shorthand notation (U, σ, V) to mean the arrow from source object (U, σ) to range $(V, \sigma|_V)$.

Cylinder sets $Z(U, \sigma, V)$ for particular morphism endings can also be written as a disjoint union

$$Z(U, \sigma, V) = \bigcup_{W \in T_X : W \subset U} \left(\bigcup_{\rho \in \mathfrak{S}(W) : \rho|_U = \sigma} Z(W, \rho, V) \right).$$

In other words, it is not enough that ρ restricts to τ when in $\mathfrak{S}(V)$; it must have restricted through the specific σ which then restricts to τ.

When F is row-finite, F^∞ is Hausdorff with the topology generated by the cylinder sets above, but more cannot be said without assumptions on the presheaf $\mathfrak{S}$ and/or the topological space X. In this formulation as in the former, F is aperiodic. To see this, let $x \in Z(V, \tau)$ and specifically $x \in Z(U, \sigma, V) \cap Z(W, \rho, V)$ for two endings. Suppose

$$\mathrm{res}_{(U,\sigma,V)}(x) = \mathrm{res}_{(W,\rho,V)}(x).$$

The left-hand side is an infinite path to object (U, σ) and the right-hand side is an infinite path to object (W,ρ), where $\rho|_V = \tau = \sigma|_V$. But cylinder sets to different objects are disjoint, so we must have had $(U,\sigma) = (W,\rho)$. In $\mathcal{A}$ there is at most one arrow between two given objects, so indeed we must have had $(U,\sigma,V) = (W,\rho,V)$. Thus F is aperiodic.

Notably, the infinite paths are exactly the global sections of the presheaf of sets over the topological space, in that an infinite path to any object in fact defines infinite paths to any object of $\mathcal{A}$.

Indeed, let $(V,\tau) \in \mathrm{Obj}(\mathcal{A})$ and fix $x \in Z(V,\tau)$. Let U be any other open in T_X. The infinite path x chooses an extension σ of τ in $\mathfrak{S}(V \cup U)$, that is, $\sigma|_{V\cup U} = \tau$. Restriction maps now define choices all the way down to the object $(U, \sigma|_U)$ in $\mathcal{A}$. Thus we can induce from x an infinite path $x_U \in Z(U,\sigma|_U)$, where σ is such that $x(V\cup U) = (V \cup U, \sigma)$. We may write

$$x_U = \mathrm{ind}_{(V\cup U,\sigma,U)} \circ \mathrm{res}_{(V\cup U,\sigma,V)}(x).$$

Thus a particular infinite path in fact determines a choice of object (U,σ) for every $U \in \mathrm{Obj}(T_X^{op})$, defining a global section.

There are several equivalent ways of writing the relationship between an x and an induced x_U. The equality above is the same as

$$\mathrm{res}_{(V\cup U,\sigma,U)}(x_U) = \mathrm{res}_{(V\cup U,\sigma,V)}(x).$$

And then since

$$\begin{aligned}
\mathrm{ind}_{(V,\tau,V\cap U)}(x) &= \mathrm{ind}_{(V,\tau,V\cap U)} \circ \mathrm{ind}_{(V\cup U,\sigma,V)} \circ \mathrm{res}_{(V\cup U,\sigma,V)}(x) \\
&= \mathrm{ind}_{(V\cup U,\sigma,V\cap U)} \circ \mathrm{res}_{(V\cup U,\sigma,V)}(x) \\
&= \mathrm{ind}_{(V\cup U,\sigma,V\cap U)} \circ \mathrm{res}_{(V\cup U,\sigma,U)}(x_U) \\
&= \mathrm{ind}_{(U,\sigma|_U,V\cap U)}(x_U),
\end{aligned}$$

it is also the same as saying

$$\mathrm{ind}_{(V,\tau,V\cap U)}(x) = \mathrm{ind}_{(U,\sigma|_U,V\cap U)}(x_U).$$

To consider G_F explicitly, its elements are equivalence classes $[(U,\sigma,V'),(U,\sigma,V),x]$ where x ends in (U,σ,V). However, the U can be replaced by an open depending only on V and V'.

Proposition 4.1.2. *The basis for G_F is given by cylinder sets*

$$Z((V\cup V',\sigma,V'),(V\cup V',\sigma,V)),$$

where the source $(V\cup V',\sigma)$ agrees between the two morphisms, and their ranges are (V,τ) and (V',τ') with $\sigma|_V=\tau,\sigma|_{V'}=\tau'$. In other words, the equivalence class

$$[(V\cup V',\sigma,V'),(V\cup V',\sigma,V),x],$$

where $\sigma\in\mathfrak{S}(V\cup V')$ and $\sigma|_V=\tau,\sigma|_{V'}=\tau'$, collects any triple $((U,\rho,V'),(U,\rho,V),x)$ such that $(V\cup V')\subset U$ and $\rho|_{V\cup V'}=\sigma$.

Proof. We wish to show that when $(V\cup V')\subset U$ and $\rho|_{V\cup V'}=\sigma$,

$$((U,\rho,V'),(U,\rho,V),x)\sim((V\cup V',\sigma,V'),(V\cup V',\sigma,V),x),$$

where $\sim$ is the equivalence relation defined in Section 1.3. Condition 1 is satisfied. To show condition 3, we wish to find $a,b\in T_X^{op}$ such that

$$F(U,\rho,V')a = F(V\cup V',\sigma,V')b,\quad F(U,\rho,V)a = F(V\cup V',\sigma,V)b,$$

that is, $a,b\in T_X^{op}$ such that $(U,V')a=(V\cup V',V')b$ and $(U,V)a=(V\cup V',V)b$. But since

$V \subset U$ and $V' \subset U$, we have $V \cup V' \subset U$, and there is an arrow $(U, V \cup V')$ such that

$$(V \cup V', V') \circ (U, V \cup V') = (U, V'), \quad (V \cup V', V) \circ (U, V \cup V') = (U, V).$$

That is, let $a = (U, V \cup V')$ and let b be trivial.

$$\begin{tikzcd}
 & & V \\
U \arrow[r, "\exists!\, a"] \arrow[urr, bend left] \arrow[drr, bend right] & V \cup V' \arrow[ur] \arrow[dr] & \\
 & & V'
\end{tikzcd}$$

Condition 2 is satisfied by taking $\lambda = (U, \rho, V)$. In that case, $Z(U, \rho, V) \subset (Z(U, \rho, V) \cap Z(V \cup V', \sigma, V))$, and since $\rho|_{V \cup V'} = \sigma$, we have

$$\begin{aligned}
\operatorname{ind}_{(U,\rho,V')} \circ \operatorname{res}_{(U,\rho,V)}(y) &= \operatorname{ind}_{(V\cup V',\rho|_{V\cup V'},V')(U,\rho,V\cup V')} \circ \operatorname{res}_{(V\cup V',\rho|_{V\cup V'},V)(U,\rho,V\cup V')}(y) \\
&= \operatorname{ind}_{(V\cup V',\sigma,V')} \circ \operatorname{res}_{(V\cup V',\sigma,V)}(y)
\end{aligned}$$

for $y \in Z(U, \rho, V)$.

Thus $[(U, \rho, V'), (U, \rho, V), x] = [(V \cup V', \sigma, V'), (V \cup V', \sigma, V), x]$ and we may write the cylinder sets in terms of the latter morphisms as representatives. $\square$

Remark 4.1.3. It follows that there is at most one arrow in G_F between any two objects (identified with) $x, y \in F^\infty$, so that G_F is a principal groupoid. It is of course étale, and since T_X^{op} is cancellative, the isomorphism $C^*(F) \cong C^*(G_F)$ holds in this example.

We conclude by explaining how Theorem 1.4.2 applies to this example. If we define a preorder on the morphisms of T_X^{op} by $(V, W) \leq (U, W)$ when $V \subset U$, then we satisfy the condition that $b_1 \leq b_2$ in $\mathcal{B}$ implies the existence of $a \in \mathcal{B}$ such that $b_1 a = b_2$. Namely, $a = (U, V)$ satisfies $(V, W)(U, V) = (U, W)$. Thus if T_X^{op} is countable, the core of $C^*(F)$ in this case is AF.

4.2 Projections on a Hilbert Space

Let $\mathcal{B}$ be a Hilbert lattice, i.e. the lattice of closed linear subspaces of a Hilbert space (with preorder given by the inclusion). This is an orthocomplemented lattice (equipped with an involution that sends each element to a complement) and in fact an orthomodular lattice. Require that the Hilbert space $\mathcal{H}$ is finite-dimensional, so that the image of a bounded linear operator $T \in B(\mathcal{H})$ is a closed subspace.

Let $\mathcal{E}$ be a category whose objects are the bounded linear operators on $\mathcal{H}$ and whose morphisms are arrows $(A, B) : A \to B$ whenever $B = P_W A$ for a projection P_W onto a subspace $W \subset \operatorname{Im}(A)$. This does form a category since $A = P_{\operatorname{Im}(A)} A$ shows the existence of identity morphisms, and composition of morphisms behaves well: If $B = P_W A$ and $C = P_{W'} B$, then $C = P_{W'}(P_W A)$. But $W' \subset W \subset \operatorname{Im}(A)$, so $P_{W'}$ is a subprojection of P_W and $C = P_{W'} A$ gives the morphism from A to C through B.

Defining a functor $F : \mathcal{E} \to \mathcal{B}$ by $F(A) = \operatorname{Im}(A)$, we claim that it is a dCf. Consider a morphism $A \longrightarrow P_W A$ in $\mathcal{E}$ and its image under F, $\operatorname{Im}(A) \longrightarrow W$. Suppose there is a factorization of the morphism in $\mathcal{B}$ through another subspace W' where $W \subset W' \subset \operatorname{Im}(A)$. Then the unique lift of the factorization to $\mathcal{E}$ is the composition $A \longrightarrow P_{W'} A \longrightarrow P_W A$.

This dCf is surjective on objects, since for any object $W \in \mathcal{B}$, we can take as preimage the projection $P_W \in \mathcal{E}$ so that $F(P_W) = W$. Moreover, it is strongly surjective, since given an operator $B \in \operatorname{Obj}(\mathcal{E})$, any morphism in $\mathcal{B}$ targeting $F(B) = \operatorname{Im}(B)$ arises from a subspace W containing $\operatorname{Im}(B)$. We can extend to an operator $A \in \mathcal{E}$ with $\operatorname{Im}(A) = W$ and $B = P_{\operatorname{Im}(B)} A$. Then $F(A, B) = (W, \operatorname{Im}(B))$.

F is also a KPf, as it admits splittings x which make the choice of operator amongst those with a certain image space. In terms of slice categories, $x : \mathcal{B}/\operatorname{Im}(B) \to \mathcal{E}/B$ is defined on morphisms $(W, \operatorname{Im}(B))$ by mapping them to (A, B) with $\operatorname{Im}(A) = W$ as in the process above. Then of course $F \circ x(W, \operatorname{Im}(B)) = F(A, B) = (\operatorname{Im}(A), \operatorname{Im}(B)) = (W, \operatorname{Im}(B))$ as desired.

F is by no means row-finite, but it is aperiodic.

Due to the bijection between projections and closed subspaces, this could be equivalently formulated by taking as $\mathcal{B}$ the lattice of projections and defining $F : \mathcal{E} \to \mathcal{B}$ by $F(A) = P_{\mathrm{Im}(A)}$. Then F fixes those operators in $\mathcal{E}$ that are already projections.

4.3 The Bicyclic Semigroup

Let $\mathcal{B}$ denote the *bicyclic semigroup*, which is in fact the free monoid on two generators a, b under the relation $ab = 1$. Due to this relation, the elements of $\mathcal{B}$ are of the form $b^p a^q$ for $p, q \in \mathbb{N}$, and may be represented as pairs (p, q) accordingly. Composition of these pairs is given by

$$(p,q)(p',q') = (p + p' - \min\{q, p'\}, q + q' - \min\{q, p'\}).$$

We will denote the identity by $(0, 0)$.

As a non-commutative monoid, it conveniently is seen as a one-object category and a good candidate for the choice of target category for a KPf $F : \mathcal{E} \to \mathcal{B}$. However, we must see that it is in fact strongly right Ore. This takes a bit of work since $\mathcal{B}$ is neither right- nor left-cancellative. Afterward, we will investigate the basic example $F = \mathrm{Id}_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$.

Proposition 4.3.1. *The bicyclic semigroup $\mathcal{B}$, seen as a one-object category, is strongly right Ore.*

Proof. Firstly, $\mathcal{B}$ has the basic right Ore condition. Let $(p, q), (p', q') \in \mathcal{B}$, which form a cospan since there is only one object. Without loss of generality, suppose that $p' \geq p$. Then since

$$(p,q)(q + p' - p, q') = (p', q')(0, 0),$$

$\mathcal{B}$ is right Ore.

Secondly, we must show the strong right Ore condition. To that end, we will show that for any other pair $(m,n),(m',n') \in \mathcal{B}$ satisfying

$$(p,q)(m,n) = (p',q')(m',n'),$$

there exist $(k_1,k_2),(k_3,k_4) \in \mathcal{B}$ such that

$$(q+p'-p,q')(k_1,k_2) = (m,n)(k_3,k_4) \quad \text{and} \quad (0,0)(k_1,k_2) = (m',n')(k_3,k_4). \qquad (\star)$$

Observe that the assumption computed componentwise implies the following relations:

$$\begin{cases} p+m-\min\{q,m\} = p'+m'-\min\{q',m'\} \\ q+n-\min\{q,m\} = q'+n'-\min\{q',m'\} \end{cases} \qquad (\star\star)$$

and subtracting these equations yields

$$p+m-q-n = p'+m'-q'-n'.$$

Case 1: Assume that $q \leq m$. We take the obvious choice for the second half of $(\star)$, so that $(k_1,k_2) = (m',n')$ and $(k_3,k_4) = (0,0)$. We wish to show that

$$(q+p'-p,q')(m',n') = (m,n)(0,0)$$

also holds.

In this case, $\min\{q,m\} = q$, so by solving $(\star\star)$ for $\min\{q',m'\}$, the first component of the left hand side is indeed

$$q+p'-p+m'-\min\{q',m'\} = q+p'-p+m'-(p'+m'-p-m+q) = m$$

and the second component is indeed

$$q' + n' - \min\{q', m'\} = q' + n' - (q' + n' - q - n + q) = n.$$

Case 2: Assume $q > m$, so that $\min\{q, m\} = m$. Then the first equation in $(\star\star)$ is in particular

$$p + m - m = p' + m' - \min\{q', m'\},$$

which rearranged is $(p - p') = m' - \min\{q', m'\}$. Now $(p - p') \leq 0$ since $p' \geq p$, but $m' - \min\{q', m'\} \geq 0$. Thus both sides must equal zero, and in fact we must have $p = p'$ and $\min\{q', m'\} = m'$. These facts in the second part of $(\star\star)$ yield

$$q + m' - n' - q' = m - n.$$

Our adjusted goal is to show that there exist $(k_1, k_2), (k_3, k_4) \in \mathcal{B}$ such that

$$(q, q')(k_1, k_2) = (m, n)(k_3, k_4) \quad \text{and} \quad (0, 0)(k_1, k_2) = (m', n')(k_3, k_4).$$

Choose $k_3 \geq \max\{q' - m' + n', n\}$, which also implies $k_3 \geq n'$ since $q' - m' \geq 0$ in this case. Then taking $k_1 = m' + k_3 - n'$ and letting $k_2 = k_4$ be arbitrary, we know $k_1 \geq$

$m' + (q' - m' + n') - n' = q'$ and we have

$$\begin{aligned}
(q, q')(k_1, k_2) &= (q + m' + k_3 - n' - \min\{q', k_1\}, q' + k_2 - \min\{q', k_1\}) \\
&= (q + m' + k_3 - n' - q', q' + k_2 - q') \\
&= (k_3 + m - n, k_2) \\
&= (m + k_3 - n, k_4) \\
&= (m + k_3 - \min\{n, k_3\}, n + k_4 - \min\{n, k_3\}) \\
&= (m, n)(k_3, k_4)
\end{aligned}$$

and

$$\begin{aligned}
(0, 0)(k_1, k_2) &= (m' + k_3 - n', k_2) \\
&= (m' + k_3 - \min\{n', k_3\}, n' + k_4 - \min\{n', k_3\}) \\
&= (m', n')(k_3, k_4).
\end{aligned}$$

Thus $\mathcal{B}$ is strongly right Ore. □

As such, $F = \mathrm{Id}_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$ is an example of a KPf, where the splitting is simply $x = F = \mathrm{Id}_{\mathcal{B}}$ and the infinite path space consists of a singleton. A Cuntz-Krieger system for F in a C*-algebra A would be generated by partial isometries $\{s_{(1,0)}, s_{(0,1)}\}$, since for any morphism $(p, q) \in \mathcal{B}$, we have $(p, q) = (p, 0)(0, q) = (1, 0)^p(0, 1)^q$ and 1.1.4 Condition 2 must hold for $s_{(p,q)}$. In addition, since there is only one object in $\mathcal{B}$ and F is the identity functor, Condition 4 and 6 imply that $s_{(0,1)}$ and $s_{(1,0)}$ are in fact unitaries.

Although in the case of a discrete group H, as discussed in [2] Example 3.11, $C^*(H) \cong C^*(\mathrm{Id}_H)$, we do not have the same result with this semigroup. The C*-algebra of $\mathcal{B}$ as a semigroup has been studied as an example of an inverse semigroup; see [19], for example. Under that treatment, $C^*(\mathcal{B}) \cong C^*(S)$, the C*-algebra generated by the unilateral shift on ℓ^2. As the unilateral shift S does not satisfy $S^*S = SS^*$, or as $s_{(1,0)} \neq s^*_{(0,1)}$, we have $C^*(\mathrm{Id}_{\mathcal{B}}) \ncong C^*(\mathcal{B})$.

Regarding the path groupoid $G_{\mathrm{Id}_{\mathcal{B}}}$, we begin with triples of the form $((p',q'),(p,q),x)$, where $x = F = \mathrm{Id}_{\mathcal{B}}$ in this case, and consider the effects of the equivalence relation conditions from Section 1.3.

Proposition 4.3.2. *Let $((p',q'),(p,q),x)$ be a triple in $\mathcal{G}_{\mathrm{Id}_{\mathcal{B}}}$, and denote by $\sim$ the equivalence relation used to form the path groupoid $G_{\mathrm{Id}_{\mathcal{B}}}$. If $q' \geq q$, then $(p',q') = (p', p+q'-q)(p,q)$ and*

$$((p',q'),(p,q),x) \sim ((p', p+q'-q),(0,0),x).$$

In particular, any $[(p',q),(p,q),x] = [(p',p),(0,0),x]$. Furthermore, these equivalence classes may be written in the form $[(k,\ell),(0,0),x]$ for $k,\ell \in \mathbb{N}$. The equivalence class $[(k,\ell),(0,0),x]$ collects any triple $((k',\ell'),(0,0),x)$ such that $k-\ell = k'-\ell'$.

If $q' < q$, then

$$((p',q'),(p,q),x) \sim ((p',0),(p,q-q'),x).$$

These equivalence classes may be written in the form $[(k,0),(\ell,m),x]$ for $k,\ell,m \in \mathbb{N}$, which will collect any triple $((k',0),(\ell',m'),x)$ such that $k-\ell+m = k'-\ell'+m'$.

Proof. In all of the above statements, Condition 1 is satisfied by $x = x$. Condition 2 is satisfied since there is only one infinite path in F^∞ in this example. To satisfy Condition 3, observe for the first statement that

$$(p',q')(0,0) = (p', p+q'-q)(p,q) \quad \text{and} \quad (p,q)(0,0) = (0,0)(p,q).$$

For the second statement, observe

$$(p',q')(q',0) = (p',0)(0,0) \quad \text{and} \quad (p,q)(q',0) = (p,q-q')(0,0).$$

Thus an equivalence class $[(p', p+q'-q), (0,0), x]$ may be represented by the two numbers $k = p', \ell = p + q' - q$. Moreover, if $k - \ell = k' - \ell'$ for $k, k', \ell, \ell' \in \mathbb{N}$ and we wish to show $((k', \ell'), (0,0), x) \in [(k, \ell), (0,0), x]$, Condition 3 requires the existence of $(i, j) \in \mathcal{B}$ such that

$$(k, \ell)(i, j) = (k', \ell')(i, j),$$

as that will obviously also satisfy $(0,0)(i,j) = (0,0)(i,j)$. But simply choose $i \geq \max\{\ell, \ell'\}$ so that computing componentwise, we have indeed

$$k + i - \ell = k' + i - \ell' \quad \text{and} \quad \ell + j - \ell = \ell' + j - \ell'.$$

The converse also holds since if $((k', \ell'), (0,0), x) \in [(k, \ell), (0,0), x]$, the same equations must hold, yielding $k - \ell = k' - \ell'$ by cancelling the i.

Finally, represent $[(p', 0)(p, q - q'), x]$ by $[(k, 0), (\ell, m), x]$ for $k, \ell, m \in \mathbb{N}$. We wish to show this equivalence class contains any $((k', 0), (\ell', m'), x)$ when $k - \ell + m = k' - \ell' + m'$. That is, we want to find (i, j) and (i', j') such that

$$(k, 0)(i, j) = (k', 0)(i', j') \quad \text{and} \quad (\ell, m)(i, j) = (\ell', m')(i', j').$$

Let $j = j'$ be an arbitrary choice and choose $i > m, i' > m'$ such that $i - i' = k - k'$. Then for the first equation we have

$$k + i - 0 = k' + i' - 0 \quad \text{and} \quad 0 + j - 0 = 0 + j - 0.$$

Due to our choice of i and i', both are true. For the second desired equation, we have

$$\ell + i - m = \ell' + i' - m' \quad \text{and} \quad m + j - m = m' + j - m'.$$

But these are also true, since we can rearrange to $\ell - \ell' + m' - m = i' - i = k - k'$, which is satisfied by $k - \ell + m = k' - \ell' + m'$.

$\square$

Briefly observe after these cospan- and span-style computations that

In $\mathcal{B}$, morphisms can be broken down into pieces in a way that appears convenient at first glance. That is, given $(p', q'), (p, q) \in \mathcal{B}$, if $p' \geq p$, we can always factor $(p', q') = (p, q)(q + p' - p, q')$, and as long as $q' \geq q$, we can factor $(p', q') = (p', p + q' - q)(p, q)$. In other words, we can find the "ending" of a morphism of any size we want whose first component is smaller, and we can find a "beginning" of a morphism of any size we want whose second component is smaller. However, the myriad factorizations available in $\mathcal{B}$ have made it difficult to invent an $F : \mathcal{E} \to \mathcal{B}$ that can satisfy the unique lifting property. We have not as yet found any examples of a KPfs targeting it beyond the trivial one described above. We hope to find more interesting examples if they exist.

Chapter 5

$\mathbb{N}$-graphs

While attempting to formulate examples of dCfs and KPfs, an obvious choice of strongly right Ore category for $\mathcal{B}$ is a familiar commutative monoid such as $(\mathbb{Z}_{>0}, \times)$. While considering this option, we realized that a good example would be a generalization of k-graphs to infinite k; that is, the degree map targets $\mathbb{N}^{\mathbb{N}}$ rather than $\mathbb{N}^k$. We searched to see if this special case had already been investigated, and discovered that Tim Schenkel had introduced a formulation, calling them $\mathbb{N}$-graphs, and showed that many features of k-graph C*-algebras still hold in the $\mathbb{N}$-graph case. Thus $\mathbb{N}$-graphs may be approached "from above" as a specific example of a KPf, or "from below" as a generalization of k-graphs. In this chapter, we show that our approach is consistent with the definitions and notation established by Schenkel, and as he did not intend to fill in the theory on the infinite path space and path groupoid for $\mathbb{N}$-graphs, we continued the work and also investigated the amenability of the path groupoid and the aperiodicity condition options.

