ESTIMATION OF AN UPPER BOUND FOR EXPECTED MAINTENANCE COST OF A SYSTEM WITH PARTIALLY KNOWN, INCREASING FAILURE RATE DISTRIBUTION

by

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CHAPTER 1

INTRODUCTION

For nearly two decades, there has been a large and continuing interest in the study of maintenance models for items with stochastic failure. This interest has its roots in many military and industrial applications. Lately, however, new applications have arisen in such areas as health, ecology, and the environment. Although it is not possible to detail these many applications of maintenance models, some of them are: the maintenance of complex electronic and/or mechanical equipment, maintenance of human body, inspection and control of pollutants in the environment, and maintenance of ecological balance in populations of plants and animals. This interest has been provoked by the high cost and extraordinary demands made of modern equipment like jet liners, electronic computers, ballistic missiles, etc. Operational requirements can be achieved only by observing relatively sophisticated maintenance policies.

The practical need for subtle and delicate maintenance policies has stimulated theoretical interest and in many cases has led to the development of policies that posses theoretical novelty and practical importance. The maintenance models can be basically divided into two types i.e, the preventive and preparedness maintenance policies. The distinctive feature of preparedness maintenance is that the state of the system (e.g. operative or failed) is at all times known during its service as opposed to the preparedness maintenance for which the state of the system is only known through inspection or checking. The preventive maintenance policies are justified when the cost of unscheduled maintenance action is higher than the cost of scheduled maintenance. The name of preparedness has been given to its policy since in some cases there are standby emergency systems which are put into operation when the main system suddenly fails. These standby systems should be checked for their preparedness; but not all of the applications of preparedness maintenance modelling lies within emergency systems. There has been already a lot of work done in both preventive maintenance [1,2,4,5,10,12,19,23,28,31] and preparedness maintenance [1,3,8,11,14,18, 23,28,29,32] areas. The basic criterion is to minimize the total cost of maintenance by optimum scheduling of the maintenance actions such as repair, inspection or replacement.

The optimization objective for preparedness inspection models is to optimally balance the cost of undetected system failure over the inspection cost.

Assuming:

$f(t)$ = The density function of the time to failure of the equipment;

I = The cost of an inspection except at time t_0 where $I_0=0$;

a = The cost per unit time associated with an undetected failed system;

and that the inspections to be performed at times t_1, t_2, t_3, ..., until the system failure is detected (See Fig 1.), then, if failure occurs between time t_0 and t_1, say at X_1, the cost of the cycle of operation (See Fig. 1.) would be:

$$I(1) + a(t_1 - x_1) \tag{1.1}$$

and the expected value of this cost in the interval $[t_0, t_1]$ is:

$$\int_{t_0}^{t_1} [I(1) + a(t_1 - x)]\, f(x)dx \,. \tag{1.2}$$

If failure occurs between t_1 and t_2, say at time x_2, the cost of the

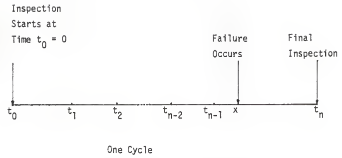

Fig. 1. The Illustration of One Time Span or One Cycle.

cycle in the interval $[t_1, t_2]$ would be:

$$I(1+1) + a(t_2 - x_2) \tag{1.3}$$

and the expected value of this cost would be:

$$\int_{t_1}^{t_2} [I(1+1) + a(t_2 - x)]\, f(x)dx \;. \tag{1.4}$$

In a manner similar to the above, the costs and probabilities of all possible cycles can be determined to give the expected cost per cycle as:

Expected cost per cycle

$$\begin{aligned} &= \int_{0}^{t_1} [I(0+1) + a(t_1 - x)]\, f(x)dx \\ &+ \int_{t_1}^{t_2} [I(1+1) + a(t_2 - x)]\, f(x)dx \\ &+ \int_{t_2}^{t_3} [I(2+1) + a(t_3 - x)]\, f(x)dx \\ &+ \text{etc.} \end{aligned} \tag{1.5}$$

Therefore if

$L(t_1, t_2, t_3, \ldots)$ = Expected Cost Per Cycle

then Eq. (1.5) can be written as

$$L(t_1, t_2, t_3, \ldots) = \sum_{k=0}^{k=n-1} \int_{t_k}^{t_{k+1}} [I(k+1) + a(t_{k+1} - x)]\, f(x)dx \;. \tag{1.6}$$

$$n = 1, 2, 3, \ldots, \infty$$

The objective function to be minimized is $L(t_1, t_2, t_3, \ldots)$ if the failed system is replaced, renewed, repaired etc. at the end of its fixed maximum life time. If the system is replaced or renewed or repaired when failed, then, the objective function to be minimized would be expected cost per unit time which is:

$$\text{Expected Cost Per Unit Time} = \frac{\text{Expected Cost Per Cycle}}{\text{Expected Cycle Length}} . \qquad (1.7)$$

The optimum solution gives the optimum number of inspections n, and optimum timing of inspections, i.e, $t_1, t_2, t_3, \ldots, t_n$. Most of the literature [6, 17, 20] is about the optimum solution which minimizes the total maintenance cost per unit time when the failure distribution of the system is completely known in advance. But there are many cases where the failure distribution of the system is completely unknown or is partially known. In these cases different methods have been devised depending on the kind and amount of information which is available about the failure characteristics of the system [26].

Minimax policies have been devised to cope with the situation in which the decision maker has virtually no information about the failure distribution. This method minimizes the maximum possible loss or maintenance cost that can occur due to the failure characteristics by designating the optimum number of inspection and optimum timing of them. If $P = \{t_1, t_2, t_3, \ldots, t_n\}$ is the space of the inspection times $t_1, t_2, t_3, \ldots, t_n$, and if F is any failure distribution within the constraints of the problem related to the failure characteristics of the system, then, the minimax policy can be formulated as

$$L^{**}(P^{**},F^{**}) = \mathrm{Min}_P \, \mathrm{Max}_F \, L(P,F), \qquad (1.8)$$

where L(P,F) is the expected maintenance cost per cycle as given by Eq. (1.6). L^{**}, P^{**} and F^{**} are the minimax expected maintenance cost, the optimum number and timing space of inspections and the failure distribution which gives the maximum possible rise to the total maintenance cost L respectively. The minimax policy is not limited to preparedness or inspection models and has been implemented in the area of preventive maintenance by Barlow [4]. But the major work in minimax policy has been done for the preparedness and inspection policies. In minimax policies applied to inspection models the time horizon T is assumed to be finite; i.e., the cost accounting stops at either the first inspection to detect failure or at time T, whichever happens first. The reason for this is that for any possible inspection schedule there exists a distribution which would induce an arbitrarily high expected cost during an infinite time horizon. Hence, a minimax solution would not then exist. The finite horizon assumption can be found to have many implications. As an example, consider the problem of detecting the occurrence of an event (say, the arrival of an enemy missile or the presence of some grave illness such as cancer) when the time of occurrence is not known in advance. Each inspection involves a cost so that we do not wish to check too often. On the other hand, there is a penalty cost associated with the lapsed time between occurrence and its detection so that we wish to check often enough to avoid a long lapse of time between failure and its detection.

Derman [9] has found analytically the optimum number of inspections and timing, applying the minimax policy, when the failure distribution is completely unknown, to the loss function, given by Eq. (1.6), under the following assumptions.

1. The maximum life or service time of the system T is limited and known, i.e., until time T the system has either failed or the system

will be put out of service or repaired, renewed etc. at the end of fixed periods of length T equal to the maximum life time of the system.

2. The failure can only be detected by inspection with a certain probability $P'(P' > 0)$ and the inspection does not affect the failure characteristics.
3. The inspection time is negligible.
4. Each check entails a cost I.
5. The time elapsed between system failure and its discovery at the next check has a cost per unit of time a.

In many cases there are estimable costs associated with the resumption in service (storage) of a unit which has failed. If the unit is a production system the costs are associated with the amount of defective product produced, if it is material in storage (e.g., certain kind of missile fuels) the costs are derived from considering the various implications of using, unknowingly, the unserviceable material; ... and the like.

Derman [9], proved that under the above assumptions the minimax schedule is given by

$$t_i = iP' \left(\frac{T}{nP'+1} + \frac{I}{2a}\left(\frac{n[(n+1)P'+2]}{nP'+1} - (i+1)\right)\right) \qquad (1.9)$$

$$i = 0, 1, \ldots, n,$$

where n, the number of inspections, is the largest integer such that

$$IP'^2n^2 + IP'(2-P')\,n + 2(I-P'aT) \leq 0. \qquad (1.10)$$

The minimax expected cost L^{**}, when $P'=1$ is given by

$$L^{**} = \frac{aT}{n+1} + \frac{I}{2}\,\frac{n(n+3)}{n+1}\,, \qquad (1.11)$$

and t_i is the time of i^{th} inspection. Roeloffs (30], obtained analytically the minimax schedule for the Derman's Problem [9] with the further assumption that the location x', of the $100.P^{th}$ percentile of the otherwise unknown cumulative failure distribution function F, of the system is known. That is,

$$F(t=x') = p \qquad x' \geq 0; \quad 0 < p < 1. \tag{1.12}$$

A surveillance or inspection schedule t, is a set of m+n points, such that

$$0 \leq t_1 \leq \cdots \leq t_m \leq x' \leq t_{m+1} \leq \cdots \leq t_{m+n} \leq T. \tag{1.13}$$

The minimax inspection schedule and the minimax expected cost analytically obtained by Roeloffs [30] does not have a simple form as Derman's does and is more sophisticated and involved. The Roeloffs' solution is also under the assumption that the probability of the detection of failure p', upon inspection is one. Roeloffs also showed that the added information about the location of a percentile of the failure distribution improves the minimax expected loss or maintenance cost function given by Eq. (1.6) and results in less maintenance cost compared to the solution given by Derman, i.e., Eqs. (1.9-11).

The implication of dynamic programming in the area of maintenance is well known to the researchers in this field. Hasting [13], Jardine [16, 17], Bellman [7] and many others have applied the dynamic programming to a variety of replacement-repair maintenance problems, but to the best of my knowledge, all of them except Kander [20,21], have assumed a known system failure distribution and in some cases even more specifically they have assumed a certain type of failure distribution. Kander solved several different kinds of inspection scheduling problems by converting the minimization of the loss function, i.e., Eq. (1.6), into optimization of recurrence relationships [21].

Introduction of the implication of dynamic programming in minimax policy for the solution of the problem of optimum inspection frequency and timing has been accomplished by Kander [22], however, the idea and theory of minimax dynamic programming was well introduced and established by Bellman [7], the pioneer in this field and later repeated by Jacobs [15]. Kander implemented the dynamic programming methodology and combined it with minimax policy to obtain numerically the optimum solution to the Roeloffs' problem, discussed earlier, with a further assumption that the system failure distribution is IFR (Increasing Failure Rate). The detail of this method has been fully explained in Chapter 2 since the present work is highly based on the Kander's solution procedure; but for the time being it should be mentioned that Kander showed that his solution reduces the total maintenance cost compared to Roeloffs' solution [30].

The purpose of the present work has been directed toward obtaining an upper bound for the expected total cost of the system maintenance when optimum policy is applied to the actual failure distribution of the system and under Roeloffs' assumptions [30] with addition to:

A. The locations x_1' and x_2', of the $100.p_1$ th and $100.p_2$ th percentiles respectively of the otherwise unknown life or failure distribution of the system are known. That is

$$F(t_1' = x_1') = p_1 \qquad x_1' \geq 0; \qquad 0 < p_1 < 1. \quad (1.14)$$

$$F(t_2' = x_2') = p_2 \qquad x_2' \geq 0; \qquad 0 < p_2 < 1. \quad (1.15)$$

In order that an IFR distribution pass through these two points it is necessary for p_1, p_2, x_1' and x_2' to satisfy the following inequality

$$\frac{\operatorname{Log}_e (1-p_1)}{x_1'} \geq \frac{\operatorname{Log}_e \left(\frac{1-p_2}{1-p_1}\right)}{x_2'-x_1'} \tag{1.16}$$

B. The inspection times can only occur at discrete points in time between time 0 to T.

Kander did not consider the restriction on the inspection time imposed by assumption B. Assumption B, which reduces the computation time when utilized in the dynamic program, is a realistic assumption in many cases, i.e., in cases where the inspector is available for inspection only at certain times and not all the times. Assumption A, which is different from Kander's by the information about the location of two percentiles of the system failure distribution instead of one, is utilized to give the relative value of the added information and the reduction in estimation of the total inspection or maintenance cost of the system.

In Chapter 2, the properties of IFR distributions in which the basic formulation lies are presented. The loss function or Eq. (1.6), has been rearranged in an order to be suitable for dynamic programming formulation. The recurrence relationships, state and control variables, stages and logic of the problem have been formed and explained in detail.

In Chapter 3, the details of the computational procedure for the computer programming of the two models, i.e., with information about one point and two points of the system failure distribution are stated.

In Chapter 4, the convergence and accuracy of the solutions are being shown through an example and then the results are presented in the form of tables and figures showing the sensitivity of the upper bound for optimum expected total system maintenance cost L**, respect to one and two known locations of the percentiles of system failure distribution together with sensitivity of L** respect to the values of the elements of the information parameter vector which consists of elements like $\underline{a}$, the cost of undetected failure, and $\underline{I}$, the cost of every inspection. The value of information v.s. the location of the known percentiles of a IFR distribution has been illustrated by an example. Also an example about the application of the models is given.

Chapter 5, gives the conclusion derived from the outputs of the computer program and results of Chapter 4 plus the comparison of the upper bound expected total system maintenance cost, having information about one point of the failure distribution (Model A) and two point of the failure distribution (Model B). This will be followed by the possibilities for further research in this area.

CHAPTER 2

FORMULATION

2.1 INTRODUCTION

Maintenance policies for stochastically failing equipment have been calculated mainly for the case where the failure distribution is assumed to be known, even though in real life situations the failure characteristics of the equipment are mostly not known in advance and the only way for finding the failure characteristics and distribution is through study and tests which can be very costly. Before introduction of a new system or equipment, the management needs a precise or at least an estimate of the costs associated with the equipment or system so that it can choose the best alternative through a cost analysis. In the case where little is known about the new system an upper bound for the total cost is very helpful in decision making. It is clear that the more information about the system the lower the upper bound for costs will be. In this work, using a minimax method, dynamic programming has been implemented to find an upper bound for the optimum cost of the system.

Minimax policies were devised with the assumption of no knowledge or at most the knowledge about the location of one percentile of the system failure distribution.

Assuming the failure or the life time distribution of a system to be F_0 and given certain costs (e.g. inspection and undetected failure costs), an optimal maintenance policy P_0 can be calculated which leads to the minimum loss or total maintenance cost L^*:

$$L^* = L(P_0, F_0) = \underset{P}{\text{Min}}\ L(P,F_0) \leq L(P,F_0), \tag{2.1.1}$$

where P is any inspection policy.

The method devised for partial knowledge (as detailed above) assumes a distribution F^{**} while subsequently deriving an optimal policy P^{**} with minimal loss L^{**}

$$L^{**} = L(P^{**},F^{**}) = \underset{P}{\text{Min}}\ L(P,F^{**}). \tag{2.1.2}$$

In general L^* can be larger or smaller than L^{**}, but the virtue of minimax policy is that L^{**} represents generally an upper bound.

For a minimax policy we have

$$L(P,F^{**}) = \underset{F}{\text{Max}}\ L(P,F) \geq L(P,F), \tag{2.1.3}$$

and
$$L^{**} = L(P^{**},F^{**}) = \underset{P}{\text{Min}}\ L(P,F^{**}) = \underset{P}{\text{Min}}\ \underset{F}{\text{Max}}\ L(P,F). \tag{2.1.4}$$

If a saddle point exists then:

$$L^{**} = \underset{F}{\text{Max}}\ L(P^{**},F).$$

Now having L^{**} defined by Eq. (2.1.4) we can write

$$L^{**} = \underset{P}{\text{Min}}\ \underset{F}{\text{Max}}\ L(P,F) \geq L(P_0,F_0) = L^* \tag{2.1.5}$$

The proof is as follows.

Proof: If (2.1.5) is not true then we should have $L^{**} < L^*$. But according to (2.1.1) we have $L^* \leq L(P,F_0)$ for any policy P including minimax policy P^{**} so that

$$L^{**} < L^* \leq L(P^{**},F_0). \tag{2.1.6}$$

On the other hand according to (2.1.3) we have $L(P,F^{**}) \geq L(P,F)$ for any policy P and failure distribution F including P^{**} and F_0 respectively so that we can write $L(P^{**},F^{**}) \geq L(P^{**},F_0)$. But $L(P^{**},F^{**})$ according to 2.1.4 is the minimax cost L^{**} so we should have $L^{**} \geq L(P^{**},F_0)$. This contradicts Eq. (2.1.6) derived on the assumption that $L^{**} < L^{*}$. So it is proved that indeed $L^{**} > L^{*}$. That is the minimax loss or cost L^{**} is an upper bound for the optimum inspection policies P_0 applied to failure distributions F_0.

In this work improved upper bounds for the optimum total expected cost will be obtained by assuming the knowledge about the location of one and two percentiles of f.d. and also that failure distributions are IFR (Increasing Failure Rate), using dynamic programming methodology.

In this chapter, first the assumptions of the present study are stated, followed by the mathematical preparation, model presentation, formulation of the loss function and finally the dynamic programming formulation of the models.

2.2. THE PROBLEM STATEMENT

The objective is to find the upper bound cost for the optimum inspection policies for known failure distributions L^{**} under the following assumptions:

(a) The probability of the failure of the system at time T is one. This means that T is the maximum life time of the system and the inspection ends some time between first inspection at time 0 and time T upon the detection of failure.

(b) The period for renewal, replacement, replenishment etc. is fixed and is equal to T.

(c) A system failure is detected only through inspection, which costs $\underline{I}$ dollars each. Inspection is carried also at times $t = 0$ and $t = T$.

(d) The time elapsed between system failure and its detection at the next inspection costs $\underline{a}$ dollars per unit time.

(e) Inspection takes negligible time, the system cannot fail during an inspection and is not degraded by inspection.

(f) The inspection can only be performed at certain discrete points in time.

(g) The failure distribution of the system is IFR (Increasing Failure Rate).

The minimax solution gives the upper bound for optimum expected total maintenance cost per cycle L** according to (2.1.5) where L is given by

Expected Maintenance Cost Per Cycle, L

$$= I \sum_{i=0}^{n-1} F^*(t_i) + a \left(\sum_{i=0}^{n-1} F^*(t_i)(t_{i+1} - t_i) - \int_0^{t_n} F^*(t)\, dt \right), \qquad (2.2.1)$$

and n is the total number of inspections and $F^*(t) = 1 - F(t)$ is the probability that the system has not failed until time t.

The failure distribution $F(t)$ is unknown except at the time T where $F(T) = 1$ and also one of the following two additional informations depending on one of the two models A or B is given. That is

For Model A

The location x', of the 100.P th percentile of the otherwise unknown failure distribution of the system is known. That is

$$F(t' = x') = p \qquad x' \geq 0; \qquad 0 < p < 1. \qquad (2.2.2)$$

For Model B

The locations x_1' and x_2', of the $100.p_1$ th and $100.p_2$ th percentiles respectively of the otherwise unknown failure distribution of the system are known. That is:

$$F(t_1' = x_1') = p_1 \qquad x_1' \geq 0; \qquad 0 < p_1 < 1, \tag{2.2.3}$$

and

$$F(t_2' = x_2') = p_2 \qquad x_2' \geq 0; \qquad 0 < p_2 < 1. \tag{2.2.4}$$

In order that IFR distributions pass through these two points, i.e., (x_1', p_1) and (x_2', p_2), it is necessary for p_1, p_2, x_1' and x_2' to satisfy the following inequality

$$-\frac{\operatorname{Log}_e (1-p_1)}{x_1'} \leq -\frac{\operatorname{Log}_e \left(\frac{1-p_2}{1-p_1}\right)}{x_2' - x_1'} \tag{2.2.5}$$

2.3 MATHEMATICAL PREPARATIONS

According to [6], a failure distribution $F(u)$, $u > 0$ which is IFR, crosses the exponential distribution $1-e^{-\alpha u}$, $\alpha > 0$ from below at most once in addition to coincidence at $u = 0$ and ∞ unless they coincide identically. This can be easily seen, that is if F is any IFR and the exponential distribution is $1-e^{-\alpha u}$ then the two functions agree at the origin (See Fig. 2) and equating the two functions

$$F(t) = 1-e^{-\alpha t} \tag{2.3.1}$$

and shifting $F(t)$ to the right and $e^{-\alpha t}$ to the left of the equal sign and calling $1-F(t)$ as $\bar{F}(t)$ and extracting the logarithms then:

$$G(t) = \log_e \bar{F}(t) - (-\alpha t). \tag{2.3.2}$$

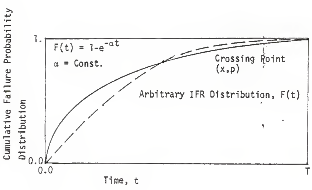

Fig. 2. Exponential and Arbitrary IFR Distribution, F(t) Crossing Exponential Distribution from Below

It is clear that $G(t)$ is zero at $t = 0$ since for $t = 0$, $\bar{F}(t) = 1$ and $\log_e 1 = 0$, so $G(t) = 0 - 0 = 0$. According to the definition of failure rate we have

$$\text{Failure Rate} = r(u) = \frac{F(u+\Delta) - F(u)}{\bar{F}(u)} \tag{2.3.3}$$

where $\Delta > 0$ is an increment of time. The definition of an IFR distribution signifies that

$$r(u_2) \geq r(u_1), \qquad u_2 \geq u_1. \tag{2.3.4}$$

Applying Eq. (2.3.3) to exponential distribution $1 - e^{-\alpha t}$, we have

$$\text{Exponential F.R.} = r_e(t) = \frac{1-e^{-\alpha(t+\Delta)} - (1-e^{-\alpha t})}{1-(1-e^{-\alpha t})} = 1 - e^{-\alpha\Delta}$$
$$= \text{constant}, \tag{2.3.5}$$

and calling the failure rate of the arbitrary IFR distribution $F(t)$ as $r(t)$, then $F(t)$ crosses $1 - e^{-\alpha t}$ at most one more time from below (See Fig. 2.). $F(t)$ cannot cross $1 - e^{-\alpha t}$ from above since if that happens then $r(t_1^*) < r_e(t_1^*) = 1 - e^{-\alpha\Delta} = \text{constant}$ where $t_1^* > 0$ is the first crossing point after the origin. But the first crossing from above cannot happen unless $r(0) > r_e(0) = 1 - e^{-\alpha\Delta} = \text{constant}$. This means that $r(0) > 1 - e^{-\alpha\Delta} > r(t_1^*)$ which violates the condition for $F(t)$ to be an IFR. As a result the first crossing of $1 - e^{-\alpha t}$ by $F(t)$ happens from below. The proof that $F(t)$ cannot cross $1 - e^{-\alpha t}$ more than once is similar to the previous one, i.e., if $F(t)$ crosses $1 - e^{-\alpha t}$ for the second time, it should be from above which means that $r_e(t_2^*) = 1 - e^{-\alpha\Delta} = \text{const.} > r(t_2^*)$, where $t_2^* > t_1^* > 0$ and t_1^* and t_2^* are first and second crossing points besides origin. But $F(t)$ has already crossed the exponential distribution from below (proved previously) at t_1^* so we already have $r(t_1^*) > r_e(t_1^*) = 1 - e^{-\alpha\Delta} = \text{const}$. This means that $r(t_1^*) > 1 - e^{-\alpha\Delta} > r(t_2^*)$ which again violates the condition for $F(t)$ to be an IFR. So no IFR distribution can cross exponential distribution more than once besides at $t = 0$ and $t = \infty$ and this crossing should take place from below. Consequently for $F(u = x) = p$ the common point, we obtain:

$$F(u) \leq 1 - e^{-\alpha u} \quad \text{if} \quad u \leq x \tag{2.3.6}$$

$$F(u) \geq 1 - e^{-\alpha u} \quad \text{if} \quad u \geq x \tag{2.3.7}$$

$$\alpha = - \frac{\mathrm{Log}_e(1 - p)}{x} . \tag{2.3.8}$$

Now let us introduce a transformed exponential distribution function - t.e.d.:

$$F(v) = \begin{cases} 0 & \text{if} \quad u < d \\ 1 - e^{-c(u-d)} & \text{if} \quad u \geq d \\ & c > 0 , \\ & d \,\&\, c = \text{const.}, \end{cases} \tag{2.3.9}$$

from Eq. (2.3.3) it follows that the transformed distribution is of constant failure rate

$$r(v) = 1 - e^{-c\Delta}, \quad u > a, \quad \Delta > 0 . \tag{2.3.10}$$

The transformed exponential distribution has the following properties [22]:

PROPERTY 1: Generalizing properties of the exponential distribution (e.d.), the t.e.d. can be crossed by an IFR distribution at most twice, the first taking place from above.

REASON: Assume that $F(t)$, which is IFR, crosses the t.e.d. $F(v)$, with constant failure rate $r(v)$, first from above and then from below. At the first coincidence $r(t_1) < r(v)$ and at the second $r(t_2) > r(v)$ so that $F(t)$ can indeed be IFR. A further crossing from above would mean $r(t_3) < r(v)$, contradicting the assumption of IFR distribution.

Further, we can uniquely find parameters c, d such that the t.e.d. passes through given two points (t_i, p_i): $p_i = F(t = t_i)$, $i = 1, 2$.

PROPERTY 2: Given three points

$$(t_i, p_i): \;\; p_i = F(w = t_i), \quad i = 0, 1, 2 \qquad t_0 < t_1 < t_2. \tag{2.3.11}$$

(a) An IFR distribution $F(w)$ passes through the three points only if (t_1, p_1) lies on or below the t.e.d. through (t_0, p_0) , (t_2, p_2).

