AN ANALYSIS OF STRESS VARIATION
IN TRACTOR AXLES BY SIMULATION METHODS

by

NAVIN PRAKASH MATHUR

B.Sc. (ag. Engg.), Allahabad Agricultural Institute, 1966, India

---

A MASTER'S THESIS

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Agricultural Engineering

KANSAS STATE UNIVERSITY

Manhattan, Kansas

1970

Approved by

Stanley J. Clark
Major Professor

TABLE OF CONTENTS

TABLE OF CONTENTS continued

ILLEGIBLE

THE FOLLOWING DOCUMENT (S) IS ILLEGIBLE DUE TO THE PRINTING ON THE ORIGINAL BEING CUT OFF

ILLEGIBLE

LIST OF FIGURES

LIST OF FIGURES continued

LIST OF TABLES

INTRODUCTION

Design of tractor drive axles is still more of an art than a science. Axle failure has drawn the attention of various tractor and power transmission manufacturers during the past ten years. Laboratory tests for the most part have not provided an adequate idea of stress fluctuations which are possible under actual field conditions. It is possible that axles could be designed more scientifically if factors such as vehicle mass, tire spring rates, tire damping co-efficients, speed range, configuration of field surfaces, material properties, and geometrical configurations were considered during design.

Random motion of tractors on highly irregular surfaces, the addition of dual wheels, and wheel tread adjustment for various field conditions are factors which have generally not received adequate attention in the design of rear axles. These factors can cause excessive axle stress and result in immediate failure or early failure due to fatigue. Axle failure presents a very dangerous situation to the tractor operator since it can cause the tractor to upset. It also results in a costly repair job for the owner.

Consideration of the dynamic condition as well as the static situation becomes very important when the vehicle oscillates at large amplitudes from its equilibrium position or when the draw bar load on the tractor fluctuates. Under these conditions, stresses can be very high and may cause the axle to fail immediately or much earlier than its expected life. Therefore, transient fluctuation of peak stresses should be analyzed before the axle is built and used for operational purposes.

A mathematical analysis of the stress variation in a tractor axle becomes quite difficult when transient stress is considered due to dynamic loading of rear tires. A mathematical solution, however, provides explicit information about stress fluctuation for the interval of time during which it is damping out. This solution also indicates the period within which the vibratory motion of the vehicle reaches a steady state situation as well as the total time during which the axle was under excessive stresses. Prediction of the number of cycles of a particular stress level to calculate fatigue life is also possible with the help of this transient stress analysis.

REVIEW OF LITERATURE

Rear axle deflection and its stresses were of interest in both road and off-the-road vehicles even before the age of steel wheels (late in the eighteenth century). Increasing speed, irregularity in road and field surfaces, and increasing torque in the axle are a few major factors which cause the axle to fail during operation.

Considerations of Vibration

The latest analysis of coupled vibration of a road vehicle axle was made by Ellis (1967). He studied a five-degrees-of-freedom model of a typical suspension, and demonstrated that a two-degrees-of-freedom model of pitching and bouncing modes is adequate for the study of axle deflection. He also emphasized that forcing functions resulting from fluctuating torques on the axle and road irregularities require that further uncoupled modes of vibration must be included if the non-linearity due to loss of contact between the road and the wheel is to be represented.

Pershing (1966) analyzed the transient motion of a tractor on side slopes. He represented the tractor by a nine-degrees-of-freedom model, and assumed the tractor was a rigid body with the exception of the tires. The front axle was considered to have tramp motion about its hinge point on the chasis. He used a single bump having a specific configuration of a half-sine period as the forcing function and plotted the rear axle tire displacement with respect to ground profile from a computer solution of the differential equations.

An analytical method was developed by Young (1948) for determining natural frequencies of a composite system which consisted of a uniform beam with a concentrated mass, spring, and dashpot that could be attached at any point along the length of the beam. Young showed that the fundamental frequency of a uniform cantilever beam can be obtained by considering the beam as a massless cantilever spring with a concentrated mass at the end equal to 24.267 percent of the beam mass. He also studied various cases of cantilever beams with masses and dashpot included in the system.

A study of secondary vibration in the rear axle of an automobile was made by Polhemus (1950). He reported that the axle can shift linearly with respect to the chassis in three directions (x, y, and z), and it can rotate around the three axes to produce the rotary motions 0_x, 0_y, and 0_z.

The translating motions are called

x . . . . Parallel hop

y . . . . Fore and aft shift

z . . . . Lateral shift

The rotary motions are

0_x. . . . . Yaw

0_y. . . . . Tramp

0_z. . . . . Windup

The amplitude and resonent frequencies of these secondary ride motions depend upon axle mass and elastic stiffness between the mass and the chassis. Parallel axle hop and tramp were the two motions directly excited by road waves.

It is not possible to accurately calculate the stress due to vibration from the amplitude of vibration since it requires double differentiation of the displacement (Hendry, 1964). Stresses can be determined more accurately by measuring the acceleration.

In the case of a vibrating beam

$$\frac{d^2M_x}{dx^2} = \frac{W_x a_x}{g}$$

Where M_x is the maximum bending moment at the distance x from the end of the beam, and

a_x is the maximum acceleration at x,

W_x is the intensity of loading at x,

g is the gravitational acceleration.

Then the stress at x is:

$$S_x = \frac{M_x C_x}{I_x} = \frac{C_x}{I_x} \int_o^x \int_o^x \frac{W_x a_x}{g} \, d_x d_x + S_o M_o$$

Where C_x is extreme fiber distance from the neutral axis of the cross-section at x, and

I_x is the area moment of intertia of the cross section at x.

Fatigue Considerations of Axle Failure

Cumulative damage theory was used to analyze tractor final drives by Graham (1961). He showed that the fatigue life for a component having cyclic loading can be calculated by the following formula:

$$N_g = \frac{N}{E\, d_i \left(\frac{S_i}{S}\right)^{1/a}}$$

Where

N_g = Fatigue life in cycles

N = Cycles to failure at stress level S

S_i = Stresses at each level observed

S = Maximum stress expected

E = Modulus of elasticity

$1/a$ = Inverse slope of S-N curve

d_i = Ratios of the cycles applied at stress level S to the total cycles applied

The fatigue characteristics of a combined stressed body are best estimated by the distortion energy theory according to Grower, Gorden, and Jackson (1960). This theory is expressed in the following equation for torsion and bending:

$$S_c = \sqrt{S_b^2 + 3S_t^2}$$

Where

S_b = Bending stress

S_t = Shear stress

S_c = Combined stress

The combined stress calculated by this equation can then be used with the bending S-N curve to estimate the fatigue life. Graham also concluded that the cumulative damage technique greatly depends upon the accuracy of the field load and fatigue data used.

Car axle fatigue testing was also made by Stott (1958). He reported that passenger car axles are subject to a combination of both torsional and bending stresses, but ultimate fatigue life can best be assessed if each factor of loading is treated separately and tested accordingly.

A modified version of a rotating cantilever machine was described by Dawtery (1946). The machine was developed to determine the fatigue life of a truck rear axle. He suggested that a target life of 10^6 cycles is desirable.

While discussing the operational stresses in automotive parts, Robert (1959) mentioned that the axle runs with a continuous stress reversal under static load. Axles are frequently subjected to larger loads due to bumps and lateral forces at the tire. They must, therefore, withstand occasional large overloads. When a car encounters a curve during cornering or on rough roads, lateral forces at the tires produce stress reversals at many sections; fatigue failure is therefore likely. Robert also reported that when the wheel strikes large bumps, the axle may fail in fatigue from the frequent application of normal forces or it may undergo permanent deformation due to excessive accelerating or decelerating loads. He reported that dynamic stresses are approximately four times the static load stress. Another critical stress in the axle is due to the torsional stress at the inner end (either in the splined section or in the circular section next to splines). In view of the stress concentration present in that location, fatigue is possible even if the stress reversal is not present.

Holfmeister (1960) emphasized that for random dynamic loading, predicting total cycles to failure using the cumulative fatigue damage theory proposed by Corten and Dolen correlates well with the experimental evidence obtained by laboratory tests. The information that is necessary to apply this theory is as follows:

1) Knowledge of endurance limit diagram for the part or machine in question.

2) Knowledge of service load spectrum.

3) The cumulative fatigue damage relationship.

The equation developed by Corton and Dolen to calculate the number of cycles to failure is:

$$N_g = \frac{N_1}{\alpha_1 + \alpha_2\left(\frac{\sigma_2}{\sigma_1}\right)^a + \alpha_3\left(\frac{\sigma_3}{\sigma_1}\right)^a + \ldots + \alpha_i\left(\frac{\sigma_i}{\sigma_i}\right)^a}$$

Where

N_g = Total cycles to failure

N_1 = Cycles to failure for continuous stressing at σ_1

N_2 = Cycles to failure for continuous stressing at σ_2

σ_1 = Maximum applied stress

σ_2 = Second largest applied stress

σ_3 = Third largest applied stress

σ_i = Minimum applied stress for which damage progresses

α_1 = Ratio of cycles N_1 at the total cycles N_g at σ_1

α_2 = Ratio of cycles N_2 to the total cycles N_g at σ_2

Some of the elements which are required to calculate the number of cycles to failure are given in Figure 1.

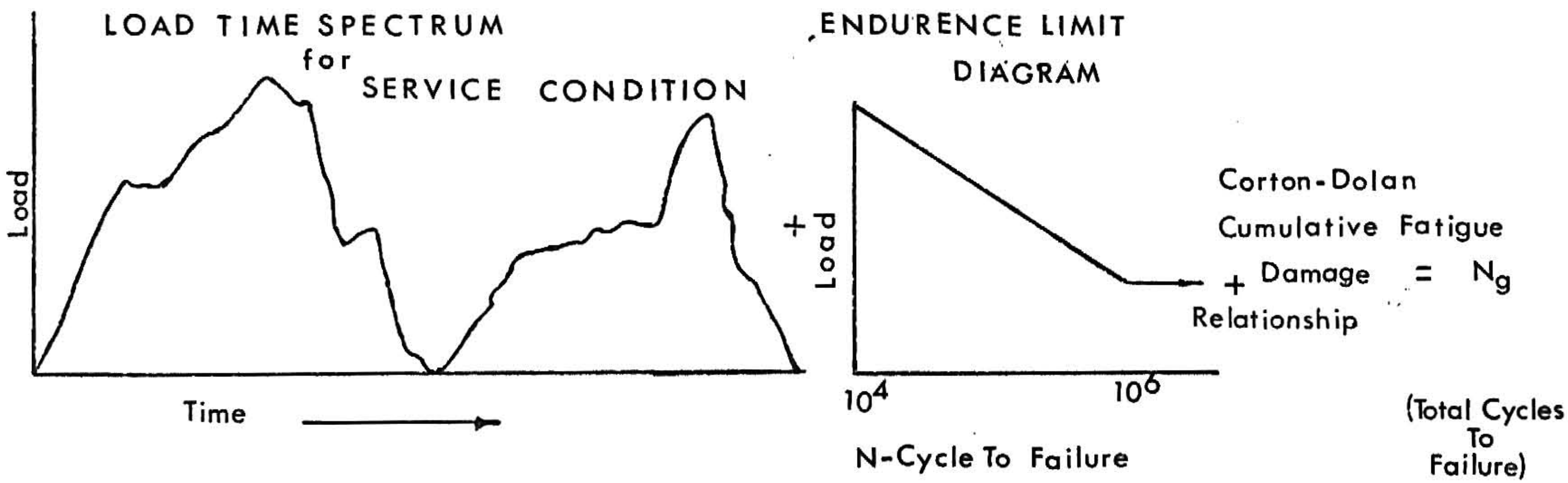

Figure 1. Elements required to calculate total cycles to failure for a machine member.

Research involving the bending of rotating beams is closely analogous to research involving axle design. Experiments on rotating beams were made by Tiedemann and Vigness (1955) on separate specimens, at different constant rotational speeds and at different constant rates of transverse bending. Rotating bars exhibited an obvious yield at much less load than did non-rotating specimens. This yield occured when the maximum stress in the rotating specimens was approximately equal to the yield stress as determined by a tensile test of the material. The load supported by a rotating specimen increased nearly linearly with deflection after yield. For small plastic strains, rotating specimens were less rigid than non-rotating specimens. For large plastic strains the reverse was true. They also emphasized that the physical strength properties of materials are sometimes greatly influenced by strain rate and the duration of load application.

Studies Made on Axle Design and Stress Analysis

Tractor axle stress variation was studied by Anderson (1966); he found that wheel spacing and dual tire attachments affect the stress variation in the rear axle of a tractor. Combined torsional and bending forces in the rear axle can be 40-50 percent greater in a tractor equipped with duals set at 60 and 120 inches respectively, compared to a single set of tires spaced at 80 inches. He also reported that magnitude and frequency of these forces are affected by terrain, type of soil, and soil conditions.

Analysis of a truck axle under dynamic conditionswas made by Gordan (1955). He reported that static load conditions are important for preliminary design and evaluation. However, they do not allow for the combined forces encountered under dynamic conditions, such as rounding a curve at high speed where weight transfer and lateral skid forces become very significant.

To improve tractor axle design, Eckert (1951) developed a non-resonant fatigue machine so that he could determine the effect of material variation, heat treatment, spline geometry, and surface treatment on the strength of the axle. The machine was capable of loading the axle up to 90,000 lb ft twice a second. Specimens could be run in torsion as well as in bending in the direction of any mean stress.

Experimental Data on Axle Loading and Its Stresses

Vehicle: Heavy-duty highway tractor trailer combination

Data: Axle ratio = 7.8

Tire size = 11.00/20

Transmission ratio = 7.5

Net engine torque = 300 lb per ft

Net engine h.p. = 150

Gross vehicle weight = 28,000 lb

Gross combination weight = 28.000 lb + 32,000 lb

Axle load = 22,000 lb

Static case

Gordon stated that the axle had a yield stress of 60,000 psi

Maximum allowable static stress = 12,000 psi

Diameter of axle = 3.668 inches

Maximum static stress in axle = 9,700 psi

Static stress at housing critical section = 14,200 psi

Static B.M. = 36,000 in lb

Dynamic case

Axle maximum stress = 70,800 psi

Maximum bending moment in housing = 41,400 in lb

Axle shaft analysis

Vehicle - Crawler tractor with loader attached

Axle stress analysis was made by James A. Graham, David K. Berns, and Duane R. Olberts.

An oscillograph record showing bending moment and torque fluctuations on the axle is shown in Figure 2.

The life of the axle was calculated by cumulative damage analysis based on the load spectrum and the S-N curve given in Figure 1. This shows the allowable stress for different ranges of life in terms of N cycles.

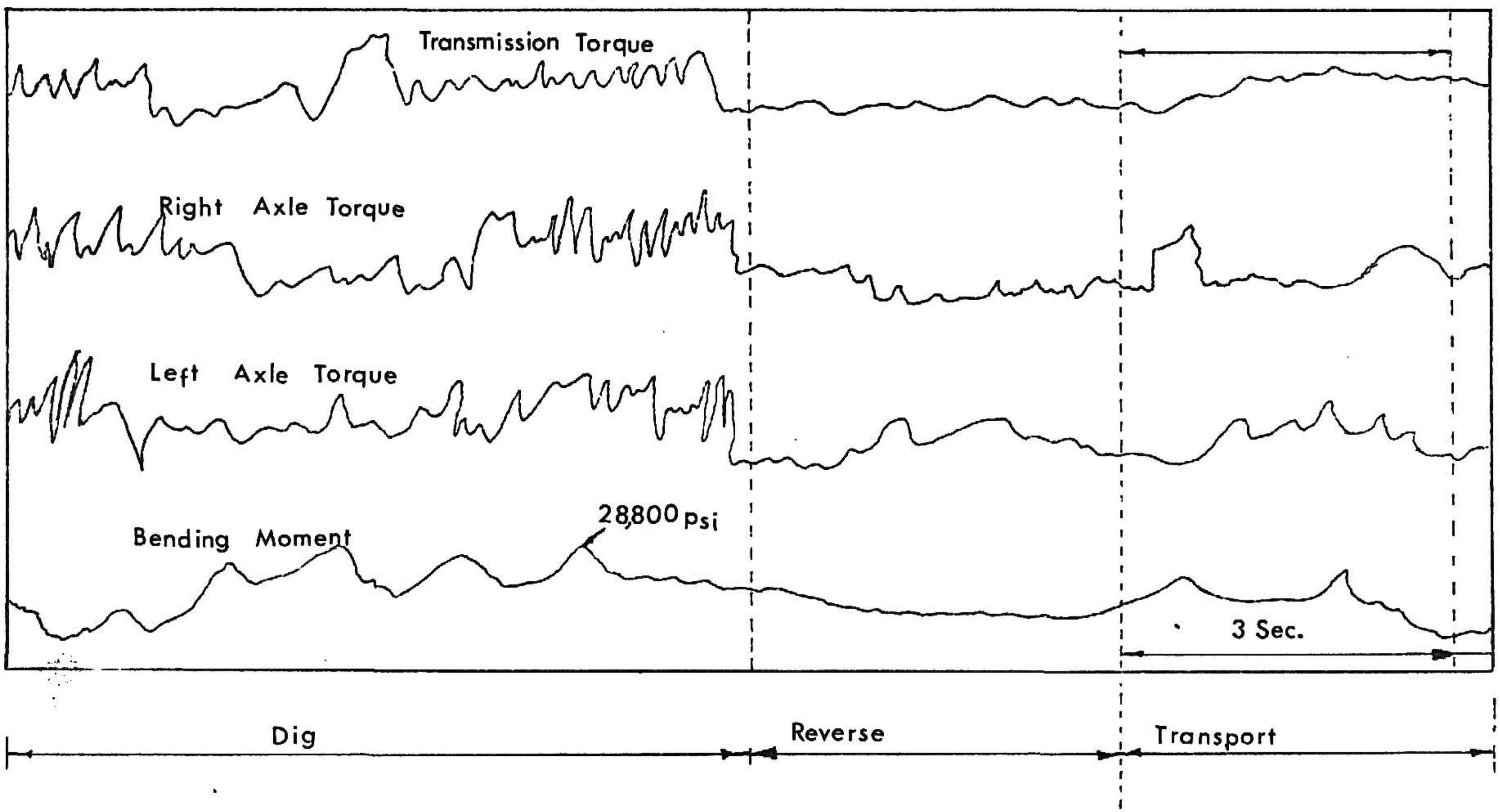

Figure 2.. A typical oscillograph trace of field load measurements obtained when digging the lime stone bank.

OBJECTIVES

1. To expand the mathematical model of the conventional agricultural tractor to include the effects of dual tire attachments.
2. To develop stress equations that provide maximum bending stress versus time relationships for various methods of attaching dual tires.
3. To determine the solution of the stress equations for conventional and dual tire configurations by digital computer simulation.

THE MATHEMATICAL MODEL

The Problem Defined

Tractor axles are generally designed on a static rather than a dynamic basis. Little consideration is given to the effects of dual wheel attachments. Methods of dual wheel attachment, loss of inside tire contact with the ground surface during operation, and vehicle speed may greatly influence the stress magnitudes in the rear axle. The maximum stress is sometimes critical in the dynamic case and thus should be studied under various dynamic conditions. Axle material properties, axle size, and methods of attaching dual wheels may be varied to obtain a design that will provide a reasonable life.

A rear axle model which describes the relationship between vehicle variables and represents the actual equivalent system is needed. This should be done so that methods for improving the design can be determined without having to build a model and test it. A fairly complete mathematical model which provides information about axle deflection as well as the information concerning vehicle stability is needed. Irregular contact of dual wheels with the ground surface effects the stress condition; therefore, a model that considers the axle as an elastic body must be considered.

The study here will be confined to expansion of a conventional tractor model to a dual wheel model with an elastic axle. In order to find axle deflection, the rear axle and dual wheels configuration will be represented as an equivalent vibratory system. The analysis of axle stress variation will be discussed in light of the results from the computer solution.

Mathematical Models

A 17 degree-of-freedom model was used to include the effect of a complex coupling present in the tractor. Ten individual bodies were included in the model: four rear wheels, two front wheels, the chassis, the front axles, and two rear axles. The wheels were represented as an equivalent system consisting of masses with linear springs and constant damping. The rear axle was modeled as a massless rotating cantilever beam.

Schematic diagrams of the various models of rear axles are shown in Figures 3 through 8. The figures consist of top, rear, and side views of equivalent vibratory systems of the chassis, wheels, and axles. All masses, equivalent springs, and dampers have been represented in terms of m, k, and c with their respective subscripts. Definitions of all the parameters have been described in Tables I through III. These models were used to calculate kinetic, potential, and dissipative energies in the system and in the development of equations of motion.

Summary of Assumptions and Their Limitations

Assumptions which were made during the analysis are as follows:

1) The dual wheel tractor was considered as nine bodies: the four rear wheels, two front wheels, the chassis, front axle, and rear axle.

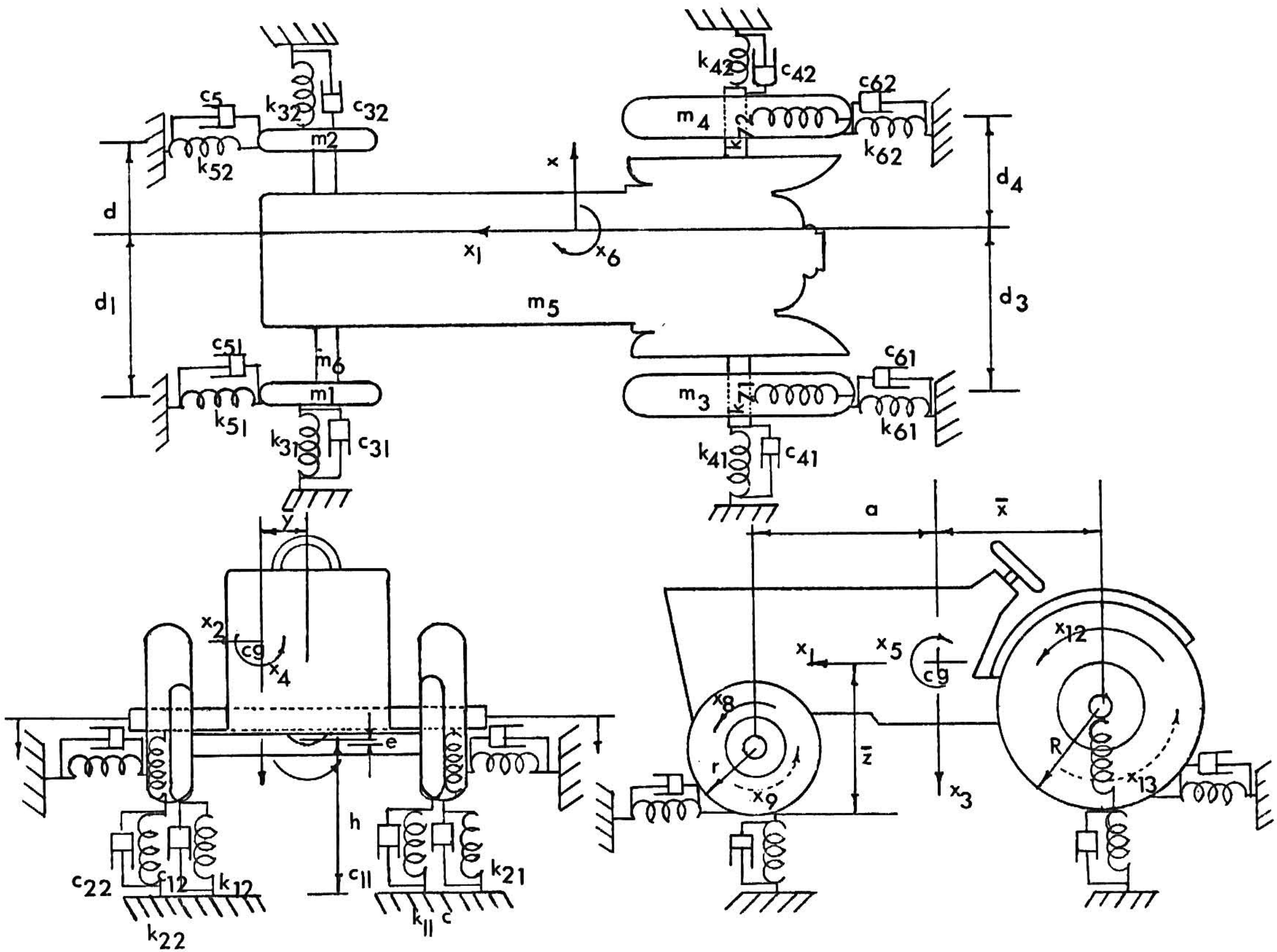

Figure 3. Mathematical Model of Single Wheel Tractor (With Elastic Axle).

