Reciprocality of p-modulus and consequences in metric spaces

by

Jared Edward Hoppis

B.S., Wichita State University, 2012

AN ABSTRACT OF A DISSERTATION

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Abstract

Arne Beurling first studied extremal length, namely the reciprocal of $2-$Modulus, in the plane, and then later studied it jointly with Lars Ahlfors. Beurling was interested in extremal length because he wanted a conformal invariant to study harmonic measure. One of the differences between $\mathbb{R}^2$ and $\mathbb{R}^N$ with $N \geq 3$, is that there are far fewer conformal maps in the latter case. This naturally suggests defining a larger class of functions that distort N-Modulus by a bounded amount. This gives rise to the notion of quasiconformal mappings, see [1].

There have been many recent developments in the discrete theory of p-Modulus, and a natural question is "Can the discrete theory tell us anything about the continuous theory?" There are two ways to try and answer this question. The first is to approximate a domain with a mesh of points and study if discrete p-Modulus of families of walks on the mesh converges to continuous p-Modulus on the domain. This line of inquiry has been pursued in the recent literature [10, 11, 21, 25, 12]. The second way to answer the question is to try to come up with a dictionary of results by developing a way to pair up results for the discrete theory and the continuous theory. This is where this thesis is developed. In [6], with Nathan Albin, Pietro Poggi-Corradini, and Nageswari Shanmugalingam, we establish a relationship between ∞-Modulus of a family of paths connecting two points in general metric spaces and the "essential" shortest path metric between two points. This result is inspired by a similar relationship in the discrete setting established in [5]. In [4] N. Albin, Jason Clemens, Nethali Fernando, and P. Poggi-Corradini show that p-Modulus, $1 \leq p < \infty$, can be related to other metrics. Using the work of Aikawa and Ohtsuka, we show that a similar modulus metric can be defined, with some slight modification, in $\mathbb{R}^2$, for $2 < p < \infty$. Note that, in the continuous setting, with N-dimensional Lebesgue measure, we cannot hope to get a metric for $1 \leq p \leq N$ because for these values the p-Modulus of the family of curves connecting

two distinct points is zero. We are currently working to adapt the argument to dimension $N \geq 3$ and in metric measure spaces (X, d, μ) where μ is a Borel regular measure.

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Acknowledgments

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Dedication

I would like to dedicate this dissertation to all of the faculty and students that suffer from mental health problems. They are all too common, and yet rarely discussed. To those that suffer from them, know that you are not alone, and that is not a sign of weakness to seek help.

Introduction

The first question this thesis seeks to answer is “What is p-Modulus?” We answer this question in each chapter in a different setting. In Chapter 1, we look at the first setting in which p-Modulus was studied, that is, the complex plane. We compute p-Modulus in a basic example, and consider Beurling's criterion as an important historical remark. In Chapter 2, we explore p-Modulus on discrete graphs with $1 \leq p \leq \infty$, specifically Fulkerson duality and some related results. The discrete case is simpler in many ways than the continuous setting or more general metric measure spaces. As such, we wish to establish a sort of “dictionary” of results as below.

Discrete	**Generalization**
$\text{Mod}_{\infty,\sigma}(\Gamma) = \frac{1}{\ell(\Gamma)}$	$\text{Mod}_{\infty}(\Gamma) = \frac{1}{\text{ess}\ell(\Gamma)}$
$\lim_{p\to\infty} \delta_p(a,b) = \ell(\Gamma(a,b))$	$d_{\text{ess}}(x,y) = \text{ess}\ell(\Gamma(x,y))$
$\text{Mod}_{p,\sigma}(\Gamma)^{1/p}\,\text{Mod}_{q,\hat{\sigma}}(\hat{\Gamma})^{1/q} = 1$	$\text{Mod}_p(\lvert d\Gamma\rvert)^{1/p}\,\text{Mod}_q(\lvert\nabla\Sigma\rvert)^{1/q} = 1$
For every $1 < p < \infty$, δ_p is a metric.	For $p > 2$, δ_p is a pseudometric on $\mathbb{R}^2$.

In Chapter 3, we explore the case $p = \infty$ in general metric measure spaces and some corresponding background material. In Chapter 4, we discuss Aikawa-Ohtsuka duality as a generalization of the reciprocal relationship found in Chapter 2. Finally, in Chapter 5, we use the Aikawa-Ohtsuka duality to develop some analogous results to those found in Chapter 2 in the continuous setting.

Chapter 1

Modulus in the Plane

The purpose of this chapter is to recall the historical case of Modulus of curves in the plane. In this section, we will introduce Modulus of families of curves in $\mathbb{C}$, and work through the classical example of computing the Modulus of the family of curves connecting two sides of a rectangle as well as show a proof of Beurling's Criterion. Throughout, Ω will be an open connected set in $\mathbb{C}$. For now, a curve is simply a function $\gamma : [a, b] \to \mathbb{C}$ which is continuous and piecewise $C^1([a, b])$. We will also only consider non-constant curves which are rectifiable, meaning their arc-length is finite. That is,

$$\ell(\gamma) := \int_a^b |\gamma'(t)| \, dt < \infty.$$

There is a reason we only consider non-constant rectifiable curves, and we will mention why once we define Modulus. In order to define Modulus, we first define a notion of admissibility.

Definition 1.0.1. Let Γ be a family of curves in Ω. Define the *admissible set* of Γ as

$$\mathrm{Adm}(\Gamma) := \left\{ \rho : \Omega \to [0, \infty) \, : \, \rho \text{ is Borel}, \int_\gamma \rho \, ds = \int_a^b \rho(\gamma(t))|\gamma'(t)| \, dt \geq 1 \right\}.$$

We refer to the Borel functions ρ as *densities*. For $\gamma \in \Gamma$, we also say the *ρ-length* of γ is $\ell_\rho(\gamma) = \int_\gamma \rho \, ds$. We now define p-Modulus for $1 \leq p < \infty$.

Definition 1.0.2. If $\mathrm{Adm}(\Gamma) \neq \varnothing$, then we define *p-Modulus* by

$$\mathrm{Mod}_p(\Gamma) = \inf_{\mathrm{Adm}(\Gamma)} \int_\Omega \rho(z)^p \, dA, \tag{1.0.1}$$

where dA is area measure, and if $\mathrm{Adm}(\Gamma) = \varnothing$, then we define $\mathrm{Mod}_p(\Gamma) = \infty$.

Sometimes we refer to the quantity $\int_\Omega \rho(z)^p \, dA$ as the *p-energy* of ρ, denoted $\mathcal{E}_p(\rho)$ or $\|\rho\|_p^p$. Also, historically, the case $p = 2$ was first because in that case, 2-Modulus is a conformal invariant. That is, if $f : \Omega \to \Omega'$ is a conformal map, then $\mathrm{Mod}_2(f(\Gamma)) = \mathrm{Mod}_2(\Gamma)$. Recall, a map f is conformal if it is holomorphic and one-to-one. See [1] for more details.

Now that we have defined p-Modulus, let us explain why we only consider non-constant rectifiable curves. If Γ contains a constant function γ_0, the $\ell(\gamma_0) = 0$, and so the admissible set is empty. Now, if Γ is the collection of all non-rectifiable curves and Ω is bounded, then letting $\rho_\varepsilon \equiv \varepsilon \in \mathrm{Adm}(\Gamma)$ for any $\varepsilon > 0$, and so $\mathrm{Mod}_p(\Gamma) \leq \varepsilon^p A(\Omega) \to 0$ as $\varepsilon \to 0$. If Ω is unbounded, then let $\rho \in L^p(\Omega)$ such that $\rho > 0$ and set $\rho_\varepsilon = \varepsilon\rho$. Then $\rho_\varepsilon \in \mathrm{Adm}(\Gamma)$ and $\mathrm{Mod}_p(\Gamma) \leq \varepsilon^p \|\rho\|_p^p \to 0$ as $\varepsilon \to 0$. We now move to our first nontrivial example of p-Modulus.

Example 1.0.3. Let Γ be the family of curves connecting the red side to the blue side in the figure above. First we will find an upper bound for $\mathrm{Mod}_p(\Gamma)$ by evaluating the p-Energy of an appropriate ρ. Let $\rho \equiv 1/W$. Then clearly $\rho \in \mathrm{Adm}(\Gamma)$ since $\int_\gamma \rho\, ds = (1/W)\ell(\gamma) \geq 1$. Then $\mathrm{Mod}_p(\Gamma) \leq \int_\Omega \rho(z)^p\, dA = (1/W)^p(HW) = H/W^{p-1}$.