(b) The IFR distribution $F(w)$ which passes through the three points and possesses maximum area $\int_{t_0}^{t_2} F(w)\, d_w$, is given by the two t.e.d.'s which meet at (t_1, p_1):

$$F_i(w) = 1 - e^{-c_i(w - d_i)} \tag{2.3.12}$$

$$c_i = \frac{1}{t_i - t_{i-1}} \operatorname{Log}_e \left(\frac{1 - p_{i-1}}{1 - p_i}\right) \tag{2.3.13}$$

$$d_i = \frac{1}{c_i} \operatorname{Log}_e (1 - p_i) + t_i \tag{2.3.14}$$

$$t_{i-1} \leq w \leq t_i, \quad i = 1, 2, \tag{2.3.15}$$

while

$$c_2 \geq c_1. \tag{2.3.16}$$

REASON (a): This property is a direct result of property 1.

REASON (b): Let us define the following distribution functions (See Fig. 3.): t.e.d. connecting points (t_2, p_2) , (t_0, p_0)

$$F_3(w) = 1 - e^{-c_3(w - d_3)}; \tag{2.3.17}$$

t.e.d. connecting points (t_1, p_1), (t_0, p_0)

$$F_1(w) = 1 - e^{-c_1(w - d_1)}; \tag{2.3.18}$$

t.e.d. connecting points (t_2, p_2), (t_1, p_1)

$$F_2(w) = 1 - e^{-c_2(w - d_2)}. \tag{2.3.19}$$

We assume according to (a) that

$$p_1 \leq 1 - e^{-c_3(t_1 - d_3)}. \tag{2.3.20}$$

Since $F_1(w)$ crosses t.e.d. $F_3(w)$ from above (See Fig. 3.) it follows directly from definition (2.3.3) that:

$$r_3(t_0) \geq r_1(t_0) , \tag{2.3.21}$$

where $r_i(\cdot)$ is the failure rate for $F_i(\cdot)$. Similarly since $F_2(w)$ crosses $F_3(w)$ from below (See Fig. 3.)

$$r_3(t_2) \leq r_2(t_2), \tag{2.3.22}$$

so that from (2.3.21), (2.3.22) we obtain:

$$r_2(t_2) \geq r_1(t_0) \tag{2.3.23}$$

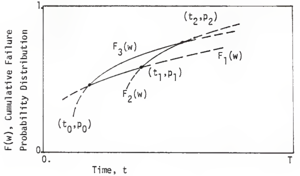

Fig. 3. The Relative Positions of the Three Points and Distributions.

from which by (2.3.10) we have:

$$c_2 \geq c_1.$$

The Eqs. (2.3.12 - 15) can easily be obtained since $F_1(w)$ and $F_2(w)$ have point (t_1, p_1) in common.

Viewing now $F_1(w)$, $w \leq t_1$ and $F_2(w) \geq t_1$ as parts of one distribution $F(w)$, we obtain for the failure rate of the latter from (2.3.3), (2.3.10) and (2.3.12 - 16):

$$F(w) = \begin{cases} 1 - e^{-c_1\Delta} & \text{if } w + \Delta < t \\ 1 - \dfrac{e^{-c_2(w+\Delta - d_2)}}{e^{-c_1(w - d_1)}} = 1 - \dfrac{e^{-c_2(\Delta - d_2)}}{e^{+c_1 d_1}} e^{-(c_2 - c_1)w} & \\ & \text{if } w \leq t \text{ or } w + \Delta \geq t \\ 1 - e^{-c_2\Delta} & \text{if } w > t \\ \Delta > 0, & \end{cases} \tag{2.3.24}$$

which is increasing in w, so that F(w) is IFR.

From Property 1 it follows that no IFR distribution can pass through points (t_i, p_i), i = 1, 2 and above F(w). Therefore indeed

$\int_{t_0}^{t_2} F(w)\,dw$ is maximum among all IFR distributions.

2.4 THE OBJECTIVE FUNCTION

In Chapter 1 it was shown that the expected total maintenance cost per cycle could be found from Eq. (1.6) which is stated as

$$L(t_0 = 0, t_1, t_2, t_3, \ldots, t_n) = \sum_{k=0}^{k=n-1} \int_{t_k}^{t_{k+1}} [I(k+1) + a(t_{k+1}-x)] \, f(x)dx,$$

$$n = 1, 2, 3, \ldots, \infty.$$

This equation is valid when there is no maximum life time for the system and inspection continues until the system fails. According to the assumption a of Section 2.2 there is a maximum life time T for the system, i.e., $F(T) = 1$ for both model A and model B presented here. Also the definition of the problem signifies that inspection is performed at time $t_0 = 0$ and $t_n = T$. With these points in mind Eq. (1.6) can be written as

$$L(t_0 = 0, t_1, t_2, \ldots, t_n = T) = \sum_{k=0}^{k=n-2} \int_{t_k}^{t_{k+1}} [I(k+1) + a(t_{k+1}-x) \,] \, f(x) \, dx + \int_{t_{n-1}}^{t_n=T} [I(n) + a(T-x) \,] \, f(x) \, dx, \qquad (2.4.1)$$

$$n = 1, 2, \ldots, N,$$

where N is the maximum possible number of inspections between $t = 0$ to $t = T$ including.

Equation (2.4.1) can be written in the following form

$$L = I \sum_{k=0}^{n-1} F^*(t_k) + a \left\{ \sum_{k=0}^{n-1} F^*(t_k)(t_{k+1} - t_k) - \int_0^{t_n} F^*(t)dt \right\}, \qquad (2.4.2)$$

where $F^*(t) = 1 - F(t)$ and $F(t)$ is the cumulative failure distribution of the system. The algebraic manipulation which transforms Eq. (2.4.1) into Eq. (2.4.2) is presented in detail in Appendix A.

2.5 FORMULATION

It is the objective of this work to maximize objective function L given by (2.4.2) for a given inspection policy $p = \{t_0 = 0, t_1, t_2, \ldots, t_n = T\}$ by searching all IFR distributions F which pass through one or two known points depending on the model, i.e., A or B respectively.

The form of the IFR distributions F(w) leading to Max. L is given, according to Property 2 by transformed exponential distribution functions (t.e.d) connecting all points $(t_i, p = F(t_i))$, $i = 0, 1, 2, \ldots, n$ and the known points.

2.5.1 Model A

In this model we assume that one point of the failure distribution (x,p) is known. If m is the number of inspections before x, we obtain the time sequence $t_0, t_1, t_2, \ldots, t_m, x, t_{m+1}, \ldots, t_n$, according to which above points are arranged. We recode this sequence as

$$\{(t_{(i)}, p_{(i)}); \quad i = 0, 1, \ldots, n+1\} \quad , \tag{2.5.1}$$

while

$$(t_0, p_0) = (0, 0) = (t_{(0)}, p_{(0)})$$

$$(x, p) = (t_{(m+1)}, p_{(m+1)}), \qquad c = c_{(m+1)}$$

$$(t_{n-1}, p_{n-1}) = (t_{(n)}, p_{(n)}) \qquad c_{n-1} = c_{(n)}$$

$$(t_n, p_n) = (T, 1) = (t_{(n+1)}, p_{(n+1)}),$$

knowing these we can define F(w) as:

$$F(w) = \begin{cases} 1 - e^{-c_{(1)}w} & \text{if } t_{(0)} = 0 \leq w \leq t_{(1)} \\ 1 - e^{-c_{(i)}(w - d_{(i)})} & \text{if } t_{(i-1)} \leq w \leq t_{(i)}, \quad 1 < i \leq n \\ 1 & \text{if } t_{(n-1)} \leq w \leq t_{(n)} = T \end{cases} \tag{2.5.2}$$

$$c_{(1)} < \alpha = - \frac{\text{Log}_e(1 - p)}{x},$$

$$c_{(1)} \leq c_{(2)} \leq \cdots \leq c_{(m+1)} \leq \cdots \leq c_{(n)} ; \qquad \alpha < c_{(m+1)},$$

$$c_{(i)} = \frac{1}{t_{(i)} - t_{(i-1)}} \text{Log}_e \left(\frac{1 - p_{(i-1)}}{1 - p_{(i)}}\right), \qquad i = 1, 2, \ldots, n$$

$$d_{(i)} = \frac{1}{c_{(i)}} \text{Log}_e \left(1 - p_{(i)}\right) + t_{(i)},$$

while we assumed

$$c_{(i)} = \begin{cases} c_1 & \text{if } 1 \leq i \leq m \\ c & \text{if } i = m+1 \\ c_{i-1} & \text{if } m+2 \leq i \leq n \end{cases} \quad . \tag{2.5.3}$$

We observe that $F(w)$ passes through points $(0,0)$, (x,p) and possesses a jump from (t_{n-1}, p_{n-1}) to $(t_{n-1}^{+}, 1)$ (as allowed for an IFR distribution in [6, 22]).

The objective function L, given feasible points (t_i, p_i), $i = 1, 2, \ldots, n-1$, is indeed maximized by function $F(w)$ since

$\int_0^{t_n=T} F^*(w)\,dw$ becomes minimum. This can be seen easily since

$$\int_0^{t_n=T} F^*(w)\,dw = \int_0^{t_n=T} (1 - F(w))\,dw$$

$$= T - \int_0^{t_n=T} F(w)\,dw, \qquad (2.5.4)$$

but $\int_0^{t_n=T} F(w)\,dw$ becomes maximized according to Property 2 (2.3) if $F(w)$ is given by the set of the transformed exponential distributions defined in (2.5.2). The value of maximum life time T is fixed so Eq. (2.5.4) gives the minimum value of the expression for all IFR distributions $F(t)$.

The feasible region of $F(t)$ v.s. t diagram is shown in Fig. 4. Between $t = 0$ and $t = x$ the region is the area confined from above by $F(t) = 1 - e^{-\alpha t}$, from below by $F(t) = 0$ and from left and right by $t = 0$ and $t = x$ respectively. The reason for this is given by Property (2-a-2.3) which states that any IFR distribution will pass through three points (e.g. $[t = 0, 0]$, $[t_i, p_i]$, $[t = x, p]$) only if (t_i, p_i) lies on or below the t.e.d through the first and third points (e.g. $[t = 0, 0]$, $[t = x, p]$). Between $t = x$ and $t = T$ the region is the area confined from above by $F(t) = 1$, from below by $F(t) = 1 - e^{-\alpha t}$ and from left and right by $t = x$ and $t = T$ respectively (See Fig. 4.). The reason that (t_i, p_i) cannot be below $F(t) = 1 - e^{-\alpha t}$ is that any IFR distribution which crosses $F(t)$ at point $(t = x, p)$ should have a higher failure rate at this point than $1 - e^{-\Delta\alpha}$. So no point (t_i, p_i) for $t_i > x$ can lie below $F(t) = 1 - e^{-\alpha t}$. In addition to this the points (t_i, p_i), must satisfy the equations given in (2.5.2).

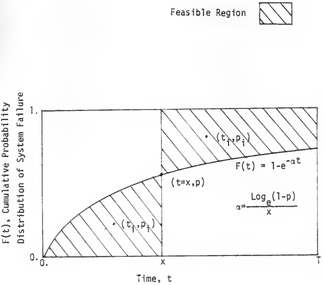

Fig. 4. The Feasible Region for Points (t_i,p_i) for Model A.

The following relationship exists between two successive points (t_i, p_i), (t_{i-1}, p_{i-1}). That is

$$F^*(t_{(i)}) = F^*(t_{(i-1)})\, e^{-c_{(i)}(t_{(i)} - t_{(i-1)})}$$

$$i = 1, 2, \ldots, n. \qquad (2.5.5)$$

This can be obtained from equations (2.5.2). That is

$$F(t_{(i)}) = 1 - e^{-c_{(i)}(t_{(i)} - d_{(i)})},$$

and

$$F(t_{(i-1)}) = 1 - e^{-c_{(i)}(t_{(i-1)} - d_{(i)})},$$

since both points $(t_{(i-1)}, p_{(i-1)})$ and $(t_{(i)}, p_{(i)})$ are the two ends of F(t). Now shifting 1 to the left side of both equations and multiplying both sides by -1 and dividing by each other then we have

$$\frac{1 - F(t_{(i)})}{1 - F(t_{(i-1)})} = e^{-c_{(i)}(t_{(i)} - d_{(i)} - t_{(i-1)} + d_{(i)})}$$

$$\frac{F^*(t_{(i)})}{F^*(t_{(i-1)})} = e^{-c_{(i)}(t_{(i)} - t_{(i-1)})}\ .$$

Maximization for given policy can now be carried out by dynamic programming methodology. The search extends over all $p_i = F(t_i)$, $i = 1, 2, \ldots, n$. We define J_i, as loss or maintenance cost on the interval between (i-1)th and (i)th inspection $[t_{i-1}, t_i]$, which is also a function of the failure probability at (i-1)th and i th inspection. That is

$$J_i = J_i(t_{i-1}, p_{i-1}, t_i, p_i) = [I + a(t_i - t_{i-1})]\, F^*(t_{i-1}) - a \int_{t_{i-1}}^{t_i} F^*(w)\, dw \qquad (2.5.6)$$

$$t_i < w < t_{i-1}, \quad i = 1, 2, \ldots, n.$$

If $J_i^* = \underset{F(w)}{\text{Max}}\ J_i$, we obtain from equations (2.5.2) that

$$J_i^* = J_i^*(t_{i-1}, p_{i-1}, t_i, p_i) = \text{Max}\ J_i\ (t_{i-1}, p_{i-1}, t_i, p_i) \qquad (2.5.7)$$

$$= \begin{cases} [I + a(t_i - t_{i-1})]F^*(t_{i-1}) + a\,\dfrac{F^*(t_i) - F^*(t_{i-1})}{c_i} & \text{if } i=1,2,\ldots,m,m+2,\ldots,n-1 \\[2ex] [I + a(t_{m+1} - t_m)]F^*(t_m) + a\left(\dfrac{(1-p) - F(t_m)}{c} + \dfrac{F^*(t_{m+1}) - (1-p)}{c_{m+1}}\right) & \text{if } i=m+1 \\[2ex] [1 + a(t_n - t_{n-1})]\, F^*(t_{n-1}) & \text{if } i=n \end{cases}$$

The first expressions on the right hand side of equations in (2.5.7), i.e., $[I + a(t_i - t_{i-1})]\, F^*(t_{i-1})$ are obviously the share of J_i from the first expression in the expected total maintenance cost L given by Eq. (2.4.2), i.e., $I \sum_{i=0}^{n-1} F^*(t_i) + a \sum_{i=0}^{n-1} F^*(t_i)(t_{i+1} - t_i)$. The second expression a $\int_{t_{i-1}}^{t_i} F^*(w)\,dw$, is also the share of J_i from L, i.e., a $\int_0^{t_n=T} F^*(w)dw$.

For values of i = 1,2, ..., m, m + 2, ..., n-1, $\int_{t_{i-1}}^{t_i} F^*(w)\,dw$ can be calculated from equations (2.5.2). That is

$$\int_{t_{i-1}}^{t_i} F^*(w)\,dw = \int_{t_{i-1}}^{t_i} e^{-c_i(w - d_i)}\,dw$$

$$= \left(- \frac{e^{-c_i(w - d_i)}}{c_i}\right)_{t_{i-1}}^{t_i}$$

$$= \frac{e^{-c_i(t_{i-1} - d_i)} - e^{c_i(t_i - d_i)}}{c_i}$$

$$= \frac{F^*(t_{i-1}) - F^*(t_i)}{c_i} . \qquad (2.5.8)$$

For i = m+1, F(w) passes through the known point (x,p), so

$$\int_{t_m}^{t_{m+1}} F^*(w)\,dw = \int_{t_m}^{x} F^*(w)\,dw + \int_{x}^{t_{m+1}} F^*(w)\,dw$$

$$= \int_{t_m}^{x} e^{-c_m(w - d_m)}\,dw + \int_{x}^{t_{m+1}} e^{-c_{m+1}(w - d_{m+1})}\,dw$$

$$= \left(- \frac{e^{-c_m(w - d_m)}}{c_m}\right)_{t_m}^{x} + \left(- \frac{e^{-c_{m+1}(w - d_{m+1})}}{c}\right)_{x}^{t_{m+1}}$$

$$= \frac{e^{-c_m(t_m - d_m)} - e^{-c_m(x - d_m)}}{c_m}$$

$$+ \frac{e^{-c_{m+1}(x - d_{m+1})} - e^{-c_{m+1}(w - d_{m+1})}}{c_{m+1}},$$

but $e^{-c_m(x - d_m)} = e^{-c_{m+1}(x - d_{m+1})} = F^*(x) = (1-p)$ since all describe the same point (x,p). Also $c_m = c$ according to recodification in Section 5 of this chapter. So

$$\int_{t_m}^{t_{m+1}} F^*(w)\, dw = \frac{F^*(t_m) - (1-p)}{c} + \frac{(1-p) - F^*(t_{m+1})}{c_{m+1}} . \qquad (2.5.9)$$

For $i = n$, i.e., to find J_n^* between one to the last inspection at t_{n-1} and last inspection at $t_n = T$, according to Eqs. (2.5.7), $F(w) = 1$ or equivalently $F^*(w) = 0$ so

$$\int_{t_{n-1}}^{t_n=T} F^*(w)\, dw = 0. \qquad (2.5.10)$$

Let us define $K_i(t_i, p_i, c_{i+1})$ as loss or expected maintenance cost during time $(0, t_i]$ with $F(t_i) = p_i$ when the parameter of the transformed failure distribution connecting point (t_i, p_i) to (t_{i+1}, p_{i+1}) is c_{i+1}. Then we have

$$K_1(t_1, p_1, c_2) = J_1(t_0, p_0, t_1, p_1) = J_1(0, 0, t_1, p_1)$$

$$K_i(t_i, p_i, c_{i+1}) = K_{i-1}(t_{i-1}, p_{i-1}, c_i) + J_i(t_{i-1}, p_{i-1}, t_i, p_i)$$

$$= \sum_{j=1}^{i} J_j(t_{j-1}, p_{j-1}, t_j, p_j). \qquad (2.5.11)$$

Let $K_n(T)$ signify the loss for the entire time period T at $F(T) = 1$ and any t.e.d. parameter c_{n+1}

$$K_n(T) = K_n(t_n = T,\ p_n = 1,\ c_{n+1})$$

$$= K_{n-1}(t_{n-1},\ p_{n-1},\ c_n) + J_n(t_{n-1},\ p_{n-1},\ t_n,\ p_n)$$

$$= L\ . \tag{2.5.12}$$

From this it should be clear that there are three state variables in the system, i.e., t_i, p_i and c_{i+1}. Optimization can now be carried out in two phases. In the first phase $K_{m+1}(t_{m+1},\ p_{m+1},\ c_{m+1})$, i.e., the loss between $t = 0$ to the time of the first inspection larger or equal to x, t_{m+1} will be optimized for all possible values of t_{m+1} and p_{m+1}. In the second and final phase $K_n(t_{n=m+j} = T,\ p_{n=m+j} = 1,\ c_{n+1=m+j+1} = \infty) = L$ will be optimized. In the second phase, the optimization will be achieved, taking the optimum K_{m+1} values for state variables at time $t = t_{m+1}$ as the K values for the first stage of the second phase and the values of L will be calculated for all possible number of stages both in phase 1 and phase 2 and the optimum value of L is obtained by search among these values.

First optimization phase - Let the minimax optimized loss for time period $(0,\ t_i]$ at $F(t_i) = p_i$ when the parameter of the transformed failure distribution connecting $(t_i,\ p_i)$ to $(t_{i+1},\ p_{i+1})$ is c_{i+1} be

$$K_i^*(t_i,\ p_i,\ c_{i+1}) = \underset{P}{\text{Min}}\ \underset{F}{\text{Max}}\ K_i(t_i,\ p_i,\ c_i)\ , \tag{2.5.13}$$

where P is the inspection policy and F is the failure distribution given by Eqs. (2.5.2). By dynamic programming procedure we then obtain

$$K_0^*(t_0 = 0,\ p_0 = 0,\ c_{0+1}) = 0. \tag{2.5.14}$$

$$K_1^*(t_1,\ p_1,\ c_2) = J_1^*(0,\ 0,\ t_1,\ p_1) = I + at_1 + a\frac{F^*(t_1)-1}{c_1},\quad c_1 < \alpha$$

$$\underset{\substack{0<t_1\leq x \\ p''(t_1) < p_1 < p'(t_1)}}{}\qquad\qquad i = 1$$

$$K_2^*(t_2,\ p_2,\ c_3) = \underset{\substack{0<t_2\leq x \\ p''(t_2)<p_2<p'(t_2)}}{\text{Min}}\quad \underset{\substack{0<t_1<t_2 \\ v(t_2,p_2,c_3,t_1)<p_1\leq G(t_2,p_2,t_1)}}{\text{Max}}\{K_1^*(t_1,\ p_1,\ c_2) + J_2^*(t_1,\ p_1,\ t_2,\ p_2)\}\qquad i = 2$$

$$\underset{\substack{0<t_i\leq x \\ p''(t_i) < p_i < p'(t_i)}}{K_i^*(t_i,\ p_i,\ c_{i+1})} = \text{Min}\quad \underset{\substack{0<t_{i-1}<t_i \\ v(t_i,p_i,c_{i+1},t_{i-1})<p_{i-1}<G(t_i,p_i,t_{i-1})}}{\text{Max}}\{K_{i-1}^*(t_{i-1},\ p_{i-1},\ c_i) + J_i^*(t_{i-1},\ p_{i-1},\ t_i,\ p_i)\},$$

$$i = 3,4,\ \ldots,\ m$$

$$\underset{\substack{x\leq t_{m+1}<T \\ p''(t_{m+1})<p_{m+1}<p'_{m+1}(t_{m+1})}}{K_{m+1}^*(t_{m+1},\ p_{m+1},\ c_{m+1})}$$

$$= \underset{0<t_m<x}{\text{Min}} \quad \underset{v(t_{m+1},p_{m+1},t_m)<p_m<G(t_m)}{\text{Max}}\{K_m^*(t_m,p_m,c) + J_{m+1}^*(t_m,p_m,t_{m+1},p_{m+1})\},$$

$$m = 0,1,2, \ldots, N_1,$$

$$\bar{K}_{n=m+1}^*(t_{m+1}=t_n=T,\ p_{m+1} = p_n = 1,\ c_{m+1} = c_n = \infty)$$

$$= \underset{0<t_m<x}{\text{Min}} \quad \underset{0<p_m<G(t_m)}{\text{Max}}\{K_m^*(t_m,p_m,c) + J_{n=m+1}^*(t_m,p_m,t_{m+1},p_{m+1})\},$$

$$m = 0,1,2, \ldots, N_1$$

$$c = -\frac{\text{Log}_e(\frac{1-p}{1-p_m})}{x-t_m} \quad .$$

N_1 is the maximum possible number of inspections in the interval $(0,x)$ and we have $c_1 \leq c_2 \leq \ldots < c < c_{m+1}$ since the failure distribution $F(w)$ is IFR. Also since at point (x,p), $F(w)$ crosses the exponential distribution $F(u)$ (2.3.6-8), we should have $\alpha < c < c_{m+1}$, as stated already in Eqs. (2.5.2). $p'(t_i)$ and $p''(t_i)$ which are the upper and lower limits for the possible values of p_i in the feasible region (See Fig. 5.) are given by

$$p'(t_i) = \begin{cases} 1-e^{-\alpha t_i} & 0 \leq t_i < x \\ 1 & x < t_i \leq T \end{cases} \quad , \tag{2.5.15}$$

and

$$p''(t_i) = \begin{cases} 0 & 0 \leq t_i < x \\ 1-e^{-\alpha t_i} & x < t_i < T \end{cases} \quad . \tag{2.5.16}$$

At $t_i = x$ and $t_i = T$ we have

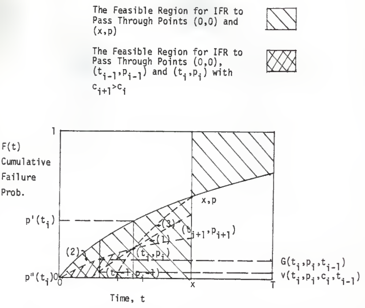

Fig. 5. The Upper and Lower Limts for p_i and p_{i-1} for $t_i \varepsilon (0,x)$.

$$p'(x) = p''(x) = p \tag{2.5.17}$$

$$p''(T) = 1. \tag{2.5.18}$$

In order for any IFR distribution to pass through points (0, 0), (t_{i-1}, p_{i-1}) and (t_i, p_i), it is necessary for point (t_{i-1}, p_{i-1}) to be located below the transformed failure distribution (t.e.d.) through points (0, 0) and (t_i, p_i) according to Property (2-a-2.3) for t_i in the interval (0,x) (See t.e.d. No 2 in Fig. 5.). The parameter of the t.e.d. No 2 called $c(t_i,p_i)$ can be calculated from

$$c(t_i,\ p_i) = -\frac{\mathrm{Log}_e(1-p_i)}{t_i}, \tag{2.5.19}$$

and the maximum possible value of p_{i-1} given by $G(t_i,p_i,t_{i-1})$ according to Eq. (2.5.5) is

$$G(t_i,p_i,t_{i-1}) = 1 - (1-p_i)e^{c(t_i,p_i)(t_i-t_{i-1})}. \tag{2.5.20}$$

$G(t_m)$ is the particular value of G function at t_m and can be found with a similar type of reasoning as for Eqs. (2.5.19-20) to be

$$G(t_m) = 1-e^{-\alpha t_m}. \tag{2.5.21}$$

In order for a transformed exponential distribution to connect points (t_{i-1},p_{i-1}) and (t_i,p_i) and have a parameter c_i so that $c_i \le c_{i+1}$, where c_{i+1} is the parameter of the transformed exponential distribution connecting points (t_i,p_i) and (t_{i+1},p_{i+1}) as required for an IFR distribution (See Fig. 5 t.e.d. No 1), the point (t_{i-1},p_{i-1}) should be above t.e.d. No 1. The reason is obvious. Let us call the t.e.d. passing through points (t_i,p_i) and

