A MATHEMATICAL ANALYSIS
OF MARRIAGE AND KINSHIP SYSTEMS

by

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TABLE OF CONTENTS

INTRODUCTION

Kinship is a complicated subject, but the tools exist for simplifying it. It can be initially and briefly defined as a study of the cultural interpretations of social relationships, social categories, and social groups that are formed among people who stand in biological relationships or chains of relationships to one another.

Generations of anthropologists have begun the study of kinship by citing that "a man marries a woman and they produce children". This information, sometimes called the facts of life is indeed a major fact of the science of social life.

Just as the roles of male and female are subject to a wide range of cultural interpretation, marriage is subject to even wider ranges of interpretation. "Marriage" is a word that, when it is used within the context of a single culture, accretes to itself a great many covert connotations of detail. However, used cross culturally, the field of meaning must be vastly reduced. For instance, the "marriage" of an Indian princess is not quite the same as the marriage of your sister.

A formal organization in our society can be regarded as a tree of cumulated primary roles; kinsmen who are equivalent to each other in the structure are designated as "alters". If certain roles have the same identification, such as wife equals mother's brother's daughter, we can simplify a given role tree. This process is known as reduction of a role tree.

Structural mathematics is essential for an exhaustive treatment of structures of compound roles. Abstract groups and matrices appear in section II, where a society is based on a given set of axioms determining its marriage and kinship structure.

Since the middle of the nineteenth century, scientific attempts have been made to classify codes in which the message of biological kinship is clothed. The modern study of kinship in this respect is called componential analysis. We discuss an example using a type of coding.

The "group partition theorem" may be regarded as the basic or most direct formulation of what is meant by marriage class system. For the theorem ot be useful, it is necessary for certain sociological interpretations to be made. For example, the assumption that a society is strongly connected can be interpreted as the society is closely knit and homogeneous. We will restrict ourselves to this small but important class of societies.

The Heuristics of Kinship

The word "kinship" has been used to mean several things. "Kinship relationships" have several referents which must be kept sharply separated analytically. There are biological referents, behavioral referents, and linguistic referents (among others).

People who are genetically related, i.e., having a common ancestor, and biologically related, (see following paragraph) to one another are "kinsmen". Kinship relationships, so viewed, form biological networks. Every living individual is a node in a biological network. He is the recipient of genes and probably a donor of genes.

There are two types of biological relationships that can exist between human beings. One of these types is a relationship of "descent"; the other is a sexual union to produce offspring. In the biological sense, all kinship relationships comprise one link or a series of links either of descent or of sexual union. We refer to these links as branches of a kinship tree.

People classify kinsmen into various categories and give these categories names called "kinship terms". Each named category will be found to have certain modes of behavior expected from the people who are fitted into it. In other words, a kinship term is a linguistic tag for a role; the role has biological criteria or admitted substitutes for admission to it. "Kinship terms" are role terms.

The diagram on the following page gives the branching rules for a biological tree of kin roles called "Eskimo" or "English".

Father	= F	Wife	= W
Mother	= M	Son	= S
Brother	= B	Daughter	= D
Sister	= Z	Husband	= H

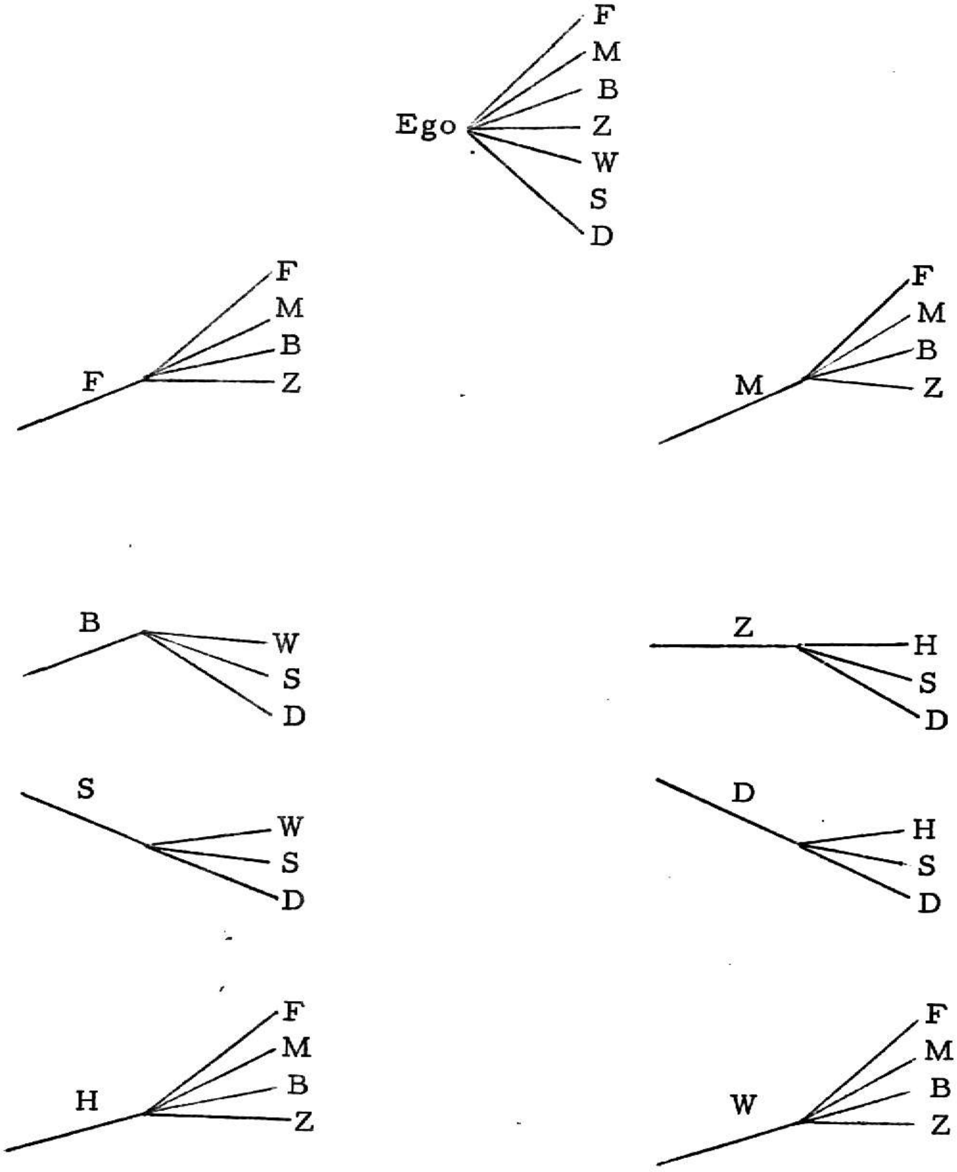


Figure 1.1

Note that Brother's father is not a distinct role in a society without polyandry, since brother's father is the same as ego's father. Similarly, Mother's husband is the same as father and hence is not a distinct role.