Now, we will show that H/W^{p-1} is also a lower bound. To do so, we start with the admissibility condition. If $\rho \in \mathrm{Adm}(\Gamma)$, then in particular, for the straight lines $\gamma(t) = t + iy$ defined on $[0, W]$, we have

$$1 \leq \int_0^W \rho(t + iy) \cdot 1\, dt.$$

Applying Hölder's inequality, we obtain

$$1 \leq \left(\int_0^W \rho(t + iy)^p\, dt \right)^{1/p} W^{1/q},$$

where q is the Hölder conjugate of p, that is $1/p + 1/q = 1$. Rearranging, we get $W^{-p/q} \leq \int_0^W \rho(t + iy)^p\, dt$. Now, integrating both sides with respect to y from 0 to H, we obtain (and using $-p/q = 1 - p$)

$$HW^{1-p} \leq \int_0^H \int_0^W \rho(t + iy)^p\, dt\, dy = \|\rho\|_p^p\,.$$

Taking an infimum with respect to ρ, we obtain $H/W^{p-1} \leq \mathrm{Mod}_p(\Gamma)$. Thus, we have $\mathrm{Mod}_p(\Gamma) = H/W^{p-1}$.

Now, we wish to point out that the same formula applies to find the q-Modulus of the family of curves connecting the top to the bottom (denoted $\hat{\Gamma}$). Then we have $\mathrm{Mod}_q(\hat{\Gamma}) = W/H^{q-1}$. Then, computing, we have the following

$$\begin{aligned}\mathrm{Mod}_p(\Gamma)^{1/p}\, \mathrm{Mod}_q(\hat{\Gamma})^{1/q} &= \left(\frac{H}{W^{p-1}}\right)^{1/p} \left(\frac{W}{H^{q-1}}\right)^{1/q} = \frac{H^{1/p}}{W^{1-1/p}} \frac{W^{1/q}}{H^{1-1/q}} \\ &= H^{1/p+1/q-1} W^{1/p+1/q-1} = 1.\end{aligned} \tag{1.0.2}$$

It turns out, (1.0.2) is a general statement. If Γ is a curve family connecting two compact sets $E, F \subset \overline{\Omega}$ and $\hat{\Gamma}$ is the family of curves $\hat{\gamma}$ that separate E from F, then (1.0.2) holds. We will state this later more formally (and in more generality) as Theorem 4.2.6.

We now turn our attention to a classical result known as Beurling's Criterion. We will show the $p = 2$ case, though Badger has generalized this in [9]. Note, we say $\rho \in \mathrm{Adm}(\Gamma)$ is *extremal* if $\mathrm{Mod}_p(\Gamma) = \mathcal{E}_p(\rho)$.

Theorem 1.0.4 (Beurling's Criterion [1]). *Let Γ be a curve family in $\Omega \subset \mathbb{C}$. Suppose there exists $\rho_0 \in \mathrm{Adm}(\Gamma)$ and $\Gamma_0 \subset \Gamma$ satisfying*

(1) $\int_\gamma \rho_0 \, ds = 1 \qquad \forall \gamma \in \Gamma_0$

(2) $\forall h : \Omega \to \mathbb{R}$ *Borel,* $\int_\gamma h \, ds \geq 0 \quad \forall \gamma \in \Gamma_0 \Rightarrow \int_\Omega h\rho_0 \, dA \geq 0,$

then ρ_0 is extremal for Γ.

Proof. Suppose $\rho \in \mathrm{Adm}(\Gamma)$. Let $h = \rho - \rho_0$. Then $\forall \gamma \in \Gamma$, we have

$$\int_\gamma h \, ds = \int_\gamma \rho \, ds - \int_\gamma \rho \, ds \geq 1 - 1 = 0.$$

Then, by assumption (2), we must have $\int_\Omega h\rho_0 \, dA \geq 0$. Hence, we have

$$\begin{aligned}
&\int_\Omega \rho_0^2 \, dA \leq \int_\Omega \rho\rho_0 \, dA \\
\Rightarrow &\int_\Omega \rho_0^2 \, dA \leq \left(\int_\Omega \rho^2 \, dA \right)^{1/2} \left(\int_\Omega \rho_0^2 \, dA \right)^{1/2} \\
\Rightarrow &\left(\int_\Omega \rho_0^2 \, dA \right)^{1/2} \leq \left(\int_\Omega \rho^2 \right)^{1/2} \\
\Rightarrow &\int_\Omega \rho_0^2 \, dA \leq \int_\Omega \rho^2 \, dA.
\end{aligned}$$

Taking the infimum over all $\rho \in \mathrm{Adm}(\Gamma)$, we get that $\mathrm{Mod}_2(\Gamma) = \mathcal{E}_2(\rho_0)$.

□

Note that in Example 1.0.3, the extremal function $\rho \equiv 1/W$ together with the family Γ_0 of straight lines given by $\gamma(t) = t + iy$ for $t \in [0, W]$ satisfy the conditions since for every

Borel function $h : \Omega \to \mathbb{R}$ such that $\int_0^W h(t + iy)\, dt \geq 0$, we have

$$\int_\Omega h\rho_0 \, dA = \int_0^H \int_0^W h \cdot (1/W)\, dx\, dy = (1/W) \int_0^H \int_0^W h(t + iy)\, dt\, dy \geq 0.$$

Chapter 2

Discrete Modulus

Discrete modulus is a powerful tool that can yield information about families of "objects" defined on a graph G. By graph, we mean a triple $G = (V, E, \sigma)$ where V is a finite set of vertices, E, a finite edge set, a subset of $V \times V$, and $\sigma : E \to \mathbb{R}_{>0}$ is a weight function. Sometimes we will simply refer to a graph $G = (V, E)$ assuming $\sigma \equiv 1$. The term object is intentionally vague. The only property needed to define an object is the existence of an associated "edge usage" function as described below. For example, we can talk about the modulus of the family of spanning trees or the family of cycles on the graph. Although we could talk about many different objects on a graph, we will focus on the case when those objects are either walks connecting two points or cuts that separate two points. We will follow the exposition in [4], where many different families of objects are considered. Further, we will only work with connected graphs and throughout, p and q are assumed to be Hölder conjugates, that is, $1/p + 1/q = 1$. Our goal is establishing Fulkerson duality in Theorem 2.2.8 and constructing some metrics on graphs as in Theorem 2.3.3.

2.1 Definition of discrete modulus

Let Γ be a family of objects such that every $\gamma \in \Gamma$ has an associated usage function $\mathcal{N}(\gamma, \cdot) : E \to \mathbb{R}_{\geq 0}$ that measures the *usage of edge e by* γ. It is convenient to think of $\mathcal{N}$ as a matrix with $|\Gamma|$ rows and $|E|$ columns. We assume that $0 < |\Gamma| < \infty$ and each row has at least one nonzero entry. Then $\mathcal{N}$ is a matrix with finite dimensions, and we can define the following useful quantity:

$$\mathcal{N}_{min} := \min_{\gamma \in \Gamma} \min_{e:\mathcal{N}(\gamma,e)\neq 0} \mathcal{N}(\gamma, e). \tag{2.1.1}$$

In words, $\mathcal{N}_{min} > 0$ is the smallest nonzero entry of the matrix. Next, we define the two types of objects we are interested in.

Definition 2.1.1. A *walk* γ is a sequence of vertices and edges $x_0e_1x_1 \dots e_nx_n$ where $e_i = \{x_{i-1}, x_i\}$. For walks, we define $\mathcal{N}(\gamma, e)$ = number of times the edge e appears in the walk γ.

Definition 2.1.2. We say Γ is a *connecting family (between two subsets $A, B \subset V$)* if Γ is a family of walks such that $x_0 \in A$ and $x_n \in B$.

Definition 2.1.3. Let $a, b \in V, a \neq b$. A subset $S \subset V$ is called an ab-cut if $a \in S$ and $b \notin S$. For ab-cuts, if $e = \{x, y\} \in E$, we define $\mathcal{N}(S, e) = 1$ if $x \in S$ and $y \notin S$ and $\mathcal{N}(S, e) = 0$ otherwise.

Note that due to the definition of a walk, the family of walks connecting two vertices a and b is an infinite family. However, in [7] it is shown that the modulus of the family of walks connecting a to b is equal to the modulus of the family of simple paths connecting a to b, which is a finite family on finite graphs. In general, we can consider $|\Gamma| = \infty$, but for this dissertation, it is enough to consider only the case where $|\Gamma| < \infty$.

Now we define modulus analogously to how we did in the plane. We first define a density on G to be any function $\rho : E \to [0, \infty)$. Since $|E|$ is assumed finite, we can think of ρ as a vector in $|E|$ dimensional Euclidean space. Intuitively speaking, we should think of $\rho(e)$ as the *cost of using edge e*. We define the *ρ-length* of an object γ by

$$\ell_\rho(\gamma) := \sum_{e\in E} \mathcal{N}(\gamma, e)\rho(e) = (\mathcal{N}\rho)(\gamma).$$

If $\rho \equiv 1$, we simply write $\ell(\gamma)$. For a walk, $\ell(\gamma)$ is the number of hops the walk takes and is analogous to the arc-length of a curve in the plane. As in the case of the plane, we say ρ is *admissible* for Γ, if for every $\gamma \in \Gamma$, we have

$$\ell_\rho(\gamma) \geq 1. \tag{2.1.2}$$

The admissible set is defined as

$$\text{Adm}(\Gamma) = \left\{ \rho \in \mathbb{R}^E_{\geq 0} : \ell_\rho(\gamma) \geq 1, \forall \gamma \in \Gamma \right\}. \tag{2.1.3}$$

As in the plane, given an exponent $p \in [1, \infty)$, we define the *p-energy* of a density ρ by

$$\mathcal{E}_{p,\sigma}(\rho) := \sum_{e \in E} \rho(e)^p \sigma(e),$$

where σ is acting as dA. For $p = \infty$ we define the ∞-energy of ρ as

$$\mathcal{E}_{\infty,\sigma}(\rho) := \max_{e \in E} \rho(e)\sigma(e).$$

Finally, we define the p-Modulus of a family of objects.