5.1 Formulations

We first summarize Schenkel's definition of an $\mathbb{N}$-graph with examples.

Definition 5.1.1 ([20], Definition 2.1 and discussion following)**.** An $\mathbb{N}$-*graph* (Λ, d) consists of a countable category Λ together with a functor $d : \Lambda \to \mathbb{N}_0^{\mathbb{N}}$ (where $\mathbb{N}_0^{\mathbb{N}}$ is the subgroup of $\mathbb{N}^{\mathbb{N}}$ whose elements have finitely many nonzero entries) satisfying the factorization property: For every $\lambda \in \Lambda$ with $d(\lambda) = m + n$ there are unique elements $\mu, \nu \in \Lambda$ such that $\lambda = \mu\nu$ and $d(\mu) = m, d(\nu) = n$. We will also denote this μ and ν by $\lambda(0, m)$ and $\lambda(m, d(\lambda))$, respectively, and for general $j < k \leq d(\lambda)$, $\lambda(j, k)$ will denote the element such that $\lambda(0, j)\lambda(j, k)\lambda(k, d(\lambda))$, which is well-defined by unique factorization.

As per usual convention, $\mathrm{Obj}(\Lambda)$ may be denoted Λ^0, the range and source maps are denoted r and s, respectively, and the notation $u\Lambda$ may be used to denote the set of morphisms in Λ with range object $u \in \Lambda^0$.

An $\mathbb{N}$-*graph morphism* between (Λ_1, d_1) and (Λ_2, d_2) is a functor $f : \Lambda_1 \to \Lambda_2$ intertwining the degree maps.

An $\mathbb{N}$-graph is called *row-finite* if for each $n \in \mathbb{N}_0^{\mathbb{N}}$ and $v \in \Lambda^0$, the set

$$\Lambda^n(v) := \{\lambda \in \Lambda : d(\lambda) = n, r(\lambda) = v\}$$

is finite. Similarly, Λ is said to have *no sources* if $\Lambda^n(v) \neq \emptyset$ for any $n \in \mathbb{N}_0^{\mathbb{N}}$.

Henceforth, assume any $\mathbb{N}$-graph mentioned is row-finite and has no sources. We will denote by e_i the element of $\mathbb{N}_0^{\mathbb{N}}$ with a 1 in the i-th place and zeros elsewhere.

Example 5.1.2 ([20], Example 2.1)**.** (i) Define $\Omega = \Omega_{\mathbb{N}}$ as the small category with objects $\mathrm{Obj}(\Omega) = \mathbb{N}_0^{\mathbb{N}}$, and morphisms $\Omega = \{(m, n) \in \mathbb{N}_0^{\mathbb{N}} \times \mathbb{N}_0^{\mathbb{N}} : m \leq n\}$. Range and source $r((m, n)) = m, s((m, n)) = n$. Define $d : \Omega \to \mathbb{N}_0^{\mathbb{N}}$ by $d((m, n)) = n - m$. Then this is an $\mathbb{N}$-graph whose $C^*(\Omega) = \mathcal{K}(\ell^2(\mathbb{N}_0^{\mathbb{N}}))$.

(ii) Viewing semigroup $\mathbb{N}_0^{\mathbb{N}}$ as a small category with degree functor $d = \mathrm{Id} : \mathbb{N}_0^{\mathbb{N}} \to \mathbb{N}_0^{\mathbb{N}}$. Schenkel shows that $C^*(\mathbb{N}_0^{\mathbb{N}}) \cong C^*(\mathbb{Z}^\infty) \cong C(\mathbb{T}^\omega)$, where $\mathbb{T}^\omega = \prod_{\mathbb{N}} \mathbb{T}$ is endowed with the product topology.

We mention a few more examples:

Example 5.1.3. Let Λ be a category with two objects u, v and morphisms generated by $a_i : u \to v$ and $a_i' : v \to u$ for $i = 1, 2, 3, ...$ such that $a_i a_j' = a_j a_i'$ and $a_i' a_j = a_j' a_i$.

$$\begin{tikzcd} u \arrow[d, bend right=50, "a_2"'] \arrow[d, bend right=20, "a_1"] \\ v \arrow[u, bend right=20, "a_1'"] \arrow[u, bend right=50, "a_2'"'] \end{tikzcd}$$

Define $d : \Lambda \to \mathbb{N}_0^{\mathbb{N}}$ by $d(a_i) = d(a_i') = e_i$. Then (Λ, d) is an $\mathbb{N}$-graph. To see the unique factorization property, note that for any $k \in \mathbb{N}_0^{\mathbb{N}}$, there exist unique morphisms $\mu \in u\Lambda$ and $\nu \in v\Lambda$ with $d(\mu) = d(\nu) = k$. Thus, given $\lambda \in \Lambda$ with $d(\lambda) = m = k_1 + k_2$ in $\mathbb{N}_0^{\mathbb{N}}$, there exist unique $\alpha, \beta \in \Lambda$ such that $r(\alpha) = r(\lambda)$, $s(\beta) = s(\lambda)$, $r(\beta) = s(\alpha)$, $\lambda = \alpha\beta$, and $d(\alpha) = k_1, d(\beta) = k_2$.

Example 5.1.4. Let Λ be the one-object category $(\mathbb{Z}_{>0}, \times)$, the free commutative monoid with infinitely many (prime number) generators, whose morphisms are integers which may be expressed in terms of the generators via prime factorization. Then choosing a listing $\{p_i\}_{i \in \mathbb{N}}$ of the prime numbers turns Λ into an $\mathbb{N}$-graph with degree map defined by

$$\lambda = p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n} \quad \mapsto \quad k = (k_1, k_2, ..., k_n, 0, ...) \in \mathbb{N}_0^{\mathbb{N}},$$

where $\lambda \in \Lambda$, the k_i powers may be zero, and all entries in $k \in \mathbb{N}_0^{\mathbb{N}}$ after the n-th position are zeros. To demonstrate the unique factorization property, suppose $d(\lambda) = k + \ell = (k_1 + \ell_1, k_2 + \ell_2, ...)$ for $k, \ell \in \mathbb{N}_0^{\mathbb{N}}$. Then since multiplication is commutative,

$$\mu = p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n}, \quad \nu = p_1^{\ell_1} p_2^{\ell_2} \cdots p_n^{\ell_n}$$

provide the factorization $\lambda = \mu\nu$ and $d(\mu) = k, d(\nu) = \ell$.

Note that $(\mathbb{Z}_{>0}, \times)$ is also a candidate for the target category $\mathcal{B}$ of a KPf $F : \mathcal{E} \to \mathcal{B}$, as it is itself strongly right Ore. Given a KPf $F : \mathcal{E} \to (\mathbb{Z}_{>0}, \times)$ and a listing of the prime generators as above inducing the degree map $d : (\mathbb{Z}_{>0}, \times) \to \mathbb{N}_0^{\mathbb{N}}$, the composition $d \circ F : \mathcal{E} \to \mathbb{N}_0^{\mathbb{N}}$ makes $\mathcal{E}$ an $\mathbb{N}$-graph. That is, as a composition of two KPfs, $d \circ F$ is a KPf by Section 2.1, and as it targets $\mathbb{N}_0^{\mathbb{N}}$, $(\mathcal{E}, d \circ F)$ is an $\mathbb{N}$-graph.

Example 5.1.5. Let $\Delta := \{(m, n) \in \mathbb{Z}^\infty \times \mathbb{Z}^\infty : m \leq n\}$ be a category with composition defined by $(m, \ell)(\ell, n) = (m, n)$. Define $d : \Delta \to \mathbb{N}_0^{\mathbb{N}}$ by $d(m, n) = n - m$, making Δ an $\mathbb{N}$-graph. Observe that $C^*(\Delta) \cong \mathcal{K}(\ell^2(\mathbb{Z}^\infty))$.

Finally, we make explicit that the degree map of an $\mathbb{N}$-graph (Λ, d) provides an example of a KPf when $\mathbb{N}_0^{\mathbb{N}}$ is seen as a one-object category (whose unique object we will denote $\star$). Clearly the unique factorization property gives the unique lifting criterion to say d is a dCf. Since $\mathbb{N}_0^{\mathbb{N}}$ is a left- and right-cancellative commutative monoid having pullbacks, it satisfies the strongly right Ore condition. Finally, $d : \Lambda \to \mathbb{N}_0^{\mathbb{N}}$ is locally split. Indeed, for an object $v \in \Lambda^0$, $d(v) = \star$, and d induces a functor $d : \Lambda/v \to \mathbb{N}_0^{\mathbb{N}}/\star$. This d admits a splitting $x : \mathbb{N}_0^{\mathbb{N}}/\star \to \Lambda/v$ which can be defined as follows. By the no sources assumption, there exists $\lambda_n \in \Lambda$ with $d(\lambda_n) = n, r(\lambda_n) = v$ for any $n \in \mathbb{N}_0^{\mathbb{N}}$. Define $x(0, n)$ to be a morphism $\lambda_n \in \Lambda$ selected such that $\lambda_0 = v$ and, whenever $k = m + n$, the unique factorization $\lambda_k = \mu\nu$ with $d(\mu) = m$ and $d(\nu) = n$ has $\mu = \lambda_m$.

It goes without saying that d is further classified as strongly surjective since it is surjective on objects and locally split. Thus, an $\mathbb{N}$-graph is an example of a KPf, and as a strongly surjective dCf, can be associated with a Cuntz-Krieger system as shown in Definition 1.3, which matches the family of partial isometries in [20] Definition 2.2.

5.2 The Infinite Path Space

We now move on to investigating the infinite path space of an $\mathbb{N}$-graph. There are two ways to approach this: first, by emulating the notation of k-graphs, and second, by treating $d : \Lambda \to \mathbb{N}_0^{\mathbb{N}}$ as a KPf.

For the first treatment, modeled after [1] Section 2, we see an infinite path $x \in \Lambda^\infty$ as an $\mathbb{N}$-graph morphism (see Definition 2.1) from $(\Omega_\mathbb{N}, d_\Omega)$ (given in Part (i) of Ex 6) to (Λ, d_Λ), defined from pairs $(m, n) \in \mathbb{N}_0^\mathbb{N} \times \mathbb{N}_0^\mathbb{N}$ where $n \geq m$ to morphisms (paths) in Λ such that $x(m, n) = \lambda$ means $d_\Lambda(\lambda) = d_\Omega(m, n) = n - m$ (although in context by abuse of notation we may write d for either). Unique factorization gives that x is completely determined by the values x takes on pairs of the form $(0, m)$ for $m \in \mathbb{N}_0^\mathbb{N}$. In fact, not all such pairs are necessary to fully determine x. Given an increasing cofinal sequence $\{n_j : j \geq 0\}$ in $\mathbb{N}_0^\mathbb{N}$ with $n_0 = 0$ (for example, $(n_j)_{j \in \mathbb{Z}_{\geq 0}}$ given by $n_j = j \sum_{i=1}^{j} e_i$), x is determined by the values $x(0, n_j)$. We say that x ends in a morphism λ if $x(0, d(\lambda)) = \lambda$.

For the second treatment, we see an infinite path to range object v as a splitting $x : \mathbb{N}_0^\mathbb{N}/\star \to \Lambda/v$, where $\mathbb{N}_0^\mathbb{N}/\star$ is the slice category of $\mathbb{N}_0^\mathbb{N}$ over its unique object $\star$ when seen as a one-object category. The objects of the slice category are the elements of $\mathbb{N}_0^\mathbb{N}$, and the morphisms are ordered pairs $(m, \ell) \in \mathbb{N}_0^\mathbb{N} \times \mathbb{N}_0^\mathbb{N}$ where ℓ is an arrow from n to m when $m + \ell = n$ forms a commuting triangle; i.e., they are pairs (m, ℓ) with range m and source $m + \ell =: n$. The figure below summarizes graphically the relation between morphisms in $\mathbb{N}_0^\mathbb{N}$ and in its slice category.

in $\mathbb{N}_0^\mathbb{N}$: in $\mathbb{N}_0^\mathbb{N}/\star$:

$$\begin{tikzcd} \star \arrow[r, "m"] \arrow[dr, "n"'] & \star \arrow[d, "\ell"] \\ & \star \end{tikzcd} \qquad \begin{tikzcd} n \arrow[d, "\ell"] \\ m \end{tikzcd}$$

This means there is a unique morphism from n to m exactly when $n \geq m$. With x being a splitting, we would have $d(x(m, \ell)) = (m, \ell) = (m, n - m)$, which maps to $n - m \in \mathbb{N}_0^\mathbb{N}$ under the forgetful functor back from the slice category. Thus the infinite paths seen in this way do correspond to the infinite paths seen in the first treatment, and a more formal explanation of the forgetful functor relationship is detailed in [2] Example 3.15 in the context of k-graphs. Recall our notational convention: the morphism $x(a, b) = (x_1(a, b), x_2(a, b))$ is sometimes denoted $x(ab) = x_1(a,b)x_2(a,b)$, and $x(a) := x_2(\mathrm{Id}, a) = x_1(a,b)$. We say x ends in a morphism $\lambda \in v\Lambda$ if $x(d(\lambda)) = \lambda$, which may also be written in the first treatment as $x(0, d(\lambda)) = \lambda$.

We denote by $Z(v)$ the cylinder set containing all infinite paths x to range object v, and by $Z(\lambda) \subset Z(v)$ the cylinder set containing all infinite paths x to v ending in λ specifically. By [2] Lemma 3.16, we have for $n \in \mathbb{N}_0^{\mathbb{N}}$,

$$Z(v) = \bigcup_{\lambda \in \Lambda : r(\lambda) = v, d(\lambda) = n} Z(\lambda),$$

and the union is disjoint. These cylinder sets form a basis for a topology on the infinite path space Λ^∞. Since $\mathbb{N}_0^{\mathbb{N}}$ is countable, the single slice category $\mathbb{N}_0^{\mathbb{N}}/\star$ is countable, and having $d : \Lambda \to \mathbb{N}_0^{\mathbb{N}}$ a row-finite, strongly surjective KPf, we may apply [2] Theorem 4.7 to say

1. $Z(\lambda)$ is compact for each $\lambda \in \Lambda$,

2. Λ^∞ is totally disconnected.

Example 5.2.1. In particular, returning to Example 5.1.3, the infinite path space in that case is the disjoint union of the two cylinder sets $Z(u), Z(v)$, having only two objects. In fact, this $\mathbb{N}$-graph has only two infinite paths: x_u targeting u and x_v targeting v. If $n = (n_1, n_2, ...) \in \mathbb{N}_0^{\mathbb{N}}$ and $x \in Z(u)$, then $x(0, n)$ is a morphism containing n_1 instances of a_1 or a_1', n_2 instances of a_2 or a_2', etc. The different orderings of these edges is irrelevant due to the swap relations, and since x targets u, endings will be of the form $a'_{i_1} a_{i_2} a'_{i_3} \cdots$.

Example 5.2.2. Returning to Example 5.1.4, having only one object, the infinite path space consists of $Z(\star)$, a cylinder set containing, in fact, only one infinite path due to the commutativity of multiplication.

Proceeding with the development of the infinite path space, we use the KPf approach to investigate the important maps $\mathrm{ind}_\lambda : Z(s(\lambda)) \to Z(\lambda)$ and $\mathrm{res}_\lambda : Z(\lambda) \to Z(s(\lambda))$ from

Proposition 1.2.1. For the latter, we consider $x \in Z(\lambda)$ and define the infinite path $\mathrm{res}_\lambda(x)$ in $Z(s(\lambda))$ by

$$\mathrm{res}_\lambda(x)(m, \ell) = (x_2(d(\lambda), m), x_2(d(\lambda) + m, \ell))$$

which corresponds to $x(m+d(\lambda), n+d(\lambda))$ in the k-graph notation. And indeed, we may as well ignore λ itself as only its degree is relevant, which we may denote $p := d(\lambda)$. And thus we may simply use the k-graph style: For $p \in \mathbb{N}_0^{\mathbb{N}}$ define $\sigma^p : \Lambda^\infty \to \Lambda^\infty$ by $\sigma^p(x)(m, n) = x(m + p, n + p)$.

As for defining $\mathrm{ind}_\lambda(x)(m, \ell)$, for simplicity we consider its values on $(0, n) \in \mathbb{N}_0^{\mathbb{N}} \times \mathbb{N}_0^{\mathbb{N}}$, since as a one-object category, $\mathbb{N}_0^{\mathbb{N}}/\star$ may be seen as a copy of $\mathbb{N}_0^{\mathbb{N}}$, and conveniently when translating notation, $x(0, n)$ in the KPf case is denoted $x(0, 0+n) = x(0, n)$ in the $\mathbb{N}$-graph case. Recall that the method of defining $\mathrm{ind}_\lambda(x)$ may be summarized as (1) by right Ore, find $a, b \in \mathcal{B}$ such that $d(\lambda) + a = n + b$, (2) take the unique lift in $\mathcal{E}$ that is $\lambda x(0, a) = \alpha\beta$ with $d(\alpha) = n, d(\beta) = b$, and (3) define $\mathrm{ind}_\lambda(x)(0, n) = (0, \alpha)$. In this case we can be explicit with two cases depending on the degree of n.

If $n \le d(\lambda)$, the choice of $d(\lambda) + a = n + b$ is taken to be $d(\lambda) = n + (d(\lambda) - n)$, the lift is $\lambda = \lambda(0, n)\lambda(n, d(\lambda) - n)$, and we have $\mathrm{ind}_\lambda(x)(0, n) = \lambda(0, n)$.

Otherwise, the choice $d(\lambda) + a = n + b$ is taken to be $d(\lambda) + (d(\lambda) \vee n) - d(\lambda) = n + (d(\lambda) \vee n) - n$, which yields a lift of $\lambda x(0, (d(\lambda) \vee n) - d(\lambda)) = \alpha\beta$, where $d(\alpha) = n$ and $d(\beta) = (d(\lambda) \vee n) - n$. For brevity, denote $M = (d(\lambda) \vee n) - d(\lambda)$. By unique factorization in the $\mathbb{N}$-graph, we see $\alpha = \mathrm{ind}_\lambda(x)(0, n) = (\lambda x(0, M))(0, n)$.

In the study of k-graphs, the notation λx is used for the unique infinite path $y \in \Lambda^\infty$ to $r(\lambda)$ such that $x = \sigma^{d(\lambda)}(y)$. Its values on pairs of the form $(0, m) \in \mathbb{N}^k \times \mathbb{N}^k$,

$$(\lambda x)(0, m) = \begin{cases} \lambda(0, m) & \text{if } m \le d(\lambda) \\ (\lambda x(0, (d(\lambda) \vee m) - d(\lambda)))(0, m) & \text{otherwise} \end{cases}$$

bear obvious correspondence to the values we found for $\mathrm{ind}_\lambda(x)$ above. Thus henceforth we utilize the k-graph inspired notation λx to denote $\mathrm{ind}_\lambda(x)$. We have as in the k-graph case that $\sigma^{d(\lambda)}(\lambda x) = x$ and $x(0,p)\sigma^p(x) = x$, consistent with the fact that ind and res are inverses as stated in Proposition 1.2.1.

We reproduce another definition established in [20] in anticipation of discussing the C*-algebra of Λ.

Definition 5.2.3 ([20], Definition 2.2)**.** Let Λ be a row-finite $\mathbb{N}$-graph with no sources. Then $C^*(\Lambda)$ is defined to be the universal C*-algebra generated by a family $\{s_\lambda : \lambda \in \Lambda\}$ of partial isometries satisfying:

(CK1) $\{s_v : v \in \Lambda^0\}$ is a family of mutually orthogonal projections;

(CK2) $s_\lambda s_\mu = s_{\lambda\mu}$ for all $\lambda, \mu \in \Lambda$ such that $s(\lambda) = r(\mu)$, and if $s(\lambda) \neq r(\mu)$, $s_\lambda s_\mu = 0$;

(CK3) $s_\lambda^* s_\lambda = s_{s(\lambda)}$ for all $\lambda \in \Lambda$;

(CK4) for all $v \in \Lambda^0$ and $n \in \mathbb{N}_0^{\mathbb{N}}$ we have $s_v = \sum_{\lambda \in v\Lambda^n} s_\lambda s_\lambda^*$.

Note that (CK1), (CK2), (CK3), and (CK4) are consistent with (1), (2), (4), and (6), respectively, from Definition 1.1.4. Having identified objects Λ^0 with the identity morphisms of the category, Condition (3) of the latter is satisfied, and Condition (5) follows from (6).

Now [20] Theorem 2.2 shows that there exists a nontrivial $*$-representation $\{S_\lambda : \lambda \in \Lambda\}$ of (Λ, d) using the Hilbert space $\ell^2(\Lambda^\infty)$.

5.3 The Infinite Paths of Finite-rank Sub-graphs of Λ

Schenkel shows that an $\mathbb{N}$-graph can be seen as a direct limit of k-graphs derived from the $\mathbb{N}$-graph via projection or truncation.

Definition 5.3.1 ([20], Definition 2.3)**.** Let Λ be an $\mathbb{N}$-graph. Then we define

$$^k\Lambda := \{\lambda \in \Lambda : d|_{e^n}(\lambda) = 0 \text{ when } n > k\},$$

where $d|_{e^n}(\lambda)$ is equal to the n-th entry of $d(\lambda)$.