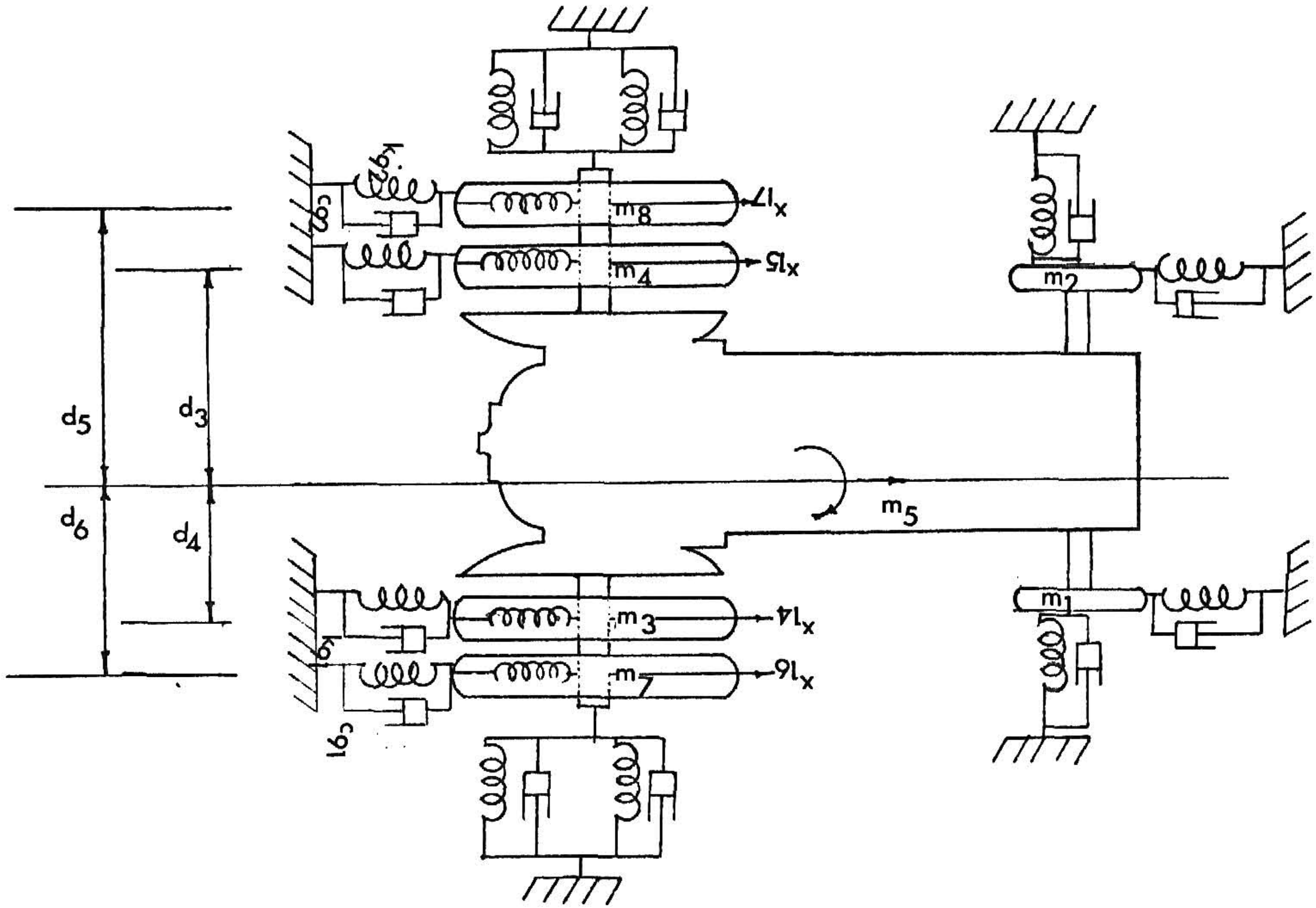

Figure 4. Mathematical Model of Dual Wheel Tractor (Top View).

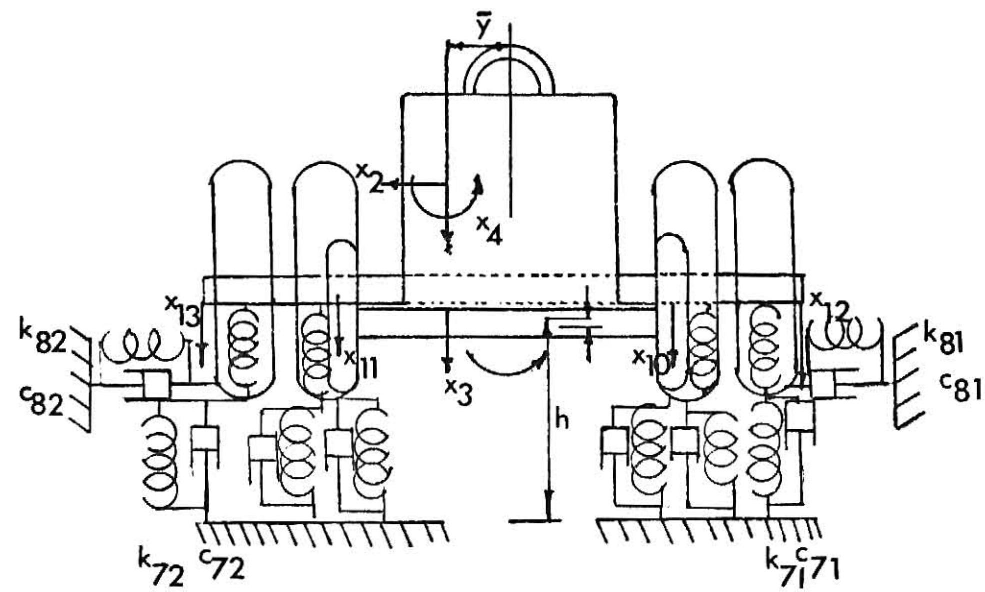

Figure 5. Mathematical Model of Dual Wheel Tractor (Rear View).

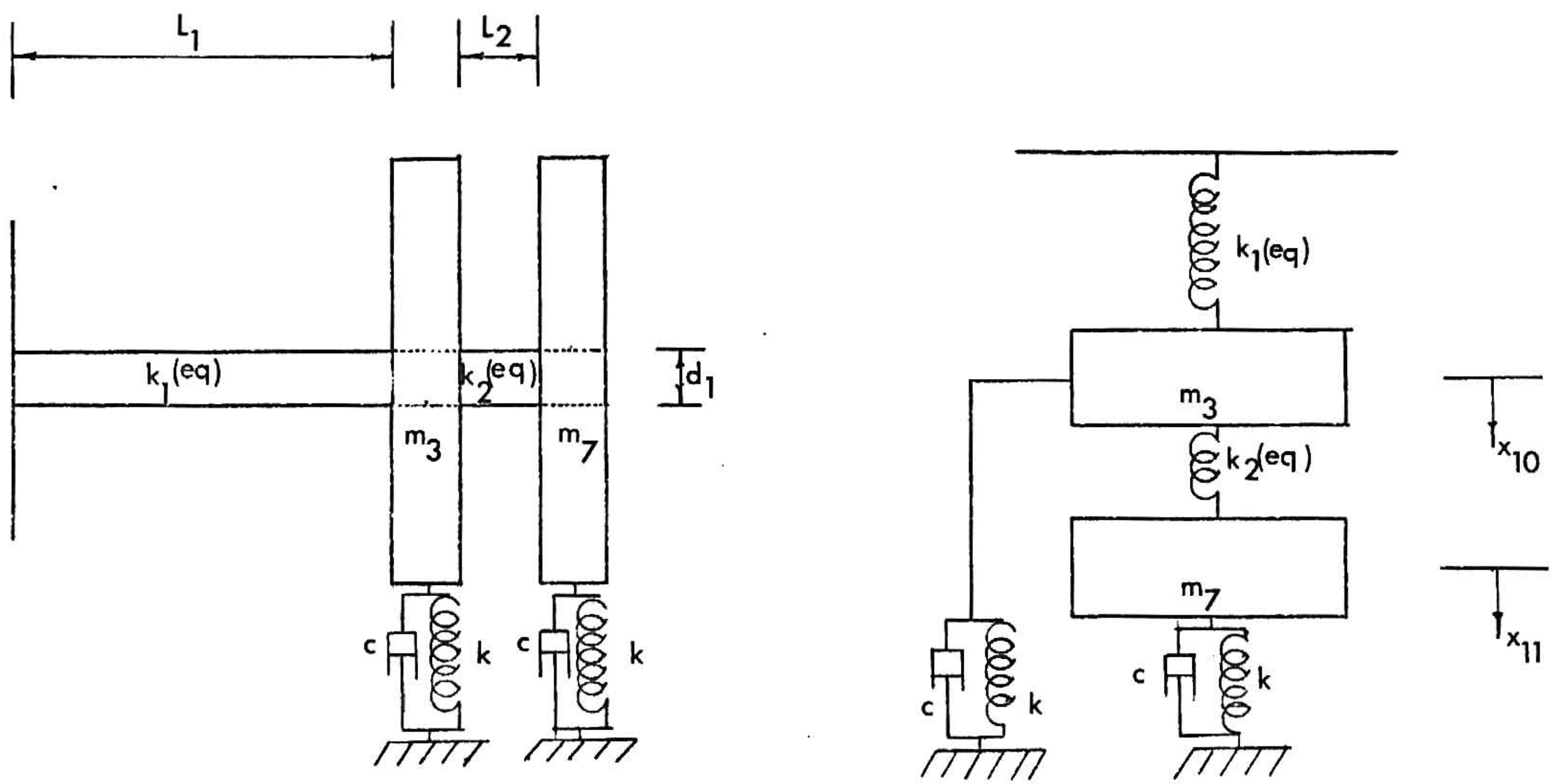

Figure 6. Dual wheel rear axle mathematical model with hub method of attachment.

TABLE I

TRACTOR DIMENSIONS

Symbol	Definition
$\bar{x}$	Perpendicular distance from the rear axle to a transverse vertical plane through the center of mass of the chassis
$\bar{y}$	Perpendicular distance from the longitudinal vertical plane through the center line of the chassis to the center of mass
$\bar{z}$	Vertical height of the center of mass of the chassis above the ground (with the tractor on level surface)
r	Rolling radius of the front wheels
R	Rolling radius of the rear wheels
d_i	Distance from the center of mass of the chassis to the wheel (i)
a	Perpendicular distance from the front axle to a transverse vertical plane through the center of mass of the chassis (wheel base - $\bar{x}$)
b	Vertical distance between the hinge point of the front axle and the center of mass of the chassis
h	Height of the front axle hinge ($\bar{z}$ - b)
e	Vertical distance between the hinge point of the front axle and the center of mass of the front axle

TABLE II

POSSIBLE DEGREES OF FREEDOM

Symbol	Definition for Positive Displacement
x_1	Forward translation of chassis
x_2	Lateral translation of chassis to the right
x_3	Downward translation of chassis
x_4	Angular motion of chassis according to the right-hand rule about the longitudinal axis (roll motion)
x_5	Angular motion of chassis according to the right-hand rule about the transverse axis (pitch motion)
x_6	Angular motion of chassis according to the right-hand rule about the vertical axis (yaw motion)
x_7	Angular motion of front axle position in the same direction as x_4 (tramp motion)
x_8	Angular motion of left front wheel, positive in the direction opposite to positive x_5
x_9	Angular motion of right front wheel, positive in the direction opposite to positive x_5
$x_{10\ \&\ 14}$	Vertical and horizontal motion of left rear axle point where the inner dual wheel is attached (positive x_3 direction)
$x_{11\ \&\ 15}$	Vertical and horizontal motion of right rear axle point where the inner dual wheel is attached (positive x_3 direction)
$x_{12\ \&\ 16}$	Vertical and horizontal motion of left rear axle point where the outer wheel is attached (positive x_3 direction)
$x_{13\ \&\ 17}$	Vertical and horizontal motion of right rear axle point where the outer dual wheel is attached (positive x_3 direction)
x_{18}	Angular motion of the left rear wheel (positive)
x_{19}	Angular motion of the right rear wheel (positive)

TABLE III

SPRING AND DAMPING CONSTANTS

Symbol	Definition
k_{11}, k_{12} and c_{11}, c_{12}	Spring and damping rate, vertical direction left and right front tires, respectively.
k_{21}, k_{22} and c_{21}, c_{22}	Spring and damping rate, vertical direction left and right inner rear wheels (for dual wheel).
k_{71}, k_{72} and c_{71}, c_{72}	Spring and damping rate, vertical direction left and right outer rear wheels (for dual wheels).
k_{31}, k_{32} and c_{31}, c_{32}	Spring and damping rate, lateral direction left and right front wheels, respectively.
k_{41}, k_{42} and c_{41}, c_{42}	Spring and damping rate, lateral direction left and right inner rear wheels, respectively.
k_{81}, k_{82} and c_{81}, c_{82}	Spring and damping rate, lateral direction left and right outer rear wheels, respectively.
k_{51}, k_{52} and c_{51}, c_{52}	Spring and damping rate, fore and aft left and right front wheels.
k_{61}, k_{62} and c_{61}, c_{62}	Spring and damping rate, fore and aft left and right inner rear wheels.
k_{91}, k_{92} and c_{91}, c_{92}	Spring and damping rate, fore and aft left outer rear wheel and right outer wheel.

2) Small angular oscillations were assumed in order to neglect the effect of nonlinearity due to loss of contact between the wheel and the ground.

3) The rear axle extending outside the axle housing was assumed as an elastic rotating cantilever beam with neglegible internal damping and mass.

4) The tires were assumed as linear springs with constant damping in all directions.

5) Torsional stresses were neglected while calculating the maximum stresses in the rear axle.

Reduction of Model

The formulated dual-wheel model with the hub method of attachment (see Fig. 6) could be reduced to several other models by making some changes in the parameters. Possible models which could be obtained are:

1) Dual wheel tractor model with rim-to-rim attachment (see Fig. 8).

This can be achieved by developing a model with two rigidly attached masses to represent the individual wheel masses. The tires are represented by individual springs and dashpots to represent the tire spring rate and damping coefficients. Mathematically, this model could be obtained as follows:

$$K_{2eq} = 0$$

$$x_{10} = x_{12} \qquad x_{14} = x_{16}$$

$$x_{11} = x_{13} \qquad x_{15} = x_{17}$$

That is, there is no relative motion between the dual

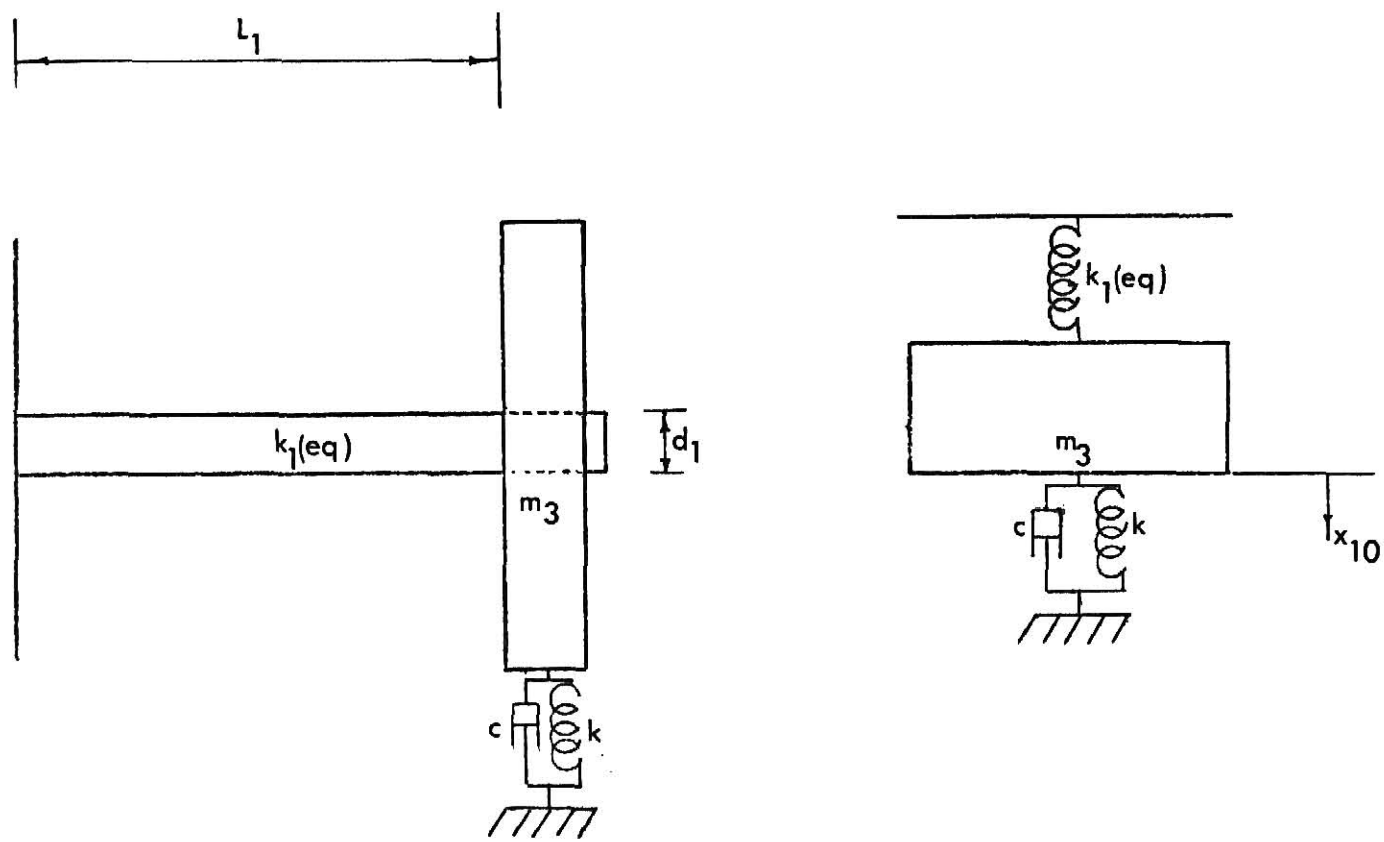

Figure 7. Single wheel rear axle mathematical model.

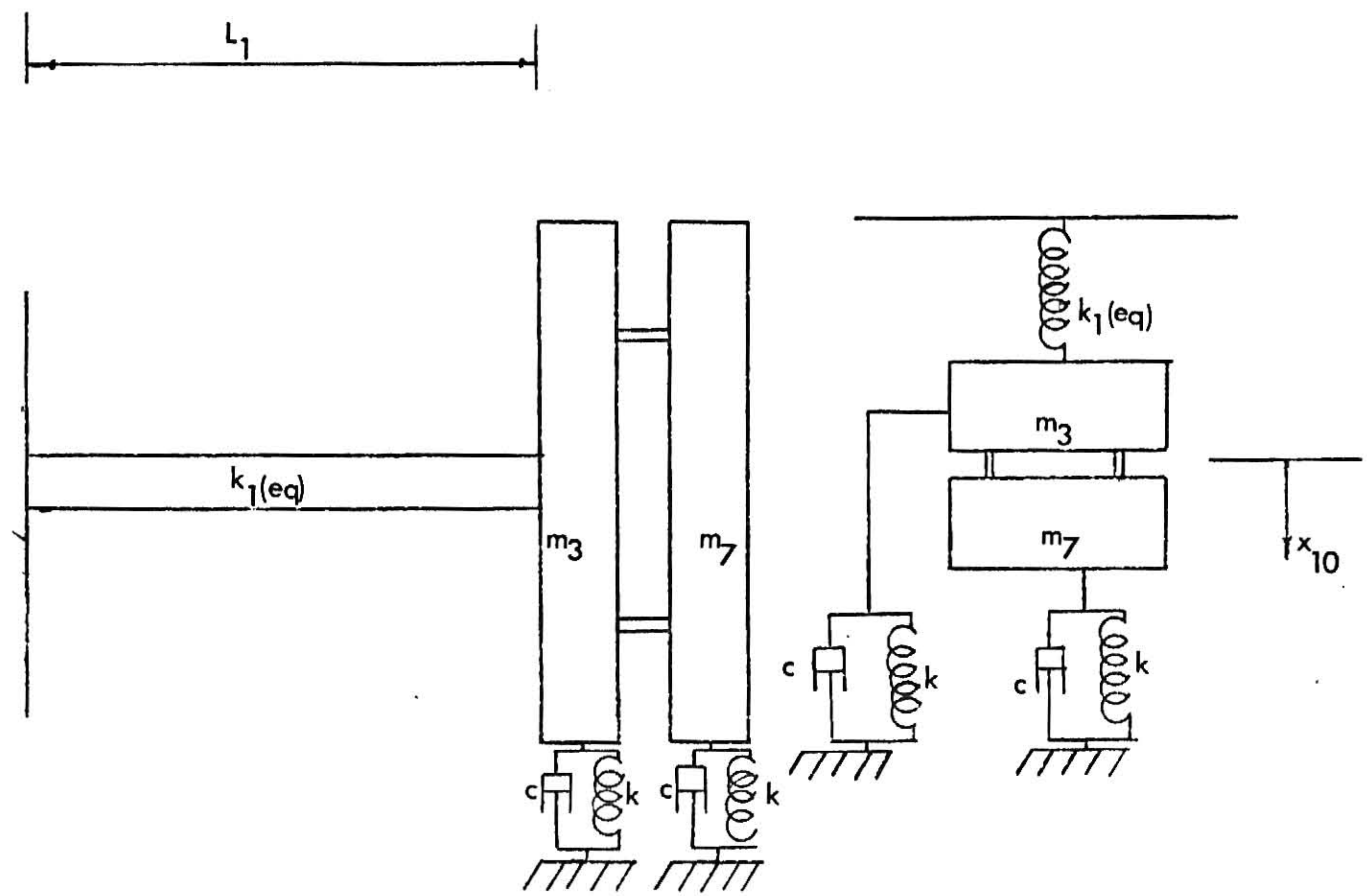

Figure 8. Dual wheel rear axle mathematical model with rim method of attachment.

wheels. This condition could be substituted into the computer program to obtain the solution of the above mentioned model.

2) Single wheel tractor model with the axle as an elastic body. This model could be obtained by making the following changes:

$$c_{71} = c_{72} = c_{81} = c_{82} = c_{91} = c_{92} = 0$$
$$k_{71} = k_{72} = k_{81} = k_{82} = k_{91} = k_{92} = 0$$
$$M_7 = M_8 = 0$$
$$k_{2eq} = 0.0$$

The model has been shown mathematically in Figure 7.

3) Single wheel tractor model with axle as a rigid body. The model could be obtained by making the following changes:

$$x_{10} - (x_3 + \bar{x}\, x_5 - d_3 x_4) = 0$$
$$x_{11} - (x_3 + \bar{x}\, x_5 + d\, x_4) = 0$$
$$x_{12} - (x_1 + \bar{z}\, x_5 - d\, x_4) = 0$$
$$x_{13} - (x_1 + \bar{z}\, x_5 + d\, x_4) = 0$$

These changes will make potential energy stored in the spring equivalent of the axle equal to zero and neglects the effect of the elastic nature of the axle.

Formulation of the Problem

An energy method was used to obtain the equations of motion for this problem. The energy approach involves Lagrange's equations which are commonly used in problems of multiple degrees of freedom. A few of the variations of Lagrange's equations are mentioned below:

In the case of a conservative system, the work done is equal to the negative of potential energy.

$$w = -u \; (q_1, q_2, \ldots, q_n)$$

and

$$\partial w = -\Sigma \frac{\partial u}{\partial_{qk}} \delta qk$$

Therefore, Lagrange's equation for the conservative system will be

$$\frac{d}{dt} \left(\frac{\partial T}{\partial_{\dot{q}k}}\right) - \frac{\partial T}{\partial_{qk}} + \frac{\partial u}{\partial_{qk}} = 0$$

For the nonconservative system, Lagrange's equation is as follows:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial_{\dot{q}k}} - \frac{\partial L}{\partial_{qk}}\right) = \bar{Q}_k$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial_{\dot{q}k}} - \frac{\partial T}{\partial_{qk}} + \frac{\partial u}{\partial_{qk}}\right) = \bar{Q}_k$$

where L = T - U; L is known as the Lagrangian Constant.

This last form enables one to entend the use of Lagrange's method to non-conservative systems. Hence, the method of Lagrange is applicable to all dynamic systems including damped vibrations.

If the non-conservative forces due to friction are proportional to velocity, a function F may be obtained so that Lagrange's equation becomes

$$\frac{d}{dt} \left(\frac{\partial L}{\partial_{\dot{q}k}}\right) - \frac{\partial L}{\partial_{qk}} - \frac{\partial F}{\partial_{\dot{q}k}} = 0$$

where L = T - v

F = Raleigh's dissipative function

Raleigh's dissipative function is defined as

$$F = 1/2\ \Sigma K_q\ \dot{q}_j^{\,2}$$

where K_j = friction in direction j

q_j = velocity in direction j

The equations of motion in this study were developed by means of the Lagrange method. Partial derivatives of kinetic energy, potential energy, and dissipative energy with respect to each coordinate in the system were obtained. These derivatives were substituted into the following equation:

$$\frac{d}{dt}\frac{\partial(K.E.)}{\partial \dot{q}i} - \frac{\partial(K.E.)}{\partial qi} + \frac{\partial(D.E.)}{\partial \dot{q}i} + \frac{\partial(P.E.)}{\partial qi} = Q_i$$

where K.E. is kinetic energy in the system

$$T = 1/2 \sum_i M_i\ \dot{q}_i^{\,2}$$

D.E. is dissipative energy in the system

$$C = 1/2 \sum_i C_i\ \dot{q}_i^{\,2}$$

and P.E. is potential energy in the system

$$U = 1/2 \sum_i K_i\ q_i^{\,2}$$

Q_i represents the external forces acting on the system.