Definition 2.1.4. Given a graph $G = (V, E, \sigma)$, a family of objects Γ with usage matrix $\mathcal{N} \in \mathbb{R}^{|\Gamma| \times |E|}$, and an exponent $1 \leq p \leq \infty$, the p-Modulus is defined by

$$\text{Mod}_{p,\sigma}(\Gamma) := \inf_{\rho \in \text{Adm}(\Gamma)} \mathcal{E}_{p,\sigma}(\rho). \tag{2.1.4}$$

Remark 2.1.5. For $1 < p < \infty$, a unique extremal density ρ^* always exists. When $p = 1$ or ∞, we lose uniqueness but still have existence. Since each row of $\mathcal{N}$ has a nonzero entry that is bounded below by $\mathcal{N}_{\min}$, we know that $\rho^* \leq \mathcal{N}_{min}^{-1}$. We reproduce Lemma 2.1 of [5] for completeness below.

Lemma 2.1.6. *Given a family of objects Γ and $1 \leq p \leq \infty$ there exists and extremal density ρ^*. If $1 < p < \infty$, then ρ^* is unique.*

Proof. For any $1 \leq p < \infty$, the function $\mathcal{E}_{p,\sigma}(\cdot)^{1/p}$ is a norm on the $|E|$ dimensional Euclidean space. When $p = \infty$, so is $\mathcal{E}_{\infty,\sigma}(\cdot)$. Thus, finding a minimizer for p-Modulus is equivalent to finding a minimizer for the norm in $\mathbb{R}^E$. Since $\mathrm{Adm}(\Gamma)$ is a closed convex subset of $\mathbb{R}^E$, such a minimizer always exists, and when $1 < p < \infty$, the minizer is unique. □

We wish to note that, in the case of connecting families, (2.1.4) is a capacity on families of objects. This is described in the following proposition which, for instance, can be found in [8]. The ultra-subadditivity can be found in [6].

Proposition 2.1.7. *Let $G = (V, E, \sigma)$ be a graph. For $p \in [1, \infty]$, the following hold:*

- **Monotonicity:** *Suppose Γ and Γ' are families of objects on G such that $\Gamma \subset \Gamma'$, meaning that the matrix $\mathcal{N}(\Gamma)$ is the restriction of the matrix $\mathcal{N}(\Gamma')$ to the rows from Γ. Then*

$$\mathrm{Mod}_{p,\sigma}(\Gamma) \leq \mathrm{Mod}_{p,\sigma}(\Gamma'). \tag{2.1.5}$$

- **Countable Subadditivity:** *Suppose $1 \leq p < \infty$, and let $\{\Gamma_j\}_{j=1}^{\infty}$ be a sequence of families of objects on G. Then*

$$\mathrm{Mod}_{p,\sigma}\left(\bigcup_{j=1}^{\infty} \Gamma_j\right) \leq \sum_{j=1}^{\infty} \mathrm{Mod}_{p,\sigma}(\Gamma_j). \tag{2.1.6}$$

- **Ultra-Subadditivity:** *Let $p = \infty$ and $\{\Gamma_j\}_{j=1}^{\infty}$ be a sequence of families of objects on G. Then*

$$\mathrm{Mod}_{\infty,\sigma}\left(\bigcup_{j=1}^{\infty} \Gamma_j\right) \leq \sup_j \mathrm{Mod}_{\infty,\sigma}(\Gamma_j) \tag{2.1.7}$$

Proof. First, we prove monotonictiy. Let $\rho \in \mathrm{Adm}(\Gamma')$. Then for each $\gamma \in \Gamma$, we have

$$1 \leq \sum_{e \in E} \mathcal{N}(\gamma, e)\rho(e)$$

since this holds for each row of $\mathcal{N}(\Gamma')$. Thus $\mathrm{Adm}(\Gamma') \subset \mathrm{Adm}(\Gamma)$. Then we have

$$\mathrm{Mod}_{p,\sigma}(\Gamma) \leq \mathrm{Mod}_{p,\sigma}(\Gamma').$$

Now we consider countable subadditivity. We assume $\sum_{j=1}^{\infty} \mathrm{Mod}_{p,\sigma}(\Gamma_j) < \infty$ because otherwise the inequality is trivial. Next, fix $p \in [1, \infty)$ and choose $\rho_j \in \mathrm{Adm}(\Gamma_j)$ such that

$$\mathcal{E}_{p,\sigma}(\rho_j) = \mathrm{Mod}_{p,\sigma}(\Gamma_j).$$

Then we have

$$\left(\min_{e\in E} \sigma(e)\right) \sum_{j=1}^{\infty} \rho_j(e)^p \leq \sum_{j=1}^{\infty} \sum_{e\in E} \rho_j(e)^p \sigma(e) = \sum_{j=1}^{\infty} \mathrm{Mod}_{p,\sigma}(\Gamma_j) < \infty.$$

Since $\sigma > 0$, we have that $\sum_{j=1}^{\infty} \rho_j(e)^p < \infty$. Define $\rho := \left(\sum_{j=1}^{\infty} \rho_j^p\right)^{\frac{1}{p}}$. For any $\gamma \in \Gamma := \bigcup \Gamma_j$, there is a $k \in \mathbb{N}$ such that $\gamma \in \Gamma_k$. Since $\rho \geq \rho_k$, we have $\rho \in \mathrm{Adm}(\Gamma_k)$. Moreover, $\rho \in \mathrm{Adm}(\Gamma)$ and

$$\begin{aligned}
\mathrm{Mod}_{p,\sigma}(\Gamma) \leq \mathcal{E}_{p,\sigma}(\rho) &= \sum_{e\in E} \rho(e)^p \sigma(e) = \sum_{e\in E} \sigma(e) \sum_{j=1}^{\infty} \rho_j(e)^p = \sum_{j=1}^{\infty} \sum_{e\in E} \rho_j(e)^p \sigma(e) \\
&= \sum_{j=1}^{\infty} \mathcal{E}_{p,\sigma}(\rho_j) = \sum_{j=1}^{\infty} \mathrm{Mod}_{p,\sigma}(\Gamma_j).
\end{aligned}$$

Now for the case $p = \infty$. Just as before, we choose $\rho_j \in \mathrm{Adm}(\Gamma_j)$ such that $\mathcal{E}_{\infty,\sigma}(\rho_j) = \mathrm{Mod}_{\infty,\sigma}(\Gamma_j)$. Define $\rho := \sup_j \rho_j$. Again, for any $\gamma \in \Gamma$, there is a $k \in \mathbb{N}$ such that $\gamma \in \Gamma_k$. Moreover, $\rho \in \mathrm{Adm}(\Gamma)$ and $\max_e \rho(e)\sigma(e) \leq \sup_i \max_e \rho_i(e)\sigma(e)$. Indeed, for each $e \in E$, we have that $\rho(e) = \sup_j \rho_j(e) \leq \sup_j \max_e \rho_j(e)\sigma(e)$. Consequently,

$$\begin{aligned}
\mathrm{Mod}_{\infty,\sigma}(\Gamma) \leq \mathcal{E}_{\infty,\sigma}(\rho) &= \max_{e\in E} \rho(e)\sigma(e) = \max_{e\in E} \sup_j \rho_j(e)\sigma(e) \leq \sup_j \max_{e\in E} \rho_j(e)\sigma(e) \\
&= \sup_j \mathrm{Mod}_{\infty,\sigma}(\Gamma_j).
\end{aligned}$$

$\square$

The concept of p-Modulus generalizes classical ways of describing graphs quantitatively. Let $G = (V, E)$ be a graph and two vertices $a, b \in V$ be given. Define $\Gamma(a, b)$ to be the family of all simple paths in G that start at a and end at b. Assign the usage function $\mathcal{N}(g,e)$ to be 1 when $e \in \gamma$ and 0 otherwise. Likewise, we can define $\Gamma(A,B)$ to be the family of all simple paths that start at some vertex in A and end at some vertex in B.

We refer to a subset $S \subset V$ such that $a \in S$ and $b \notin S$ as an *ab-cut*. Given an ab-cut S, we define the *edge boundary* as

$$\partial S := \{\{x,y\} \in E : x \in S, y \notin S\}.$$

We say an ab-cut is minimal if its edge boundary does not contain the edge boundary of any other ab-cut as a strict subset.

Definition 2.1.8. The *min cut* between a and b is defined as

$$MC(a,b) := \min\{|\partial S| : S \text{ is an } ab\text{-cut}\}$$

Definition 2.1.9. Thinking of G as an electrical network with edge conductances given by σ as in [], we define the *effective resistance* $\mathcal{R}_{eff}(a,b)$ as the voltaage drop necessary to pass 1 Amp of current between a and b through G. In this case, define the *effective conductance* by $\mathcal{C}_{eff}(a,b) := \mathcal{R}_{eff}(a,b)^{-1}$.

