# A DERIVATION OF THE PROBABILITY DISTRIBUTION FUNCTION OF THE OUTPUT OF A SQUARE-LAW DETECTOR OPERATING IN A JAMMING ENVIRONMENT

by

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# CHAPTER I

# INTRODUCTION

In any type of a communication link the major role of the receiver is to precisely recover the signal being transmitted. The received input usually consists of the desired signal plus noise added in transmission. It is also possible to have intentional interference, i.e., the possibility of a jammer exists with the objective of disrupting the link. To achieve its objective the jammer can introduce a signal, add more noise to that already added by the channel, or introduce a combination of these.

Little or nothing at all it is known of the jammer. The presence of the jammer and the noise introduced by the channel are responsible for the random nature of certain parameters in the received signal. Under such adverse circumstances a reference signal for detection frequently cannot be derived. Therefore, a non-coherent detection scheme is best recommended. The detection system of interest here consists of a bandpass pre-filter followed by a square-law envelope detector and a zonal low-pass filter. A block diagram of the system is shown in Figure 1.

This work is concerned with the derivation of a probability distribution function (pdf) of the receiver output. The pdf derived is very general. Few restrictions have been set in the characteristics of either the signal or jammer. This noise added by the channel or the jammer is assumed to be additive,

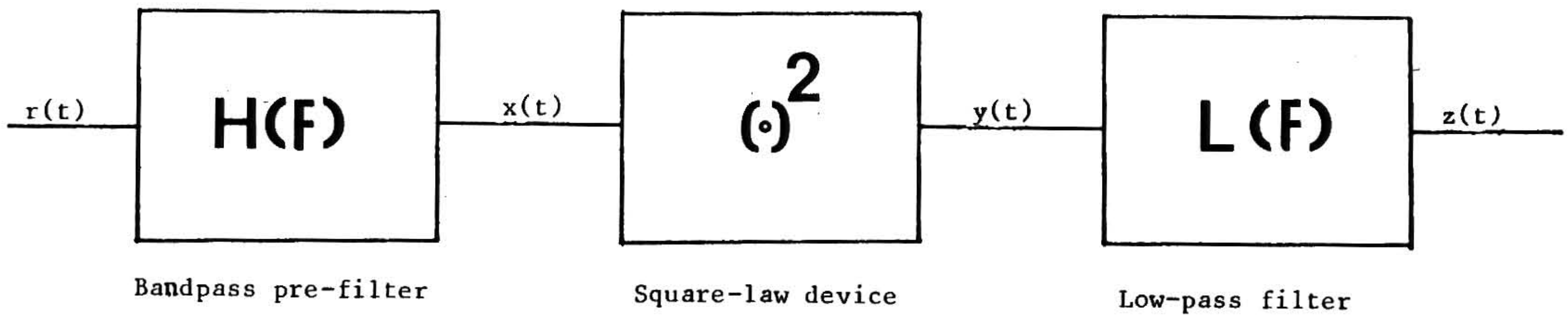


Figure 1. Block Diagram of the System

Gaussian, and zero-mean. This kind of channel will be referred as the Additive White Gaussian Noise (AWGN) channel.

The importance of the pdf derived herein resides in the following fact. With the development of appropriate numerical methods, the probability of detection for a very general square-law detection system can be found for a variety of jamming environments. This is the main goal of this report. Appropriate mathematical models for the receiver are developed in the next chapter. The derivation of the resulting pdf for the receiver output is presented in Chapter III. Some examples are given in Chapter IV.

## CHAPTER II

## MATHEMATICAL MODELS

### Representation of a Bandpass Signal

Signals and channels or systems which satisfy the condition that their bandwidth is much smaller than the carrier frequency are called bandpass signals and channels, or simply bandpass systems. One representation of a bandpass signal is obtained by using the properties of complex envelopes [1,2,4]. Some important properties of bandpass systems are summarized in the following paragraphs.

A bandpass signal, x(t), can be represented by its complex envelope, $\tilde{x}(t)$, defined as

$$\tilde{x}(t) = V(t) \exp [j\psi(t)] , \tag{1}$$

where V(t) denotes the slowly varying amplitude (envelope) of x(t), and $\psi(t)$ denotes the phase angle. The complex-valued waveform, $\tilde{x}(t)$, is basically a low-pass signal waveform. Hence, it is called the equivalent, low-pass signal. The real-valued bandpass signal, x(t), can be expressed in the form

$$x(t) = \text{Real } [\tilde{x}(t) \exp(j2\pi fct)] , \tag{2}$$

with fc being the carrier frequency. Note that the complex envelope is independent of fc.