(t_{i+1}, p_{i+1}) as $v_1(t)$ then if point (t_{i-1}, p_{i-1}) is located below $v_1(t)$ we should have $v_1(t_{i-1}) > p_{i-1}$. Now we have

$$c_i = -\frac{\mathrm{Log}_e\left(\frac{1-p_i}{1-p_{i-1}}\right)}{t_i - t_{i-1}}, \tag{2.5.22}$$

and assuming $\bar{c}$ = parameter of $v_1(t) = -\dfrac{\mathrm{Log}_e\left(\frac{1-p_i}{1-F(t_{i-1})}\right)}{t_i - t_{i-1}}$, (2.5.23)

we can write $\quad c_i - \bar{c} = \dfrac{-1}{t_i - t_{i-1}}\left(\mathrm{Log}_e\left(\frac{1-p_i}{1-p_{i-1}}\right) - \mathrm{Log}_e\left(\frac{1-p_i}{1-F(t_{i-1})}\right)\right)$ (2.5.24)

$$= \frac{-1}{t_i - t_{i-1}} \mathrm{Log}_e\left(\frac{1-F(t_{i-1})}{1-p_{i-1}}\right),$$

but $\frac{-1}{t_i - t_{i-1}}$ is negative since $t_i > t_{i-1}$ and $\log_e\left(\frac{1-F(t_{i-1})}{1-p_{i-1}}\right)$ is also negative since $v_1(t_{i-1}) > p_{i-1}$ according to the assumption and so $1-F(t_{i-1}) < 1 - p_{i-1}$ which shows that $\log_e(A<1)$. But $\log_e(A<1)$ is always negative. This means that $c_i - \bar{c} = B$ where B is a positive amount so $c_i \geq \bar{c} = c_{i+1}$. But we should have $c_i \leq c_{i+1}$ in order that the failure distribution to be IFR. As a result point (t_{i-1}, p_{i-1}) should be above $v_1(t)$. The lower limit on p_{i-1} is given by v function which its value is determined by the above mentioned $v_1(t)$ distribution or regional boundaries and has different forms at different stages and different points. $v(t_i, p_i, c_{i+1}, t_{i-1})$ according to Eq. (2.5.5) is given by

$$v(t_i, p_i, c_{i+1}, t_{i-1}) = \mathrm{Max}\{1. - (1-p_i)e^{c_{i+1}(t_i - t_{i-1})} \text{ or } 0.0\}. \tag{2.5.25}$$

$v(t_{m+1},p_{m+1},t_m)$ is the particular value of v function at t_{m+1}, i.e., the t value at (m+1)th stage and is given by

$$v(t_{m+1},p_{m+1},t_m) = \text{Max}\{1-(1-p)e^{c_{m+1}(x-t_m)} \text{ or } 0.0\} , \qquad (2.5.26)$$

where

$$c_{m+1} = -\frac{\text{Log}_e(\frac{1-p_{m+1}}{1-p})}{t_{m+1}-x} . \qquad (2.5.27)$$

The value of c_i, i.e., the parameter for the t.e.d. connecting points (t_i,p_i) and (t_{i-1},p_{i-1}) is given according to Eq. (2.5.5) by

$$c_i = -\frac{\text{Log}_e(\frac{1-p_i}{1-p_{i-1}})}{t_i-t_{i-1}} . \qquad (2.5.28)$$

We now choose for each (t_{m+1},p_{m+1}) that m=m* which renders K^*_{m+1} a minimum. That is

$$K^{**}_{m^*+1}(t_{m^*+1},p_{m^*+1},c_{m^*+1}) = \text{Min}\{K^*_{m+1}(t_{m+1},p_{m+1},c_{m+1})\}$$

$$x \le t_{m^*+1} < T$$

$$p''(t_{m^*+1}) < p_{m^*+1} < p'(t_{m^*+1}) \qquad m = 0,1,2, \ldots, N_1$$

$$\bar{K}^{**}_{n=m^*+1}(t_{m^*+1}=t_n=T, p_{m^*+1}=p_n=1, c_{m^*+1}=c_n=\infty) \qquad (2.5.29)$$

$$= \text{Min}\ \{\bar{K}^*_{n=m+1}(T,1,\infty)\} \qquad m = 0,1,2, \ldots, N_1 . \qquad (2.5.30)$$

$\bar{K}^*_{n=m+1}(T,1,\infty)$, the total expected loss when there is only one inspection in the interval (x,T] is given by the last equation in (2.5.14). $K^{**}_{m^*+1}$ is the minimax expected total maintenance cost in the time interval $(0,t_{m*+1})$ where $x \leq t_{m*+1} < T$. $m*$ is the optimum number of inspections in the interval $(0,x)$. From the above formulations it becomes clear that the control variables are the previous inspection time t_{i-1} and previous cumulative failure probability p_{i-1}. The stage number m represents the number of inspections. Second optimization phase - we continue the iterative procedure, taking the K^{**}_{m*+1} values at each point (t_{m*+1},p_{m*+1}) as the minimax expected maintenance cost up to that point or state for the first value of K^{**} at the first stage in the second phase of the optimization. That is

$$K^{**}_{m*+j}(t_{m*+j},p_{m*+j},c_{m*+j+1})$$

$$x \leq t_{m*+j} < T$$

$$p''(t_{m*+j}) < p_{m*+j} < p'(t_{m*+j})$$

$$= \underset{x \leq t_{m*+j-1} \leq t_{m*+j}}{\mathrm{Min}} \quad \underset{\substack{v(t_{m*+j},p_{m*+j},c_{m*+j+1},t_{m*+j}) < p_{m*+j-1} < \\ < G(t_{m*+j},p_{m*+j},t_{m*+j-1})}}{\mathrm{Max}} \{K^{**}_{m*+j-1}(t_{m*+j-1},p_{m*+j-1},c_{m*+j})$$

$$+ J^{*}_{m*+j}(t_{m*+j-1},p_{m*+j-1},t_{m*+j},p_{m*+j})\}, \tag{2.5.31}$$

$$j = 2,3,4, \ldots, N_2$$

$$K^{**}_{n=m*+j}(t_{m*+j}=t_n=T,p_{m*+j}=p_n=1,c_{m*+j}=c_n=\infty)$$

$$= \underset{x < t_{m*+j-1} < T}{\mathrm{Min}} \quad \underset{v(t_{m*+j-1}) < p_{m*+j-1} < 1}{\mathrm{Max}} \{K^{**}_{m*+j-1}(t_{m*+j-1},p_{m*+j-1},\infty)$$

$$+ J_{n=m^*+j}(t_{m^*+j-1}, p_{m^*+j-1}, t_{n=m^*+j}=T, p_{n=m^*+j}=1)\},$$

$$j = 2,3, \ldots, N_2 \tag{2.5.32}$$

where N_2 is the maximum possible number of inspections or equivalently maximum number of stages in the second phase. Also we should have $\alpha \le c_{m^*+1} \le c_{m^*+2} \le \cdots \le c_{n-1}$ according to the definition of the failure distribution F(w) given in Eqs. (2.5.2-3). $p'(t_{m^*+j})$ and $p''(t_{m^*+j})$ are given in Eqs. (2.5.15-18). Function G with a similar reasoning as for phase one is given by

$$G(t_{m^*+j}, p_{m^*+j}, t_{m^*+j-1}) = 1 - (1-p)e^{-c(t_{m^*+j}, p_{m^*+j})(t_{m^*+j-1}-x)}, \tag{2.5.33}$$

where

$$c(t_{m^*+j}, p_{m^*+j}) = - \frac{\operatorname{Log}_e\left(\frac{1-p_{m^*+j}}{1-p}\right)}{t_{m^*+j-x}} .$$

Function v with a similar reasoning as stated in phase one for Eqs. (2.5.22-24) is given by

$$v(t_{m^*+j}, p_{m^*+j}, c_{m^*+j+1}, t_{m^*+j})$$

$$= \operatorname{Max}\{1 - (1-p_{m^*+j})e^{c_{m^*+j+1}(t_{m^*+j}-t_{m^*+j-1})} \text{ or } 1 - e^{-\alpha t_{m^*+j-1}}\}, \tag{2.5.34}$$

and function $v(t_{m^*+j-1})$ is a particular value of v function at t_{m^*+j-1}.

That is

$$v(t_{m^*+j-1}) = 1 - e^{-\alpha t_{m^*+j-1}} . \quad (2.5.35)$$

The value of the parameter of the transformed exponential distribution which passes through $(t_{m^*+j-1}, p_{m^*+j-1})$ and (t_{m^*+j}, p_{m^*+j}), i.e., c_{m^*+j} is given by

$$c_{m^*+j} = - \frac{\mathrm{Log}_e\left(\frac{1-p_{m^*+j}}{1-p_{m^*+j-1}}\right)}{t_{m^*+j} - t_{m^*+j-1}} . \quad (2.5.36)$$

$K^{**}_{n=m^*+j}(T,1,\infty)$, given by Eq. (2.5.32) is the minimax total expected cost in the interval $[0,T]$ when m^* inspections are performed in the interval $[0,x)$ and j inspection performed in the interval $[x,T]$. Now the optimal loss L^{**} at $n^* = m^*+j^*$ is:

$$L^{**} = \mathrm{Min}\{K^{**}_{n=m^*+j}(T,1,\infty) \quad \text{or} \quad \bar{K}^{**}_{n=m^*+1}(T,1,\infty)\}, \quad (2.5,37)$$
$$j = 2,3,4, \ldots, N_2$$

where $\bar{K}^{**}_{n=m^*+1}(T,1,\infty)$ is the minimax total expected cost in the interval $[0,T]$ with m^* inspections in the interval $[0,x)$ and one inspection at time T and is given by Eq. (2.5.30). j^* is the optimal number of inspections in the interval $[x,T]$.

The policy $p^{**} = \{t_1^*, t_2^*, \ldots, t_n^*\}$ and the failure distribution F^{**} defined by (2.1.3-4) can also be found if needed by a backward recursive procedure from the computer printout (as explained in Chapter 3).

2.5.2 Model B

In this model we assume that the location x_1 and x_2 of the 100.p_1 th and 100.p_2 th percentiles respectively of the otherwise unknown failure distribution of the system are known. If m_1 is the number of checks or inspections before x_1 and m_2 is the number of inspections before x_2, we obtain the time sequence $t_0, \ldots, t_{m_1}, x_1, t_{m_1+1}, \ldots, t_{m_2}, x_2, t_{m_2+1}, \ldots, t_n$, according to which above points are arranged. We recode this sequence as

$$\{(t_{(i)}, p_{(i)}); \quad i = 0, 1, \ldots, n+2\} , \tag{2.5.38}$$

while

$$(t_0, p_0) = (0, 0) = (t_{(0)}, p_{(0)})$$

$$(x_1, p_1) = (t_{(m_1+1)}, p_{(m_1+1)}) \qquad c_1^* = c_{(m_1+1)}$$

$$(t_{n'-1}, p_{n'-1}) = (t_{(n')}, p_{(n')}) \qquad c_{n'-1} = c_{(n')}$$

$$(x_2, p_2) = (t_{(m_2+2)}, p_{(m_2+2)}), \qquad c_2^* = c_{(m_2+2)}$$

$$(t_{n-1}, p_{n-1}) = (t_{(n+1)}, p_{(n+1)}) \qquad c_{n-1} = c_{(n+1)}$$

$$(t_n, p_n) = (T, 1) = (t_{(n+2)}, p_{(n+2)}),$$

knowing these we can define F(w) with a series of transformed exponential distributions connected together as

$$F(w) = \begin{cases} 1 - e^{-c_{(1)} w} & \text{if } t_{(0)} = 0 \leq w \leq t_{(1)} \\ 1 - e^{-c_{(i)} (w-d_{(i)})} & \text{if } t_{(i-1)} \leq w \leq t_{(i)}, \ i<i\leq n+1 \\ 1 & \text{if } t_{(n+1)} \leq w \leq t_{(n+2)} = T \end{cases} \tag{2.5.39}$$

$$c_{(1)} < \alpha_1 = -\frac{\mathrm{Log}_e(1-p_1)}{x_1},$$

$$\alpha_2 = -\frac{\mathrm{Log}_e(\frac{1-p_2}{1-p_1})}{x_2-x_1} \geq \alpha_1 = -\frac{\mathrm{Log}_e(1-p_1)}{x_1}.$$

$$c_{(1)} \leq \cdots < c_{(m_1)} \leq c_{(m_1+1)} \leq \cdots \leq c_{(m_{2+1})} \leq c_{(m_2+2)} \leq \cdots \leq c_{(n+1)}$$

$$\alpha_1 < c_{(m_1+1)} < \alpha_2 \qquad c_{(m_2+2)} > \alpha_2$$

$$c_{(i)} = \frac{1}{t_{(i)} - t_{(i-1)}} \log_e(\frac{1-p_{(i-1)}}{1-p_{(i)}}), \quad i = 1,2, \ldots, n$$

$$d_{(i)} = \frac{1}{c_{(i)}} \log_e(1-p_{(i)}) + t_{(i)},$$

while we assumed

$$c_{(i)} = \begin{cases} c_i & \text{if} \quad 1 \leq i \leq m_1 \\ c_1^* & \text{if} \quad i = m_1 + 1 \\ c_{i-1} & \text{if} \quad m_1+2 \leq i \leq m_2+1 \\ c_2^* & \text{if} \quad i = m_2+2 \\ c_{i-2} & \text{if} \quad m_2+3 \leq i \leq n+2 \end{cases} . \qquad (2.5.40)$$

We observe that $F(w)$ passes through points $(0, 0)$, (x_1, p_1), (x_2, p_2) and possesses a jump from (t_{n-1}, p_{n-1}) to $(t_{n-1}^+, 1)$ (as allowed for an IFR distribution in [6, 22]).

As it was stated and proved form model A, Eq. (2.5.4), objective function L (2.4.2), given feasible points (t_i, p_i), i = 1, 2, ..., n-1, is indeed maximized by function F(w) defined above by Eqs. (2.5.39-40) which is a set of the transformed exponential distributions.

The feasible region of F(t) v.s.t. diagram for the set of IFR distributions which pass through points (0,0), (x_1, p_1), and (x_2, p_2) is shown in Fig. 6. Between t = 0 to t = x_1 the region is the area confined from above by $F_1(t) = 1 - e^{-\alpha_1 t}$, for the same reason stated for Model A. The region is confined from below by Max$\{F_2(t), 0.\}$, where $F_2(t)$ is the t.e.d. connecting ponts (x_1, p_1) and (x_2, p_2) and according to Eq. (2.5.5) is given by

$$F_2(t) = 1 - (1-p_1)\, e^{-\alpha_2(t-x_1)}, \tag{2.5.41}$$

$$\alpha_2 = -\frac{\mathrm{Log}_e\left(\frac{1-p_2}{1-p_1}\right)}{x_2 - x_1} \quad . \tag{2.5.42}$$

The reason for this is that any point (t_i, p_i) below $F_2(t)$ will be connected to point (x_1, p_1) by an IFR distribution which crosses t.e.d. $F_2(t)$ from below. But according to property 1 (2.3) no IFR distribution can cross a t.e.d. at two points (e.g. (x_1, p_1), (x_2, p_2)) first from below. From left and right the region is confined by t = 0 and t = x_1 respectively. Between t > x_1 and t = x_2, the region is confined from above by t.e.d. $F_2(t)$ since according to property 2-a (2.3), an IFR passes through three points (e.g. (x_1, p_1), (t_i, p_i), (x_2, p_2)) only if (t_i, p_i) lies on or below the t.e.d. through (x_1, p_1) and (x_2, p_2), i.e., $F_2(t)$. The region is confined from below by t.e.d. $F_1(t)$ which connects points (0, 0) to (x_1, p_1). The reason that (t_i, p_i) cannot be below $F_1(t) = 1 - e^{-\alpha_1 t}$ is that any IFR distribution which crosses at point $(t = x_1, p_1)$ should have a higher failure rate at this point than $1 - e^{-\alpha_1 \Delta}$. So no point (t_i, p_i) for $t > x_1$ can lie below $F_1(t)$. The region is confined from left and right by $t = x_1$ and $t = x_2$ respectively. Between $t > x_2$ and $t = T$ the region is confined from above by $F(t) = 1$ as it is obviously clear and from below by $F_2(t)$. The reason that (t_i, p_i) cannot be below $F_2(t)$ is that any IFR distribution which crosses at point (x_2, p_2) should have a higher failure rate at this point than $1 - e^{-\alpha_2 \Delta}$. So any point (t_i, p_i) for $t_i > x_2$ cannot lie below t.e.d. $F_2(t)$. The region is confined from left and right by $t = x_2$ and $t = T$ respectively. In addition to the above limitations the point (t_i, p_i), must satisfy the equations given in (2.5.39)

Maximization for given policy can now be carried out by dynamic programming methodology. The search extends over all $p_i = F(t_i)$, $i = 1, 2, \ldots, n$. As for Model A, we define J_i, as loss or maintenance cost on the interval between (i-1)th and (i)th inspection $[t_{i-1}, t_i]$, which is also a function of the failure probability at (i-1)th and ith inspection. That is

$$J_i = J_i\,(t_{i-1}, p_{i-1}, t_i, p_i) = [I + a(t_i - t_{i-1})]\, F^*(t_{i-1}) - a \int_{t_{i-1}}^{t_i} F^*(w)\, dw \qquad (2.5.43)$$

$$t_i < w < t_{i-1}, \quad i = 1,2,\ldots, n.$$

If $J_i^* = \underset{F(w)}{\text{Max}}\, J_i$, we obtain from Eqs. (2.5.39) that

$$J_i^* = J_i^*(t_{i-1}, p_{i-1}, t_i, p_i) = \text{Max}\, J_i(t_{i-1}, p_{i-1}, t_i, p_i) \qquad (2.5.44)$$

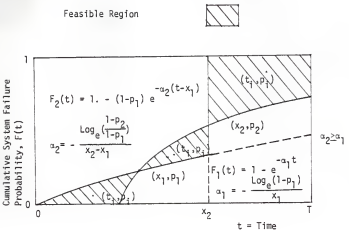

Fig. 6. The Feasible Region for Points (t_i, p_i) for Model B.

$$= [I + a(t_i - t_{i-1})]\, F^*(t_{i-1}) + a\, \frac{F^*(t_i) - F^*(t_{i-1})}{c_i}$$

if $i = 1, 2, \ldots, m_1, m_1+2, \ldots, m_2, m_2+2, \ldots, n-1$,

$$= [I + a(t_i - t_{i-1})]F^*(t_{i-1}) + a\left(\frac{(1-p_1) - F^*(t_{m_1})}{c_1^*} + \frac{F^*(t_{m_1+1}) - (1-p_1)}{c_{m_1+1}}\right)$$

if $i = m_1+1$,

$$= [I + a(t_{m_2+1} - t_{m_1})]\, F^*(t_{m_1}) + a\left(\frac{(1-p_1) - F^*(t_{m_1})}{c_1^*} + \frac{p_1-p_2}{\alpha_2} + \frac{F^*(t_{m_2+1}) - (1-p_2)}{c_{m_2+1}}\right)$$

if $i = m_2+1$ and $m_2 = m_1$,

$$= [I + a(t_i - t_{i-1})]F^*(t_{i-1}) + a\left(\frac{(1-p_2)-F^*(t_{m_2})}{c_2^*} + \frac{F^*(t_{m_2+1}) - (1-p_2)}{c_{m_2+1}}\right)$$

if $i = m_2+1$ and $m_2 \neq m_1$,

$$= [I + a(t_i - t_{i-1})]\, F^*(t_{i-1}) \qquad \text{if } i = n.$$

As mentioned for Model A, the first expressions on the right side of equations in (2.5.44), i.e., $[I + a(t_i - t_{i-1})]\, F^*(t_{i-1})$ are obviously the share of J_i from the first expression in the expected total maintenance cost L given by Eq. (2.4.2), i.e., $I \sum_{i=0}^{n-1} F^*(t_i) + a \sum_{i=0}^{n-1} F^*(t_i)(t_{i+1} - t_i)$. The second expression is, $a \int_{t_{i-1}}^{t_i} F^*(w)\, dw$, which is also the share of J_i from L, i.e.,

$a \int_0^{t_n=T} F^*(w)\, dw$. For values of i, except for $i = m_2+1$ when $m_2=0$, the calculation of the values of $a \int_{t_{i-1}}^{t_i} F^*(w)\, dw$ is the same as for model A. But for $i = m_1+1$ when $t_i \geq x_2$ we have

$$\int_{t_{i-1}}^{t_i} F^*(w)\, dw = \int_{t_{i-1}}^{x_1} F^*(w)\, dw + \int_{x_1}^{x_2} F^*(w)\, dw + \int_{x_2}^{t_i} F^*(w)\, dw. \tag{2.5.45}$$

Substituting the values of $F^*(w)$ from Eqs. (2.5.38-40) into equation (2.5.45) we have

$$\int_{t_{i-1}}^{t_i} F^*(w)\, dw = \int_{t_{i-1}}^{x_1} e^{-c_{(m_1+1)}(w - d_{(m_1+1)})}\, dw +$$

$$\int_{x_1}^{x_2} (1-p_1)\, e^{-\alpha_2(w - x_1)}\, dw +$$

$$\int_{x_2}^{t_i} e^{-c_{(m_2+3)}(w - d_{(m_2+3)})}\, dw$$

$$= \left(- \frac{e^{-c_{(m_1+1)}(w - d_{(m_1+1)})}}{c_{(m_1+1)}} \right)_{t_{i-1}}^{x_1} +$$

$$\left(- \frac{(1-p_1)\, e^{-\alpha_2(w - x_1)}}{\alpha_2} \right)_{x_1}^{x_2} +$$

$$\left\{-\frac{e^{-c_{(m_2+3)}(w-d_{(m_2+3)})}}{c_{(m_2+3)}}\right\}_{x_2}^{t_i} .$$

$$= \frac{e^{-c_{(m_1+1)}(t_{i-1}-d_{(m_1+1)})} - e^{-c_{(m_1+1)}(x_1-d_{(m_1+1)})}}{c_{(m_1+1)}}$$

$$+ \frac{(1-p_1)\, e^{-\alpha_2(x_1-x_1)} - (1-p_1)\, e^{-\alpha_2(x_2-x_1)}}{\alpha_2}$$

$$+ \frac{e^{-c_{(m_2+3)}(x_2-d_{(m_2+3)})} - e^{-c_{(m_2+3)}(t_i-d_{(m_2+3)})}}{c_{(m_2+3)}} .$$

$$= \frac{F^*(t_{(m_1)}) - F^*(t_{(m_1+1)})}{c_{(m_1+1)}} + \frac{(1-p_1)-(1-p_2)}{\alpha_2} +$$

$$\frac{F^*(t_{(m_2+2)}) - F^*(t_{(m_2+3)})}{c_{(m_2+3)}} ,$$

since transformed exponential distribution $F(w)$, which connects (p_1, x_1) and (p_2, x_2) has already been given by $F_2(t)$ as

$$F_2(t) = 1 - (1-p_1)\, e^{-\alpha_2(t-x_1)} ,$$

and also according to recodification (2.5.38) and (2.5.40) we have

$$F^*(t_{(m_1)}) = F^*(t_m)$$

$$F^*(t_{(m_1+1)}) = F^*(x_1) = (1-p_1)$$

$$F^*(t_{(m_2+2)}) = F^*(x_2) = (1-p_2)$$

$$F^*(t_{(m_2+3)}) = F^*(t_{m_2+1})$$

$$c(m_1+1) = c_1^*$$

$$c_{(m_2+3)} = c_{m_2+1},$$

so that

$$\int_{t_{i-1}}^{t_i} F^*(w) = \frac{F^*(t_{m_1}) - (1-p_1)}{c_1^*} + \frac{p_2-p_1}{\alpha_2} + \frac{(1-p_2) - F^*(t_{m_2+1})}{c_{m_2+1}}$$

In Model B also we define $K_i\ (t_i,\ p_i,\ c_{i+1})$ as minimax expected maintenance cost during time interval $[0,\ t_i]$ with $F(t_i) = p_i$ when the parameter of the t.e.d. connecting $(t_i,\ p_i)$ to $(t_{i+1},\ p_{i+1})$ is less than or equal to a certain amount c_{i+1}. We can write

$$K_1\ (t_1,\ p_1,\ c_2) = J_1^*(t_0,\ p_0,\ t_1,\ p_1) = J_1^*\ (0,\ 0,\ t_1,\ p_1)$$

$$K_i(t_i,\ p_i,\ c_{i+1}) = K_{i-1}\ (t_{i-1},\ p_{i-1},\ c_i) + J_i^*\ (t_{i-1},\ p_{i-1},\ t_i,\ p_i)$$

$$= \sum_{j=1}^{i} J_j^*\ (t_{j-1},\ p_{j-1},\ t_j,\ p_j). \qquad (2.5.46)$$

Let $K_n(T)$ signify the loss for the entire time period T at F(T) = 1 and t.e.d. with parameter $c_{n+1} = \infty$

$$K_n(T) = K_n\ (t_n = T,\ p_n = 1,\ c_{n+1} = \infty)$$

$$= K_{n-1}(t_{n-1},\ p_{n-1},\ c_n) + J_n^*(t_{n-1},\ p_{n-1},\ t_n,\ p_n)$$

$$= L. \qquad (2.5.47)$$

In this model as for Model A, we have three state variable, i.e., t_i, p_i and c_{i+1}. There are two control variable, i.e., t_{i-1}, p_{i-1}. Minimaxation will be carried out in three phases. In the first phase the minimax expected maintenance cost from time $t_0 = 0$ to the time of first inspection in the interval $[x_1, x_2)$, i.e., $K_{m_1+1}\ (t_{m_1+1},\ p_{m_1+1},\ c_{m_1+2})$ and the minimax expected maintenance cost from time $t_0 = 0$ to the time of first inspection in the interval $[x_2, T]$, i.e., $K_{m_2+1}\ (t_{m_2+1},\ p_{m_2+1},\ c_{m_2+2})$ will be obtained.

In the second phase $K_{m_2+1}\ (t_{m_2+1},\ p_{m_2+1},\ c_{m_2+2})$, i.e., the minimax loss from $t_0 = 0$ to the time of the first inspection $t_{m_2+1} \geq x_2$ will be optimzied for all possible values of t_{m_2+1}, p_{m_2+1} and $c_{m_2+2} = c_{m_2+1}$. In the third and final phase $K_n\ (t_{n=m_2+j} = T,\ p_{n=m_2+j} = 1,\ c_{n+1} = \infty) = L$ will be minimaxed. In the second phase the minimaxation will be achieved, taking the minimax K_{m_1+1} values for state variables at time $t = t_{m_1+1}$ as the K values for the first stage of the second phase and in third phase also in the same way the minimax K_{m_2+1} values for state variables at time $t = t_{m_2+1}$ are taken as the K values for the first stage of the third phase. The value of L will be calculated for all possible number of stages in phase 1, phase two and phase 3 and the optimum value of L will be found by a simple search among these values.