Kinetic Energy:

Total kinetic energy in the system of the dual wheel tractor with the hub method of tire attachment was formulated as follows:

$$T_1 = 1/2\ m_1[(\dot{x}_1 + \bar{z} - r - (d_1-\bar{x})x_7)(\dot{x}_5 + d_1\dot{x}_6)]^2$$

$$+ (\dot{x}_2 - b\dot{x}_4 - (h - r - (d_1-\bar{x})x_7)\ \dot{x}_7 + a\dot{x}_6)^2$$

$$+ (\dot{x}_3 - a\dot{x}_5 - \bar{x}\dot{x}_4 - (d_1-\bar{x})\dot{x}_7)^2 + 1/2\ I_{111}\ \dot{x}_7^{\,2}$$

$$+ 1/2\ I_{122}\ \dot{x}_8{}^2 + 1/2\ I_{133}\ \dot{x}_6{}^2$$

$$T_2 = 1/2\ m_2[(\dot{x}_1 + (\bar{z} - r + (d_2 + \bar{y})\ x_7)\ \dot{x}_5 - d_2\dot{x}_6)^2$$

$$+ (\dot{x}_2 - b\dot{x}_4 - (h - r + (d_2 + \bar{y})\ x_7)\ \dot{x}_7 + a\dot{x}_6)^2$$

$$+ (\dot{x}_3 - a\dot{x}_5 - \bar{y}\dot{x}_4 + (d_2 + \bar{y})\ \dot{x}_7)^2] + 1/2\ I_{211}\dot{x}_7{}^2$$

$$+ 1/2\ I_{222}\dot{x}_9{}^2 + 1/2\ I_{233}\dot{x}_6{}^2$$

$$T_3 = 1/2\ m_3\dot{x}_{10}{}^2 + 1/2\ m_3\dot{x}_{14}{}^2 + 1/2\ m_3(\dot{x}_2 - (\bar{z}-R)\ \dot{x}_4$$

$$- \bar{x}\dot{x}_6)^2 + I_{311}(\dot{x}_4 + \dot{x}_6)^2 + 1/2\ I_{322}(-\dot{x}_5)^2$$

$$T_4 = 1/2\ m_4\dot{x}_{11}{}^2 + 1/2 m_4\dot{x}_{15}{}^2 + 1/2\ m_4(\dot{x}_2 - (\bar{z}-R)\ \dot{x}_4$$

$$- \bar{x}\dot{x}_6)^2 + 1/2\ I_{411}(\dot{x}_4 + \dot{x}_6)^2 + 1/2\ I_{422}(-\dot{x}_5)^2$$

$$T_5 = 1/2\ m_5[\dot{x}_1{}^2 + \dot{x}_2{}^2 + \dot{x}_3{}^2]$$

$$+ 1/2[I_{511}\dot{x}_4{}^2 + I_{522}\dot{x}_5{}^2 + I_{533}\dot{x}_6{}^2] - I_{512}\dot{x}_4\dot{x}_5$$

$$- I_{513}\dot{x}_4\dot{x}_6 - I_{523}\dot{x}_5\dot{x}_6$$

$$T_6 = 1/2\ m_6[(\dot{x}_1 + (b+e)\ \dot{x}_5 + \bar{y}\dot{x}_6)^2 + (\dot{x}_2 - b\dot{x}_4 + a\dot{x}_6$$

$$- e\dot{x}_7)^2 + (\dot{x}_3 - \bar{y}\dot{x}_4 - a\dot{x}_5)^2] + 1/2\ I_{611}\dot{x}_7{}^2$$

$$+ 1/2\ I_{622}\dot{x}_5{}^2 + 1/2\ I_{633}\dot{x}_6{}^2$$

$$T_7 = 1/2\ m_7\dot{x}_{12}{}^2 + 1/2\ m_7\dot{x}_{15}{}^2 + 1/2\ m_7(\dot{x}_2 - (\bar{z}-R)\ \dot{x}_4$$

$$- \bar{x}\dot{x}_6)^2 + I_{711}(\dot{x}_4 + \dot{x}_6)^2 + 1/2\ I_{722}(-\dot{x}_5)^2$$

$$T_8 = 1/2\ m_8\dot{x}_{13}{}^2 + 1/2\ m_8\dot{x}_{17}{}^2 + 1/2\ m_8(\dot{x}_2 - (\bar{z}-R)\ \dot{x}_4$$

$$- \bar{x}\dot{x}_6)^2 + 1/2\ I_{811}(\dot{x}_4 + \dot{x}_6)^2 + 1/2\ I_{822}(-\dot{x}_5)^2$$

Potential and Dissipative Energy:

Energy stored in the equivalent springs of the system and energy dissipated due to velocity in the equivalent dampers of the system were formulated as follows:

Total dissipative energy of the system due to the tire damping coefficients:

$$
\begin{aligned}
F = & \tfrac{1}{2} c_{11}[\dot{x}_3 - a\dot{x}_5 - \bar{y}\dot{x}_4 - (d_1-\bar{y})\,\dot{x}_7]^2 \\
& + 1/2\ c_{12}[\dot{x}_3 - a\dot{x}_5 - \bar{y}\dot{x}_4 + (d_2+\bar{y})\,\dot{x}_7]^2 \\
& + 1/2\ c_{31}[\dot{x}_2 + a\dot{x}_6 - b\dot{x}_4 - h\dot{x}_7]^2 \\
& + 1/2\ c_{32}[\dot{x}_2 + a\dot{x}_6 - b\dot{x}_4 - h\dot{x}_7]^2 \\
& + 1/2\ c_{51}[\dot{x}_1 + d_1\dot{x}_6 + (\bar{z}-r)\,\dot{x}_5 - r\dot{x}_8]^2 \\
& + 1/2\ c_{52}[\dot{x}_1 - d_2\dot{x}_6 + (\bar{z}-r)\,\dot{x}_5 - r\dot{x}_9]^2 \\
& + c_{21}\dot{x}_{10}^2 + 1/2\ c_{22}\dot{x}_{11}^2 + 1/2\ c_{71}\dot{x}_{12}^2 + 1/2\ c_{72}\dot{x}_{13}^2 \\
& + 1/2\ c_{81}[\dot{x}_2 - \bar{x}\dot{x}_6 - \bar{z}\dot{x}_4]^2 + 1/2\ c_{82}[\dot{x}_2 - \bar{x}\dot{x}_6 - \bar{z}\dot{x}_4]^2 \\
& + 1/2\ c_{41}[\dot{x}_2 - \bar{x}\dot{x}_6 - \bar{z}\dot{x}_4]^2 + 1/2\ c_{42}[\dot{x}_2 - \bar{x}\dot{x}_6 - \bar{z}\dot{x}_4]^2 \\
& + 1/2\ c_{61}\dot{x}_{14}^2 + 1/2\ c_{62}\dot{x}_{15}^2 + 1/2\ c_{91}\dot{x}_{16}^2 \\
& + 1/2\ c_{92}\dot{x}_{17}^2
\end{aligned}
$$

Total potential energy of the system due to the tire equivalent springs:

$$
\begin{aligned}
V = & 1/2\ k_{11}[x_3 - ax_5 - \bar{y}x_4 - (d_1-\bar{y})\,x_7]^2 \\
& + 1/2\ k_{12}[x_3 - ax_5 - \bar{y}x_4 + (d_2+\bar{y})\,x_7]^2 \\
& + 1/2\ k_{31}[x_2 + ax_6 - bx_4 - hx_7]^2
\end{aligned}
$$

$$+ 1/2\ k_{32}[x_2 + ax_6 - bx_4 - hx_7]^2$$
$$+ 1/2\ k_{51}[x_1 + d_1x_6 + (\bar{z}-r)\ x_5 - rx_8]^2$$
$$+ 1/2\ k_{52}[x_1 - d_2x_6 + (\bar{z}-r)\ x_5 - rx_9]^2$$
$$+ 1/2\ k_{21}x_{10}^2 + 1/2\ k_{22}x_{11}^2 + 1/2\ k_{71}x_{12}^2$$
$$+ 1/2\ k_{72}x_{13}^2 + 1/2\ k_{81}[x_2 - \bar{x}x_6 - \bar{z}x_4]^2$$
$$+ 1/2\ k_{82}[x_2 - \bar{x}x_6 - \bar{z}x_4]^2$$
$$+ 1/2\ k_{41}[x_2 - \bar{x}x_6 - \bar{z}x_4]^2$$
$$+ 1/2\ k_{42}[x_2 - \bar{x}x_6 - \bar{z}x_4]^2$$
$$+ 1/2\ k_{61}x_{14}^2 + 1/2\ k_{62}x_{15}^2 + 1/2\ k_{91}x_{16}^2$$
$$+ 1/2\ k_{92}x_{17}^2$$

Potential energy due to axle equivalent spring for the dual wheel model:

$$V = 1/2\ k_1[x_3 + \bar{x}x_5 - d_3x_4 - x_{10}]^2$$
$$+ 1/2\ k_1[x_3 + \bar{x}x_5 + d_4x_4 - x_{11}]^2$$
$$+ 1/2\ k_2[x_3 + \bar{x}x_5 - d_5x_4 - x_{12}]^2$$
$$+ 1/2\ k_2[x_3 + \bar{x}x_5 + d_6x_4 - x_{13}]^2$$
$$+ 1/2\ k_1[x_1 + \bar{z}x_5 + d_3x_6 - x_{14}]^2$$
$$+ 1/2\ k_1[x_1 + \bar{z}x_5 - d_4x_6 - x_{15}]^2$$
$$+ 1/2\ k_2[x_1 + \bar{z}x_5 + d_5x_6 - x_{16}]^2$$
$$+ 1/2\ k_2[x_1 + \bar{z}x_5 - d_5x_6 - x_{17}]^2$$

Equations of Motion

The equations of motion differ considerably from one model to another. More equations are required for the dual wheel models since the degrees of freedom are increased compared to the single wheel model. In order to keep the study more general, the equations for the dual wheel model with the hub method of tire attachment were formulated first. These are:

Equations due to freedom in x_1 direction:

$$
\begin{aligned}
&[(m_1 + m_2 + m_5 + m_6)]\ddot{x}_1 + [(m_1 + m_2)(\bar{z} - r) \\
&+ (b + e)m_6]\ddot{x}_6 + [c_{51} + c_{52}]\dot{x}_1 + [(c_{51} + c_{52})(\bar{z} - r)]\dot{x}_5 \\
&+ [c_{51}d_1 - c_{52}d_2]\dot{x}_6 + [-c_{51}r]\dot{x}_8 + [-c_{52}r]\dot{x}_9 \\
&+ [k_{51} + k_{52} + 2k_1 + 2k_2]x_1 \\
&+ [(k_{51} + k_{52})(\bar{z} - r) + 2(k_1 + k_2)\bar{z}]x_5 \\
&+ [k_{51}d_1 - k_{52}d_2]x_6 + [-k_{51}r]x_8 + [-k_{52}r]x_9 + [-k_1]x_{14} \\
&+ [-k_1]x_{15} + [-k_2]x_{16} + [-k_2]x_{17} = f_1(t)
\end{aligned}
$$

Equations of motion due to freedom in x_2 direction:

$$
\begin{aligned}
&[(m_1 + m_2 + m_3 + m_4 + m_5 + m_6 + m_7 + m_8)]\ddot{x}_2 \\
&+ [-(m_1 + m_2 + m_6)b - (m_3 + m_4 + m_7 + m_8)(\bar{z} - r)]\ddot{x}_4 \\
&+ \ (m_1 + m_2 + m_6)a - (m_3 + m_4)\bar{x} - (m_7 + m_8)\bar{x}\ \ddot{x}_6 \\
&+ [-(m_1 + m_2)(h - r) - m_6 e]\ddot{x}_7 \\
&+ [c_{31} + c_{32} + c_{41} + c_{42} + c_{81} + c_{82}]\dot{x}_2 \\
&+ [-(c_{31} + c_{32})b - (c_{41} + c_{42} + c_{81} + c_{82})\bar{z}]\dot{x}_4
\end{aligned}
$$

$$+ [(c_{31} + c_{32})z - (c_{41} + c_{42} + c_{81} + c_{82})x]\dot{x}_6$$

$$+ [-(c_{31} + c_{32})h]\dot{x}_7 + [k_{31} + k_{32} + k_{41} + k_{42} + k_{81} + k_{82}]$$

$$x_2 + [-(k_{31} + k_{32})b - (k_{41} + k_{42} + k_{81} + k_{82})\bar{z}]x_4$$

$$+ [(k_{31} + k_{32})a - (k_{41} + k_{42} + k_{81} + k_{82})\bar{x}]x_6$$

$$+ [-(k_{31} + k_{32})h]x_7 = f_2(t)$$

Equations of motion due to freedom in x_3 direction:

$$[m_1 + m_2 + m_5 + m_6]\ddot{x}_3 + [-(m_1 + ,_2 + m_6)\bar{y}]\ddot{x}_4$$

$$+ [-(m_1 + m_2 + m_6)a]\ddot{x}_5 + [-m_1(d_1 - \bar{y}) + m_2(d_2 + \bar{y})]\ddot{x}_7$$

$$+ [(c_{11} + c_{12})]\dot{x}_3 + [-(c_{11} + c_{12})\bar{y}]\dot{x}_4$$

$$+ [-(c_{11} + c_{12})a]\dot{x}_5 + [-c_{11}(d_1 - \bar{y}) + c_{12}(d_2 + \bar{y})]\dot{x}_7$$

$$+ [k_{11} + k_{12} + 2k_1 + 2k_2]x_3 + [-(k_{11} + k_{12})a + 2k_1\bar{x}$$

$$+ 2k_2\bar{x}]x_5 + [-k_1\ x_{10} + [-k_1]x_{11} + [-k_2]x_{12} + [-k_2]x_{13}$$

$$= f_3(t)$$

Equations of motion due to freedom in x_4 direction:

$$[(m_1 + m_2 + m_6)(b^2 + \bar{y}^2) + (m_3 + m_4 + m_7 + m_8)(\bar{z} - R)^2$$

$$+ I_{511} + I_{311} + I_{411} + I_{711} + I_{811}]\ddot{x}_4$$

$$+ [(m_1 + m_2 + m_6)\bar{y}a - I_{512}]\ddot{x}_5$$

$$+ [-(m_1 + m_2 + m_6)b - (m_3 + m_4 + m_7 + m_8)(z - R)]\ddot{x}_2$$

$$+ [-(m_1 + m_2 + m_6)\bar{y}]\ddot{x}_3$$

$$+ [-(m_1 + m_2 + m_6)ba + (m_3 + m_4 + m_7 + m_8)(\bar{z} - R) - I_{513}]\ddot{x}_6$$

$$+ [m_1\{(h-r)b + (d_1-\bar{y})\bar{y}\} + m_2\{(h-r)b - (d_2+\bar{y})\bar{y}\} + m_6eb]\ddot{x}_7$$

$$+ [-(c_{31} + c_{32})b - (c_{41} + c_{42} + c_{81} + c_{82})\bar{z}]\dot{x}_2$$

$$+ [-(c_{11} + c_{12})\bar{y}]\dot{x}_3 + [(c_{11} + c_{12})\bar{y}^2 + (c_{31} + c_{32})b^2$$

$$+ (c_{41} + c_{42} + c_{81} + c_{82})\bar{z}^2]\dot{x}_4 + [(c_{11} + c_{12})\bar{y}a]\dot{x}_5$$

$$+ [-(c_{31} + c_{32})ab + (c_{41} + c_{42} + c_{81} + c_{82})\bar{x}\bar{z}]\dot{x}_6$$

$$+ [c_{11}\bar{y}(d_1 - \bar{y}) - c_{12}\bar{y}(d_2 + \bar{y}) + (c_{31} + c_{32})bh]\dot{x}_7$$

$$+ [-(k_{11} + k_{12})\bar{y}]x_3 + [(k_{11} + k_{12})\bar{x}^2 + (k_{31} + k_{32})b^2$$

$$+ (k_{41} + k_{42} + k_{81} + k_{82})\bar{z}^2 + k_1 d_3{}^2 + k_1 d_4{}^2 + k_2 d_5^2 + k_2 d_6{}^2]x_4$$

$$+ [(k_{11} + k_{12})\bar{y}a]x_5 + [-(k_{31} + k_{32})ab + (k_{41} + k_{42} + k_{81}$$

$$+ k_{82})\bar{x}\bar{z}]x_6 + [k_{11}\bar{x}(d_1 - \bar{y}) - k_{12}\bar{y}(d_2 + \bar{y}) + (k_{31}$$

$$+ k_{32})bh]x_7 + [d_3 k_1]x_{10} + [-d_4 k_1]x_{11} + [d_5 k_2]x_{12}$$

$$+ [-d_6 k_2]x_{13} = f_4(t)$$

Equations of motion due to freedom in x_5 direction:

$$[(m_1 + m_2)(\bar{z} - r) + m_6 be]\ddot{x}_1 + [-(m_1 + m_2)a - m_6 a]\ddot{x}_3$$

$$+ [(m_1 + m_2 + m_6)\bar{y}a - I_{512}]\ddot{x}_4 + [(m_2 + m_2)\{(\bar{z}-r)^2 + a^2\}$$

$$+ I_{322} + I_{422} + I_{622} + m_6\{(b+e)^2 + a^2\} + I_{522} + I_{722}$$

$$+ I_{822}]\ddot{x}_5 + [m_1 d_1(\bar{z} - r) - m_2 d_2(\bar{z} - r) - I_{523}$$

$$+ m_6\bar{y}(b+e)]\ddot{x}_6 + [m_1 a(d_1 - \bar{y}) - m_2 a(d_2 + \bar{y})]\ddot{x}_7$$

$$+ [(c_{51} + c_{52})(\bar{z} - r)]\dot{x}_1 + [-(c_{11} + c_{12})a]\dot{x}_3$$

$$+ [(c_{11} + c_{12})\bar{y}a]\dot{x}_4 + [(c_{11} + c_{12})a^2 + (c_{51} + c_{52})$$

$$(\bar{z} - r)^2]\dot{x}_5 + [c_{11}a(d_1 - \bar{y}) - c_{12}a(d_2 + \bar{y})]\dot{x}_7$$

$$+ [-c_{51}r(\bar{z} - r)]\dot{x}_8 + [-c_{52}(\bar{z} - r)r]\dot{x}_9$$

$$+ [(k_{51} + k_{52})(\bar{z} - r) + 2(k_1 + k_2)\bar{z}]x_1$$

$$+ [-(k_{11} + k_{12})a + 2k_1\bar{x} + 2k_2\bar{x}]x_3 + [(k_{11} + k_{12})\bar{y}a]x_4$$

$$+ [(k_{11} + k_{12})a^2 + (k_{51} + k_{52})(\bar{z} - r)^2$$

$$+ 2(k_1 + k_2)(\bar{x}^2 + \bar{z}^2)]x_5 + [k_{11}a(d_1 - \bar{y}) - k_{12}a(d_2 + \bar{y})]x_7$$

$$+ [-k_{51}r(\bar{z} - r)]x^8 + [-k_{52}(\bar{z} - r)r]x_9 + [-k_1\bar{x}]x_{10}$$

$$+ [-k_1\bar{x}]x_{11} + [-k_2\bar{x}]x_{12} + [-k_2\bar{x}]x_{13} + [-k_1\bar{z}]x_{14}$$

$$+ [-k_1\bar{z}]x_{15} + [-k_2\bar{z}]x_{16} + [-k_2\bar{z}]x_{17} = f_5(t)$$

Equations of motion in x_6 direction:

$$(m_1d_1 - m_2d_2)\ddot{x}_1 + [(m_1 + m_2)a - (m_3 + m_4)\bar{x} - (m_7 + m_8)\bar{x}$$

$$+ m_6a]\ddot{x}_2 + [-(m_1 + m_2)ab + (m_3 + m_4 + m_7 + m_8)(\bar{z} - R)\bar{x}$$

$$- m_6ab - I_{513}]\ddot{x}_4 + [m_1d_1(\bar{z} - r) - m_2d_2(\bar{z} - r) - I_{523}$$

$$+ m_6\bar{y}(b + e)]\ddot{x}_5 + [(m_1 + m_2)(d_1{}^2 + a^2)$$

$$+ (m_3 + m_4 + m_7 + m_8)\bar{x}^2 + m_6(\bar{y}^2 + a^2) + I_{111} + I_{211}$$

$$+ I_{311} + I_{411} + I_{533} + I_{633} + I_{711} + I_{811}]\ddot{x}_6$$

$$+ [-(m_1 + m_2)(h - r)a - m_6ae]\ddot{x}_7 + [c_{51}d_1 - c_{52}d_2]\dot{x}_1$$

$$+ [(c_{31} + c_{32})a - (c_{41} + c_{42} + c_{81} + c_{82})\bar{x}]\dot{x}_2$$

$$+ [-(c_{31} + c_{32})ab - (c_{41} + c_{42})\bar{z} - (c_{81} + c_{82})\bar{x}\bar{z}]\dot{x}_4$$

$$+ [c_{51}d_1(\bar{z} - r) - c_{52}d_2(\bar{z} - r)]\dot{x}_5$$

$$+ [(c_{31} + c_{32})a^2 + (c_{41} + c_{42})\bar{x}^2 + c_{51}d_1{}^2 + c_{52}d_2{}^2$$

$$+ (c_{81} + c_{82})\bar{x}^2]\dot{x}_6 + [-(c_{31} + c_{32})ah]\dot{x}_7 + [-c_{51}d_1r]\dot{x}_8$$

$$+ [c_{52}d_2r]\dot{x}_9 + [k_{51}d_1 - k_{52}d_2]x_1 + [(k_{31} + k_{32}a$$

$$- (k_{41} + k_{42} + k_{81} + k_{82})\bar{x}]x_2 + [-(k_{31} + k_{32})ab$$
$$- (k_{41} + k_{42})\bar{x}\bar{z} - (k_{81} + k_{82})\bar{x}\bar{z}]x_4 + [k_{51}d_1(\bar{z} - r)$$
$$- c_{52}d_2(\bar{z} - r)]x_5 + [(k_{31} + k_{32})a^2 + (k_{41} + k_{42})\bar{x}^2$$
$$+ k_{51}d_1{}^2 + k_{52}d_2{}^2 + (k_{81} + k_{82})\bar{x}^2 + k_1d_3{}^2 + k_1d_4{}^2$$
$$+ k_2d_5{}^2 + k_2d_6^2]x_6 + [-(k_{31} + k_{32})ah]x_7 + [-k_{51}d_1r]x_8$$
$$+ [k_{52}d_2r]x_9 + [-d_3k_1]x_{14} + [d_4k_1]x_{15} + [-k_2d_5]x_{16}$$
$$+ [d_6k_2]x_{17} = f_6(t)$$

Equations of motion due to freedom in the x_7 direction:

$$[-(m_1 + m_2)(h - r) - m_6e]\ddot{x}_2 + [-m_1(d_1 - \bar{y}) + m_2(d_2 + \bar{y})]\ddot{x}_3$$
$$+ [m_1\{(h-r)b + (d_1-\bar{y})\bar{y}\} + m_2\{(h-r)b - (d_2+\bar{y})\} + m_6eb]\ddot{x}_4$$
$$+ [m_1a(d_1 - \bar{y}) - m_2a(d_2 + \bar{y})]\ddot{x}_5$$
$$+ [-(m_1 + m_2)(h - r)a - m_6ea]\ddot{x}_6$$
$$+ [m_1\{(h-r)^2 + (d_1-\bar{y})^2\} + m_2\{(h-r)^2 + (d_2+\bar{y})^2\} + m_6e^2$$
$$+ I_{111} + I_{211} + I_{611}]\ddot{x}_7 + [-(c_{31} + c_{32})h]\dot{x}_2$$
$$+ [-c_{11}(d_1 - \bar{y}) + c_{12}(d_2 + \bar{y})]\dot{x}_3$$
$$+ [c_{11}\bar{y}(d_1 - \bar{y}) - c_{12}\bar{y}(d_2 + \bar{y}) + (c_{31} + c_{32})bh]\dot{x}_4$$
$$+ [c_{11}a(d_1 - \bar{y}) - c_{12}a(d_2 + \bar{y})]\dot{x}_5$$
$$+ [-(c_{31} + c_{32})ah]\dot{x}_6 + [c_{11}(d_1 - \bar{y})^2 + c_{12}(d_2 + \bar{y})^2$$
$$+ (c_{31} + c_{32})h^2]\dot{x}_7 + [-(k_{31} + k_{32})h]x_2$$
$$+ [- k_{11}(d_1 - \bar{y}) + k_{12}(d_2 + \bar{y})]x_3$$
$$+ [k_{11}\bar{y}(d_1 - \bar{y}) - k_{12}\bar{y}(d_2 + \bar{y}) + (k_{31} + k_{32})bh]x_4$$

$$+ [k_{11}a(d_1 - \bar{y}) - k_{12}a(d_2 + \bar{y})]x_5$$

$$+ [-(k_{31} + k_{32})ah]x_6 + [k_{11}(d_1 - \bar{y})^2 + k_{12}(d_2 + \bar{y})^2$$

$$+ (k_{31} + k_{32})h^2]x_7 = f_7(t)$$

Equations of motion due to freedom in the x_8 direction:

$$[I_{122}]\ddot{x}_8 + [-c_{51}r]\dot{x}_1 + [-c_{51}r(\bar{z} - r)]\dot{x}_5 + [-c_{51}rd_1]\dot{x}_6$$

$$+ [(c_{51}r^2]\dot{x}_8 + [-k_5r]x_1 + [-k_{51}r(\bar{z} - r)]x_5$$

$$+ [-k_{51}rd_1]x_6 + [k_{51}r^2]x_8 = f_8(t)$$

Equations of motion due to freedom in the x_9 direction:

$$[I_{222}]\ddot{x}_9 + [-c_{52}r]\dot{x}_1 + [-c_{52}r(\bar{z} - r)]\dot{x}_5 + [c_{52}rd_2]\dot{x}_6$$

$$+ [c_{52}r^2]\dot{x}_9 + [-k_{52}r]x_1 + [-k_{52}r(\bar{z} - r)]x_5$$

$$+ [k_{52}rd_2]x_6 + [k_{52}r^2]x_9 = f_9(t)$$

Equations of motion due to freedom in the x_{10} direction:

$$m_3\ddot{x}_{10} + c_{21}\dot{x}_{10} + [k_{21} + k_1]x_{10} + [-k_1]x_3 + [-k_1\bar{x}]x_5$$

$$+ [d_3k_1]x_4 = f_{10}(t)$$

Equations of motion due to freedom in the x_{11} direction:

$$m_4\ddot{x}_{11} + c_{22}\dot{x}_{11} + [k_{22} + k_1]x_{11} + [-k_1]x_3 + [-k_1\bar{x}]x_5$$

$$+ [-d_4k_1]x_4 = f_{11}(t)$$

Equations of motion due to freedom in the x_{12} direction:

$$m_7\ddot{x}_{12} + c_{71}\dot{x}_{12} + [k_{71} + k_2]x_{12} + [-k_2]x_3 + [-k_2\bar{x}]x_5$$

$$+ [d_5k_2]x_4 = f_{12}(t)$$

Equations of motion due to freedom in the x_{13} direction:

$$m_8\ddot{x}_{13} + c_{72}\dot{x}_{13} + [k_{72} + k_2]x_{13} + [-k_2]x_3 + [-k_2\bar{x}]x_5 + [-d_6k_2]x_4 = f_{13}(t)$$

Equations of motion due to freedom in the x_{14} direction:

$$m_3\ddot{x}_{14} + c_{61}\dot{x}_{14} + [k_{61} + k_1]x_{14} + [-k_1]x_1 + [-k_1\bar{z}]x_5 + [-d_3k_1]x_6 = f_{14}(t)$$

Equation of motion due to freedom in the x_{15} direction:

$$m_4\ddot{x}_{15} + c_{62}\dot{x}_{15} + [k_{62} + k_1]x_{15} + [-k_1]x_1 + [-k_1\bar{z}]x_5 + [d_4k_1]x_6 = f_{15}(t)$$

Equations of motion due to freedom in the x_{16} direction:

$$m_7\ddot{x}_{16} + c_{91}\dot{x}_{16} + [k_{91} + k_2]x_{16} + [-k_2]x_1 + [-k_2\bar{z}]x_5 + [-d_5k_2]x_6 = f_{16}(t)$$

Equations of motion due to freedom in the x_{17} direction:

$$m_8\ddot{x}_{17} + c_{92}\dot{x}_{17} + [k_{92} + k_2]x_{17} + [-k_2]x_1 + [-k_2\bar{z}]x_5 + [d_6k_2]x_6 = f_{17}(t)$$

Forcing of Model

Forcing of model was made by utilizing a half sine period bump, which is similar to bumps encountered under field conditions (0.416 ft high and 3 ft in length). All models solved in this study used this prescribed bump.

The forcing function for each equation was evaluated by the principle of virtual work. The virtual displacements (δ_x, δ_o, δ_q, etc.) are an infintesimal change in the coordinates which may be conceived irrespective of time, and which must be compatible with the constraints in the system. The principle of virtual work states that if a system is in equilibrium, the work done by the applied forces in a virtual displacement is zero.