Theorem 5.3.2 ([20], Theorem 2.4). *${}^k\Lambda$ as defined above is a k-graph, with degree functor $d|_{\mathbb{N}^k} = \pi_k \circ d$, where π_k is the canonical projection of $\mathbb{N}_0^{\mathbb{N}} \to \mathbb{N}^k$. The range and source maps are $r|_{{}^k\Lambda}, s|_{{}^k\Lambda}$ respectively.*

Hereafter, we will often use the bar notation $\overline{n} \in \mathbb{N}^k$ to emphasize when an element is a finite sequence versus an infinite sequence $n \in \mathbb{N}_0^{\mathbb{N}}$. Using the k-graph notation style, we show the following:

Proposition 5.3.1. *Let Λ be an $\mathbb{N}$-graph and let ${}^k\Lambda$ be the k-graphs defined above. An infinite path $x \in \Lambda^\infty$ induces a canonical infinite path $x_k \in ({}^k\Lambda)^\infty$ for each $k \in \mathbb{N}$. The values of $x_k(\overline{m}, \overline{n})$ for $\overline{m}, \overline{n} \in \mathbb{N}^k$ are defined by setting $m = (\overline{m}, 0, ...), n = (\overline{n}, 0, ...)$ in $\mathbb{N}_0^{\mathbb{N}}$ and letting $x_k(\overline{m}, \overline{n}) = x(m, n)$.*

Proof. We first show that the proposed definition yields an infinite path in $({}^k\Lambda)^\infty$ by showing that the paths $x_k(\overline{m}, \overline{n})$ as defined above are in the subgraph ${}^k\Lambda$ and that we have $d|_{\mathbb{N}^k}(x_k(\overline{m}, \overline{n})) = \overline{n} - \overline{m}$. To show the former, notice that since $d(x(m, n)) = n - m = (\overline{n} - \overline{m}, 0, ...)$, we have $d|_{e^j}(x(m, n)) = 0$ for all $j > k$, and by definition $x(m, n) \in {}^k\Lambda$. The latter is clear since

$$d|_{\mathbb{N}^k}(x_k(\overline{m}, \overline{n})) = \pi_k \circ d(x(m, n)) = \pi_k(n - m) = \overline{n} - \overline{m}.$$

To form this canonical derivation, we chose particular m, n that append infinitely many zeros to $\overline{m}$ and $\overline{n}$. In fact, given any m' such that $\pi_k(m') = \overline{m}$, we may decompose $m' = m + \hat{m}$, where $m = (\overline{m}, 0, ...)$ as used above and $\hat{m} = (0, ..., 0, m'_{k+1}, m'_{k+2}, ...)$. Similarly, $n' = n + \hat{n}$ when $\pi_k(n') = \overline{n}$. That is, $d(x(m', n')) = n' - m' = (n - m) + (\hat{n} - \hat{m})$, and thus by unique factorization, $x(m', n') = x(m, n)x(\hat{m}, \hat{n})$, and the $x(m, n) \in {}^k\Lambda$ portion is unaffected by choice of m', n', even though $x(m', n')$ may not be in ${}^k\Lambda$ itself. Note that in the special case where $n_j = m_j$ for all $j \geq k + 1$, we do have $x(m', n') \in {}^k\Lambda$, but it should not be considered the canonical choice. □

Thus it is possible to derive infinite paths for the finite-rank sub-graphs from an infinite path $x \in \Lambda^\infty$. However, individual infinite paths $x_k \in ({}^k\Lambda)^\infty$ cannot be seen as infinite paths in Λ^∞, as they provide no information on degrees $n = (n_1, n_2, ...) \in \mathbb{N}_0^{\mathbb{N}}$ when there exists an $n_j \neq 0$ for $j > k$.

In other words, given a particular $y \in ({}^k\Lambda)^\infty$, it is not possible to canonically select a particular infinite path $x \in \Lambda^\infty$ such that $y = x_k$. However, note that any $x \in \Lambda^\infty$ yields a family $\{x_k : k \in \mathbb{N}\}$ satisfying $(x_{k+1})_k = x_k$. Any such compatible countably infinite family does yield a limiting infinite path in Λ^∞.

Proposition 5.3.2. *Suppose $\{y_i : i \in \mathbb{N}\}$ is a sequence of infinite paths such that $y_i \in ({}^i\Lambda)^\infty$ and $(y_{i+1})_i = y_i$ for all $i \in \mathbb{N}$. Together they define an infinite path $y \in \Lambda^\infty$ such that for all i, $(y)_i = y_i$.*

Proof. Suppose $\{y_i : i \in \mathbb{N}\}$ is a sequence of infinite paths such that $y_i \in ({}^i\Lambda)^\infty$ and $(y_{i+1})_i = y_i$ for all $i \in \mathbb{N}$. We define their limit as the infinite path $y \in \Lambda^\infty$ determined by its values $y(0, n)$ for $n \in \mathbb{N}_0^{\mathbb{N}}$ as follows: Since $n \in \mathbb{N}_0^{\mathbb{N}}$ must have finitely many nonzero entries, there is an entry at position j such that all entries afterward are zeros. Define $y(0, n) = y_j(0, \pi_j(n))$, where π_j is the canonical projection from $\mathbb{N}_0^{\mathbb{N}}$ to $\mathbb{N}^j$. As this can be done for any $(0, n)$ by taking j large enough, we have established values $y(0, n)$ for all defining pairs. To find $y(m, n)$ for $n \geq m$, let its value be the unique path such that $y(0, m) = y(0, m)y(m, n)$ in Λ. □

We now consider the effect of σ^p in this context. For $p \in \mathbb{N}_0^{\mathbb{N}}$ and $k \in \mathbb{N}$, $(\sigma^p x)_k$ is defined by

$$(\sigma^p x)_k(0, \overline{n}) = (\sigma^p x)(0, n) = x(p, n + p),$$

where $\overline{n} \in \mathbb{N}^k$ and $n = (\overline{n}, 0, ...)$. If it happens that p can be written $p = (\overline{p}, 0, ...)$ for $\overline{p} \in \mathbb{N}^k$, then in fact

$$(\sigma^p x)_k(0, \overline{n}) = x(p, n + p) = x_k(\overline{p}, \overline{n + p}) = x_k(\overline{p}, \overline{n} + \overline{p}),$$

but this is not necessarily true if p has nonzero entries past the k-th position.

Similarly, we look at λx for $\lambda \in \Lambda$. Again let $M = (d(\lambda) \vee n) - d(\lambda)$, and consider

$$(\lambda x)_k(0,\overline{n}) = (\lambda x)(0,n) = \begin{cases} \lambda(0,n) & \text{if } n \leq d(\lambda) \\ (\lambda x(0,M))(0,n) & \text{otherwise.} \end{cases}$$

The first case only holds if $\lambda \in {}^k\Lambda$ already, and if that is the case, its degree may be written $d(\lambda) = (\overline{d(\lambda)}, 0, \ldots)$ where $\overline{d(\lambda)} \in \mathbb{N}^k$. Then if $\lambda \in {}^k\Lambda$ and in particular $n \geq d(\lambda)$, we have $M = n - d(\lambda)$, and

$$(\lambda x(0,M))(0,n) = (\lambda x_k(0, \overline{n - d(\lambda)}))(0,n) = \lambda x_k(0, \overline{n} - \overline{d(\lambda)}),$$

but again this relationship does not hold for general λ.

Next, recall the construction given by Kumjian and Pask below.

Definition 5.3.3 ([1], Definition 1.9)**.** Let $f : \mathbb{N}^\ell \to \mathbb{N}^k$ be a monoid morphism, then if (Λ, d) is a k-graph we may form the ℓ-graph $f^*(\Lambda)$ as follows: (the objects of $f^*(\Lambda)$ may be identified with those of Λ and) $f^*(\Lambda) = \{(\lambda, n) : d(\lambda) = f(n)\}$ with $d(\lambda, n) = n$, $s(\lambda, n) = s(\lambda)$, and $r(\lambda, n) = r(\lambda)$.

They give an example of the above for a particular choice of f.

Example 5.3.4. ([1], Example 1.10(i)) Let Λ be a k-graph and put $\ell = 1$, then if we define the morphism $f_i(n) = ne_i$ for $1 \leq i \leq k$, we obtain the coordinate graphs $\Lambda_i := f_i^*(\Lambda)$ of Λ (these are 1-graphs).

It is straightforward to perform a similar construction for $\mathbb{N}$-graphs, yielding the following definition.

Definition 5.3.5. Let $f : \mathbb{N}^\ell \to \mathbb{N}_0^{\mathbb{N}}$ be a monoid morphism. Then if Λ is an $\mathbb{N}$-graph, we form an ℓ-graph $f^*(\Lambda) = \{(\lambda, \overline{n}) \in \Lambda \times \mathbb{N}^\ell : d_\Lambda(\lambda) = f(\overline{n})\}$. The objects of $f^*(\Lambda)$ may be identified with those of Λ by $(v, 0) \mapsto v$, the degree map is given by $d(\lambda, \overline{n}) = \overline{n}$, and the range and source maps are $r(\lambda, \overline{n}) = r(\lambda), s(\lambda, \overline{n}) = s(\lambda)$, respectively.

Having introduced $f^*(\Lambda)$, consider the map $f^* : \Lambda^\infty \to (f^*(\Lambda))^\infty$ given by

$$f^*(x)(\overline{m}, \overline{n}) = (x(f(\overline{m}), f(\overline{n})), \overline{n} - \overline{m}),$$

where $\overline{m}, \overline{n} \in \mathbb{N}^\ell$. This map is continuous and surjective, and if the image of f is cofinal, then f^* is a homeomorphism. The proof is essentially the same as that of [1] Proposition 2.9.

For coordinate graphs like the ones in the example above, $\Lambda_1 \cong {}^1\Lambda$ as in Definition 5.3.1. That is, $(\lambda, m) \in \Lambda_i$ has $m \in \mathbb{N}$ such that $f(m) = me_1 = d(\lambda)$, which is the same as requiring $d|_{e^n}(\lambda) = 0$ when $n > 1$, placing $\lambda \in {}^1\Lambda$. Conversely, if $\lambda \in {}^1\Lambda$, $d(\lambda)$ has its only nonzero entry in the first position, which means it can be written me_1 where $m \in \mathbb{N}$ is the first entry. Then the pair (λ, m) appears in Λ_1. However, clearly this notational coincidence does not hold for $i > 1$.

Rather, to see ${}^k\Lambda$ in terms of such a construction, simply let $f : \mathbb{N}^k \to \mathbb{N}_0^{\mathbb{N}}$ be defined by $f(\overline{n}) = (\overline{n}, 0, ...)$, the same appending of zeros as we did before. Then the $(\lambda, \overline{n}) \in f^*(\Lambda)$ have $d(\lambda) = f(\overline{n}) = (\overline{n}, 0, ...)$ meaning $\lambda \in {}^k\Lambda$. Likewise, if $\lambda \in {}^k\Lambda$ with $d(\lambda) = (\overline{n}, 0, ...)$, then $(\lambda, \overline{n})$ appears in $f^*(\Lambda)$. Thus for this choice of f, the k-graph $f^*(\Lambda) \cong {}^k\Lambda$.

Letting $n = (\overline{n}, 0, ...)$ and $m = (\overline{m}, 0, ...)$, if f is specifically the one defined above, this is exactly

$$f^*(x)(\overline{m}, \overline{n}) = (x(m, n), \overline{n} - \overline{m}) = (x_k(\overline{m}, \overline{n}), \overline{n} - \overline{m}),$$

allowing us to see $(f^*(\Lambda))^\infty \cong ({}^k\Lambda)^\infty$ by the map $(x_k(\overline{m}, \overline{n}), \overline{n} - \overline{m}) \mapsto x_k(\overline{m}, \overline{n})$.

Remark 5.3.6. If $\mathbb{N}_0^{\mathbb{N}}$ replaces $\mathbb{N}^\ell$ rather than $\mathbb{N}^k$ in the construction, then a monoid morphism $g : \mathbb{N}_0^{\mathbb{N}} \to \mathbb{N}^k$ together with a k-graph Λ_k yields an $\mathbb{N}$-graph $g^*(\Lambda_k) = \{(\lambda, n) \in \Lambda_k \times \mathbb{N}_0^{\mathbb{N}} : d_{\Lambda_k}(\lambda) = g(n)\}$ with the same definitions of structure maps and degree map $d(\lambda, n) = n$ targeting $\mathbb{N}_0^{\mathbb{N}}$. In this case the map $g^* : \Lambda_k^\infty \to (g^*(\Lambda_k))^\infty$ is defined for $x \in \Lambda_k^\infty$ and $m, n \in \mathbb{N}_0^{\mathbb{N}}$ by

$$g^*(x)(m, n) = (x(g(m), g(n)), n - m).$$

This map is still continuous and surjective as in the k-graph case from [1] Proposition 2.9.

Example 5.3.7. If in the above remark g is taken to be $\pi_k : \mathbb{N}_0^{\mathbb{N}} \to \mathbb{N}^k$ the canonical projection, then the rank-k sub-graph ${}^k(g^*(\Lambda_k))$ is isomorphic to Λ_k. Indeed, by definition

$$^k(g^*(\Lambda_k)) = \{(\lambda, n) \in g^*(\Lambda_k) : d|_{e^j}(\lambda, n) = 0 \text{ when } j > k\}$$

has degree functor $\pi_k \circ d$ where d is the degree functor for $g^*(\Lambda_k)$, but $d_{\Lambda_k}(\lambda) = g(n) = \pi_k(n) = \pi_k \circ d(\lambda, n)$. In other words, for all $\lambda \in \Lambda_k$, (λ, n) appears in ${}^k(g^*(\Lambda_k))$ with the unique $n \in \mathbb{N}_0^{\mathbb{N}}$ such that $\pi_k(n) = d_{\Lambda_k}(\lambda)$ and $n = (\pi_k(n), 0, ...)$.

Recall in general that Proposition 2.3.1 yielded a similar result to [1] Proposition 1.11. Below, we tailor this to the $\mathbb{N}$-graph case and notation.

Proposition 5.3.3. *Let Λ be an $\mathbb{N}$-graph and $f : \mathbb{N}^\ell \to \mathbb{N}_0^{\mathbb{N}}$ a monoid morphism. Then there is a $*$-homomorphism $\pi_f : C^*(f^*(\Lambda)) \to C^*(\Lambda)$ given by $s_{(\lambda,\overline{n})} \mapsto s_\lambda$, and if f is surjective, then so is π_f. The similar result holds for Λ_k a k-graph and $g : \mathbb{N}_0^{\mathbb{N}} \to \mathbb{N}^k$, with the $*$-homomorphisms $\pi_g : C^*(g^*(\Lambda_k)) \to C^*(\Lambda_k)$ given by $s_{(\lambda,n)} \mapsto s_\lambda$.*

Proof. We show that a (Cuntz-Krieger) family $\{s_\lambda : \lambda \in \Lambda\}$ for the $\mathbb{N}$-graph Λ is also a Cuntz-Krieger family in $C^*(f^*(\Lambda))$, and apply the universal property.

(CK1) For objects $(u, 0) \neq (v, 0)$ in $f^*(\Lambda)$, $u \neq v$ implies $s_u \perp s_v$ still holds.

(CK2) If $(\lambda, n), (\mu, m)$ are composable in $f^*(\Lambda)$, then λ, μ are composable in Λ, and $s_\lambda s_\mu = s_{\lambda\mu}$ holds.

(CK3) For all $(\lambda, n) \in f^*(\Lambda)$, since $\lambda \in \Lambda$ we still have $s_\lambda^* s_\lambda = s_{s(\lambda)}$.

(CK4) For all objects $(v, 0)$ in $f^*(\Lambda)$ and $\overline{n} \in \mathbb{N}^\ell$, $(\lambda, \overline{n})$ targeting $(v, 0)$ implies $\lambda \in v\Lambda$ and $d(\lambda) = f(\overline{n})$, and so

$$\sum_{(\lambda,\overline{n}) \in (v,0)(f^*(\Lambda))^{\overline{n}}} s_\lambda s_\lambda^* = \sum_{\lambda \in v\Lambda^{f(\overline{n})}} s_\lambda s_\lambda^* = s_v$$

still holds. Then the universal property of $C^*(f^*(\Lambda)) = C^*(\{s_{(\lambda,\overline{n})}\})$ gives the existence of π_f. Analogous checking yields the existence of π_g for the second case. If f is surjective, then for all $n \in \mathbb{N}_0^{\mathbb{N}}$, there exist $\overline{m} \in \mathbb{N}^\ell$ and $\lambda \in \Lambda$ such that $f(\overline{m}) = n = d(\lambda)$. Thus the generators $\{s_\lambda : \lambda \in \Lambda\}$ of $C^*(\Lambda)$ are in the image of π_f. Similarly, if g is surjective onto $\mathbb{N}^k$, there exists $\lambda \in \Lambda_k$ and $n \in \mathbb{N}_0^{\mathbb{N}}$ such that $g(n) = d(\lambda)$ and any generator s_λ is in the image of π_g. $\square$

5.4 The path groupoid

Under the KPf treatment, elements of the path groupoid G_Λ are equivalence classes $[\alpha, \beta, x]$, where $x \in Z(\beta)$, $s(\alpha) = s(\beta)$, and $(\alpha', \beta', x') \in [\alpha, \beta, x]$ if the following conditions are satisfied:

1. $x = x'$
2. $\exists\, \lambda \in \Lambda$ with $x \in Z(\lambda) \subset Z(\beta) \cap Z(\beta')$ and $\mathrm{ind}_\alpha \circ \mathrm{res}_\beta |_{Z(\lambda)} = \mathrm{ind}_{\alpha'} \circ \mathrm{res}_{\beta'} |_{Z(\lambda)}$,
3. there exist $a, b \in \mathbb{N}_0^{\mathbb{N}}$ with $d(\alpha) + a = d(\alpha') + b$ and $d(\beta) + a = d(\beta') + b$.

Note that subtracting the equations in Condition 3 yields

$$d(\alpha) + a - d(\beta) - a = d(\alpha') + b - d(\beta') - b \quad \Longrightarrow \quad d(\alpha) - d(\beta) = d(\alpha') - d(\beta')$$

so in the $\mathbb{N}$-graph case, it is still reasonable to describe the elements of the path groupoid as triples of the form $(\alpha z, d(\alpha) - d(\beta), \beta z)$ as in the k-graph case. The explicit association is $[\alpha, \beta, x] \mapsto (\alpha\sigma^{d(\beta)}(x), d(\alpha) - d(\beta), x)$, and we will proceed using the k-graph style of notation.

The source of $(\alpha z, d(\alpha) - d(\beta), \beta z)$ as an arrow of the groupoid is βz, and the range is αz. The objects of the groupoid, $G_\Lambda^0 = \{(x, 0, x) : x \in \Lambda^\infty\}$, can be identified with Λ^∞. Two elements $(\alpha z, d(\alpha) - d(\beta), \beta z), (\mu y, d(\mu) - d(\nu), \nu y)$ in G_Λ are composable when $\mu y = \beta z$, in which case the result is $(\alpha x, d(\alpha) + d(\mu) - d(\beta) - d(\nu), \nu y)$.

Theorem 5.4.1. *A basis for a topology on the path groupoid is given by the cylinder sets*

$$Z(\alpha, \beta) = \{(\alpha z, d(\alpha) - d(\beta), \beta z) : z \in Z(s(\alpha))\},$$

and with respect to this topology, G_Λ is an étale topological groupoid. In addition, G_Λ is totally disconnected locally compact Hausdorff.

Proof. The set of cylinder sets is a basis for a topology on G_Λ by [2] Corollary 4.13. G_Λ is an étale topological groupoid by [2] Lemmas 4.14 and 4.15. It is totally disconnected locally compact Hausdorff by noting that the slice category $\mathbb{N}_0^{\mathbb{N}}/\star$ is countable since $\mathbb{N}_0^{\mathbb{N}}$ is, and we may then apply [2] Theorem 4.16. $\square$

There is another notation for the triples which will be convenient later on, so we introduce it now. Namely, elements of G_Λ are triples $(x, n, y) \in \Lambda^\infty \times \mathbb{Z}^\infty \times \Lambda^\infty$ such that $\sigma^p(x) = \sigma^q(y)$ for some $p, q \in \mathbb{N}_0^{\mathbb{N}}$ and $n = p - q$. Here, $\mathbb{Z}^\infty$ is the subgroup of $\mathbb{Z}^{\mathbb{N}}$ whose elements have only finitely many nonzero entries.

Considering the C*-algebra of the path groupoid, [2] Section 5 showed that when $\mathcal{B}$ is left-cancellative, we have the isomorphism $C^*(G_F) \cong C^*(F)$ for a KPf $F : \mathcal{E} \to \mathcal{B}$. Since $\mathbb{N}_0^{\mathbb{N}}$ as our choice of $\mathcal{B}$ is left- and right-cancellative, we have $C^*(G_\Lambda) \cong C^*(\Lambda)$. Schenkel proved that the latter is the inductive limit of the C*-algebras $C^*({}^k\Lambda)$:

Theorem 5.4.2 ([20], Theorem 2.8)**.** *Let Λ be a row-finite $\mathbb{N}$-graph with no sources. Then $C^*(\Lambda) = \overline{\bigcup_{k \in \mathbb{N}} C^*({}^k\Lambda)}$.*

Thus it seems intuitive to consider the path groupoids $G_{{}^k\Lambda}$ of the finite-rank sub-graphs. We show that it is possible to cut down elements from G_Λ to $G_{{}^k\Lambda}$ as we did for infinite paths alone.

Theorem 5.4.3. *If $(x, n, y) \in G_\Lambda$, then $(x_k, \pi_k(n), y_k) \in G_{{}^k\Lambda}$.*

Proof. By using the above notation, we are saying that there exist $p, q \in \mathbb{N}_0^{\mathbb{N}}$ such that $\sigma^p(x) = \sigma^q(y)$ and $n = p - q$. Clearly $\pi_k(n) = \pi_k(p) - \pi_k(q)$. For notation, denote

$\pi_k(p), \pi_k(q)$ by $\overline{p}, \overline{q}$ respectively, define $p_0 = (\overline{p}, 0, ...)$ and $q_0 = (\overline{q}, 0, ...)$, and let $\hat{p}, \hat{q}$ be such that $p = p_0 + \hat{p}$ and $q = q_0 + \hat{q}$. Then for $\overline{m} \in \mathbb{N}^k$, writing $m_0 = (\overline{m}, 0, ...)$, we have

$$\sigma^{\overline{p}}(x_k)(0, \overline{m}) = x_k(\overline{p}, \overline{m} + \overline{p}) = x(p_0, m_0 + p_0) = \sigma^{p_0}(x)(0, m_0),$$

and similarly $\sigma^{\overline{q}}(y)(0, \overline{m}) = y(q_0, m_0 + q_0) = \sigma^{q_0}(y)(0, m_0)$. we have

$$\sigma^p(x)(0, m_0) = x(p, m_0 + p) = x(p_0 + \hat{p}, m_0 + p_0 + \hat{p}) = x(p_0, m_0 + p_0)x(\hat{p}, \hat{p}).$$

Likewise, $\sigma^q(y)(0, m_0) = y(q, m_0 + q) = y(q_0, m_0 + q_0)y(\hat{q}, \hat{q})$. Thus, given that $\sigma^p(x)(0, m_0) = \sigma^q(y)(0, m_0)$,

$$x(p_0, m_0 + p_0)x(\hat{p}, \hat{p}) = y(q_0, m_0 + q_0)y(\hat{q}, \hat{q}),$$

and both sides show a factorization in which the pieces have the same degree (namely, $d(m_0)$ and 0). Then unique factorization gives that $x(p_0, m_0 + p_0) = y(q_0, m_0 + q_0)$; that is, $\sigma^{p_0}(x)(0, m_0) = \sigma^{q_0}(y)(0, m_0)$, and thus

$$\sigma^{\overline{p}}(x_k)(0, \overline{m}) = \sigma^{\overline{q}}(y_k)(0, \overline{m})$$

for any $\overline{m} \in \mathbb{N}^k$. Thus $\sigma^{\overline{p}}(x) = \sigma^{\overline{q}}(y)$ and $(x_k, \pi_k(n), y_k) \in G_{^k\Lambda}$. □

With $\mathbb{N}_0^{\mathbb{N}}$ left-cancellative, we have $C^*(\Lambda) \cong C^*(G_\Lambda)$, and in the k-graph case having $C^*(^k\Lambda) \cong C^*(G_{^k\Lambda})$, we have the following.

Theorem 5.4.4. *For Λ an $\mathbb{N}$-graph,*

$$C^*(G_\Lambda) \cong C^*(\Lambda) = \overline{\bigcup_{k \in \mathbb{N}} C^*(^k\Lambda)} \cong \overline{\bigcup_{k \in \mathbb{N}} C^*(G_{^k\Lambda})}.$$

Moreover, since the path groupoids $G_{^k\Lambda}$ of k-graphs are amenable, it is known ([8, 21]) that they have the *weak containment property*: their full and reduced C*-algebras coincide.