We can sum up the formulated tree in a new diagram. Since most societies emphasize patrilineal descent, F, B, Z, W, H, and S are a convenient minimal set of roles to use.

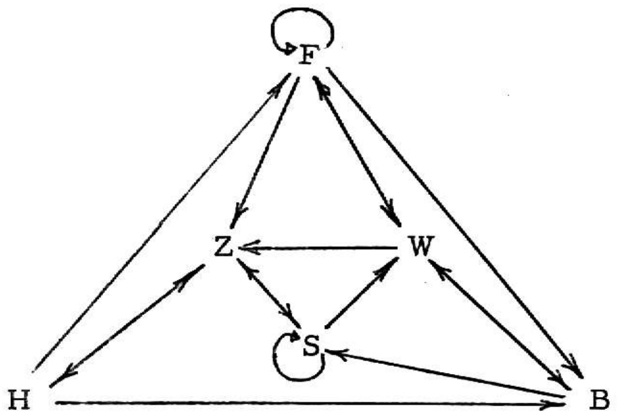


Figure 1.2

Ego (not shown) has a direct relation with all but the Husband role (if ego is male), or the Wife role (if ego is female). An arrow connecting two nodes indicates the existence of a relationship from the tail of the arrow to the head of the arrow (but not necessarily from the head to the tail). Compound role relations can be shown using this diagram. For example, Father's sister's daughter, FZ's daughter, is the same as father's sister's son's sister, that is, FZSZ.

Consider a tribe through the eyes of one man taken as ego. He wishes to place members at the nodes of his tree. A new role relation called "alter" describes the relation ego preceives between two persons in the same role with respect to ego, i.e., Mac and Sam, two father's brothers have identical roles in ego's view.

Recall, that a set of axioms are consistent if the truth of all but one does not imply the falsity of the remaining one. Then, if kin roles are to place members of a tribe in a clear and mutually consistent structure, the following properties are essential.

P1. In the kin-role tree of a given person ego, if two people are alters in one node of the tree, then in any other node in which one appears the other must also appear as an alter to the first.

P2. The tree of kinship roles must be such that all persons who are alters with respect to one person as ego must be alters with respect to any other person as ego.

P3. If the same person can occupy two roles with respect to an ego, the content of the prescriptions of the two roles should be consistent.

The kinship tree of Western society described does not satisfy the properties above. Consider the case of three brothers, John, Jim and Joe. In John's role tree Jim and Joe are alters in the node for brother, but in Jim's role tree John and Jim are alters. Hence all three must view each other as alters rather than brothers if property one, P1, is to be satisfied. In "English" there is not a clear and mutually consistent society in the above sense.

If we accept a set of axioms for a consistent kinship system, then brother is not a distinct role with respect to any male role; nor sister with respect to any female role, i.e., siblings of the same sex must be treated as alter egos. Anthropologists call this term the principle of "brother equivalence".

Using this fact, the role tree discussed above can be simplified. We will construct a tree of male nodes only, consisting of the following minimal set of male-kin roles for these trees.

Father	= F	Wife's brother	= WB
Son	= S	Sister's husband	= ZH

The new construction then becomes:

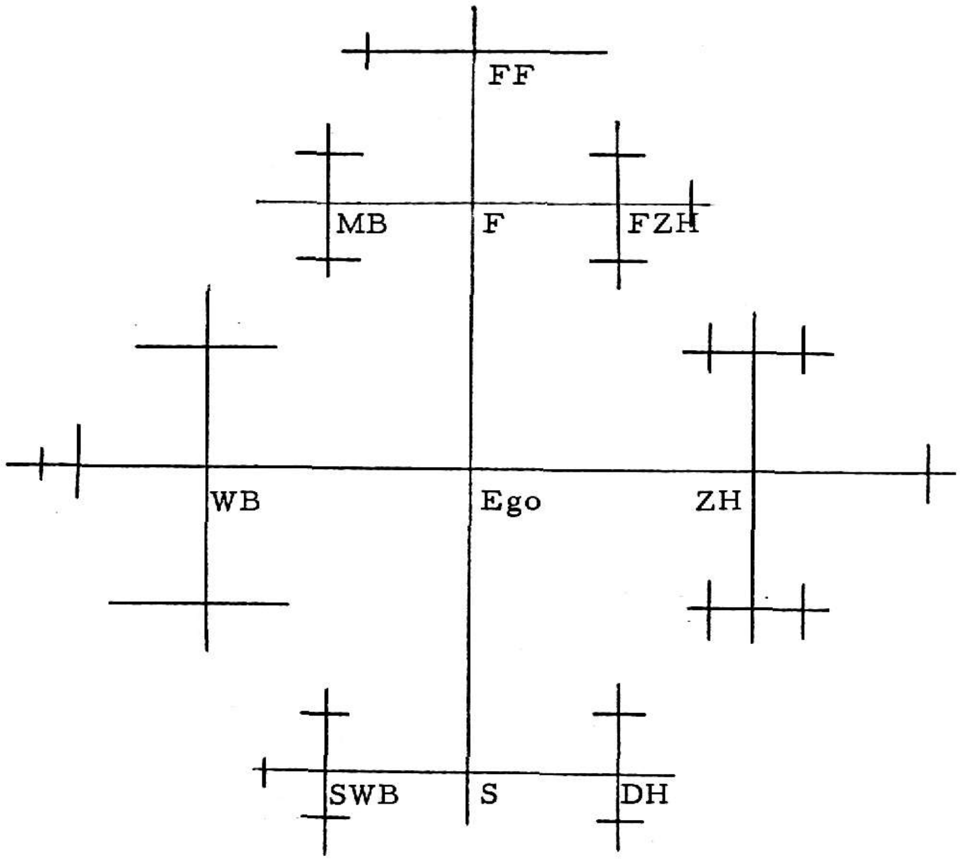
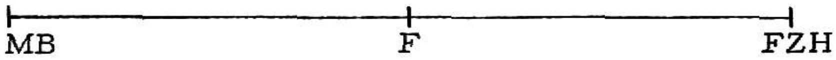


Figure 1.3

The female roles can be inserted midway on the horizontal lines, so that



becomes,

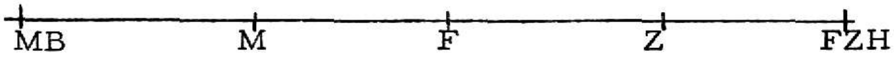


Figure 1.4

If we make some roles indistinguisable from others reduction of this tree occurs. For example, let it be the law of the society that each male marries only his MBD, i.e., wife(W) and Mother's brother's daughter, (MBD) are identical roles. Then Figure 1.3 becomes the two dimensional grid,

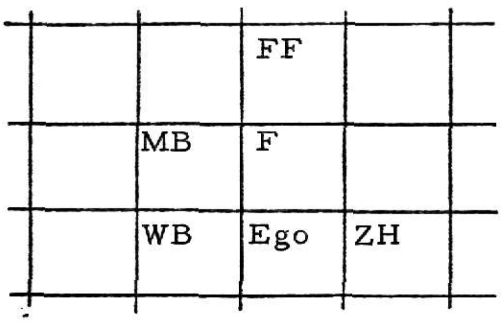


Figure 1.5

where the node just above a given node is related to it as Father to Son; the node just to the left is related to its neighbor to the right as Wife's Brother to Sister's Husband.