To complete the development of equations of motion through use of Lagrange's method, the work done by the applied forces in the virtual displacement is written as:

$$\delta_w = \sum_i f_i \cdot \delta_{ri} = \sum_i f_i \cdot \sum_{k=1}^{n} \frac{\partial_{ri}}{\partial_{qk}} q_k$$

$$= \sum_{k=1}^{n} \left(\sum_i f_i \cdot \frac{\partial_{ri}}{\partial_{qk}}\right) \delta_{qk}$$

$$= \sum_{k=1}^{n} q_k \, \delta_{qk}$$

where

$$q_k = \sum_i f_i \cdot \frac{\partial_{ri}}{\partial_{qk}}$$

is called the generalized force associated with the coordinate qk.

In the system under study, this function was obtained as follows:

$$f_i = ku + c\dot{u}$$

Where k and c are the spring and damping rates of the rear tires. This force acts in the direction of the coordinates which represent rear wheel motion. For the dual wheel model where the four rear wheels have vertical displacements x_{10}, x_{11}, x_{12}, and x_{13}, respectively, the function is:

$$qk = \sum_i (ku + c\dot{u}) \frac{\partial(x_{10} + x_{11} + x_{12} + x_{13})}{\partial xk}$$

where k = 1,..., n (number of degrees of freedom).

Differentiating the last term of the equation,

$$q_1 = \sum_i (ku + c\dot{u}) \frac{\partial(x_{10} + x_{11} + x_{12} + x_{13})}{\partial x_1} = 0$$

$$q_2 = \sum_i (ku + c\dot{u}) \frac{\partial(x_{10} + x_{11} + x_{12} + x_{13})}{\partial x_2} = 0$$

results in all functions becoming equal to zero except q_{10}, q_{11}, q_{12}, and q_{13}. Therefore,

$$q_{10} = ku + c\dot{u}$$

$$q_{11} = ku + c\dot{u}$$

$$q_{12} = ku + c\dot{u}$$

$$q_{13} = ku + c\dot{u}$$

where k and c are the spring and damping rates of the tire which contacts the bump.

The function of the bump is:

$$u = h \sin wt$$

where

h = height of the bump

$w = \frac{\pi x s}{L}$

t = time interval

s = speed of the vehicle

L = length of the bump

Therefore,

$$f_{10(t)} = q_{10} = k_{10}\ h \sin wt + wc_{10}h \cos wt$$

$$f_{11(t)} = q_{11} = k_{11}\ h \sin wt + wc_{11}h \cos wt$$

$$f_{12(t)} = q_{12} = k_{12}\ h \sin wt + c_{12}hw \cos wt$$

$$f_{13(t)} = q_{13} = k_{13}\ h \sin wt + c_{13}hw \cos wt$$

This is the general form of the forcing functions, but their direction and magnitude will depend upon tire configuration and the manner in which the bump is contacted. These forcing functions were evaluated and are included in the section on computor solutions and discussion.

A physical representation of the forcing function is shown in Figures 9 and 10 for rolling and pitching modes, respectively.

Solution of Model

The system was described by means of 17 second order differential equations. Constants involved in the equations were vehicle parameter such as: mass, moment of inertia, spring and damping coefficients of tire, and various tractor dimensions. The parameters used in this study were from an IHC-340 utility tractor.

The equations were solved through simultaneous integration on an IBM 360/50 digital computer. A numerical technique of solving differential equations called the Runge-Kutta method was used. In this method, the solution is initiated by placing initial values of all the known variables into the given differential equations. The principal advantage of this method is that it is self starting which means that only the functional value

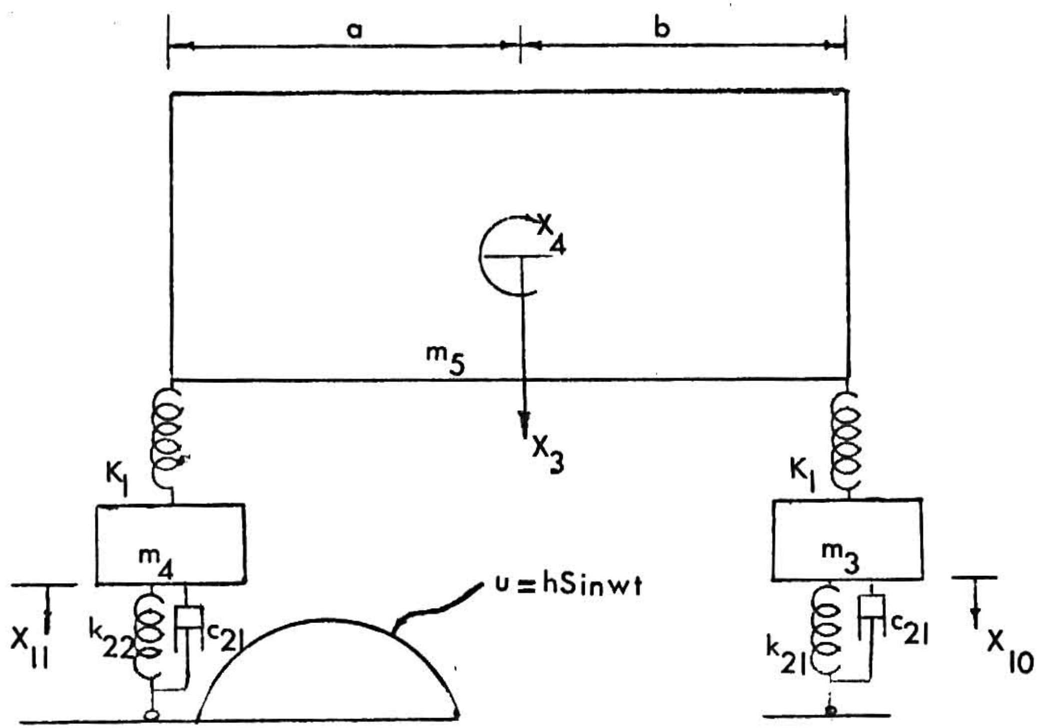

Figure 9. Forcing of model (single wheel rear view).

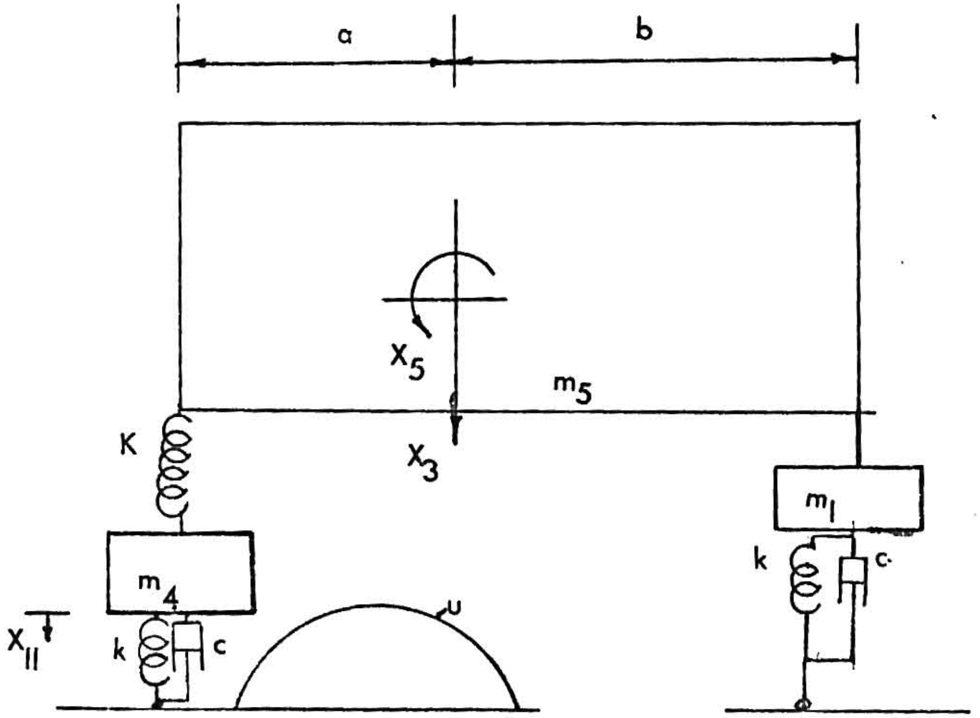

Figure 10. Forcing of model (single wheel side view).

at a single initial point is required to start the solution. The detailed process of this method is given in Appendix A.

To utilize the Runge-Kutta method of solutions, the system of equations must be in the form of

$$\dot{X} = A_X + F$$

where

$\dot{X}$ = Column matrix of $\dot{X}_i$

X = Column matrix of X_i

A = Square matrix of all the coefficients of X_i

F = Forcing function in form of column matrix

To obtain this form, the following procedure was used:

Since the equations of motion were written in the form of

$$M\ddot{x} + C\dot{x} + Kx = f_i(t) \qquad (1)$$

where

M = Mass matrix represented by matrix AA in this study

C = Damping matrix represented by matrix BB

K = Spring matrix represented by matrix DD

f = Forcing matrix represented by F matrix

Therefore, the complete system was as follows:

$$\begin{bmatrix} AA(1,1) \ldots\ldots AA(1,13) \\ \vdots \qquad \vdots \\ AA(13,1) \ldots AA(13,13) \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \vdots \\ \ddot{x}_{17} \end{bmatrix} + \begin{bmatrix} BB(1,1) \ldots\ldots BB(1,13) \\ \vdots \qquad \vdots \\ BB(13,1) \ldots BB(13,13) \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{17} \end{bmatrix}$$

$$+ \begin{bmatrix} DD(1,1) \ldots\ldots DD(1,17) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ DD(17,1) \ldots\ldots DD(17,17) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_{17} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ f_{17} \end{bmatrix}$$

$$AA\ \ddot{x} + BB\ \dot{x} + DD\ x = F(t) \qquad (2)$$

or

$$AA\ \ddot{x} = -BB\ \dot{x} - DD\ x + F(t) \qquad (3)$$

By premultiplying all the matrices by the inverse of mass matrix AA

$$AA^{-1} * AA\ \ddot{x} = -AA^{-1} * BB\ \dot{x} - AA^{-1} * DD\ x + AA^{-1} * F(+)$$

or

$$\ddot{x}_i = +R\ \dot{x}_i + R1x_i + F1_i(t) \qquad (4))$$

where

$R \quad = -AA^{-1} * BB$

$R1 \quad = -AA^{-1} * DD$

$F1(t) = +AA^{-1} * F(t)$

$i \quad = 1, 2, 3, \ldots, 17$

The computer program for matrix inversion and multiplication is given in Appendix B. The 17 second order differential equations were transformed into 34 first order differential equations by letting

$\dot{x}_1 = x_{18}$

$\dot{x}_2 = x_{19}$

$\dot{x}_3 = x_{20}$

$\dot{x}_4 = x_{21}$

$$\dot{x}_5 = x_{22}$$
$$\dot{x}_6 = x_{23}$$
$$\dot{x}_7 = x_{24}$$
$$\dot{x}_8 = x_{25}$$
$$\dot{x}_9 = x_{26}$$
$$\dot{x}_{10} = x_{27}$$
$$\dot{x}_{11} = x_{28}$$
$$\dot{x}_{12} = x_{29}$$
$$\dot{x}_{13} = x_{30}$$
$$\dot{x}_{14} = x_{31}$$
$$\dot{x}_{15} = x_{32}$$
$$\dot{x}_{16} = x_{33}$$
$$\dot{x}_{17} = x_{34}$$

Therefore

$$\ddot{x}_1 = \dot{x}_{18}$$
$$\ddot{x}_2 = \dot{x}_{19}$$
$$\ddot{x}_3 = \dot{x}_{20}$$
$$\ddot{x}_4 = \dot{x}_{21}$$
$$\ddot{x}_5 = \dot{x}_{22}$$
$$\ddot{x}_6 = \dot{x}_{23}$$
$$\ddot{x}_7 = \dot{x}_{24}$$
$$\ddot{x}_8 = \dot{x}_{25}$$
$$\ddot{x}_9 = \dot{x}_{26}$$
$$\ddot{x}_{10} = \dot{x}_{27}$$
$$\ddot{x}_{11} = \dot{x}_{28}$$
$$\ddot{x}_{12} = \dot{x}_{29}$$

$$\ddot{x}_{13} = \dot{x}_{30}$$
$$\ddot{x}_{14} = \dot{x}_{31}$$
$$\ddot{x}_{15} = \dot{x}_{32}$$
$$\ddot{x}_{16} = \dot{x}_{33}$$
$$\ddot{x}_{17} = \dot{x}_{34}$$

All these values were substituted into the equation whereby a complete system of first order differential equations was obtained:

$$\dot{x}_i = R\,x_j + R1x_i + F1_i(t)$$

where

$i = 14, 15, 16, \ldots, 34$

$j = 1, 2, 3, 4, 5, \ldots, 17$

These first order differential equations were programmed into the function subroutine of the main computer program included in Appendix B. The RKGs subroutine was used along with the following parameters:

Parameter (1) = 0.00 [Initial time interval]

Parameter (2) = 2.00 [Final time interval]

Parameter (3) = .005 [Time increment]

Parameter (4) = .00001 [Accuracy limit]

The complete program was written in FORTRAN IV language for an IBM 360/50 computer. A sample of the computer solution of the differential equations, along with axle deflections and transient stresses has been included in Appendix C.

Axle Stress Calculations

Dynamic Stress

Axle stresses were calculated by determining axle deflection in the horizontal and vertical directions with respect to axle axis (undeflected). Axle deflection in the horizontal and vertical directions was computed from the following equations:

$$\Delta_1 = x_{10} + \bar{x}\, x_5 - d_3\, x_4 - x_3 \quad \text{(for a single-wheel model)}$$

$$\Delta_2 = x_{12} + \bar{x}\, x_5 - d_3\, x_6 - x_1$$

where

Δ_1 = Vertical deflection of axle

Δ_2 = Horizontal deflection of axle

x_{10}= Vertical displacement wheel mass

x_{12}= Horizontal displacement wheel mass

x_5 = Pitch motion of vehicle

x_4 = Roll motion of vehicle

x_6 = Yaw motion of vehicle

x_1 = Fore and aft motion of tractor body

x_3 = Bouncing motion of tractor body

These horizontal and vertical deflections were used in calculating the force, bending moment, and stress developed. The dynamic force acting at the end of the axle was

$$F_1 = \Delta_1 * K_1$$

$$F_2 = \Delta_2 * K_1$$

where K_1 is equivalent spring constant of the axle due to its material properties and its area moment of inertia, and

$K_1 = \frac{3EI}{l_1^3}$ for a cantilever beam with force applied at the end

E = Modulus of elasticity = 30×10^6 psi

I = Moment of inertia = $\frac{\pi}{64} \cdot d_1^4$ in^4

L_1 = Length of stub axle = 15 in.

$$K_1 = 3 \times 30 \times 10^6 \times \frac{\pi}{64} \times \frac{3^4}{15 \times 15 \times 15} \times 12 \text{ lb/ft} = 10^6 \text{ lb/ft}$$

$K_2 = \frac{3EI}{l_1^3 + l_2^3}$ (for parallel springs)

$$= \frac{3 \times 30 \times 10^6 \times \pi/64 \times 3^4}{15^3 + 8^3} \times 12 \text{ lb/ft} = .99 \times 10^6 \text{ lb/ft}$$

$$M_R = \sqrt{M_H^2 + M_V^2}$$

$$S = \frac{M_R \times C}{I}$$

where S is equal to stress due to bending.

Using these relationships, maximum bending stress versus time, relationships were obtained for the various models.

Static Stresses

Static stresses should also be considered in the design of axles. In the static situation, the stresses due to horizontal forces become zero and only vertical stresses are taken into consideration. The reaction at the rear wheel was calculated as follows:

Let R_1 be the reaction at the front end

R_2 be the reaction at the rear end

By referring to Figure 3 and taking moments about point A

$$R_2 = \frac{Wa}{a + \bar{x}}$$

$$R_2 = \frac{mga}{a + \bar{x}}$$

m = mass of chassis + rear wheels + front wheels + axle

= 1.83 + 1.83 + 11.33 + 11.33 + 128 + 5.10

= 159.42 slugs

$$R_2 = \frac{159.42 \times 32.2 \times 3.67}{3.67 + 2.33} = 3139.88 \text{ lbs}$$

$R_2/2$ = 1569.94159 [for the single wheel model]

$$M = \frac{R_2}{2} \times L_1$$

$$S = \frac{R_2}{2} \times L_1 \times \frac{d}{2} \times \frac{1}{\pi/64} \times d^4$$

$$= \frac{3139.88}{2} \times 15.0 \times \frac{3}{2} \times \frac{64}{\pi \times 3^4} = 8884.05 \text{ psi}$$

In the case of the dual wheel model, the weight of the complete vehicle increases due to the two additional rear wheels.

Thus,

= 1.83 + 1.83 + 2 x 11.33 + 128 + 5.10 = 182.08 slugs

$$R_2 = \frac{mga}{a + \bar{x}} = \frac{m \times 32.2 \times 3.67}{3.67 + 2.33} = \frac{182.08 \times 32.2 \times 3.67}{6.00}$$

= 3586.186 lbs.

$M_v = R_2/2 \times L_1$ [for cantilever beam]

$$= \frac{3586.186}{2} \times 15 = 26896.395 \text{ lb/in}$$

$$S_v = \frac{M_v \times c}{I} = \frac{26896.395 \times 3 \times 64}{2 \times \pi \times 81} = 10{,}146.83930 \text{ psi}$$

where S_v is equal to stress due to static vertical loading in the case of dual wheels.

These static stresses should be added to respective dynamic stresses to obtain the maximum total stress.

Dynamic stress for a hub-attached dual wheel tractor axle was computed as follows:

$$\Delta_1 = x_{10} + \bar{x}\, x_5 - d_3 x_4 - x_3$$

$$\Delta_2 = x_{12} + \bar{x}\, x_5 - d_5 x_4 - x_3$$

$$\Delta_3 = x_{14} + \bar{x}\, x_5 - d_3 x_6 - x_1$$

$$\Delta_4 = x_{16} + \bar{x}\, x_5 - d_5 x_6 - x_1$$

Dynamic vertical bending moment

$$M_V = k_1 \Delta_1 \times L_1 + k_2 \Delta_2 L_2$$

Dynamic horizontal bending moment

$$M_H = k_1 \times \Delta_3 L_1 + k_2 \Delta_4 \times L_2$$

and

$$M_R = \sqrt{M_V{}^2 + M_H{}^2}$$

The resultant bending stress

$$S_R = \frac{M_R \times c}{I}$$

where L_1 = 15.0"

L_2 = 24.0"

$k_1 = 10 \times 10^6$ lb/ft

$k_2 = .99 \times 10^6$ lb/ft

Δ_1 and Δ_2 = vertical deflection of axle at the points where the inner and outer dual wheels are attached respectively.

Δ_3 and Δ_4 = horizontal deflection of axle at the points where the inner and outer dual wheels are attached respectively.

TABLE IV

LIST OF STANDARD CONDITIONS

Physical dimensions (ft)

$\bar{x} = 2.33$	$a = 3.67$	$R = 2.00$
$\bar{y} = 0.00$	$b = 1.00$	$d_1 = 2.21$
$\bar{z} = 2.67$	$e = 0.00$	$d_2 = 2.21$
$L_1 = 1.25$	$h = 1.58$	$d_3 = 2.83$
$L_2 = 0.75$	$r = 1.12$	$d_4 = 2.83$
		$d_5 = 4.33$
		$d_6 = 4.33$

Mass (lb - sec^2/ft)

$m_1 = 1.83$	$m_4 = 11.33$	$m_7 = 11.33$
$m_2 = 1.83$	$m_5 = 128$	$m_8 = 11.33$
$m_3 = 11.33$	$m_6 = 5.1$	

Inertia tensor components (lb - ft - sec^2)

$I_{111} = 0.57$	$I_{122} = 1.14$	$I_{133} = 0.57$
$I_{222} = 0.57$	$I_{222} = 1.14$	$I_{233} = 0.57$
$I_{311} = 11.33$	$I_{322} = 22.66$	$I_{333} = 11.33$
$I_{411} = 11.33$	$I_{422} = 22.66$	$I_{433} = 11.33$
$I_{511} = 375$	$I_{522} = 900$	$I_{533} = 1050$
$I_{13} = 200$	$I_{531} = 200$	
$I_{611} = 9.0$	$I_{622} = 2.0$	$I_{633} = 10.0$
$I_{711} = 11.33$	$I_{722} = 22.66$	$I_{733} = 11.33$
$I_{811} = 11.33$	$I_{822} = 22.66$	$I_{833} = 11.33$

(all other components = 0)

Damping coefficients (lb - sec/ft) and spring rates (lb/ft)

c_{11} = 186	c_{12} = 186	k_{11} = 22,600	k_{12} = 22,600
c_{21} = 248	c_{22} = 248	k_{21} = 20,500	k_{22} = 20,500
c_{31} = 25	c_{32} = 225	k_{31} = 10,700	k_{32} = 10,700
c_{41} = 32	c_{42} = 32	k_{41} = 11,900	k_{42} = 11,900
c_{51} = 88	c_{52} = 88	k_{51} = 16,000	k_{52} = 16,000
c_{61} = 134	c_{62} = 134	k_{61} = 18,000	k_{62} = 18,000
c_{71} = 25	c_{72} = 25	k_{71} = 10,700	k_{72} = 10,700
c_{81} = 32	c_{82} = 32	k_{81} = 11,900	k_{82} = 11,900
c_{91} = 134	c_{92} = 134	k_{91} = 18,000	k_{92} = 18,000

Stub axle

$$k_1(E_q) = \frac{3EI}{l_1^3} \times 12 = 10^6 \text{ lb/ft}$$

$$k_2(E_q) = 0.99 \times 10^6 \text{ lb/ft}$$

L_1 = 15 in, L_2 = 7.5 in, E = 30 x 10^6 lb/in

Forward speed = 4.4 ft/sec (3 mph)

Bump size

Height H = 0.416 ft (5in)

Length L = 3.00 ft

For the International Harvester 340 utility tractor.

TABLE V

REAR WHEEL CONFIGURATIONS STUDIED

Case No.	Abbreviation	Description of Wheel Configuration
I	SW	Single wheel model. Left rear wheel traverses the standard forcing function.*
II	DWH-I	Dual wheel model with individual hub method of wheel attachment. Each of the left rear wheels traverses the standard forcing function* simultaneously.
III	DWH-II	Dual wheel model with hub method of dual wheel attachment. Only the left outer rear wheel traverses the standard forcing function* and the corresponding inner wheel remains out of contact with the bump until the outer wheel encounters smooth surface.
IV	DWR-I	Dual wheel model with rim-to-rim rigid method of wheel attachment. Each of the left rear wheels traverses the standard forcing function* simultaneously.
V	DWR-II	Dual wheel model with rim-to-rim rigid method of wheel attachment. Only outer left rear wheel traverses over the standard forcing function* and corresponding inner wheel remains out of contact with the bump until outer wheel encounters smooth surface.

*The standard forcing function is

$U = h \sin wt$

$W = \pi S/L$

h = height of bump

This function should be multiplied by the corresponding tire spring and damping rates to obtain the total force acting on each wheel.

RESULTS AND DISCUSSION OF COMPUTER SOLUTION

The tractor mathematical model as defined in earlier chapters consists of a set of simultaneous differential equations. This could be considered as a continuous system in which time is the independent variable and motions in all the coordinates are the dependent variables. The standard conditions listed in Table IV, and the standard sinusoidal forcing function previously described in the section on Forcing of Model, were used to find the solution for the system of equations.

A digital solution of different models listed under Table V was obtained for a time interval of 0 to 2.0 seconds, a time increment of 0.005 seconds was used to obtain the required accuracy. A sample of computer output for the generalized case of the dual wheel model with the hub method of attachment has been given in Appendix C.

Different cases of rear wheel contact with the bump were studied in order to determine the conditions which cause the highest stress level due to bending moment. The solution of the model described in Case I (SW) was obtained and is illustrated in Figure 11. The discussion that follows explains how the axle stress fluctuates if the tractor encounters the bump for Case I (SW). This concept will be valid for all the models, but results are different for each case due to changes in values of stiffness, damping, mass, and wheel configuration used in each system.

The tire contacts the bump and deflects in a negative direction until 0.0625 second. This compression of the tire causes

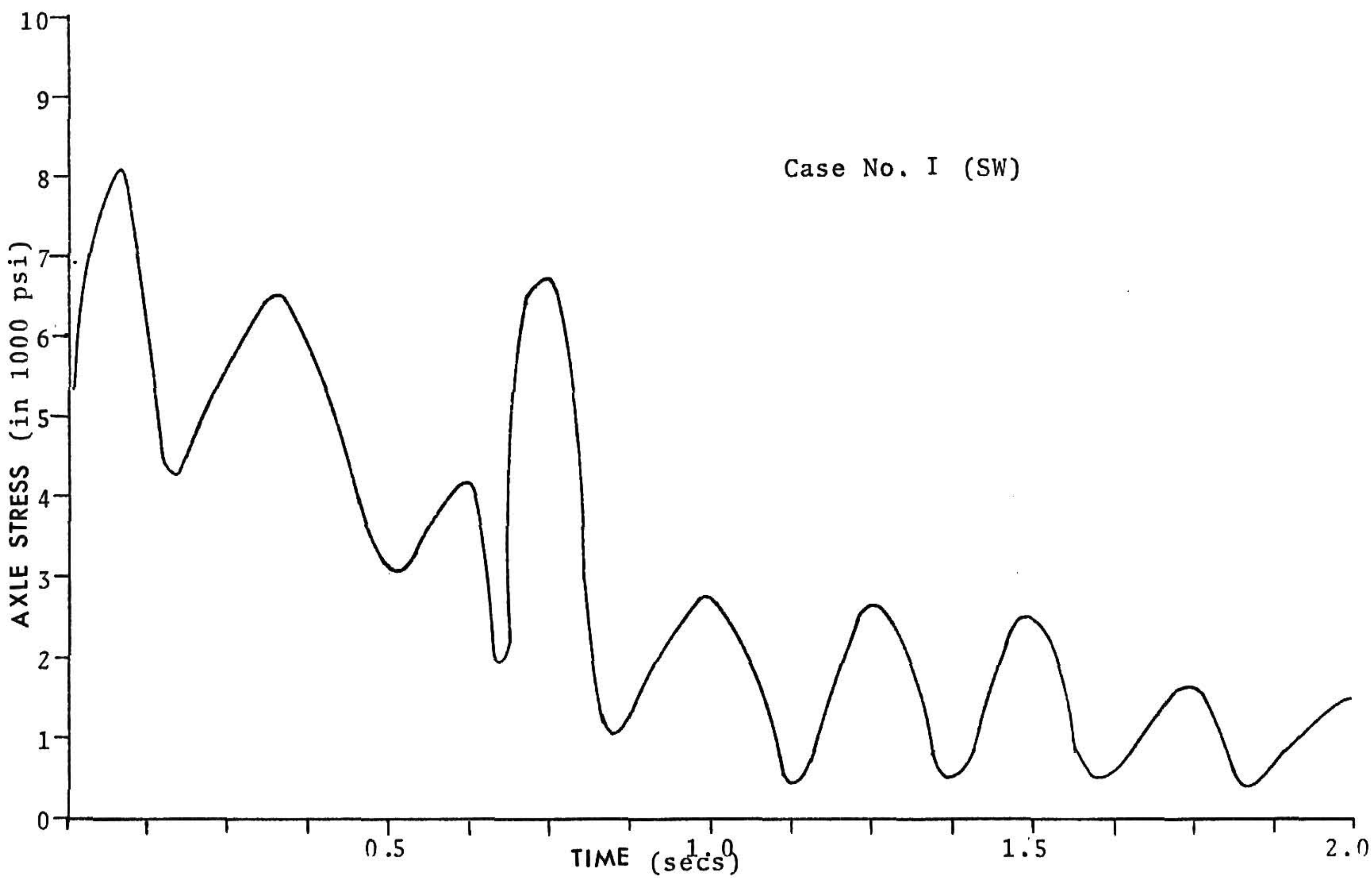

Figure 11. Rear axle stress variation for single wheel model (while left rear wheel traverses the bump).

axle deflection since axle deflection is a function of tire deflection, tractor bounce, and rigid body rotation of chassis about longitudinal and transverse axis. At 0.0625, the stress level in the axle was 8,118.00 psi (above static). After this time, the deflection decreased. The point of zero deflection (neglecting static deflection) occurred at 0.175 second. The bending resultant stress in the axle at this time (4386.0 psi) was due to horizontal forces.