That is, $C^*(G_{^k\Lambda}) \cong C^*_r(G_{^k\Lambda})$ in the usual notation. The direct limit of nuclear C*-algebras with injective connecting maps is also a nuclear C*-algebra [22]. Thus we have by

$$C^*(G_\Lambda) = \overline{\bigcup_{k\in\mathbb{N}} C^*(G_{^k\Lambda})} = \overline{\bigcup_{k\in\mathbb{N}} C^*_r(G_{^k\Lambda})} = C^*_r(G_\Lambda)$$

that G_Λ has the weak containment property when Λ is an $\mathbb{N}$-graph.

However, in general for groupoids this is not enough to state that G_Λ is amenable; [23] gives an example of a non-amenable groupoid whose full and reduced C*-algebras nonetheless coincide. As summarized in [24], it is an open problem whether an exactness condition on a general etale groupoid in conjunction with the weak containment property does imply the amenability of the groupoid. For the $\mathbb{N}$-graph case, the path groupoid is shown to be amenable in the the next section following k-graph case methods.

As in the k-graph case, the isotropy of the path groupoid is abelian. Indeed, since $\mathrm{Iso}(G_\Lambda)$ consists of elements (x, n, x) having the same source and range, they will be composable only when of the form $(x, n, x), (x, m, x)$, for $x \in \Lambda^\infty$, $n, m \in \mathbb{N}_0^{\mathbb{N}}$. The composition is then $(x, n+m, x)$ in either order by the commutativity of addition in $\mathbb{N}_0^{\mathbb{N}}$.

As the definition of cycline pairs has already been generalized in Section 3.2 to KPfs targeting left-cancellative categories, we simply state the definition in simplified form here.

Definition 5.4.5. A pair $(\alpha, \beta) \in \Lambda \times \Lambda$ with $s(\alpha) = s(\beta)$ is called *cycline* if $\alpha x = \beta x$ for all $x \in Z(s(\alpha))$.

A minor adjustment to the proof of [25] Theorem 4.4 yields a similar result regarding the isotropy of the path groupoid for the $\mathbb{N}$-graph case.

Theorem 5.4.6. *Let Λ be a row-finite $\mathbb{N}$-graph with no sources. Denote $x(p) := x(0, p)$ for infinite path $x \in \Lambda^\infty$ and $p \in \mathbb{N}_0^{\mathbb{N}}$. For $p \neq q \in \mathbb{N}_0^{\mathbb{N}}$, denote by $\Lambda^\infty_{p,q}$ the set of $x \in \Lambda^\infty$ satisfying*

1. *$\sigma^p(x) = \sigma^q(x)$;*

2. *for any $p', q' \in \mathbb{N}_0^{\mathbb{N}}$ with $p' - p = q' - q \in \mathbb{N}_0^{\mathbb{N}}$, the pair $(x(p'), x(q'))$ is not cycline, and $(x(p')\mu, x(q')\mu)$ is a cycline pair for some $\mu \in \Lambda$.*

Then $\operatorname{Int}\operatorname{Iso}(G_\Lambda)$ *is closed if and only if $\Lambda_{p,q}^{\infty} = \emptyset$ for all $p \neq q \in \mathbb{N}_0^{\mathbb{N}}$.*

Proof. The proof follows that of [25] Theorem 4.4, with elements p, q, p_n, q_n, etc. taken in $\mathbb{N}_0^{\mathbb{N}}$, and with an adjustment to the cofinal sequence used. Namely, let $j_n = \sum_{i=1}^{n} ne_i$, and let $p_n = p + j_n$ and $q_n = q + j_n$. □

Since $\operatorname{Int}\operatorname{Iso}(G_\Lambda)$ is abelian, when it is closed we can apply [14] Corollary 4.5 to find that $\iota_r(C^*(\operatorname{Int}\operatorname{Iso}(G_F))) \subset C_r^*(G_F)$ is a Cartan subalgebra in the sense of [15] Definition 4.5.

We conclude this section by revisiting the construction of Remark 5.3.6 and generalizing [1] Proposition 2.10.

Proposition 5.4.1. *Let Λ_k be a k-graph and let $g : \mathbb{N}_0^{\mathbb{N}} \to \mathbb{N}^k$ be a surjective morphism. Then the path groupoid $G_{g^*(\Lambda_k)}$ decomposes as*

$$G_{g^*(\Lambda_k)} \cong G_{\Lambda_k} \times \mathbb{Z}^{\infty},$$

recalling that $\mathbb{Z}^{\infty}$ is the subgroup of $\mathbb{Z}^{\mathbb{N}}$ whose elements have only finitely many nonzero entries.

Proof. As in the proof of [1] Proposition 2.10, extend g to a surjective morphism $g : \mathbb{Z}^{\infty} \to \mathbb{Z}^k$. Let j be a section for g, and let i be an identification of $\mathbb{Z}^{\infty}$ with $\ker(g)$. Again the isomorphism is given by

$$((x, \overline{n}, y), m) \mapsto (g^*x, j(\overline{n}) + i(m), g^*y),$$

for $((x, \overline{n}, y), m) \in G_{\Lambda_k} \times \mathbb{Z}^{\infty}$. □

Example 5.4.7. Returning to Example 5.3.7, the kernel of $g = \pi_k$ consists of elements with zeros in the first k entries, and at least one nonzero entry afterward. A natural choice of section j is $\overline{n} \mapsto (\overline{n}, 0, ...)$. Identify $\mathbb{Z}^\infty$ with the kernel of π_k by $i(m) = (0, ..., 0, m)$, that is, preceding $m \in \mathbb{Z}^\infty$ with k zeros. Then we see

$$((x, \overline{n}, y), m) \mapsto (g^*x, j(n) + i(m), g^*y) = (g^*x, (\overline{n}, 0, ...) + (0, ..., 0, a_1, a_2, ...), g^*y)$$

or in the other direction,

$$(g^*x, (n_1, n_2, ...), g^*y) \mapsto ((x, (n_1, ..., n_k), y), (0, ..., 0, n_{k+1}, n_{k+2}, ...)).$$

5.5 Amenability of G_Λ

In this section, we generalize arguments from [1] Section 5 using skew products and group actions. As investigated in Section 2.4, similar results are possible up to a point in the fully general KPf case, but as the $\mathbb{N}$-graph case can be carried much further, we specialize here using the k-graph style notation. We neglect to maintain a KPf treatment alongside since the notation is unwieldy and will be left behind shortly.

Definition 5.5.1. Let Γ be a discrete group and let (Λ, d) be an $\mathbb{N}$-graph. Given a functor $c : \Lambda \to \Gamma$, define the *skew product* $\Gamma \times_c \Lambda$ as follows: the objects are identified with $\Gamma \times \Lambda^0$ and the morphisms are identified with $\Gamma \times \Lambda$ with the following structure maps:

$$s(g, \lambda) = (gc(\lambda), s(\lambda)) \quad \text{and} \quad r(g, \lambda) = (g, r(\lambda)).$$

If $s(\lambda) = r(\mu)$ then (g, λ) and $(gc(\lambda), \mu)$ are composable in $\Gamma \times_c \Lambda$ and the result is

$$(g, \lambda)(gc(\lambda), \mu) = (g, \lambda\mu).$$

The degree map is given by $d(g, \lambda) = d(\lambda)$, abusing notation.

We confirm that $\Gamma \times_c \Lambda$ is a $\mathbb{N}$-graph by checking the unique factorization property: for every $(g,\lambda) \in \Gamma \times_c \Lambda$ and $m,n \in \mathbb{N}_0^{\mathbb{N}}$ with $d(g,\lambda) = d(\lambda) = m+n$, there exist unique $\mu,\nu \in \Lambda$ such that $\lambda = \mu\nu$ and $d(\mu) = m, d(\nu) = n$. Then the unique factorization in $\Gamma \times_c \Lambda$ is $(g,\mu\nu) = (g,\mu)(gc(\mu),\nu)$.

The infinite path space $(\Gamma \times_c \Lambda)^\infty$ may be identified with $\Gamma \times \Lambda^\infty$. Given a pair $(g,x) \in \Gamma \times \Lambda^\infty$, define the infinite path $(g,x) \in (\Gamma \times_c \Lambda)^\infty$ on pairs $(m,n) \in \Omega$ by

$$(g,x)(m,n) = (gc(x(0,m)), x(m,n)).$$

Since $d(gc(x(0,m)), x(m,n)) = d(x(m,n)) = n - m = d_\Omega(m,n)$, this is indeed degree-preserving. In addition,

$$\begin{aligned}
\sigma^p(g,x)(m,n) &= (g,x)(m+p,n+p) \\
&= (gc(x(0,m+p)), x(m+p,n+p)) \\
&= (gc(x(0,p))c(x(p,m+p)), \sigma^p x(m,n)) \\
&= (gc(x(0,p))c(\sigma^p x(0,m)), \sigma^p x(m,n)) \\
&= (gc(x(0,p)), \sigma^p x)(m,n).
\end{aligned}$$

As in the k-graph case, following in the footsteps of [12] Section 2, the functor c induces a cocycle $\tilde{c} : G_\Lambda \to \Gamma$ defined by

$$\tilde{c}(x, \ell - m, y) = c(x(0,\ell))c(y(0,m))^{-1},$$

where $(x, l-m, y) \in G_\Lambda$ so that $\sigma^l x = \sigma^m y$. Alternatively and sometimes more conveniently,

$$\tilde{c}(\alpha z, d(\alpha) - d(\beta), \beta z) = c(\alpha)c(\beta)^{-1}.$$

In the general KPf case, this map is not at all well-defined (see Section 2.4). However, in the $\mathbb{N}$-graph case, if $(\alpha z, d(\alpha) - d(\beta), \beta z) = (\gamma y, d(\gamma) - d(\delta), \delta y)$ so that $\alpha z = \gamma y$, $\beta z = \delta y$, and $d(\alpha) - d(\beta) = d(\gamma) - d(\delta)$, then there exists $\mu, \mu' \in \Lambda$ such that $\alpha\mu' = \gamma\mu$ and $\beta\mu' = \delta\mu$. Then

$$c(\alpha)c(\beta)^{-1} = (c(\gamma)c(\mu))(c(\delta)c(\mu))^{-1} = c(\gamma)c(\delta)^{-1}.$$

As in the k-graph case, $\tilde{c}$ is a continuous homomorphism, being constant on cylinder sets. In particular, seeing $d : \Lambda \to \mathbb{N}_0^{\mathbb{N}} \subset \mathbb{Z}^\infty$, we have $\tilde{d}(\alpha z, d(\alpha) - d(\beta), \beta z) = d(\alpha) - d(\beta)$ for $(\alpha z, d(\alpha) - d(\beta), \beta z) \in G_\Lambda$.

Given our homomorphism $\tilde{c} : G_\Lambda \to \Gamma$ from a groupoid to a group, [8] I.1.6 constructs a *skew-product groupoid* which we denote $G_\Lambda(\tilde{c})$. The following generalizes [1] Theorem 5.2.

Theorem 5.5.2. *Let Γ be a discrete group, Λ an $\mathbb{N}$-graph, and $c : \Lambda \to \Gamma$ a functor. Then $G_{\Gamma\times_c\Lambda} \cong G_\Lambda(\tilde{c})$.*

Proof. The identification of $\Gamma \times \Lambda^\infty$ with $(\Gamma \times_c \Lambda)^\infty$ has already been discussed above. We now consider the path groupoids to see the same relationship as in the proof of [1] Theorem 5.2.

The path groupoid $G_{\Gamma\times_c\Lambda}$ consists of triples of the form $((g, x), n, (h, y)) \in (\Gamma \times_c \Lambda)^\infty \times \mathbb{Z}^\infty \times (\Gamma \times_c \Lambda)^\infty$ such that $\sigma^\ell(g, x) = \sigma^m(h, y)$ for $n = \ell - m$. If that is to be the case, we must have $(gc(x(0, \ell)), \sigma^\ell x) = (hc(y(0, m)), \sigma^m y)$; that is, we must have $\sigma^\ell x = \sigma^m y$ so $(x, \ell - m, y) \in G_\Lambda$, and $h = gc(x(0, \ell))c(y(0, m))^{-1} = g\tilde{c}(x, \ell - m, y)$, implying that

$$G_{\Gamma\times_c\Lambda} = \{((g, x), \ell - m, (g\tilde{c}(x, \ell - m, y), y)) : (x, \ell - m, y) \in G_\Lambda, g \in \Gamma\}.$$

Then the map $\phi : G_\Lambda(\tilde{c}) \to G_{G\times_c\Lambda}$ given by $\phi((x, \ell - m, y), g) = ((g, x), \ell - m, (g\tilde{c}(x, \ell - m, y), y))$ is shown to be an isomorphism of groupoids by following the proof of [12] Theorem 2.4. $\square$

The gauge action of [1] Section 3 has been generalized in [20] Definition 2.4, and likewise we can generalize [1] Corollary 5.3, whose proof references [12] Corollary 2.5. For convenience, we explicitly go through the steps in the proof of the following result.

Corollary 5.5.3. *Let Γ be a discrete abelian group, Λ an $\mathbb{N}$-graph, and $c : \Lambda \to \Gamma$ a functor. There is an action of the dual group of Γ on $C^*(\Lambda)$, denoted $\alpha^c : \widehat{\Gamma} \to \operatorname{Aut} C^*(\Lambda)$, such that for $\chi \in \widehat{\Gamma}$ and $\lambda \in \Lambda$,*

$$\alpha^c_\chi(s_\lambda) = \langle \chi, c(\lambda) \rangle s_\lambda.$$

Moreover $C^*(\Lambda) \rtimes_{\alpha^c} \widehat{\Gamma} \cong C^*(\Gamma \times_c \Lambda)$. In particular, $\alpha^d = \alpha$, the gauge action in [20] Definition 2.4, and then $C^*(\Lambda) \rtimes_\alpha \mathbb{T}^\omega \cong C^*(\mathbb{Z}^\infty \times_d \Lambda)$.

Proof. First note that as in the k-graph case, α^c preserves the Cuntz-Krieger relations when applied to the generators s_λ of $C^*(\Lambda)$. That is, since χ is a homomorphism from Γ to $\mathbb{T}$,

(CK1) $\alpha^c_\chi(s_v)$ and $\alpha^c_\chi(s_u)$ are orthogonal projections for objects $v \neq u$,

(CK2) $\alpha^c_\chi(s_\lambda)\alpha^c_\chi(s_\mu) = \langle \chi, c(\lambda)\rangle\langle \chi, c(\mu)\rangle s_\lambda s_\mu = \langle \chi, c(\lambda\mu)\rangle s_\lambda s_\mu = \alpha^c_\chi(s_{\lambda\mu})$ for $\lambda, \mu \in \Lambda$ composable,

(CK3) $\alpha^c_\chi(s^*_\lambda)\alpha^c_\chi(s_\lambda) = \langle \chi, c(\lambda)^{-1}\rangle\langle \chi, c(\lambda)\rangle s^*_\lambda s_\lambda = \langle \chi, 1_\Gamma\rangle s_{s(\lambda)} = \langle \chi, c(s(\lambda))\rangle s_{s(\lambda)} = \alpha^c_\chi(s_{s(\lambda)})$,

(CK4) for all $v \in \Lambda^0$ and $n \in \mathbb{N}_0^{\mathbb{N}}$,

$$\sum_{\lambda \in v\Lambda^n} \alpha^c_\chi(s_\lambda)\alpha^c_\chi(s^*_\lambda) = \sum_{\lambda \in v\Lambda^n} \langle \chi, c(r(\lambda))\rangle s_\lambda s^*_\lambda = \langle \chi, c(v)\rangle s_v = \alpha^c_\chi(s_v).$$

Thus α^c defines an action of $\widehat{\Gamma}$ on $C^*(\Lambda)$. Recall that $C^*(\Lambda) \cong C^*(G_\Lambda)$ via $s_\lambda \mapsto 1_{Z(\lambda, s(\lambda))}$.

Next, we check that α^c is given by $\tilde{c}$ in the sense that if $f \in C_c(G_\Lambda) \subset C^*(\Lambda)$, then $(\alpha^c_\chi f)((x,n,y)) = \langle \chi, \tilde{c}(x,n,y)\rangle f((x,n,y))$. To see this, approximate f using simple functions, on which

$$\alpha^c_\chi(1_{Z(\lambda,\mu)}) = \alpha^c_\chi(s_\lambda s^*_\mu) = \langle \chi, c(\lambda)c(\mu)^{-1}\rangle s_\lambda s^*_\mu = \langle \chi, c(\lambda)c(\mu)^{-1}\rangle 1_{Z(\lambda,\mu)}.$$

When applied to a triple (x,n,y) these will vanish unless $(x,n,y) \in Z(\lambda,\mu)$, in which case $\tilde{c}(x,n,y) = c(\lambda)c(\mu)^{-1}$.

Now, since $G_{\Gamma\times_c\Lambda} \cong G_\Lambda(\tilde{c})$, it suffices to show that the latter is a crossed product of $C^*(\Lambda)$ by $\widehat{\Gamma}$. For this we apply [8] II.5.7 since G_Λ is a locally compact groupoid and c a continuous homomorphism.

Finally, using in particular $\alpha = \alpha^d$ and recalling $\tilde{d}(x, n, y) = n$, since in this case $\Gamma = \mathbb{Z}^\infty$ whose dual group is $\mathbb{T}^\omega$ [26],

$$\alpha_t^d(s_\lambda) = t^{d(\lambda)}s_\lambda = \alpha_t(s_\lambda).$$

Thus we have $C^*(\Lambda) \rtimes_\alpha \mathbb{T}^\omega \cong C^*(\mathbb{Z}^\infty \times_d \Lambda)$. □

Note that the following lemma is slightly adjusted from [1] Lemma 5.4 for clarity.

Lemma 5.5.4. *Let Λ be a $\mathbb{N}$-graph and suppose there is a map $b : \Lambda^0 \to \mathbb{Z}^\infty$ such that $d(\lambda) = b(s(\lambda)) - b(r(\lambda))$ for all $\lambda \in \Lambda$, then $C^*(\Lambda)$ is AF. Moreover, in this case G_Λ is AF and hence amenable.*

The proof is the same as that of [1] Lemma 5.4, taking $n \in \mathbb{Z}^\infty$ instead, including the subsequent discussion.

At this point, we are able to show that the path groupoid of an $\mathbb{N}$-graph is amenable.

Theorem 5.5.5. *Let Λ be an $\mathbb{N}$-graph. Then $C^*(\Lambda) \rtimes_\alpha \mathbb{T}^\omega$ is AF and the groupoid $\mathcal{G}_\Lambda$ is amenable.*

Proof. As in the proof of [1] Theorem 5.5, observe that the map $b : (\mathbb{Z}^\infty \times_d \Lambda)^0 \to \mathbb{Z}^\infty$ given by $b(n, v) = n$ satisfies

$$b(s(n, \lambda)) - b(r(n, \lambda)) = b(n + d(\lambda), \lambda) - b(n, r(\lambda)) = n + d(\lambda) - n = d(n, \lambda).$$

Thus Lemma 5.5.4 and Corollary 5.5.3 applied to the skew product $\mathbb{Z}^\infty \times_d \Lambda$ give that $C^*(\mathbb{Z}^\infty \times_d \Lambda) \cong C^*(\Lambda) \rtimes_\alpha \mathbb{T}^\omega$ is AF, and $G_{\mathbb{Z}^\infty\times_d\Lambda} \cong G_\Lambda(\tilde{d})$ is amenable. Also, since $\mathbb{Z}^\infty$ is an abelian group, it is amenable. Now [8] II.3.8 yields the amenability of G_Λ. □

We proceed to consider free actions of groups on $\mathbb{N}$-graphs as done for k-graphs in [1] Section 5 and for KPfs in general in Section 2.4. Letting Λ be an $\mathbb{N}$-graph and Γ a countable group, Γ acts on Λ if there is a group homomorphism from Γ to $\mathrm{Aut}(\Lambda)$; as in the k-graph case, we write $(g, \lambda) \mapsto g\lambda$. The action of Γ on Λ is free if it is free on Λ^0; i.e. for all $v \in \Lambda^0$, $gv = v$ implies that g is the identity element of Γ. We know that an action of Γ on Λ induces an action on $C^*(\Lambda)$ such that $\beta_g s_\lambda = s_{g\lambda}$ since the same result holds in the general case (see Proposition 2.4.1).

As in the k-graph case, when Γ acts freely on an $\mathbb{N}$-graph Λ, we form the quotient Λ/Γ using the equivalence relation $\lambda \sim \mu$ if $\lambda = g\mu$ for some $g \in \Gamma$. That is, elements of Λ/Γ are equivalence classes $[\lambda] = \{\lambda' : \lambda' = g\lambda \text{ for some } g \in \Gamma\}$. The structure maps are compatible with $\sim$ since $d(\lambda) = d(g\lambda) = d(\lambda')$, and Λ/Γ is also an $\mathbb{N}$-graph.

Note that for Γ a countable group and $c : \Lambda \to \Gamma$ a functor, Γ acts freely on $\Gamma \times_c \Lambda$ by $g(h, \lambda) = (gh, \lambda)$ since if $g(h, \lambda) = (h, \lambda)$ for all $(h, \lambda) \in \Gamma \times_c \Lambda$, we have $gh = h$ for all $h \in \Gamma$, and g must be the identity of Γ. Also, $(\Gamma \times_c \Lambda)/\Gamma \cong \Lambda$ by applying Theorem 2.4.2.

In fact, if Γ acts freely on Λ with quotient Λ/Γ, we see as for k-graphs that Λ is isomorphic to a skew product of Λ/Γ for a particular $c : \Lambda/\Gamma \to \Gamma$. Denote the quotient map by $q : \Lambda \to \Lambda/\Gamma$. Choose a representative $v' \in \Lambda^0$ for each $[v] \in (\Lambda/\Gamma)^0$, and then for each $[\lambda] \in \Lambda/\Gamma$, let λ' denote the unique element in Λ such that $\lambda' \in [\lambda]$ and $r(\lambda') = r(\lambda)'$. Then $c : \Lambda/\Gamma \to \Gamma$ is defined by $c([\lambda]) = g$ where $g \in \Gamma$ is the element such that $s(\lambda') = gs(\lambda)'$. The same argument as in the k-graph case gives that $(g, [\lambda]) \mapsto g\lambda'$ defines an equivariant isomorphism between $\Gamma \times_c (\Lambda/\Gamma)$ and Λ.

We are now able to reproduce [27] Theorem 5.7 for $\mathbb{N}$-graphs.

Theorem 5.5.6. *Let Λ' be an $\mathbb{N}$-graph, Γ a countable group, and $c : \Lambda' \to \Gamma$ a functor. Then*

$$C^*(\Gamma \times_c \Lambda') \rtimes_\beta \Gamma \cong C^*(\Lambda') \otimes \mathcal{K}(\ell^2(\Gamma)),$$

where β, the action of Γ on $C^*(\Gamma \times_c \Lambda')$, is induced by the natural action on $\Gamma \times_c \Lambda'$, $g(h, \lambda) = (gh, \lambda)$. If Γ is abelian this action is dual to α^c under the identification of Corollary

5.5.3.