Definition 2.1.10. The (unweighted) *shortest-path distance* between a and b refers to the length of the shortest walk from a to b, where the length of a walk γ is $\ell(\gamma)$. Thus we write

$$\ell(\Gamma(a,b)) := \inf_{\gamma \in \Gamma(a,b)} \ell(\gamma)$$

for the shortest length of a family $\Gamma(a,b)$.

The theorem below describes how p-Modulus generalizes the above quantities.

Theorem 2.1.11 ([5])**.** *Let $G = (V, E, \sigma)$ be a graph with edge weights σ. Let Γ bea nontrivial family of objects on G with usage matrix $\mathcal{N}$ and let $\sigma(E) := \sum_{e \in E} \sigma(e)$. Then the function $p \mapsto \mathrm{Mod}_{p,\sigma}(\Gamma)$ is continuous for $1 \le p < \infty$, and the following two monotonictiy properties* *hold for* $1 \leq p \leq p' < \infty$.

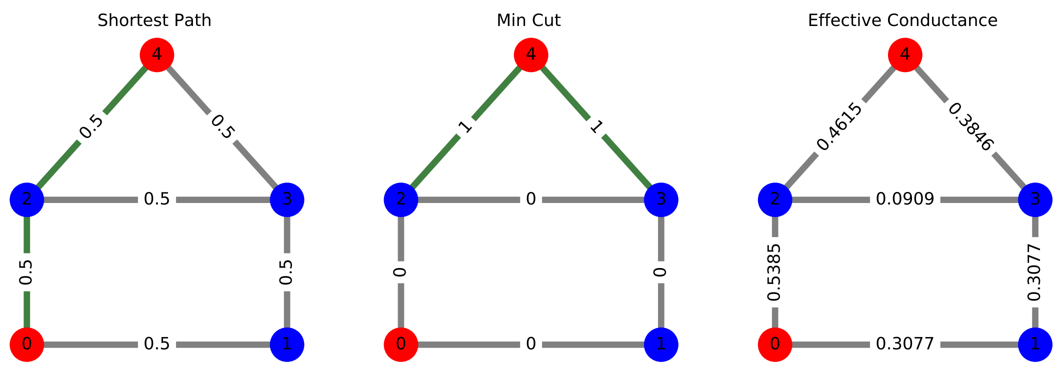
$$\mathcal{N}_{\min}^{p} \operatorname{Mod}_{p,\sigma}(\Gamma) \geq \mathcal{N}_{\min}^{p'} \operatorname{Mod}_{p',\sigma}(\Gamma), \tag{2.1.8}$$

$$\left(\sigma(E)^{-1} \operatorname{Mod}_{p,\sigma}(\Gamma)\right)^{1/p} \leq \left(\sigma(E)^{-1} \operatorname{Mod}_{p',\sigma}(\Gamma)\right)^{1/p'}. \tag{2.1.9}$$

Moreover, let $a \neq b$ *in* V *be given and let* Γ *be the connecting family* $\Gamma(a, b)$*. Then we have the following.*

- *For* $p = 1$*,* $\operatorname{Mod}_{1,\sigma}(\Gamma) = \min\{|\partial S| : S \text{ an } ab\text{-cut}\} = MC(a, b)$*.*
- *For* $p = 2$*,* $\operatorname{Mod}_{2,\sigma}(\Gamma) = \mathcal{C}_{eff}(a, b)$*.*
- *For* $p = \infty$*,* $\operatorname{Mod}_{\infty,\sigma}(\Gamma) = \lim_{p\to\infty} \operatorname{Mod}_{p,\sigma}(\Gamma)^{1/p} = \ell(\Gamma)^{-1}$*.*

Let us now consider an example that applies the above theorem.



Example 2.1.12. Consider the house graph above. Let $\sigma \equiv 1$ and Γ be the set of all simple paths connecting node 0 to node 4. Then $\operatorname{Mod}_1(\Gamma) = 2$ because we need to remove at least 2 edges to separate node 0 from node 4. We also have $\operatorname{Mod}_\infty(\Gamma) = 1/2$ because the length of the shortest path from node 0 to node 4 is 2. Finally, we have $\operatorname{Mod}_2(\Gamma) \approx .8462$. In the figure above, highlighted in green is the shortest path on the left and a minimal cut in the middle. Note, in the case where $p = 1$ or $p = \infty$, there may be multiple shortest paths or minimal cuts respectively. As a result, in those cases there can be multiple extremal functions. Note the edges have been labeled with an example extremal function.

2.2 Blocking Duality

Having defined p-Modulus for a family of objects on a graph G, we will now describe a notion of duality. To do so, for a non-trivial family of objects Γ on a graph G, we can think of the objects as elements of $\mathbb{R}^E_{\geq 0}$ by looking at the rows of the usage matrix $\mathcal{N}(\gamma,\cdot)$ corresponding to each object $\gamma \in \Gamma$. We will define duality in terms of vectors in $\mathbb{R}^E_{\geq 0}$. Let us recall some of the definitions we will use.

Definition 2.2.1. Let $\mathcal{K}$ be the set of all closed convex sets $K \subset \mathbb{R}^E_{\geq 0}$ that are *recessive*, that is $K + \mathbb{R}^E_{\geq 0} = K$. We will assume $\varnothing \neq K \subsetneq \mathbb{R}^E_{\geq 0}$ to avoid trivial cases.

Definition 2.2.2. For each $K \in \mathcal{K}$ there is an associated *blocking polyhedron*, or *blocker*,

$$\mathrm{BL}(K) := \{\eta \in \mathbb{R}^E_{\geq 0} : \eta^T \rho \geq 1 \forall \rho \in K\}.$$

Definition 2.2.3. Given $K \in \mathcal{K}$ and a point $x \in K$ we say that x is an *extreme point* of K if $x = tx_1 + (1-t)x_2$ for some $x_1, x_2 \in K$ and some $t \in (0,1)$, implies $x_1 = x_2 = x$. Moreover, we let $\mathrm{ext}(K)$ be the set of all extreme points of K.

Definition 2.2.4. The *dominant* of a set $P \subset \mathbb{R}^E_{\geq 0}$ is the recessive closed convex set

$$\mathrm{Dom}(P) := \mathrm{co}(P) + \mathbb{R}^E_{\geq 0},$$

where $\mathrm{co}(P)$ is the convex hull of P.

Since we are considering finite graphs, it is easy to see that $\mathrm{Adm}(\Gamma)$ is a closed set. Further, it is a convex set since

$$\sum_{e\in E} \mathcal{N}(\gamma,e)(t\rho_1(e) + (1-t)\rho_2(e)) = t\ell_{\rho_1}(\gamma) + (1-t)\ell_{\rho_2}(\gamma) \geq t + (1-t) = 1.$$

Thus, $\mathrm{Adm}(\Gamma)$ is the dominant of its extreme points, see Theorem 18.5 of [23].

Definition 2.2.5. Suppose $G = (V,E)$ is a finite graph and Γ is a finite non-trivial family of objects on G. We define the *Fulkerson blocker of* Γ, denoted $\hat{\Gamma}$, by

$$\hat{\Gamma} := \text{ext}(\text{Adm}(\Gamma)) = \{\hat{\gamma}_1, \hat{\gamma}_2, \ldots, \hat{\gamma}_s\} \subset \mathbb{R}^E_{\geq 0},$$

Define the $\hat{\mathcal{N}} \in \mathbb{R}^{\hat{\Gamma}\times E}_{\geq 0}$ to be the matrix whose rows are the vectors $\hat{\gamma}^T$ for $\hat{\gamma} \in \hat{\Gamma}$.

We now state the main Theorem we are interested in that makes use of the above definitions.

Theorem 2.2.6 (Fulkerson [18])**.** *Let $G = (V, E)$ be a graph and let Γ be a non-trivial family of objects on G. Let $\hat{\Gamma}$ be the Fulkerson blocker of Γ. Then*

(1) $\text{Adm}(\Gamma) = \text{Dom}(\hat{\Gamma}) = \text{BL}(\text{Adm}(\hat{\Gamma}))$*;*

(2) $\text{Adm}(\Gamma) = \text{Dom}(\Gamma) = \text{BL}(\text{Adm}(\Gamma))$*;*

(3) $\hat{\hat{\Gamma}} \subset \Gamma$.

Corollary 2.2.7 ([4])**.** *Let $G = (V, E)$ be a graph and let Γ be a non-trivial finite family of objects on G. Then,*

(1) $\text{BL}(\text{BL}(\text{Adm}(\Gamma))) = \text{Adm}(\Gamma)$ *and* $\text{BL}(\text{BL}(\text{Dom}(\Gamma))) = \text{Dom}(\Gamma)$

(2) $\text{Adm}(\Gamma) = \text{BL}(\text{Dom}(\Gamma))$ *and* $\text{BL}(\text{Adm}(\Gamma)) = \text{Dom}(\Gamma)$.

We now establish the main Blocking Duality result.