Equation (2) can be further arranged leading to a second representation

$$x(t) = V(t) \cos [wct + \psi(t)]. \tag{3}$$

With the aid of trigonometric identities, one may write

$$x(t) = Xc(t)\cos(wct) - Xs(t)\sin(wct) \quad , \tag{4}$$

where the signals Xc(t) and Xs(t), termed the quadrature components of x(t), are defined as

$$\begin{aligned} Xc(t) &= V(t)\cos \psi(t) = Xc, \\ Xs(t) &= V(t)\sin \psi(t) = Xs. \end{aligned} \tag{5}$$

The frequency content of Xc(t) and Xs(t) is concentrated at low frequencies. Hence, these components are called low-pass signals. Therefore, by either representation (1) or (5) we have obtained a low-pass model for the signal.

## Representation of a Bandpass Random Process

Suppose now that n(t) is a sample function from a wide-sense stationary (WSS) random process with zero mean and power spectral density, Snn(f). The power spectral density is assumed to be zero outside of an interval of frequencies centered about fc. The process n(t) is said to be a narrowband, bandpass process if the width of the spectral density is much smaller than fc. Under such conditions, the random process can also be represented by any of the two representations (1) or (5) already mentioned [1,2,4].

Let us consider the representation of n(t) by its quadrature components in more detail. If

$$n(t) = nc(t)\cos(wct) - ns(t)\sin(wct), \tag{6}$$

and n(t) is a stationary process with zero-mean, then nc(t) and ns(t) are zero-mean, low-pass processes. Also, the autocorrelation functions of nc(t) and ns(t) are equal, i.e.,

$$Rnc(\zeta) = Rns(\zeta) \ . \tag{7}$$

In the special case where n(t) is a Gaussian process, the quadrature components are jointly Gaussian. For this case, their joint probably density function is

$$f(nc,ns) = \frac{1}{2\pi\sigma^2} \exp \left| \frac{-(nc^2 + ns^2)}{2\sigma^2} \right| \ , \tag{8}$$

where nc and ns are samples of nc(t) and ns(t) respectively and the variance $\sigma^2$ is given by

$$Rnc(0) = Rns(0) = Rnn(0) . \tag{9}$$

Comparing Equations (4) and (6) a correspondence between nc(t) and Xc(t), and between ns(t) and Xs(t) can be established, where for this particular case nc(t) and ns(t) are Gaussian low-pass processes. Thus, the processes nc(t) and ns(t) form the basis for a low-pass model for both the channel noise and the jammer noise.

## Representation of a Bandpass Linear System

A bandpass, linear system can be modeled as an equivalent low-pass system in the same way as a bandpass signal by using complex envelopes [1,2,4]. A system with impulse response, h(t), has an equivalent, low-pass, impulse response, $\tilde{h}(t)$, defined by the relationship

$$h(t) = 2 \text{ Real } [\tilde{h}(t) \exp(j 2\pi f c t)] , \tag{10}$$

where fc is the center frequency. Taking the Fourier transform of (10) yields,

$$H(f) = \tilde{H}(f+fc) + \tilde{H}^*(-f-fc) , \tag{11}$$

where H(f) and $\tilde{H}(f)$ are the Fourier transforms of h(t) and $\tilde{h}(t)$ respectively. $\tilde{H}(f)$ is the equivalent low-pass transfer function of the system.

Inverting the relationship of Equation (10) the equivalent, low-pass transfer function in terms of the bandpass transfer function is found to be

$$\tilde{H}(f) = [H(f+fc)]_{\text{low-pass term}} . \tag{12}$$

It is evident from (12) that a low-pass equivalent model will have a bandwidth equal to one half the bandwidth of the bandpass system.

## Response of a Bandpass System to a Bandpass Signal

All transform relations for linear systems hold for low-pass complex envelope models [1,2]. In the frequency domain, the input-output relation of a bandpass system is

$$Y(f) = H(f)X(f) \tag{13}$$

where Y(f), H(f), and X(f) are the transform representations of the output, the impulse response of the system, and the input respectively.