Consequently, if Λ is a $\mathbb{N}$-graph and the countable group Γ acts freely on Λ, then

$$C^*(\Lambda) \rtimes_\beta \Gamma \cong C^*(\Lambda/\Gamma) \otimes \mathcal{K}(\ell^2(\Gamma)).$$

Proof. As in the proof of [27] Theorem 5.7, we apply [12] Proposition 3.7 to the natural Γ-action on $G_{\Gamma \times_c \Lambda'} \cong G_{\Lambda'}(\tilde{c})$. That is, utilizing the isomorphism between the C*-algebra of the $\mathbb{N}$-graph and the C*-algebra of its path groupoid, we have

$$C^*(G_{\Lambda'}(\tilde{c})) \rtimes_\beta \Gamma \cong C^*(\Gamma \times_c \Lambda') \rtimes_\beta \Gamma \cong C^*(G_{\Lambda'}) \otimes \mathcal{K}(\ell^2(\Gamma)) \cong C^*(\Lambda') \otimes \mathcal{K}(\ell^2(\Gamma)).$$

If Γ is abelian, then $C^*(\Lambda') \rtimes_{\alpha^c} \widehat{\Gamma} \cong C^*(\Gamma \times_c \Lambda') \cong C^*(G_{\Lambda'}(\tilde{c}))$ where α^c is the action of the dual group of Γ on $C^*(\Lambda')$ as given in Corollary 5.5.3. Now, taking $\Lambda' = \Lambda/\Gamma$ since it is an $\mathbb{N}$-graph by the preceding discussion, we let $c : \Lambda/\Gamma \to \Gamma$ be the functor such that $\Gamma \times_c (\Lambda/\Gamma) \cong \Lambda$. Then applying the first statement yields

$$C^*(\Gamma \times_c (\Lambda/\Gamma)) \rtimes_\beta \Gamma \cong C^*(\Lambda) \rtimes_\beta \Gamma \cong C^*(\Lambda/\Gamma) \otimes \mathcal{K}(\ell^2(\Gamma)).$$

□

5.6 Aperiodicity

Schenkel defines an aperiodicity condition for $\mathbb{N}$-graphs without reference to infinite paths:

Definition 5.6.1 ([20], Definition 3.1)**.** We say Λ is *aperiodic* if for each vertex $v \in \Lambda^0$ and each pair $n \neq m \in \mathbb{N}_0^{\mathbb{N}}$, there is a path $\lambda \in v\Lambda$ such that $d(\lambda) \geq m \vee n$ and

$$\lambda(m, m + d(\lambda) - (m \vee n)) \neq \lambda(n, n + d(\lambda) - (m \vee n)).$$

Let (B) represent the condition in this definition. Using the infinite path space to follow the k-graph case in [1] Section 4, we give the alternate definition:

Definition 5.6.2. We say $x \in \Lambda^\infty$ is an *aperiodic path* if for all $p, q \in \mathbb{N}_0^{\mathbb{N}}$, $\sigma^p(x) = \sigma^q(x)$ implies $p = q$. Λ is said to satisfy the aperiodicity condition (A) if for every $v \in \Lambda^0$, there exists an aperiodic $x \in Z(v)$.

By way of defining a few more terms, say $x \in \Lambda^\infty$ is *periodic* with nonzero *period* $p \in \mathbb{Z}^\infty$ (meaning infinite sequences of integers with finitely many nonzero entries) if for every $(m, n) \in \Omega$ with $m + p \geq 0$, we have $x(m + p, n + p) = x(m, n)$. In other words, p is a period of x if and only if $\sigma^m(x) = \sigma^n(x)$ for all $m, n \in \mathbb{N}_0^{\mathbb{N}}$ such that $m - n = p$. If $\sigma^n(x)$ is periodic for some $n \in \mathbb{N}_0^{\mathbb{N}}$, we say x is *eventually periodic.*

The two conditions are equivalent:

Theorem 5.6.3. *Λ satisfies the aperiodicity condition (A) if and only if Λ is aperiodic in the sense of condition (B).*

Proof. ($\Longrightarrow$) Assume Λ satisfies aperiodicity condition (A). Let $v \in \Lambda^0$ and $n \neq m \in \mathbb{N}_0^{\mathbb{N}}$. Then there exists an aperiodic infinite path $x \in Z(v)$, and if x is aperiodic, we must have $m \neq n$ implies $\sigma^m(x) \neq \sigma^n(x)$. Then there exist $\ell \in \mathbb{N}_0^{\mathbb{N}}$ such that $\sigma^m(x)(0, \ell) \neq \sigma^n(x)(0, \ell)$. Note that we must also have that $\sigma^m(x)(0, \ell') \neq \sigma^n(x)(0, \ell')$ whenever $\ell' \geq \ell$, so if necessary adjust ℓ so that it is greater than or equal to $m \vee n$. Then letting $\lambda := x(0, \ell)$ will satisfy condition (B).

($\Longleftarrow$) (Modeled after [28] Lemma 3.2) Assume Λ is aperiodic as defined by (B) and let $v \in \Lambda^0$. We wish to find a path $x \in Z(v)$ such that $\sigma^p(x) = \sigma^q(x)$ implies $p = q$, or equivalently, such that $p \neq q$ implies $\sigma^p(x) \neq \sigma^q(x)$. For any particular choice of $p \neq q$ we are able to find a finite path $\lambda_{v,p,q}$ targeting v by assuming Λ is aperiodic. Let $(p^i, q^i)_{i \in \mathbb{N}}$ be a listing of $\{(p, q) \in \mathbb{N}_0^{\mathbb{N}} \times \mathbb{N}_0^{\mathbb{N}} : p \neq q\}$. Let $\lambda_1 = \lambda_{v,p^1,q^1}$, and inductively define $\lambda_i := \lambda_{s(\lambda_{i-1}),p^i,q^i}$ by applying the same condition at each $s(\lambda_i)$. Let $\eta_i := \lambda_1\lambda_2 \cdots \lambda_i$ for each $i \geq 1$. Then $Z(\eta_{i+1}) \subset Z(\eta_i)$ for each $i \geq 1$, and as $i \to \infty$, the desired infinite path is found in the nonempty intersection

$$x := \bigcap_{i \in \mathbb{N}} Z(\eta_i).$$

To see that x is an aperiodic path, let $p \neq q$ be arbitrary, and note that we must have $(p, q) = (p^j, q^j)$ for some j. Then

$$\begin{aligned}\lambda_j(p^j, p^j + d(\lambda_j) - (p^j \vee q^j)) &= \sigma^{d(\eta_{j-1})}(x)(p^j, p^j + d(\lambda_j) - (p^j \vee q^j)) \\ &= \sigma^{d(\eta_{j-1})+p}(x)(0, d(\lambda_j) - (p \vee q))\end{aligned}$$

and

$$\begin{aligned}\lambda_j(q^j, q^j + d(\lambda_j) - (p^j \vee q^j)) &= \sigma^{d(\eta_{j-1})}(x)(q^j, q^j + d(\lambda_j) - (p^j \vee q^j)) \\ &= \sigma^{d(\eta_{j-1})+q}(x)(0, d(\lambda_j) - (p \vee q)),\end{aligned}$$

so x is an aperiodic path by the definition of λ_j. □

Note that in addition to the properties of G_Λ listed in Theorem 5.4.1, G_Λ is topologically principal if and only if Λ is aperiodic (one direction being given by [2] Theorem 5.6, and the other shown in Theorem 3.3.1). Since G_Λ is amenable, we are also free to apply [2] Corollary 5.7 in full, Part (1) of which generalizes [1] Proposition 2.11 and justifies [20] Theorem 2.2, and Part (2) of which is consistent with [20] Theorem 3.5.

We say Λ is *cofinal* if for every $x \in \Lambda^\infty$ and $v \in \Lambda^0$ there is $\lambda \in \Lambda$ and $n \in \mathbb{N}_0^{\mathbb{N}}$ such that $s(\lambda) = s(x(0, n))$ and $r(\lambda) = v$.

Proposition 5.6.1 ([1], Proposition 4.7)**.** *If Λ satisfies the aperiodicity condition, then $C^*(\Lambda)$ is simple if and only if Λ is cofinal.*

The proof is the same as the proof of [1] Proposition 4.7 by generalizing the elements from $\mathbb{N}^k$ to elements of $\mathbb{N}_0^{\mathbb{N}}$. It also follows from the similar result in the more general case, Proposition 3.3.1, since the $\mathbb{N}$-graph case satisfies the necessary conditions.

Chapter 6

$\mathbb{Q}$-KPfs

In this chapter, we consider the choice of $\mathcal{B} = (\mathbb{Q}_{\geq 0}, +)$, the commutative monoid of non-negative rational numbers with addition. Since this $\mathcal{B}$ is cancellative and right Ore, it is strongly right Ore. Seen as a one-object category, it is not finitely generated since the generating set is $\{0\} \cup \{\frac{1}{n} : \forall n \in \mathbb{Z}_{>0}\}$. Strictly speaking, all of the elements of $\mathbb{Q}_{\geq 0}$, including those, are also non-generators, since any finite subset of a generating set can be removed and it will still be a generating set. We will use the above standard generating set, and may impertinently call such elements "generators" anyway. We will call a KPf targeting this choice of $\mathcal{B}$ a $\mathbb{Q}$-KPf, giving the formal definition below.

6.1 Definition and Notation

Definition 6.1.1. A $\mathbb{Q}$-*KPf* consists of a small category $\mathcal{E}$ and a KPf $d : \mathcal{E} \to \mathcal{B}$, where $\mathcal{B} = (\mathbb{Q}_{\geq 0}, +)$. That is, d is a dCf having the unique lifting property: for $\lambda \in \mathcal{E}$, if $d(\lambda) = r_1 + r_2$ with $r_1, r_2 \in \mathbb{Q}_{\geq 0}$, then there exist unique $\mu, \nu \in \mathcal{E}$ such that $\lambda = \mu\nu$, $d(\mu) = r_1$, and $d(\nu) = r_2$, and d is locally split.

The C*-algebra for a $\mathbb{Q}$-KPf is the universal C*-algebra $C^*(F)$ when $F = d$ is row-finite and strongly surjective, generated by a Cuntz-Krieger system for d as in Definition 1.1.4.

Below, we recall and condense this definition tailored to the case of a row-finite, strongly surjective $\mathbb{Q}$-KPf.

Definition 6.1.2. Given a strongly surjective, row-finite $\mathbb{Q}$-KPf $(\mathcal{E}, d)$, a *Cuntz-Krieger system associated to* $(\mathcal{E}, d)$ is a collection of partial isometries $\{s_\lambda : \lambda \in \mathcal{E}\}$ in a C*-algebra satisfying the relations

1. The collection $\{s_X : X \in \mathrm{Obj}(\mathcal{E})\}$ consists of mutually orthogonal projections.
2. If α and β are composable, then $s_{\alpha\beta} = s_\alpha s_\beta$.
3. For all $\alpha : Y \to X$, $s_\alpha^* s_\alpha = s_{s(\alpha)} = s_Y$.
4. For all $X \in \mathrm{Obj}(\mathcal{E})$, and for all rational numbers $r \in \mathcal{B}$,

$$\sum_{\{\alpha \in X\mathcal{E} \,:\, d(\alpha)=r\}} s_\alpha s_\alpha^* = s_X.$$

Although we have often throughout this work denoted $s_X =: p_X$ to emphasize the projection, in this chapter we will usually use s_X to avoid confusion given the frequent use of p to denote a rational number.

Example 6.1.3. When d is the identity map, $d : \mathcal{B} \to \mathcal{B}$ is a KPf and thus a $\mathbb{Q}$-KPf. Since it is also row-finite and strongly surjective, we consider the Cuntz-Krieger system, where we have $s_{r_1+r_2} = s_{r_1}s_{r_2} = s_{r_2}s_{r_1}$ and $s_{r_1}^* s_{r_1} = s_{r_1}s_{r_1}^* = s_0 = \mathrm{Id}$, where 0 denotes the identity morphism on the single object of $\mathcal{B}$ and Id the identity projection. Then a set of generators of the Cuntz-Krieger system is given by $\{s_0, s_{1/n} : n \in \mathbb{Z}_{>0}\}$, where the partial isometries $s_{1/n}$ are in fact unitaries. The C*-algebra generated by this Cuntz-Krieger family $C^*(d)$ is the group C*-algebra of $\mathbb{Q}$ when seen as a topological group under the discrete topology by $s_{1/n} \mapsto \frac{1}{n}$ and $s_{1/n}^* \mapsto -\frac{1}{n}$. This will also be clear via the isomorphism with the C*-algebra of the path groupoid, dealt with in the next section. Since $\mathbb{Q}$ is an abelian group, we have by [29] Proposition VII.1.1 that $C^*(d) = C^*(\mathbb{Q}) = C_0(\widehat{\mathbb{Q}})$. Here $C^*(\mathbb{Q})$ must be formed from

$\mathbb{Q}$ with the discrete topology, and the dual $\widehat{\mathbb{Q}}$ is $\mathbf{A}_{\mathbb{Q}}/\mathbb{Q}$, where $\mathbf{A}_{\mathbb{Q}}$ is the adele ring of $\mathbb{Q}$ viewed as an additive group (see [30] Theorem 3.2).

Remark 6.1.4. In general, when analyzing splittings or infinite paths under the KPf treatment, we let $r \in \mathrm{Obj}(\mathcal{E})$ and consider the slice category $\mathcal{E}/r$ whose objects are morphisms $\lambda \in \mathcal{E}$ with range r. Morphisms in the slice category arise from commutative triangles $\lambda' = \lambda\mu$ in $\mathcal{E}$, and the morphism in $\mathcal{E}/r$ with range λ and source λ' is denoted by the pair (λ, μ). For the slice category $\mathcal{B}/\star$ where $\star$ is the single object of $\mathcal{B}$, like $\mathbb{N}^k$ for the k-graph case, the objects are elements of $\mathbb{Q}$, while morphisms are ordered pairs (p, q), with range p and source $p + q$.

If p, q are two objects of $\mathcal{B}/\star$, since $\mathcal{B} = (\mathbb{Q}_{\geq 0}, +)$, there is a unique morphism from q to p exactly when $q \geq p$ so that the commutative triangle $q = p + (q - p)$ exists in $\mathcal{B}$. Thus if we define the category $\Omega_{\mathbb{Q}} := \{(p, q) \in \mathbb{Q}_{\geq 0} \times \mathbb{Q}_{\geq 0} : q \geq p\}$ similarly to the category Ω_k from [1] Example 1.7.ii, we have an analogue to the isomorphism $\mathbb{N}^k/\star \cong \Omega_k$ from the k-graph case. We may associate $(p, q - p) \in \mathcal{B}/\star$ as described above with $(p, q) \in \Omega_{\mathbb{Q}}$. The infinite path space for a row-finite $\mathbb{Q}$-KPf may be described using the KPf treatment, but for convenience as in the case of $\mathbb{N}$-graphs, we will end up developing a k-graph-reminiscent notation due to this correspondence.

As the second typical example, $\Omega_{\mathbb{Q}}$ can be seen as a row-finite $\mathbb{Q}$-KPf itself.

Example 6.1.5. Let $\mathcal{E} = \Omega_{\mathbb{Q}}$, the category whose objects are $\mathbb{Q}_{\geq 0}$ and whose morphisms are $\{(p, q) \in \mathbb{Q}_{\geq 0} \times \mathbb{Q}_{\geq 0} : q \geq p\}$ having range p, source q, and compositions $(p, q)(q, r) = (p, r)$. Define the functor $d : \Omega_{\mathbb{Q}} \to \mathcal{B}$ by $d(p, q) = q - p$. If $q - p = r_1 + r_2$ for some $r_1, r_2 \in \mathcal{B}$, then $(p, q) = (p, p + r_1)(p + r_1, q)$ is the lift satisfying $d(p, p + r_1) = r_1$ and $d(p + r_1, q) = q - p - r_1 = r_2$. Thus $(\Omega_{\mathbb{Q}}, d)$ is a $\mathbb{Q}$-KPf, and it is row-finite since for an object $r \in \Omega_{\mathbb{Q}}$ and a degree $q \in \mathcal{B}$, the only morphism in $\Omega_{\mathbb{Q}}$ targeting r with degree q is $(r, r + q)$. In this case $C^*(d)$ is generated by the Cuntz-Krieger system $\{s_r, s_{(r_1,r_2)} : r \in \mathrm{Obj}(\Omega_{\mathbb{Q}}), (r_1, r_2) \in \Omega_{\mathbb{Q}}\}$. In fact, considering $\ell^2(\mathbb{Q}_{\geq 0}) = \{f : \mathbb{Q}_{\geq 0} \to \mathbb{C} \text{ such that } \sum_{r \in \mathbb{Q}_{\geq 0}} |f(r)|^2 < \infty\}$, mapping projections s_r to generators δ_r and partial isometries $s_{(r_1,r_2)}$ to the rank-one maps $\delta_{r_1} \mapsto \delta_{r_2}$ shows that $C^*(d) \cong \mathcal{K}(\ell^2(\mathbb{Q}_{\geq 0}))$.

Example 6.1.6. As in Example 5.1.5, consider the category $\Delta_{\mathbb{Q}} := \{(p,q) \in \mathbb{Q}\times\mathbb{Q} : p \leq q\}$ with composition rule $(p,r)(r,q) = (p,q)$. Define $d : \Delta_{\mathbb{Q}} \to \mathbb{Q}_{\geq 0}$ by $d(p,q) = q - p$. Then $(\Delta_{\mathbb{Q}}, d)$ is a $\mathbb{Q}$-KPf with $C^*(d) \cong \mathcal{K}(\ell^2(\mathbb{Q}))$.

We conclude this section by applying Theorem 1.4.2 to say that the core of the C*-algebra of a $\mathbb{Q}$-KPf is AF, since $\mathcal{B} = \mathbb{Q}_{\geq 0}$ is countable, and when $q_1 \leq q_2$, we have the factorization $q_1 + (q_2 - q_1) = q_2$.

6.2 Infinite Path Spaces and Path Groupoids

Since the infinite path space of $(\mathcal{E}, d)$ seen as a KPf consists of the splittings of $d : \mathcal{E}/X \to \mathcal{B}/\star$ for all $X \in \mathrm{Obj}(\mathcal{E})$, we may actually utilize the correspondence from Remark 6.1.4 to define them following the k-graph convention, as we did for $\mathbb{N}$-graphs.

Definition 6.2.1. With $\mathcal{B} = (\mathbb{Q}_{\geq 0}, +)$, let $d : \mathcal{E} \to \mathcal{B}$ be a row-finite, strongly surjective KPf yielding the $\mathbb{Q}$-KPf $(\mathcal{E}, d)$. We consider an infinite path to be a functor $x : \Omega_{\mathbb{Q}} \to \mathcal{E}$ intertwining the degree maps. That is, x is defined on pairs $(p,q) \in \mathbb{Q}_{\geq 0} \times \mathbb{Q}_{\geq 0}$ where $q \geq p$ and takes values in $\mathcal{E}$ such that $x(p,q) = \lambda$ means $d(\lambda) = q - p$. Denote the infinite path space by $(\mathcal{E}, d)^\infty$ for clarity. The range object of x is given by $x(0)$, and x ends in a morphism λ if $x(0, d(\lambda)) = \lambda$.

In fact, an infinite path $x \in (\mathcal{E}, d)^\infty$ is determined by the values it takes on $(0,q)$ for $q \in \mathbb{Q}_{\geq 0}$, since by unique lifting, $x(p,q)$ is the morphism that satisfies $x(0,q) = x(0,p)x(p,q)$. Moreover, given a cofinal sequence $(p_i)_{i\in\mathbb{N}} \subset \mathbb{Q}_{\geq 0}$, x is determined by its values on $(0, p_i)$ for all $i \in \mathbb{N}$. For example, we could take $(n)_{n\in\mathbb{N}}$ since for any rational number p, there exists an $n \geq p$, namely $n = \lceil p \rceil$.

As usual, we have cylinder sets $Z(X) = \{x \in (\mathcal{E}, d)^\infty : r(x) = X\}$ for all $X \in \mathrm{Obj}(\mathcal{E})$ and $Z(\lambda) = \{x \in (\mathcal{E}, d)^\infty : x(0, d(\lambda)) = \lambda\}$ for all $\lambda \in \mathcal{E}$. Observe that since the rationals are countable, these form a basis of compact open sets for a locally compact Hausdorff topology on $(\mathcal{E},d)^\infty$ when $(\mathcal{E},d)$ is a row-finite, strongly surjective $\mathbb{Q}$-KPf, by [2] Proposition 4.6 and Theorem 4.7.

As for the maps ind and res, we will use the k-graph style notation $\mathrm{res}_\lambda(x) = \sigma^{d(\lambda)}(x)$ and $\mathrm{ind}_\lambda(x) = \lambda x$ for the same reasons as in the $\mathbb{N}$-graph case, owing to the commutativity and "factorability" of addition as the operation in $(\mathbb{Q}_{\geq 0}, +)$.

For similar reasons, the path groupoid for a $\mathbb{Q}$-KPf may use the notation

$$G_d := \{(\alpha z, d(\alpha) - d(\beta), \beta z) \in (\mathcal{E},d)^\infty \times \mathbb{Q} \times (\mathcal{E},d)^\infty : s(\alpha) = s(\beta), z \in Z(s(\beta))\}$$

or alternatively when convenient as

$$G_d = \{(x,r,y) \in (\mathcal{E},d)^\infty \times \mathbb{Q} \times (\mathcal{E},d)^\infty : \sigma^p(x) = \sigma^q(y) \text{ for some } p,q \in \mathbb{Q}_{\geq 0} \text{ and } r = p-q\}.$$

We have cylinder sets $Z(\alpha,\beta) := \{(\alpha z, d(\alpha)-d(\beta), \beta z) \in G_d : z \in Z(s(\beta))\}$, which form a basis for a totally disconnected locally compact Hausdorff topology on G_d when $(\mathcal{E},d)$ is a row-finite $\mathbb{Q}$-KPf since $\mathcal{B}$ is countable by [2] Theorem 4.16. With respect to this topology, G_d is a topological groupoid, and G_d is étale. In addition, since $\mathcal{B}$ is cancellative, we have $C^*(G_d) \cong C^*(d)$ for a row-finite $\mathbb{Q}$-KPf $(\mathcal{E},d)$, and the map $\alpha \mapsto s_\alpha$ from $\mathcal{E} \to C^*(d)$ is injective.

Finally, returning in particular to the examples from Section 6.1, we consider their path groupoids in particular.

When $d = \mathrm{Id}_\mathcal{B} : \mathcal{B} \to \mathcal{B}$ is the KPf in question, there is exactly one infinite path, $x = \mathrm{Id}_\mathcal{B}$. Its path groupoid is simply $\mathbb{Q}$ with the discrete topology, since the elements are (x,r,x) with $r \in \mathbb{Q}$, and the cylinder sets $Z(p,q)$ are singletons. That is, the path groupoid is in fact a group, and $C^*(G_d) \cong C^*(d)$ is separable and nuclear.

When $d : \Omega_\mathbb{Q} \to \mathcal{B}$ is defined $d(p,q) = q-p$, there is exactly one infinite path $x_r \in Z(r)$ for each object r of $\Omega_\mathbb{Q}$. The path groupoid G_d has cylinder sets of the form $Z((p,q), s(p,q))$ for

$(p, q) \in \Omega_{\mathbb{Q}}$, and each is a singleton since there is exactly one path x_q with range $s(p, q) = q$. The map $s_{(p,q)} \mapsto 1_{Z((p,q),s(p,q))}$ is the isomorphism from $C^*(d)$ to $C^*(G_d)$.