First optimization phase - Let the minimax optimized loss for time period $(0, t_i]$ at $F(t_i)=p_i$ when the parameter of the t.e.d. connecting (t_i, p_i) to (t_{i+1}, p_{i+1}) is equal to a certain amount c_{i+1}, be

$$K_i^*(t_i, p_i, c_{i+1}) = \underset{P}{\text{Min}}\ \underset{F}{\text{Max}}\ K_i\ (t_i, p_i, c_i) \tag{2.5.48}$$

where P is the inspection policy (2.2.1) and F is failure distribution given by Eqs. (2.5.39). Now by using dynamic programming procedure we obtain

$$K_0^*\ (t_0=0,\ p_0=0,\ c_{0+1}) = 0, \tag{2.5.49}$$

$$\underset{0<t_1\le x_1}{K_1^*\ (t_1, p_1, c_2)} = J_1^*\ (0,0,t_1,p_1) = I + at_1 + a\frac{F^*(t_1) - 1}{c_1^*}\ ,\ c_1^* < \alpha_1$$

$$p''(t_1) < p_1 < p'(t_1)$$

$$K_2^*(t_2, p_2, c_3) = \underset{0<t_2\le x_1}{\text{Min}}\quad \underset{\substack{0<t_1<t_2 \\ v(t_2,\, p_2,\, c_3,\, t_1)\ \le\ p_1\ \le\ G(t_2,\, p_2,\, t_1)}}{\text{Max}}\{K_1^*(t_1, p_1, c_2) + J_2^*(t_1, p_1, t_2, p_2)\}$$

$$p''(t_2) \le p_2 \le p'(t_2)$$

$$K_i^*\ (t_i, p_i, c_{i+1}) = \underset{0<t_i\le x_1}{\text{Min}}\quad \underset{\substack{0<t_{i-1}<t_i \\ v(t_i,p_i,c_{i+1},t_{i-1})<p_{i-1}\le G(t_i,p_i,t_{i-1})}}{\text{Max}}\{K_{i-1}^*(t_{i-1},p_{i-1},c_i) + J_i^*(t_{i-1},p_{i-1},t_i,p_i)\}$$

$$p''(t_i)<p_i<p'(t_i) \qquad\qquad i = 3,4,\ \ldots,\ m_1,$$

$$\underset{x_1\le t_{m_1+1}\le x_2}{K_{m_1+1}^*\ (t_{m_1+1}, p_{m_1+1}, c_{m_1+1})}$$

$$p''(t_{m_1+1}) < p_{m_1+1} < p'(t_{m_1+1})$$

$$= \underset{\substack{0<t_{m_1}<x_1}}{\text{Min}} \quad \underset{\substack{v(t_{m_1+1}, p_{m_1+1}, t_{m_1}) \leq p_{m_1} \leq G_0(t_{m_1})}}{\text{Max}} \{K^*_{m_1}(t_{m_1}, p_{m_1}, c_1^*) + J^*_{m_1+1}(t_{m_1}, p_{m_1}, t_{m_1+1}, p_{m_1+1})\}$$

$$m_1 = 0, 1, 2, \ldots, N_1,$$

$$c_1^* = -\frac{\text{Log}_e\left(\frac{1-p_1}{1-p_{m_1}}\right)}{x_1 - t_{m_1}}$$

$$\bar{K}_{m_2+1}(t_{m_2+1}, p_{m_2+1}, c_{m_2+1})$$

$$x_2 \leq t_{m_2+1} \leq T$$

$$p''(t_{m_2+1}) < p_{m_2+1} < p'(t_{m_2+1})$$

$$= \underset{\substack{0<t_{m_1}<x_1}}{\text{Min}} \quad \underset{\substack{v(t_{m_j+1}, p_{m_1+1}, t_{m_1}) \leq p_{m_1} \leq G_0(t_{m_1})}}{\text{Max}} \{K^*_{m_1}(t_{m_1}, p_{m_1}, c_1^*) + J^*_{m_2+1}(t_{m_1}, p_{m_1}, t_{m_1+1}, p_{m_1+1})\}$$

$$m_1 = 0, 1, 2, \ldots, N_1,$$

where N_1 is the maximum possible number of inspections in the interval $(0, x_1)$ and $c_1 \leq c_2 \leq \cdots \leq c_1^* \leq c_{m_1+1}, \cdots \leq c_{m_2} \leq c_2^* \leq c_{m_2+1} \leq \cdots \leq c_{n-1}$ since the F(w) distribution is IFR. Also because at point (x_1, p_1), F(w) crosses $F(u) = 1 - e^{-\alpha_1 u}$ from below, we should have $\alpha_1 < c_1^* \leq c_{m_1+1}$. $p'(t_i)$ and $p''(t_i)$ are defined the same way as in Model A (See Fig. 5.). But their values are given by (See Fig. 6.)

$$p'(t_i) = \begin{cases} 1 - e^{-\alpha_1 t_i} & 0 \leq t_i < x_1 \\ 1-p\ (1-p_1)e^{-\alpha_2(t_i-x_1)} & x_1 \leq t_i \leq x_2\ , \\ 1 & x_2 \leq t_i \leq T \end{cases} \tag{2.5.50}$$

$$p''(t_i) = \begin{cases} \mathrm{Max}\{0,\ 1 - (1-p_1)\ e^{-\alpha_2(t_i-x_1)}\} & 0 \leq t_i \leq x_1 \\ 1 - e^{-\alpha_1 t_i} & x_1 \leq t_i \leq x_2 \\ 1 - (1-p_1)\ e^{-\alpha_2(t_i-x_1)} & x_2 \leq t_i < T \end{cases} \tag{2.5.51}$$

and at $t_i = T$ we have $p''(T) = 1$.

The definition of G and v functions for Eqs. (2.5.49) are the same as for Model A and with a similar reasoning we have (See Fig. 5-6):

$$G(t_i,\ p_i,\ t_{i-1}) = 1. - (1 - p_i)\ e^{c(t_i,p_i)(t_i - t_{i-1})} \quad , \qquad 0 < t_i \leq x_1 \tag{2.5.52}$$

where

$$c(t_i,\ p_i) = - \frac{\mathrm{Log}_e(1-p_i)}{t_i} \quad . \tag{2.5.53}$$

$G_0(t_{m1})$ is the particular value of G function at $t_{i-1} = t_{m_1}$ and its value is (See Fig. 5-6).

$$G_0(t_{m_1}) = 1 - e^{-\alpha_1 t_{m_1}} , \tag{2.5.54}$$

also

$$v(t_i, p_i, c_{i+1}, t_{i-1}) = \text{Max}\{1 - (1 - p_i)\, e^{c_{i+1}(t_i - t_{i-1})} \text{ or } 0\} ,$$

$$0 < t_i \leq x_1 \tag{2.5.55}$$

and $v(t_{m_1+1}, p_{m_1+1}, t_{m_1})$ is the particular value of v function at t_{m_1} and it's value is

$$v(t_{m_1+1}, p_{m_1+1}, t_{m_1}) = \text{Max}\{1 - (1 - p_1)e^{-c_{m_1+1}(t_{m_1} - x_1)} , 0\} . \tag{2.5.57}$$

The value of c_i, i.e., the parameter of the t.e.d. connecting point (t_i, p_i) to (t_{i-1}, p_{i-1}), according to Eq. (2.5.5) is given by

$$c_i = - \frac{\text{Log}_e(\frac{1-p_i}{1-p_{i-1}})}{t_i - t_{i-1}} , \tag{2.5.58}$$

and the value of c_{m_1+1} is given by

$$c_{m_1+1} = - \frac{\text{Log}_e(\frac{1-p_{m_1+1}}{1-p_1})}{t_{m_1+1} - x_1} . \tag{2.5.59}$$

We now choose for each (t_{m_1+1}, p_{m_1+1}) that $m_1 = m_1^*$ which renders $K^*_{m_1+1}$ a minimum. That is

$$K^{**}_{m_1^*+1}(t_{m_1^*+1}, p_{m_1^*+1}, c_{m_1^*+1}) = \text{Min}\{K^*_{m_1+1}(t_{m_1+1}, p_{m_1+1}, c_{m_1+1})\} .$$

$$m_1 = 0, 1, \ldots, N_1. \tag{2.5.60}$$

$K^{**}_{m_1^*+1}(t_{m_1^*+1}, p_{m_1^*+1}, c_{m_1^*+1})$ is the minimax expected total maintenance cost from time $t_0 = 0$ to $t_{m_1^*+1}$ where $t_{m_1^*+1}$ is larger or equal to x_1 with $F(t_{m_1^*+1}) = p_{m_1^*+1}$. Also m_1^* is the optimum number of checks or inspections before time $t = x_1$.

Second optimization phase - The iterative procedure continues taking the $K^{**}_{m_1^*+1}$ values at each point $(t_{m_1^*+1}, p_{m_1^*+1}), x_1 < t_{m^*+1} < x_2$, as the minimax expected maintenance costs up to that point for the first stage of the second phase of the optimization. That is

$$K^{**}_{m_1^*+2}(t_{m_1^*+2}, p_{m_1^*+2}, c_{m_1^*+3}) \tag{2.5.61}$$

$$x_1 < t_{m_1^*+2} < x_2$$

$$p''(t_{m_1^*+2}) < p_{m_1^*+2} < p'(t_{m_1^*+2})$$

$$= \min_{x_1 \le t_{m_1^*+1} < t_{m_1^*+2}} \ \max_{v(t_{m_1^*+2}, p_{m_1^*+2}, c_{m_1^*+3}, t_{m_1^*+1}) < p_{m_1^*+1} < G(t_{m_1^*+2}, p_{m_1^*+2}, t_{m_1^*+1})} \{K^{**}_{m_1^*+1}(t_{m_1^*+1}, p_{m_1^*+1}, c_{m_1^*+1})$$

$$+ J^*_{m_1^*+2}(t_{m_1^*+1}, p_{m_1^*+1}, t_{m_1^*+2}, p_{m_1^*+2})\}$$

$$c^*_{m_1+1} < \alpha_2$$

$$K^{**}_{m_1^*+j}(t_{m_1^*+j}, p_{m_1^*+j}, c_{m_1^*+j+1})$$

$$x_1 < t_{m_1^*+1} < x_2$$

$$p''(t_{m_1^*+j}) < p_{m_1^*+j} < p'(t_{m_1^*+j})$$

$$= \min_{x_1 < t_{m_1^*+j-1} < t_{m_1^*+j}} \ \max_{v(t_{m_1^*+j}, p_{m_1^*+j}, c_{m_1^*+j+1}, t_{m_1^*+j-1}) < p_{m_1^*+j-1} < G(t_{m_1^*+j}, p_{m_1^*+j}, t_{m_1^*+j-1})} \{K^{**}_{m_1^*+j-1}(t_{m_1^*+j-1}, p_{m_1^*+j-1}, c_{m_1^*+j})$$

$$+ J^{*}_{m_1^*+j}(t_{m_1^*+j-1},\ p_{m_1^*+j-1},\ t_{m_1^*+j},\ p_{m_1^*+j})\} \qquad j = 3,4,\ \ldots,\ m_2-m_1^* ,$$

$$K^{**}_{m_2+1}(t_{m_2+1},\ p_{m_2+1},\ c_{m_2+1}) \tag{2.5.62}$$

$$x_2 \le t_{m_2+1} \le T$$

$$= \underset{x_1 \le t_{m_2} < x_2}{\text{Min}} \quad \underset{v(t_{m_2+1},\ p_{m_2+1},\ c_{m_2+1},\ t_{m_2}) < p_{m_2} < G_0(t_{m_2})}{\text{Max}} \{K^{**}_{m_2}(t_{m_2}, p_{m_2}, c_2^*) + J^{*}_{m_2+1}(t_{m_2}, p_{m_2}, t_{m_2+1},\ p_{m_2+1})\}$$

$$m_2 = 1 + m_1^*,\ 2 + m_1^*,\ \ldots,\ N_2 + m_1^*,$$

$$c_2^* = -\frac{\text{Log}_e(\frac{1-p_2}{1-p_{m_2}})}{x_2 - t_{m_2}} \ .$$

N_2 is the maximum possible number of inspections in the interval $[x_1,\ x_2]$ and $c_{m_1^*+1} \le c_{m_1^*+2},\ \ldots,\ \le c_{m_2+1}$ since failure distribution $F(w)$ is IFR. $p'(t_i)$ and $p''(t_i)$ are given by (2.5.2.13) and (2.5.2.14) respectively. The value of G function is given by (See Fig. 5.6)

$$G(t_{m_1^*+j},\ p_{m_1^*+j},\ t_{m_1^*+j-1}) = 1 - (1-p_1)\ e^{-c(t_{m_1^*+j},\ p_{m_1^*+j})(t_{m_1^*+j-1} - x)} ,$$

$$x_1 < t_{m_1^*+j-1} < t_{m_1^*+j} < x_2 \tag{2.5.63}$$

where

$$c(t_{m_1^*+j},\ p_{m_1^*+j}) = -\frac{\text{Log}_e(\frac{1 - p_{m_1^*+j}}{1 - p_1})}{t_{m_1^*+j} - x_1} \ . \tag{2.5.64}$$

$G_0(t_{m_2})$ is the particular value of G function at point t_{m_2} (See Fig. 5.6.)

$$G_0(t_{m_2}) = 1 - (1 - p_2)\, e^{-\alpha_2(t_{m_2} - x_1)} . \qquad (2.5.65)$$

The value of v function is given by

$$v(t_{m_1^*+j},\ p_{m_1^*+j},\ c_{m_1^*+j+1},\ t_{m_1^*+j-1}) = \text{Max}\{1 - (1-p_{m_1^*+j})\, e^{c_{m_1^*+j+1}(t^*_{m_1+j} - t_{m_1^*+j-1})}$$

$$\text{or } 1 - e^{-\alpha_1 t_{m_1^*+j-1}}\} \qquad (2.5.66)$$

$$x_1 < t_{m_1^*+j} < x_2.$$

$v(t_{m_2+1},\ p_{m_2+1},\ t_{m_2})$ is the particular value of v function at t_{m_2} and its value is (See Fig. 5.6.)

$$v(t_{m_2+1},\ p_{m_2+1}, t_{m_2}) = \text{Max}\{1 - (1 - p_2)\, e^{c_{m_2+1}(x_2 - t_{m_2})} \text{ or } 1 - e^{-\alpha_1 t_{m_2}}\},$$

$$(2.5.67)$$

where

$$c_{m_2+1} = - \frac{\text{Log}_e\left(\dfrac{1 - p_{m_2+1}}{1 - p_2}\right)}{t_{m_2+1} - x_2} . \qquad (2.5.68)$$

The value of $c_{m_1^*+j}$, i.e., the parameter of t.e.d. connecting point $(t_{m_1^*+j},\ p_{m_1^*+j})$ to $(t_{m_1^*+j-1},\ p_{m_1^*+j-1})$, according to Eq. (2.5.5) is given by

$$c_{m_1^*+j} = -\frac{\operatorname{Log}_e\left(\dfrac{1-p_{m_1^*+j}}{1-p_{m_1^*+j-1}}\right)}{t_{m_1^*+j} - t_{m_1^*+j-1}} \quad . \qquad (2.5.69)$$

Now we choose for each $(t_{m_2+1},\ p_{m_2+1})$ that $m_2 = m_2^* = m_1^* + j^*$ which renders $K^{**}_{m_2+1}$ a minimum. That is

$$\begin{aligned} K^{***}_{m_2^*+1}(t_{m_2^*+1},\ p_{m_2^*+1},\ c_{m_2^*+1}) &= \operatorname{Min}\{K^{**}_{m_2+1}(t_{m_2+1},\ p_{m_2+1},\ c_{m_2+1}) \\ x_2 \le t_{m_2^*+1} < T \qquad & \qquad m_2 = 1 + m_1^*,\ 2 + m_1^*,\ \ldots,\ N_2 + m_1^* \\ & \text{or } \bar{K}_{m_2+1}(t_{m_2+1},\ p_{m_2+1},\ c_{m_2+1})\} \\ & \qquad m_2 = m_1^* \text{ or } j = 0, \end{aligned} \qquad (2.5.70)$$

$$\begin{aligned} \bar{K}_{m_2^*+1} = {}_n(T,1,\infty) &= \operatorname{Min}\{K^{**}_{m_2+1}(T,1,\infty) \\ & \qquad m_2 = 1 + m_1^*, 2 + m_1^*, \ldots,\ N_2 + m_1^* \\ \text{or} \quad & \bar{K}_{m_2+1}(T,\ 1,\infty)\}\ , \\ & \qquad m_2 = m_1 = 0,1,2,\ldots,\ N_1 \end{aligned} \qquad (2.5.71)$$

where $\bar{K}_{m_2+1}$ is calculated in the first optimization phase for $m_2 = m_1$ or equivalently when no inspection is performed in the interval $[x_1,\ x_2)$. $K^{***}_{m_2^*+1}\ (t_{m_2^*+1},\ p_{m_2^*+1},\ c_{m_2^*+1})$ is the minimax expected total maintenance cost in the interval $(0, t_{m_2^*+1}]$ where $T > t_{m_2^*+1} \ge x_2$ with $F(t_{m_2^*+1}) = p_{m_2^*+1}$. Also $m_2^* = m_1^* + j^*$ where m_2^* is the optimal total number of inspections at the end of second stage and j^* is the optimal number of inspections only in the second stage.

Third optimization phase - The iterative procedure continues taking the $K^{***}_{m_2^*+1}$ values at each point $(t_{m_2^*+1}, p_{m_2^*+1}), x_2 \leq t_{m_2^*+1} < T$ as the minimax expected total maintenance cost up to that point for the first stage of the third phase of optimization. That is

$$K^{***}_{m_2^*+2}\ (t_{m_2^*+2},\ p_{m_2^*+2},\ c_{m_2^*+3})$$

$$x_2 < t_{m_2^*+2} < T$$

$$p''(t_{m_2^*+2}) < p_{m_2^*+2} < p'(t_{m_2^*+2})$$

$$= \min_{x_2 \leq t_{m_2^*} < t_{m_2^*+2}} \quad \max_{v(t_{m_2^*+2},\ p_{m_2^*+2},\ c_{m_2^*+3},\ t_{m_2^*+1}) < p_{m_2^*+1} < G(t_{m_2^*+2},\ p_{m_2^*+2},\ t_{m_2^*+1})} \{K^{***}_{m_2^*+1}(t_{m_2^*+1}, p_{m_2^*+1},\ c_{m_2^*+1})$$

$$+\ J^*_{m_2^*+2}\ (t_{m_2^*+1},\ p_{m_2^*+1},\ t_{m_2^*+2},\ p_{m_2^*+2})\},$$

and

$$K^{***}_{m_2^*+\ell}\ (t_{m_2^*+\ell},\ p_{m_2^*+\ell},\ c_{m_2^*+\ell+1})$$

$$x_2 < t_{m_2^*+\ell} < T$$

$$p''(t_{m_2^*+\ell}) < p_{m_2^*+\ell} < p'(t_{m_2^*+\ell})$$

$$= \min_{x_2 < t_{m_2^*+\ell-1} < t_{m_2^*+\ell}} \quad \max_{v(t_{m_2^*+\ell}, p_{m_2^*+\ell}, c_{m_2^*+\ell+1}, t_{m_2^*+\ell-1}) < p_{m_2^*+\ell-1} < G(t_{m_2^*+\ell}, p_{m_2^*+\ell}, t_{m_2^*+\ell-1})} \{K^{***}_{m_2^*+\ell-1}(t_{m_2^*+\ell-1},\ p_{m_2^*+\ell-1},\ c_{m_2^*+\ell})$$

$$+ J^{*}_{m_2^*+\ell}(t_{m_2^*+\ell-1}, p_{m_2^*+\ell-1}, t_{m_2^*+\ell}, p_{m_2^*+\ell})\}$$

$$\ell = 3,4,\ldots,\ n-m_2^*-1. \qquad (2.5.72)$$

In particular, the term for the total optimal loss with $n = m_2^*+\ell$ inspections, is

$$K^{**}_{n=m_2^*+\ell}\ (T,1,\infty)$$

$$= \min_{\substack{x_2<t_{m_2^*+\ell-1}<T \\ v(t_{m_2^*+\ell-1}) < p_{m_2^*+\ell-1} < G(t_{m_2^*+\ell-1})}} \operatorname{Max}\{K^{***}_{m_2^*+\ell-1}(t_{m_2^*+\ell-1}(t_{m_2^*+\ell-1}, p_{m_2^*+\ell-1}, \infty)$$

$$+ J^{*}_{n=m_2^*+\ell}\ (t_{m_2^*+\ell-1},\ p_{m_2^*+\ell-1}, t_{n=m_2^*+\ell}=T, p_{n=m_2^*+\ell}=1)\}$$

$$\ell = 1,3,\ldots,N_3, \qquad (2.5.73)$$

where N_3 is the maximum possible number of inspections and G and v functions are given by (See Fig. 5-6.)

$$G(t_{m_2^*+\ell}, p_{m_2^*+\ell}, t_{m_2^*+\ell-1}) = 1 - (1-p_2)e^{-c(t_{m_2^*+\ell}, p_{m_2^*+\ell})(t_{m_2^*+\ell-1}-x_2)}, \qquad (2.5.74)$$

where

$$c(t_{m_2^*+\ell}, p_{m_2^*+\ell}) = -\frac{\operatorname{Log}_e\left(\frac{1-p_{m_2^*+\ell}}{1-p_2}\right)}{t_{m_2^*+\ell}-x_2}\ , \qquad (2.5.74)$$

and $G(t_{m_2^*+\ell-1})$ is particular value of G function at $t_{n-1} = t_{m_2^*+\ell-1}$ and is given by

$$G(t_{m_2^*+\ell-1}) = 1, \qquad (2.5.75)$$

and

$$v(t_{m_2^*+\ell}, p_{m_2^*+\ell}, c_{m_2^*+\ell+1}, t_{m_2^*+\ell-1})$$

$$= \text{Max}\{1 - (1 - p_{m_2^*+\ell})\, e^{c_{m_2^*+\ell+1}(t_{m_2^*+\ell} - t_{m_2^*+\ell-1})} \quad \text{or} \quad 1 - (1 - p_2)\, e^{-\alpha_2(t_{m_2^*+\ell-1} - x_2)}\}, \qquad (2.5.76)$$

and at $t_{n-1} = t_{m_2^*+\ell-1}$ we have

$$v(t_{m_2^*+\ell-1}) = 1 - (1-p_2)\, e^{-\alpha_2(t_{m_2^*+\ell-1} - x_2)} .$$

The value of $c_{m_2^*+\ell}$ can be found according to Eq. (2.5.5) for $c_n \neq c_{m_2^*+\ell}$ by

$$c_{m_2^*+\ell} = -\frac{\text{Log}_e\left(\frac{1-p_{m_2^*+\ell}}{1-p_{m_2^*+\ell-1}}\right)}{t_{m_2^*+\ell} - t_{m_2^*+\ell-1}} \qquad \ell = 2, \ldots, N_3, \qquad (2.5.77)$$

subject to $\alpha_2 < c_{m_2^*+1} \leq c_{m_2^*+2} \leq \cdots \leq c_{n-1=m_2^*+\ell-1}$ since the failure distribution should be IFR and $c_{m_2^*+1}$ is given by Eq. (2.5.68) for $m_2 = m_2^*$.

The optimal loss L^{**} at $n^* = m_2^* + \ell^* = m_1^* + j^* + \ell^*$ where n^* is the optimal total number of inspection in the interval $[0,T]$ and ℓ^* is the optimum number of inspections in the interval $[x_2,T]$ is then found from

$$L^{**} = \text{Min}\left\{ \begin{matrix} K^{**}(T,1,\infty)_{n=m_2^*+\ell} & \text{or} & \bar{K}(T,1,\infty)_{n=m_2^*+1} \\ & \ell = 2,3, \ldots, N3 & \end{matrix} \right\}, \qquad (2.5.2.78)$$

where $\bar{K}(T,1,\infty)_{n=m_2^*+1}$, the total expected loss if $\ell = 1$, is given by Eq. (2.5.71).

The P** and failure distribution F** defined by Eqs. (2.1.3.4) can also be found if needed by a backward recursive procedure from the computer print out.

2.6 DISCUSSION

In this chapter the mathematical meaning of a minimax policy was stated followed by the statement of the present problem which consists of two models A and B according to having information about one point or two points of an increasing failure rate (IFR) distribution of a system respectively. The special properties of IFR distribution utilized in the logic of the formulation of the model A and B were proved and explained by property No 1 and property No 2.a and 2.b. The form of the objective function suitable for a recursive relationship was presented and derived. Both model A and model B have been formulated by functional equations of dynamic programming. Three state variables, i.e., the last inspection time t_i, the cumulative failure probability of the system $F(t_i) = p_i$ up to time t_i and the parameter of a transformed failure distribution passing through points (t_i, p_i) and (t_{i+1}, p_{i+1}) or c_{i+1} represent the state of the system. The control variables are t_{i-1}, the timing of the previous inspection and p_{i-1}, the failure probability of the system at t_{i-1}. The stages represent the number of inspections which ranges from 2 to maximum possible number of inspections in the interval [0,T]. The minimax policy first maximizes at each stage and for a fixed state vector (t_i, p_i, c_{i+1}), the total expected loss from initial state vector $(t_0=0, p_0=0, c_{0+1})$ to the final state (t_i, p_i, c_{i+1}) by the choice of the control variable p_{i-1} and then minimizes the total expected loss by the choice of control variable t_{i-1}. As it is clear the number of stages should be also optimized.

This has been done in model A in two phases, i.e., in phase one the optimal number of inspections or stages in the interval, $[0,x_1)$ is obtained and in the second and final phase the total optimum number of inspections and their timing is obtained. In model B optimization has been accomplished in three phases where the optimal number of inspections in the intervals $[0,x_1)$, $[x_1,x_2)$, $[x_2,T]$ are obtained.

In Chapter 3 the computational details and procedures will be presented.

CHAPTER 3

COMPUTATIONAL PROCEDURE

3.1 INTRODUCTION

In this chapter the numerical procedure for the computer program is presented for each of the two models, i.e., Model A and Model B. For each model first the range of the possible values of state and control variables is stated and then the computational and numerical procedures is followed step by step with reference to the computer program in Appendix B.