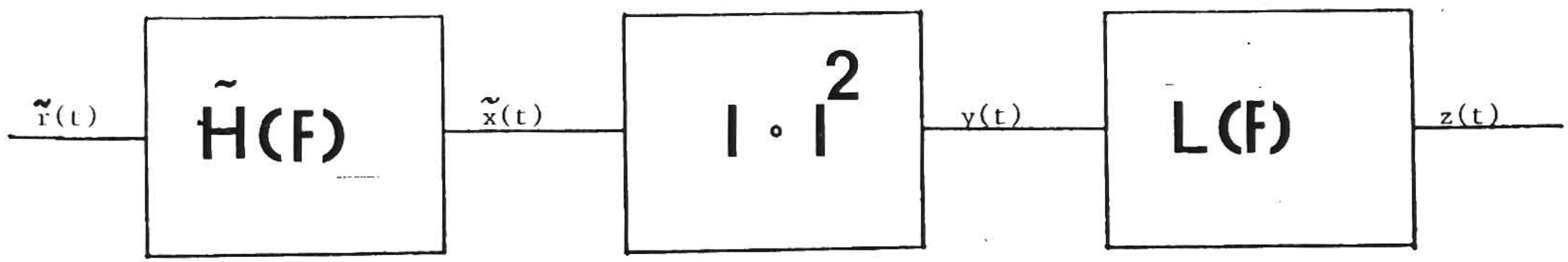


Figure 2. Low-pass, equivalent model of the system

When using the low-pass, equivalent models, the input-output relation is given by

$$\tilde{Y}(f) = \tilde{H}(f)\tilde{X}(f), \tag{14}$$

where $\tilde{Y}(f)$, $\tilde{H}(f)$, and $\tilde{X}(f)$ are the low-pass transforms of the output, transfer function of the system, and the input respectively. The overall equivalent, low-pass model for the system is shown in Figure 2.

## Square-law Device

Detection is accomplished through the use of square-law device. The square-law device is responsible for producing a voltage y(t) proportional to the square of the input signal, x(t), i.e.,

$$y(t) = x^2(t) \ . \tag{15}$$

With the aid of Equations (1) and (2), the input signal, x(t), can be written as

$$x(t) = V(t)\cos[wct + \psi(t)] \ . \tag{16}$$

The output, y(t), is then given by

$$y(t) = V^2(t)\cos^2[wct + \psi(t)] \ . \tag{17}$$

Replacing the squared cosine function with its trigonometric identity yields

$$y(t) = \frac{V^2(t)}{2} + \frac{V^2(t)}{2}\cos[2wct + 2\psi(t)] \ . \tag{18}$$

Since the square-law device is followed by a zonal low-pass filter, the second term of the right-hand side of Equation (18) will be eliminated. Thus, the low-pass model for the square-law device is defined by

$$z(t) = \frac{V^2(t)}{2} , \tag{19}$$

or; in terms of the input complex envelope, $\tilde{x}(t)$,

$$z(t) = \frac{|\tilde{x}(t)|^2}{2} . \tag{20}$$

That is, z(t) is proportional to the magnitude squared of the input complex envelope.

# CHAPTER III

## DERIVATION OF THE PROBABILITY DENSITY FUNCTION OF THE OUTPUT

### Characteristics of the Input Signal to the Envelope Detector

Prior to deriving the pdf of the output, we need first to lay out in some detail the signal of interest, as well as the channel noise and the jammer.

For the sake of simplicity we will assume the pre-filter to be ideal. Thus, the input signal to the square-law detector, x(t), is given by

$$x(t) = s(t) + n(t) + j(t) + jn(t) \quad , \tag{21}$$

where s(t), n(t), j(t), and jn(t) represent the signal of interest, channel noise, jammer as a signal and jammer noise, respectively.

First, let us represent the signal of interest, s(t), by expanding s(t) into its quadrature components:

$$\begin{aligned} s(t) &= P(t)\cos[(wc + \Delta wc)t + \theta] \\ &= P(t)\cos[(wc+\Delta wc)t]\cos\theta - P(t)\sin[(wc+\Delta wc)t]\sin\theta \quad , \end{aligned} \tag{22}$$

with P(t) being the slowly varying envelope, $\Delta wc$ the frequency shift with respect to wc, and $\theta$ the random phase angle of the signal.