Finally, observe that as in the case of k-graphs and $\mathbb{N}$-graphs, the isotropy of the path groupoid for a $\mathbb{Q}$-KPf is abelian, following from the commutativity of addition in this choice of $\mathcal{B}$.

Remark 6.2.2. We recall the skew product construction from Section 2.4 to mention that with the choice of group $\mathbb{Q}$ and choice of functor $d : \mathcal{E} \to \mathbb{Q}_{\geq 0} \subset \mathbb{Q}$, we find the skew product KPf $d : (\mathbb{Q} \times_d \mathcal{E})$ and associated $\tilde{d} : G_d \to \mathbb{Q}$ defined by $\tilde{d}(x, p, y) = p$. While the Pontryagin dual of $\mathbb{Q}$ with the discrete topology is much more complicated than the duality of $\mathbb{Z}^k$ and $\mathbb{T}^k$, we suspect that there may be a way to circumvent the lack of a gauge action to yield results about the amenability of particulated $\mathbb{Q}$-KPfs.

6.3 Particulated $\mathbb{Q}$-KPfs

In this section, we introduce a family of examples of $\mathbb{Q}$-KPfs derived from directed graphs. We begin with a simple example and discussion.

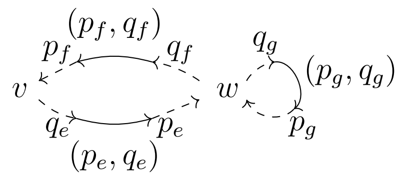
Example 6.3.1. Consider the directed graph below, which is row-finite and has no sources.

$$\begin{tikzcd} v \arrow[r, bend right, "e"'] & w \arrow[l, bend right, "f"'] \arrow[loop right, "g"] \end{tikzcd}$$

The general idea for $\mathcal{E}$ is that the rational numbers between 0 and 1 will be superimposed on each edge of the graph. We have as objects the "joint" vertices v and w, and also for all rational $r \in [0, 1]$, we have objects lying on e, f, and g labeled by r_e, r_f, and r_g, respectively. We have $0_f = v = 1_e$ and $0_e = w = 0_g = 1_g$. We also have as generating morphisms (p_f, q_f) for all rational $0 \leq p_f \leq q_f \leq 1$ with range p_f and source q_f, subject to the relation

$$(0_f, p_f)(p_f, q_f)(q_f, 1_f) = f.$$

We have the same morphisms and rules for edges e and g again. The degree functor $d : \mathcal{E} \to \mathcal{B}$ is defined on the generating morphisms by $d(p_*, q_*) = q - p$, extended to compositions, and in particular $d(f) = d(e) = d(g) = 1$. The figure below gives a visual idea of the details described.



This KPf is row-finite due the lack of interaction between morphisms contained in distinct edges and the row finiteness of the original directed graph. Its infinite path space is more interesting than that of the previous two examples, since although the endings chosen by infinite paths are dictated while within e, f, or g, choices can be made at the joint objects v and w. The cylinder sets for the joint objects are $Z(v) = Z(f)$, $Z(w) = Z(g) \cup Z(e)$, and for any other object r_*, $Z(r_*) = Z((r_*, 1_*))$. More specifically,

$$Z(r_e) = Z((r_e, 1_e)) = Z((r_e, v)f),$$

$$Z(r_f) = Z((r_f, 1_f)) = Z((r_f, w)g) \cup Z((r_f, w)e),$$

$$Z(r_g) = Z((r_g, 1_g)) = Z((r_g, w)g) \cup Z((r_g, w)e).$$

In general, this superposition of rationals will work for any directed graph E to produce a $\mathbb{Q}$-KPf $d : \mathcal{E} \to \mathcal{B}$.

Definition 6.3.2. Given a directed graph $E = (E^0, E^1, r, s)$, let $\mathcal{E}$ be the category with $\mathrm{Obj}(\mathcal{E}) = E^0 \cup \{q_e : q \in [0, 1] \cap \mathbb{Q}, \text{ for all } e \in E^1\}$, where $0_e = r(e)$ and $1_e = s(e)$, and with morphisms generated by (p_e, q_e) with $p, q \in \mathbb{Q}$, $0 \leq p \leq q \leq 1$, $r(p_e, q_e) = p_e$, $s(p_e, q_e) = q_e$; and for all $e_1, e_2, e_3 \in E^1$, $(p_{e_1}, r_{e_2})(r_{e_2}, q_{e_3}) = (p_{e_1}, q_{e_3})$, and $(0_e, 1_e) = e$. Note that morphisms of the form (p_e, p_e) are the identity morphism on the object p_e and identified with the object as usual.

Then defining the degree functor on generating morphisms $d : \mathcal{E} \to \mathcal{B}$ by $d(p_e, q_e) = q - p$ and extending to general (p_{e_1}, q_{e_2}) by addition, we have a $\mathbb{Q}$-KPf $(\mathcal{E}, d)$. We call this the $\mathbb{Q}$-KPf *particulated* from the directed graph E.

Observe that the $\mathbb{Q}$-KPf $(\Omega_{\mathbb{Q}}, d)$ from Example 6.1.5 may also be seen as the $\mathbb{Q}$-KPf particulated from the directed graph

$$\bullet \xleftarrow[e_0]{} \bullet \xleftarrow[e_1]{} \bullet \xleftarrow[e_2]{} \bullet \xleftarrow[e_3]{} \cdots$$

by the correspondence of morphism (p_{e_i}, q_{e_j}) with morphism $(i + p, j + q) \in \Omega_{\mathbb{Q}}$.

Similarly, Example 6.1.6 may be considered as the $\mathbb{Q}$-KPf particulated from the directed graph

$$\cdots \xleftarrow[e_{-2}]{} \bullet \xleftarrow[e_{-1}]{} \bullet \xleftarrow[e_0]{} \bullet \xleftarrow[e_1]{} \bullet \xleftarrow[e_2]{} \cdots$$

by the same correspondence.

Proposition 6.3.1. *Let $(\mathcal{E}, d)$ be the $\mathbb{Q}$-KPf particulated from a directed graph E. If E is row-finite, then so is $(\mathcal{E}, d)$. If E, seen as a 1-graph, has the no sources condition defined in [1], then $(\mathcal{E}, d)$ is strongly surjective. If E satisfies the aperiodicity condition as given in the same reference, then $(\mathcal{E}, d)$ satisfies the aperiodicity condition given by Definition 3.1.1.*

Proof. Assume that $E = (E^0, E^1, r, s)$ is a row-finite directed graph; that is, $r^{-1}(v)$ is a finite set for every $v \in E^1$. Now let $e \in E^0$ and let $q \in \mathcal{B}$ and consider the set of morphisms λ in $\mathcal{E}$ targeting p_e with $d(\lambda) = q$. If $q \leq 1 - p$, this set is a singleton $\{(p_e, (p + q)_e)\}$. If $1 - p < q < 2 - p$, then the set consists of paths $(p_{e_1}, 1_{e_1})(1_{e_1}, (q - 1 + p)_{e_2})$, where $e_2 \in E^1$ such that $r(e_2) = s(e_1)$. But since E was row-finite, there are only finitely many such e_2. This argument extends for q greater than $2 - p$ in the obvious way.

Assume E has no sources; that is, for every $v \in E^0$ and $m \in \mathbb{N}$, $\{\lambda \in E : d(\lambda) = m, r(\lambda) = v\} \neq \emptyset$. Now let $p_e \in \mathrm{Obj}(\mathcal{E})$ (which is mapped by d to the single object of $\mathcal{B}$) and let $q \in \mathcal{B}$. Decompose $q = (1 - p) + m + (q - m - 1 + p)$ for some $m \in \mathbb{N}$. Since E has no sources, there exists a path λ with degree $m + 1 \in \mathbb{N}$ consisting of edges from E^1

with $r(\lambda) = 1_e$; denote $\lambda = f_1 f_2 \cdots f_{m+1}$ where $f_j \in E^1$. Then the morphism in $\mathcal{E}$ given by $(p_e, 1_e) f_1 f_2 \cdots f_m (1_{f_m}, (q - m - 1 + p)_{f_{m+1}})$ has degree q. Thus the map from the set of morphisms in $\mathcal{E}$ targeted at p_e to the set of morphisms in $\mathcal{B}$ is surjective, and $(\mathcal{E}, d)$ is strongly surjective.

If E satisfies the aperiodicity condition, for every $v \in E^0$ there is an aperiodic infinite path $x \in E^\infty(v)$ in the sense that $\sigma^m(x) = \sigma^n(x)$ implies $m = n$, for $m, n \in \mathbb{N}$. Now if p_e is any object in $\mathcal{E}$, then we simply consider the aperiodic path x guaranteed by the assumption with range $r(x) = 0_e$, which extends naturally and uniquely to an aperiodic path in $(\mathcal{E}, d)^\infty$. Then $\mathrm{res}_{(0_e, p_e)}(x) = \sigma^p(x)$ is an infinite path to range object p_e, and by Proposition 3.1.1, it is an aperiodic path since $\mathcal{B}$ is left-cancellative.

□

Remark 6.3.3. As mentioned in the proof above, an infinite path for E seen as a 1-graph also defines an infinite path for $(\mathcal{E}, d)$ with the same range, since endings within edges are fully determined. Furthermore, if $x \in Z(e) \subset E^\infty$ is an infinite path ending in edge $e \in E^1$, then $\mathrm{res}_{(0_e, p_e)}(x) = \sigma^p(x) =: (x, p)$ for all $p \in [0, 1) \cap \mathbb{Q}$ defines a collection of infinite paths in $(\mathcal{E}, d)^\infty$. Indeed, taking $\{(x, p) : x \in E^\infty, p \in [0, 1) \cap \mathbb{Q}\}$, we find all infinite paths in $(\mathcal{E}, d)^\infty$. If $y \in (\mathcal{E}, d)^\infty$ has range $r(y) = p_{e_i}$ for some e_i, then $y = (x, p)$ if $x = \mathrm{ind}_{(0, p_{e_i})}(y) = (0, p_{e_i}) y$. Thus there is a bijection between $(\mathcal{E}, d)^\infty$ and $E^\infty \times ([0, 1) \cap \mathbb{Q})$ via

$$E^\infty \times ([0, 1) \cap \mathbb{Q}) \ni (x, p) \mapsto \sigma^p(x) \in (\mathcal{E}, d)^\infty.$$

Proposition 6.3.2. *Let $\{s_{r_e}, s_{(p_e, q_e)} : r, p, q \in \mathbb{Q} \cap [0, 1), \forall e \in E^1\}$ be a Cuntz-Krieger system for a $\mathbb{Q}$-KPf $(\mathcal{E}, d)$ particulated from a directed graph $E = (E^0, E^1, r, s)$. Then the subset $A := \{s_{0_e}, s_{1_e}, s_{(0_e, 1_e)} : e \in E^1\}$ is a Cuntz-Krieger E-family in $C^*(d)$ generating a C^*-subalgebra isomorphic to the universal C^*-algebra $C^*(E)$ for directed graph E.*

Proof. All that is needed to show that A is a Cuntz-Krieger E-family in $C^*(d)$ is to check the conditions given in [3]. The subset contains a projection p_v for every joint object $v \in E^0$

since those appear in $(\mathcal{E}, d)$ as 0_e or 1_e for some $e \in E^1$, and let us define $t_e := s_{(0_e, 1_e)}$. We wish to show that $\{p_v, t_e\}$ satisfies the Cuntz-Krieger relations.

(CK1) We inherit the property $t_e^* t_e = p_{s(e)}$ since $s^*_{(0_e,1_e)} s_{(0_e,1_e)} = s_{1_e}$ in the original Cuntz-Krieger system for $(\mathcal{E}, d)$.

(CK2) It follows from Condition 4 of Definition 6.1.2, with X chosen to be $v = 0_f$ for some $f \in E^1$, and r chosen to be 1, that

$$p_v = s_{0_f} = \sum_{(0_e,1_e)\in(\mathcal{E},d)\,:\,0_e=0_f} s_{(0_e,1_e)} s^*_{(0_e,1_e)} = \sum_{e\in E^1\,:\,r(e)=v} t_e t_e^*$$

when v is not a source.

Then the universal property gives that $C^*(A) \cong C^*(E)$, where $C^*(E)$ is the C*-algebra of the graph E as a 1-graph. $\square$

Remark 6.3.4. In alignment with the above fact, the isomorphism $C^*(d) \cong C^*(G_d)$, and Remark 6.3.3, the infinite path space E^∞ appears as a subset of $(\mathcal{E}, d)^\infty$, and the path groupoid G_E appears as a subgroupoid of G_d; namely, the elements $(x, r, y) \in G_d$ where $r(x), r(y) \in E^0$ and $r \in \mathbb{Z}$. We notice a similar interaction between path groupoid elements from G_E and G_d as we did for the infinite path spaces E^∞ and $(\mathcal{E}, d)^\infty$. If $(x, r, y) \in G_d$, then there exist $p, q \in \mathbb{Q}_{\geq 0}$ such that $\sigma^p(x) = \sigma^q(y)$ and $r = p - q$. Note that the range of $\sigma^p(x) = \sigma^q(y)$ is either a joint vertex of E, or else there exists a joint vertex v such that $\lambda\sigma^p(x) = \lambda\sigma^q(y)$ for some $\lambda \in vE$, so without loss of generality take the representative (x, r, y) to be minimal in the sense that the range of $\sigma^p(x) = \sigma^q(y)$ is $v \in E^0$. Now, $x = \sigma^{\lceil p\rceil - p}(x')$ and $y = \sigma^{\lceil q\rceil - q}(y')$ for some $x', y' \in E^\infty$. We can see that

$$\sigma^{\lceil p\rceil}(x') = \sigma^{p+\lceil p\rceil - p}(x') = \sigma^p(x) = \sigma^q(y) = \sigma^{q+\lceil q\rceil - q}(y') = \sigma^{\lceil q\rceil}(y'),$$

and thus corresponds to $(x', \lceil p\rceil - \lceil q\rceil, y') \in G_E$. In other words, every element of G_d arises from choice of an element of G_E and two rational numbers in $[0, 1)\cap\mathbb{Q}$. Conversely, any such choices will result in an element of G_d: for $(x', n, y') \in G_E$ with $\sigma^m(x') = \sigma^\ell(y')$, $m - \ell = n$, and choices $r_1, r_2 \in [0, 1) \cap \mathbb{Q}$, we let $x = \sigma^{r_1}(x')$ and $y = \sigma^{r_2}(y')$, and then

$$\sigma^{m-r_1}(x) = \sigma^{m-r_1+r_1}(x') = \sigma^m(x') = \sigma^\ell(y') = \sigma^{\ell-r_2+r_2}(y') = \sigma^{\ell-r_2}(y),$$

so $(x, m - r_1 - \ell + r_2, y) \in G_d$. In this way,

$$G_E \times \left([0, 1)^2 \cap \mathbb{Q}^2\right) \ni ((x', n, y'), (r_1, r_2)) \mapsto (\sigma^{r_1}x', n - r_1 + r_2, \sigma^{r_2}y') \in G_d$$

yields an isomorphism.

6.4 Higher-Rank $\mathbb{Q}$-KPfs

As directed graphs are seen as 1-graphs and generalized to k-graphs whose degree functors target $\mathbb{N}^k$, we may create higher-rank $\mathbb{Q}$-KPfs by allowing the degree map d to target $\mathbb{Q}^k_{\geq 0}$. Seen as a one-object category, $\mathbb{Q}^k_{\geq 0}$ has a generating set of morphisms $e_{i,n} := \left(0, ..., \frac{1}{n}, ..., 0\right)$, with $\frac{1}{n}$ in the i-th position and zeros elsewhere, $1 \leq i \leq k$, $n \in \mathbb{Z}_{>0}$. The definition is natural.

Definition 6.4.1. A rank-k $\mathbb{Q}$-*KPf* consists of a small category $\mathcal{E}$ and a KPf $d : \mathcal{E} \to \mathbb{Q}^k_{\geq 0}$. That is, d has the unique lifting property and is locally split.

Example 6.4.2. We give the basic examples.

(i) As always, the identity functor $\mathrm{Id} : \mathbb{Q}^k_{\geq 0} \to \mathbb{Q}^k_{\geq 0}$ is a KPf.

(ii) the category $\Omega := \{(\vec{p}, \vec{q}) \in \mathbb{Q}^k_{\geq 0} \times \mathbb{Q}^k_{\geq 0} : \vec{q} \geq \vec{p}\}$ can be constructed as usual and $d : \Omega \to \mathbb{Q}^k_{\geq 0}$ defined by $d(\vec{p}, \vec{q}) = \vec{q} - \vec{p}$ is a KPf.

This definition allows us to create rank-k $\mathbb{Q}$-KPfs particulated from k-graphs. We mention as in Section 6.2 that for higher-rank $\mathbb{Q}$-KPfs as well, the isotropy of the path groupoid is abelian.

6.4.1 $\mathbb{Q}$-KPfs particulated from 2-graphs

It is possible to represent a 2-graph on paper by drawing its 1-skeleton (see [3] Chapter 10 for a brief overview of 1-skeletons). A 1-skeleton alone need not define a unique 2-graph, as some choices may need to be made regarding factorizations, but in this chapter we will assume a particular k-graph and use its skeleton to help visualize the process. We begin with examples.

Example 6.4.3. The familiar category $\Omega_2 = \{(m,n) \in \mathbb{N}^2 \times \mathbb{N}^2 : m \leq n\}$, with $d : \Omega_2 \to \mathbb{N}^2$ defined by $d(m,n) = n - m$, is a 2-graph with 1-skeleton

$$\begin{tikzcd}
{} \arrow[d, dashed] & {} \arrow[d, dashed] & {} \arrow[d, dashed] & {} \arrow[d, dashed] & \\
\bullet \arrow[d, dashed] & \bullet \arrow[l] \arrow[d, dashed, "e"] & \bullet \arrow[l, "f"'] \arrow[d, dashed, "k"] & (n_1, n_2) \arrow[l, "g"'] \arrow[d, dashed, "\ell"] & {} \arrow[l] \\
\bullet \arrow[d, dashed] & (m_1, m_2) \arrow[l] \arrow[d, dashed] & \bullet \arrow[l, "h"'] \arrow[d, dashed] & \bullet \arrow[l, "j"'] \arrow[d, dashed] & {} \arrow[l] \\
\bullet & \bullet \arrow[l] & \bullet \arrow[l] & \bullet \arrow[l] & {} \arrow[l]
\end{tikzcd}$$

where the edges with degree $(1,0)$ are in black and the dashed edges with degree $(0,1)$ are called red, and the origin is the lower left corner. Viewed this way, a path in Ω_2 is a rectangle with source (n_1, n_2) and range (m_1, m_2) and degree $(n_1 - m_1, n_2 - m_2) \in \mathbb{N}^2$. We have by unique factorization that $efg = hkg = hj\ell$.

Any bi-colored path of two edges defines a commutative square, which we will call a face, denoted F. We create $\mathcal{E}$ by adding objects and morphisms to each face and enforcing agreement along shared edges. Let $e_1 f_1$ be a bi-colored path with $e_1 \in \Omega^{(0,1)}$ and $f_1 \in \Omega^{(1,0)}$, so that the square $e_1 f_1 = f_2 e_2$ is induced with $f_2 e_2$ the reverse coloring path, and call this square the face F. We include in $\mathcal{E}$ the objects $\begin{bmatrix} p \\ q \end{bmatrix}_F$, where $p, q \in \mathbb{Q}_{\geq 0}$, enforcing the following agreement rules: the objects of the forms $\begin{bmatrix} 0 \\ q \end{bmatrix}_F$, $\begin{bmatrix} 1 \\ q \end{bmatrix}_F$, $\begin{bmatrix} p \\ 0 \end{bmatrix}_F$, and $\begin{bmatrix} p \\ 1 \end{bmatrix}_F$ lie on edges $e_1, e_2, f_2,$ and f_1, respectively; the objects $\begin{bmatrix} 0 \\ 0 \end{bmatrix}_F$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}_F$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}_F$, and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}_F$ are $r(e_1 f_1), r(f_1), r(e_2)$, and $s(e_1 f_1)$, respectively; finally, when two faces F_1 and F_2 share an edge, require the agreement of objects and morphisms. For example, say that $e_2 \in F_1$ takes the place of $e_1 \in F_2$, we require that $\begin{bmatrix}1\\q\end{bmatrix}_{F_1} = \begin{bmatrix}0\\q\end{bmatrix}_{F_2}$. In summary, we fill in each face with all points on the plane with rational coordinates between 0 and 1, not allowing any duplicates to be produced. In the figure below, the face is a square lifted from Ω_2, the morphism known as $(\vec{m}, \vec{m}+\vec{1})$ in the usual pairs notation.

$$\begin{tikzcd}
\bullet \arrow[d, dashed, "e_1"'] & & \vec{m} + \begin{bmatrix}1\\1\end{bmatrix} \arrow[ll, "f_1"'] \arrow[d, dashed, "e_2"] \\
& \bullet \begin{bmatrix}p\\q\end{bmatrix}_F \in \mathbb{Q}^2_{\geq 0} & \\
\mathbb{N}^2 \ni \vec{m} & & \bullet \arrow[ll, "f_2"]
\end{tikzcd}$$

Now the morphisms of $\mathcal{E}$ will be defined using these objects, as being generated by the arrows $\left(\begin{bmatrix}p\\q\end{bmatrix}_F, \begin{bmatrix}p'\\q\end{bmatrix}_F\right)$ with $p' \geq p$, or $\left(\begin{bmatrix}p\\q\end{bmatrix}_F, \begin{bmatrix}p\\q'\end{bmatrix}_F\right)$ with $q' \geq q$. The degree map $d : \mathcal{E} \to \mathbb{Q}^2_{\geq 0}$ is defined on these generating arrows by subtraction again, so that

$$d\left(\begin{bmatrix}p\\q\end{bmatrix}_F, \begin{bmatrix}p'\\q\end{bmatrix}_F\right) = \begin{pmatrix}p'-p\\0\end{pmatrix} \quad \text{and} \quad d\left(\begin{bmatrix}p\\q\end{bmatrix}_F, \begin{bmatrix}p\\q'\end{bmatrix}_F\right) = \begin{pmatrix}0\\q'-q\end{pmatrix},$$

where we use round parentheses to emphasize the vector indicating the degree in $\mathbb{Q}^2_{\geq 0}$. We enforce the same composition rule on edges as we did for 1-graphs: for example

$$\left(\begin{bmatrix}0\\0\end{bmatrix}_F, \begin{bmatrix}p\\0\end{bmatrix}_F\right)\left(\begin{bmatrix}p\\0\end{bmatrix}_F, \begin{bmatrix}p'\\0\end{bmatrix}_F\right)\left(\begin{bmatrix}p'\\0\end{bmatrix}_F, \begin{bmatrix}1\\0\end{bmatrix}_F\right) = f_2$$

in the figure above. We also enforce the rectangular unique factorization rule, so that in the figure below, given a path ij in the face F, we have the square $ij = k\ell$.

$$
\begin{tikzcd}
\bullet \arrow[dddd, dashed, "e_1"'] & & & \arrow[l, "f_1"'] & \vec{m}+\vec{1} \arrow[llll, phantom] \arrow[dddd, dashed, "e_2"] \\
& & & \arrow[d, dashed] & \\
& \begin{bmatrix} p_1 \\ q_2 \end{bmatrix}_F \arrow[d, dashed, "i"'] & & \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}_F \arrow[ll, "j"'] \arrow[d, dashed, "\ell"] & \arrow[l] \\
\arrow[u, phantom] & \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}_F \arrow[l] \arrow[d, dashed] & & \begin{bmatrix} q_1 \\ p_2 \end{bmatrix}_F \arrow[ll, "k"] & \\
\vec{m} & \arrow[l] & & & \bullet \arrow[lll, "f_2"]
\end{tikzcd}
$$

Due to the composition and unique factorization conditions, note that all of the infinitely many zigzag paths (having finitely many turns) from $\vec{m}+\vec{1}$ to $\vec{m}$ in Ω_2^0 are equal to the compositions $e_1 f_1 = f_2 e_2$ and have total degree $\begin{pmatrix}1\\1\end{pmatrix}$. Just as objects on shared edges between faces were forced to be equal, we require the same of the arrows lying on edges, so that no duplicates are allowed.