Theorem 2.2.8 ([4])**.** *Let $G = (V, E)$ be a graph and let Γ be a non-trivial finite family of objects on G with Fulkerson blocker $\hat{\Gamma}$. Let the exponent $1 < p < \infty$ be given, with $q := p/(p-1)$ it Hölder conjugate exponent. For any set of weights $\sigma \in \mathbb{R}^E_{>0}$, define the dual set of weights $\hat{\sigma}$ as $\hat{\sigma} := \sigma(e)^{-q/p}$ for all $e \in E$. Then*

$$\text{Mod}_{p,\sigma}(\Gamma)^{1/p}\, \text{Mod}_{q,\hat{\sigma}}(\hat{\Gamma})^{1/q} = 1. \tag{2.2.1}$$

Moreover, the extremal $\rho^* \in \text{Adm}(\Gamma)$ and $\eta^* \in \text{Adm}(\hat{\Gamma})$ are unique and are related as follows:

$$\eta^*(e) = \frac{\rho(e)^{p-1}\sigma(e)}{\text{Mod}_{q,\sigma}(\Gamma)} \quad \forall e \in E. \tag{2.2.2}$$

Proof. Let $\rho \in \mathrm{Adm}(\Gamma)$ and $\eta \in \mathrm{Adm}(\hat{\Gamma})$. The Hölder's inequality implies

$$\begin{aligned} 1 \leq \sum_{e\in E} \rho(e)\eta(e) &= \sum_{e\in E} (\sigma(e)^{1/p}\rho(e))(\sigma(e)^{-1/p}\eta(e)) \\ &\leq \left(\sum_{e\in E} \rho(e)^p\sigma(e)\right)^{1/p} \left(\sum_{e\in E} \eta(e)^q\hat{\sigma}(e)\right)^{1/q}. \end{aligned} \tag{2.2.3}$$

Let $\alpha := \mathrm{Mod}_{q,\hat{\sigma}}(\hat{\Gamma})^{-1}$, $\eta^* \in \mathrm{Adm}(\hat{\Gamma})$ be the minimizer for $\mathrm{Mod}_{q,\hat{\sigma}}(\hat{\Gamma})$, and ρ^* be defined by

$$\rho^*(e) := \alpha \left(\frac{\hat{\sigma}(e)}{\sigma(e)}\eta^*(e)^q\right)^{1/p} = \alpha\hat{\sigma}(e)\eta^*(e)^{q/p}. \tag{2.2.4}$$

Note that (2.2.4) is another way of writing (2.2.2). Further, by (2.2.3), we have $\mathrm{Mod}_{p,\sigma}(\Gamma) \geq \alpha^{p/q} = \alpha^{p-1}$. Hence, if $\rho^* \in \mathrm{Adm}(\Gamma)$, then $\mathcal{E}_{p,\sigma}(\rho^*)$ gives us an upper bound for $\mathrm{Mod}_{p,\sigma}(\Gamma)$, and

$$\mathcal{E}_{p,\sigma}(\rho^*) = \sum_{e\in E} \rho(e)^p\sigma(e) = \alpha^p \sum_{e\in E} \eta^*(e)^q\hat{\sigma}(e) = \alpha^{p-1},$$

and (2.2.1) follows.

To verify $\rho^* \in \mathrm{Adm}(\Gamma)$, in light of (1) of Theorem 2.2.6, we will verify $\sum_{e\in E} \rho^*(e)\eta(e) \geq 1$ for each $\eta \in \mathrm{Adm}(\hat{\Gamma})$. The first case is when $\eta^* = \eta$. Then we have

$$\sum_{e\in E} \rho^*(e)\eta^*(e) = \alpha \sum_{e\in E} \eta^*(e)^q\hat{\sigma}(e) = 1.$$

Now let $\eta \in \mathrm{Adm}(\hat{\Gamma})$ be arbitrary. Since $\mathrm{Adm}(\hat{\Gamma})$ is convex, we have $(1-\theta)\eta^* + \theta\eta \in \mathrm{Adm}(\hat{\Gamma})$ for all $\theta \in [0, 1]$. Using Taylor's theorem, we have

$$
\begin{aligned}
\alpha^{-1} = \mathcal{E}_{q,\hat{\sigma}}(\eta^*) \leq \mathcal{E}_{q,\hat{\sigma}}((1-\theta)\eta^* + \theta\eta) &= \sum_{e\in E} \hat{\sigma}(e)[(1-\theta)\eta^*(e) + \theta\eta(e)]^q \\
&= \sum_{e\in E} \hat{\sigma}(e)\eta^*(e)^q \left[1 + \theta\left(\frac{\eta(e)-\eta^*(e)}{\eta^*(e)}\right)\right]^q \\
&= \sum_{e\in E} \hat{\sigma}(e)\eta^*(e)^q \left[1 + q\theta\left(\frac{\eta(e)-\eta^*(e)}{\eta^*(e)}\right) + O(\theta^2)\right] \\
&= \alpha^{-1} + q\theta \sum_{e\in E} \hat{\sigma}(e)\eta^*(e)^{q-1}(\eta(e)-\eta^*(e)) + O(\theta^2) \\
&= \alpha^{-1} + \alpha^{-1}q\theta \sum_{e\in E} \rho^*(e)(\eta(e)-\eta^*(e)) + O(\theta^2).
\end{aligned}
\tag{2.2.5}
$$

Hence, we obtain

$$
1 = \sum_{e\in E} \rho^*(e)\eta^*(e) \leq \sum_{e\in E} \rho^*(e)\eta(e) + O(\theta),
$$

and letting $\theta \to 0$, we obtain the $\rho^* \in \mathrm{Adm}(\hat{\Gamma})$. □

The main example we are interested in is in the case when $\Gamma = \Gamma(a,b)$ is the family of all walks from a to b. In this case, the Fulkerson Blocker $\hat{\Gamma}$ is the family of all minimal ab-cuts. Recall, an ab-cut is minimal if its edge boundary does not contain the edge boundary of any other ab-cut as a strict subset. For more discussion, see 4.2 of [4]. Let us now see an example.

Example 2.2.9. Consider the Path graph P_3 consisting of two edges $e_1 = \{a,b\}$ and $e_2 = \{b,c\}$. Let Γ be the family containing a single object, namely the simple path $\gamma = a\ b\ c$. In this case the usage matrix is

$$
\mathcal{N} = [\,1 \quad 1\,]
$$

The admissible set is

$$
\mathrm{Adm}(\Gamma) = \{(\rho_1,\rho_2) : \rho_1 + \rho_2 \geq 1\} \subset \mathbb{R}^2_{\geq 0}.
$$

In particular, $\mathrm{Adm}(\Gamma)$ has two extreme points $\hat{\gamma}_1 = (0,1)^T$ and $\hat{\gamma}_2 = (1,0)^T$. Therefore,

$$\mathrm{Adm}(\hat{\Gamma}) = \{(\eta_1, \eta_2) : \eta_1 \geq 1 \text{ and } \eta_2 \geq 1\}.$$

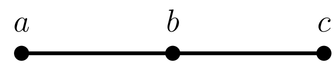


Figure 2.1: *The Path graph P_3 with $\gamma = a\ b\ c$*

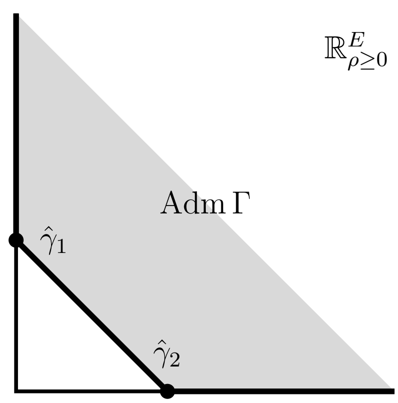


Figure 2.2: $\mathrm{Adm}(\Gamma)$ *in ρ-space*

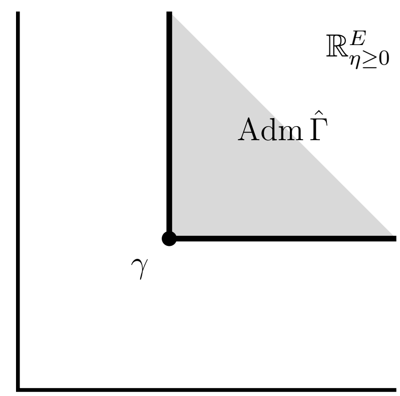


Figure 2.3: $\mathrm{Adm}(\hat{\Gamma})$ *in η-space*

Thus, we can see geometrically that the extremal $\rho^* \in \mathrm{Adm}(\Gamma)$ is $\rho = (1/2, 1/2)^T$. Hence $\mathrm{Mod}_p(\Gamma(a,c)) = (1/2)^p + (1/2)^p = 2^{1-p}$. Further, $\eta^* = (1,1)^T$, so $\mathrm{Mod}_q(\hat{\Gamma}(a,c)) = 1^q + 1^q = 2$, and we have

$$2^{\frac{1-p}{p}} 2^{\frac{1}{q}} = 2^{\frac{1}{p} - 1 + \frac{1}{q}} = 2^0 = 1,$$

which is exactly the conclusion our Theorem yields.