Second, let us represent the noise added by the channel, n(t), in terms of its quadrature components, i.e., let

$$n(t) = nc(t)\cos(wct) - ns(t)\sin(wct) \quad , \tag{23}$$

where nc(t) and ns(t) are jointly Gaussian, zero-mean, low-pass, independent processes with variance $\sigma_n^2$ [2,3,4].

Third, let us also represent the jammer signal, j(t), by its quadrature components:

$$j(t) = J(t)\cos[(wc + \Delta wl)t + \theta l] = J(t)\cos[(wc+\Delta wl)t]\cos\theta l - J(t)\sin[(wc+\Delta wl)t]\sin\theta l \ , \quad (24)$$

where J(t) is the slowly varying envelope, Δwl is the frequency shift with respect to wc, and θl the random phase angle of the jammer signal.

Finally, the noise introduced by the jammer, jn(t), may also be represented in quadrature form:

$$jn(t) = njc(t)\cos(wct) - njs(t)\sin(wct) , \quad (25)$$

with njc(t) and njs(t) being jointly Gaussian, zero-mean, low-pass, independent processes with variance $\sigma jn^2$.

Now substituting these representations for the appropriate terms in Equation (21) and grouping common terms we find

$$x(t) = [P(t)\cos[\Delta wct+\theta] + J(t)\cos[\Delta wlt+\theta l] + nc(t) + nc(t) + njc(t)]\cos(wct) - [P(t)\sin[\Delta wct+\theta] + J(t)\sin[\Delta wlt+\theta l] + ns(t) + njs(t)] \sin(wct) \ . \quad (26)$$

Comparing Equations (4) and (26) the following relationships can be established:

$$Xc = P(t)\cos[\Delta wct+\theta] + J(t)\cos[\Delta wlt+\theta l] + nc(t) + njc(t),$$
$$Xs = P(t)\sin[\Delta wct+\theta] + J(t)\sin[\Delta wlt+\theta l] + ns(t) + njs(t), \quad (27)$$

or;

$$Xc = P(t)\cos[\Delta wct+\theta] + J(t)\cos[\Delta w1t+\theta 1] + nc',$$
$$Xs = P(t)\sin[\Delta wct+\theta] + J(t)\sin[\Delta w1t+\theta 1] + ns', \tag{28}$$

where,

$$nc' = nc(t) + njc(t),$$
$$ns' = ns(t) + njs(t). \tag{29}$$

Note that nc'(t) and ns'(t) are jointly Gaussian, zero-mean, low-pass, independent processes where the variance, $\sigma t^2$, is given by

$$\sigma t^2 = \sigma n^2 + \sigma jn^2 . \tag{30}$$

<u>Determination of the Pdf</u>

In order to determine the pdf of the output signal, several changes of variables and transformations of random variables are necessary. The general form for a transformation of random variables is obtained from [3]. That is, given n functions $g_i(x1,x2,...,xn)$, $1 \leq i \leq n$, of the random variables x1,x2,...,xn, we can form a new set of random variables

$$z_i = g_i(x1,x2,...,xn); \quad 1 \leq i \leq n.$$

To determine the joint distribution of f(z1, z2, ..., zn) of the $z_i$, we solve the latter equations for x1, x2, ..., xn in terms of z1, z2, ..., zn. If all sets of solutions for these equations are real, then the distribution for f(z1, z2, ..., zn) is given by

$$f(z1,z2,\ldots zn) = \frac{f(x11,x12,\ldots,x1n)}{|J(x11,x12,\ldots,x1n)|} + \frac{f(x21,x22,\ldots,x2n)}{|J(x21,x22,\ldots,x2n)|} + \ldots + \frac{f(xn1,xn2,\ldots,xnn)}{|J(xn1,xn2,\ldots,xnn)|},$$

where $J(xi1,xi2,\ldots,xin)$ is the Jacobian of the transformation. If any set of solutions is not real in some region, then

$$f(z1,z2,\ldots,zn) = 0$$

over that region.