In summation, with $\mathcal{E}$ created this way and degree map $d : \mathcal{E} \to \mathbb{Q}^2_{\geq 0}$ defined as above within faces and extended naturally to compositions of morphisms, $(\mathcal{E}, d)$ is a rank-2 $\mathbb{Q}$-KPf particulated from the 2-graph Ω_2. We observe that in this example, the existence of minimal object $(0,0) \in \Omega_2^0$ lends itself to more convenient notations, in that faces may be named F_{ij} according to the row and column in which they appear, and rather than labeling objects by face, one could denote $\begin{bmatrix}p\\q\end{bmatrix}_{F_{ij}} = \begin{bmatrix}i+p\\j+q\end{bmatrix}$, allowing for convenient computation of the degree of a path that contains multiple faces, and direct correspondence with the traditional names of objects in Ω_2 when p, q are integers.

Example 6.4.4. The same process can be performed for the 2-graph with 1-skeleton given below, once we explain how to find the faces that were used to define $\mathcal{E}$.

$$
\begin{tikzcd}
u \arrow[r, bend right, "a"'] \arrow[loop left, dashed, "\alpha"] & v \arrow[l, bend right, "b"'] \arrow[loop right, dashed, "\beta"]
\end{tikzcd}
$$

By considering possible two-colored paths, we can visually represent the commutative squares in the 2-graph by drawing a grid.

$$
\begin{tikzcd}
 & \vdots \arrow[d, dashed, "\alpha"'] & \vdots \arrow[d, dashed, "\beta"'] & \vdots \arrow[d, dashed, "\alpha"'] & \\
\cdots & u \arrow[l, "a"] \arrow[d, dashed, "\alpha"'] & v \arrow[l, "b"] \arrow[d, dashed, "\beta"'] & u \arrow[l, "a"] \arrow[d, dashed, "\alpha"'] & \cdots \arrow[l, "b"] \\
\cdots & u \arrow[l, "a"] \arrow[d, dashed, "\alpha"'] & v \arrow[l, "b"] \arrow[d, dashed, "\beta"'] & u \arrow[l, "a"] \arrow[d, dashed, "\alpha"'] & \cdots \arrow[l, "b"] \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

There are two squares or faces determining this 2-graph, F_1 defined by $b\beta = \alpha b$ and F_2 defined by $a\alpha = \beta a$. If we think of populating both faces as one-by-one planes with objects and morphisms, requiring agreement between F_1 and F_2 on shared edges α and β, and so on, we can create category $\mathcal{E}$.

To generalize and formalize the definition, we utilize the ideas and notation of [31], Section 2. However, we refer to squares as faces, since we will be building on these concepts in the next subsection.

Definition 6.4.5. Let (Λ, d) be a 2-graph determined uniquely by a two-color 1-skeleton and a collection of faces $S = \Lambda^{(1,1)}$ in which every bi-colored path of two edges appears exactly once. Let $\mathcal{E}$ be a category created as follows. For each face $F \in S$ defined by edges $e_1 f_1 = f_2 e_2$ with $e_i \in \Lambda^{(1,0)}$ called black and $f_i \in \Lambda^{(0,1)}$ called red, include in $\mathcal{E}$ the objects $\begin{bmatrix} p \\ q \end{bmatrix}_F$, for all $p, q \in [0,1] \cap \mathbb{Q}$, such that $\begin{bmatrix} 0 \\ 0 \end{bmatrix}_F = r(e_1 f_1)$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}_F = s(e_1 f_1)$. Include in $\mathcal{E}$ the generating morphisms between these objects of the form $\left(\begin{bmatrix} p \\ q \end{bmatrix}_F, \begin{bmatrix} p' \\ q \end{bmatrix}_F \right)$ with $p' \geq p$, or $\left(\begin{bmatrix} p \\ q \end{bmatrix}_F, \begin{bmatrix} p \\ q' \end{bmatrix}_F \right)$ with $q' \geq q$, under the identification that when a composition of morphisms has degree $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, i.e. lies entirely and only on a single edge, then the composition is equal to the edge. Enforce the requirement that when two faces share an edge, all objects and morphisms lying within that edge agree, such that no duplicate objects or morphisms are created in $\mathcal{E}$. Finally, considering generating morphisms changing in the first component as "black" and those changing in the second component as "red," enforce that a bi-colored path of two morphisms equals the reverse-colored path with the same respective degrees. That is, with $p \leq p'$ and $q \leq q'$,

$$\left(\begin{bmatrix} p \\ q \end{bmatrix}_F, \begin{bmatrix} p' \\ q \end{bmatrix}_F\right)\left(\begin{bmatrix} p' \\ q \end{bmatrix}_F, \begin{bmatrix} p' \\ q' \end{bmatrix}_F\right) = \left(\begin{bmatrix} p \\ q \end{bmatrix}_F, \begin{bmatrix} p \\ q' \end{bmatrix}_F\right)\left(\begin{bmatrix} p \\ q' \end{bmatrix}_F, \begin{bmatrix} p' \\ q' \end{bmatrix}_F\right).$$

Now define the degree functor $d : \mathcal{E} \to \mathbb{Q}^2_{\geq 0}$ (by slight abuse of notation) on the generating morphisms by subtraction,

$$d\left(\begin{bmatrix} p \\ q \end{bmatrix}_F, \begin{bmatrix} p' \\ q \end{bmatrix}_F\right) = \begin{pmatrix} p' - p \\ 0 \end{pmatrix} \quad \text{and} \quad d\left(\begin{bmatrix} p \\ q \end{bmatrix}_F, \begin{bmatrix} p \\ q' \end{bmatrix}_F\right) = \begin{pmatrix} 0 \\ q' - q \end{pmatrix},$$

which agrees with the degree map from Λ whenever p and q are integers. Now $(\mathcal{E}, d)$ is the rank-2 $\mathbb{Q}$-KPf particulated from the 2-graph Λ.

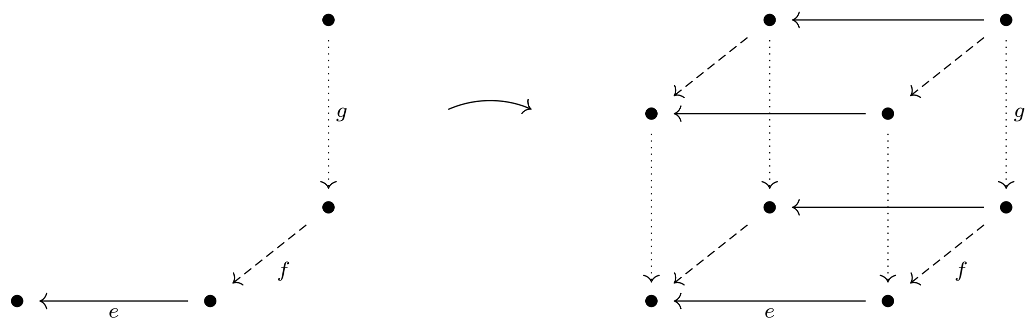
As in the one-dimensional case, $C^*(\Lambda)$ is isomorphic to a C*-subalgebra of $C^*(d)$, but this will be stated in the next subsection for general k.

Infinite paths in Λ^∞ determine infinite paths in $(\mathcal{E}, d)^\infty$ due to the restrictions on morphisms inside the squares or faces. That is, an $x \in \Lambda^\infty$ induces an $x' \in (\mathcal{E}, d)^\infty$ defined on pairs $(\vec{p}, \vec{q}) \in \mathbb{Q}^2_{\geq 0} \times \mathbb{Q}^2_{\geq 0}$ such that $x'(\vec{p}, \vec{q}) = x(\vec{p}, \vec{q})$ when $\vec{p}, \vec{q}$ happen to have integer components. Analogous facts to those in Remarks 6.3.3 and 6.3.4 hold with the appropriate notational adjustments and will be formalized in the next subsection for general k.

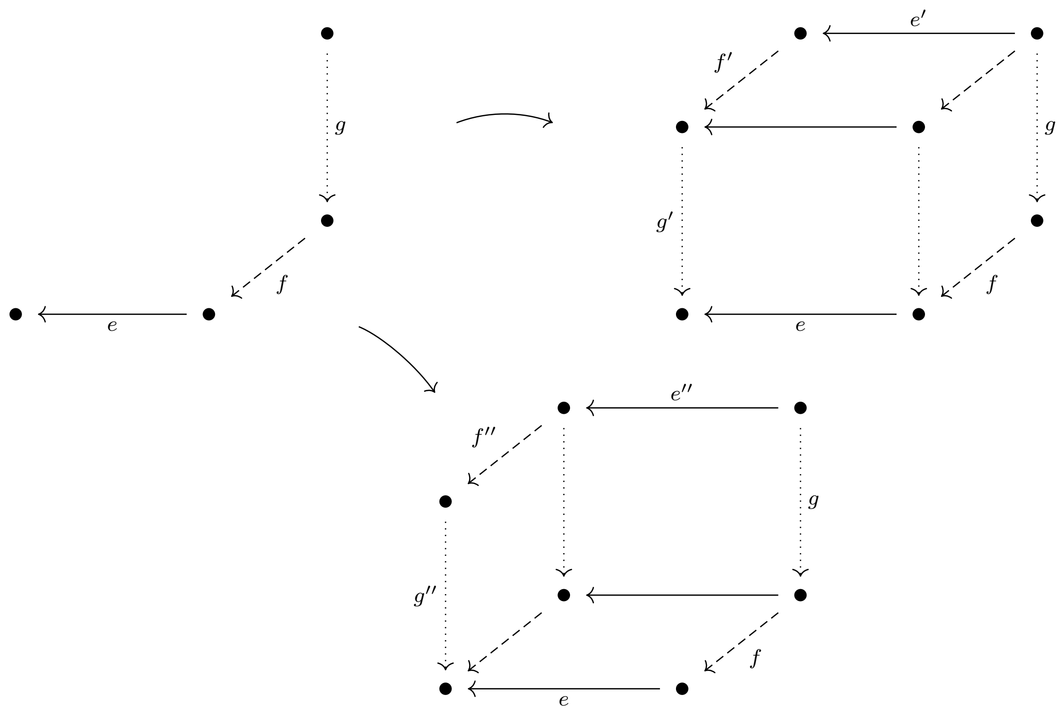
6.4.2 $\mathbb{Q}$-KPfs particulated from k-graphs with $k \geq 3$

As summarized in Section 2 of [31], and detailed in Remark 2.3 of [32], when $k \geq 3$, for a k-graph to arise from a 1-skeleton, it is necessary not only to have a suitable collection of squares as in the previous subsection, but also satisfying a certain condition on cubes in which the squares appear as faces (hence our choice of term).

That is, given a k-graph, any tri-colored path efg in the 1-skeleton should determine a cube representing a well-defined path of degree $(1, 1, 1)$.



Thus we need to know that the reverse-colored path $g'f'e'$ induced by extrapolating the three visible faces according to unique factorization matches the path $g''f''e''$ induced by extrapolating the three invisible faces.



A collection of faces S containing every bi-colored path of two edges exactly once and satisfying this relation on cubes determines a k-graph.

Thus the idea of a higher-rank $\mathbb{Q}$-KPf particulated from a k-graph will involve populating the faces of a cube with objects and morphisms as in the previous subsection, enforcing similar agreements, and populating the interior of the cube with points with rational coordinates, enforcing the same condition on resulting interior cubes. The coordinates of interior points can be given in a k-vector, which helps show that objects in a shared face or shared cube are the same, demonstrated in the cases $k = 4$ and $k = 5$ below.

When $k = 3$, it is intuitively easy to see that cubes determine the $\mathbb{Q}$-KPf. To make $\mathcal{E}$, for every cube C determined by a tri-colored path efg of three edges, include the objects

$$\left\{ \begin{bmatrix} p \\ q \\ r \end{bmatrix}_C : p, q, r \in [0,1] \cap \mathbb{Q} \right\},$$

considering $r(efg) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_C$ as the origin corner of the cube, and $s(efg) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_C$. Thus, for example, objects contained in the face determined by fg with face-wise coordinates (p, q) will be of the form $\begin{bmatrix} 1 \\ p \\ q \end{bmatrix}_C$. Include generating morphisms of the forms

$$\left(\begin{bmatrix} p \\ q \\ r \end{bmatrix}_C, \begin{bmatrix} p' \\ q \\ r \end{bmatrix}_C \right) \text{ with } p \leq p', \quad \left(\begin{bmatrix} p \\ q \\ r \end{bmatrix}_C, \begin{bmatrix} p \\ q' \\ r \end{bmatrix}_C \right) \text{ with } q \leq q',$$

$$\text{and } \left(\begin{bmatrix} p \\ q \\ r \end{bmatrix}_C, \begin{bmatrix} p \\ q \\ r' \end{bmatrix}_C \right) \text{ with } r \leq r'.$$

As before, require that when a composition of arrows λ lies entirely on an edge e and $s(\lambda) = s(e)$, $r(\lambda) = r(e)$, and $d(\lambda) = 1$, then $\lambda = e$. Considering morphisms of the first type above as black, the second type as red, and the third type as blue, we require that cubes arising from a tri-colored path internal to C satisfy the same condition on reverse coloring. When two cubes C_1 and C_2 share a face, require that corresponding objects in the shared face are equal such that no duplicates arise. As before, define the degree map

$d : \mathcal{E} \to \mathbb{Q}^3_{\geq 0}$ on generating morphisms by subtraction and extend to compositions. Then $(\mathcal{E}, d)$ is a rank-three $\mathbb{Q}$-KPf.

When $k = 4$: In a 4-colored path $efgh$ with one edge of each color, factorization into cubes determined by efg and fgh results in a shared face, that which is determined by fg. We would like the coordinate vectors of a point (p, q) on the shared face to be consistent between these two cubes. The 4-vector representing $\begin{bmatrix} p \\ q \end{bmatrix}_{F_{fg}}$ is as follows: if $f \in \Lambda^{e_i}$ for the particular standard basis vector e_i, the i-th coordinate will be p, and if $g \in \Lambda^{e_j}$, the j-th coordinate of the vector will be q. For $e \in \Lambda^{e_k}$, the k-th coordinate of the vector representing point will be 1, and for $h \in \Lambda^{e_\ell}$, the ℓ-th coordinate of the vector will be 0. The figure below demonstrates this in the case that $e \in \Lambda^{e_1}$, $f \in \Lambda^{e_2}$, $g \in \Lambda^{e_3}$, and $h \in \Lambda^{e_4}$.

C_{efg}

$$\begin{tikzcd}
 & & \bullet \arrow[dl] \arrow[dd, "g"] \\
 & \bullet \arrow[dd] & \\
 & & \bullet \arrow[dl, "f"] \\
\bullet & \bullet \arrow[l, "e"] &
\end{tikzcd}$$

C_{fgh}

$$\begin{tikzcd}
 & & & \bullet \arrow[d, "h"] \\
 & \bullet \arrow[dl] & & \bullet \arrow[ll] \arrow[dl, "g"] \\
\bullet & & \bullet \arrow[ll, "f"] &
\end{tikzcd}$$

$$\begin{bmatrix} p \\ q \end{bmatrix}_{F_{fg}} = \begin{bmatrix} 1 \\ p \\ q \end{bmatrix}_{C_{efg}} \to \begin{bmatrix} 1 \\ p \\ q \\ 0 \end{bmatrix}_{efgh} \leftarrow \begin{bmatrix} p \\ q \\ 0 \end{bmatrix}_{C_{fgh}} = \begin{bmatrix} p \\ q \end{bmatrix}_{F_{fg}}$$

The coordinates of interior points of the cubes can also be written in 4-vectors for similar clarity.

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix}_{C_{efg}} \to \begin{bmatrix} p \\ q \\ r \\ 0 \end{bmatrix}_{efgh} \quad \text{and} \quad \begin{bmatrix} p \\ q \\ r \end{bmatrix}_{C_{fgh}} \to \begin{bmatrix} 1 \\ p \\ q \\ r \end{bmatrix}_{efgh}$$

When $k = 5$: In a 5-colored path $defgh$ with one edge of each color, we want consistency between objects incuded in $\mathcal{E}$ considered on cubes def, efg, and fgh. Suppose for example that $d \in \Lambda^{e_1}, e \in \Lambda^{e_2}, f \in \Lambda^{e_3}, g \in \Lambda^{e_4}, h \in \Lambda^{e_5}$. The same strategy encodes the coordinates of objects within cubes into 5-vectors for clarity. For example, consider an object $\begin{bmatrix} p \\ q \end{bmatrix}_{F_{ef}}$ on the shared face between C_{def} and C_{efg}:

$$\begin{bmatrix} p \\ q \end{bmatrix}_{F_{ef}} = \begin{bmatrix} 1 \\ p \\ q \end{bmatrix}_{C_{def}} = \begin{bmatrix} p \\ q \\ 0 \end{bmatrix}_{C_{efg}} = \begin{bmatrix} 1 \\ p \\ q \\ 0 \end{bmatrix}_{defg} = \begin{bmatrix} 1 \\ p \\ q \\ 0 \\ 0 \end{bmatrix}_{defgh}.$$

An interior point in the shared cube efg between $defg$ and $efgh$ would be represented by

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix}_{C_{efg}} = \begin{bmatrix} 1 \\ p \\ q \\ r \end{bmatrix}_{defg} = \begin{bmatrix} p \\ q \\ r \\ 0 \end{bmatrix}_{efgh} = \begin{bmatrix} 1 \\ p \\ q \\ r \\ 0 \end{bmatrix}_{defgh}$$

Definition 6.4.6. Let (Λ, d) be a k-graph with $k \geq 3$. Then (Λ, d) is determined by a collection of faces S in which every bi-colored path of two edges appears exactly once, such that the faces satisfy the condition detailed above on cubes determined by tri-colored paths of three edges. Create the category $\mathcal{E}$ as follows. For each tri-colored path $mn\ell$ with $m \in \Lambda^{e_{i_m}}$, $n \in \Lambda^{e_{i_n}}$, $\ell \in \Lambda^{e_{i_\ell}}$ where $\{e_i\}_{i=1}^k$ are standard basis vectors for $\mathbb{N}^k$, a cube is fully determined, denoted $C_{mn\ell}$. Include as objects in $\mathcal{E}$ the set of points with rational coordinates in $C_{mn\ell}$,

$$\left\{ \vec{p} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}_{C_{mn\ell}} : p, q, r \in [0,1] \cap \mathbb{Q} \right\},$$

requiring the agreement of all objects that appear on shared faces between cubes and shared cubes between longer paths in Λ. (In context, this can be clarified by encoding coordinates into k-vectors.) Include in $\mathcal{E}$ generating morphisms between these objects, such that coordinates change in only one dimension; i.e.

$$\left(\begin{bmatrix} p \\ q \\ r \end{bmatrix}_{C_{mn\ell}}, \begin{bmatrix} p' \\ q \\ r \end{bmatrix}_{C_{mn\ell}}\right) \text{ with } p \leq p', \quad \left(\begin{bmatrix} p \\ q \\ r \end{bmatrix}_{C_{mn\ell}}, \begin{bmatrix} p \\ q' \\ r \end{bmatrix}_{C_{mn\ell}}\right) \text{ with } q \leq q',$$

$$\text{and } \left(\begin{bmatrix} p \\ q \\ r \end{bmatrix}_{C_{mn\ell}}, \begin{bmatrix} p \\ q \\ r' \end{bmatrix}_{C_{mn\ell}}\right) \text{ with } r \leq r'.$$

These morphisms can be considered as matching the color of the relevant edge; e.g. if the changing coordinate is in the position corresponding to $m \in \Lambda^{e_{i_m}}$, the morphism is the color associated to edges in $\Lambda^{e_{i_m}}$. Generating morphisms contained in shared edges between cubes and shared cubes between longer paths are also required to agree. Inside cubes, tri-colored paths determining rectangular prisms must satisfy the same reverse-coloring condition that the cube does. The degree map $d : \mathcal{E} \to \mathbb{Q}^k_{\geq 0}$ is defined on generating morphisms by subtraction as usual, and extended to compositions of morphisms by addition as usual. Then $(\mathcal{E}, d)$ is a rank-k $\mathbb{Q}$-KPf particulated from the k-graph Λ.

Proposition 6.4.1. *Let $\{s_X, s_\lambda : X \in \mathrm{Obj}(\mathcal{E}), \lambda \in \mathcal{E}\}$ be a Cuntz-Krieger system for a rank-k $\mathbb{Q}$-KPf $(\mathcal{E}, d)$ particulated from a k-graph Λ. If $B = \Lambda \cap \mathcal{E}$, then $A := \{s_v, s_f : v \in \mathrm{Obj}(B), f \in B\}$ is a Cuntz-Krieger family for Λ in $C^*(d)$ generating a C^*-subalgebra isomorphic to the universal C^*-algebra $C^*(\Lambda)$ for Λ.*

Proof. We check that the subset A satisfies the conditions for being a Cuntz-Krieger family for Λ.

1. $\{s_v : v \in (\Lambda \cap \mathcal{E})^0\} = \{s_v : v \in \Lambda^0\}$ is a family of mutually orthogonal projections since $\{s_X : X \in \mathrm{Obj}(\mathcal{E})\}$ is;

2. $s_{\lambda\mu} = s_\lambda s_\mu$ for all $\lambda, \mu \in \Lambda$ s.t. $s(\lambda) = r(\mu)$ still holds;

3. $s^*_\lambda s_\lambda = s_{s(\lambda)}$ for all $\lambda \in \Lambda$ follows from the definition of the Cuntz-Krieger system for $(\mathcal{E}, d)$;

4. for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$ we have $s_v = \sum_{\lambda \in \Lambda^n(v)} s_\lambda s_\lambda^*$, since $\{\lambda \in \Lambda^n(v)\}$ is exactly $\{\lambda \in v\mathcal{E} : d(\lambda) = n\}$ since the degree map d for $\mathcal{E}$ coincides with the degree map for Λ whenever $\vec{p} \in \mathbb{Q}^k_{\geq 0}$ has integer entries.

□

Denote $\mathbb{Q}^k_{[0,1)} := [0,1)^k \cap \mathbb{Q}^k$ for convenience. Also, $(\mathbb{Q}^k_{[0,1)})^2$ is used to denote the groupoid $\mathbb{Q}^k_{[0,1)} \times \mathbb{Q}^k_{[0,1)}$ in which elements are composable if they are of the form $(p_1, p_2)(p_2, p_3) = (p_1, p_3)$ with $p_1, p_2, p_3 \in \mathbb{Q}^k_{[0,1)}$.