We now turn to the main application we are interested in of the above duality. That is, the establsihment of so called “Modulus Metrics.”

2.3 Modulus Metrics on Graphs

We start by recalling what a metric space is. We will then proceed to show that, given a graph G, we can define a metric on G so that G becomes a metric space.

Definition 2.3.1. A *metric space* (X, d) is a set X equipped with a *metric* d. That is, d is a function $d : X \times X \to \mathbb{R}$ satisfying the following conditions for all $x, y, z \in X$:

1. $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$ (Non-degeneracy)

2. $d(x, y) = d(y, x)$ (Symmetry)

3. $d(x, y) \leq d(x, z) + d(z, y)$ (Triangle inequality)

If instead of item 3. above we have

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}, \tag{2.3.1}$$

then we call d an *ultrametric*. Clearly, every ultrametric is a metric.

We now define our metrics.

Definition 2.3.2. Let $G = (V, E, \sigma)$ be a weighted, connected, simple graph. Given $a, b \in V$, let $\Gamma(a, b)$ be the family of all paths between a and b. Fix $1 < p < \infty$ and let $q := p/(p-1)$ be the Hölder conjugate exponent. Then we define

$$\delta_p(a, b) := \begin{cases} 0 & \text{if } a = b \\ \mathrm{Mod}_{p,\sigma}(\Gamma(a, b))^{-q/p} & \text{if } a \neq b. \end{cases} \tag{2.3.2}$$

Theorem 2.3.3 ([4])**.** *Suppose $G = (V, E, \sigma)$ is a weighted, connected, simple graph. Then δ_p is a metric on V. Moreover,*

(a) $\lim_{p\to\infty} \delta_p(a, b) = \ell(\Gamma(a, b))$*;*

(b) $\delta_2(a, b) = \mathcal{C}_{eff}(a, b)$

(c) For $1 < p < 2$, $\mathrm{Mod}_{p,\sigma}(\Gamma(a, b))^{-1}$ is a metric and it tends to $MC(a, b)^{-1}$ as $p \to 1$.

Lastly, for every $\varepsilon > 0$ and every $p \in [1, \infty]$, there is a connected graph for which $\delta_p^{1+\varepsilon}$ is not a metric.

Proof. Let us prove the 'Moreover' parts assuming δ_p is a metric on V. Parts (a) and (b) follow from Theorem 2.1.11. For (c), since $1 < p < 2$, we have $p/q = p - 1$ satisfies $0 < p/q < 1$, and hence $\mathrm{Mod}_{p,\sigma}(\Gamma(a,b))^{-1} = \delta_p^{p/q}$ is a metric. Then using continuity in p, we obtain the result. Clearly $\delta_p(a,b) \geq 0$ since p-Modulus is an infimum of non-negative quantities and since we assume G is connected, there is always at least one path from a to b, whence $\mathrm{Mod}_{p,\sigma}(\Gamma(a,b)) > 0$. Further, since the usage matrix is the same for the curves connecting b to a as it is for connecting a to b, we have $\delta_p(a,b) = \delta_p(b,a)$.

Now we show the triangle inequality. First, we note that although $\hat{\Gamma}(a,b)$ is the family of all minimal ab-cuts, we can look at the larger set of all ab-cuts without changing the value of the q-Modulus. To see this, consider $\eta \in \mathrm{Adm}(\hat{\Gamma})$. Since every ab-cut that isn't minimal contains a minimal ab-cut, we must have η is admissible with respect to the larger family. So, without loss of generality, we assume $\hat{\Gamma}(a,b)$ is the family of all ab-cuts. By Theorem 2.2.8, we have $\delta_p(a,b) = \mathrm{Mod}_{q,\hat{\sigma}}(\hat{\Gamma}(a,b))$. Further, if $c \in V$, then for each ab-cut S, either $c \in S$ or $c \notin S$. If $c \in S$, then S is a cb-cut, and otherwise, S is an ac-cut. Thus $\hat{\Gamma}(a,b) \subset \hat{\Gamma}(a,c) \cup \hat{\Gamma}(c,b)$. Then we have

$$\begin{aligned}
\delta_p(a,b) &= \mathrm{Mod}_{p,\sigma}(\Gamma(a,b))^{-q/p} \\
&= \mathrm{Mod}_{q,\hat{\sigma}}(\hat{\Gamma}(a,b) \\
&\leq \mathrm{Mod}_{q,\hat{\sigma}}(\hat{\Gamma}(a,c) \cup \hat{\Gamma}(c,b)) \\
&\leq \mathrm{Mod}_{q,\hat{\sigma}}(\hat{\Gamma}(a,c)) + \mathrm{Mod}_{q,\hat{\sigma}}(\hat{\Gamma}(c,b)) \\
&= \delta_p(a,c) + \delta_p(c,b).
\end{aligned}$$

Now we turn to the last bit of the Theorem. Consider the path graph in Example 2.2.9. Let us fix $p \in (1,\infty)$. We have shown that $\mathrm{Mod}_p(\Gamma(a,c)) = 2^{1-p}$. Further, it is easy to see that $\mathrm{Mod}_p(\Gamma(a,b)) = \mathrm{Mod}_p(\Gamma(b,c)) = 1$. Computing, we have

$$\delta_p(a,c) = 2^{\frac{q(p-1)}{p}} = 2 = \delta_p(a,b) + \delta_p(b,c).$$

Thus, $\delta_p(a,c)^{1+\varepsilon} = 2^{1+\varepsilon} > 2 = \delta_p(a,b)^{1+\varepsilon} + \delta_p(b,c)^{1+\varepsilon}$ and $\delta_p^{1+\varepsilon}$ does not satisfy the triangle inequality. $\square$

Chapter 3

Modulus in Metric Spaces

Here we will present some of the results found in [6], but first we present some of the basic definitions and tools we use in metric spaces that can be found in, for instance, [20]. The goal of this chapter is to establish Theorem 3.3.9 and Theorem 3.4.1.

3.1 Metric Space Preliminaries

We recall the contents of Definition 2.3.1. A *metric space* (X, d) is a set X equipped with a metric d. That is, d is a function $d : X \times X \to \mathbb{R}$ satisfying the following conditions for all $x, y, z \in X$:

1. $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$ (Non-degeneracy)

2. $d(x, y) = d(y, x)$ (Symmetry)

3. $d(x, y) \leq d(x, z) + d(z, y)$ (Triangle inequality)

A *path* in X is a continuous function $\gamma : I \to (X, d)$ for some interval $I = [a, b] \subset \mathbb{R}$. In general, the interval I can be closed, open, or half-open, and either bounded or unbounded. We will assume I is closed and bounded. The *length* of a path γ is its total variation:

$$\text{length}(\gamma) := \sup_{a=t_0 \leq t_1 \leq \ldots \leq t_N = b} \sum_{k=1}^{N} d(\gamma(t_{k-1}), \gamma(t_k)) \tag{3.1.1}$$

where the supremum is taken over all possible partitions of the interval $[a, b]$ with N arbitrary subintervals.

The path γ is said to be *rectifiable* if $\text{length}(\gamma) < \infty$. If γ is a rectifiable path, we can define the *length function* $s_\gamma : I \to [0, \text{length}(\gamma)]$ by

$$s_\gamma(t) := \text{length}(\gamma|_{I_t}), \tag{3.1.2}$$

where $I_t := [a, t]$. Clearly, s_γ is weakly increasing. Also, it is easily seen that for any $t, s \in I$, $t < s$, we have

$$d(\gamma(t), \gamma(s)) \leq \text{length}(\gamma|_{[t,s]}) = s_\gamma(t) - s_\gamma(s). \tag{3.1.3}$$

Lemma 3.1.1 ([20]). *If γ is a rectifiable path in X, then s_γ is continuous.*

Proof. Since s_γ is increasing, it can fail to be continuous only if there is $a < t_0 \leq b$ and $\delta > 0$ such that

$$s_\gamma(t_0) > s_\gamma(t) + \delta \qquad \text{for all } a < t < t_0, \tag{3.1.4}$$

or a similar statement for $t_0 < t < b$.

Since γ is continuous at t_0, there is $t_1 < t_0$ so that

$$d(\gamma(t), \gamma(t_0)) < \delta/2 \qquad \text{for every } t_1 < t < t_0. \tag{3.1.5}$$

Now, since $\text{length}\left(\gamma \mid_{[t_1,t_0]}\right) = s_\gamma(t_0) - s_\gamma(t_1)$, by (3.1.4), we can pick a partition $t_1 = s_0 < s_1 < \cdots < s_{N-1} < s_N = t_0$ so that

$$\sum_{j=0}^{N-1} d(\gamma(s_j), \gamma(s_{j+1})) > \delta \tag{3.1.6}$$

and using (3.1.5) with $t = s_{N-1}$, we get that

$$\sum_{j=0}^{N-2} d(\gamma(s_j), \gamma(s_{j+1})) > \delta/2. \tag{3.1.7}$$

We now let $t_2 := s_{N-1}$ and repeat what we just did with t_2 in place of t_1, noting that both (3.1.4) and (3.1.5) remain valid. The process can be iterated indefinitely and it gives rise to a sequence t_k such that length $\left(\gamma \mid_{[t_1,t_k]}\right) \to \infty$. This contradicts rectifiability. $\square$

The length function s_γ is continuous and increasing, but not necessarily strictly increasing. We can define a right inverse as follows:

$$s_\gamma^{-1}(t) = \max\{s : s_\gamma(s) = t\} \qquad \forall t \in [0, \text{length}(\gamma)] \tag{3.1.8}$$

Then s_γ^{-1} is increasing and right-continuous.