We can still make further restrictions such as FZD, MBD, and W are all distinguishable from each other. The tree is condensed to a simple ladder shown in Figure 1.6. Societies with this role tree can be called bilateral cross-cousin marriage societies, since one's wife is also both Mother's Brother's and Father's Sister's Daughter.

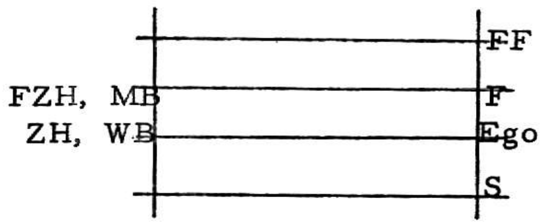


Figure 1.6

A node is related to the node below as Father to Son, and a node is related to the other node at the same level as WB to ZH and as ZH to WB.

It is possible to reduce a tree of compound roles into a small closed set of linked nodes, solely as the result of historial patterns in the choice of wives by particular men. In the limit, the closure of a reduced kin tree can result in a single node. To trace a non-trivial case here would take up too much space. Nevertheless, I refer the enthusiastic reader to White's book (pages 22-27).

Suppose we wish to construct a typology of all marriage systems based on the following axioms:

Axiom Schema I

1. The entire population of the society is divided into mutually exclusive groups, which we call clans. The identification of a person with a

clan is permanent. Hereafter, n denotes the number of clans.

2. There is a permanent rule fixing the single clan among whose women the men of a given clan must find their wives.

3. By rule 2, men from two different clans cannot marry women of the same clan.

4. All children of a couple are assigned to a single clan, uniquely determined by the clans of their mother and father.

5. Children whose fathers are in different clans must themselves be in different clans.

6. A man can never marry a woman of his own clan.

7. Every person in the society has some relative by marriage and descent in each other clan, i.e., the society is not split into groups not related to each other.

8. Whether two people who are related by marriage and descent links are in the same clan depends only on the kind of relationship, not on the clan either one belongs to.

Axioms (2) and (3) allow us to represent the system in terms of a permutation matrix, i.e., a matrix with exactly one entry of unity in each row and column and all other entries zero. For convenience, number the clans from 1 to n. The ith row and the ith column of the matrix corresponds to the ith clan. Let the rows of the matrix correspond to the husband's clan and the columns to the wife's clan. Then the (i, j)th entry of this matrix is 1 if men in clan i marry women in clan j and 0 otherwise. Denote this matrix by W (Wife's matrix).

The clan of a couple's children is determined by the clan of the father since the mother's clan is determined by that of her husband as seen above. Let $D_{ij} = 1$ if fathers of clan i have children in clan j. Denote this permutation matrix by D.

Given the clan of, say a son, the transpose of D, denoted by D^T, tells us to which clan the father belongs. Thus $(D^T)_{ji} = D_{ij}$.

Compound role relationships are formed, then, by matrix multiplication. For example:

$(WD)_{ij} = 1$ if and only if men in the ith clan marry women whose clan brothers are the fathers of children in the jth clan.

We now make the following definitions:

(a) When the two siblings who are parents of the first cousins are of the same sex the latter are termed parallel cousins, otherwise cross-cousins.

(b) When the parent of male ego is female the cousins are said to be <u>matrilateral</u> cousins; and <u>patrilateral</u> cousins if it is the father of male ego who is a sibling of one of the girl's parents.

Notice, then, that we can have patrilateral or matrilateral parallel cousins.

These concepts lead to the following questions:

(1) What are the conditions which permit matrilateral cross cousin marriage?

(2) What are the conditions which permit patrilateral parallel cousin marriages?

If a boy can marry a girl, her clan must be indicated by a unity entry in a row of W corresponding to the boy's clan.

Marriage between matrilateral cross cousins results in the following definition for W:

$$W = D^{-1}WD$$

In other words, the D and W matrices must commute if marriages of this type are permitted.

Similarly, patrilateral parallel marriages require that the clan of the boy's father, say j, is specified in the ith row of D^{-1}; the father's brother being in the same clan. Father's brother's children are in a clan specified in the jth row of D, that is, in the clan specified in the ith row of $(D^{-1}D)$. Hence,

$$W = D^{-1}D = I$$

which contradicts axiom 6. So in no society satisfying axioms 1-8 can a boy marry his patrilateral parallel cross cousin.

A more general mathematical model for a society based on the above axioms is discussed in the next section.

A Mathematical Formulation of Marriage and Kinship Systems

In this section, we present a mathematical formulation of marriage and kinship systems. It should be noted to the reader that another presentation using permutations of the integers N is found in the appendix.

For the purpose of this formulation, we make the following definitions:

<u>Definition 2.1.</u> Let S be a population of males S_1 and females S_2 so that $S = S_1 \cup S_2$ and $S_1 , S_2 = \emptyset$. If $\sim$ is an equivalence relation on S, then $\sim$ induces an equivalence relation $\sim_1$ and $\sim_2$ on S_1 and S_2, respectively, by defining $\sim_1 = \{(a, b) \in S_1 \times S_1 \mid a \sim b\}$. Similarly, $\sim_2 = \{(c, d) \in S_2 \times S_2 \mid c \sim d\}$. If $\mathcal{C} \underset{\text{def.}}{=} S/\sim$ is the partition on S determined by $\sim$, then we let $\mathcal{C}_1$, and $\mathcal{C}_2$ be the corresponding partitions determined by $\sim_1$ and $\sim_2$, respectively, that is, $\mathcal{C} = \{C \cap S_i \mid C \in \mathcal{C}\}$, $i = 1, 2$.

<u>Definition 2.2.</u> A society is an ordered quintuple, $\mathfrak{S} = (S_1, S_2, \sim, \mu, \Delta)$, where S_1 and S_2 are two disjoint, nonempty sets, $\sim$ is an equivalence relation on $S \underset{\text{def.}}{=} S_1 \cup S_2$, μ is a function mapping $\mathcal{C}_1$ onto $\mathcal{C}_2$. (The set $\mathfrak{M}$ of ordered pairs $(C^j, C^k) \in \mu$ are called marriage types, and Δ is a mapping from $\mathfrak{M}$ onto $\mathcal{C}$.)

Note: Technically $\mu = \mathfrak{M}$. However, we write "μ" when regarding μ as a mapping and we write $\mathfrak{M}$ when regarding μ as a set.

It should also be noted that there are examples of societies where μ is a relation. The following books refer to such examples:

Leach, E. R., <u>Political Systems of Highland Burma</u>, Massachusetts: Harvard University Press, 1954.

and

Meek, C. K., <u>Law and Authority in a Nigerian Tribe</u>, London: Oxford University Press, 1937.

For the purposes of our analysis, however, we assume μ is a function.