Suppose that $p_x(x1,x2,\ldots,xn)$ is a pdf of interest and a new set of random variables, y1 through yn, is defined according to the rules

$$x_i = f_i(y1,y2,\ldots,yn); \quad 1 \le i \le n .$$

The probability that x is within some N-dimensional volume, A, is given by

$$\int \ldots \int_A p_x(x1,x2,\ldots,xn)\, dx1\, dx2 \ldots dxn.$$

Making the change of variables, this intergal becomes

$$\int \ldots \int_B p_x[f1(y1,y2,\ldots,yn),\ldots,fn(y1,y2,\ldots,yn)]\,|J|\,dy1dy2\ldots dyn,$$

where J is the Jacobian of the transformation, given by

$$\text{Jacobian} = \begin{vmatrix} \frac{\partial f1}{\partial y1} & \cdots & \frac{\partial f1}{\partial yn} \\ \vdots & & \vdots \\ \frac{\partial fn}{\partial y1} & \cdots & \frac{\partial fn}{\partial yn} \end{vmatrix}$$

and B is the volume in the Y-space corresponding to the volume A in the X-space. Understanding the principles of a general transformation and change of variables allows us to proceed with the derivation of the desired pdf.

With the aid of Equation (8), the pdf of the jointly Gaussian random processes nc' and ns' is

$$f(nc',ns') = \frac{1}{2\pi\sigma t^2} \exp\left[\frac{-(nc'^2 + ns'^2)}{2\sigma t^2}\right] , \tag{31}$$

with the variance, $\sigma t^2$, being defined by Equation (30). From (28) we find

$$nc' = Xc - P(t)\cos[\Delta wct+\theta] - J(t)\cos[\Delta w1t+\theta 1] ,$$
$$ns' = Xs - P(t)\sin[\Delta wct+\theta] - J(t)\sin[\Delta w1t+\theta 1] . \tag{32}$$

Substituting the latter relations into Equation (31), and also considering the distributions of $\theta$ and $\theta 1$, we may make a change of random variables to determine the joint pdf of Xc, Xs, $\theta$ and $\theta 1$. The pdf is then

$$f(Xc,Xs,\theta,\theta 1) = \frac{1}{2\pi}\,\frac{1}{2\pi}\,\frac{1}{2\pi\sigma t^2} *$$
$$* \exp\left[\frac{-\left(Xc-P(t)\cos(\Delta wct+\theta)-J(t)\cos(\Delta w1t+\theta 1)\right)^2}{2\sigma t^2}\right] *$$
$$* \exp\left[\frac{-\left(Xs-P(t)\sin(\Delta wct+\theta)-J(t)\sin(\Delta w1t+\theta 1)\right)^2}{2\sigma t^2}\right] \tag{33}$$

or,

$$f(Xc,Xs,\theta,\theta 1) = \frac{1}{4\pi^2}\frac{1}{2\pi\sigma t^2} *$$

$$* \exp\left[\frac{-(Xc^2 + Xs^2 + P^2(t) + J^2(t))}{2\sigma t^2}\right] *$$

$$* \exp\left[\frac{+2P(t)[Xc\cos(\Delta wct+\theta) + Xs\sin(\Delta wct+\theta)]}{2\sigma t^2}\right] *$$

$$* \exp\left[\frac{+2J(t)[Xc\cos(\Delta w1t+\theta 1) + Xs\sin(\Delta w1t+\theta 1)]}{2\sigma t^2}\right] *$$

$$* \exp\left[\frac{-2P(t)J(t)\cos[(\Delta wc-\Delta w1)t+\theta-\theta 1]}{2\sigma t^2}\right] . \tag{34}$$

We know that the input signal to the square-law device, $x(t)$, can be represented by Equation (3). With the aid of a trigonometric identity, $x(t)$ can be written as follows

$$x(t) = V(t)\cos[wct + \psi(t)] \tag{3}$$

$$= V(t)\cos\psi(t)\cos(wct) - V(t)\sin\psi(t)\sin(wct), \tag{35}$$

where $V(t)$ denotes the envelope and, $\psi(t)$ denotes the phase angle.

Recall from Equation (5) that

$$Xc(t) = V(t)\cos\psi(t) = Xc,$$

$$Xs(t) = V(t)\sin\psi(t) = Xs. \tag{5}$$

Inverting the relationships for $V(t)$ and $\psi(t)$ yields

$$V(t) = [Xc^2 + Xs^2]^{1/2} , \tag{36}$$

and,

$$\psi(t) = \tan^{-1} [Xs/Xc] . \tag{37}$$

Now, the joint pdf in terms of the envelope, V(t), and the phase, $\psi(t)$, random processes, can be determined.