Theorem 6.4.7. *Let Λ be a k-graph and let $(\mathcal{E}, d)$ be the $\mathbb{Q}$-KPf particulated from Λ. The infinite path space Λ^∞ appears as a subset of $(\mathcal{E}, d)^\infty$, and the map*

$$\Lambda^\infty \times \mathbb{Q}^k_{[0,1)} \ni (x, \vec{p}) \mapsto \sigma^{\vec{p}}(x) \in (\mathcal{E}, d)^\infty$$

provides an bijection. Moreover, when $x \in \Lambda^\infty$ is aperiodic, so are $\sigma^{\vec{p}}(x)$ and λx in $(\mathcal{E}, d)^\infty$, where $\lambda \in \mathcal{E}$ with $s(\lambda) = r(x)$.

Similarly, the path groupoid G_Λ appears as a subgroupoid of G_d, and the map

$$G_\Lambda \times (\mathbb{Q}^k_{[0,1)})^2 \ni ((x, \vec{n}, y), (\vec{r_1}, \vec{r_2})) \mapsto (\sigma^{\vec{r_1}} x, \vec{n} - \vec{r_1} + \vec{r_2}, \sigma^{\vec{r_2}} y) \in G_d$$

provides an isomorphism.

Proof. The first map in question is well-defined. It is injective: if $\sigma^{\vec{p}}(x) = \sigma^{\vec{q}}(y)$ in $(\mathcal{E}, d)^\infty$, then since both $r(x), r(y) \in \Lambda^0$ and $\vec{0} \leq \vec{p}, \vec{q} < \vec{1}$, we must have had $x = y$ and $\vec{p} = \vec{q}$. It is surjective: if $z \in (\mathcal{E}, d)^\infty$ with $r(z) \in \Lambda^0$, then the preimage is $(z, \vec{0})$. Otherwise, if $r(z) = \vec{r}_C$ is an object in some cube C, then if $y = \mathrm{ind}_{(\vec{0}_C, \vec{r}_C)}(z)$ where $\vec{0}_C = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_C$, there exist $\vec{r'} \in \mathbb{Q}^k_{\geq 0}$ such that $\sigma^{\vec{r'}}(y) = \mathrm{res}_{(\vec{0}_C, \vec{r}_C)} \circ \mathrm{ind}_{(\vec{0}_C, \vec{r}_C)}(z) = z$. (This $\vec{r'}$ can be found by embedding the coordinates from $\vec{r}$ into a k-tuple using the method described in the examples for $k = 4$ and

$k = 5$. Namely, the coordinates for the other colors will be zeroes since we are observing the final tri-colored cube in the infinite path.) Then the preimage is $(y, \vec{r'})$.

Suppose that $x \in \Lambda^\infty$ is an aperiodic path. Seen as an infinite path determined in $(\mathcal{E}, d)^\infty$, it is still aperiodic. Indeed, if $\sigma^{\vec{q_1}}(x) = \sigma^{\vec{q_2}}(x)$ in $(\mathcal{E}, d)^\infty$, then they have the same range, and by stripping off an ending of x for some $\vec{r} \in [0, q)^k \cap \mathbb{Q}^k$ such that $\vec{q_1} + \vec{r} =: m$ and $\vec{q_2} + \vec{r} =: n$ are in $\mathbb{N}^k$, we have that $\sigma^n(x) = \sigma^m(x)$. Since x was aperiodic in Λ^∞, $n = m$ and hence $\vec{q_1} = \vec{q_2}$. Then by the general case (see Proposition 3.1.1), $x \in (\mathcal{E}, d)^\infty$ aperiodic implies $\sigma^{\vec{p}}(x)$ aperiodic since our choice of $\mathcal{B}$ is left-cancellative. On the other hand, consider $\lambda x \in (\mathcal{E}, d)^\infty$ where $\lambda \in \mathcal{E}$. There exist $y \in \Lambda^\infty$ and $\vec{r} \in \mathbb{Q}^k_{[0,1)}$ such that $(y, \vec{r})$ is the corresponding pair to λx under the isomorphism. Since $d(\lambda) + \vec{r}$ is in $\mathbb{N}^k$, there exists $\mu \in \Lambda$ such that $y = \mu x$ and in k-graphs, x aperiodic implies $y = \mu x$ aperiodic. Then apply the same reasoning to say that $\lambda x = \sigma^{\vec{r}}(y)$ is aperiodic.

The second map in question is also well-defined. It is injective: if

$$(\sigma^{\vec{r_1}} x, \vec{n} - \vec{r_1} + \vec{r_2}, \sigma^{\vec{r_2}} y) = (\sigma^{\vec{r_1}'} x', \vec{n}' - \vec{r_1}' + \vec{r_2}', \sigma^{\vec{r_2}'} y'),$$

then we must have $\sigma^{\vec{r_2}} y = \sigma^{\vec{r_2}'} y'$ so that the sources agree. Then $\vec{r_2}' = \vec{r_2}$ and $y = y'$ by the same reasoning as above. The ranges must also agree, so $\sigma^{\vec{r_1}} x = \sigma^{\vec{r_1}'} x'$ and $\vec{r_1} = \vec{r_1}'$, $x = x'$. Finally,

$$\vec{n} - \vec{r_1} + \vec{r_2} = \vec{n}' - \vec{r_1}' + \vec{r_2}' \quad \Longrightarrow \quad \vec{n} = \vec{n}'.$$

It is surjective: if $(x, \vec{p}, y) \in G_d$, then there exist $\vec{q}, \vec{q'} \in \mathbb{Q}^k_{\geq 0}$ such that $\sigma^{\vec{q}} x = \sigma^{\vec{q'}} y$ and $\vec{p} = \vec{q} - \vec{q'}$, which we can choose such that $s(\vec{q}) = s(\vec{q'}) \in \Lambda^0$. If it so happens that $r(x), r(y) \in \Lambda^0$, then $\vec{q}, \vec{q'} \in \mathbb{N}^k$ already, and the preimage is simply $((x, \vec{p}, y), (\vec{0}, \vec{0}))$. If not, say $r(x) \notin \Lambda^0$, then $r(x)$ is an object $\vec{r}_C$ for some cube C. Embed the coordinates of $\vec{r}_C$ as usual into a k-vector $\vec{r} \in \mathbb{Q}^k_{[0,1)]}$ by setting the coordinates for unused colors to be 0, which is standard since we are dealing with the ending cube of an infinite path. Then define $x' := \mathrm{ind}_{(\vec{0}_C, \vec{r}_C}(x)$, which is in Λ^∞ and satisfies $\sigma^{\vec{r}} x' = x$. Do likewise if $r(y) \notin \Lambda^0$, so that we have $\vec{r_1}, \vec{r_2} \in \mathbb{Q}^k_{[0,1)}$ and $x', y' \in \Lambda^\infty$ with $\sigma^{\vec{r_1}} x' = x$ and $\sigma^{\vec{r_2}} y' = y$. Note that $\vec{q} + \vec{r}_1, \vec{q'} + \vec{r}_2 \in \mathbb{N}^k$, and that

$$\sigma^{\vec{q}} x = \sigma^{\vec{q}+\vec{r_1}} x' = \sigma^{\vec{q'}} y = \sigma^{\vec{q'}+\vec{r_2}} y'.$$

Then $(x', \vec{q} + \vec{r_1} - \vec{q'} - \vec{r_2}, y') \in G_\Lambda$. The preimage desired for the map is

$$((x', \vec{p} + \vec{r_1} - \vec{r_2}, y'), (\vec{r_1}, \vec{r_2})),$$

since its image is

$$(\sigma^{\vec{r_1}} x', \vec{p} + \vec{r_1} - \vec{r_2} - \vec{r_1} + \vec{r_2}, \sigma^{r_2} y') = (x, \vec{p}, y).$$

It is a homomorphism, since elements in $G_\Lambda \times (\mathbb{Q}^k_{[0,1)})^2$ are composable if they are of the form

$$((z, \vec{n_2}, y), (\vec{r_3}, \vec{r_2}))((y, \vec{n_1}, x), (\vec{r_2}, \vec{r_1})) = ((z, \vec{n_1} + \vec{n_2}, x), (\vec{r_3}, \vec{r_1})).$$

Under the map, we do indeed have

$$(\sigma^{\vec{r_3}} z, \vec{n_2} + \vec{r_3} - \vec{r_2}, \sigma^{\vec{r_2}} y)(\sigma^{\vec{r_2}} y, \vec{n_1} + \vec{r_2} - \vec{r_1}, \sigma^{\vec{r_1}} x) = (\sigma^{\vec{r_3}} z, \vec{n_1} + \vec{n_2} + \vec{r_3} - \vec{r_1}, \sigma^{\vec{r_1}} x).$$

□

Proposition 6.4.2. *Let (Λ, d_Λ) be a k-graph and let $(\mathcal{E}, d)$ be the $\mathbb{Q}$-KPf particulated from Λ. If Λ is row-finite, then so is $d : \mathcal{E} \to \mathbb{Q}^k_{\geq 0}$. If Λ has no sources, then $d : \mathcal{E} \to \mathbb{Q}^k_{\geq 0}$ is strongly surjective. If Λ satisfies the aperiodicity condition, then $d : \mathcal{E} \to \mathbb{Q}^k_{\geq 0}$ is aperiodic.*

Proof. For the first statement, assume that Λ is row-finite, so for each $m \in \mathbb{N}^k$ and $v \in \Lambda^0$ the set $\Lambda^m(v) := \{\mu \in \Lambda^m : r(\mu) = v\}$ is finite. Let $\vec{p}_{C_0} \in \mathrm{Obj}(\mathcal{E})$ in some ending cube C_0, and encode the coordinates in a k-tuple $\vec{p}$ such that if no edge of degree e_i appears in cube C_0, the i-th coordinate is zero. Let $\vec{q} \in \mathbb{Q}^k_{\geq 0}$ and consider the set of morphisms λ in $\mathcal{E}$ targeting $\vec{p}_{C_0}$ with $d(\lambda) = \vec{q}$. If any such λ exists, then it may be decomposed into $\lambda = \lambda_1\lambda_2\lambda_3$ such that $\lambda_1 = (\vec{p}_{C_0}, \vec{1}_{C_0}$, the range and source of λ_2 are in Λ^0 so that $d(\lambda_2) \in \mathbb{N}^k$, and if we denote $r(\lambda) = \vec{p}'_{C_f}$ lying in a cube C_f, then $\lambda_3 = (\vec{0}_{C_0}, \vec{p}'_{C_f})$. Note that since endings are fully determined within cubes, all such $\lambda \in \mathcal{E}$ will share the same λ_1. Now taking $m = d(\lambda_2)$, we apply the row-finite condition of Λ to observe that there are finitely many options for λ_2. Finally, observe that each choice of λ_2 has source object in Λ^0, at which object we apply the row-finiteness of Λ to say there are finitely many options for λ_3. Thus the set of morphisms $\lambda \in \mathcal{E}$ targeting $\vec{p}_{C_0}$ with $d(\lambda) = \vec{q}$ is a finite set.

For the second statement, assume Λ has no sources. Consider an object $\vec{p}_C \in \mathrm{Obj}(\mathcal{E})$ and consider some $\vec{q} \in \mathbb{Q}^k_{\geq 0}$. We wish to show that there exists some morphism $\lambda \in \mathcal{E}$ targeting $\vec{p}_C$ such that $d(\lambda) = \vec{q}$. By considering the morphism $(\vec{0}_C, \vec{p}_C)$, whose range $\vec{0}_C$ is a joint vertex from the underlying k-graph, we know there exists a path in Λ targeting $\vec{0}_C$ with degree $\lceil \vec{q} \rceil$, implying the existence of cubes defined by three-colored paths of its edges. Since each of those cubes is filled with morphisms of all possible rational lengths, we can find λ consisting of generating morphisms from each cube such that $d(\lambda) = \vec{q}$ which targets the range object $\vec{p}_C$.

For the third statement, assume that Λ satisfies the aperiodicity condition and let $\vec{p}_C$ be any object in $\mathcal{E}$. Then since $\vec{1}_C$ is a joint object from Λ, there exists an aperiodic path $x \in \Lambda^\infty$ with range $\vec{1}_C$, which determines an infinite path also denoted $x \in (\mathcal{E}, d)^\infty$. Then applying Theorem 6.4.7, $\mathrm{ind}_{(\vec{p}_C, \vec{1}_C)}(x) = (\vec{p}_C, \vec{1}_C)x$ is an aperiodic infinite path in $(\mathcal{E}, d)^\infty$ with range $\vec{p}_C$ as desired.

□

Proposition 6.4.3. *Let Λ be a k-graph and let $(\mathcal{E}, d)$ be the $\mathbb{Q}$-KPf particulated from Λ. Λ is cofinal in the sense of [1] if and only if $(\mathcal{E}, d)$ is cofinal in the sense of Definition 3.3.2. In addition, G_Λ is a transitive groupoid if and only if G_d is.*

Proof. Since Λ^∞ appears as a subset of $(\mathcal{E}, d)^\infty$, it is cofinal when $(\mathcal{E}, d)$ is according to the definition. Conversely, observe that any infinite path $y \in (\mathcal{E}, d)^\infty$ may be written as $\sigma^{\vec{r}}(x)$ for some $\vec{r} \in \mathbb{Q}^k_{[0,1)}$ and some $x \in \Lambda^\infty$. Let $p \in \mathrm{Obj}(\mathcal{E})$. Then there exists $\mu \in \mathcal{E}$ such that

$r(\mu) = p$ and $s(\mu) = v$ for some $v \in \Lambda^0$. Since Λ is cofinal, there exists $\lambda \in \Lambda$ such that $s(\lambda) = x(n)$ for some $n \in \mathbb{N}^k$ and $r(\lambda) = v$. Since $y(n) = x(n)$, we have in $\mathcal{E}$ the morphism $\mu\lambda$ which has $r(\mu\lambda) = p$ and $s(\mu\lambda) = y(n)$.

For the second statement, observe again that since G_Λ appears as a subset of G_d, it will be transitive if G_d is. Conversely, we wish to show that if G_Λ is transitive, then G_d has only one orbit. Let $\sigma^{\vec{r_1}}(x) \in (\mathcal{E}, d)^\infty$, where $\vec{r_1} \in \mathbb{Q}^k_{[0,1)}$ and $x \in \Lambda^\infty$. Let $\sigma^{\vec{r_2}}(y)$ be another infinite path in $(\mathcal{E}, d)^\infty$ where $\vec{r_2} \in \mathbb{Q}^k_{[0,1)}$. Since G_Λ is transitive, there exists $(y, n, x) \in G_\Lambda$ for some $n \in \mathbb{N}^k$. Then $(\sigma^{\vec{r_2}}(y), n + \vec{r_1} - \vec{r_2}, \sigma^{\vec{r_1}}(x))$ is in G_d and G_d is also transitive. $\square$

As a consequence of the previous proposition, note that if G_Λ is transitive and thus G_d is, we can apply Proposition 2.75 of [33] to say that $C^*(G_d) \cong C^*_r(G_d)$ since any isotropy group of G_d is abelian and hence amenable. Then if Λ is cofinal and thus $(\mathcal{E}, d)$ is, we can apply Proposition 3.3.1 to state that $C^*(G_d)$ is simple.

6.5 A Finitely Aligned Example

In this section, we consider a more interesting example, but one which fails to be row-finite.

Example 6.5.1. Let $\mathcal{C}$ be a category whose objects are given by $\mathbb{Q}_{\geq 0}$ and whose morphisms are generated by arrows between any two rational numbers r_1 and r_2. Denote by (r_2, r_1) the arrow with source r_1 and range r_2. Thus if $r_1 < r_2$, (r_2, r_1) is an arrow pointing to the right on a number line, while (r_1, r_2) is the arrow between them pointing to the left. When $r_1 = r_2$, identify (r_1, r_1) with the identity on the object r_1. Compositions are defined as follows: for $(r_1, r_2), (r_2, r_3) \in \mathcal{E}$, when $r_1 \leq r_2 \leq r_3$ or $r_1 \geq r_2 \geq r_3$, we have

$$(r_3, r_2)(r_2, r_1) = (r_3, r_1),$$

but if r_2 does not fall between r_1 and r_3, the composition $(r_3, r_2)(r_2, r_1)$ is defined but not collapsible. Thus "paths" may be thought of as zigzags between rational numbers on a number line, and may be written in a fully reduced zigzag form consisting of alternating right and left arrows by collapsing any compositions going in the same direction.

Let the KPf or degree functor $d : \mathcal{C} \to \mathcal{B}$ be defined by mapping all objects to the single object of $\mathcal{B}$, and for a fully-reduced zigzag path $\lambda = (r_k, r_{k-1})(r_{k-1}, r_{k-2}) \cdots (r_2, r_1)(r_1, r_0)$,

$$d(\lambda) = \sum_{i=1}^{k} |r_i - r_{i-1}|,$$

the sum of the unsigned lengths of the line segments. We wish to show that this functor is a locally split dCf.

To show unique lifting, suppose $d(\lambda) = q_1 + q_2$ for $q_1, q_2 \in \mathbb{Q}_{\geq 0}$. There are two cases to consider. If $q_1 = \sum_{i=1}^{N} |r_i - r_{i-1}|$ for some integer $1 \leq N \leq k$, then take $\nu = (r_N, r_{N-1})(r_{N-1}, r_{N-2}) \cdots (r_1, r_0)$ and $\mu = (r_k, r_{k-1}) \cdots (r_{N+1}, r_N)$. Otherwise, there exists an integer M such that q_1 falls between the two partial sums

$$\sum_{i=1}^{M} |r_i - r_{i-1}| < q_1 < \sum_{i=1}^{M+1} |r_i - r_{i-1}|.$$

Denote such partial sums by S_M and S_{M+1}. Let $\varepsilon = q_1 - S_M$ and factor $(r_{M+1}, r_M) = (r_{M+1}, r_M + \varepsilon)(r_M + \varepsilon, r_M)$ since $r_M < r_M + \varepsilon < r_{M+1}$. Then take $\nu = (r_M + \varepsilon, r_M) \cdots (r_1, r_0)$ and $\mu = (r_k, r_{k-1}) \cdots (r_{M+1}, r_M + \varepsilon)$.

To see that d is locally split under the KPf treatment, let $r \in \mathrm{Obj}(\mathcal{C})$ and consider the slice category $\mathcal{C}/r$ whose objects are zigzag paths λ with range r. Morphisms in the slice category arise from commutative triangles $\lambda' = \lambda\mu$ in $\mathcal{C}$, and the morphism in $\mathcal{C}/r$ with range λ and source λ' is denoted by the pair (λ, μ). The slice category $\mathcal{B}/\star$ where $\star$ is the single object of $\mathcal{B}$, like $\mathbb{N}^k$ for the k-graph case, the objects are elements of $\mathbb{Q}$, while morphisms are ordered pairs (p, q), with range p and source $p + q$. The induced functor $d : \mathcal{C}/r \to \mathcal{B}/\star$ admits a splitting $x : \mathcal{B}/\star \to \mathcal{C}/r$ since for any $(p, q) \in \mathcal{B}/\star$, $x(p, q) = (x_1(p, q), x_2(p, q))$ with $x_1(p, q) = (r, r + p)$ and $x_2(p, q) = (r + p, r + p + q)$ satisfy

$d(x(p,q)) = (|r - r - p|, |r + p - r - p - q|) = (p,q)$ as desired.

Thus $(\mathcal{C}, d)$ is a $\mathbb{Q}$-KPf.

Observe that this d is not row-finite, since infinitely many distinct zigzag paths of a certain length may target an object in $\mathcal{C}$ (simply add in more turns). However, the C*-algebras of finitely aligned k-graphs have been studied ([34], [35], [36]), and we can show that this example does provide a finitely-aligned $\mathbb{Q}$-KPf. To that end, we establish definitions and notation.

Let $d : \mathcal{E} \to \mathcal{B}$ be a KPf targeting $\mathcal{B} = (\mathbb{Q}_{\geq 0}, +)$, so that $(\mathcal{E}, d)$ is a $\mathbb{Q}$-KPf. Let $r \in \mathcal{B}$ and define

$$\mathcal{E}^r := \{\lambda \in \mathcal{E} : d(\lambda) = r\}, \text{ and } \mathcal{E}^{\leq r} := \{\lambda \in \mathcal{E} : d(\lambda) \leq r\}.$$

For $E \subset \mathcal{E}$ and $\lambda \in \mathcal{E}$, define notation

$$\lambda E := \{\lambda\mu : \mu \in E, r(\mu) = s(\lambda)\}, \text{ and } E\lambda := \{\mu\lambda : \mu \in E, s(\mu) = r(\lambda)\},$$

which extends to vE and Ev for $v \in \mathrm{Obj}(\mathcal{E})$ in the obvious way.

Definition 6.5.2. For $\lambda, \mu \in \mathcal{E}$, we define the collection of *minimal common extensions* of λ and μ as

$$\mathcal{E}^{\min}(\lambda, \mu) := \{(\alpha, \beta) : \lambda\alpha = \mu\beta, d(\lambda\alpha) = \max(d(\lambda), d(\mu))\}.$$

A $\mathbb{Q}$-KPf $(\mathcal{E}, d)$ is called *finitely aligned* if $\mathcal{E}^{\min}(\lambda, \mu)$ is finite for all $\lambda, \mu \in \mathcal{E}$.

In Example 6.5.1, given two paths $\lambda, \mu \in \mathcal{C}$ with $r(\lambda) = r(\mu)$, there is an extension $\lambda\alpha = \mu\beta$ only if $\lambda = \mu\beta$ or $\mu = \lambda\alpha$ for some α or β in $\mathcal{C}$. And indeed, when that is the case, that extension is the (only) minimal one, so the set $\mathcal{C}^{\min}(\lambda, \mu)$ is either empty or a singleton. Hence this $d : \mathcal{C} \to \mathcal{B}$ is finitely aligned.

In future work, we would like to develop the theory of dCfs and KPfs in the absence of the row-finite condition, loosening to a finitely-aligned condition as defined above.

Further Study

As mentioned in the previous section, the C*-algebras of k-graphs have been analyzed in the absence of the row-finite condition. The Cuntz-Krieger conditions were reworked by a non-trivial process detailed in Appendix A of [34] so as to achieve the desired properties of the universal C*-algebra and its core. An adjustment to the definition of the Cuntz-Krieger system for a KPf F was suggested by [2], but as it does not result in similarly desirable properties for $C^*(F)$, we did not continue an exploration in this direction.

The question of amenability for a KPf's path groupoid is obviously still open. We believe that for particulated $\mathbb{Q}$-KPfs especially, it may follow from the amenability of k-graph path groupoids, despite the lack of a gauge action argument. Further investigation into non-particulated $\mathbb{Q}$-KPfs and more general KPfs is needed, but it seems that progress in these areas would require quite a few assumptions of nicer properties on category $\mathcal{B}$.

Most interestingly to us, the topic of K-theory appears to be accessible for the C*-algebras of KPfs with convenient qualities. Since K-theory is of interest and well-explored for k-graphs, we anticipate that analyzing the K_0 and K_1 groups for $\mathbb{N}$-graphs would be an excellent next project.

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