Deflection in a positive direction continued to increase until the wheel reached the top of the bump. At this time, the horizontal force was negligible since the wheel was at the peak of the bump. The resultant stress at 0.335 second (when the wheel was at the top of the bump) was 6,698.00 psi. The tire was then forced back into the bump and the axle was deflected (toward the negative direction) again before the tractor reached the smooth surface. Once again the axle intersected the time of zero dynamic deflection at 0.5 second, and the axle deflection continued in the negative direction until the bump ceased. It caused the axle stress to increase to 4,570 psi at 0.60 second. A sharp increase in stress occurred at 0.725 second due to the abrupt change in terrain profile where the bump ends. This model had a stress value of 6,666.0 psi at 0.725 second. The transient stress due to axle vibration decayed within a period of 2.00 seconds.

Figure 12 illustrates the rear axle stress variation for Case II (DWH-I). A very sharp increase in stress (to 13,050.00 psi) occurred soon after the wheel hit the bump (at 0.045 second).

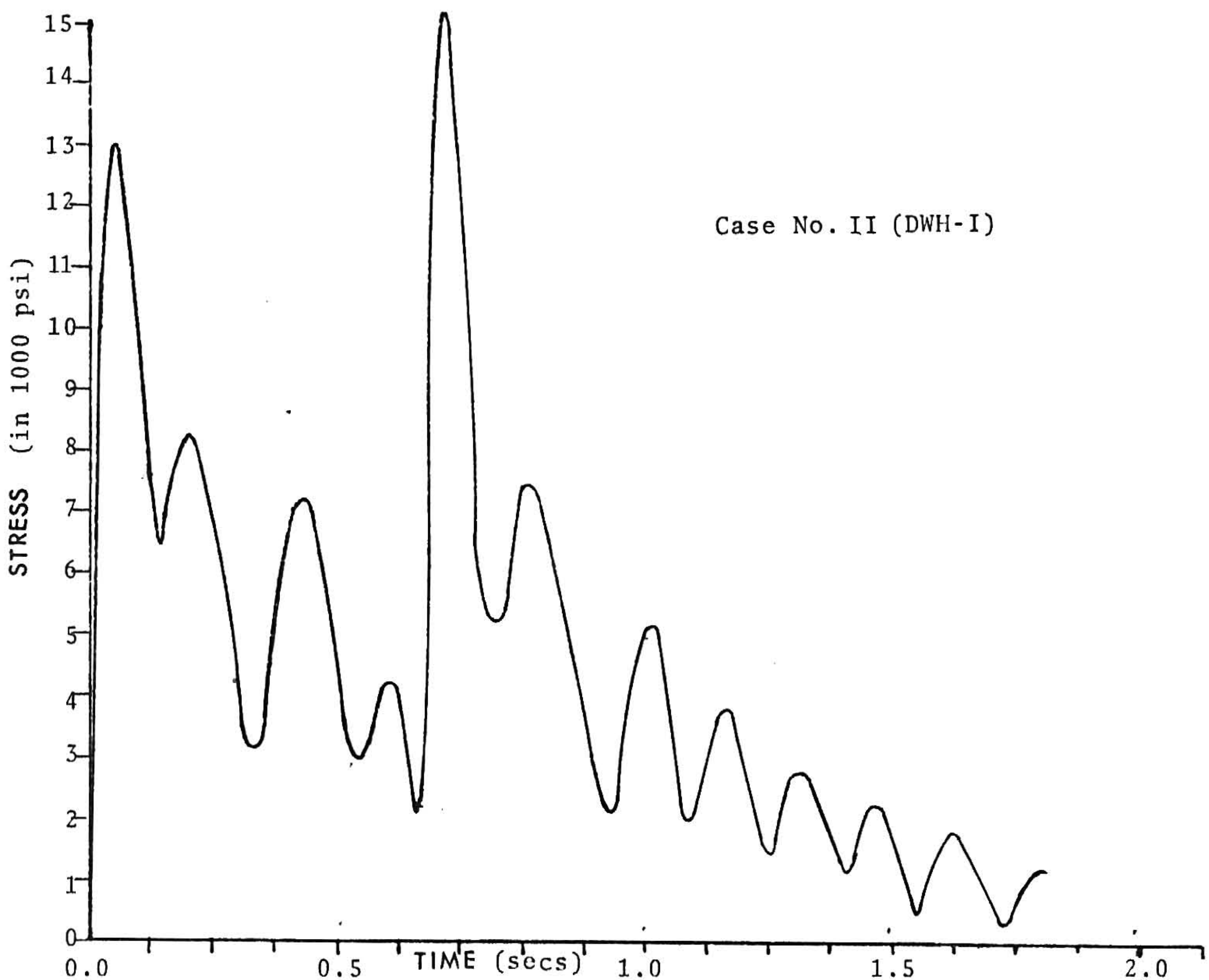

Figure 12. Rear axle stress fluctuation for individual- hub method of attachment (left rear wheels strike the bump).

In this case, forces are applied to the axle at two different points since the dual wheels were attached individually by means of rigid hubs. A comparison of computer solutions for Case I and Case II showed that the frequency of vibration for Case II was higher, but its amplitude of bounce was lower than Case I (SW) at all time intervals. The increase in positive deflection resulted in a stress magnitude of only 2,900 psi at the time when the tractor's wheels were at the top of the bump (0.335 second).

A peak stress of 15,150.0 psi caused by the abrupt change in terrain profile as the wheels left the bump occurred at 0.745 second. This was higher than the stress level reached immediately after the wheel hit the bump. Thereafter, the stress level decreased rapidly to 4,745.0 psi and damped out to a steady-state condition after 2.00 seconds of time.

Figure 13 depicting the transient stress variation for Case IV (DWR-I) could be compared with Case II (DWH-I). The rear axle was loaded at only one point because of the rigid rim-to-rim attachment of the dual wheels. In Case II (DWH-I) the axle had two point loading due to the separate hubs. At 0.057 second after the tractor encountered the bump, the stress level reached a value of 11,670.00 psi. This was lower by 1,250.00 psi in comparison to Case II (DWH-I). This was due to the higher bending moment resulting from the additional distance between the wheels of the hub attached duals. The pattern of stress variation was quite similar, but there was a frequency shift due to the addition of the extra equivalent spring for the system of Case II

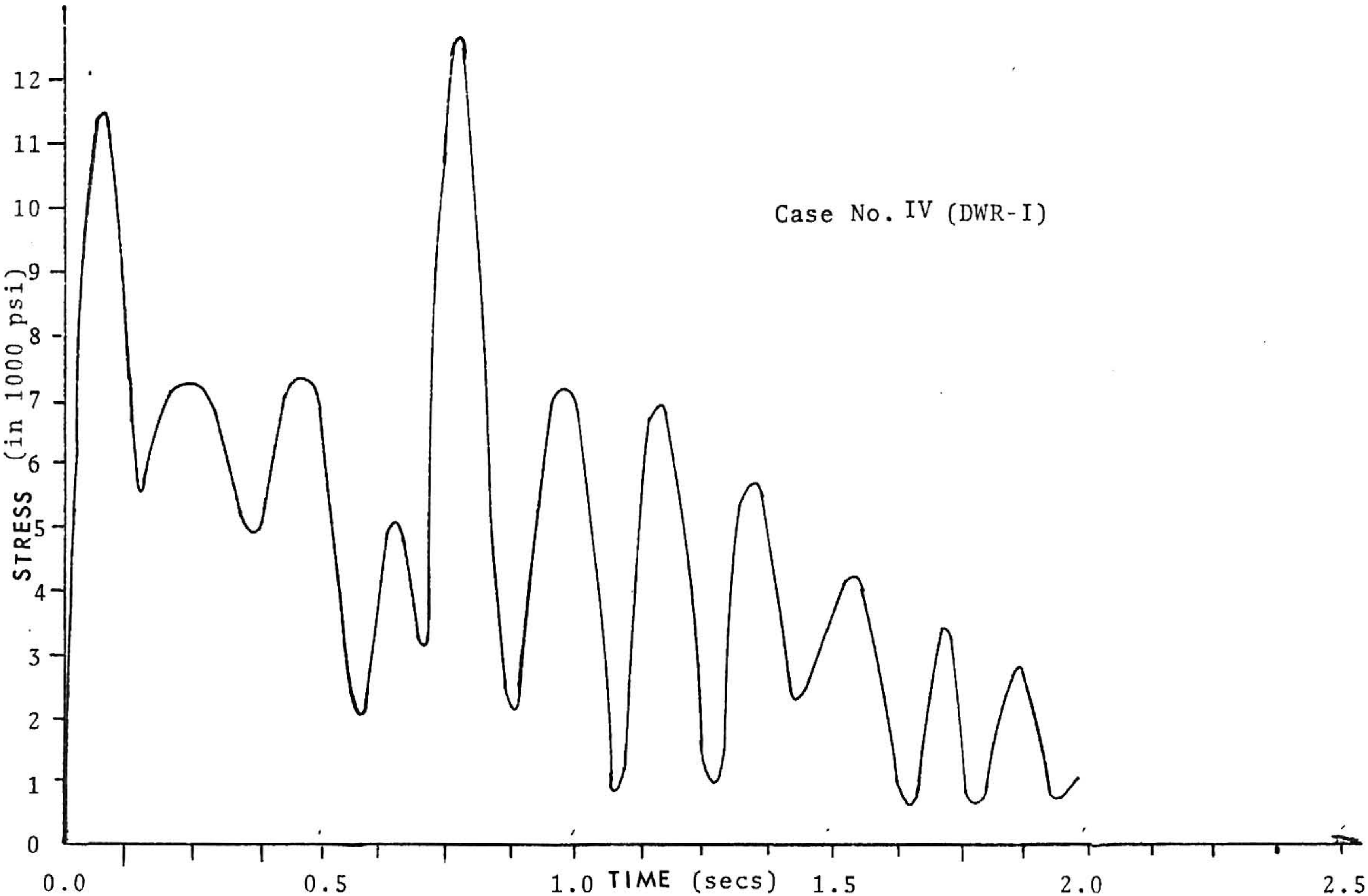

Figure 13. Rear axle stress fluctuation for rim to rim method of attachment (left rear wheels strike the bump).

(DWH-I). The stresses remained very low just as the wheel left the bump, but increased immediately afterwards to a peak value of 12,650.0 psi (0.10 second later). The stress amplitudes after the vehicle encountered the smooth surface were higher, but the frequency of the natural damped oscillations was lower than Case II (DWH-I).

The resultant bending stresses versus time relationship is shown in Figure 14 for Case III (DWH-II). A rapid increase of stress up to 11,200.00 psi at a time of 0.10 second could be compared with a stress level of 9,167.00 psi for Case V (DWR-II). This difference was mainly because of greater axle bending which was due to increased acceleration and rigid body rotation about the longitudinal axis due to additional rotation (roll) for the extra hub length. This resultant stress level of 11,200.00 psi was quite high in comparison to Case I (SW) due to higher amplitude of oscillation of the rigid chassis about the longitudinal axis since the outer dual wheels were spaced at a distance of 4.33 ft from the C.G. of the chassis which contacts the bump.

The situation was less critical than Case II (DWH-I) which had a resultant bending stress of 15,000 psi immediately after the bump had been traversed where as the stress level for Case III (DWH-II) was only 9,345.00 psi (at 0.745 second). The stress amplitude damped out with higher frequency, but with a lower logarithmic decrease in comparison to Case V (DWR-II).

The computer solution of the resultant bending stresses for the model described in Case V (DWR-II) was plotted versus time in

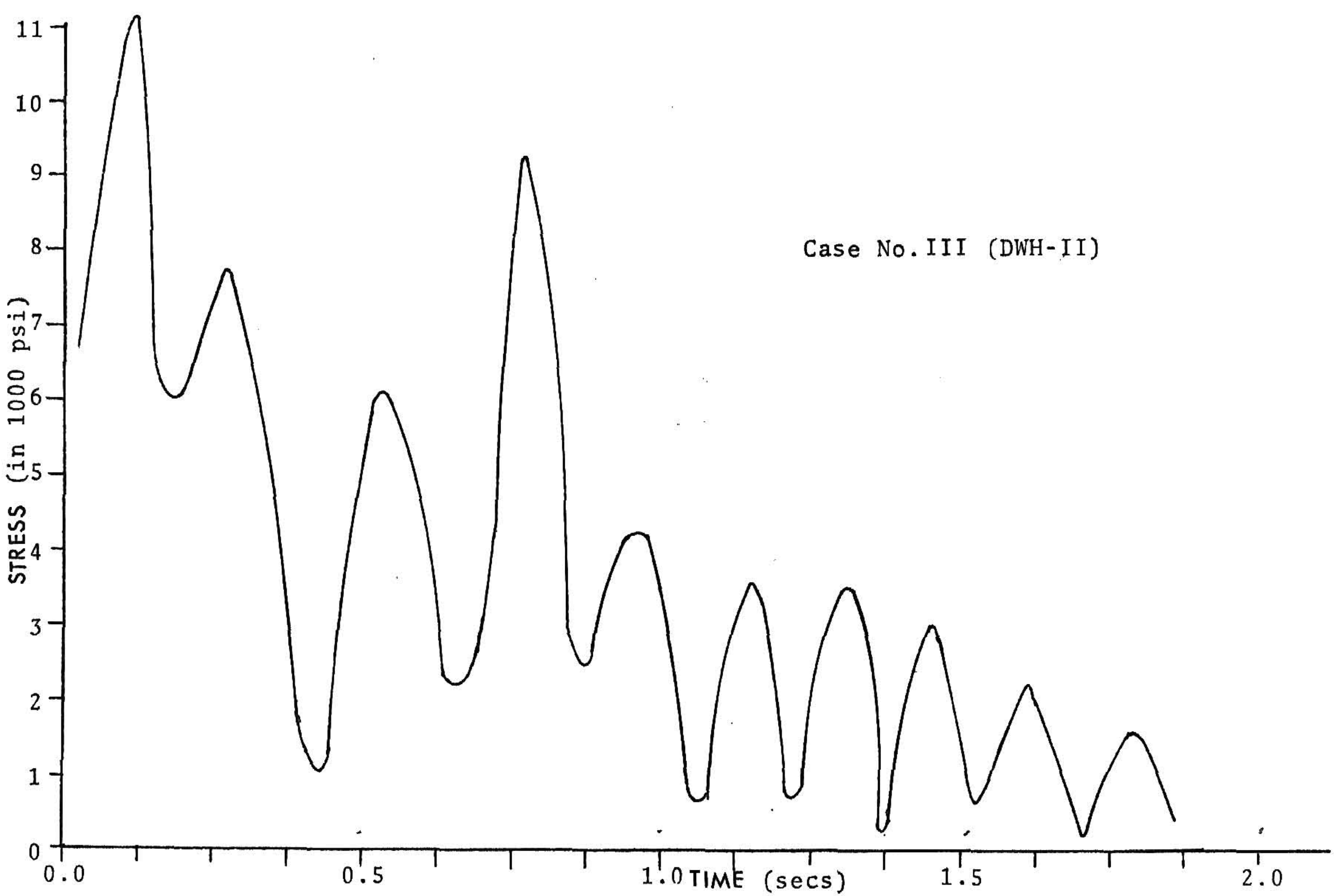

Figure 14, Rear axle stress fluctuation for individual-hub method of attachment (outer left rear wheel strikes the bump).

Figure 15. The behavior of this solution was very similar to Case I (SW). The peak stress immediately after it hits the bump was 9,167.00 psi (at 0.097 second) whereas the level of this stress in Case I (SW) was 9,160.00 psi. This difference of 1,000 psi was due to the extra wheel mass of the outside dual tire. The stress level at the time when the tractor's tire was at the top of the bump was 9,216.0 psi while in Case I (SW) the stress was 6,698.00 psi. This was due to the fact that the forcing function which acted on Case V (DWR-II) was at the distance of 4.33 ft from the C.G. of the chassis; but in the case of the single wheel, it was 2.83 ft from the C.G. of the chassis. This produced a larger force for the X_4 coordinate (roll) and caused more axle bending. The peak stress at the time when the bump had been traversed (0.7125 second) was 7,466.0 psi, which was no greater than the value obtained just after the tractor hit the bump (at 0.0975 second). This was due to the sudden increase in spring rate that the inside tire picked up as soon as it contacted the smooth surface. The amplitude of stress in this case reduced more rapidly after the wheel left the bump than in Case I (SW); however, the frequency of vibration was quite high.

The results obtained and discussed above may be varified with some of the experimental statistics available on axle stress, in the literature. Field measurement of axle stress (Anderson, 1966) showed that combined torsional and bending stress in the rear axle were 40-50 percent greater in a tractor equipped with dual wheels set at 60 and 120 inches, respectively, compared to a

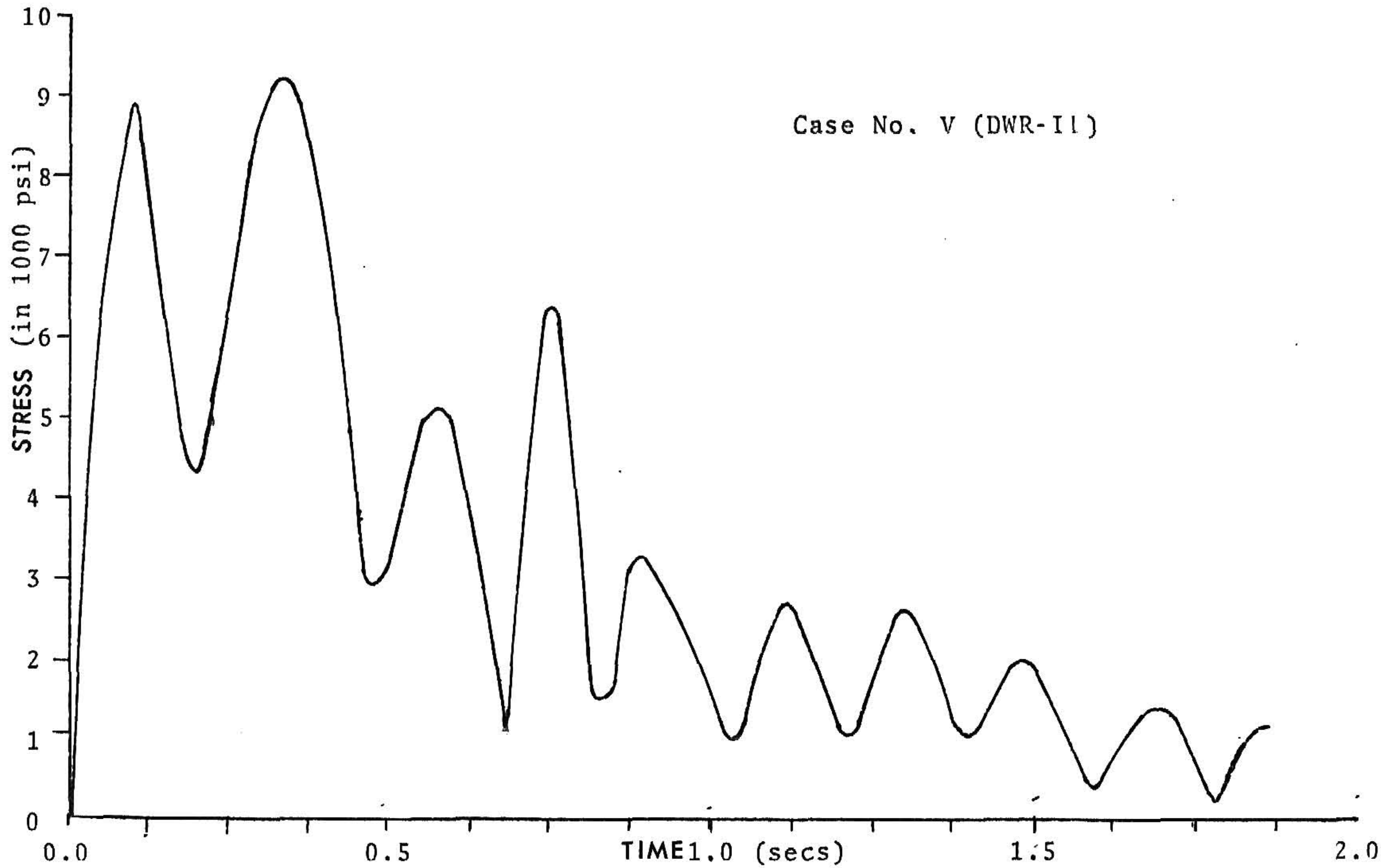

Figure 15. Rear axle stress fluctuation for rim-to-rim method of attachment (outer left rear wheel strikes the bump).

single tire spaced at 80 inches. The results in the presented analysis indicate that maximum bending stresses were 50 percent greater for a rim-to-rim attached set of dual wheels spaced at 70 and 100 inches, respectively, compared to a single tire spaced at 70 inches.

A typical oscillograph trace obtained by Graham (1961) for a tractor rear axle showed that the maximum bending moment (due to dynamic loading) was 28,800 in.lb., which causes a bending stress of 11,500 psi; where-as in presented analysis a maximum bending stress of 9,150 psi was obtained for a single wheel tractor. The difference in these two stress levels was due to different dynamic conditions under which these stress values were recorded.

An increase in axle length increases the stress due to bending was reported by Anderson (1966) and similar evidences were observed in this analysis when the outer duals were attached with hubs and axle length was extended.

SUMMARY AND CONCLUSIONS

This analysis was made to develop a set of axle stress equations to provide dynamic maximum bending stress versus time relationships for various methods of attaching dual tires on farm tractors. Solutions were determined for conventional and dual-tire configurations with the help of a digital computer.

Rear axle models for individual-hub-attached, rim-attached dual wheels, and single wheels were developed. An energy approach was used to formulate stress equations for the models. Linearization of the systems and consideration of the rear axle as a rotating cantilever beam were two major assumptions that were made during this analysis. The effect of tractor vibration, fore and aft motion, and rigid body rotation of the chassis for yaw, roll, and pitch motion were considered in the formulation to include the effect of complex coupling present in the system on the axle deflection.

The set of stress equations thus obtained were solved using a numerical integration technique and vehicle parameters for an IHC-340 utility tractor were used for the solutions. A standard sinusoidal bump (5 in. high and 3 ft. long) was used as the forcing function.

A FORTRAN IV program was made for a generalized dual-wheel tractor model to run on an IBM 360/50 digital computer. The solution for all three models mentioned above along with cases where only the outer dual strikes the bump were obtained. The resultant maximum bending stresses for each model were plotted versus time and their behaviors were discussed. The conclusions which were made after this analysis are as follows:

1) The maximum axle stress (due to bending) for the case of the hub-attached dual wheel was twice as high as that of the conventional tractor with single rear tires.

2) The maximum axle stress (due to bending) in the case of rim-attached dual wheels was 50 percent greater than that of single wheel tractors.

3) The maximum axle stress (due to bending) for the hub-attached dual wheels was 2,000 psi higher than that of the rim-attached dual wheels where the bump was traversed by the outer wheel.

4) In all cases there were two times when the stress increased to very high magnitudes. The first was immediately after the tractor hit the bump and the other was just after the bump had been traversed.

5) The axle stiffness (which depends upon modulus of elasticity of material, area moment of inertia, and length of axle) variation changes the stress situation significantly for a particular set of conditions.

6) The axle dynamic stresses also depend upon the tire configurations that a tractor encounters during operation.

SCOPE OF FURTHER RESEARCH

This analysis on transient stress variation of a tractor rear axle was presented with certain assumptions and limitations. A more accurate and complete model based on the present study could be obtained if the system were defined by more variables. The following items are some of the suggestions which might be incorporated to make this study more realistic and accurate.

1) The stresses due to torsion of the axle could be included in order to complete the information required to calculate the maximum normal stresses. A set of torsional stress equations could be formulated for single and dual wheel models by treating the axle and the rear wheel as a separate system consisting of a rotating cantilever beam with a tipped mass assuming no constraint present between the axle angular twist and the angular motion of the chassis about the transverse axis (pitch motion). A solution of these models should be obtained introducing a torque forcing function to represent the torque transmitted from the final drives to the axle.

2) An experimental setup is needed to verify the theoretical analysis for bending and torsional stresses. Strain gauge techniques for analyzing the maximum stress could be used on tractor axles that extend outside the housing.

3) The tire spring and damping rates should be considered non-linear to include the effect of large oscillations and nonlinearity due to loss of contact between the ground and the wheel.

4) The wheels could be excited with more random and complex functions by generating them by means of simulation methods.

5) Effect of draw-bar pull, soil condition, torque transmitted from final drives to rear axles, and slippage could also be included in this study to make the model more complete.

SELECTED BIBLIOGRAPHY

Anderson, K. W., "Narrow Crop and Tractor Design," SAE Journal, May 1966.

Boden, E. G., "Deflection Test of Axles and Transmission," SAE Transactions, Dec. 1968.