3.2 MODEL A

In Chapter 2, there are three state variables and two control variables in the formulation of the problem. The state variables are:

1. The inspection or checking time of the ith inspection or checking denoted by t_i.
2. The cumulative failure probability of the system at ith inspection denoted by p_i.
3. The parameter of the transformed exponential failure distribution connecting point (t_i, p_i) to point (t_{i+1}, p_{i+1}) on the $\underline{F}$ v.s. $\underline{t}$ diagram denoted by c_{i+1}.

The control variables are:

1. The inspection time of the (i-1)th inspection denoted by t_{i-1}.
2. The cumulative failure probability of the system at (i-1)th inspection denoted by p_{i-1}.

t_i and t_{i-1} according to the assumption f in (2.2) are discrete values ranging from 0 to T the maximum life time of the system. p_i and p_{i-1} have continuous values whose range is given by p' and p" functions defined in Chapter 2. c_{i+1} is also continuous and the range of its values is given by

$$R_1 \leq c_{i+1} \leq R_2 \quad , \tag{3.2.1}$$

where the values of R_1 and R_2 are given by

$$R_1(t_i,p_i) = \begin{cases} -\dfrac{\mathrm{Log}_e(1-p_i)}{t_i} & 0 < t_i \leq x \\[2ex] \dfrac{\mathrm{Log}_e(\frac{1-p_i}{1-p})}{t_i-x} & x < t_i \leq T \end{cases} \quad , \tag{3.2.2}$$

and

$$R_2(t_i,p_i) = \begin{cases} -\dfrac{\mathrm{Log}_e(\frac{1-p}{1-p_i})}{x-t_i} & 0 < t_i < x \\[2ex] -\dfrac{\mathrm{Log}_e(\frac{\varepsilon}{1-p_i})}{t_{i+1}-t_i} & x \leq t_i < T \\ & \varepsilon = 1. \times 10^{-25} \end{cases} \quad . \tag{3.2.3}$$

The reason for the above values for R_1 and R_2 is based on the properties of exponential and IFR failure distributions explained in (2.3) and on the same line of reasoning used in (2.5.2) for stablishing the feasible region for IFR distributions passing through points (0,0), (x_1,p_1) and (x_2,p_2) on the F v.s. t diagram (See Fig. 6) with the difference that here the feasible region is given for the IFR distributions which pass through points (0,0), (t_i,p_i) and (x,p) for t_i in the time interval (0,x) and through points (x,p), (t_i,p_i) and (T,1) for t_i in the time interval (x,T).

The input vector elements are:

a = The cost of undetected failure per unit time.

I = The cost of every inspection with the exception of the inspection cost at t = 0 where I(0) = 0.

x = the time at which the cumulative failure probability of the system is known.

p = The known cumulative failure probability of the system at time x

$T' = \{t_0 = 0, t_1, t_2, \ldots, x, \ldots t_N = T\}$

= The set of possible inspection times.

PN_1 = Number of increments of p_i & p_{i-1} in the feasible region at t_i & t_{i-1} respectively

AL_1 = Number of increments of c_{i+1} in the feasible range given by Eqs. (3.1.1-3).

The computational procedure consist of the following steps;

Phase 1:

1. Divide the feasible range of p_i values, given by p' and p" functions in Eqs. (2.5.15-18), at each time t_i where t_i is an element of the set T', into PN_1 increments.

2. Divide the feasible range of c_{i+1} values, given by R_1 and R_2 functions in Eqs. (3.2.1-3), at each point (t_i, p_i) on the F v.s. t diagram, in to AL_1 increments.

3. Assume no inspection in the time interval (0,T) with only one inspection at time T, i.e., set the stage number m equal to 0.

4. Calculate the maximum expected total maintenance cost when m inspections are perfromed in the interval (0,x), i.e., $\bar{K}^*_{n=m+1}(T,1,\infty)$ and

$K^*_{m+1}(t_{m+1}, p_{m+1}, c_{m+1})$ for all possible values of t_{m+1}, p_{m+1} and c_{m+1} from Eq. (2.5.7) for $i = m+1$ and set of equations (2.5.14).

5. Set current m = old m+1 and calculate $K^*_i(t_i, p_i, c_{i+1})$ given by Eq. (2.5.14) when $i = m$ and for all possible values of t_i, p_i and c_{i+1}.

6. Repeat step 4 and compare the $\bar{K}^*_{n=m+1}(T,1,\infty)$ and $K^*_{m+1}(t_{m+1}, p_{m+1}, c_{m+1})$ values for the old m values with their values for current m respectively and choose that value of $\bar{K}^*_{n=m+1}$ and K^*_{m+1} for each point (t_m, p_m) which is smaller and discard the larger values.

7. Compare the maximum value of m, i.e., N_1 with the current m. If $N_1 \geq m+1$, go back to step 5 and if $N_1 < m+1$ go to step 8.

8. Take that value of $\bar{K}^*_{n=m+1}(T,1,\infty)$ and $K^*_{m+1}(t_{m+1}, p_{m+1}, c_{m+1})$ at each point (t_{m+1}, p_{m+1}) which is minimum for all values of $m = 0, 1, \ldots, N_1$ and denote them as $\bar{K}^{**}_{n=m^*+1}(T,1,\infty)$ and $K^{**}_{m^*+1}(t_{m^*+1}, p_{m^*+1}, c_{m^*+1})$ respectively. m^* is the optimum number of inspections in the time interval $(0, x)$.

Phase 2

9. Consider the $K^{**}_{m+1}(t_{m^*+1}, p_{m^*+1}, c_{m^*+1})$ values as the minimax expected maintenance cost upto the time of the first inspection in the time interval $[x,T)$ or the first stage $j = 1$ of the second phase.

10. Calculate $K^{**}_{n=m^*+j}(T,1,\infty)$ from Eq. (2.5.32) for $j = 1$.

11. Set current j = Old j+1. Compare the maximum possible number of inspections in the interval $[x,T)$, i.e., N_2 with the current j. If $N_2 \geq j$ go to step 12 and if $N_2 < j$ go to step 14.

12. Calculate $K^{**}_{m^*+j}(t_{m^*+j}, p_{m^*+j}, c_{m^*+j+1})$ from Eq. (2.5.31) for all feasible values of t_{m^*+j}, p_{m^*+j} and c_{m^*+j+1}.

13. Calculate $K^{**}_{n=m^*+j}(T,1,\infty)$ from Eq. (2.5.32). Compare $K^{**}_{n=m^*+j}$ values for old j values with its value for current j and keep the smaller value and discard the larger values. Then go back to step 11.

14. Find L^{**}, the upper bound for the optimum expected total maintenance cost with known system failure distribution from Eq. (2.5.37).

At each stage i and each state (t_i, p_i, c_{i+1}) the values of the t_{i-1}, p_{i-1}, i.e., the control variables of stage i which minimaxes $K_i(t_i, p_i, c_{i+1})$ are recorded and printed out. The value of c_i is taken as the first larger value of $R_1 + \Delta . r$ where r is the number of increments and Δ is the size of the increment obtained from

$$\Delta = \frac{R_2 - R_1}{AL_1}, \tag{3.2.4}$$

respect to cc given according to Eq. (2.5.5) by

$$cc = \frac{\operatorname{Log}_e\left(\frac{1-p_i}{1-p_{i-1}}\right)}{t_i - t_{i-1}}. \tag{3.2.5}$$

The m^* values for the first stage of the second phase are being recorded and printed out at the beginning of the second phase, i.e., when $j = 1$. Finally the optimum total number of stages or $n^* = m^* + j^*$ is recorded together with L^{**}. The policy p^{**} and failure distribution F^{**} defined by Eqs. (2.1.3-4) can then be found easily if needed by searching backward from stage $n^* = m^* + j^*$ to n^*-1, n^*-2, ..., until m^*+1. Then the search continues from stage m^* to m^*-1, m^*-2, ..., until the first stage.

3.3 MODEL B

In this model also there are three state variables, i.e., t_i, p_i, c_{i+1} and two control variables, i.e., t_{i-1}, p_{i-1} which are defined in the same way as for Model A. Here also t_i and t_{i-1} are discrete values according to assumption f in (2.2) and range from 0 to T the maximum life time of the system. p_i and p_{i-1} have continuous values whose range is given by p' and p" functions defined in Chapter 2. c_{i+1} is also continuous and its range is given by

$$R_1 \leq c_{i+1} \leq R_2 , \tag{3.3.1}$$

where the values of R_1 and R_2 are given by

$$R_1(t_i, p_i) = \begin{cases} -\dfrac{\mathrm{Log}_e(1-p_i)}{t_i} & 0 < t_i \leq x_1 \\[2ex] -\dfrac{\mathrm{Log}_e(\frac{1-p_i}{1-p_1})}{t_i - x_1} & x_1 < t_i \leq x_2 , \\[2ex] -\dfrac{\mathrm{Log}_e(\frac{1-p_i}{1-p_2})}{t_i - x_2} & x_2 < t_i \leq T \end{cases} \tag{3.3.2}$$

and

$$R_2(t_i, p_i) = \begin{cases} -\dfrac{\mathrm{Log}_e(\frac{1-p_1}{1-p_i})}{x_1 - t_i} & 0 < t_i < x_1 \\[2ex] -\dfrac{\mathrm{Log}_e(\frac{1-p_2}{1-p_i})}{x_2 - t_i} & x_1 \leq t_i < x_2 . \\[2ex] -\dfrac{\mathrm{Log}(\frac{\varepsilon}{1-p_i})}{t_{i+1} - t_i} & x_2 \leq t_i < T \end{cases} \tag{3.3.3}$$

The reason for the above values for R_1 and R_2 is based on the properties of exponential and IFR failure distributions and on the same line of reasoning utilized in (2.5.2) for establishing the feasible region for IFR distributions passing through points (0, 0) (x_1, p_1) and (x_2, p_2) on the F v.s. t diagram (See Fig. 6.) with the difference that here the feasible region is given for IFR distributions which pass through points (0, 0), (t_i, p_i) and (x_1, p_1) for t_i in the interval $(0, x_1)$, through points (x_1, p_1), (t_i, p_i) and (x_2, p_2) for t_i in the interval (x_1, x_2) and through points (x_2, p_2) (t_i, p_i) and (T,1) for t_i in the interval (x_2, T). The input vector elements consists of a, I, PN_1 and AL_1 as defined in (3.2) plus

x_1 and x_2 = The times at which the cumulative failure distribution of the system are known where $x_1 < x_2$.

p_1 and p_2 = The known cumulative failure probabilities of the system at times x_1 and x_2 respectively where $p_1 < p_2$.

$$T' = \{t_0 = 0,\ t_1,\ \ldots,\ x_1,\ \ldots,\ x_2,\ \ldots,\ T_N = T\}$$

= The set of possible inspection times.

The computational procedure consists of the following steps;

Phase 1:

1. Divide the feasible range of p_i values given by p' and p" functions in Eqs. (2.5.50-51), at each time t_i where t_i is an element of the set T', into PN_1 increments.

2. Divide the feasible range of c_{i+1} values, given by R_1 and R_2 functions in Eqs. (3.3.1-3), at each point (t_i, p_i) on the F v.s. t diagram, into AL_1 increments.

3. Assume no inspection in the time interval $(0,T)$ with only one inspection at time T, i.e., set the stage numbers $m_1 = m_2 = 0$.

4. Calculate the minimax cost $K^*_{m_1+1}(t_{m_1+1}, p_{m_1+1}, c_{m_1+1})$ for all possible values of t_{m_1+1}, p_{m_1+1} and c_{m_1+1} from Eqs. (2.5.44) for $i = m_1+1$, and the set of equations (2.5.49). Also calculate the minimax cost $\bar{K}_{m_2+1}(t_{m_2+1}, p_{m_2+1}, c_{m_2+1})$ for all possible values of t_{m_2+1}, p_{m_2+1} and c_{m_2+1} from Eq. (2.5.44) for $i = m_2+1$, $m_2 = m_1$ and the set of equations (2.5.49).

5. Set current m_1 = Old m_1+1 and calculate $K^*_i(t_i, p_i, c_{i+1})$ given by Eq. (2.5.49) when $i = m_1$ and for all possible values of t_i, p_i and c_{i+1}.

6. Repeat step 4 and compare the $K^*_{m_1+1}(t_{m_1+1}, p_{m_1+1}, c_{m_1+1})$ and $\bar{K}_{m_2+1}(t_{m_2+1}, p_{m_2+1}, c_{m_2+1})$ values for old m_1 values with their values for current m_1 respectively and choose that value of $K^*_{m_1+1}$ and $\bar{K}_{m+1}$ for each point (t_{m_1+1}, p_{m_1+1}) and (t_{m_2+1}, p_{m_2+1}) which is smaller and discard the larger values.

7. Compare the maximum value of m_1, i.e., N_1 with the current value of m_1. If $N_1 \geq m_1+1$, go back to step 5 and if $N_1 < m_1 + 1$ go to Step 8.

8. Take that value of $K^*_{m_1+1}$ at each point (t_{m_1+1}, p_{m_1+1}) which is minimum for all values of $m_1 = 0, 1, \ldots, N_1$ and denote it as $K^{**}_{m_1^*+1}(t_{m_1^*+1}, p_{m_1^*+1}, c_{m_1^*+1})$ where m_1^* is the optimum number of inspections in the time interval $(0, x_1)$.

Phase 2

9. Consider the $K^{**}_{m_1^*+1}\ (t_{m_1^*+1}, p_{m_1^*+1}, c_{m_1^*+1})$ values as the minimax expected maintenance cost upto the time of the first inspection in the time interval $[x_1, x_2)$ or the first stage $j = 1$ of the second phase.

10. Calculate $K^{**}_{m_2+1}(t_{m_2+1}, p_{m_2+1}, c_{m_2+1})$ values for all feasible points (t_{m_2+1}, p_{m_2+1}) on the F v.s. t diagram from Eq. (2.5.44) setting $i = m_2+1$ and $m_2 = m_1^*+1$ and from Eq. (2.5.62).

11. Set current m_2 = old m_2+1. Compare the maximum possible number of inspections in the interval $[x_1, x_2)$, i.e., $N_2 + m_1^*$ with the current value of m_2. If $N_2+m_1^* \geq m_2$ go to step 12. If $N_2 < m_2$ go to step 15.

12. Calculate $K^{**}_{m_1^*+j}(t_{m_1^*+j}, p_{m_1^*+j}, c_{m_1^*+j})$ for all feasible values of $t_{m_1^*+j}$, $p_{m_1^*+j}$ and $c_{m_1^*+j}$ from Eq. (2.5.44) for $i = m_1^*+j = m_2$ and from Eq. (2.5.61).

13. Calculate $K^{**}_{m_2+1}(t_{m_2+1}, p_{m_2+1}, c_{m_2+1})$ values for all feasible points (t_{m_2+1}, p_{m_2+1}) from Eq. (2.5.44) setting $i = m_2+1$ and from Eq. (2.5.62).

14. Compare $K^{**}_{m_2+1}$ values for old m_2 values with $K^{**}_{m_2+1}$ values for current m_2 and for all feasible points (t_{m_2+1}, p_{m_2+1}) and keep the smaller values and discard the larger ones. Go back to step 11.

15. Take that value of $K^{**}_{m_2+1}(t_{m_2+1}, p_{m_2+1}, c_{m_2+1})$ at each point (t_{m_2+1}, p_{m_2+1}) except point (T,1) which is minimum for all values of $m_2 = m_1^*+1$, $m_1^*+2, \ldots, m_1^*+N_2$ as found in step 14 and compare it with the minimum value of $\bar{K}_{m_2+1}(t_{m_2+1}, p_{m_2+1}, c_{m_2+1})$ as found in step 6. Denote the smaller value of them at each point (t_{m_2+1}, p_{m_2+1}) except point (T,1) as $K^{***}_{m_2^*+1}(t_{m_2^*+1}, p_{m_2^*+1}, c_{m_2^*+1})$ where m_2^* is the optimum number of inspection in the interval $(0,x_2)$.

Phase 3

16. Consider $K^{***}_{m_2^*+1}(t_{m_2^*+1}, p_{m_2^*+1}, c_{m_2^*+1})$ values as the minimax expected maintenance cost upto the time of the first inspection in the interval $[x_2,T)$ or the first stage $\ell = 1$ of the third phase.

17. Calculate $K^{**}(T,1,\infty)$ from Eq. (2.5.73) for $\ell = 1$

18. Set current ℓ = Old $\ell + 1$. Compare the maximum possible number of inspections in the interval $[x_2, T)$, i.e., N_3 with the current ℓ. If

$N_3 \geq \ell$ go to step 19 and if $N_3 < \ell$ go to step 21.

19. Calculate $K^{***}_{m_2^*+\ell}\ (t_{m_2^*+\ell},\ p_{m_2^*+\ell},\ c_{m_2^*+\ell+1})$ from Eq. (2.5.72) and Eq. (2.5.44) for $i = m_2^* + \ell$ and $m_2^* \neq m_1^*$ for all feasible values of $t_{m_2^*+\ell}$, $p_{m_2^*+\ell}$ and $c_{m_2^*+\ell}$ and $c_{m_2^*+\ell}$.

20. Calculate $k^{**}_{n=m_2^*+\ell}(T,1,\infty)$ from Eq. (2.5.73) and from Eq. (2.5.44) for $i = n$. Compare $k^{**}_{n=m_2^*+\ell}$ values for old ℓ values with its value for current ℓ and keep the smaller value and discard the larger values. Then go back to step 18.

21. Find L^{**} the upper bound for the optimum expected total maintenance cost with known system failure distribution passing through points (0,0), $(x_1,\ p_1)$, $(x_2,\ p_2)$ and (T,1) from Eq. (2.5.78) where $\bar{K}_{n=m_2^*+1}(T,1,\infty)$ is given by Eq. (2.5.71). The value of ℓ^*, i.e., the optimum number of inspections in the interval $[x_2,\ T)$ is then obtained.

As for Model A, at each stage i and each state $(t_i,\ p_i,\ c_{i+1})$, the values of the t_{i-1}, p_{i-1}, i.e., the control variables of stage i which minimaxes $K_i(t_i,\ p_i,\ c_{i+1})$ are recorded and printed out. The value of c_i is taken as the first larger value of $R_1 + \Delta.r$ where r is the number of increments and Δ is the size of the increment obtained from

$$\Delta = \frac{R_2-R_1}{AL_1}\ , \tag{3.3.4}$$

respect to cc given according to Eq. (2.5.5) by

$$cc = \frac{\mathrm{Log}_e(\frac{1-p_i}{1-p_i-1})}{t_i - t_{i-1}}\ . \tag{3.3.5}$$

The m_1^* values for the first stage of the second phase is being recorded and printed out at the beginning of the second phase, i.e., when

$j = 1$. The m_2^* values for the first stage of the third phase are also being recorded and printed out at the beginning of the third phase, i.e., when $\ell = 1$. Finally the optimum total number of stages or $n^* = m_2^* + \ell^* = m_1^* + j^* + \ell^*$ is recorded together with value of c^{**}. The policy p^{**} and failure distribution F^{**} defined by Eqs. (2.1.3-4) can then be found easily if needed by searching the printout backwardly from stage $n^* = m_1^*+j^* + \ell^*$ to $n^*-1, n^*-2, \ldots,$ until $m_1^*+j^*+1$. Then the search continues from stage $m_1^* + j^*$ to $m_1^* + j^*-1, m_1^* + j^*-2, \ldots,$ until $m_1^* + 1$. Now the search continues from stage m_1^* to $m_1^* - 1, m_1^* - 2, \ldots,$ until the first stage.

In appendix B the computer program together with its discription is presented. In Chapter 4 the result for both Model A and Model B are presented and comparison is made between the two models on the basis of L^{**} values obtained by each model.

CHAPTER 4

THE RESULTS & APPLICATION

4.1 INTRODUCTION

In this chapter the results obtained from computer programming for several input data has been presented and the application of the results has been discussed through an example problem utilizing the information obtained from it. The results are presented according to the following format.

1. The convergence and accuracy of the results.
2. Evaluation of the value of information about the failure distribution of a system through comparison of the maximum expected total maintenance cost L**, applying Model A (location of one percentile known) and Model B (location of the two percentiles known).
3. Comparison of the actual optimum expected total maintenance cost for several IFR failure distributions with the maximum of the optimum total expected maintenance cost L** for both Model A and Model B.
4. Evaluation of the value of information about a system failure distribution with regard to the relative location of the known points.
5. Variation of L** with the inspection Cost I for Model A and Model B.
6. Variation of L** with the cost per unit time of undetected failure a for Model A and Model B.
7. Example problem.

4.2 CONVERGENCE AND ACCURACY

FOR MODEL A

The following input data

$a = 35.$ \$/unit time,

$I = 1.4$ \$/inspection,

$x = 0.4$ unit time,

$p = 0.180$

$T' = \{0,0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1.,1.1,1.2,1.3,1.4\}$,

plus each of the following sets of values:

$$S_1 = \begin{cases} PN_1 = 10 \\ AL_1 = 5 \end{cases} ; \quad S_2 = \begin{cases} PN_1 = 20 \\ AL_1 = 9 \end{cases} ; \quad S_3 = \begin{cases} PN_2 = 25 \\ AL_1 = 10 \end{cases} ,$$

were given for the computer program in Appendix B. The values of F**, P** and maximum expected total maintenance cost for different numbers of inspections in each of the subsets of inspection times

$T_1' = \{0,.1,.2,.3\}$ and $T_2' = \{.4,.5,.6,.7,.8,.9,1.,1.1,1.2,1.3\}$

are given in Table 1 where SS_1, SS_2 and SS_3 refer to the values of F**, P** and maximum expected total cost v.s. number of inspections when the number of increments of state variables, $\underline{c}$ and $\underline{p}$ are given by sets S_1, S_2 and S_3 respectively. Table 1 also shows that the minimum value with respect to number of inspections of the maximum with respect to system failure distributions of IFR type passing through point ($x = .4$, $p = 0.180$), i.e., L** is equal to 10.1 for $M_2 = 5$. P** shows a good convergence and accuracy since shifting from set S_2 to S_3 with higher number of state increments has not changed the solution for P**. However for F** and expected total cost the table shows that although shifting from S_1 to S_2 and from S_2 to S_3 increases the convergence and accuracy but not to more than one digit of

Table 1. The Accuracy v.s. the Number of Increments for the State Variables c and p for Model A.

$F^{**}=\{p_1,p_2,\ldots,p_N\}$				$P^{**}=\{t_1,t_2,\ldots,t_n\}$				Max. Expected Total Cost v.s. M_i, the Total No. of Inspections in Subset T_i			
p	SS_1	SS_2	SS_3	t	SS_1	SS_2	SS_3		SS_1	SS_2	SS_3
p_1	0.000	0.000	0.000	t_1	0.0	0.0	0.0	$M_1=1$	37.20	37.20	37.20
p_2	0.048	0.048	0.048	t_2	0.4	0.4	0.4	=2	37.12	37.38	37.43
p_3	0.094	0.094	0.094	t_3	0.5	0.7	0.7	=3	37.91	38.49	38.40
p_4	0.138	0.138	0.138	t_4	0.6	0.9	0.9	=4	38.70	39.61	39.78
p_5	0.180	0.180	0.180	t_5	0.7	1.1	1.1	$M_2=1$	33.72	33.75	33.75
p_6	0.306	0.233	0.231	t_6	0.8	1.3	1.3	=2	15.92	16.40	16.55
p_7	0.422	0.283	0.278	t_7	0.9	1.4	1.4	=3	12.14	12.46	12.61
p_8	0.529	0.330	0.323	t_8	1.0			=4	10.28	10.38	10.50
p_9	0.626	0.380	0.370	t_9	1.1			=5	9.67	10.03	10.10
p_{10}	0.716	0.427	0.414	t_{10}	1.2			=6	9.65	10.37	10.50
p_{11}	0.797	0.471	0.455	t_{11}	1.3			=7	9.65	10.74	10.94
p_{12}	0.871	0.512	0.493	t_{12}	1.4			=8	9.59	11.12	11.41
p_{13}	0.939	0.551	0.529	t_{13}				=9	9.47	11.51	11.60
p_{14}	1.00	0.585	0.563	t_{14}				=10	9.36	11.92	11.98
p_{15}	1.00	1.00	1.00	t_{15}							

accuracy has been obtained. Surely, increasing the number of increments of state variables increases the accuracy but the computational time also increases very rapidly since there are two continuous state variable, i.e., $\underline{c}$ and p. But since this method will give an upper bound for the total expected cost, in many cases the level of accuracy on this upper bound of the order obtained for this problem (i.e, $\frac{0.1}{10.1} \times 100 = 99\%$) can be sufficient.

FOR MOOEL B

The following input data

$a = 35.$ $/Unit Time,

$I = 1.4$ $/Inspection,

$x_1 = 0.4$ Unit Time,

$x_2 = 0.9$ Unit Time,

$p_1 = 0.180$,

$p_2 = 0.560$,

$T' = \{0,0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1.0,1.1,1.2,1.3,1.4\}$,

plus each of the following sets of values:

$$S_1 = \begin{cases} PN_1 = 10 \\ AL_1 = 5 \end{cases} ; \quad S_2 = \begin{cases} PN_1 = 20 \\ AL_1 = 9 \end{cases} ; \quad S_3 = \begin{cases} PN_1 = 25 \\ SL_1 = 15 \end{cases} ,$$

were given for the computer program in Appendix B. The values of F**, P** and maximum total expected maintenance cost for different numbers of inspections in each of the subsets of inspections $T_1' = \{0,0.1,0.2,0.3\}$,

$T_2' = \{0.4, 0.5, 0.6, 0.7, 0.8\}$ and $T_3' = \{0.9, 1.0, 1.1, 1.2, 1.3\}$ are given in Table 2 where SS_1, SS_2 and SS_3 are defined in the same way as for Model A. Table 2 also shows that the minimum value with respect to number of inspections of the maximum with respect to system failure distributions of IFR type passing through points ($x_1 = 0.4$, $p_1 = 0.180$) and ($x_2 = 0.9$, $p_2 = 0.560$), i.e., L** is equal to 9.61 for $M_3 = 3$. Again here P** shows a good convergence and accuracy since shifting from input set S_2 to S_3 with higher number of state increments has not changed the solution for P**, F** and maximum expected cost. Values in Table 2 show that an accuracy of up to one decimal point can be obtained by shifting from S_2 to S_3.