<u>Definition 2.3.</u> The cells C^i of $\mathcal{C}$ are called <u>clans.</u>

<u>Definition 2.4.</u> Let $\beta_1 : \mathcal{C} \to \mathcal{C}_1$ be defined by $\beta_1(C^i) = C^i \cap S_1$; $\beta_2 : \mathcal{C} \to \mathcal{C}_2$ be defined by $\beta_2(C^j) = C^j \cap S_2$. Then $\beta : \mathcal{C}_2 \to \mathcal{C}_1$ is defined by $\beta(C^i_2) = \beta^{-1}_2 \, \beta_1 \, (C^j_2)$.

<u>Definition 2.5.</u> Let $1 \times \mu$ map from $\mathcal{C}_1$ to $\mathcal{M}$ be defined by $(1 \times \mu)(C^i_1) = (C^i_1, (\mu(C^i_1))$; $\mu^{-1} \times 1$ map from $\mathcal{C}_2$ onto $\mathcal{M}$ be defined by $(\mu^{-1} \times 1)(C^i_2) = (\mu^{-1}(C^i_2), C^i_2)$.

Axiom Schema I'

2.1 μ is 1-1 (and onto)

2.2 Δ is 1-1 (and onto)

2.3 $\beta_1 \mu \beta^{-1}_2$ is fixed point free, i. e., $\beta_1 \mu \beta^{-1}_2 (C^i) \neq C^i$ for any $C \in \mathcal{C}$.

2.4 A society is connected.

Recalling the definitions of μ, $1 \times \mu$, $\mu^{-1} \times 1$, and β, we make the following diagram.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\beta_2"] \arrow[d, "\beta_1"'] & \mathcal{C}_2 \arrow[d, "\mu^{-1} \times 1"] \arrow[dl, shift right, "\beta"'] \\
\mathcal{C}_1 \arrow[r, "\mu \times 1"'] \arrow[ur, shift right, "\mu"'] & \mathcal{M} \arrow[ul, "\Delta"']
\end{tikzcd}$$

Figure 2.1

Remark: $1 \times \mu = (\mu^{-1} \times 1)\mu$ for $(1 \times \mu)\, C_2^i = (C_2^i,\ \mu(C_2^i))$ and $(\mu^{-1} \times 1)\,(\mu(C_2^i)) = (\mu^{-1}\mu(C_2^i),\ \mu(C_2^i)) = (C_2^i,\ \mu(C_2^i))$.

This means a woman's marriage type is completely determined by that of her husband's.

Let W and D be n X n matrices defined as follows:

$$W_{ij} = \begin{cases} 1 \text{ if and only if } \beta_1\mu\beta_2^{-1}\,(\mathcal{C}^i) = \mathcal{C}^j \\ 0 \text{ otherwise} \end{cases}$$

$$D_{ij} = \begin{cases} 1 \text{ if and only if } \beta_i(1 \times \mu)\,\Delta\,(\mathcal{C}^i) = \mathcal{C}^j \\ 0 \text{ otherwise} \end{cases}$$

Lemma: W is a permutation matrix.

Proof: Since the domain of $\mu = \mathcal{C}$, then for all $C^i \in \mathcal{C}$ there exists a $C^j \in \mathcal{C}$ which implies $\beta_1\mu\beta_2^{-1}(C^i) = C^j$. Hence every row of W contains a 1. Now $\beta_1\mu\beta_2^{-1}$ is a function which implies C^j is unique so there exists only one 1 in each row. Furthermore, $\beta_1\mu\beta_2^{-1}$ is one-to-one so there exists only one 1 in each column of W. Hence, W is a permutation matrix.

Note: D is a permutation matrix for the same reasoning since the composition of bijections is a bijection.

We note that matrix multiplication corresponds to composition of functions. For example:

$$(WD)_{ij} = 1 \text{ is equivalently written}$$
$$\beta_1\mu\beta_2^{-1}\beta_1(1 \times \mu)\,\Delta\,(\mathcal{C}^i) = \mathcal{C}^j$$

Using our representation of permutation matrices as functions, we can investigate what marriages a given society will permit. Recall that when the parent of male ego is female, the cousins are said to be matrilateral; male ego and the girl are patrilateral cousins if it is the father of male ego who is a sibling of the girl's parents. Then using Axiom Schema I', is a marriage between matrilateral cross cousins possible? If so, then W must equal the matrix describing the clan of the girl related to male ego, i.e., W = ego's mother's brother's daughter.

By Figure 2.1, we can determine by composition of functions ego's mother's brother's child. Ego is located in some clan $C \in \mathcal{C}$. We find his mother by first going from C to his mother's clan via $\Delta^{-1}(\mu^{-1} \times 1)^{-1}$. The clan of ego's mother's brother is found by the mapping β_2^{-1}; he has a child in some clan C^k given by the mapping $\beta_1(1 \times \mu)\,\Delta$. Composing these functions, we find that W, which is identified with $\beta_1\mu\beta_2^{-1}$, becomes

$$\beta_1\mu\beta_2^{-1} = \Delta^{-1}(\mu^{-1} \times 1)^{-1}\beta_2^{-1}\beta_1(1 \times \mu)\,\Delta = \Delta^{-1}(1 \times \mu)^{-1}\mu\beta_2^{-1}\beta_1(1 \times \mu)\,\Delta$$
$$= [\Delta^{-1}(1 \times \mu)^{-1}\beta_1^{-1}]\,[\beta_1\mu\beta_2^{-1}]\,[\beta_1(1 \times \mu)\,\Delta].$$

Equivalently, in matrix notation, $W = D^{-1}WD$ or $WD = DW$. Then a necessary condition for matrilateral cross cousin marriage in a society based on Axiom Schema I' is that W and D commute.

Observe now the case of marriage between patrilateral parallel cousins. Then W must equal ego's father's brother's child. Again, using the functions described previously, we have,

$$\beta_1\mu\beta_2^{-1} = \Delta^{-1}(1 \times \mu)^{-1}\beta_1^{-1}\beta_1(1 \times \mu)\,\Delta = I.$$

This implies $W = I$, which contradicts axiom 2.3. Hence under this set of axioms, marriage between patrilateral parallel cousins is never permitted.

Analysis of Marriage Types By Coding

Marriage types can be analyzed using a concept of coding. Suppose in a society S we have eight classes: A1, A2, B1, B2, C1, C2, D1, D2. Allowing two formulae for marriage applied alternately:

(I)

$$\begin{array}{c} A1 \leftrightarrow B1 \\ A2 \leftrightarrow B2 \\ C1 \leftrightarrow D1 \\ C2 \leftrightarrow D2 \end{array}$$

(II)

$$\begin{tikzcd} A1 \arrow[dr, leftrightarrow] & B1 \arrow[dl, leftrightarrow] \\ A2 & B2 \\ C1 \arrow[dr, leftrightarrow] & D1 \arrow[dl, leftrightarrow] \\ C2 & D2 \end{tikzcd}$$

where an arrow denotes allowable marriage between classes.