Making the change of variables defined by Equation (5) and taking into account the Jacobian, we obtain

$$f(Vt,\psi,\theta,\theta 1) = \frac{Vt}{4\pi^2}\frac{1}{2\pi\sigma t^2} *$$

$$* \exp\left[\frac{-\left(Vt^2 + P^2(t) + J^2(t)\right)}{2\sigma t^2}\right] *$$

$$* \exp\left[\frac{+2P(t)Vt\cos[\psi-\Delta wct+\theta]}{2\sigma t^2}\right] *$$

$$* \exp\left[\frac{+2J(t)Vt\cos[\psi-\Delta w1t-\theta 1]}{2\sigma t^2}\right] *$$

$$* \exp\left[\frac{-2P(t)J(t)\cos[(\Delta wc-\Delta w1)t+\theta-\theta 1]}{2\sigma t^2}\right] \qquad (38)$$

and,

$$\text{Jacobian} = \begin{vmatrix} \cos\psi(t) & -V(t)\sin\psi(t) \\ \sin\psi(t) & V(t)\cos\psi(t) \end{vmatrix} = Vt. \qquad (39)$$

Determination of the final pdf requires the further change of variables

$$\begin{aligned} \alpha &= \psi - \theta - \Delta wct, \\ \beta &= \psi - \theta 1 - \Delta w1t, \\ \psi &= \psi \; . \end{aligned} \qquad (40)$$

The Jacobian for this change of variables is found to be

$$\text{Jacobian} = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{vmatrix} = 1. \qquad (41)$$

Observe that the cosine function in the last exponential term can be written as

$$\begin{aligned}\cos[(\Delta wc-\Delta wl)t+\theta-\theta l] &= \cos[\theta+\Delta wct-\psi+\psi-\Delta wlt-\theta l] \\ &= \cos[(\psi-\theta l-\Delta wlt) - (\psi-\theta-\Delta wct)] \\ &= \cos[\beta-\alpha] \,. \end{aligned} \tag{42}$$

Completing the transformation yields the joint pdf as

$$\begin{aligned} f(Vt,\psi,\alpha,\beta) &= \frac{Vt}{4\pi^2}\,\frac{1}{2\pi\sigma t^2} * \\ &* \exp\left[\frac{-\left(Vt^2 + P^2(t) + J^2(t)\right)}{2\sigma t^2}\right] \exp\left[\frac{+2P(t)Vt\cos(\alpha)}{2\sigma t^2}\right] * \\ &* \exp\left[\frac{+2VtJ(t)\cos(\beta)}{2\sigma t^2}\right] \exp\left[\frac{-2P(t)J(t)\cos[\beta-\alpha]}{2\sigma t^2}\right] \end{aligned} \tag{43}$$

To eliminate dependence on $\psi$ we need to evaluate the following integral

$$\begin{aligned} f(Vt,\alpha,\beta) &= \int_0^{2\pi} f(Vt,\psi,\alpha,\beta)\, d\psi \\ &= \frac{Vt}{4\pi^2}\,\frac{1}{2\pi\sigma t^2}\, 2 * \\ &* \exp\left[\frac{-\left(Vt^2 + P^2(t) + J^2(t)\right)}{2\sigma t^2}\right] \exp\left[\frac{+2P(t)Vt\cos(\alpha)}{2\sigma t^2}\right] * \\ &* \exp\left[\frac{+2J(t)Vt\cos(\beta)}{2\sigma t^2}\right] \exp\left[\frac{-2P(t)J(t)\cos[\beta-\alpha]}{2\sigma t^2}\right]. \end{aligned} \tag{44}$$

The resulting pdf is then given by the following expression

$$f(Vt) = \int_0^{2\pi}\int_0^{2\pi} f(Vt,\alpha,\beta)\ d\alpha d\beta$$

$$= \frac{Vt}{4\pi^2}\frac{1}{\sigma t^2}\exp\left[\frac{-(Vt^2 + P^2(t) + J^2(t))}{2\sigma t^2}\right] *$$

$$* \int_0^{2\pi}\int_0^{2\pi} \exp\left[\frac{+2P(t)Vt\cos(\alpha)}{2\sigma t^2}\right]\exp\left[\frac{+2J(t)Vt\cos(\beta)}{2\sigma t^2}\right] *$$

$$* \exp\left[\frac{-2P(t)J(t)\cos[\beta-\alpha]}{2\sigma t^2}\right] d\alpha d\beta\ , \tag{45}$$

which has no closed-form solution. However, it is interesting to observe in the last exponential term of the above expression, the interaction between the signal and the CW (continuous wave) jammer. This is given by the expression

$$P(t)J(t)\cos[\beta-\alpha]\ , \tag{46}$$

where from Equations (40), it is evident that $\alpha$ is the phase angle of the signal and $\beta$ is the phase angle of the jammer.