Definition 3.1.2. The *arc-length parametrization* of a rectifiable path $\gamma : [0,1] \to X$ is the curve $\gamma_s : [0, \text{length}(\gamma)] \to X$ defined by

$$\gamma_s(t) := \gamma(s_\gamma^{-1}(t)).$$

In particular, $\gamma(u) = \gamma_s(s_\gamma(u))$, and

$$\text{length}\left(\gamma_s \mid_{[t,u]}\right) = \text{length}\left(\gamma \mid_{[s_\gamma^{-1}(t), s_\gamma^{-1}(u)]}\right) = s_\gamma(s_\gamma^{-1}(t)) - s_\gamma(s_\gamma^{-1}(u)) = t - u. \tag{3.1.9}$$

Definition 3.1.3. $\gamma : [0,1] \to X$ is *absolutely continuous* if for all $\epsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that whenever $\{(a_i, b_i)\}_{i=1}^N$ are disjoint intervals in $[0,1]$:

$$\sum_{i=1}^N |b_i - a_i| < \delta \Longrightarrow \sum_{i=1}^N d(\gamma(a_i), \gamma(b_i)) < \varepsilon.$$

Proposition 3.1.4 ([20])**.** *Suppose $\gamma : [0,1] \to X$ is rectifiable. Then γ is absolutely continuous if and only if s_γ is absolutely continuous.*

Proof. ($\Leftarrow$) This direction is clear by (3.1.3).

($\Rightarrow$) Given $\varepsilon > 0$, find $\delta = \delta(\varepsilon) > 0$ as in Definition 3.1.3. Let $\{(a_i, b_i)\}_{i=1}^N$ be disjoint intervals in $[0,1]$ with

$$\sum_{i=1}^{N} |b_i - a_i| < \delta.$$

Then, as we have seen, $s_\gamma(b_i) - s_\gamma(a_i) = \text{length}\left(\gamma\mid_{[a_i,b_i]}\right) < \infty$. By (3.1.1), there are disjoint intervals $\{(a_i^j, b_i^j)\}_{j=1}^{N_i}$ contained in (a_i, b_i) such that

$$\sum_{j=1}^{N_i} d(\gamma(a_i^j), \gamma(b_i^j)) \geq s_\gamma(b_i) - s_\gamma(a_i) - \frac{\varepsilon}{N}.$$

By absolute continuity of γ, since the disjoint intervals $\{(a_i^j, b_i^j)\}_{i,j}$ also have length adding up to less than δ, we get

$$\sum_{i=1}^{N} s_\gamma(b_i) - s_\gamma(a_i) \leq \sum_{i=1}^{N}\sum_{j=1}^{N_i} d(\gamma(a_i^j), \gamma(b_i^j)) + \varepsilon \leq 2\varepsilon.$$

$\square$

Next we recall Theorem 4.4.8, Proposition 5.1.8 of [20]. Here, if O is an open subset of $[a,b]$, then $V(\gamma, O)$ is defined as follows. If $O = (c,d)$, then $V(\gamma, O)$ is the total variation of γ over O as defined in (3.1.1), with a, b replaced with c, d, respectively. If O is a disjoint union of open intervals I_j, then $V(\gamma, O) = \sum_j V(\gamma, I_j)$. Further, we set

$$V(\gamma, [c,d]) = V(\gamma, (c,d]) = V(\gamma, [c,d)) = V(\gamma, (c,d)).$$

Note that using this notation, $V(\gamma, [a,b]) = \text{length}(\gamma)$.

Theorem 3.1.5 ([20]). *Let $\gamma : [a,b] \to X$ be a continuous map of bounded variation. Then we can associate a unique Radon measure ν_γ on $[a,b]$ such that $\nu_\gamma(O) = V(\gamma, O)$ for each open $O \subset [a,b]$ and*

$$\frac{d\nu_\gamma}{dm_1}(t) = \lim_{u\to t, u\neq t} \frac{d(\gamma(t), \gamma(u))}{|t-u|} =: |\gamma'(t)| \tag{3.1.10}$$

for $m_1-a.e.$ $t \in [a,b]$.

We refer to the limit above as the *metric differential* of γ. We omit the proof here because it is quite long and uses tools that would take us astray. However, when the variation is equal to Lebesgue measure, we have the following corollary.

Corollary 3.1.6 ([20])**.** *Let $\gamma : [a,b] \to X$ be a continuous map of bounded variation such that $V(\gamma, [t,u]) = u - t$ whenever $a \leq t \leq u \leq b$. Then*

$$\lim_{u \to t, u \neq t} \frac{d(\gamma(t), \gamma(u))}{|t-u|} = 1 \tag{3.1.11}$$

for m_1 a.e. $t \in [a,b]$. Moreover, for every set $E \subset [a,b]$ we have that $\mathcal{H}^1(E) > 0$ if $\mathcal{H}^1(\gamma(E)) > 0$.

Combining (3.1.1) and Corollary 3.1.6, we get the following proposition.

Proposition 3.1.7 ([20])**.** *Let $\gamma : [0,1] \to X$ be a compact rectifiable path. Then, its arc-length parametrization γ_s is always absolutely continuous. Indeed, γ_s is 1-Lipschitz and*

$$\lim_{u \to t, u \neq t} \frac{d(\gamma_s(t), \gamma_s(u))}{|t-u|} = 1 \qquad \textit{for a.e. } t.$$

We will only be interested in rectifiable, non-constant paths because the collection of non-rectifiable paths will be shown to be negligible in some sense. We will refer to such paths as *curves*. Because the a rectifiable curve parameterized by arc length satisfies $|\gamma_s'(t)| = 1$, we make the following definition.

Definition 3.1.8. Suppose $\gamma : [0,1] \to X$ is rectifiable and $\rho : X \to [0,\infty]$ is Borel. Then the *line integral* of ρ along γ is

$$\int_\gamma \rho \, ds := \int_0^{\text{length}(\gamma)} \rho(\gamma_s(t)) \, dt.$$

Also if $F \subset X$ is a Borel set,

$$\int_{\gamma \cap F} \rho \, ds := \int_0^{\text{length}(\gamma)} \mathbb{1}_{\gamma_s^{-1}(F)}(t) \rho(\gamma_s(t)) \, dt.$$

Definition 3.1.9. Suppose $\gamma : [0,1] \to X$ is a curve and $F \subset X$ is a Borel set. We say that γ *spends positive time* in F if

$$\int_{\gamma \cap F} ds = \int_0^{\text{length}(\gamma)} \mathbb{1}_{\gamma_s^{-1}(F)}(t)dt = m_1\left(\gamma_s^{-1}(F)\right) > 0.$$

where m_1 is the Lebesgue measure on $\mathbb{R}$. We write Γ_F^+ for the family of all curves that spend positive time in F.

Remark 3.1.10. If γ is absolutely continuous, then for every Borel set $F \subset X$ we have

$$m_1\left(\gamma^{-1}(F)\right) = 0 \Longrightarrow m_1\left(\gamma_s^{-1}(F)\right) = 0.$$

To see this, let $f = s_\gamma^{-1}$. Suppose $A \subset [0,1]$ is measurable. Then for $t \in f^{-1}(A)$, we have $f(t) = s_\gamma^{-1}(t) \in A$. By definition, $s_\gamma^{-1}(t)$ is the maximal number $s \in [0,1]$ such that $s_\gamma(s) = t$. Now, $s = f(s_\gamma(s)) = f(t) \in A$. Thus $t = s_\gamma(s) \in s_\gamma(A)$, and so we have the inclusion $f^{-1}(A) \subset s_\gamma(A)$. By definition of γ_s, we have

$$\gamma_s^{-1}(F) = (\gamma \circ f)^{-1}(F) = f^{-1}\left(\gamma^{-1}(F)\right)$$

By Proposition 3.1.4, s_γ is absolutely continuous, and hence,

$$m_1\left(\gamma^{-1}(F)\right) = 0 \Rightarrow m_1\left(s_\gamma\left(\gamma^{-1}(F)\right)\right) = 0 \Rightarrow m_1\left(f^{-1}\left(\gamma^{-1}(F)\right)\right) = 0 \Rightarrow m_1\left(\gamma_s^{-1}(F)\right) = 0$$

So if γ spends positive time in F, then $m_1\left(\gamma^{-1}(F)\right) > 0$.