4.3 COMPARISON OF MODEL A WITH MODEL B

Figure 7 shows the variation of maximum expected cost with the number of inspections. It can easily be seen that the inspections which are performed after the time for which cumulative probability of failure is known are more important and their number determines the upper bound on the expected total optimum cost of the maintenance. Table 3 shows the value of information for IFR distributions which pass through points (0.4, 0.180) and (0.9, 0.560). If both informations are utilized then Model B gives the values for L** which is equal to 9.61. If only point (0.4, 0.180) is known then Model A gives L_0^{**} equal to 10.10. Similarly if only point (0.9, 0.560) is known then Model A gives L** equal to 9.64. Table 3 also shows that the additional knowledge about point (0.9, 0.560) improves the upper bound cost L** for about 4.85%. Another important conclusion is that even if only information about one point is available the closer this point to the final point (T,1) the more improved upper bound value, L** can be obtained. This

Table 2. The Accuracy v.s. the Number of Increments for the State Variables c and p for Model B.

$F^{**}=\{p_1,p_2,p_3,\ldots,p_N\}$

p	SS_1	SS_2	SS_3
p_1	0.000	0.000	0.000
p_2	0.048	0.048	0.048
p_3	0.094	0.094	0.094
p_4	0.138	0.138	0.138
p_5	0.180	0.180	0.180
p_6	0.263	0.223	0.222
p_7	0.337	0.318	0.322
p_8	0.404	0.401	0.408
p_9	0.494	0.474	0.487
p_{10}	0.560	0.560	0.560
p_{11}	0.643	0.632	0.628
p_{12}	0.710	0.923	0.930
p_{13}	0.764	0.984	0.987
p_{14}	1.000	1.000	1.000
p_{15}	1.000	1.000	1.000

$p^{**}=\{t_1,t_2,\ldots,t_n\}$

t	SS_1	SS_2	SS_3
t_1	0.0	0.0	0.0
t_2	0.4	0.5	0.5
t_3	0.7	0.8	0.8
t_4	0.9	1.0	1.0
t_5	1.2	1.2	1.2
t_6	1.3	1.3	1.3
t_7	1.4	1.4	1.4
t_8			
t_9			
t_{10}			

Max. Expected Total Cost v.s. M_i, the Total No. of Inspections in Subset T_i

	SS_1	SS_2	SS_3
$M_1=1$	26.88	26.88	26.88
$=2$	24.72	24.84	24.86
$=3$	25.85	25.99	26.02
$=4$	27.20	27.36	27.39
$M_2=1$	18.48	18.53	18.54
$=2$	17.66	17.80	17.82
$=3$	17.74	18.09	18.16
$=4$	18.14	18.74	18.85
$=5$	18.66	19.48	19.64
$M_3=1$	15.41	15.50	15.52
$=2$	10.43	10.46	10.46
$=3$	9.43	9.60	9.61
$=4$	9.74	9.66	9.69
$=5$	9.51	9.86	9.83

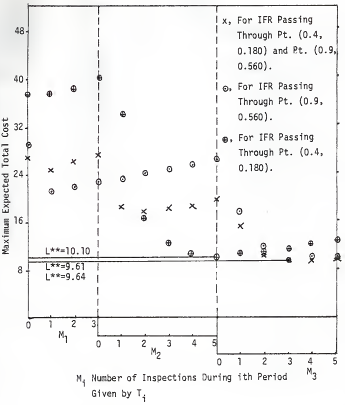

Fig. 7. Variation of Upper Bound for the Expected Total Cost with M_i Number of Inspections.

Table 3. The Value of Information about IFR Distributions Passing Through Points (0.4, 0.180) and (0.9, 0.560).

Utilizing The Information(s) That	L^{**}	$\frac{L^{**}-L_0^{**}}{L_0^{**}} \times 100$
$p_1 = 0.180;\ x_1 = 0.4$ $p_2 = 0.560;\ x_2 = 0.9$ (Using Model B)	9.61	4.85
$p = 0.180;\ x = 0.4$ (Using Model A)	10.10	0.00
$p = 0.560;\ x = 0.9$ (Using Model A)	9.64	4.55

For:

$a = 35.$ \$/Unit Time,

$I = 1.4$ \$/Inspection,

$T' = \{0,0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1.0,1.1,1.2,1.3\}$,

and AL_1 & PN_1 given by set S_3.

is clear again from Table 3 since having the knowledge about point (0.9,0.560) is 4.55% more valuable than the knowledge about point (0.4, 0.180) which is closer to point (0,0). In section 4.5 more results are presented regarding the location of the known point or points.

4.4 ACUTAL OPTIMAL COST V.S. L**

The optimal policy $\bar{P}_i = \{t_1, t_2, \ldots, t_n\}$ and the optimal cost R_i has been found numerically by using computer program of Appendix C for several different IFR failure distributions F_i which consist of points $(\bar{t}_i, p_i)$ connected by transformed exponential distributions. Tables 4 and 5 give the policies $\bar{P}_i$ and optimal costs R_i and also the ratio of R_i to L**. It can be seen that in both tables this ratio is less than one as could have been expected since by definition L** is an upper bound for the optimal expected maintenance cost for known distributions.

4.5 VALUE OF INFORMATION V.S. RELATIVE LOCATION OF THE KNOWN POINT(S).

A certain IFR failure distribution F_0 has been selected for which:

$p(t = 0.0) = 0.000$; $p(t = 0.2) = 0.150$; $p(t = 0.3) = 0.216$;

$p(t = 0.4) = 0.278$; $p(t = 0.5) = 0.535$; $p(t = 0.6) = 0.700$.

The set of possible inspection times T' is given by

$T' = \{0,0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8\}$.

Now for the above failure distribution the values of L** are plotted in Fig. 8 and Fig. 9 assuming that only one point (x, p_x) or two points (x_1, p_{x_1}) and (x_2, p_{x_2}) of the distribution F_0 is known. Figure 8 shows that the value of L** improves as the known location x of the $100.p_x$ th percentile of the failure distribution moves toward the maximum life time

Table 4. Optimal Policy $\bar{P}_i$ and Cost R_i for Some IFR Distributions F_i which Pass Through Points (0.4, 0.180) and (0.9, 0.560) and Their Comparison with L**

	F_1	F_2	F_3	F_4	F_5	$F^{**}=F_6$
p_1	0.000	0.000	0.000	0.000	0.000	0.000
p_5	0.180	0.180	0.180	0.180	0.180	0.180
p_7	0.305	0.322	0.270	0.270	0.356	0.305
p_9	0.444	0.480	0.364	0.444	0.494	0.487
p_{10}	0.560	0.560	0.560	0.560	0.560	0.560
p_{11}	0.822	0.644	0.822	0.692	0.966	0.628
p_{13}	0.975	0.848	1.000	0.874	1.000	0.987
p_{14}	1.000	0.978	1.000	1.000	1.000	1.000
p_{15}	1.000	1.000	1.000	1.000	1.000	1.000
t_1	0.0	0.0	0.0	0.0	0.0	0.0
t_2	0.4	0.4	0.5	0.4	0.5	0.4
t_3	0.7	0.7	0.9	0.7	0.7	0.7
t_4	0.9	0.9	1.0	0.9	1.0	0.9
t_5	1.0	1.1	1.1	1.0	1.1	1.1
t_6	1.1	1.2	1.4	1.1	1.4	1.2
t_7	1.2	1.3		1.2		1.2
t_8	1.3	1.4		1.3		1.4
t_9	1.4			1.4		
t_{10}						
R_i	8.28	8.81	8.16	8.77	7.99	8.32
R_i/L^{**}	0.861	0.916	0.849	0.912	0.831	0.866

$L^{**} = 9.61\$$ $F_i = \{F(\bar{t}_i) = p_i$ for $i = 1,5,7,9,10,11,13,14,15\}$

a = 35. \$/Unit Time

I = 1.4 \$/Inspection $\bar{t}_1 = 0.0$ Unit Time; $\bar{t}_5 = 0.4$ Unit Time

Table 4. (continued)

$T' = \{0, 0.1, 0.2, 0.3, 0.4, 0.5,$	$\bar{t}_7$ = 0.6 Unit Time; $\bar{t}_9$ = 0.8 Unit Time
$0.6, 0.7, 0.8, 0.9, 1.0, 1.1,$	$\bar{t}_{10}$ = 0.9 Unit Time; $\bar{t}_{11}$ = 1.0 Unit Time
$1.2, 1.3\}$	$\bar{t}_{13}$ = 1.2 Unit Time; $\bar{t}_{14}$ = 1.3 Unit Time
AL_1 & AL_2 given by Set S_3	$\bar{t}_{15}$ = 1.4 Unit Time;

Table 5. Optimal Policy $\bar{P}_i$ and Cost R_i for Some IFR Distributions F_i which Pass Through Point (0.4, 0.180) and Their Comparison with L**

	F_1	F_2	F_3	F_4	F_5	$F^{**}=F_6$
p_1	0.000	0.000	0.000	0.000	0.000	0.000
p_5	0.180	0.180	0.180	0.180	0.180	0.180
p_8	0.382	0.441	1.000	0.412	0.553	0.323
p_{10}	0.680	0.760	1.000	0.653	0.813	0.413
p_{12}	0.903	0.976	1.000	0.827	0.956	0.493
p_{14}	0.978	1.000	1.000	0.918	1.000	0.563
p_{15}	1.000	1.000	1.000	1.000	1.000	1.000
t_1	0.0	0.0	0.0	0.0	0.0	0.0
t_2	0.5	0.3	0.5	0.5	0.5	0.4
t_3	0.8	0.6	1.4	0.8	0.7	0.8
t_4	1.0	0.8		1.0	0.8	1.1
t_5	1.1	0.9		1.1	0.9	1.4
t_6	1.2	1.0		1.2	1.0	
t_7	1.3	1.1		1.3	1.1	
t_8	1.4	1.2		1.4	1.2	
t_9		1.4			1.4	
t_{10}						
R_i	8.22	7.83	6.20	8.70	7.80	9.51
R_i/L^{**}	0.814	0.775	0.614	0.861	0.772	0.942

L** = 10.10 \$ $F_i = \{F(\bar{t}_i) = p_i$ for $i = 1,5,8,10,12,14,15\}$

a = 35. \$Unit Time

I = 1.4 \$/Inspection $\bar{t}_1$ = 0.0 Unit Time; $\bar{t}_5$ = 0.4 Unit Time

T'= Defined in Table 4. $\bar{t}_8$ = 0.7 Unit Time; $\bar{t}_{10}$ = 0.9 Unit Time

AL_1 & AL_2 given by Set S_3 $\bar{t}_{12}$= 1.1 Unit Time; $\bar{t}_{14}$ = 1.3 Unit Time

$\bar{t}_{15}$= 1.4 Unit Time

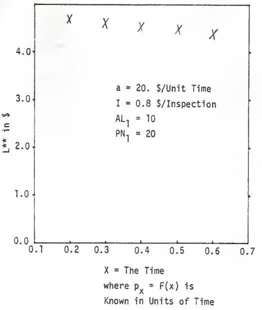

Fig. 8. Variation of L** with the Time x at which $p_x = F(x)$ is Known.

of the system. The same trend can be seen in Fig. 9 for constant value of x_2. In addition to this, Fig. 9 shows that L** improves as the location of x_2 moves toward the maximum life time of the system.

4.6 VARIATION OF L** WITH I

Figure 10 shows the variation of L**, the upper bound total expected cost with I the inspection cost for each inspection. As it can be seen the rate of increase of L** with the increase of I is higher for lower values of I for both cases, i.e., when the information about two points of distribution are known or the information about only one point is known. Table 6 gives the values of L** and I which are plotted in Fig. 10 where L_1^{**} and L_2^{**} stand for the values of L** when two points and one point of the distribution are known respectively. The third column of Table 6 shows the percent improvement in L** values when additional point of distribution is known for different values of I.

4.7 VARIATION OF L** WITH a

Figure 11 shows the variation of L** with a the cost per unit time of undetected failure. It can be seen that the rate of increase of L** with the increase of a is higher for lower values of a for both cases, i.e., when the information about two points of the distribution are known or the information about only one point is known. Table 7 gives the values of L** and a which are plotted in Fig. 11 where L_1^{**} and L_2^{**} stand for the values of L** when two points and one point of the distribution are known respectively. The third column of Table 7 shows the percent improvement in L** values when additional point of distribution is known for different values of a. The value of L** at a = 0 is equal to 0.8 or the cost of one inspection which

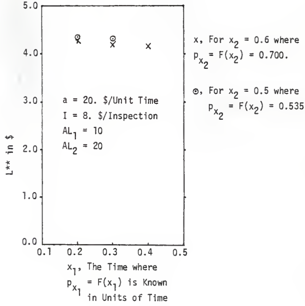

Fig. 9. Variation of L** with the Times x_1 and x_2 at which p_{x_1} and p_{x_2} are Known.

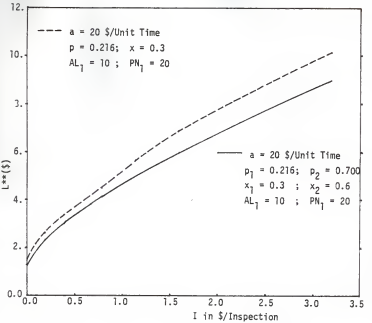

Fig. 10. Variation of L** with Cost of an Inspection I.

Table 6. Percent Improvement in L** for Various Values of I

I	L_1^{**}	L_2^{**}	$\frac{L_2^{**}-L_1^{**}}{L_2^{**}} \times 100$
0.0	1.34	1.51	11.3
0.2	2.34	2.58	9.3
0.4	3.10	3.48	10.9
0.8	4.24	4.60	7.8
1.6	6.07	6.93	12.4
3.2	8.98	10.22	12.1

$a = 20$

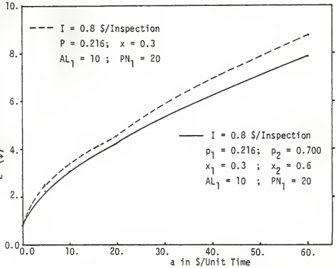

Fig. 11. Variation of L** with Cost per Unit Time of Undetected Failure a.

Table 7. Percent Improvement in L** for Various Values of a

a	L_1^{**}	L_2^{**}	$\frac{L_2^{**}-L_1^{**}}{L_2^{**}} \times 100$
0.0	0.80	0.80	0.0
5.0	2.25	2.55	11.7
10.0	3.03	3.47	12.7
20.0	4.24	4.60	7.8
40.0	6.20	6.96	10.9
60.0	7.89	8.73	9.6

$I = 0.8$

at least is needed to be performed in order to detect the failure of the system.

4.8 EXAMPLE PROBLEM

Problem: The management of an organization wants to introduce a new deteriorating item into the system. There are three deteriorating items that have the same function but have different failure and cost characteristics. Because of the shortage of investing capital the company wants to choose the alternative which has the lowest possible expected cost. The characteristics of the items A, B and C are as follows:

	Item A	Item B	Item C
1. Average inventory cost, $/day	140	200	100
2. Cost per inspection, $	56	20	80
3. Probability of failure p before time x.	0.180	0.216	0.535
4. x, is equal to	4th day	3rd day	10th day
5. Maximum life time	14 days	3 days	16 days.

Besides these the inspection can only be performed at certain time and only once a day for example at 3 p.m. every day. The replenishment of the item will occur at the end of periods equal to the maximum life time of the item. Which alternative should be chosen with the above limited information?

Solution: Since the failure characteristics of the system are not completely known then the best estimate of the costs would be the maximum possible expected cost L^{**}. In many cases the systems show an IFR failure characteristic so it is assumed that failure characteristic of the items are IFR. Average inventory cost becomes equivalent to the cost of undetected failure $\underline{a}$ since a failed item will remain in the stock and assumes cost until its failure is detected through inspection. We assume that the inspection is perfect and it does not take time and does not degrade the item. It can be shown from Eq. (2.4.2) that if L is multiplied by a scale factor u, and t, the time, multiplied by another scale factor v then new L called L' = (u)L is the expected cost of the system provided that

$$I' = (u)I,$$

$$a' = \frac{(u)a}{v} = \left(\frac{u}{v}\right) a,$$

and

$$t'_{max} = (v)\ t_{max}$$

where I' and a' are new scaled inspection and undetected failure costs. t'_{max} and t_{max} are the scaled and not scaled values of the maximum life time of the system.

For Item A we have:

$$I' = (40)(1.4) = 56$$

$$a' = \left(\frac{40}{10}\right) (35) = 140$$

$$x' = (10)(.4) = 4$$

$$t'_{max} = (10)\ (1.4) = 14.$$

So for u = 40 and v = 10, we have I = 1.4, a = 35., x = 0.4, p = 0.180 and t_{max} = 1.4. From Table 3 the value of L** for above values of input parameters is L** = 10.10. So

$$L^{**'} = (40)(10.10) = 404 \text{ \$/cycle or } 404/14 = 28.86 \text{ \$/day}.$$

For Item B we have:

$$I' = (100)(0.2) = 20$$

$$a' = (\frac{100}{10})(20) = 200$$

$$x' = (10)(0.3) = 3$$

$$t'_{max} = (10)(0.8) = 8.$$

So for $u = 100$ and $v = 10$, we have $I = 0.2$, $a = 20$, $x = 0.3$, $p = 0.216$ and $t_{max} = 0.8$. From Table 6 the value of L^{**} for above values of input parameters is $L^{**} = 2.58$. So

$$L^{**'} = (100)(2.58) = 258 \text{ \$/cycle or } 258/8 = 32.25 \text{ \$/day}.$$

For Item C we have

$$I' = (100)(0.8) = 80$$

$$a' = (\frac{100}{20})(20) = 100$$

$$x' = (20)(0.5) = 10$$

$$t'_{max} = (20)(0.8) = 16.$$

So for $u = 100$ and $v = 20$, we have $I = 0.8$, $a = 20$, $x = 0.5$, $p = 0.535$ and $t_{max} = 0.8$. From Fig. 8 the value of L^{**} for above values of input parameters is $L^{**} = 4.45$. So

$$L^{**'} = (100)(4.45) = 445 \text{ \$/cycle or } 445/16 = 27.81 \text{ \$/day}.$$

Comparison between the cost per day for the three items shows that introduction of item C is more economical in a long run than the two other items.

CHAPTER 5

CONCLUSION

Minimax policy has been devised to cope with the problem of partial knowledge about the failure distribution of systems. In this work this policy has been adopted together with dynamic programming methodology to find the upper bound on the optimal total expected maintenance cost of a system subject to deterioration with an increasing failure rate distribution. The major difference with the previous work in this area is that here a procedure and a computer program has been devised which finds the upper bound cost not only when one point (x, p) of the failure distribution is known but also when two points (x_1, p_1) and (x_2, p_2) of the failure distribution are known.

The basic findings can be summarized as:

1. The convergence and accuracy of the upperbound on the expected optimal total maintenance cost depends on the number of increments into which the feasible range of the state variables c and p are divided and for reasonable amount of computer time a fairly accurate upper bound value can be obtained.
2. Additional knowledge about a second point (x_2, p_2) improves (decreases) the upper bound in general. But it improves the upper bound cost especially if the second point is closer to the end point $(T_{max}, 1)$ where T_{max} is the maximum life time of the system.
3. The closer the two known points (x_1, p_1) and (x_2, p_2) to the end point $(T_{max}, 1)$ the more improved upper bound cost can be obtained.

4. The upper bound cost seems to increase faster with the increase in the cost per inspection, I, at lower values of I.
5. The upper bound cost seems to increase faster with the increase in the cost per unit time of undetected failure, a, at lower values of a.
6. Several optimal expected total costs which have been found for several different IFR distributions showed that all were lower than upper bound cost as it could have been expected.

The basic disadvantage of the present computational procedure is due to the existance of three state variables. In this work the state variable, time, has been assumed to be discrete in order to increase the computational feasibility. This assumption although limits the scope of the applicability of the procedure but nonetheless it can be a practical assumption since in many cases the maintenance and inspection actions can only be done at discrete points in time. The other disadvantage is that it only provides for the replenishment to take place at the end of its maximum life time. In general this work and the previous works in this area are worthwhile to be continued in a wider sense since in many cases when a new system or deteriorating item is introduced into the existing system the complete knowledge on the failure characteristic of it is rare and it takes time to build up knowledge about the system failure characteristics and the management needs to have some estimate of the cost even before introducing a new system in order to compare different options or alternatives.

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APPENDICES

APPENDIX A

MATHEMATICAL TRANSFORMATION OF THE EXPECTED TOTAL MAINTENANCE COST

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MATHEMATICAL TRANSFORMATION OF THE EXPECTED TOTAL MAINTENANCE COST

As stated in (2.4), Eq. (2.4.1), i.e.,

$$L(t_0 = 0, t_1, t_2, \ldots, t_n = T) = \sum_{k=0}^{n-2} \int_{t_k}^{t_{k+1}} [I(k+1)+a(t_{k+1}-x)]f(x)\, dx$$

$$+ \int_{t_{n-1}}^{t_n=T} [I(n) + a(T-x)]f(x)\, dx$$

$$n = 1,2, \ldots, N, \tag{1}$$

can be transformed into Eq. (2.4.2), i.e.,

$$L = I \sum_{k=0}^{n-1} F^*(t_k) + a\left(\sum_{k=0}^{n-1} F^*(t_k)(t_{k+1}-t_k) - \int_0^{t_n} F^*(t)\, dt\right)$$

$$n = 1,2, \ldots, N. \tag{2}$$

PROOF:

$$\sum_{k=0}^{n-2} \int_{t_k}^{t_{k+1}} [I(k+1) + a(t_{k+1}-x)]f(x)\, dx = \sum_{k=0}^{n-2} \left(\int_{t_k}^{t_{k+1}} I(k+1)f(x)\, dx + \right.$$

$$\left. a \int_{t_k}^{t_{k+1}} t_{k+1}f(x)dx - a \int_{t_k}^{t_{k+1}} x\, f(x)dx \right), \tag{3}$$

but

$$\int_{t_k}^{t_{k+1}} I(k+1)f(x)dx = I(k+1)(F(t_{k+1}) - F(t_k)), \tag{4}$$

and

$$\sum_{k=0}^{n-2} \int_{t_k}^{t_{k+1}} I(k+1)f(x)dx = \sum_{k=0}^{n-2} I(k+1)(F(t_{k+1}) - F(t_k))$$

$$= I(0+1)(F(t_1) - F(t_0)) + I(1+1)(F(t_2) - F(t_1))$$

$$+ I(2+1)(F(t_3) - F(t_2)) + \ldots + I(n-2+1)$$

$$(F(t_{n-1}) - F(t_{n-2}))$$

$$= -IF(t_0) - IF(t_1) - IF(t_2) \ldots - IF(t_{n-2})$$

$$+ I(n-1)F(t_{n-1})$$

$$= - I \sum_{k=0}^{n-2} F(t_k) + I(n-1)F(t_{n-1}), \tag{5}$$

also

$$a \sum_{k=0}^{n-2} \int_{t_k}^{t_{k+1}} t_{k+1}f(x)dx = a \sum_{k=0}^{n-2} t_{k+1}(F(t_{k+1}) - F(t_k)). \tag{6}$$

We know that according to the integeration by part formula

$$\int_a^b vu'du = (vu)_a^b - \int_a^b v'u\ du, \tag{7}$$

where v' and u' are the derivatives of functions v and u. Assuming $u'du = f(x)dx$ and $v = x$ and using Eq. (7) we can write

$$\int_{t_k}^{t_{k+1}} xf(x)dx = (xF(x))_{t_k}^{t_{k+1}} - \int_{t_k}^{t_{k+1}} F(x)dx$$

$$= t_{k+1} F(t_{k+1}) - t_k F(t_k) - \int_{t_k}^{t_{k+1}} F(x)\,dx, \tag{8}$$

and

$$\sum_{k=0}^{n-2} - a \int_{t_k}^{t_{k+1}} xf(x)dx = - a \sum_{k=0}^{n-2} (t_{k+1}F(t_{k+1}) - t_kF(t_k))$$

$$+ a \int_{t_0=0}^{t_{n-1}} F(x)dx. \tag{9}$$

Now substituting the equivalent values from Eqs. (5), (6) and (9) in Eq. (3) we have

$$\begin{aligned}
\sum_{k=0}^{n-2} \int_{t_k}^{t_{k+1}} [I(k+1) + a(t_{k+1}-x)]f(x)dx = {} & - I \sum_{k=0}^{n-2} F(t_k) + \\
& I(n-1)F(t_{n-1}) + \\
& a \sum_{k=0}^{n-2} t_{k+1}(F(t_{k+1}) - F(t_k)) \\
& - a \sum_{k=0}^{n-2} (t_{k+1}F(t_{k+1}) - t_kF(t_k)) \\
& + a \int_0^{t_{n-1}} F(x)dx \\
= {} & - I \sum_{k=0}^{n-2} F(t_k) + I(n-1)F(t_{n-1}) \\
& + a \left(- \sum_{k=0}^{n-2} F(t_k)(t_{k+1} - t_k) + \right. \\
& \left. \int_0^{t_{n-1}} F(x)dx \right). \qquad (10)
\end{aligned}$$