Further assume that the class of children is determined by that of their mother:

mother's class:	A1	A2	B1	B2	C1	C2	D1	D2
children's class:	C2	C1	D2	D1	A1	A2	B1	B2

We can designate each class by a triple (a, b, c) where each of these indices a, b, and c is a number of the arithmetic mod 2, i.e.,

1. a = 0 if the class is A or B; 1 if it is C or D.
2. b = 0 if the class is A or C; 1 if it is B or D.
3. c = 0 for the subclass 1, and 1 for the subclass 2.

For example, if a man or woman is of class C2, in this notation, we say he or she is of class (1, 0, 1).

Each type of marriage can be designated by a quartuple (a, b, c, d), where (a, b, c) designates the class of the husband and d = 0 if the marriage follows formula (I) and 1 otherwise. Suppose in a marriage we have the quadruple (1, 0, 1, 1). The husband is of class (1, 0, 1), i.e., of type C2 and the marriage follows formula (II); the wife is of Class D1, that is to say (1, 1, 0); moreover, the children are of clan B1, that is (0, 1, 0).

A case by case verification shows that in formula (I) marriages, if the husband is of class (a, b, c), the wife is of class (a, b + 1, c); in formula (II) marriages, if the husband is of class (a, b, c), the wife is of class (a, b + 1, c + 1). Then, in a marriage (a, b, c, d) the husband is of class (a, b, c), and the wife is of class (a, b + 1, c + d). *Direct* verification shows that if a wife is of class (x, y, z), her children are of (x + 1, y, z + 1). It follows that in a marriage (a, b, c, d) the children are of class (a + 1, b + 1, a + c + d + 1).

The hypothesis on alternation between formulae (I) and (II) is now made more precise.

Consider the following four cases:

(1) The children always follow the parents formula

(2) The children always follow the opposite formula to that of the parents, with the result that the formulae will alternate from generation to generation

(3) Sons follow the parents' formula; daughters, the opposite of the parents' formula

(4) daughters follow the formula of the parents, and sons the opposite formula

Let f and g be permutations among the marriage types of the parents, denoting the possible marriage types of the son and daughter, respectively.

Assume the marriage laws satisfy the following conditions:

(A) For each individual, male or female, there is one and only one type of marriage which he (or she) has the right to contract.

(B) For each individual, the type of marriage which he (or she) is capable of contracting depends solely on his sex and the type of marriage of which he (or she) is the issue.

(C) All males ought to be able to marry the daughter of the mother's brother.

Let (a, b, c, d) represent the marriage type of the parents. Then $(a + 1, b + 1, a + c + d + 1)$ represents the class of the children and

$$f(a, b, c, d) \equiv (a + 1, b + 1, a + c + d + 1, d + p)_{\text{mod } 2},$$

that is, the marriage type of the son. Now the marriage type of the daughter is strictly determined by her future husband. So, if the daughter's marriage type is $(a + 1, b + 1, a + c + d + 1, d + q)$ the daughter as a wife has marriage type $(a + 1, b\ c + d + q + 1 + a + d, d + q) = (a + 1, b, a + c + q + 1, d + q) \equiv g(a, b, c, d)_{\text{mod } 2}$.

If these permutations commute, we must have $g \circ f \equiv f \circ g$, that is, $g \circ f(a, b, c, d) \equiv g(a + 1, b + 1, c + d + a + 1, d + q) = (a, b + 1, c + d + q + 1, d + p + q)_{\text{mod } 2}$

and

$f \circ g(a, b, c, d) \equiv f(a + 1, b, a + c + q + 1, d + q) = (a, b + 1, c + d + 1, d + p + q)_{\text{mod } 2}$

or

$$(a, b + 1, c + d + 1) \equiv (a, b + 1, c + d + q + 1)_{\text{mod } 2}$$

This shows that q cannot be 1; the cases (2) and (3) are thus excluded by condition (C), and the only other possible cases are (1) and (4) where p = 1, q = 0. The functions f and g are

$$f(a, b, c, d) \equiv (a + 1, b + 1, a + c + d + 1, d + 1)_{\text{mod } 2}$$

and

$$g(a, b, c, d) \equiv (a + 1, b, a + c = 1, d)_{\text{mod } 2}.$$

The Arunta Permutation Group

As we know, a more common "prefix" notation is used in writing functions, i.e., if φ is a function, we usually write $\varphi(x)$ rather than $x\varphi$, the "suffix" notation. Suffix notation will be used throughout this discussion, since the English-speaking anthropologist will write ♂MoBr in kinship terms for "a man's mother's brother," not BrMo(♂).

We now describe the Arunta Kinship system using this convention. Let E be the set of people belonging to the Arunta tribe. The E is partitioned into eight marriage classes. Let I be the equivalence relation associated with this partition E/I. Recall the cultural norms with respect to these classes, where $(1 \times \mu)$, β_1 and Δ are defined in section II, i.e.,

(a) $\beta_1(1 \times \mu)\,\Delta$ is 1 - 1

(b) $\beta_1(1 \times \mu)\,\Delta$ is well-defined

Let F and M be permutation matrices on E/I and X, Y classes of E.

XF = Y iff there is a father of X in Y. Similarly,

XM = Y iff there is a mother of X in Y.

Identify F and M as follows:

$$F_{ij} = \begin{cases} 1 & \text{iff } \Delta^{-1}(1 \times \mu)^{-1}\beta_1^{-1}(C^i) = C^j \\ 0 & \text{otherwise} \end{cases}$$

and

$$M_{ij} \begin{cases} 1 & \text{iff } \Delta^{-1}(\mu^{-1} \times 1)^{-1}\beta_2^{-1}(C^i) = C^j \\ 0 & \text{otherwise} \end{cases}$$

The W and D matrices discussed in section II can be represented in terms of F and M as follows:

$$\Delta^{-1}(\mu^{-1} \times 1)^{-1}\beta_2^{-1} = \Delta^{-1}(1 \times \mu)^{-1}\mu\beta_2^{-1} = \Delta^{-1}(1 \times \mu)^{-1}\mu\mu^{-1}\beta_1^{-1}$$

$$= \Delta^{-1}(1 \times \mu)^{-1}\beta_1^{-1} = \Delta^{-1}(1 \times \mu)^{-1}\mu\beta_2^{-1} = \Delta^{-1}(1 \times \mu)^{-1}\beta_1^{-1}\beta_1\mu\beta_2^{-1}$$

$$= (\beta_1(1 \times \mu)\Delta)^{-1}(\beta_1\mu\beta_2^{-1}),$$

which implies $M = F^{-1}W$ or $W = F^{-1}M$. Clearly, $D = F^{-1}$ by definition.

Remark: It may be helpful for the reader to refer to Figure 2.1 when carrying out the above calculations.

The set of all (finite) compositions of the permutations F and M form a set G of permutations and is called the permutation group generated by F and M.