Buckwalter, T. V., and O. J. Horger, "Stress Analysis of Locomotives and Other Larger Axles," Iron and Steel Engineer, Nov. 1936.

Chiesa, A., and L. Tanbureni, "Transmission of Tire Vibrations," Automotive Engineer, Dec. 1964.

Crandall, S. H., Engineering Analysis, McGraw-Hill Book Co., Inc. 1956.

Ellis, J. R., "Beam Axle Suspension," Automobile Engineer, Jan. 1967.

Eckert, E. J., "Improvement in Design of Tractor Axles," SAE Journal, Jan. 1951.

Ficher, E. G., "Ateral Vibrations and Stresses in a Beam Under Shock Machine Loading," Journal of Experimental Stress Analysis, vol. 5, no. 1.

Gorden, K. W., "Design Evaluation and Selection of Heavy Duty Rear Axles," SAE Journal, vol. 63, 1955.

Hendry, A. W., Elements of Experimental Stress Analysis, McGraw-Hill Book Co., New York. 1964.

Hofmeister, W. F., "Application of Cumulative Fatigue Damage Theory to Practical Problems," SAE Transactions, vol. 68, 1960.

Lewis, R. P., and L. J. O'Brien, "Rear Axles Today and Tomorrow," SAE Transactions, vol. 66, 1958.

Pershing, R. L., Transient Motion of Tractors on Side Slopes, unpublished Ph.D. thesis, University of Illinois, 1966.

Polhemus, V. D., "Secondard Vibrations in Rear Suspensions," SAE Journal, July 1950.

Ralston and Wilf, Mathematical Methods for Digital Computor, Wilex Company, New York, 1960.

Robert, S. R., "Operational Stresses in Automobile Parts," SAE Quarterly Transactions, April 1959.

Stott, T. C. F., "Fatigue Testing of Vehicle Components," Proceedings of Automobile Division, 1958-59.

Tiedeman, J. B., and T. E. Pardue, "Bending of Rotating Beams," Proceedings of the Society of Experimental Stress Analysis, vol. XI, no. 2, pp. 189-200.

APPENDIX A

THE RUNGE-KUTTA METHOD OF THE SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

The Runge-Kutta method can be used to solve a set of simultaneous ordinary differential equations of the initial value type.

This method can be demonstrated by considering a pair of simultaneous first order differential equations of the form

$$\frac{dx}{dt} = f_1(t, x(t), y(t))$$

$$\frac{dy}{dt} = f_2(t, x(t), y(t))$$

with the initial conditions

$$x = x_o,\ y = y_o,\ \text{at } t = t_o$$

using the forth order Runge-Kutta method, the increments in x and y for the first interval are found by the relations:

$$\Delta_x = 1/6(k_1 + 2k_2 + 2k_3 + k_4)$$
$$\Delta_y = 1/6(\ell_1 + 2\ell_2 + 2\ell_3 + \ell_4)$$
$$k_1 = hf_1(t_o, x_o, y_o)$$
$$k_2 = hf_1(t_o + h/2, x_o + k_1/2, y_o + \ell_1/2)$$
$$k_3 = hf_1(t_o + h/2, x_o + k_2/2, y_o + \ell_2/2)$$
$$k_4 = hf_1(t_o + h, x_o + k_3, y_o + \ell_3)$$
$$\ell_1 = hf_2(t_o, x_o, y_o)$$
$$\ell_2 = hf_2(t_o + h/2, x_o + k_1/2, y_o + \ell_1/2)$$
$$\ell_3 = hf_2(t_o + h/2, x_o + k_2/2, y_o + \ell_2/2)$$
$$\ell_4 = hf_2(t_o + h, x_o + k_3, y_o + \ell_3)$$

The increment for the succeeding intervals are computed in exactly the same way except that t_o, x_o, and y_o are replaced by t_1,

x_1, y_1, etc. Thus,

$$x_{i+1} = x_i + 1/6(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y_{i+1} = y_i + 1/6(\ell_1 + 2\ell_2 + 2\ell_3 + \ell_4)$$

The solution is initated by substituting the known initial values of x and y into the differential equations. This step provides initial values of the functions f_1 and f_2. Values of k_1 and ℓ_1 are obtained next by multiplying the initial values of f_1 and f_2 by h. With values of k_1 and ℓ_1 known, k_2 and ℓ_2 are next calculated followed by k_3 and ℓ_3, and finally k_4 and ℓ_4. Then, from the recurrence equation, values of x_{i+1} and y_{i+1} are found at $t = t_{i+h}$. These new values of x and y are used as the beginning values for the next step to obtain values of x_{i+2} and y_{i+2} at $t = t_{i+2h}$ and so on until the desired integration interval has been run through.

The above method which was demonstrated for two equations can be generalized to any number of equations by merely using a set of equations for each dependent variable appearing in the system of simultaneous equations.

Error and Step Size Control

The local truncation error, e_t, for Taylor's expansion of the solution function, y(t), of the forth order Runge-Kutta method is of the form

$$e_t = kh^5 + \phi(h^6) \tag{3}$$

where k depends upon f(t,y) and its higher order partial derivatives. If h is sufficiently small so that the error is dominated by the first term, then it is possible to find bounds for k. In general, such bounds depend upon bounds for f(t,y) and its various partial derivatives and upon the order of Runge-Kutta method used.

Selection of step size depends upon the accuracy that is desired. The step size should be small enough to achieve required accuracy, yet it should be as large as possible in order to keep rounding errors under control and to avoid excessive numbers of derivative evaluations to save computer time. This consideration is very important since, in a system like the one under study, it is quite complicated and consumes a lot of computer time to solve each step.

Simultaneous Solution of Ordinary Differential Equations

The system was defined by n simultaneous first order ordinary differential equations in the dependent variables y_1, y_2, ..., y_n; or

$$\frac{d_{x1}}{d_x} = f_1(x, y_1, y_2, \ldots, y_n),$$

$$\frac{d_{y2}}{d_x} = f_2(x, y_1, y_2, \ldots, y_n),$$

$$\vdots$$

$$\frac{d_{yn}}{d_x} = f_n(x, y_1, y_2, y_3, \ldots, y_n),$$

with initial conditions given at a common point x_o, that is

$$y_1(x_o) = y_{1,0}$$
$$y_2(x_o) = y_{2,0}$$
$$\vdots$$
$$y_n(x_o) = y_{n,0}$$

The solution of such systems is similar, in principle, to a single first order equation. The algorithm selected is applied to each of the n equations in parallel to each step.

The complete computer programming for forth order integration processes has been included in Appendix B. Since the forcing function is of the form

$$F = F_1 \sin wt + F_2 \cos wt,$$

the angle wt varies from 0 to $\pi/2$ for a half sine period bump. The force is maximum when $\partial F/\partial t = 0$, or

$$F_1 w \cos wt - F_2 w \sin wt = 0$$

$$\tan wt = F_1/F_2$$

$$wt = \tan^{-1} F_1/F_2$$

$$t = 1/w \tan^{-1} F_1/F_2$$

$$F = F_1 \sin (1/w \tan^{-1} F_1/F_2) + F_2 \cos (1/w \tan^{-1} F_1/F_2)$$

This is a maximum force applied to the vehicle system in the direction of pitch, bounce, and roll motion.