The second portion of the right hand side of Eq. (1) can be written in the same way as

$$\int_{t_{n-1}}^{t_n=T} [I(n) + a(T-x)]f(x)dx = I(n)(F(T) - F(t_{n-1})) +$$

$$+ a\ T(F(T) - F(t_{n-1})) + a \int_{t_{n-1}}^{t_n=T} F(x)\ dx$$

$$- a\ (TF(T) - t_{n-1}F(t_{n-1}))$$

$$= I(n)(1 - F(t_{n-1})) +$$

$$a\Big(- F(t_{n-1})(T - t_{n-1}) +$$

$$\int_{t_{n-1}}^{t_n=T} F(x)dx\Big), \tag{11}$$

now substituting Eq. (10) and Eq. (11) in Eq. (1) and naming t = x we have

$$L = - I \sum_{k=0}^{n-2} F(t_k) + I(n-1)F(t_{n-1}) + a\Big(- \sum_{k=0}^{n-2} F(t_k)(t_{k+1} - t_k)$$

$$+ \int_0^{t_{n-1}} F(t)dt + I(n)(1-F(t_{n-1})) + a\Big(-F(t_{n-1})(T - t_{n-1}) +$$

$$\int_{t_{n-1}}^{t_n=T} F(t)dt$$

$$= I \sum_{k=0}^{n-1} (1 - F(t_k)) + a\Big(- \sum_{k=0}^{n-1} F(t_k)(t_{k+1} - t_k) + \int_0^{t_n=T} F(t)dt\Big), \tag{12}$$

denoting $F^*(t)$ as $F^*(t) = 1-F(t)$ and substituting it instead of $1 - F(t)$ in Eq. (12) we have

$$L = I \sum_{k=0}^{n-1} F^*(t_k) + a\left\{-\sum_{k=0}^{n-1} (1 - F^*(t_k))(t_{k+1} - t_k) + \int_0^{t_n=T} (1 - F^*(t))dt\right\}$$

$$= I \sum_{k=0}^{n-1} F^*(t_k) + a\left\{-\sum_{k=0}^{n-1} (t_{k+1} - t_k) + \sum_{k=0}^{n-1} F^*(t_k)(t_{k+1} - t_k)\right.$$

$$\left. + \int_0^{t_n=T} dt - \int_0^{t_n=T} F^*(t)dt\right\}$$

$$= I \sum_{k=0}^{n-1} F^*(t_k) + a\left\{\sum_{k=0}^{n-1} F^*(t_k)(t_{k+1}-t_k) - \int_0^{t_n=T} F^*(t)dt\right\}.$$

APPENDIX B

COMPUTER PROGRAM FOR CALCULATING AN UPPER BOUND FOR THE EXPECTED TOTAL COST FOR BOTH MODELS A AND B

```
C     ****************************************************************
C
C     CALCULATION OF THE UPPER BOUND FOR THE OPTIMAL EXPECTED TOTAL
C     MAINTENANCE COST OF A SYSTEM WITH PARTIALLY KNOWN FAILURE
C     DISTRIBUTION
C
C     NOTATIONS
C     NB=1--CALCULATES UPPER BOUND COST WHEN ONLY ONE POINT OF THE
C     DISTRIBUTION IS KNOWN--MODEL A
C     NB=2--CALCULATES UPPER BOUND COST WHEN TWO POINTS OF THE
C     DISTRIBUTION ARE KNOWN--MODEL B
C     P1---THE KNOWN CUMULATIVE FAILURE PROBABILITY AT TIME D1
C     P2---THE KNOWN CUMULATIVE FAILURE PROBABILITY AT TIME D2
C     AI---INSPECTION COST PER INSPECTION
C     AA---COST OF UNDETECTED FAILURE PER UNIT TIME
C     NP---NUMBER OF INCREMENTS INTO WHICH THE FEASIBLE RANGE OF
C     FAILURE PROBABILITY P IS DIVIDED MINUS ONE
C     LA---NUMBER OF INCREMENTS INTO WHICH THE FEASIBLE RANGE OF C
C     PARAMETER OF TRANFORMED EXPONENTIAL DISTRIBUTION IS
C     DIVIDED MINUS ONE
C     E1---MAXIMUM POSSIBLE NUMBER OF INSPECTIONS
C     RKK(I,J,K),RBB(I,J,K)---THE MAXIMUM EXPECTED MAINTENANCE COST
C     UP TO TIME T(I) WITH P(J),C(K) AND AT STAGE N-1 AND N
C     RESPECTIVELY
C     THIS PROGRAM WAS WRITTEN BY KOUROSH-KARAMPISHEH, DEPARTMENT OF
C     INDUSTRIAL ENGINEERING KANSAS STATE UNIVERSITY, MANHATTAN
C     KANSAS, MAY, 1979
C     ****************************************************************
C
      INTEGER D1,D2,E1,DD1,DDD1,DD2,DDD2,EE1
      DIMENSION RKK(10,20,10),C(10,20,10),MF(10,20,10),MT(10,20,10),
     1RBB(10,20,10),MK(10,20,10),P(10,20),PK2(10),JJR(10),MMF1(10),
     2T(10)
      NB=2
      IF(NB-1)100,821,820
  821 READ(5,822)P1,AI,AA,NP,E1,D1,LA
  822 FORMAT(3F10.6,4I5)
      D2=D1
      P2=P1
      GO TO 823
  820 READ(5,1)P1,P2,AI,AA,NP,E1,D1,D2,LA
    1 FORMAT(4F10.6,5I5)
  823 PAC=1.E-25
      DO 2 LI=1,E1
      READ(5,600)T(LI)
  600 FORMAT(F10.6)
    2 CONTINUE
      CIC1=-ALOG(1.-P1)/T(D1)
      IF(NB-1)100,824,825
      STEP 1 FOR BOTH MODELS A&B
  824 CIC2=CIC1
      GO TO 831
  825 CIC2=-ALOG((1.-P2)/(1.-P1))/(T(D2)-T(D1))
```

```
831 P(1,1)=0.
    NP1=NP-1
    PN1=NP-1
    DD1=D1-1
    DDD1=D1+1
    DD2=D2-1
    DDD2=D2+1
    LA1=LA-1
    AL1=LA-1
    EE1=E1-1
    DO 10 L2=2,DD1
    PP1=1.-EXP(-CIC1*T(L2))
    IF(NB-1)100,3,828
828 PP2=1.-(1.-P1)*EXP(CIC2*(T(D1)-T(L2)))
    IF(PP2)3,4,4
  3 DP=PP1/PN1
    GO TO 6
  4 DP=(PP1-PP2)/PN1
  6 P(L2,1)=PP1
    DO 7 L3=2,NP1
  7 P(L2,L3)=P(L2,L3-1)-DP
    IF(NB-1)100,8,886
886 IF(PP2)8,9,9
  8 P(L2,NP)=0.0
    GO TO 10
  9 P(L2,NP)=PP2
 10 CONTINUE
    P(D1,1)=P1
    IF(NB-1)100,829,830
830 DO 12 L4=DDD1,DD2
    PP1=1.-(1.-P1)*EXP(-CIC2*(T(L4)-T(D1)))
    PP2=1.-EXP(-CIC1*T(L4))
    DP=(PP1-PP2)/PN1
    P(L4,1)=PP1
    DO 11 L5=2,NP1
 11 P(L4,L5)=P(L4,L5-1)-DP
    P(L4,NP)=PP2
 12 CONTINUE
    P(D2,1)=P2
829 DO 14 L6=DDD2,E1
    PP1=1.-PAC
    PP2=1.-(1.-P2)*EXP(-CIC2*(T(L6)-T(D2)))
    DP=(PP1-PP2)/PN1
    P(L6,1)=1.-PAC
    DO 13 L7=2,NP1
 13 P(L6,L7)=P(L6,L7-1)-DP
    P(L6,NP)=PP2
 14 CONTINUE

    STEP 2 FOR BOTH MODELS A&B

    DC=CIC1/AL1
    C(1,1,1)=0.
    DO 27 K=2,LA1
 27 C(1,1,K)=C(1,1,K-1)+DC
    C(1,1,LA)=CIC1
    DO 17 L9=2,DD1
    DO 17 L10=2,NP1
    C1=-ALOG(1.-P(L9,L10))/T(L9)
    C2=-ALOG((1.-P1)/(1.-P(L9,L10)))/(T(D1)-T(L9))
```

```
      DC=(C2-C1)/AL1
      C(L9,L10,1)=C1
      DO 16 K=2,LA1
   16 C(L9,L10,K)=C(L9,L10,K-1)+DC
      C(L9,L10,LA)=C2
   17 CONTINUE
      IF(NB-1)100,832,833
  833 DC=(CIC2-CIC1)/AL1
      C(D1,1,1)=CIC1
      DO 28 K=2,LA1
   28 C(D1,1,K)=C(D1,1,K-1)+DC
      C(D1,1,LA)=CIC2
      DO 20 L9=DDD1,DD2
      DO 20 L10=2,NP1
      C1=-ALOG((1.-P(L9,L10))/(1.-P1))/(T(L9)-T(D1))
      C2=-ALOG((1.-P2)/(1.-P(L9,L10)))/(T(D2)-T(L9))
      DC=(C2-C1)/AL1
      C(L9,L10,1)=C1
      DO 19 K=2,LA1
   19 C(L9,L10,K)=C(L9,L10,K-1)+DC
      C(L9,L10,LA)=C2
   20 CONTINUE
  832 C(D2,1,1)=CIC2
      PP1=1.-(1.-P2)*EXP(-CIC2*(T(DDD2)-T(D2)))
      DP=(1.-PAC-PP1)/AL1
      DO 21 K=2,LA1
      AK=K
   21 C(D2,1,K)=-ALOG((1.-PP1-(AK-1.)*DP)/(1.-P2))/(T(DDD2)-T(D2))
      C(D2,1,LA)=-ALOG(PAC/(1.-P2))/(T(DDD2)-T(D2))
      DO 24 L11=DDD2,E1
      DO 512 L12=1,NP1
      IF(L12-1)100,707,708
  707 C1=-ALOG(PAC/(1.-P2))/(T(L11)-T(D2))
      GO TO 709
  708 C1=-ALOG((1.-P(L11,L12))/(1.-P2))/(T(L11)-T(D2))
  709 C(L11,L12,1)=C1
      IF(L11-E1)512,24,24
  512 CONTINUE
   24 CONTINUE
      DO 610 L13=DDD2,EE1
      DO 610 L14=2,NP1
      PP1=1.-(1.-P2)*EXP(-C(L13,L14,1)*(T(L13+1)-T(D2)))
      DP=(1.-PAC-PP1)/AL1
      DO 700 K=2,LA1
      AK=K
  700 C(L13,L14,K)=-ALOG((1.-PP1-(AK-1.)*DP)/(1.-P(L13,L14)))/(T(L
     1L13+1)-T(L13))
  610 C(L13,L14,LA)=-ALOG(PAC/(1.-P(L13,L14)))/(T(L13+1)-T(L13))

      STEPS 3&4 FOR BOTH MODELS A&B

      M=1
      I=D1
      RKK(D1,1,2)=AI+AA*T(D1)-AA*P(D1,1)/CIC1
      DO 25 K=2,LA
      RKK(D1,1,K)=RKK(D1,1,2)
      MF(D1,1,K)=1
      MT(D1,1,K)=1
   25 MK(D1,1,K)=1
      IF(NB-1)100,835,836
```

```
836 DO 26 I=DDD1,DD2
    DO 26 J=2,NP1
    RKK(I,J,1)=AI+AA*T(I)-AA*P(D1,1)/CIC1+AA*(P(D1,1)-P(I,J))/C(
   1I,J,1)
    MF(I,J,1)=1
    MT(I,J,1)=1
 26 MK(I,J,1)=1
    I=D2
    RKK(D2,1,2)=AI+AA*T(D2)-AA*P(D1,1)/CIC1+AA*(P(D1,1)-P(D2,1))
   1/CIC2
    DO 601 K=2,LA
    RKK(D2,1,K)=RKK(D2,1,2)
    MF(D2,1,K)=1
    MT(D2,1,K)=1
601 MK(D2,1,K)=1
835 DO 29 I=DDD2,E1
    DO 29 J=1,NP1
    RKK(I,J,1)=AI+AA*T(I)-AA*P(D1,1)/CIC1+AA*(P(D1,1)-P(D2,1))/
   1CIC2+AA*(P(D2,1)-P(I,J))/C(I,J,1)
    MF(I,J,1)=1
    MT(I,J,1)=1
    MK(I,J,1)=1
    IF(I-E1)29,570,100
 29 CONTINUE
570 WRITE(6,804)RKK(E1,1,1)
804 FORMAT(4X,'RO=',F12.6)

    STEP 5 FOR BOTH MODELS A&B WHEN M=2

    M=2
    WRITE(6,32)M
 32 FORMAT(2X,'M=',I5)
    DO 30 I=2,DD1
    DO 30 J=2,NP1
    RKK(I,J,LA)=AI+AA*T(I)-AA*P(I,J)/C(I,J,1)
    MFM2=1
    MTM1=1
    WRITE(6,602)RKK(I,J,LA),P(MTM1,MFM2),MTM1,I,P(I,J),C(I,J,1)
602 FORMAT(2X,'R=',F12.6,2X,'F=',F12.6,2X,'T=',I5,2X,'I=',I5,2X,'
   1J=',F12.6,2X,'C=',F14.6)
 30 CONTINUE

    STEP 6 FOR BOTH MODELS A&B

166 DO 31 II=M,DD1
    PK2(II)=8000.
 31 JJR(II)=2
    DO 47 K=2,LA
    LL=1
    II=M
 42 IF(JJR(II)-NP1)40,40,41
 40 JR=JJR(II)
    PPP1=1.-(1.-P(D1,1))*EXP(C(D1,1,K)*(T(D1)-T(II)))
 33 IF(PPP1-P(II,JR))34,34,35
 34 IF(RKK(II,JR,LA)-4000.)380,39,39
380 PPK1=(AI+AA*(T(D1)-T(II)))*(1.-P(II,JR))+AA*(P(II,JR)-P(D1,1
   1))/C(II,JR,LA)+RKK(II,JR,LA)
    IF(PK2(II)-7000.)36,36,38
 36 IF(PK2(II)-PPK1)38,39,39
 39 JR=JR+1
```

```
      IF(JR-NP1)33,33,35
   38 PK2(II)=PPK1
      MMF1(II)=JR
      JR=JR+1
      IF(JR-NP1)33,33,35
   35 JJR(II)=JR
   41 IF(LL-1)100,515,43
  515 LL=2
      PK3=PK2(II)
      IF(PK2(II)-7000.)145,146,146
  145 MT2=II
      MF2=MMF1(II)
  146 II=II+1
      IF(II-DD1)42,42,44
   43 IF(PK2(II)-PK3)45,46,46
   46 II=II+1
      IF(II-DD1)42,42,44
   45 PK3=PK2(II)
      IF(PK2(II)-7000.)147,148,148
  147 MT2=II
      MF2=MMF1(II)
  148 II=II+1
      IF(II-DD1)42,42,44
   44 IF(RKK(D1,1,K)-PK3)47,47,48
   48 RKK(D1,1,K)=PK3
      MF(D1,1,K)=MF2
      MT(D1,1,K)=MT2
      MK(D1,1,K)=M
   47 CONTINUE
      IF(NB-1)100,860,880
  880 MN=DD2
      JVR=2
      GO TO 861
  860 MN=E1
      JVR=1
  861 DO 66 I=DDD1,MN
      DO 66 J=JVR,NP1
      II=M
      LL=1
   56 PPK2=8000.
      JR=2
      L=1
      FP=C(I,J,1)*(T(D1)-T(II))
      IF(FP-100.)868,869,869
  869 PPP1=0.0
      GO TO 49
  868 PPP1=1.-(1.-P(D1,1))*EXP(C(I,J,1)*(T(D1)-T(II)))
   49 IF(PPP1-P(II,JR))50,50,51
   50 IF(RKK(II,JR,LA)-4000.)381,53,53
  381 PPK1=(AI+AA*(T(I)-T(II)))*(1.-P(II,JR))+AA*((P(II,JR)-P(D1,1))/
     1C(II,JR,LA)+(P(D1,1)-P(I,J))/C(I,J,1))+RKK(II,JR,LA)
      IF(L-1)100,68,52
   68 L=2
      PPK2=PPK1
      MFM1=JR
      JR=JR+1
      IF(JR-NP1)49,49,51
   52 IF(PPK2-PPK1)54,53,53
   53 JR=JR+1
      IF(JR-NP1)49,49,51
```

```
 54 PPK2=PPK1
    MFM1=JR
    JR=JR+1
    IF(JR-NP1)49,49,51
 51 IF(LL-1)100,55,56
 55 LL=2
    PPK3=PPK2
    IF(PPK2-7000.)61,62,62
 61 MFM2=MFM1
    MTM1=II
 62 II=II+1
    IF(II-DD1)56,56,57
 58 IF(PPK2-PPK3)59,60,60
 60 II=II+1
    IF(II-DD1)56,56,57
 59 PPK3=PPK2
    IF(PPK2-7000.)64,65,65
 64 MFM2=MFM1
    MTM1=II
 65 II=II+1
    IF(II-DD1)56,56,57
 57 IF(RKK(I,J,1)-PPK3)870,870,67
 67 RKK(I,J,1)=PPK3
    MF(I,J,1)=MFM2
    MT(I,J,1)=MTM1
    MK(I,J,1)=M
870 IF(N8-1)100,862,66
862 IF(I-E1)66,530,100
 66 CONTINUE
    I=D2
    II=M
    LL=1
 79 PPK2=8000.
    JR=2
    L=1
 72 IF(RKK(II,JR,LA)-4000.)382,74,74
382 PPK1=(AI+AA*(T(D2)-T(II)))*(1.-P(II,JR))+AA*((P(II,JR)-P(D1,1))/
   1C(II,JR,LA)+(P(D1,1)-P(D2,1))/CIC2)+RKK(II,JR,LA)
    IF(L-1)100,70,71
 70 L=2
    PPK2=PPK1
    MFM1=JR
    JR=JR+1
    IF(JR-NP1)72,72,75
 71 IF(PPK2-PPK1)73,74,74
 74 JR=JR+1
    IF(JR-NP1)72,72,75
 73 PPK2=PPK1
    MFM1=JR
    JR=JR+1
    IF(JR-NP1)72,72,75
 75 IF(LL-1)100,76,77
 76 LL=2
    PPK3=PPK2
    IF(PPK2-7000.)385,386,386
385 MFM2=MFM1
    MTM1=II
386 II=II+1
    IF(II-DD1)79,79,80
 77 IF(PPK3-PPK2)82,82,83
```

```
 82 II=II+1
    IF(II-DD1)79,79,80
 83 PPK3=PPK2
    IF(PPK2-7000.)387,388,388
387 MFM2=MFM1
    MTM1=II
388 II=II+1
    IF(II-DD1)79,79,80
 80 IF(RKK(D2,1,2)-PPK3)85,85,86
 86 RKK(D2,1,2)=PPK3
    MF(D2,1,2)=MFM2
    MT(D2,1,2)=MTM1
    MK(D2,1,2)=M
    DO 87 K=2,LA
    RKK(D2,1,K)=RKK(D2,1,2)
    MF(D2,1,K)=MF(D2,1,2)
    MT(D2,1,K)=MT(D2,1,2)
 87 MK(D2,1,K)=M
 85 DO 105 I=DDD2,E1
    DO 105 J=1,NP1
    II=M
    LL=1
 99 PPK2=8000.
    JR=2
    L=1
 92 IF(RKK(II,JR,LA)-4000.)389,94,94
389 PPK1=(AI+AA*(T(I)-T(II)))*(1.-P(II,JR))+AA*((P(II,JR)-P(D1,1))/
   1C(II,JR,LA)+(P(D1,1)-P(D2,1))/CID2+(P(D2,1)-P(I,J))/C(I,J,1))
   2+RKK(II,JR,LA)
    IF(L-1)100,90,91
 90 L=2
    PPK2=PPK1
    MFM1=JR
    JR=JR+1
    IF(JR-NP1)92,92,95
 91 IF(PPK2-PPK1)93,94,94
 94 JR=JR+1
    IF(JR-NP1)92,92,95
 93 PPK2=PPK1
    MFM1=JR
    JR=JR+1
    IF(JR-NP1)92,92,95
 95 IF(LL-1)100,96,97
 96 LL=2
    PPK3=PPK2
    IF(PPK2-7000.)390,391,391
390 MFM2=MFM1
    MTM1=II
391 II=II+1
    IF(II-DD1)99,99,101
 97 IF(PPK3-PPK2)102,102,103
102 II=II+1
    IF(II-DD1)99,99,101
103 PPK3=PPK2
    IF(PPK2-7000.)392,393,393
392 MFM2=MFM1
    MTM1=II
393 II=II+1
    IF(II-DD1)99,99,101
101 IF(RKK(I,J,1)-PPK3)535,535,106
```

```
106 RKK(I,J,1)=PPK3
    MF(I,J,1)=MFM2
    MT(I,J,1)=MTM1
    MK(I,J,1)=M
535 IF(I-E1)105,530,100
105 CONTINUE
530 WRITE(6,805)PPK3,P(MTM1,MFM2),MTM1,M
805 FORMAT(4X,'RO=',F12.6,2X,'J=',F12.6,2X,'T=',I5,2X,'MM=',I5)

    STEP 7 FOR BOTH MODELS A&B

    IF(M-DD1)180,845,100
845 IF(NB-1)100,353,218

    STEP 5 FOR BOTH MODELS A&B WHEN M>2

180 M=M+1
    WRITE(6,125)M
125 FORMAT(2X,'M=',I5)
    DO 129 I=M,DD1
    DO 129 J=2,NP1
    I1=M-1
    I2=I-1
    DO 111 II=I1,I2
    PK2(II)=8000.
    PPP1=1.-(1.-P(I,J))*EXP(C(I,J,1)*(T(I)-T(II)))
    DO 109 JR=2,NP1
    IF(PPP1-P(II,JR))109,110,110
109 CONTINUE
    JJR(II)=NP
    GO TO 111
110 JJR(II)=JR
111 CONTINUE
    DO 129 K=2,LA
    IF(I-DD1)523,524,100
524 IF(K-LA)129,523,100
523 LL=1
    I1=M-1
    I2=I-1
    DO118 II=I1,I2
    IF(JJR(II)-NP)112,547,100
112 PPP2=1.-(1.-P(I,J))*EXP(C(I,J,K)*(T(I)-T(II)))
    JR=JJR(II)
117 IF(PPP2-P(II,JR))114,114,113
114 C1=-ALOG((1.-P(I,J))/(1.-P(II,JR)))/(T(I)-T(II))
    IF(M-3)100,171,172
172 DO 173 KP=2,LA
    IF(C(II,JR,KP)-C1)173,174,174
173 CONTINUE
    KKP=LA
    GO TO 520
174 KKP=KP
520 IF(RKK(II,JR,KKP)-4000.)394,116,116
394 PPK1=(AI+AA*(T(I)-T(II)))*(1.-P(II,JR))+AA*(P(II,JR)-P(I,J))/C1
   1+RKK(II,JR,KKP)
    GO TO 521
171 IF(RKK(II,JR,LA)-4000.)395,116,116
395 PPK1=(AI+AA*(T(I)-T(II)))*(1.-P(II,JR))+AA*(P(II,JR)-P(I,J))/C1
   1+RKK(II,JR,LA)
521 IF(PK2(II)-7000.)150,115,115
```

```
150 IF(PK2(II)-PPK1)115,116,116
116 JR=JR+1
    IF(JR-NP1)117,117,113
115 PK2(II)=PPK1
    MMFI(II)=JR
    JR=JR+1
    IF(JR-NP1)117,117,113
113 JJR(II)=JR
547 IF(LL-1)100,119,120
119 LL=2
    PPK3=PK2(II)
    IF(PK2(II)-7000.)121,118,118
121 MFM2=MMFI(II)
    MTM1=II
    GO TO 118
120 IF(PPK3-PK2(II))118,118,123
123 PPK3=PK2(II)
    IF(PK2(II)-7000.)124,118,118
124 MFM2=MMF1(II)
    MTM1=II
118 CONTINUE
    RBB(I,J,K)=PPK3
    IF(PPK3-7000.)127,128,128
127 WRITE(6,126)RBB(I,J,K),P(MTM1,MFM2),MTM1,I,P(I,J),C(I,J,K)
126 FORMAT(2X,'R=',F12.6,2X,'F=',F12.6,2X,'T=',I5,2X,'I=',I5,2X,'J='
   1,F12.6,2X,'C=',F14.6)
    GO TO 129
128 WRITE(6,130)RBB(I,J,K),I,J
130 FORMAT(2X,'R=',F12.6,2X,'I=',I5,2X,'J=',I5)
129 CONTINUE
    DO 730 I=M,DD1
    DO 730 J=2,NP1
    DO 730 K=2,LA
    IF(I-DD1)731,732,100
732 IF(K-LA)730,731,100
731 RKK(I,J,K)=RBB(I,J,K)
730 CONTINUE
    GO TO 100

    STEPS 8&9 FOR MODEL B

218 M=1
    WRITE(6,219)M
219 FORMAT(2X,'N=',I5)
    I=D1
    J=1
    DO 220 K=2,LA
    MV1=MF(D1,1,K)
    MV2=MT(D1,1,K)
    WRITE(6,221)RKK(D1,1,K),P(MV2,MV1),MT(D1,1,K),MK(D1,1,K),I,
   1P(I,J),C(I,J,K)
221 FORMAT(2X,'R=',F12.6,2X,'F=',F12.6,2X,'T=',I5,2X,'M=',I5,2X,
   1'I=',I5,2X,'J=',F12.6,2X,'C=',F14.6)
220 CONTINUE
    DO 223 I=DDD1,DD2
    DO 223 J=2,NP1
    RKK(I,J,LA)=RKK(I,J,1)
    MV1=MF(I,J,1)
    MV2=MT(I,J,1)
    WRITE(6,224)RKK(I,J,1),P(MV2,MV1),MT(I,J,1),MK(I,J,1),I,P(I,
```

```
     1J),C(I,J,1)
 224 FORMAT(2X,'R=',F12.6,2X,'F=',F12.6,2X,'T=',I5,2X,'M=',I5,2X,
     1'I=',I5,2X,'J=',F12.6,2X,'C=',F14.6)
 223 CONTINUE
     GO TO 300

     STEP 12 FOR MODEL B

 350 WRITE(6,225)M
 225 FORMAT(2X,'M=',I5)
     I1=DD1+M
     DO 226 I=I1,DD2
     DO 226 J=2,NP1
     I3=M+DD1-1
     I2=I-1
     DO 227 II=I3,I2
     PK2(II)=8000.
     IF(II-D1)100,227,313
 313 PPP1=1.-(1.-P(I,J))*EXP(C(I,J,1)*(T(I)-T(II)))
     DO 229 JR=2,NP1
     IF(PPP1-P(II,JR))229,210,210
 229 CONTINUE
     JJR(II)=NP
     GO TO 227
 210 JJR(II)=JR
 227 CONTINUE
     DO 226 K=2,LA
     IF(I-DD2)550,551,100
 551 IF(K-LA)226,550,100
 550 LL=1
     I3=M+DD1-1
     I2=I-1
     DO 228 II=I3,I2
     IF(II-D1)100,230,231
 231 IF(JJR(II)-NP)232,548,100
 232 PPP2=1.-(1.-P(I,J))*EXP(C(I,J,K)*(T(I)-T(II)))
     JR=JJR(II)
 249 IF(PPP2-P(II,JR))236,236,233
 236 C1=-ALOG((1.-P(I,J))/(1.-P(II,JR)))/(T(I)-T(II))
     IF(M-2)100,237,238
 238 DO 239 KP=2,LA
     IF(C(II,JR,KP)-C1)239,240,240
 239 CONTINUE
     KP=LA
 240 IF(RKK(II,JR,KP)-4000.)402,248,248
 402 PKK1=(AI+(T(I)-T(II))*AA)*(1.-P(II,JR))+AA*(P(II,JR)-P(I,J))/C1
    1+RKK(II,JR,KP)
     GO TO 241
 237 IF(RKK(II,JR,1)-4000.)403,243,248
 403 PKK1=(AI+AA*(T(I)-T(II)))*(1.-P(II,JR))+AA*(P(II,JR)-P(I,J))/C1
    1+RKK(II,JR,1)
     GO TO 241
 230 DO 242 KP=2,LA
     IF(C(D1,1,KP)-C(I,J,1))242,243,243
 242 CONTINUE
     KP=LA
 243 PKK3=(AI+AA*(T(I)-T(D1)))*(1.-P(D1,1))+AA*(P(D1,1)-P(I,J))/C(I,J
    1,1)+RKK(D1,1,KP)
     MFM2=1
     MTM1=D1
```

```
      LL=2
      GO TO 228
  241 IF(PK2(II)-7000.)316,247,247
  316 IF(PK2(II)-PKK1)247,248,248
  248 JR=JR+1
      IF(JR-NP1)249,249,233
  247 PK2(II)=PKK1
      MMF1(II)=JR
      JR=JR+1
      IF(JR-NP1)249,249,233
  233 JJR(II)=JR
  548 IF(LL-1)100,252,253
  252 LL=2
      PKK3=PK2(II)
      IF(PK2(II)-7000.)254,228,228
  254 MFM2=MMF1(II)
      MTM1=II
      GO TO 228
  253 IF(PKK3-PK2(II))228,228,255
  255 PKK3=PK2(II)
      IF(PK2(II)-7000.)256,228,228
  256 MFM2=MMF1(II)
      MTM1=II
  228 CONTINUE
      RBB(I,J,K)=PKK3
      IF(PKK3-7000.)257,259,259
  257 WRITE(6,258)RBB(I,J,K),P(MTM1,MFM2),MTM1,I,P(I,J),C(I,J,K)
  258 FORMAT(2X,'R=',F12.6,2X,'F=',F12.6,2X,'T=',I5,2X,'I=',I5,2X,'J=',
     1F12.6,2X,'C=',F14.6)
      GO TO 226
  259 WRITE(6,260)RBB(I,J,K),I,J
  260 FORMAT(2X,'R=',F12.6,2X,'I=',I5,2X,'J=',I5)
  226 CONTINUE
      I1=DD1+M
      DO 735 I=I1,DD2
      DO 735 J=2,NP1
      DO 735 K=2,LA
      IF(I-DD2)736,737,100
  737 IF(K-LA)735,736,100
  736 RKK(I,J,K)=RBB(I,J,K)
  735 CONTINUE

      STEPS 10,13&14 FOR MODEL B

  300 I=D2
      I1=M+DD1
      DO 301 II=I1,DD2
      IF(II-D1)100,301,303
  303 PK2(II)=8000.
      JJR(II)=2
  301 CONTINUE
      DO 304 K=2,LA
      LL=1
      I1=M+DD1
      DO 305 II=I1,DD2
      IF(II-D1)100,306,307
  307 IF(JJR(II)-NP1)308,308,309
  308 JR=JJR(II)
      FP=C(D2,1,K)*(T(D2)-T(II))
      IF(FP-100.)811,812,812
```

```
812 PPP1=0.0
    GO TO 321
811 PPP1=1.-(1.-P(D2,1))*EXP(C(D2,1,K)*(T(D2)-T(II)))
321 IF(PPP1-P(II,JR))310,310,311
310 IF(RKK(II,JR,LA)-4000.)407,320,320
407 PPK1=(AI+AA*(T(D2)-T(II)))*(1.-P(II,JR))+AA*(P(II,JR)-P(D2,1))/
   1C(II,JR,LA)+RKK(II,JR,LA)
    GO TO 312
306 PPK1=(AI+(T(D2)-T(D1))*AA)*(1.-P(D1,1))+AA*(P(D1,1)-P(D2,1))
   1/CIC2+RKK(D1,1,LA)
    MFM2=1
    MTM1=D1
    LL=2
    GO TO 305
312 IF(PK2(II)-7000.)318,319,319
318 IF(PK2(II)-PPK1)319,320,320
320 JR=JR+1
    IF(JR-NP1)321,321,311
319 PK2(II)=PPK1
    MMF1(II)=JR
    JR=JR+1
    IF(JR-NP1)321,321,311
311 JJR(II)=JR
309 IF(LL-1)100,322,323
322 LL=2
    PPK3=PK2(II)
    IF(PK2(II)-7000.)324,305,305
324 MTM1=II
    MFM2=MMF1(II)
    GO TO 305
323 IF(PK2(II)-PPK3)325,305,305
325 PPK3=PK2(II)
    IF(PK2(II)-7000.)326,305,305
326 MTM1=II
    MFM2=MMF1(II)
305 CONTINUE
    IF(RKK(D2,1,K)-PPK3)304,304,327
327 RKK(D2,1,K)=PPK3
    MF(D2,1,K)=MFM2
    MT(D2,1,K)=MTM1
    MK(D2,1,K)=M
304 CONTINUE
    DO 330 I=DDD2,E1
    DO 330 J=1,NP1
    LL=1
    I1=M+DD1
    DO 331 II=I1,DD2
    IF(II-D1)100,332,333
333 PPK2=8000.
    JR=2
    L=1
    FP=C(I,J,1)*(T(D2)-T(II))
    IF(FP-100.)808,809,809
809 PPP1=0.0
    GO TO 338
808 PPP1=1.-(1.-P(D2,1))*EXP(C(I,J,1)*(T(D2)-T(II)))
338 IF(PPP1-P(II,JR))334,334,335
334 IF(RKK(II,JR,LA)-4000.)408,341,341
408 PPK1=(AI+AA*(T(I)-T(II)))*(1.-P(II,JR))+AA*((P(II,JR)-P(D2,1))/
   1C(II,JR,LA)+(P(D2,1)-P(I,J))/C(I,J,1))+RKK(II,JR,LA)
```

```
      IF(L-1)100,336,337
 336  L=2
      PPK2=PPK1
      MFM1=JR
      JR=JR+1
      IF(JR-NP1)338,338,335
 337  IF(PPK2-PPK1)340,341,341
 341  JR=JR+1
      IF(JR-NP1)338,338,335
 340  PPK2=PPK1
      MFM1=JR
      JR=JR+1
      IF(JR-NP1)338,338,335
 335  IF(LL-1)100,343,344
 343  LL=2
      PPK3=PPK2
      IF(PPK2-7000.)345,331,331
 345  MFM2=MFM1
      MTM1=II
      GO TO 331
 344  IF(PPK2-PPK3)346,331,331
 346  PPK3=PPK2
      IF(PPK2-7000.)347,331,331
 347  MFM2=MFM1
      MTM1=II
      GO TO 331
 332  PPK3=(AI+AA*(T(I)-T(D1)))*(1.-P(D1,1))+AA*((P(D1,1)-P(D2,1))/CIC
     12+(P(D2,1)-P(I,J))/C(I,J,1))+RKK(D1,1,LA)
      LL=2
      MFM2=1
      MTM1=D1
 331  CONTINUE
      IF(RKK(I,J,1)-PPK3)532,532,603
 603  RKK(I,J,1)=PPK3
      MF(I,J,1)=MFM2
      MT(I,J,1)=MTM1
      MK(I,J,1)=M
 532  IF(I-E1)330,352,100
 330  CONTINUE
 352  WRITE(6,301)PPK3,P(MTM1,MFM2),MTM1,M
 301  FORMAT(4X,'RO=',F12.6,2X,'J=',F12.6,2X,'T=',I5,2X,'MM=',I5)

      STEP 11 FOR MODEL B

      M=M+1
      IF(M-D2+D1)350,350,353

      STEPS 8&9 FOR MODEL A, STEPS 15&16 FOR MODEL B

 353  M=1
      WRITE(6,358)M
 358  FORMAT(2X,'V=',I5)
      I=E2
      J=1
      DO 359 K=2,LA
      MV1=MF(D2,1,K)
      MV2=MT(D2,1,K)
      WRITE(6,360)RKK(D2,1,K),P(MV2,MV1),MT(D2,1,K),MK(D2,1,K),I,
     1P(I,J),C(I,J,K)
 360  FORMAT(2X,'R=',F12.6,2X,'F=',F12.6,2X,'T=',I5,2X,'M=',I5,2X,
```

```
   1'I=',I5,2X,'J=',F12.6,2X,'C=',F14.6)
359 CONTINUE
    DO 362 I=DDD2,E1
    DO 362 J=1,NP1
    RKK(I,J,LA)=RKK(I,J,1)
    MV1=MF(I,J,1)
    MV2=MT(I,J,1)
    WRITE(6,363)RKK(I,J,1),P(MV2,MV1),MT(I,J,1),MK(I,J,1),I,P(I,
   1J),C(I,J,1)
363 FORMAT(2X,'R=',F12.6,2X,'F=',F12.6,2X,'T=',I5,2X,'M=',I5,2X,
   1'I=',I5,2X,'J=',F12.6,2X,'C=',F14.6)
    IF(I-E1)362,430,100
362 CONTINUE

    STEP 10 FOR MODEL A, STEP 17 FOR MODEL B

430 I1=M+DD2
    DO 372 II=I1,EE1
    LL=1
    PKK2=8000.
    IF(II-D2)100,370,371
370 PKK3=(AI+AA*(T(E1)-T(D2)))*(1.-P(D2,1))+RKK(D2,1,LA)
    MFM2=1
    MTM1=D2
    LL=2
    GO TO 372
371 DO 414 JK=1,NP1
    IF(RKK(II,JK,LA)-4000.)410,414,414
410 PKK1=(AI+AA*(T(E1)-T(II)))*(1.-P(II,JK))+RKK(II,JK,LA)
    IF(PKK2-7000.)412,413,413
413 PKK2=PKK1
    MFM1=JK
    GO TO 414
412 IF(PKK2-PKK1)415,414,414
415 PKK2=PKK1
    MFM1=JK
414 CONTINUE
    IF(LL-1)100,416,417
416 PKK3=PKK2
    IF(PKK2-7000.)418,372,372
418 MFM2=MFM1
    MTM1=II
    GO TO 372
417 IF(PKK3-PKK2)372,372,419
419 PKK3=PKK2
    IF(PKK2-7000.)420,372,372
420 MFM2=MFM1
    MTM1=II
372 CONTINUE
    WRITE(6,800)PKK3,P(MTM1,MFM2),MTM1,M
800 FORMAT(4X,'RO=',F12.6,2X,'J=',F12.6,2X,'T=',I5,2X,'MM=',I5)
    IF(PKK3-RKK(E1,1,LA))422,423,423
422 RKK(E1,1,LA)=PKK3
    MF(E1,1,1)=MFM2
    MT(E1,1,1)=MTM1
    MK(E1,1,1)=M
423 CONTINUE

    STEP 11 FOR MODEL A, STEP 18 FOR MODEL B
```

```
      M=M+1
      IF(M-E1+D2)431,431,432

      STEP 12 FOR MODEL A, STEP 19 FOR MODEL B

  431 WRITE(6,470)M
  470 FORMAT(2X,'M=',I5)
      I1=M+DD2
      DO 434 I=I1,EE1
      J=1
      LL=1
      I2=M+DD2-1
      I3=I-1
      DO 435 II=I2,I3
      PKK2=8000.
      IF(II-D2)100,436,437
  436 LL=2
      DO 501 K=2,LA
      IF(C(D2,1,K)-C(I,J,1))501,503,503
  501 CONTINUE
      K=LA
  503 PKK3=(AI+AA*(T(I)-T(D2)))*(1.-P(D2,1))+AA*(P(D2,1)-P(I,J))/
     1C(I,J,1)+RKK(D2,1,K)
      MFM2=1
      MTM1=D2
      GO TO 435
  437 PPP1=1.-(1.-P(I,1))*EXP(C(I,1,1)*(T(I)-T(II)))
      JR=2
  441 IF(PPP1-P(II,JR))440,439,439
  440 JR=JR+1
      IF(JR-NP1)441,441,459
  439 JJV=JR
  450 IF(M-2)100,690,620
  620 C1=-ALOG(PAC/(1.-P(II,JJV)))/(T(I)-T(II))
      DO 621 KP=2,LA
      IF(C(II,JJV,KP)-C1)621,622,622
  621 CONTINUE
      KP=LA
  622 IF(RKK(II,JJV,KP)-4000.)623,445,445
  623 PKK1=(AI+AA*(T(I)-T(II)))*(1.-P(II,JJV))+AA*(P(II,JJV)-P(I,J))
     1/C1+RKK(I1,JJV,KP)
      GO TO 625
  690 IF(RKK(II,JJV,LA)-4000.)443,445,445
  443 C1=-ALOG(PAC/(1.-P(II,JJV)))/(T(I)-T(II))
      PKK1=(AI+AA*(T(I)-T(II)))*(1.-P(II,JJV))+AA*(P(II,JJV)-P(I,J
     1))/C1+RKK(II,JJV,LA)
  625 IF(PKK2-7000.)447,448,448
  448 PKK2=PKK1
      MFM1=JJV
  445 JJV=JJV+1
      IF(JJV-NP1)450,450,459
  447 IF(PKK2-PKK1)452,456,456
  452 PKK2=PKK1
      MFM1=JJV
  456 JJV=JJV+1
      IF(JJV-NP1)450,450,459
  459 IF(LL-1)100,460,461
  460 LL=2
      PKK3=PKK2
      IF(PKK3-7000.)462,435,435
```

```
462 MFM2=MFM1
    MTM1=II
    GO TO 435
461 IF(PKK3-PKK2)435,435,464
464 PKK3=PKK2
    IF(PKK3-7000.)465,435,435
465 MFM2=MFM1
    MTM1=II
435 CONTINUE
    RKK(I,J,LA)=PKK3
    IF(PKK3-7000.)472,473,473
472 WRITE(6,474)RKK(I,J,LA),P(MTM1,MFM2),MTM1,I,P(I,J),C(I,J,1)
474 FORMAT(2X,'R=',F12.6,2X,'F=',F12.6,2X,'T=',I5,2X,'I=',I5,2X,'J='
   1,F12.6,2X,'C=',F14.6)
    GO TO 705
473 WRITE(6,475)RKK(I,J,LA),I,J
475 FORMAT(2X,'R=',F12.6,2X,'I=',I5,2X,'J=',I5)
705 DO 434 J=2,NP1
    DO 630 II=I2,I3
    PK2(II)=8000.
    IF(II-D2)100,630,631
631 PPP1=1.-(1.-P(I,J))*EXP(C(I,J,1)*(T(I)-T(II)))
    DO 632 JR=2,NP1
    IF(PPP1-P(II,JR))632,633,633
632 CONTINUE
    JJR(II)=NP
    GO TO 630
633 JJR(II)=JR
630 CONTINUE
    DO 434 K=2,LA
    IF(I-EE1)637,638,100
638 IF(K-LA)434,637,100
637 LL=1
    DO 639 II=I2,I3
    IF(II-D2)100,640,641
641 IF(JJR(II)-NP)642,643,100
642 FP=C(I,J,K)*(T(I)-T(II))
    IF(FP-100.)712,710,710
710 PPP2=0.
    GO TO 711
712 PPP2=1.-(1.-P(I,J))*EXP(C(I,J,K)*(T(I)-T(II)))
711 JR=JJR(II)
663 IF(PPP2-P(II,JR))644,644,645
644 C1=-ALOG((1.-P(I,J))/(1.-P(II,JR)))/(T(I)-T(II))
    IF(M-2)100,647,648
648 DO 649 KP=2,LA
    IF(C(II,JR,KP)-C1)649,650,650
649 CONTINUE
    KP=LA
650 IF(RKK(II,JR,KP)-4000.)651,662,662
651 PKK1=(AI+(T(I)-T(II))*AA)*(1.-P(II,JR))+AA*(P(II,JR)-P(I,J))/C1
   1+RKK(II,JR,KP)
    GO TO 653
647 IF(RKK(II,JR,1)-4000.)654,662,662
654 PKK1=(AI+AA*(T(I)-T(II)))*(1.-P(II,JR))+AA*(P(II,JR)-P(I,J))/C1
   1+RKK(II,JR,1)
    GO TO 653
640 DO 655 KP=2,LA
    IF(C(D2,1,KP)-C(I,J,1))655,656,656
655 CONTINUE
```

```
      KP=LA
  656 PKK3=(AI+AA*(T(I)-T(D2)))*(1.-P(D2,1))+AA*(P(D2,1)-P(I,J))/C(I,J
     1,1)+RKK(D2,1,KP)
      MFM2=1
      MTM1=D2
      LL=2
      GO TO 639
  653 IF(PK2(II)-7000.)660,661,661
  660 IF(PK2(II)-PKK1)661,662,662
  662 JR=JR+1
      IF(JR-NP1)663,663,645
  661 PK2(II)=PKK1
      MMF1(II)=JR
      JR=JR+1
      IF(JR-NP1)663,663,645
  645 JJR(II)=JR
  643 IF(LL-1)100,667,668
  667 LL=2
      PKK3=PK2(II)
      IF(PK2(II)-7000.)669,639,639
  669 MFM2=MMF1(II)
      MTM1=II
      GO TO 639
  668 IF(PKK3-PK2(II))639,639,670
  670 PKK3=PK2(II)
      IF(PK2(II)-7000.)671,639,639
  671 MFM2=MMF1(II)
      MTM1=II
  639 CONTINUE
      RBB(I,J,K)=PKK3
      IF(PKK3-7000.)672,673,673
  672 WRITE(6,674)RBB(I,J,K),P(MTM1,MFM2),MTM1,I,P(1,J),C(I,J,K)
  674 FORMAT(2X,'R=',F12.6,2X,'F=',F12.6,2X,'T=',I5,2X,'I=',I5,2X,'J='
     1,F12.6,2X,'C=',F14.6)
      GO TO 434
  673 WRITE(6,675)RBB(I,J,K),I,J
  675 FORMAT(2X,'R=',F12.6,2X,'I=',I5,2X,'J=',I5)
  434 CONTINUE
      I1=M+DD2
      DO 740 I=I1,EE1
      DO 740 J=2,NP1
      DO 740 K=2,LA
      IF(I-EE1)741,742,100
  742 IF(K-LA)740,741,100
  741 RKK(I,J,K)=RBB(I,J,K)
  740 CONTINUE

      STEP 13 FOR MODEL A, STEP 20 FOR MODEL B

      GO TO 430
      STEP 14 FOR MODEL A, STEP 21 FOR MODEL B---PRINTS THE UPPER
      BOUND EXPECTED TOTAL COST RKK(E1,1,LA)

  432 MV1=MF(E1,1,1)
      MV2=MT(E1,1,1)
      WRITE(6,500)RKK(E1,1,LA),P(MV2,MV1),MT(E1,1,1),MK(E1,1,1)
  500 FORMAT(2X,'ROPT=',F12.6,2X,'F=',F12.6,2X,'T=',I5,2X,'NN=',I5)
  100 STOP
      END
```