Recall, for $X, Y \in E/I$, $XF = Y$ iff there is a father of X in Y, and similarly, $XM = Y$ iff there is a mother of X in Y. So the Arunta permutation group G has the property that for any $X, Y \in E/I$ there is exactly one permutation $P \in G$ such that $XP = Y$. Such a permutation group is called (right) regular and has the property that every element $P \in G$ can be uniquely written in the form XP for some fixed $X \in E/I$. Let $\cdot$ be a binary operation defined on E/I by $(XP_1) \cdot (XP_2) = X(P_1P_2)$.

Lemma I. $(E/I, \cdot)$ is a group.

<u>Proof:</u> Fix $X \in E/I$. Since G is regular, for all $Y \in E/I$ there exists a unique P such that $Y = XP$.

Let $Y, Z \in E/I$. Then $Y \cdot Z = (XP_1 \cdot XP_2) = X(P_1P_2) \in E/I$. Hence the operation $\cdot$ is closed.

Claim: E/I has X as a right identity. Let $Y \in E/I$ such that $Y = XP_1$ for some $P_1 \in G$. Then,

$$(XP_1) = (XP_1) \cdot (X1) = X(P_1 \cdot 1) = XP_1 = Y.$$

Hence the claim is true.

Claim: Every element of E/I has an inverse in E/I. Let $Z = XP \in E/I$ and $Z' = XP^{-1}$. Then $ZZ' = (XP \cdot XP^{-1}) = X(PP^{-1}) = X$, proving the claim.

Lemma II. $(E/I, \cdot)$ is isomorphic to G.

<u>Proof:</u> Let $Y \in E/I$ so that $Y = XP$ for some $P \in G$ and define $\varphi(Y) = \varphi(XP) = P$.

φ is well-defined since G is a regular permutation group.

Claim: φ is a homomorphism. Let $Y, Z \in E/I$ such that $Y = XP_1$ and $Z = XP_2$, $P_1, P_2 \in G$. Then $\varphi(Y \cdot Z) = \varphi(XP_1 \cdot XP_2) = \varphi(X(P_1P_2)) = P_1P_2$ and $\varphi(XP_1)\varphi(XP_2) = P_1P_2$. Hence $\varphi(Y \cdot Z) = \varphi(Y)\varphi(Z)$.

Claim: φ is 1-1. Let $Y, Z \in E/I$ such that $Y = XP_1$ and $Z = XP_2$, $P_1, P_2 \in G$. Then, $\varphi(XP_1) = \varphi(XP_2)$ which implies $P_1 = P_2$. So $XP_1 = XP_2$ or equivalently, $Y = Z$.

Claim: φ is onto. Let $P \in G$ and let $Y = XP$ for $X, Y \in E/I$. Then $\varphi(Y) = P$. Thus proving the lemma.

The Arunta marriage class system is graphed in Figure 4.1. The dots represent the elements of E/I. An arrow is drawn from point X to point Y with its head at Y and labeled with F (or M) iff $XF = Y$ (or $XM = Y$). We will denoted M-arrows by a solid line and F-arrows by a dotted line.

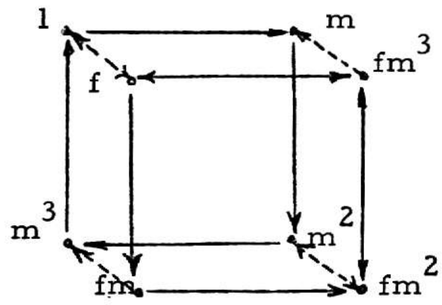


Figure 4.1. The graph of the Arunta marriage class system.

We can get the "Cayley multiplication table" from this configuration. For example, $(fm^2)(m) = fm^3$ by following the M-arrow from fm^2 to its point at fm^3.

TABLE I

The Arunta Multiplication Table

	1	M	M^2	M^3	F	FM	FM^2	FM^3
1	1	M	M^2	M^3	F	FM	FM^2	FM^3
M	M	M^2	M^3	1	FM^3	F	FM	FM^2
M^2	M^2	M^3	1	M	FM^2	FM^3	F	FM
M^3	M^3	M	M	M^2	FM	FM^2	FM^3	F
F	F	FM	FM^2	FM^3	1	M	M^2	M^3
FM	FM	FM^2	FM^3	F	M^3	1	M	M^2
FM^2	FM^2	FM^3	F	FM	M^2	M^3	1	M
FM^3	FM^3	F	FM	FM^2	M	M^2	M^3	1

Note: This is the dihedral group D_4.

Since only two of the set $\mathfrak{G}$ of basic Arunta Kinship terms have been used thus far, we must investigate how the entire network of relations in E relates to G. The set $\mathfrak{B}$ (E) of relations on E forms a monoid under the operation of composition since composition is associative and the identity relation 1_E is the identity element. A submonoid $\mathfrak{G}$ of a monoid $\mathfrak{B}$ is a subset $\mathfrak{B}$ such that $\mathfrak{G}$ contains the identity and is closed under composition.

Define $\mathfrak{G}^* = \bigcap \{B \mid A \subseteq B,\ B \text{ a monoid}\}$. Then $\mathfrak{G}^*$ is called the submonoid generated by $\mathfrak{G}$. $\mathfrak{G}^*$ may be constructed by taking all possible compositions of elements in $\mathfrak{G}$ and adding the identity.

Let $\mathfrak{G}^*$ be the submonoid of $\mathfrak{B}$(E) generated by $\mathfrak{G}$ and φ a homorphism from $\mathfrak{G}^* - \emptyset$ onto G.

The following is a table of the homorphism φ mapping $\mathfrak{G}^* - \emptyset$ onto the Arunta group G.

TABLE 2

$A \in \mathfrak{G}^* - \emptyset$		$A\varphi \in G$
brother or sister	=	1, the group identity
a man's father or a man's child	=	$F = F^{-1}$
mother	=	M
son or daughter	=	M^{-1}
Husband or wife	=	$F^{-1}M = M^{-1}F = FM$

If φ is a homorphism from $G^* - \emptyset$ onto G, by Table 2, "a man's father" φ = "a woman's father" φ = F and "a man's mother" φ = "a woman's mother" φ = M.

This discussion bridges the gap for presentation of the group partition theorem in the next section.

The Group Partition Theorem

In the previous section φ mapped $\mathcal{G}^* - \emptyset$ onto a group G. The question now arises as to how vital is it that G be a group rather than, say, a monoid. It is conceivable that social structures of the form $\varphi: \mathcal{G}^* - \emptyset \to G$ can exist, where G is a monoid and φ is a homomorphism. The "group partition theorem" to be proven below shows under what conditions G must be a group.

Before the statement of the theorem we make the following definitions:

Definition: A line is called positive if it is mapped onto the identity 1.

Definition: Let $\mathcal{G} = \{N, P\}$ be the set of positive and negative lines, respectively. A walk from x to y is a relation $A \in \{N, P\}^*$ such that xAy. A closed walk at x is a walk from x to x, i.e., xAx.

Definition: A digraph G is a finite nonempty set of points P and a set E whose members are ordered pairs of points of P.

Definition: The sign of a walk is the product of the signs of its lines.