APPENDIX B

COMPUTER PROGRAM TO SOLVE STRESS EQUATIONS FOR GENERALIZED DUAL WHEEL MODEL

```
.......................................................................
.......................................................................
.......................................................................
.......................................................................
.......................................................................
ANALYSIS OF TRACTOR AXLE STRESS VARIATION BY SIMULATION METHOD
                        FOR
GENERALIZED CASE OF DUAL WHEEL TRACTOR MODEL DISCRIBED IN-
CHAPTER III
         PROGRAMMED IN FORTRAN IV
                      BY
                N. P. MATHUR
.......................................................................
.......................................................................
    REAL M,K,IA,L ,MI,K1,K2,L1,L2
    EXTERNAL FCT,OUTP
    DOUBLE PRECISION DERY,Y,AUX,X,PRMT,DSIN,DCOS
    DOUBLE PRECISION DABS
    COMMON R(17,17),R1(17,17),F1(17,2),W, BB(17,17),CC(17,17)
   1,DD(17,17),XBAR,ZBAR,K(9,2),C(9,2),VALUE(6,2001),D(6),AN,INDEX
    DIMENSION AA(17,34),M(8),IA(9,3,3),AINV(17,17),Y(34),DERY(34)
   1,F(17,2),PRMT(5),AUX(8,34),C1(9,2)
.00 FORMAT (8F10.5)
.01 FORMAT(9F7.2)
.02 FORMAT (8F10.0)
.03 FORMAT(1H ,13E10.3)
.04 FORMAT (I5)
.05 FORMAT (2E14.5)
.06 FORMAT (///10X, 'THE MASS MATRIX AA'/)
.07 FORMAT (///10X, 'THE DAMPING MATRIX BB'/)
.08 FORMAT (///10X, 'THE SPRING  MATRIX DD'/)
.09 FORMAT (///10X, 'THE FORCING MATRIX F'/)
.11 FORMAT (///10X, 'THE MATRIX AINV'/)
.12 FORMAT (///10X, 'THE MATRIX R'/)
.13 FORMAT (///10X, 'THE MATRIX R1'/)
.14 FORMAT (///10X, 'THE MATRIX F1'/)
.15 FORMAT (3F10.5)
 16 FORMAT(1H ,11E11.4)
 17 FORMAT(1H ,D10.3,10X,D11.4,10X,D11.4)
 18 FORMAT(1H ,D10.3,10X,D11.4)
 19 FORMAT(///10X, 'THE AXLE DEFLECTION'/)
 21 FORMAT (///10X, 'THE RESULTANT AXLE STRESS VARIATION'/)
 22 FORMAT (///10X, 'THE AXLE STRESS VARIATION'/)
.......................................................................
    READ THE DATA
    DATA IS TAKEN FROM STANDARD CONDITION LISTED UNDER TABLE NO. IV
    AND THEIR DEFNITION IS GIVEN IN TABLE NO. I THROUGH III
.......................................................................
    READ (1,104)N
    READ (1,100)(M(I),I=1,8)
    READ (1,100)((C(I,J),I=1,9),J=1,2)
    READ (1,102)((K(I,J),I=1,9),J=1,2)
```

```
      READ (1,100)((IA(I1,I2,I2),I2=1,3),I1=1,8)
      READ (1,100) IA(5,1,3),IA(5,3,1),XBAR,YBAR,ZBAR,A,B,E
      READ (1,100) H,RSMALL,RBIG,(D(I),I=1,6)
      READ (1,115) H1,S,L
      WRITE THE DATA
      WRITE(3,100)(M(I),I=1,8)
      WRITE(3,100)((C(I,J),I=1,9),J=1,2)
      WRITE(3,102)((K(I,J),I=1,9),J=1,2)
      WRITE(3,100)((IA(I1,I2,I2),I2=1,3),I1=1,8)
      WRITE(3,100) IA(5,1,3),IA(5,3,1),XBAR,YBAR,ZBAR,A,B,E
      WRITE(3,100) H,RSMALL,RBIG,(D(I),I=1,6)
      WRITE(3,115) H1,S,L
      D1=3.0
      L2=9.0
      L1=15.0
      MI=(3.14159/64.0)*D1**4
      E1=30000000.00
      K1=1000000.0
      K2=990000.0
      DO 110 I=1,17
      DO 110 J=1,17
      AA(I,J)=0.00
      BB(I,J)=0.00
      DD(I,J)=0.00
  110 CONTINUE
      IA(5,1,2)=0.0
      IA(5,2,3)=0.0
      ....................................................................
      CALCULATION FOR COEFICIENTS OF MASS MATRIX. THE FIRST SUBSCRIPT
      DENOTES THE NUMBER OF EQUATION AND SECOND SUBSCRIPT DENOTES THE
      NUMBER OF VARIABLE  OF THE SYSTEM
      ....................................................................
      AA(1,1)=M(1)+M(2)+M(5)+M(6)
      AA(1,5)=(M(1)+M(2))*(ZBAR-RSMALL)+M(6)*(B+E)
      AA(1,6)=M(1)*D(1)-M(2)*D(2)+M(6)*YBAR
      AA(2,2)=M(1)+M(2)+M(3)+M(4)+M(5)+M(6)+M(7)+M(8)
      AA(2,4)=-(M(1)+M(2)+M(6))*B-(M(3)+M(4)+M(7)+M(8))*(ZBAR-RBIG)
      AA(2,6)=(M(1)+M(2)+M(6))*A-(M(3)+M(4))*XBAR -(M(7)+M(8))*XBAR
      AA(2,7)=-(M(1)+M(2))*(H-RSMALL)-M(6)*E
      AA(3,3)=M(1)+M(2)           +M(5)+M(6)
      AA(3,4)=-(M(1)+M(2)+M(6))*YBAR
      AA(3,5)=-(M(1)+M(2)+M(6))*A
      AA(3,7)=-M(1)*(D(1)-YBAR)+M(2)*(D(2)+YBAR)
      AA(4,2)=-(M(1)+M(2)+M(6))*B-(M(3)+M(4))*(ZBAR-RBIG)
     1-(M(7)+M(8))*(ZBAR-RBIG)
      AA(4,3)=-(M(1)+M(2)+M(6))*YBAR
      AA(4,4)=(M(1)+M(2)+M(6))*(B**2+YBAR**2)+M(3)*((ZBAR-RBIG)**2)
     1      +M(4)*((ZBAR-RBIG)**2)+IA(4,1,1)+IA(5,1,1)+IA(3,1,1)
     1+M(7)*((ZBAR-RBIG)**2)+M(8)*((ZBAR-RBIG)**2          )+
     1IA(7,1,1)+IA(8,1,1)
      AA(4,5)=(M(1)+M(2)+M(6))*YBAR*A-IA(5,1,2)
      AA(4,6)=-(M(1)+M(2)+M(6))*B*A+(M(3)+M(4))*(ZBAR-RBIG)*XBAR
     1-IA(5,1,3)+(M(7)+M(8))*XBAR*(ZBAR-RBIG)
```

```
      AA(4,7)=M(1)*((H-RSMALL)*B+(D(1)-YBAR)*YBAR)+M(2)*((H-RSMALL)*B-
     1(D(2)+YBAR)*YBAR)+M(6)*E*B
      AA(5,1)=(M(1)+M(2))*(ZBAR-RSMALL)+M(6)*(B+E)
      AA(5,3)=-(M(1)+M(2))*A-M(6)*A
      AA(5,4)=(M(1)+M(2)+M(6))*YBAR*A-IA(5,1,2)
      AA(5,5)=(M(1)+M(2))*((ZBAR-RSMALL)**2+A**2)+
     1 IA(3,2,2)+IA(4,2,2)+IA(6,2,2)+M(6)*(B+E)**2+M(6
     1)*A**2+IA(5,2,2)
     1+IA(7,2,2)+IA(8,2,2)
      AA(5,6)=M(1)*C(1)*(ZBAR-RSMALL)-M(2)*D(2)* (ZBAR-RSMALL)
     1-IA(5,2,3)+M(6)*YBAR*(B+E)
      AA(5,7)=M(1)*A*(D(1)-YBAR)-M(2)*A*(D(2)+YBAR)
      AA(6,1)=M(1)*C(1)-M(2)*D(2)+M(6)*YBAR
      AA(6,2)=(M(1)+M(2))*A-(M(3)+M(4))*XBAR+M(6)*A
     1-(M(7)+M(8))*XBAR
      AA(6,4)=-(M(1)+M(2))*A*B+(M(3)+M(4))*(ZBAR-RBIG)*XBAR-M( )*A*B
     1-IA(5,1,3)+(M(7)+M(8))*XBAR*(ZBAR-RBIG)
      AA(6,5)=M(1)*C(1)*(ZBAR-RSMALL)-M(2)*C(2)* (ZBAR-RSMALL)
     1-IA(5,2,3)+M(6)*YBAR*(B+E)
      AA(6,6)=M(1)*(D(1)**2+A**2)+M(2)*(D(2)**2+A**2)+M(3)* (   XB
     1AR**2)+M(4)*(          XBAR**2)+M(6)*(YBAR**2+A**2)+IA(1,1,1)+IA(
     12,1,1)+IA(3,1,1)+IA(4,1,1)+IA(5,3,3)+IA(6,3,3)+
     1(M(8)+M(7))*XBAR**2+IA(8,1,
     11)+IA(7,1,1)
      AA(6,7)=-(M(1)+M(2))*(H-RSMALL)*A-M(6)*A*E
      AA(7,2)=-(M(1)+M(2))*(H-RSMALL)-M(6)*E
      AA(7,3)=-M(1)*(C(1)-YBAR)+M(2)*(D(2)+YBAR)
      AA(7,4)=M(1)*((H-RSMALL)*B+(D(1)-YBAR)*YBAR)+M(2)*((H-RSMALL)*B-
     1(D(2)+YBAR)*YBAR)+M(6)*E*B
      AA(7,5)=M(1)*A*(D(1)-YBAR)-M(2)*A*(D(2)+YBAR)
      AA(7,6)=-(M(1)+M(2))*(H-RSMALL)*A-M(6)*E*A
      AA(7,7)=M(1)*((H-RSMALL)**2+(D(1)-YBAR)**2)+M(2)*((H-RSMALL)**2+(
     1D(2)+YBAR)**2)+M(6)*E**2+IA(1,1,1)+IA(2,1,1)+IA(6,1,1)
      AA(8,8)=IA(1,2,2)
      AA(9,9)=IA(2,2,2)
      AA(10,10)=M(3)
      AA(14,14)=M(3)
      AA(11,11)=M(4)
      AA(15,15)=M(4)
      AA(12,12)=M(7)
      AA(16,16)=M(7)
      AA(13,13)=M(8)
      AA(17,17)=M(8)
C ......................................................................
C     CALCULATION FOR COEFICIENTS OF DAMPING MATRIX. THE FIST SUBSCRIPT
C     DENOTES THE NUMBER OF EQUATION AND THE SECOND SUBSCRIPT DENOTES
C     THE NUMBER OF VARIABLE  OF THE SYSTEM
C ......................................................................
  125 BB(1,1)=C(5,1)+C(5,2)
      BB(1,5)=(C(5,1)+C(5,2))*(ZBAR-RSMALL)
      BB(1,6)=C(5,1)*C(1)-C(5,2)*D(2)
      BB(1,8)=-C(5,1)*RSMALL
      BB(1,9)=-C(5,2)*RSMALL
```

```
      BB(2,2)=C(3,1)+C(3,2)+C(4,1)+C(4,2)+C(8,1)+C(8,2)
      BB(2,4)=-(C(3,1)+C(3,2))*B-(C(4,1)+C(4,2))*ZBAR-(C(8,1)+C(8,2))*ZB
     1AR
      BB(2,6)=(C(3,1)+C(3,2))*A-(C(4,1)+C(4,2))*XBAR-(C(8,1)+C(8,2))*XB
     1AR
      BB(2,7)=-(C(3,1)+C(3,2))*H
      BB(3,3)=C(1,1)+C(1,2)
      BB(3,4)=-(C(1,1)+C(1,2))*YBAR
      BB(3,5)=-(C(1,1)+C(1,2))*A
      BB(3,7)=-C(1,1)*(D(1)-YBAR)+C(1,2)*(D(2)+YBAR)
      BB(4,2)=-(C(3,1)+C(3,2))*B-(C(4,1)+C(4,2))*ZBAR-(C(8,1)+C(8,2))*Z
     1BAR
      BB(4,3)=-(C(1,1)+C(1,2))*YBAR
      BB(4,4)= (C(1,1)+C(1,2))*YBAR**2  +(C(3,1)+C(3,2))*B**2
     1 +(C(4,1)+C(4,2))*ZBAR**2+(C(8,1)+C(8,2))*ZBAR**2
      BB(5,4)=(C(1,1)+C(1,2))*YBAR*A
      BB(4,5)=(C(1,1)+C(1,2))*YBAR*A
      BB(4,6)=-(C(3,1)+C(3,2))*A*B+(C(4,1)+C(4,2))*XBAR*ZBAR
     1+(C(8,1)+C(8,2))*XBAR*ZBAR
      BB(4,7)=C(1,1)*YBAR*(D(1)-YBAR)-C(1,2)*YBAR*(D(2)+YBAR)+(C(3,1)+C
     1(3,2))*B*H
      BB(5,1)=(C(5,1)+C(5,2))*(ZBAR-RSMALL)
      BB(5,3)=-(C(1,1)+C(1,2))*A
      BB(5,5)=(C(1,1)+C(1,2))*A**2+(C(5,1)+C(5,2))*(ZBAR-RSMALL)**2
      BB(5,7)=C(1,1)*A*(D(1)-YBAR)-C(1,2)*A*(D(2)+YBAR)
      BB(5,8)=-C(5,1)*RSMALL*(ZBAR-RSMALL)
      BB(5,9)=-C(5,2)*RSMALL*(ZBAR-RSMALL)
      BB(6,1)=C(5,1)*D(1)-C(5,2)*D(2)
      BB(6,2)=(C(3,1)+C(3,2))*A-(C(4,1)+C(4,2)+C(8,1)+C(8,2))*XBAR
      BB(6,4)=-(C(3,1)+C(3,2))*A*B+(C(4,1)+C(4,2))*XBAR*ZBAR
     1+(C(8,1)+C(8,2))*XBAR*ZBAR
      BB(5,6)=C(5,1)*D(1)*(ZBAR-RSMALL)-C(5,2)*D(2)*(ZBAR-RSMALL)
      BB(6,5)=C(5,1)*D(1)*(ZBAR-RSMALL)-C(5,2)*D(2)*(ZBAR-RSMALL)
      BB(6,6)=(C(3,1)+C(3,2))*A**2+(C(4,1)+C(4,2))*XBAR**2+C(5,1)*D(1)*
     1*2+C(5,2)*D(2)**2+
     1        (C(8,1)+C(8,2))*XBAR**2
      BB(6,7)=-(C(3,1)+C(3,2))*A*H
      BB(6,8)=-C(5,1)*D(1)*RSMALL
      BB(6,9)=C(5,2)*D(2)* RSMALL
      BB(7,2)=-(C(3,1)+C(3,2))*H
      BB(7,3)=-C(1,1)*(D(1)-YBAR)+C(1,2)*(D(2)+YBAR)
      BB(7,4)=C(1,1)*YBAR*(D(1)-YBAR)-C(1,2)*YBAR*(D(2)+YBAR)+C(3,1)*B*
     1H+C(3,2)*B*H
      BB(7,5)=C(1,1)*A*(D(1)-YBAR)-C(1,2)*A*(D(2)+YBAR)
      BB(7,6)=-(C(3,1)+C(3,2))*A*H
      BB(7,7)=C(1,1)*(D(1)-YBAR)**2+C(1,2)*(D(2)+YBAR)**2+C(3,1)*H**2+C
     1(3,2)*H**2
      BB(8,1)=-C(5,1)*RSMALL
      BB(8,5)=-C(5,1)*RSMALL*(ZBAR-RSMALL)
      BB(8,6)=-C(5,1)*RSMALL*D(1)
      BB(8,8)=C(5,1)*RSMALL**2
      BB(9,1)=-C(5,2)*RSMALL
      BB(9,5)=-C(5,2)*RSMALL*(ZBAR-RSMALL)
      BB(9,6)=C(5,2)*RSMALL*D(2)
      BB(9,9)=C(5,2)*RSMALL**2
      BB(10,10)=C(2,1)
      BB(11,11)=C(2,2)
```

```
      BB(12,12)=C(7,1)
      BB(13,13)=C(7,2)
      BB(14,14)=C(6,1)
      BB(15,15)=C(6,2)
      BB(16,16)=C(9,1)
      BB(17,17)=C(9,2)
      WRITE (3,105) BB(5,6),BB(6,5)
      WRITE (3,105) BB(5,4),BB(4,5)
      IF(C(1,1).EQ.K(1,1)) GO TO 129
      DO 123 I=1,17
      DO 123 J=1,17
  123 CC(I,J)=BB(I,J)
      ...............................................................
      THIS EQUALIZES THE VALUE OF 'C' WITH VALUE OF 'K' AND GOES BACK-
      TO STATEMENT NUMBER 125 TO COMPUTES THE COEFFICIENTS FOR SPRING-
      MATRIX DUE TO EQUIVALENT TIRE SPRING RATES.
      ...............................................................
      DO 124 I=1,9
      DO 124 J=1,2
      C1(I,J)=C(I,J)
      C(I,J)=K(I,J)
  124 CONTINUE
      GO TO 125
  129 DO 126 I=1,17
      DO 126 J=1,17
  126 DD(I,J)=BB(I,J)
      ...............................................................
      THIS CALCULATES  THE VALUES OF SPRING MATRIX DUE TO EQUIVALENT-
      AXLE SPRING RATES AND ADDS IT TO RESPECTIVE TERM OF SPRING MATRIX-
      DEVELOPED DUE TC TIRE SPRING CONSTANTS.
      ...............................................................
      DD(5,5)=DD(5,5)+2*(K1+K2)*(XBAR**2+ZBAR**2)
      DD(10,10)=  K1      +DD(10,10)
      DD(4,4)=DD(4,4)+K1*D(3)**2+K1*D(4)**2+K2*D(5)**2+K2*D(6)**2
      DD(11,11)=DD(11,11)+  K1
      DD(6,6)=DD(6,6)+K1*D(3)**2+K2  *D(5)**2+K1*D(4)**2 +K2*D(6)**2
      DD(5,1)=DD(5,1)+2*(+K1+K2)*ZBAR
      DD(1,5)=DD(1,5)+2*(+K1+K2)*ZBAR
      DD(1,1)=2*K1+2*K2+DD(1,1)
      DD(3,3)=2*K1+2*K2+DD(3,3)
      DD(5,3)=+2*K1*XBAR+2*K2*XBAR+DD(5,3)
      DD(3,5)=+2*K1*XBAR+2*K2*XBAR+DD(3,5)
      DD(5,10)=DD(5,10)-K1*XBAR
      DD(16,5)=DD(16,5)-K2*ZBAR
      DD(17,5)=DD(17,5)-K2*ZBAR
      DD(12,12)=DD(12,12)+K2
      DD(13,13)=DD(13,13)+K2
      DD(14,14)=DD(14,14)+K1
      DD(15,15)=DD(15,15)+K1
      DD(16,16)=DD(16,16)+K2
      DD(17,17)=DD(17,17)+K2
      DD(1,14)=DD(1,14)-K1
```

```
      DD(14,1)=DD(14,1)-K1
      DD(1,17)=DD(1,17)-K2
      DD(17,1)=DD(17,1)-K2
      DD(1,15)=DD(1,15)-K1
      DD(15,1)=DD(15,1)-K1
      DD(1,16)=DD(1,16)-K2
      DD(16,1)=DD(16,1)-K2
      DD(10,3)=DD(10,3)-K1
      DD(11,3)=DD(11,3)-K1
      DD(3,11)=DD(3,11)-K1
      DD(3,10)=DD(3,10)-K1
      DD(12,3)=DD(12,3)-K2
      DD(13,3)=DD(13,3)-K2
      DD(3,13)=DD(3,13)-K2
      DD(3,12)=DD(3,12)-K2
      DD(4,11)=-D(4)*K1
      DD(11,4)=-D(4)*K1
      DD(4,12)=D(5)*K2
      DD(12,4)=D(5)*K2
      DD(4,13)=-D(6)*K2
      DD(13,4)=-D(6)*K2
      DD(6,14)=-D(3)*K1
      DD(14,6)=-D(3)*K1
      DD(6,15)=D(4)*K1
      DD(15,6)=D(4)*K1
      DD(6,16)=-D(5)*K2
      DD(16,6)=-D(5)*K2
      DD(6,17)=D(6)*K2
      DD(17,6)=D(6)*K2
      DD(5,11)=DD(5,11)-K1*XBAR
      DD(10,5)=DD(10,5)-K1*XBAR
      DD(11,5)=DD(11,5)-K1*XBAR
      DD(12,5)=DD(12,5)-K2*XBAR
      DD(13,5)=DD(13,5)-K2*XBAR
      DD(5,12)=DD(5,12)-K2*XBAR
      DD(5,13)=DD(5,13)-K2*XBAR
      DD(5,14)=DD(5,14)-K1*ZBAR
      DD(5,15)=DD(5,15)-K1*ZBAR
      DD(14,5)=DD(14,5)-K1*ZBAR
      DD(15,5)=DD(15,5)-K1*ZBAR
      DD(5,17)=DD(5,17)-K2*ZBAR
      DD(5,16)=DD(5,16)-K2*ZBAR
      DD(4,10)=D(3)*K1+DD(4,10)
      DD(10,4)=D(3)*K1+DD(10,4)
      ......................................................................
      THIS TRANSFER THE SPRING AND DAMPING MATRIX TO RIGHT HAND SIDE OF
      THE EQUATION.
      ......................................................................
      DO 200 I=1,17
      DO 200 J=1,17
   CC CC(I,J)=-CC(I,J)
      DO 300 I=1,17
```

```
DO 300 J=1,17
DD(I,J)=-DD(I,J)
PI=3.14159
W=PI*S/L
.......................................................................
 THIS INITIALIZES THE VALUES OF FORCING FUNCTION FOR DIFFERENT
COORDINATES.
.......................................................................
DO 120 I=1,17
DO 120 J=1,2
F(I,J)=0.0
F(10,1)=-8528.0
F(10,2)=-475.36366
F(12,1)=-8528.0
F(12,2)=-475.36366
WRITE (3,106)
WRITE(3,103)((AA(I,J),J=1,13),I=1,13)
WRITE (3,107)
WRITE(3,103)((CC(I,J),J=1,13),I=1,13)
WRITE (3,108)
WRITE(3,103)((DD(I,J),J=1,13),I=1,13)
WRITE (3,109)
WRITE(3,105)((F(I,J),J=1,2),I=1,13)
.......................................................................
.......................................................................
CALLING OF SUBPROGRAMME MATRIX INVERSION TO INVERT MASS MARRIX.-
THE RESULTANT  MATRIX IS 'AINV'.
.......................................................................
CALL MATINV(AA,AINV,N,N2)
WRITE (3,111)
WRITE (3,103)((AINV(I,J),J=1,13),I=1,13)
.......................................................................
CALLING OF SUBPROGRAMME MATRIX MULTIPLICATION TO MULTIPLY THE
INVERSED MASS MATRIX WITH DAMPING MATRIX. THE RESULTANT MATRIX
IS 'R'
.......................................................................
CALL MATMUL (AINV,CC,R ,N,N,17)
WRITE (3,112)
WRITE (3,103)((R(I,J),J=1,13),I=1,13)
.......................................................................
CALLING OF SUBPROGRAMME MATRIX MULTIPLICATION TO MULTIPLY THE
INVERSED MASS MATRIX WITH SPRING  MATRIX. THE RESULTANT MATRIX
IS 'R1'
.......................................................................
CALL MATMUL (AINV,DD,R1,N,N,17)
WRITE (3,113)
WRITE(3,103)((R1(I,J),J=1,13),I=1,13)
.......................................................................
CALLING OF SUBPROGRAMME MATRIX MULTIPLICATION TO MULTIPLY THE
INVERSED MASS MATRIX WITH FORCING MATRIX. THE RESULTANT MATRIX
IS 'F1'
```

```
••••••••••••••••••••••••••••••••••••••••••••••••••••••••••••••••••••••
  CALL MATMUL (AINV,F,F1,17,17,2)
  WRITE (3,114)
  WRITE(3,105)((F1(I,J),J=1,2),I=1,13)
••••••••••••••••••••••••••••••••••••••••••••••••••••••••••••••••••••••
  PREPERATION FOR CALLING SUBPROGRAMME RKGS FOR NUMARICAL INTEGRATI-
  ON OF DIFFERENTIAL EQUATIONS OF THE SYSTEM. THE DEFNITION OF ALL
  THE ARGUMENTS USED IS GIVEN IN SUBROUTINE RKGS
••••••••••••••••••••••••••••••••••••••••••••••••••••••••••••••••••••••
  PRMT(1)=0.0
  PRMT(2)=2.0
  PRMT(3)=0.005
  PRMT(4)=.0001
  DO 150 I=1,34
C Y(I)=0.0
  DO 160 I=1,34
0 DERY(I)=1./34.
  NDIM=34
  AN=0.0
  INDEX=0
  CALL DRKGS (PRMT,Y,DERY,NDIM,IHLF,FCT,OUTP,AUX)
  WRITE (3,119)
  WRITE(3,117)((VALUE(I,J),I=1,3),J=1,401)
  WRITE (3,122)
  WRITE(3,118)(VALUE(1,J),VALUE(4,J),J=1,2001)
  STOP
  END
••••••••••••••••••••••••••••••••••••••••••••••••••••••••••••••••••••••
  SOBROUTINE  FOR MATRIX INVERSION
••••••••••••••••••••••••••••••••••••••••••••••••••••••••••••••••••••••
  SUBROUTINE MATINV(AA,AINV,N,N2)
  DIMENSION AA(17,34),AINV(17,17)
  N1=N+1
  N2=2*N
  DO300I=1,N
  DO301J=N1,N2
1 AA(I,J)=0.0
  M=N+I
C AA(I,M)=1.0
  DO200J=1,N
  DIV=AA(J,J)
  S=1.0/DIV
  DO201K=J,N2
1 AA(J,K)=AA(J,K)*S
  DO202I=1,N
  IF(I-J)203,202,203
3 AAIJ=-AA(I,J)
  DO204K=J,N2
4 AA(I,K)=AA(I,K)+AAIJ*AA(J,K)
2 CONTINUE
C CONTINUE
  DO400I=1,N
```

```
      DO400J=1,N
      M=N+J
 400  AINV(I,J)=AA(I,M)
      RETURN
      END
C ...................................................................
C     SUBROUTINE FOR MATRIX MULTIPLICATION
C ...................................................................
      SUBROUTINE MATMUL(A,B,C,M,N,L)
      DIMENSION A(M,N),B(N,L),C(M,L)
      DO 102 I=1,M
      DO 101 J=1,L
      SUM=0.0
      DO 100 K=1,N
 100  SUM=SUM+A(I,K)*B(K,J)
 101  C(I,J)=SUM
 102  CONTINUE
      RETURN
      END
C ...................................................................
C     OUTPUT SUBROUTINE
C ...................................................................
      SUBROUTINE OUTP (X,Y,DERY,IHLF,NDIM,PRMT)
      DOUBLE PRECISION DABS
      DOUBLE PRECISION DERY,Y,AUX,X,PRMT
      REAL MH,MV,MI,MR,L1
      COMMON R(17,17),R1(17,17),F1(17,2),W, BB(17,17),CC(17,17)
     1,DD(17,17),XBAR,ZBAR,K(9,2),C(9,2),VALUE(6,2001),D(6),AN,INDEX
      DIMENSION Y(34),DERY(34),PRMT(5)
 103  FORMAT(1H ,F6.3,2(10X,E11.4))
 104  FORMAT (1H ,D10.3,11(1X,D10.3))
      VI=ICCC000.0
      V2=99C000.0
      T=0.67
      L1=15.
      L2=24.0
      D1=3.0
      C1=D1/2.
      PI=3.14159
      MI=(PI/64.0)*81.0
C ...................................................................
C     THIS EQUATES THE FORCING FUNCTION FOR ALL THE COORDINATES TO ZERO
C     SOON AFTER THE BUMP DISCONTINUES
C ...................................................................
      IF (X.LE.T) GO TO 110
 165  DO 170 I= 1,17
      DO 166 J=1,2
 166  F1(I,J)=0.0
 170  CONTINUE
C ...................................................................
C     CALCULATION OF VERTICAL AND HORIZANTAL DEFLECTION.
C ...................................................................
```

```
11C DELTA1=Y(3)+XBAR*Y(5)-D(3)*Y(4)-Y(10)
    DELTA2=Y(1)+ZBAR*Y(5)+D(3)*Y(6)-Y(14)
    DELTA3=Y(3)+XBAR*Y(5)-D(5)*Y(4)-Y(12)
    DELTA4=Y(1)+ZBAR*Y(5)+D(5)*Y(6)-Y(16)
    MV=V1*DELTA1*L1+V2*DELTA3*L2
    MH=V1*DELTA2*L1+V2*DELTA4*L2
    ..................................................................
    CALCULATION OF RESULTANT STRESS IN THE AXLE
    ..................................................................
    A=MV**2
    B=MH**2
    P=A+B
    MR=SQRT(P)
    SR=MR*(C1/MI)
    SH=MH*(C1/MI)
    SV=MV*(C1/MI)
    IF(X.EQ.0.0) INDEX=0
    INDEX=INDEX+1
    VALUE(1,INDEX)=X
    VALUE(2,INDEX)=DELTA1
    VALUE(3,INDEX)=DELTA2
    VALUE(4,INDEX)=SR
    VALUE(5,INDEX)=SH
    VALUE(6,INDEX)=SV
    WRITE(3,104)X,(Y(I),I=1,11)
00C RETURN
    END
    ..................................................................
    THIS FUNCTION SUBROUTINE DEFINES FIRST ORDER DIFFERENTIAL EQUATION
    OF THE SYSTEM
    ..................................................................
    SUBROUTINE FCT(X,Y,DERY)
    DOUBLE PRECISION DABS
    DOUBLE PRECISION DERY,Y,AUX,X,PRMT,DSIN,DCOS
    COMMON R(17,17),R1(17,17),F1(17,2),W, BB(17,17),CC(17,17)
   1,DD(17,17),XBAR,ZBAR,K(9,2),C(9,2),VALUE(6,2001),D(6),AN,INDEX
    DIMENSION DERY(34),Y(34)
    DO 120 I=1,17
120 DERY(I)=Y(I+17)
    DO 125 J=1,17
    SUMJ=0.0
    DO 124 N=1,17
124 SUMJ=SUMJ+R(J,N)*Y(N+17)+R1(J,N)*Y(N)
125 DERY(J+17)=SUMJ+F1(J,1)*DSIN(W*X)+F1(J,2)*DCOS(W*X)
    RETURN
    END
                                                                      DRKG
    ..................................................................DRKG
                                                                      DRKG
       SUBROUTINE DRKGS                                               DRKG
                                                                      DRKG
       PURPOSE                                                        DRKG
```

```
   TO SOLVE A SYSTEM OF FIRST ORDER ORDINARY DIFFERENTIAL      DRKG
   EQUATIONS WITH GIVEN INITIAL VALUES.                         DRKG
                                                                DRKG
USAGE                                                           DRKG
   CALL DRKGS (PRMT,Y,DERY,NDIM,IHLF,FCT,OUTP,AUX)              DRKG
   PARAMETERS FCT AND OUTP REQUIRE AN EXTERNAL STATEMENT.       DRKG
                                                                DRKG
DESCRIPTION OF PARAMETERS                                       DRKG
   PRMT    - DOUBLE PRECISION INPUT AND OUTPUT VECTOR WITH      DRKG
             DIMENSION GREATER THAN OR EQUAL TO 5, WHICH        DRKG
             SPECIFIES THE PARAMETERS OF THE INTERVAL AND OF    DRKG
             ACCURACY AND WHICH SERVES FOR COMMUNICATION BETWEENDRKG
             OUTPUT SUBROUTINE (FURNISHED BY THE USER) AND      DRKG
             SUBROUTINE DRKGS. EXCEPT PRMT(5) THE COMPONENTS    DRKG
             ARE NOT DESTROYED BY SUBROUTINE DRKGS AND THEY ARE DRKG
   PRMT(1)- LOWER BOUND OF THE INTERVAL (INPUT),                DRKG
   PRMT(2)- UPPER BOUND OF THE INTERVAL (INPUT),                DRKG
   PRMT(3)- INITIAL INCREMENT OF THE INDEPENDENT VARIABLE       DRKG
             (INPUT),                                           DRKG
   PRMT(4)- UPPER ERROR BOUND (INPUT). IF ABSOLUTE ERROR IS     DRKG
             GREATER THAN PRMT(4), INCREMENT GETS HALVED.       DRKG
             IF INCREMENT IS LESS THAN PRMT(3) AND ABSOLUTE     DRKG
             ERROR LESS THAN PRMT(4)/50, INCREMENT GETS DOUBLED.DRKG
             THE USER MAY CHANGE PRMT(4) BY MEANS OF HIS        DRKG
             OUTPUT SUBROUTINE.                                 DRKG
   PRMT(5)- NO INPUT PARAMETER. SUBROUTINE DRKGS INITIALIZES    DRKG
             PRMT(5)=0. IF THE USER WANTS TO TERMINATE          DRKG
             SUBROUTINE DRKGS AT ANY OUTPUT POINT, HE HAS TO    DRKG
             CHANGE PRMT(5) TO NON-ZERO BY MEANS OF SUBROUTINE  DRKG
             OUTP. FURTHER COMPONENTS OF VECTOR PRMT ARE        DRKG
             FEASIBLE IF ITS DIMENSION IS DEFINED GREATER       DRKG
             THAN 5. HOWEVER SUBROUTINE DRKGS DOES NOT REQUIRE  DRKG
             AND CHANGE THEM. NEVERTHELESS THEY MAY BE USEFUL   DRKG
             FOR HANDING RESULT VALUES TO THE MAIN PROGRAM      DRKG
             (CALLING DRKGS) WHICH ARE OBTAINED BY SPECIAL      DRKG
             MANIPULATIONS WITH OUTPUT DATA IN SUBROUTINE OUTP. DRKG
   Y       - DOUBLE PRECISION INPUT VECTOR OF INITIAL VALUES    DRKG
             (DESTROYED). LATERON Y IS THE RESULTING VECTOR OF  DRKG
             DEPENDENT VARIABLES COMPUTED AT INTERMEDIATE       DRKG
             POINTS X.                                          DRKG
   DERY    - DOUBLE PRECISION INPUT VECTOR OF ERROR WEIGHTS     DRKG
             (DESTROYED). THE SUM OF ITS COMPONENTS MUST BE     DRKG
             EQUAL TO 1. LATERON DERY IS THE VECTOR OF          DRKG
             DERIVATIVES, WHICH BELONG TO FUNCTION VALUES Y AT  DRKG
             INTERMEDIATE POINTS X.                             DRKG
   NDIM    - AN INPUT VALUE, WHICH SPECIFIES THE NUMBER OF      DRKG
             EQUATIONS IN THE SYSTEM.                           DRKG
   IHLF    - AN OUTPUT VALUE, WHICH SPECIFIES THE NUMBER OF     DRKG
             BISECTIONS OF THE INITIAL INCREMENT. IF IHLF GETS  DRKG
             GREATER THAN 10, SUBROUTINE DRKGS RETURNS WITH     DRKG
             ERROR MESSAGE IHLF=11 INTO MAIN PROGRAM. ERROR     DRKG
             MESSAGE IHLF=12 OR IHLF=13 APPEARS IN CASE         DRKG
```

```
              PRMT(3)=0 OR IN CASE SIGN(PRMT(3)).NE.SIGN(PRMT(2)-DRKG
              PRMT(1)) RESPECTIVELY.                              DRKG
   FCT      - THE NAME OF AN EXTERNAL SUBROUTINE USED. THIS       DRKG
              SUBROUTINE COMPUTES THE RIGHT HAND SIDES DERY OF    DRKG
              THE SYSTEM TO GIVEN VALUES X AND Y. ITS PARAMETER   DRKG
              LIST MUST BE X,Y,DERY. SUBROUTINE FCT SHOULD        DRKG
              NOT DESTROY X AND Y.                                DRKG
   OUTP     - THE NAME OF AN EXTERNAL OUTPUT SUBROUTINE USED.     DRKG
              ITS PARAMETER LIST MUST BE X,Y,DERY,IHLF,NDIM,PRMT,DRKG
              NONE OF THESE PARAMETERS (EXCEPT, IF NECESSARY,     DRKG
              PRMT(4),PRMT(5),...) SHOULD BE CHANGED BY           DRKG
              SUBROUTINE OUTP. IF PRMT(5) IS CHANGED TO NON-ZERO,DRKG
              SUBROUTINE DRKGS IS TERMINATED.                     DRKG
   AUX      - DOUBLE PRECISION AUXILIARY STORAGE ARRAY WITH 8     DRKG
              ROWS AND NDIM COLUMNS.                              DRKG
                                                                  DRKG
REMARKS                                                           DRKG
   THE PROCEDURE TERMINATES AND RETURNS TO CALLING PROGRAM, IF    DRKG
   (1) MORE THAN 10 BISECTIONS OF THE INITIAL INCREMENT ARE       DRKG
       NECESSARY TO GET SATISFACTORY ACCURACY (ERROR MESSAGE      DRKG
       IHLF=11),                                                  DRKG
   (2) INITIAL INCREMENT IS EQUAL TO 0 OR HAS WRONG SIGN          DRKG
       (ERROR MESSAGES IHLF=12 OR IHLF=13),                       DRKG
   (3) THE WHOLE INTEGRATION INTERVAL IS WORKED THROUGH,          DRKG
   (4) SUBROUTINE OUTP HAS CHANGED PRMT(5) TO NON-ZERO.           DRKG
                                                                  DRKG
SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED                     DRKG
   THE EXTERNAL SUBROUTINES FCT(X,Y,DERY) AND                     DRKG
   OUTP(X,Y,DERY,IHLF,NDIM,PRMT) MUST BE FURNISHED BY THE USER.DRKG
                                                                  DRKG
METHOD                                                            DRKG
   EVALUATION IS DONE BY MEANS OF FOURTH ORDER RUNGE-KUTTA        DRKG
   FORMULAE IN THE MODIFICATION DUE TO GILL. ACCURACY IS          DRKG
   TESTED COMPARING THE RESULTS OF THE PROCEDURE WITH SINGLE      DRKG
   AND DOUBLE INCREMENT.                                          DRKG
   SUBROUTINE DRKGS AUTOMATICALLY ADJUSTS THE INCREMENT DURING DRKG
   THE WHOLE COMPUTATION BY HALVING OR DOUBLING. IF MORE THAN     DRKG
   10 BISECTIONS OF THE INCREMENT ARE NECESSARY TO GET            DRKG
   SATISFACTORY ACCURACY, THE SUBROUTINE RETURNS WITH             DRKG
   ERROR MESSAGE IHLF=11 INTO MAIN PROGRAM.                       DRKG
   TO GET FULL FLEXIBILITY IN OUTPUT, AN OUTPUT SUBROUTINE        DRKG
   MUST BE FURNISHED BY THE USER.                                 DRKG
   FOR REFERENCE, SEE                                             DRKG
   RALSTON/WILF, MATHEMATICAL METHODS FOR DIGITAL COMPUTERS,      DRKG
   WILEY, NEW YORK/LONDON, 1960, PP.110-120.                      DRKG
                                                                  DRKG
..................................................................DRKG
                                                                  DRKG
SUBROUTINE DRKGS(PRMT,Y,DERY,NDIM,IHLF,FCT,OUTP,AUX)              DRKG
                                                                  DRKG
                                                                  DRKG
```

```
      DIMENSION Y(1),DERY(1),AUX(8,1),A(4),B(4),C(4),PRMT(1)
      DOUBLE PRECISION PRMT,Y,DERY,AUX,A,B,C,X,XEND,H,AJ,BJ,CJ,R1,R2,
     1DELT
      DOUBLE PRECISION DABS
      DO 1 I=1,NDIM
    1 AUX(8,I)=.066666666666666667D0*DERY(I)
      X=PRMT(1)
      XEND=PRMT(2)
      H=PRMT(3)
      PRMT(5)=0.D0
      CALL FCT(X,Y,DERY)

      ERROR TEST
      IF(H*(XEND-X))38,37,2

      PREPARATIONS FOR RUNGE-KUTTA METHOD
    2 A(1)=.5D0
      A(2)=.29289321881345248D0
      A(3)=1.7071067811865475D0
      A(4)=.16666666666666667D0
      B(1)=2.D0
      B(2)=1.D0
      B(3)=1.D0
      B(4)=2.D0
      C(1)=.5D0
      C(2)=.29289321881345248D0
      C(3)=1.7071067811865475D0
      C(4)=.5D0

      PREPARATIONS OF FIRST RUNGE-KUTTA STEP
      DO 3 I=1,NDIM
      AUX(1,I)=Y(I)
      AUX(2,I)=DERY(I)
      AUX(3,I)=0.D0
    3 AUX(6,I)=0.D0
      IREC=0
      H=H+H
      IHLF=-1
      ISTEP=0
      IEND=0


      START OF A RUNGE-KUTTA STEP
    4 IF((X+H-XEND)*H)7,6,5
    5 H=XEND-X
    6 IEND=1

      RECORDING OF INITIAL VALUES OF THIS STEP
    7 CALL OUTP(X,Y,DERY,IREC,NDIM,PRMT)
      IF(PRMT(5))40,8,40
    8 ITEST=0
    9 ISTEP=ISTEP+1
```

```
   START OF INNERMCST RUNGE-KUTTA LOOP
   J=1
10 AJ=A(J)
   BJ=B(J)
   CJ=C(J)
   DO 11 I=1,NDIM
   R1=H*DERY(I)
   R2=AJ*(R1-BJ*AUX(6,I))
   Y(I)=Y(I)+R2
   R2=R2+R2+R2
11 AUX(6,I)=AUX(6,I)+R2-CJ*R1
   IF(J-4)12,15,15
12 J=J+1
   IF(J-3)13,14,13
13 X=X+.5D0*H
14 CALL FCT(X,Y,DERY)
   GOTO 10
   END OF INNERMOST RUNGE-KUTTA LCOP


   TEST OF ACCURACY
15 IF(ITEST)16,16,20

   IN CASE ITEST=0 THERE IS NO POSSIBILITY FOR TESTING OF ACCURACY
16 DO 17 I=1,NDIM
17 AUX(4,I)=Y(I)
   ITEST=1
   ISTEP=ISTEP+ISTEP-2
18 IHLF=IHLF+1
   X=X-H
   H=.5D0*H
   DO 19 I=1,NDIM
   Y(I)=AUX(1,I)
   DERY(I)=AUX(2,I)
19 AUX(6,I)=AUX(3,I)
   GOTO 9

   IN CASE ITEST=1 TESTING OF ACCURACY IS POSSIBLE
20 IMOD=ISTEP/2
   IF(ISTEP-IMOD-IMOD)21,23,21
21 CALL FCT(X,Y,DERY)
   DO 22 I=1,NDIM
   AUX(5,I)=Y(I)
22 AUX(7,I)=DERY(I)
   GOTO 9

   COMPUTATION OF TEST VALUE DELT
23 DELT=0.D0
   DO 24 I=1,NDIM
24 DELT=DELT+AUX(8,I)*DABS(AUX(4,I)-Y(I))
```

```
      IF(DELT-PRMT(4))28,28,25
C
C     ERROR IS TOO GREAT
   25 IF(IHLF-10)26,36,36
   26 DO 27 I=1,NDIM
   27 AUX(4,I)=AUX(5,I)
      ISTEP=ISTEP+ISTEP-4
      X=X-H
      IEND=0
      GOTO 18
C
C     RESULT VALUES ARE GOOD
   28 CALL FCT(X,Y,DERY)
      DO 29 I=1,NDIM
      AUX(1,I)=Y(I)
      AUX(2,I)=DERY(I)
      AUX(3,I)=AUX(6,I)
      Y(I)=AUX(5,I)
   29 DERY(I)=AUX(7,I)
      CALL OUTP(X-H,Y,DERY,IHLF,NDIM,PRMT)
      IF(PRMT(5))40,30,40
   30 DO 31 I=1,NDIM
      Y(I)=AUX(1,I)
   31 DERY(I)=AUX(2,I)
      IREC=IHLF
      IF(IEND)32,32,39
C
C     INCREMENT GETS DOUBLED
   32 IHLF=IHLF-1
      ISTEP=ISTEP/2
      H=H+H
      IF(IHLF)4,33,33
   33 IMOD=ISTEP/2
      IF(ISTEP-IMOD-IMOD)4,34,4
   34 IF(DELT-.02D0*PRMT(4))35,35,4
   35 IHLF=IHLF-1
      ISTEP=ISTEP/2
      H=H+H
      GOTO 4
C
C
C     RETURNS TO CALLING PROGRAM
   36 IHLF=11
      CALL FCT(X,Y,DERY)
      GOTO 39
   37 IHLF=12
      GOTO 39
   38 IHLF=13
   39 CALL OUTP(X,Y,DERY,IHLF,NDIM,PRMT)
   40 RETURN
      END
```

-**--**--**--**--**-

APPENDIX C

A SAMPLE OF COMPUTOR OUTPUT

A sample of the computor output for a generalized case of hub-attached dual wheel farm tractors follows:

Time (sec.)	Resultant Bending Stress (psi)
0.00	0.00
0.025	9,874.00
0.050	12,490.00
0.075	11,240.00
0.100	10,510.00
0.125	7,590.00
0.150	6,445.00
0.175	7,945.00
0.200	8,352.00
0.225	7,288.00
0.250	6,508.00
0.275	5,677.00
0.300	4,203.00
0.325	3,198.00
0.350	3,284.00
0.375	5,162.00
0.400	6,413.00
0.425	7,209.00
0.450	7,224.00
0.475	6,307.00
0.500	4,892.00
0.525	3,484.00
0.550	3,052.00
0.575	3,681.00
0.600	4,246.00
0.625	4,129.00
0.650	3,270.00
0.675	2,018.00
0.700	8,002.00
0.725	13,560.00
0.750	14,510.00
0.775	11,650.00
0.800	9,646.00