Three particular cases of importance arise from this observation:

1) If $[\beta-\alpha]$ is a multiple of $\pm n\pi/2$ the scalar product is null, implying that no interaction exists between the two signals. The exponential term under consideration becomes unity and a closed-form solution can be found for the marginal pdf.

2) If $[\beta-\alpha]$ is a multiple of $\pm n\pi$ with n even, a reinforcement from the part of the jammer is made on the desired signal. The last exponential term of (45) becomes

$$\exp\left[\frac{-2P(t)J(t)}{2\sigma t^2}\right]\ . \tag{47}$$

3) If $[\beta-\alpha]$ is a multiple of $\pm n\pi$ with n odd, the jammer signal tends to cancel out the desired signal. The exponential term considered becomes

$$\exp\left[\frac{2P(t)J(t)}{2\sigma t^2}\right] . \tag{48}$$

Analyzing the three cases, the worst situation is presented by case three. That is, when the jammer opposes the signal in strength and thus, makes the detection of the signal less likely. This is true because a great reduction of the desired signal strength may cause it to become embedded in the existing noise.

A common engineering practice, when designing or analyzing a system, is to establish as a reference the worst conditions of operation under which the system may operate. For the detection system of interest the most adverse conditions of operation is case three. Having adopted the above assumption, the double integral expression in Equation (45) can now be solved. The resulting marginal pdf is then

$$f(Vt) = \frac{Vt}{\sigma t^2} \exp\left[\frac{-\left(Vt^2 + P^2(t) + J^2(t) - 2P(t)J(t)\right)}{2\sigma t^2}\right] *$$

$$* \frac{1}{2\pi} \int_0^{2\pi} \exp\left[\frac{P(t)Vt\cos(\beta)}{2\sigma t^2}\right] d\beta \; *$$

$$* \frac{1}{2\pi} \int_0^{2\pi} \exp\left[\frac{J(t)Vt\cos(\beta)}{2\sigma t^2}\right] d\beta$$

$$= \frac{Vt}{\sigma t^2} \exp\left[\frac{-\left(Vt^2 + P^2(t) + J^2(t) - 2P(t)J(t)\right)}{2\sigma t^2}\right] *$$

$$* \; Io \left[\frac{P(t)Vt}{\sigma t^2}\right] \; Io \left[\frac{J(t)Vt}{\sigma t^2}\right] , \tag{49}$$

where Io is the modified Bessel function of order zero.

As a last step, the transformation of random variables defined by Equation (19) allows us to obtain the distribution of interest. This pdf corresponds to the output of the zonal, low-pass filter, i.e.,

$$z(t) = \frac{V(t)^2}{2} \quad . \tag{19}$$

Solving the above relation for V(t) in terms of z(t) yields

$$V(t) = [2z(t)]^{1/2}. \tag{50}$$

The Jacobian can be found to be

$$\text{Jacobian} = V(t), \tag{51}$$

or, in terms of z(t)

$$\text{Jacobian} = [2z(t)]^{1/2}. \tag{52}$$

Therefore, the probability distribution function of the output envelope is

$$f(Zt) = \frac{1}{\sigma t^2} \exp \left[ \frac{-\left(Zt + \frac{P^2(t)}{2} + \frac{J^2(t)}{2} - P(t)J(t)\right)}{\sigma t^2} \right] *$$

$$* \ Io \left[ \frac{P(t)(2Zt)^{1/2}}{\sigma t^2} \right] \ Io \left[ \frac{J(t)(2Zt)^{1/2}}{\sigma t^2} \right] \ . \tag{53}$$

In fact, we have determined a distribution function that takes into account, first, the presence of an intentional jammer which introduces more noise to that already existing in the channel, and secondly, a CW jamming signal that opposes the desired signal in strength. Also, for the derivation of the pdf, both the CW jammer signal and the desired signal have been considered to have random phase angles. This condition makes the proposed detection scheme the most practical for implementation.