Definition: A digraph is balanced iff $\alpha(A) = 1$ for every closed walk A in G.

Definition: A signed digraph is a pair (G, α) where G is a digraph and α a mapping of G onto $S_2 = \{1, a\}$.

Comment: The elements 1 and a stand for ± 1.

Recall that a set $\mathcal{G}$ of relations on E is said to be strongly connected on E iff $\bigcup \mathcal{G}^* = E \times E$.

GROUP PARTITION THEOREM: Let $\mathcal{G}$ be a set of relations on a finite set E that is strongly connected. Let φ be a homorphism from $\mathcal{G}^* - \emptyset$ onto a monoid G. Then the following statements are equivalent:

(i) For any $A \in \mathcal{G}^*$, if xAx for some x in E, then $A\varphi = 1$, the identity element of the monoid G.

(ii) The relation $\cup\, 1\, \varphi^{-1}$, which we shall denote by I, is an equivalence relation on E, and $A - \emptyset / \varphi\varphi^{-1}$ is a regular permutation group on the I-classes with the definition

$$(Ix)(A\varphi\varphi^{-1}) = I(x(\cup\, A\varphi\varphi^{-1}))$$

(iii) For all $A, B \in \mathcal{G}^*$, if xAy and xBy for some $x, y \in E$, then $A\varphi = B\varphi$.

(i) implies (ii)

Lemma (I). I is an equivalence relation on E.

<u>Proof:</u> I is <u>reflexive</u> by the following argument: Since E is strongly connected, there exists an $A \in \mathcal{G}^*$ for every $x \in E$ such that xAx holds. So $A\varphi = 1$ implies $A \subseteq I$. Hence xIx.

I is <u>symmetric</u> by the following: Suppose xIy so that xAy holds for some A such that $A\varphi = 1$. By strong connectivity, there exists a $B \in \mathcal{G}^*$ such that yBx. So $xABx$ implies by (i) that $(AB)\varphi = 1$, which implies $B\varphi = 1$. So $B \subseteq I$ which implies yIx.

<u>Transitivity</u> is by the following argument: If xIy and yIz then there are $A, B \in \mathcal{G}^*$ such that xAy and yBz hold and that $A\varphi = B\varphi = 1$. So $xABz$. But $(AB)\varphi = 1$. Hence $AB \subseteq I$ which implies xIz.

Lemma II. $z \in (x(\cup\, [A]))$ iff zIy, i.e., $I(x(\cup\, [A]))$ is an equivalence class.

<u>Proof:</u> ($\Rightarrow$) Define $[A] = A\varphi\varphi^{-1}$ and suppose $z \in (x(\cup\, [A]))$. Let

$y \in (x(\cup [A]))$ be fixed. Then there exists an $A_1 \in [A]$ such that xA_1y and an $A_2 \in [A]$ such that xA_2z.

By strong connectivity, there exists a $B \in G^* - \emptyset$ such that yBx. Then xA_1Bx implies $(A_1B)\varphi = 1$ by part (i). So $A_1\varphi = (B\varphi)^{-1}$.

Consider $(BA_2)\varphi = (B\varphi)(A_2\varphi) = (A\varphi)^{-1}(A\varphi) = 1$. So $BA_2 \subseteq I$ which implies yIz. But I is symmetric by lemma I, so zIy.

Suppose zIy. Then there exists $I_1 \in G^* - \emptyset$ such that yI_1z and $I_1\varphi = 1$. Since $y \in (x(\cup [A]))$ there exists an $A_1 \in [A]$ such that xA_1y. Let $A_2 = A_1I_1$. Then $A_2\varphi = (A_1I_1)\varphi = A_1\varphi = A\varphi$. So $A_2 \in [A]$ and xA_1I_1z, implies xA_2z. Therefore $z \in (x(\cup [A]))$.

Lemma III. $(Ix) \mapsto (Ix)[A]$ is well defined, i.e., $Ix = Iy$ implies $Ix[A] = Iy[A]$ or equivalently, $(x(\cup[A])) = (y(\cup [A]))$.

Proof: Suppose $Ix = Iy$. Then there exists an $I_1 \in G^* - \emptyset$ such that $I_1\varphi = 1$ and xI_1y. By strong connectivity, there exists an $I_2 \in I$ such that yI_2x.

Assume $z \in (x(\cup [A]))$. [We want to show $z \in (y(\cup [A]))$]. There exists an $A_1 \in [A]$ such that xA_1z. Let $A_2 = I_2A_1$. So yA_2z. But by the previous argument

$$A_2 = I_2A_1 \in [A].$$

Therefore $y \in (y(\cup [A])$.

There reverse inclusion follows by symmetry.

Lemma IV. $(Ix) \mapsto (Ix[A])$ is 1 - 1, i.e., if $(x(\cup [A])) = (y(\cup [A]))$ then $Ix = Iy$.

Proof: Let $z \in (x(\cup [A])) = (y(\cup [A]))$. Then there exists an $A_1 \in [A]$; $A_2 \in [A]$ such that xA_1z and yA_2z. By strong connectivity there exists a $B \in \mathfrak{G}^* - \emptyset$ such that zBy. $yA_2By \Rightarrow (A_2B)\varphi = 1$. So $A_2\varphi = A_1\varphi = (B\varphi)^{-1}$. Now $(A_1B)\varphi = A_1\varphi B\varphi = 1$. So $A_1B \subseteq I$ which implies $x(A_1B)y$ or xIy.

Lemma V. $\mathfrak{G}^* - \emptyset / \varphi\varphi^{-1}$ is regular, i.e., given $Ix \in E/I$ and $Iy \in E/I$ there exists a unique $[A] \in \mathfrak{G}^* - \emptyset / \varphi\varphi^{-1}$ such that $Ix[A] = Iy$.

Proof: *We have from a previous lemma, that* $(Ix)[A] = Iy$ for some y; so we need only show the uniqueness of $[A]$.

Suppose $Ix[A] = Iy$ *and* $Ix[B] = Iy$. *There exists an* $A_1 \in [A]$ such that xA_1y and $B_1 \in [B]$ such that xB_1y. By strong connectivity, there exists a $B_2 \in \mathfrak{G}^* - \emptyset$ *such that* yB_2x. *So* xB_1B_2x *implies* $B_1\varphi = (B_2\varphi)^{-1}$ and xB_1B_2x implies $(B_2\varphi)^{-1} = A_1\varphi$. Then $A_1\varphi = B_1\varphi$ which implies $[A] = [B]$. Hence (i) implies (ii) is complete.

(ii) implies (iii). Suppose xAy and xBy hold for some $x, y \in E$ and $A, B \in \mathfrak{G}^*$. Then $(Ix)[A] = Iy$ and $(Ix)[B] = Iy$ and by regularity $[A] = [B]$ implies $A\varphi = B\varphi$.

(iii) $\Rightarrow$ (i) Because G is onto there exists an $A \in \mathfrak{G}^*$ such that $A\varphi = 1$ and so that $1_E\varphi = 1$. Now $x1_Ex$ for all $x \in E$. If xAx then by (iii) $A\varphi = 1_E\varphi = 1$. This completes the proof of the theorem.