APPENDIX C

COMPUTER PROGRAM FOR CALCULATING THE OPTIMAL EXPECTED TOTAL COST WITH COMPLETE INFORMATION ABOUT FAILURE DISTRIBUTION OF THE SYSTEM.

```
C     ****************************************************************
C
C     CALCULATION OF THE OPTIMAL EXPECTED TOTAL COST WITH KNOWN
C     IFR FAILURE DISTRIBUTION F0
C
C     NOTATIONS
C     AI---INSPECTION COST PER INSPECTION
C     AA---COST OF UNDETECTED FAILURE PER UNIT TIME
C     T(TF)---MAXIMUM LIFE TIME OF THE SYSTEM
C     RK(I),RB(I)---OPTIMUM EXPECTED MAINTENANCE COST UP TO THE
C    1TIME T(I) AND AT STAGE M-1 AND M RESPECTIVELY
C     THIS PROGRAM WAS WRITTEN BY KOUROSH-KARAMPISHEH, DEPARTMENT
C     OF INDUSTRIAL ENGINEERING KANSAS STATE UNIVERSITY, MANHATTAN
C     KANSAS, MAY, 1979
C     ****************************************************************
C
      INTEGER TF,TF1,TMT
      DIMENSION RK(15),RB(15),C(15),P(15),T(15)
      READ(5,1)AA,AI,TF
    1 FORMAT(2F10.6,I5)

      INPUT DATA ABOUT IFR FAILURE DISTRIBUTION F0

      DO 2 I=1,TF
      READ(5,40)P(I),C(I),T(I)
   40 FORMAT(3F10.6)
    2 CONTINUE

      CALCULATES RK(TF) FOR M=1

      M=1
      TF1=TF-1
      V=0.0
      DO 4 L=2,TF1
    4 V=V+(P(L-1)-P(L))/C(L)
      RK(TF)=AI+AA*T(TF)+AA*V
      MT=1
      MM=1
      WRITE(6,7)RK(TF)
    7 FORMAT(2X,'RO=',F12.6)

      CALCULATES RK(I) FOR M=2 AND I<TF

      M=2
      WRITE(6,12)M
   12 FORMAT(2X,'M=',I5)
      DO 10 L1=2,TF1
      V=0.0
      DO 9 L2=2,L1
    9 V=V+(P(L2-1)-P(L2))/C(L2)
      RK(L1)=AI+AA*T(L1)+AA*V
      TMT=1
      WRITE(6,11)RK(L1),L1,TMT
   11 FORMAT(2X,'R=',F12.6,2X,'I=',I5,2X,'IT=',I5)
```

```
10 CONTINUE

   CALCULATES RK(TF) FOR M>2

30 DO 13 II=M,TF1
   V=0.0
   DO 31 LL=II,TF1
31 V=V+(P(LL)-P(LL+1))/C(LL+1)
   PK=(AI+AA*(T(TF)-T(II)))*(1.-P(II))+V*AA+RKIII)

   COMPARES RK(TF) FOR OLDER AND CURRENT VALUES OF M AND CHOOSES
   THE SMALLER VALUE AND DISCARDS THE LARGER VALUES

   IFIRK(TF)-PK)13,13,14
14 RK(TF)=PK
   MT=II
   MM=M
13 CONTINUE

   SETS NEW M = OLD M + 1 AND COMPARES IT WITH M(MAX)

   M=M+1
   IF(TF1-M)90,15,15

   IF M<M(MAX) + 1, CALCULATES RK(I) FOR TF>M>2

15 WRITE(6,16)M
16 FORMAT(2X,'M=',I5)
   DO 18 LV=M,TF1
   LV1=LV-1
   M1=M-1
   LN=1
   DO 19 LP=M1,LV1
   V=0.0
   DO 20 LE=LP,LV1
20 V=V+(P(LE)-P(LE+1))/C(LE+1)
   PF=(AI+AA*(T(LV)-T(LP)))*(1.-PILP))+AA*V+RK(LP)
   IF(LN-1)100,23,24
23 LN=2
   RB(LV)=PF
   TMT=M-1
24 IF(RB(LV)-PF)19,19,25
25 RB(LV)=PF
   TMT=LP
19 CONTINUE
   WRITE(6,27)RB(LV),LV,TMT
27 FORMAT(2X,'R=',F12.6,2X,'T=',I5,2X,'TT=',I5)
18 CONTINUE
   DO 28 LC=M,TF1
28 RK(LC)=RB(LC)
   GO TO 30

   PRINTS THE OPTIMAL EXPECTED TOTAL COST RK(I) AND OPTIMAL
   NUMBER OF INSPECTIONS M

90 WRITEI6,91)RK(TF),MT,MM
91 FORMAT(2X,'ROPT=',F12.6,2X,'TT=',I5,2X,'M=',I5)
100 STOP
   END
```

ESTIMATION OF AN UPPER BOUND FOR EXPECTED MAINTENANCE COST OF A SYSTEM WITH PARTIALLY KNOWN, INCREASING FAILURE RATE DISTRIBUTION

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A considerable work has been done in the field of maintenance, but except for a few almost all of this work has been based on the complete knowledge about the characteristics of the deteriorating system or equipment especially about the failure characteristics of it. But in real situations the complete information about failure distribution of a system is rarely available especially when a new one is introduced into the existing system. Minimax policy has been devised already to cope with this problem. Basically it gives the best maintenance or inspection policy or timing when the information about the system failure distribution is incomplete. It utilizes the information which is available, to minimize the total cost and at the same time maximizes the total cost with respect to all possible and feasible values of the unknown portion of the failure characteristics. The present work utilizes the information about one and two points of increasing failure rate distributions to find an upper bound for the optimal expected total maintenance cost. The basic difference of this work with previous work in this area is that a procedure and computer program has been divised which utilizes the available information not only about one point of the failure distribution but also searches for improved upper bound for the total cost when information about two points is available.

The comparison has been made on the basis of the available knowledge about a failure distribution between the upper bound total costs and the value of information has been discussed through an example. The variation of the upper bound cost with changes in the cost per inspection and the cost per unit time of undetected failure has been discussed through an example. It has been found basically that having information about two points of the distribution improves (decreases) the upper bound total cost. Also it has been found that the closer the known time of failure probability to the maximum life time, the higher the value of information and the lower the upper bound would be. It has been found that the upperbound total cost is more sensitive to changes of the inspection and undetected failure costs at lower values of these costs. Computer program has been written and used for several different increasing failure rate distributions to find the optimal total costs and policies. These optimal costs were compared with the upper bound total costs and all were lower than the upper bound cost as it could have been expected. Finally an example problem has been worked out to illustrate the application of the results of this work in industry. There is still a lot of study needed in the field of maintenance with partial knowledge to find efficient and computationally feasible methods to optimally utilize the available partial knowledge about the system failure characteristics for finding improved upper bounds.