## CHAPTER IV

## RESULTS OF INTEREST

Depending on the combination of signals present at the input of the receiver, the distribution for the output envelope will vary. The pdf derived, Equation (53), can be used as multiple-purpose equation to individually find the pdfs for the four cases where the input is 1) channel noise, 2) signal and channel noise, 3) signal, channel noise, and jammer noise, or 4) signal, channel noise, and jammer signal simply by eliminating the signal or signals not present for the particular case. In addition to these four cases of particular interest for the study of jamming environments, other eliminations will further multiply the use of this main result equation.

Some resulting pdfs of interest are:

1) Channel noise only (AWGN channel)

The input signal to the envelope detector, x(t), is given by

$$x(t) = n(t).$$

The distribution of the output will be given by

$$f(Zt) = \frac{1}{\sigma n^2} \exp\left[\frac{-Zt}{\sigma n^2}\right],$$

where $\sigma n^2$ is the channel noise variance.

2) Signal and channel noise

The input signal to the square-law device is

$$x(t) = s(t) + n(t),$$

where the signal, $s(t)$, is

$$s(t) = P(t)\cos[(wc+\Delta wc)t+\theta] \ .$$

For this case the pdf is found to be

$$f(Zt) = \frac{1}{\sigma n^2} \exp\left[\frac{-\left(Zt + \frac{P^2(t)}{2}\right)}{\sigma n^2}\right] Io \left[\frac{P(t)(2Zt)^{1/2}}{\sigma n^2}\right] ,$$

where $\sigma n^2$ is the channel noise variance.

3) Signal, channel noise, and jammer noise

The input signal to the envelope detector is

$$x(t) = s(t) + n(t) + jn(t) \ ,$$

where the signal, $s(t)$, is

$$s(t) = P(t)\cos[(wc+\Delta wc)t+\theta] \ ,$$

and the channel noise, $n(t)$, jammer noise, $jn(t)$, are independent, additive, white Gaussian, random processes with variances $\sigma n^2$ and $\sigma jn^2$, respectively.

The pdf can be found to be

$$f(Zt) = \frac{1}{\sigma t^2} \exp\left[\frac{-Zt}{\sigma t^2}\right] ,$$

where the variance $\sigma t^2$ is

$$\sigma t^2 = \sigma n^2 + \sigma jn^2 .$$

4) Signal, channel noise and jammer signal

The input signal is now given by

$$x(t) = s(t) + n(t) + j(t),$$

where the signal, s(t), and jammer signal, j(t), are

$$s(t) = P(t)\cos[(wc+\Delta wc)t+\theta],$$

$$j(t) = J(t)\cos[wl+\Delta wl)t+\theta l].$$

The marginal pdf can be found to be

$$f(Zt) = \frac{1}{\sigma n^2} \exp\left[\frac{-\left(Zt + \frac{P^2(t)}{2} + \frac{J^2(t)}{2} - 2P(t)J(t)\right)}{\sigma n^2}\right] *$$

$$* \; Io \left[\frac{P(t)\,(2Zt)^{1/2}}{\sigma n^2}\right] Io \left[\frac{J(t)\,(2Zt)^{1/2}}{\sigma n^2}\right],$$

where $\sigma n^2$ is the channel noise variance. Similar to Equation (53) except for the noise variance.

# CHAPTER V

## CONCLUSION AND FUTURE APPLICATIONS

In this report a probability distribution function of the output signal for a square-law receiver has been successfully derived. The pdf is general in the sense that no restrictions have been set apriori on the characteristics of the signal or the jammer as a signal. This means both can be continuous wave signals, any pulse type signal, or any combination of these. Furthermore, this distribution provides a general model from which the probability density functions of various special cases can be derived. These pdfs will depend on the combination of signals present at the receiver input.

As a suggestion for future work, the application of numerical methods will enable interested researchers to evaluate the probability of detection for the various jamming environments. The proposed method would be sufficiently general to allow consideration of a wide range of signal formats and filter characteristics. The resulting procedure should be validated by comparisons with known results for a non-jamming environment.

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A DERIVATION OF THE PROBABILITY DISTRIBUTION FUNCTION OF THE OUTPUT OF A SQUARE-LAW DETECTOR OPERATING IN A JAMMING ENVIRONMENT

by

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AN ABSTRACT OF A MASTER'S REPORT

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## ABSTRACT

This document involves the derivation of the output distribution function for a square-law detector system under an international jamming environment. The pdf derived is general in the sense that few restrictions have been set on the characteristics of the signal or the intentional jammer. Also, it provides a general model from which the probability density functions of various special cases can be derived.