Time	Resultant Bending Stress
0.825	5,512.00
0.850	5,239.00
0.875	6,051.00
0.900	7,556.00
0.925	7,312.00
0.950	6,948.00
0.975	6,535.00
1.000	4,780.00

TABLE VI

Motion in First Eight Coordinates for a Generalized Case of Dual Wheel Tractor (Hub Attached)

(Time)	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0025	0.255D-07	0.310D-06	-0.107D-05	0.116D-05	-0.312D-06	0.143D-06	-0.243D-07	0.0
0.025	0.403D-03	0.475D-03	-0.216D-02	0.142D-02	-0.530D-03	0.102D-03	-0.479D-04	-0.121D-03
0.050	0.244D-02	0.334D-02	-0.112D-01	0.618D-02	-0.269D-02	0.196D-03	-0.260D-03	-0.103D-02
0.100	0.158D-01	0.265D-01	-0.495D-01	0.199D-01	-0.116D-01	-0.178D-02	-0.319D-03	-0.558D-02
0.150	0.445D-01	0.742D-01	-0.925D-01	0.336D-01	-0.218D-01	-0.536D-02	0.188D-02	-0.154D-02
0.200	0.822D-01	0.123D 00	-0.115D 00	0.481D-01	-0.297D-01	-0.531D-02	0.592D-02	0.217D-01
0.250	0.113D 00	0.150D 00	-0.118D 00	0.566D-01	-0.354D-01	-0.330D-02	0.936D-02	0.459D-01
0.300	0.124D 00	0.149D 00	-0.121D 00	0.577D-01	-0.389D-01	-0.316D-02	0.953D-02	0.513D-01
0.350	0.114D 00	0.128D 00	-0.129D 00	0.557D-01	-0.386D-01	-0.513D-02	0.639D-02	0.387D-01
0.400	0.927D-01	0.107D 00	-0.132D 00	0.516D-01	-0.347D-01	-0.705D-02	0.322D-02	0.207D-01
0.450	0.712D-01	0.982D-01	-0.119D 00	0.466D-01	-0.295D-01	-0.640D-02	0.285D-02	0.981D-02
0.500	0.572D-01	0.984D-01	-0.962D-01	0.414D-01	-0.246D-01	-0.360D-02	0.478D-02	0.985D-02
0.550	0.505D-01	0.923D-01	-0.715D-01	0.341D-01	-0.198D-01	-0.143D-02	0.621D-02	0.150D-01
0.600	0.442D-01	0.678D-01	-0.479D-01	0.238D-01	-0.143D-01	-0.107D-02	0.500D-02	0.177D-01
0.650	0.304D-01	0.260D-01	-0.214D-01	0.101D-01	-0.746D-02	-0.134D-02	0.162D-02	0.143D-01
0.700	0.595D-02	-0.199D-01	0.845D-02	-0.477D-02	0.629D-03	-0.607D-03	-0.183D-02	0.508D-02
0.750	-0.209D-01	-0.480D-01	0.184D-01	-0.892D-02	0.503D-02	0.105D-02	-0.417D-02	-0.951D-02
0.800	-0.343D-01	-0.363D-01	0.234D-02	-0.649D-02	0.430D-02	-0.229D-03	-0.395D-02	-0.255D-01
0.850	-0.253D-01	0.301D-02	-0.120D-01	-0.340D-03	0.229D-02	-0.175D-02	-0.779D-03	-0.235D-01
0.900	-0.874D-03	0.345D-01	-0.694D-02	0.653D-02	0.243D-03	0.640D-04	0.314D-02	-0.318D-03
0.950	0.217D-01	0.366D-01	0.574D-02	0.616D-02	-0.229D-02	0.207D-02	0.475D-02	0.208D-01
1.000	0.283D-01	0.104D-01	0.786D-02	0.137D-02	-0.383D-02	0.161D-02	0.228D-02	0.236D-01
1.050	0.186D-01	-0.191D-01	0.430D-03	-0.264D-02	-0.290D-02	-0.493D-03	-0.184D-02	0.118D-01
1.100	-0.252D-02	-0.333D-01	-0.596D-02	-0.497D-02	0.170D-03	-0.239D-02	-0.457D-02	-0.691D-02
1.150	-0.194D-01	-0.174D-01	-0.264D-02	-0.304D-02	0.267D-02	-0.127D-02	-0.271D-02	-0.164D-01

(Time)	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
1.200	-0.218D-01	0.102D-01	0.279D-02	0.925D-03	0.301D-02	0.114D-02	0.132D-02	-0.132D-01
1.250	-0.108D-01	0.268D-01	0.302D-02	0.372D-02	0.161D-02	0.198D-02	0.371D-02	-0.347D-02
1.300	0.544D-02	0.204D-01	-0.394D-03	0.64D-02	-0.560D-03	0.721D-03	0.264D-02	0.562D-02
1.350	0.166D-01	-0.119D-02	-0.192D-02	0.740D-03	-0.228D-02	-0.932D-03	-0.393D-03	0.992D-02
1.400	0.166D-01	-0.191D-01	-0.364D-03	-0.262D-02	-0.247D-02	-0.128D-02	-0.253D-02	0.896D-02
1.450	0.670D-02	-0.200D-01	0.109D-02	-0.361D-02	-0.103D-02	-0.384D-03	-0.232D-02	0.386D-02
1.500	-0.619D-02	-0.530D-02	0.484D-03	-0.161D-02	0.903D-03	0.489D-03	-0.425D-03	-0.334D-02
1.550	-0.144D-01	0.128D-01	-0.825D-03	0.179D-02	0.202D-02	0.580D-03	0.152D-02	-0.903D-02
1.600	-0.126D-01	0.175D-01	-0.752D-03	0.312D-02	0.172D-02	0.266D-03	0.191D-02	-0.853D-02
1.650	-0.365D-02	0.922D-02	0.246D-03	0.199D-02	0.513D-03	-0.105D-04	0.101D-02	0.262D-02
1.700	0.663D-02	-0.509D-02	0.792D-03	-0.594D-03	-0.832D-03	-0.142D-03	-0.485D-03	0.457D-02
1.750	0.118D-01	-0.138D-01	0.363D-03	-0.235D-02	-0.158D-02	-0.274D-03	-0.149D-02	0.792D-02
1.800	0.921D-02	-0.109D-01	-0.306D-03	-0.202D-02	-0.133D-02	-0.331D-03	-0.132D-02	0.581D-02
1.850	0.138D-02	0.715D-04	-0.461D-03	-0.167D-03	-0.282D-03	-0.104D-03	-0.152D-03	0.631D-03
1.900	-0.641D-02	0.974D-02	-0.158D-03	0.153D-02	0.858D-03	0.308D-03	0.107D-02	-0.399D-02
1.950	-0.950D-02	0.108D-01	0.113D-03	0.182D-02	0.135D-02	0.487D-03	0.136D-02	-0.574D-02
2.000	-0.655D-02	0.335D-02	0.131D-03	0.701D-03	0.949D-03	0.192D-03	0.531D-03	-0.421D-02

NOTE: Move decimal point to the left by the number of places given after D.

APPENDIX D

METHODS OF EVALUATING VARIOUS VEHICLE CONSTANTS

Determination of Dynamic Characteristics of Tires

It is known from the experiments that tire spring rate is non-linear. Similarly, the damping coefficients of tires are not entirely a linear function of velocity. However, consideration of tires as a mass connected to the surface traversed by a system of linear springs and viscous dampers gives quite accurate results. Thus, the tire measurements to determine the parameters are needed. Tractor tires have spring rates and damping coefficients in three directions:

1) Vertical direction
2) Lateral direction
3) Fore and aft direction

Stiffness and viscous damping constants vary significantly in the three directions. Therefore, it is necessary to determine these parameters separately for all three directions. Methods for measuring these parameters in the vertical direction are described below. The same methods could be used to determine the parameters in the other directions.

Equipment

To determine the tire spring rates and damping constants, Pershing mounted the tractor wheels rigidly on a long arm which pivoted from a fixed vertical member. The wheel was loaded for dynamic measurement by clamping weights to the wheel mounting structure. The supported weight was then varied from zero to 3000 lb of load. Static deflection was recorded by means of potentiometer displacement transducers. Dynamic transient recording was made with a carrier-amplifier and galvanometer recorder.

Calculation of Dynamic Characteristics of Tires

To determine the tire constants, it is necessary to make the following assumptions:

1) The system can be treated as a single degree of freedom model with a concentrated mass, stiffness coefficient, and damping coefficient.

2) The non-linear stiffness of the tire may be approximated by an equivalent linear value.

3) Damping is small and purely a function of velocity.

4) Tire oscillation is small and is always in contact with the ground.

The equation of motion of a free vibrating body having mass, spring, and damping is represented by:

$$M\ddot{x} + C\dot{x} + Kx = 0 \quad (1)$$

$$R \sim 0.1\ (1 - R^2) \approx \rho$$

Thus, the stiffness coefficient K can be obtained from

$$K = (2\pi\ F_R)^2 M$$

The damping ratio R is small to allow transient motion.

$$C < 2\sqrt{K_m}$$

$$\frac{C}{C_c} < 1$$

Therefore, the solution of the above equation of motion will be

$$x = B\ e^{(-c/2m)}t\cos(\omega t + \phi)$$

Where B and ϕ are constant, the solution can be found by utilizing the initial conditions:

ω = angular frequency

With this the peak displacement at A_n, A_{n+1} is given by

$$x_n = B\ e^{-c/2mt_n}$$

$$x_{n+1} = B\ e^{-c/2mt_{n+1}}$$

Since

$$[\cos(\ t_n + \phi) = \cos(\ t_{n+1} + \phi) = 1 \text{ at peaks}]$$

the ratio of two consecutive amplitudes is:

$$\frac{x_n}{x_{n+1}} = \frac{e^{-c/2Mt_n}}{e^{-c/2Mt_{n+1}}} = e^{ct/2M}$$

$$\therefore C = \frac{2M}{T} \log_e \frac{x_n}{x_{n+1}}$$

where

x = displacement

C = damping coefficient of the tire

K = tire stiffness

M = mass supported by tire

T = period of one cycle of oscillation

The transient vibration frequency F(r) is

$$F_R = \frac{1}{2}\sqrt{\frac{K(1 - R^2)}{M}}$$

$$R = \text{damping ratio} = C/2\sqrt{KM}$$

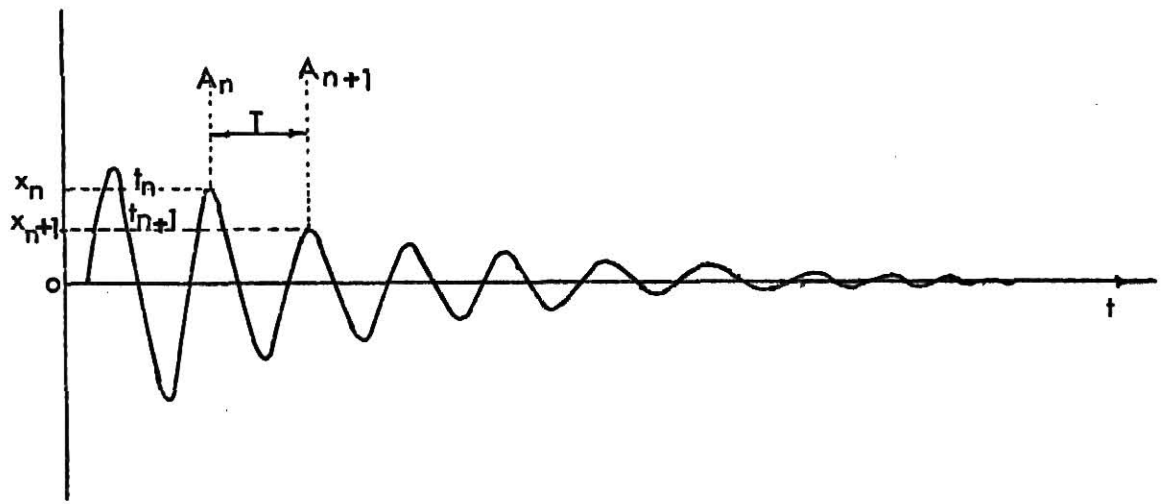

Figure 16. Transient vibration of tires.

Moment of Inertia Determinations

Moment of inertia of the tractor about the three different axes (pitch, roll, and yaw) through the center of gravity of the vehicle can be obtained by several techniques. Several of these methods are described below:

Pitch moment of inertia

This constant can be determined from the period of swing of the tractor when it is mounted on a suspended platform. The radius of gyration of the platform itself would have to be determined separately so that it could be deducted.

The period of swing is obtained by means of an electronic timing apparatus.

Calculation for moment of inertia

The period of swing (T) of a rigid compound pendulum system is given by

$$T = 2\pi\sqrt{\frac{h^2 + k^2}{h\ g}}$$

where

h = distance from the pivot to center of gravity (C.G.) of the complete system

k = radius of gyration of the system

Length (L) of equivalent single pendulum is given by

$$T = 2\pi\sqrt{\frac{L}{g}}$$

$$\therefore L = \frac{h^2 + k^2}{h}$$

or

$$h^2 - h + k^2 = 0$$

$$h = \frac{\ell \pm \sqrt{\ell^2 - 4k^2}}{2}$$

or

$$h_1 = \frac{\ell + \sqrt{\ell^2 - 4k^2}}{2}$$

$$h_2 = \frac{\ell - \sqrt{\ell^2 - 4k^2}}{2}$$

and

$$h_1 \times h_2 = k^2$$

A graph plotting of h vs. T will give values of h_1 and h_2 for any period T.

Let k be the radius of gyration of the tractor and the platform over which it has been mounted, and

J = pitch moment of inertia of the tractor about its center of gravity

k_1 = radius of gyration of the platform

w_1 = weight of the platform

w_2 = weight of the tractor

d_1 = distance from the pivot to C.G. of the platform (C_1)

d_2 = distance from the pivot to C.G. of the tractor (C_2)

C = position of the combined C.G. of tractor and platform

then

$$(w_1 + w_2)k^2 = J + w_1 k_1{}^2 + w_2 \text{ (distance } CC_1)^2$$
$$+ w_1 \text{ (distance } C_2C)^2$$

therefore

$$(w_1 + w_2)k^2 = J + w_1 k_1{}^2 + \frac{w_2 w_1{}^2 (d_1 - d_2)^2}{(w_1 + w_2)^2} + \frac{w_1 w_1{}^2 (d_1 - d_2)^2}{(w_1 + w_2)^2}$$

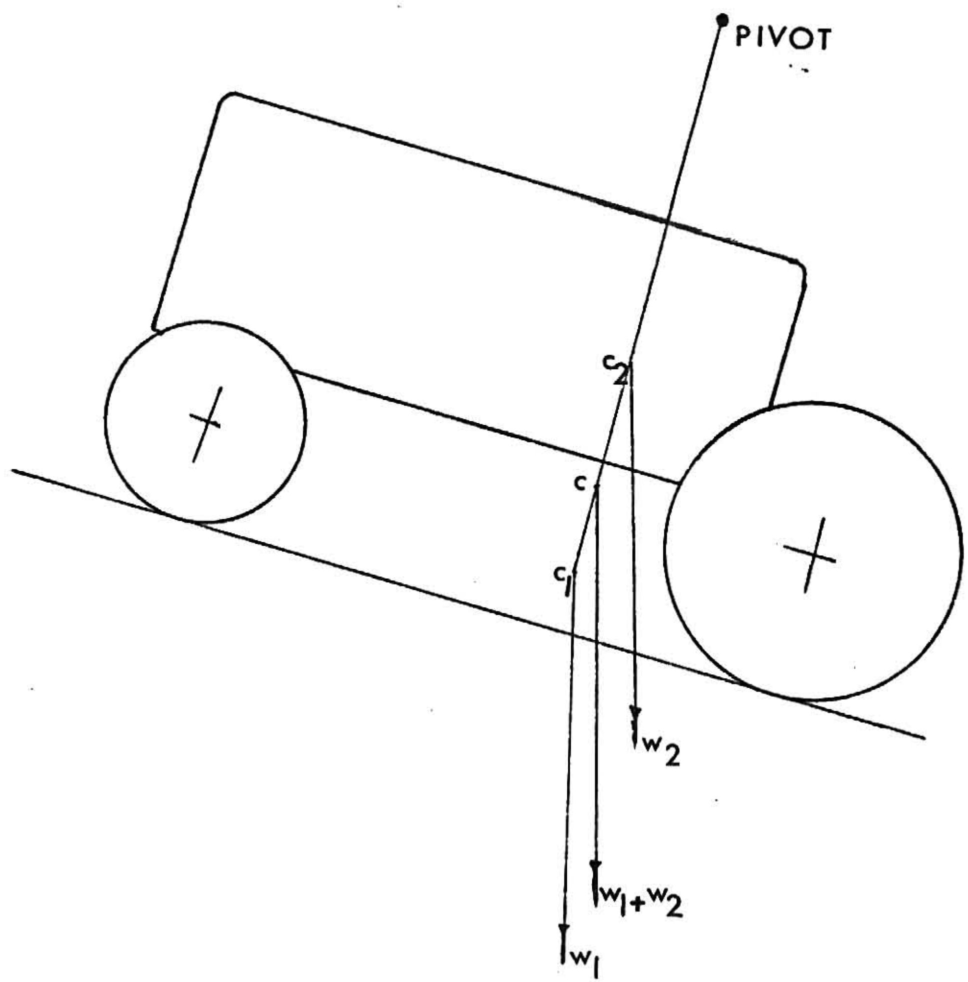

Figure 17. Determination of moment of intertia by suspension method.

from this

$$J = (w_1+w_2)k^2 - \frac{w_1k_1{}^2(w_1+w_2) + w_1w_2(d_1-d_2)^2}{w_1+w_2}$$

Pitch moment of inertia of vehicles can also be calculated using the natural frequencies of vibration. This method is as follows.

Moment of Inertia by Vibration Method

Let

m = mass of complete vehicle

I_θ = mass moment of inertia about a traverse axis through C.G.

y = vertical displacement of the C.G.

θ = pitch about the C.G.

ϕ = roll around the C.G.

k_1 = spring rate of left front tires

k_2 = spring rate of right front tires

k_3 = spring rate of left rear wheel

k_4 = spring rate of right rear wheel

C_1 = damping coefficient of left front wheel

C_2 = damping coefficient of right front wheel

C_3 = damping coefficient of left rear wheel

C_4 = damping coefficient of right rear wheel

a = distance from C.G. to front axle

b = distance from C.G. to rear axle

c = distance from C.G. to center of left rear tire

d = distance from C.G. to center of right rear tire

w_y = natural angular frequency associated with the y

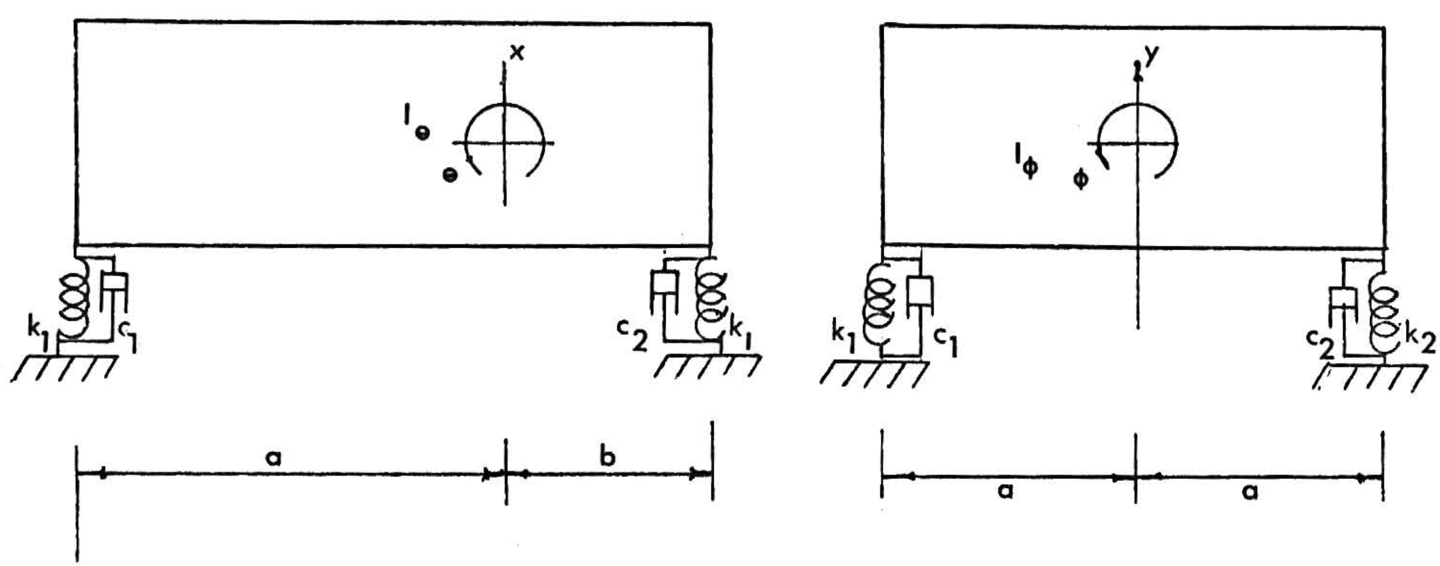

Figure 18. Mathematical representation of vehicle side and rear view for pitching and rolling modes.

coordinate

w_θ = natural angular frequency associated with the θ coordinate

w_ϕ = natural angular frequency associated with the ϕ coordinate

By referring to Figure 18, the equations of motion (neglecting damping for practical purposes) are as follows:

$$m\ddot{x} + (k_1 + k_2)x - (k_1 a - k_2 b)\theta = 0$$

$$I_\theta \ddot{\theta} + (k_1 a^2 + k_2 b^2)\theta - (k_1 L_1 - k_2 L_2)x = 0$$

by letting

$$x = A \sin wt$$

$$\theta = B \sin wt$$

$$-mw^2 A \sin wt + (k_1 + k_2)A \sin wt - (k_1 a - k_2 b)B \sin wt = 0$$

$$-IBw^2 \sin wt + (k_1 a^2 + k_2 b^2)B \sin wt - (k_1 a_1 - k_2 b_2) \cdot$$

$$A \sin wt = 0$$

$$[-mw^2 + k_1 + k_2]A - (k_1 a - k_2 b)B = 0$$

$$[-I_\theta w^2 + k_1 a^2 + k_2 b^2]B - (k_1 a - k_2 b)A = 0$$

$$\begin{bmatrix} k_1 + k_2 - mw^2 & -(k_1 a - k_2 b) \\ -(k_1 a - k_2 b) & k_1 a^2 + k_2 b^2 - I_\theta w^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0$$

For non-trivial solution

$$\begin{bmatrix} k_1 + k_2 - mw^2 & k_2 b - k_1 a \\ k_2 b - k_1 a & k_1 a^2 + k_2 b^2 - I_\theta w^2 \end{bmatrix} = 0$$

$$(k_1 + k_2 - mw^2)(k_1 a^2 + k_2 b^2 - I_\theta w^2) - (k_2 b - k_1 a)^2 = 0$$

$$(k_1 + k_2 - mw^2)(k_1a^2 + k_2b^2 - I_\theta w^2) - (k_2b - k_1a)^2 = 0$$

This is a frequency equation for the resulting two degrees of freedom problem. Either of the two natural frequencies can be used to find the I_θ in this equation. Since the damping is small in the system, the damped and undamped natural frequencies can be assumed equal for practical purposes.

By referring to Figure , the equation of motion is:

$$I_\phi\ddot{\phi} + (k_1 + k_2)a^2\phi = 0$$

Since there is no coupling, the equation of motion due to freedom in the y direction can be neglected.

Let

$$\phi = A \sin wt$$

Therefore

$$-I_\phi Aw^2 \sin wt + (k_1 + k_2)a^2 A \sin wt = 0$$

or

$$(k_1 + k_2)a^2 - I_\phi w^2 = 0$$

So, by using natural frequency due to roll, I_ϕ can be determined.

ACKNOWLEDGMENTS

The author wishes to express his sincere gratitude and thanks to his major professor, Dr. Stanley J. Clark, for his challenging guidance and continuous encouragement throughout this study.

The author also gratefully acknowledges Dr. G. H. Larson, Head of the Department of Agricultural Engineering, and the members of his graduate committee, Professor Gustave E. Fairbanks and Dr. Hugh S. Walker, for their counseling and guidance.

AN ANALYSIS OF STRESS VARIATION IN TRACTOR AXLES BY SIMULATION METHODS

by

NAVIN PRAKASH MATHUR

B.Sc. (Ag. Engg.), Allahabad Agricultural Institute, India, 1966

AN ABSTRACT OF A MASTER'S THESIS

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Agricultural Engineering

KANSAS STATE UNIVERSITY

Manhattan, Kansas

1970

ABSTRACT

This analysis presents the method of formulating stress equations for rear axles of conventional single and dual wheel farm tractors in order to investigate the transient variation of bending stresses. The mathematical models for single wheel, individual-hub-attached, and rim-to-rim attached dual wheel rear axles were derived and equations were developed with the help of the energy method. The derivation included the effect of tractor vibration for bounce, pitch, roll, and yaw coordinates about which the tractor may possibly vibrate while it traverses a particular sinusoidal bump. The cases in which the left outer wheels hit the bump were also included in this study to find the effect on axle stress.

The stress response curves for all the different models using a standard set of conditions for an IHC-340 utility tractor were computed. The Runge-Kutta method of numerical integration was used to solve the sets of simultaneous differential equations of the system.

The analysis showed that the maximum bending stress in the rear axle of an individual-hub-attached dual wheel was twice as high as that of the single wheel tractor. However, the stress in rim-to-rim attached dual wheels on the rear axle was only 50 percent greater than that of single wheels. The solution indicated that the stresses were significantly higher at two different points in the time domain during which the tractor traverses over the bump and returns to a steady-state condition. The stress level

in the rear axle of individual-hub-attached dual wheels was quite high in comparison to rim-to-rim attached dual wheels when the tractor hit the bump with only the outer dual wheels. It was also observed that the axle material properties and area moment of inertia greatly influence the stress variation.

A comparison of all the solutions showed that wheel configuration and the manner in which the wheel hits the bump change the stress situation significantly.