There are certain sociological interpretations of the group partition theorem. The requirement that E be strongly connected can be interpretated as the society is closely knit and homogeneous. That is, everybody is related to everybody else. This assumption would not be met in societies which do not intermarry.

It should be noted that when G is a group of two elements, the graph of $\mathcal{G}$ is what Cartwright and Harary call a balanced graph, where a line is called *positive if it* is mapped onto the identity 1.

We can consider the ordered pair (E, $\mathcal{G}$) as a digraph D by identifying the set of points of D with E. We say there is a line from point x of E to point y of E if and only if there exist an $A \in \mathcal{G}$ such that $(x, y) \in A$ or, equivalently, xAy.

Similarly, we identify walks and closed walks with $\mathcal{G}^*$. That is, there is a walk from x to y if and only if there exists a $B \in \mathcal{G}^*$ such that xBy.

Let G be a monoid with one generator (a). With the homorphism φ mapping from the lines of the graph D onto G, which associates with each line in D one of the signs $\{1, a\}$, D is a signed digraph.

Call $\varphi^{-1}(a) = N$, the negative lines of D and $\varphi^{-1}(1) = P$, *the positive lines* of D.

If φ is a homomorphism then the sign of a walk is the product of the signs of its component lines. With these identifications, we can restate the group partition theorem for this simplified case.

If E is strongly connected and $\varphi : \{N, P\} - \emptyset \to G$ is a homomorphism, the following are equivalent:

(i) Every closed walk is positive.

(ii) G is a group; and the society can be partitioned into two non-empty subsets such that negative relations hold only between persons of different subsets, and positive relations hold only between persons of the same subset.

(iii) Any two walks between points x and y have the same sign.

The proof of this theorem is an immediate consequence of the preceding theorem.

An example of balanced graphs in kinship systems is the "moiety" system, that is, partitioning a system into two parts; marriage being restricted to couples belonging to different moieties. If a child belongs to the same moiety as his father, the system is called patrilineal. Similarly, if he belongs to the same moiety as his mother, the system is called matrilineal. In a patrilineal system, the relation between father and child is "positive"; and the relation between mother and child "negative".

CONCLUSIONS

The intention of this paper was to show the importance of group theory in studying marriage and exchange systems. Certain sociological interpretation were underlying factors in this study.

Researchers are optimistic that continued study of such systems will yield suggestions for new mathematical results. This brings to mind the following questions that became apparent during the writing of this report.

(1) Is the exchange based on (kinship) terminology or geneology? If it is based on terminology, a more complicated mathematical system (which is nonassociative) may be a more appropriate model.

(2) Which groups, generated by two elements, provide a model for a (potential) marriage exchange system? Restated, why do the groups which appear in the real world (as models for exchange systems) actually appear? This question is related to (algebraic) group extensions.

(3) What mathematical structures model systems in which μ(defined in Section II) is not a function but merely a relation?

These questions and the development which motivated them indicate only one possible area of application of abstract algebra to the social sciences. It is believed that behavioral science applications using abstract algebra will prove a more valuable tool than group theory, since groups as models of behavior can have extreme limitations.

APPENDIX

In this presentation, we consider the abstract group G, generated by two elements. An identification is made between the functions described in section II. Rather than considering a mapping involving clans, we define the nXn matrices W and D in terms of permutations of the integers.

Before we begin, the following definition is necessary.

Definition: A society is an ordered quadruple $\mathcal{S}' = (S_1, S_2, \sim, \mu')$ where S_1 and S_2 are two disjoint, nonempty sets, $\sim$ is an equivalence relation on S where $S = S_1 \cup S_2$, and $\mu' : N \to N$ by $\mu'(i) = j$ iff $\beta(C_2^i) = C_1^j$.

Define $\alpha_1 \colon \mathcal{C} \to \mathcal{C}_1$ by $\alpha_1(C^i) = C_1^i$

$\alpha_2 \colon \mathcal{C} \to \mathcal{C}_2$ by $\alpha_2(C^i) = C_2^i$

α_1 and α_2 are bijections and hence α_1^{-1}, α_2^{-1} exist.

Recall, $\beta \colon \mathcal{C}_2 \to \mathcal{C}_1$ by $\beta(C_2^i) = C_1^j$ and

$$\Delta : \mathcal{M} \to \mathcal{C} \text{ by } \Delta(C_1^i, \mu(C_1^i)) = C^j = \Delta_1(C^i)$$

Define $\Delta' \colon N \to N$ by $\Delta'(i) = j$ iff $\Delta_1(C^i) = C^j$ and $\mu' : N \to N$ by $\mu'(i) = j$ iff $\beta(C_2^i) = C_1^j$.

Δ' and μ' are bijections and hence are permutations on N.

The axioms on page 9 reduces to the following:

Axiom Schema I''

1.1'' μ' is fixed point free, i.e., $\mu'(i) \neq i$ for all $i \in N$.

1.2'' The society $\mathcal{S}'$ is connected.

Let W and D be nXn matrices defined as follows:

$$W_{ij} = \begin{cases} 1 & \text{iff } \mu'(i) = j \\ 0 & \text{otherwise} \end{cases}$$

and

$$D_{ij} = \begin{cases} 1 & \text{iff } \Delta'(i) = j \\ 0 & \text{otherwise} \end{cases}$$

We now investigate the possibilities for male ego to marry his matrilateral cross cousin, that is, his mother's brother's daughter. Let e be the identity map from N onto N. The wife's matrix W must equal to the matrix describing the clan of a girl relation of male ego. Then, W must equal FWBD, i.e., father's wife's brother's daughter. So,

$$\mu' = \Delta'^{-1} \mu e \Delta' = \Delta'^{-1} \mu \Delta'$$

Hence a necessary condition for marriage between male ego and his matrilateral cross cousin is for

$$\Delta'\mu' = \mu'\Delta' \quad \text{or equivalently } DW = WD.$$

Similarly, for male ego to marry his patrilateral cross cousin, i.e., his father's brother's daughter,

$$\mu' = \Delta'^{-1}(e)\Delta' = e,$$

which contradicts axiom 2.6. Hence marriage between cousins of this type is not permitted in a society based on the previous axioms.

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A MATHEMATICAL ANALYSIS OF MARRIAGE AND KINSHIP SYSTEMS

by

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ABSTRACT

The purpose of this report is to formulate a mathematical approach to marriage and kinship systems. The main tool for this formulation is group theory.

Given a set of axioms governing marriage and descent relations of a society, bijections or, equivalently, permutation matrices are utilized to describe allowable marriages within the society. Compound kin relations are described by composition of functions or, equivalently, matrix multiplication.

Marriage class systems are modeled abstractly in the "group partition theorem". A corollary of this general theorem is the well known graph theorectic theorem characterizing balanced graphs.

Observations and open questions are raised at the end of this report. They serve as motivation for further mathematical modeling of marriage and kinship systems.