However, if γ is not absolutely continuous, then there are examples where this fails. For instance, consider the curve

$$\gamma(t) = (t, C(t)), \qquad t \in [0,1],$$

where $C(t)$ is the usual Cantor step-function. Let C be the middle-third Cantor set, D be the dyadic rationals in $[0,1]$, and $F = [0,1] \times ([0,1] \backslash D)$. Then $\gamma^{-1}(F) = C$, and so

$m_1\left(\gamma^{-1}(F)\right) = 0$. Now, letting $E = [0,1] \times D$, we have

$$\sqrt{2} \leq \operatorname{length}(\gamma) = m_1\left(\gamma_s^{-1}([0,1]\times[0,1])\right) = m_1\left(\gamma_s^{-1}(F)\right) + m_1\left(\gamma_s^{-1}(E)\right)$$

A simple computation shows $m_1\left(\gamma_s^{-1}(E)\right) = 1$, and hence, $m_1\left(\gamma_s^{-1}(F)\right) \geq \sqrt{2} - 1 > 0$.

3.2 Supremum Modulus

Just like in the plane and on graphs, we begin defining Modulus by defining the notion of *admissible* functions. In metric spaces, a Borel function $\rho : X \to [0,\infty]$ is admissible if for every $\gamma \in \Gamma$, we have

$$\ell_\rho(\gamma) := \int_\gamma \rho\, ds \geq 1,$$

and we denote the set of all such Borel functions by $\operatorname{Adm}(\Gamma)$. We now define Supremum Modulus as

$$\operatorname{Mod}_{\sup}(\Gamma) := \inf_{\rho\in\operatorname{Adm}(\Gamma)} \sup_X(\rho),$$

where $\sup_X(\rho) = \sup\{\rho(x) : x \in X\}$.

The notion of Supremum Modulus is entirely metric in nature. We will now demonstrate this fact by establishing

$$\operatorname{Mod}_{\sup}(\Gamma) = \frac{1}{\ell(\Gamma)}, \tag{3.2.1}$$

where $\ell(\Gamma) := \inf_{\gamma\in\Gamma} \operatorname{length}(\gamma)$. Let ρ_0 be the constant function $\frac{1}{\ell(\Gamma)}$. Clearly, ρ_0 is admissible since

$$\int_\gamma \rho_0\, ds = \frac{\operatorname{length}(\gamma)}{\ell(\Gamma)} \geq 1.$$

Thus, $\operatorname{Mod}_{\sup}(\Gamma) \leq \frac{1}{\ell(\Gamma)}$. To show the converse, suppose $\rho \in \operatorname{Adm}(\Gamma)$. Then we have

$$1 \leq \int_\gamma \rho\, ds \leq \sup_X \rho(x)\operatorname{length}(\gamma).$$

Since $\gamma \in \Gamma$ and $\rho \in \mathrm{Adm}(\Gamma)$ are arbitrary, taking the infimum over Γ and supremum over $\mathrm{Adm}(\Gamma)$, we get

$$1 \leq \mathrm{Mod}_{\sup}(\Gamma)\ell(\Gamma).$$

Hence, we conclude $\mathrm{Mod}_{\sup}(\Gamma) = \frac{1}{\ell(\Gamma)}$.

3.3 ∞-Exceptional Families of Curves and ∞-Modulus

A very important fact (that we will later show) about ∞-Modulus[1] in general metric measure spaces is that it is an outer measure on families of curves. Namely, if $\Gamma_0 \subset \Gamma$ and $\Gamma = \cup_{j=1}^{\infty}\Gamma_j$, then

$$\mathrm{Mod}_\infty(\varnothing) = 0 \tag{3.3.1}$$

$$\mathrm{Mod}_\infty(\Gamma_0) \leq \mathrm{Mod}_\infty(\Gamma) \tag{3.3.2}$$

$$\mathrm{Mod}_\infty(\Gamma) \leq \sum_{j=1}^{\infty} \mathrm{Mod}_\infty(\Gamma_j). \tag{3.3.3}$$

Suppose further that $\mathrm{Mod}_\infty(\Gamma_0) = 0$. Then we would have

$$\mathrm{Mod}_\infty(\Gamma) = \mathrm{Mod}_\infty((\Gamma\backslash\Gamma_0) \cup \Gamma_0) \leq \mathrm{Mod}_\infty(\Gamma\backslash\Gamma_0) + \mathrm{Mod}_\infty(\Gamma_0) = \mathrm{Mod}_\infty(\Gamma\backslash\Gamma_0).$$

Combining this with monotonicity, we have that $\mathrm{Mod}_\infty(\Gamma) = \mathrm{Mod}_\infty(\Gamma\backslash\Gamma_0)$. Hence, when defining admissibility, we really have no need for the line integral of a function to be at least 1 for *every* curve, but rather for *almost every* curve, that is up to a family of curves of ∞-Modulus zero. This gives us a notion of *weak admissibility*. Thus, first we will define a strong ∞-Modulus denoted $\mathrm{Mod}_\infty^*(\Gamma)$. We will then use this to define a weak ∞-Modulus. Finally, we will show the two notions are equal.

Assume now that a regular Borel measure μ is given on (X, d). Then it makes sense to

[1] This holds for p-Modulus with $1 \leq p \leq \infty$, however, in the setting of general metric spaces, we will not show this. See [20].

talk about the essential supremum

$$\|\rho\|_\infty = \inf\{a \geq 0 : \mu(\{x : \rho(x) > a\}) = 0\}.$$

Given a curve family Γ, define

$$\mathrm{Mod}_\infty^*(\Gamma) := \inf_{\rho \in \mathrm{Adm}(\Gamma)} \|\rho\|_\infty.$$

Now we establish that Mod_∞^* is an outer measure.

Lemma 3.3.1 ([20]). *$\mathrm{Mod}_\infty^*(\Gamma)$ has the following properties:*

(i) If $\Gamma' \subset \Gamma$, then $\mathrm{Mod}_\infty^*(\Gamma') \leq \mathrm{Mod}_\infty^*(\Gamma)$ **(monotone)***;*

(ii) $\mathrm{Mod}_\infty^*(\Gamma) \leq \|\rho\|_\infty \leq \sum_{j=1}^\infty \mathrm{Mod}_\infty^*(\Gamma_j)$ **(subadditive)***;*

(iii) If $\Gamma_1 \subset \Gamma_2 \subset \cdots$ and $\Gamma = \cup_{j=1}^\infty \Gamma_j$, then $\lim_{j\to\infty} \mathrm{Mod}_\infty^*(\Gamma_j) = \mathrm{Mod}_\infty^*(\Gamma)$ **(continuous from below)***;*

(iv) If for every $\gamma \in \Gamma_1$ there is a subcurve $\sigma \subset \gamma$ such that $\sigma \in \Gamma_2$, we write $\Gamma_2 \preceq \Gamma_1$, and then $\mathrm{Mod}_\infty^*(\Gamma_1) \leq \mathrm{Mod}_\infty^*(\Gamma_2)$ **(shorter walks)***.*

Proof. (i) Suppose that $\Gamma' \subset \Gamma$. Then $\mathrm{Adm}(\Gamma) \subset \mathrm{Adm}(\Gamma')$. Thus, $\mathrm{Mod}_\infty^*(\Gamma') \leq \mathrm{Mod}_\infty^*(\Gamma)$.

(ii) Suppose that $\Gamma = \cup_{j=1}^\infty \Gamma_j$. For each j, find $\rho_j \in \mathrm{Adm}(\Gamma_j)$ such that

$$\|\rho_j\|_\infty \leq \mathrm{Mod}_\infty^*(\Gamma_j) + \frac{\varepsilon}{2^j}.$$

Let $\rho = \sum_j \rho_j$. Then $\rho \in \mathrm{Adm}(\Gamma)$ and

$$\mathrm{Mod}_\infty^*(\Gamma) \leq \|\rho\|_\infty \leq \sum_{j=1}^\infty \mathrm{Mod}_\infty^*(\Gamma_j) + \varepsilon$$

Now let ε tend to zero.

(iii) Let $M = \lim_{j\to\infty} \mathrm{Mod}_\infty^*(\Gamma_j)$. For every $j = 1, 2, \dots$, there is $\rho_j \in \mathrm{Adm}(\Gamma_j)$ such that $\|\rho_j\|_\infty \le M + 1/j$. Define

$$\rho^{(k)} := \sup_{j\ge k} \rho_j. \tag{3.3.4}$$

For every j, there is a set N_j, with $\mu(N_j) = 0$, such that $\rho_j < M + 2/j$ outside of N_j. Then $N = \cup_j N_j$ also has measure zero, and outside of N we have $\rho^{(k)} < M + 2/k$. In particular, $\|\rho^{(k)}\|_\infty \le M + 2/k$.

Now fix $k = 1, 2, \dots$ and fix $\gamma \in \Gamma$. Then $\gamma \in \Gamma_j$ for some j. Let $i = \max\{j, k\}$. Then since $i \ge j$, we have $\rho_i \in \mathrm{Adm}(\Gamma_j)$ and $\ell_{\rho_i}(\gamma) \ge 1$. Thus, since $k \le i$, by (3.3.4), $\ell_{\rho^{(k)}}(\gamma) \ge 1$. So we have shown that $\rho^{(k)} \in \mathrm{Adm}(\Gamma)$, which implies that $\mathrm{Mod}_\infty^*(\Gamma) \le M + 2/k$. Letting k tend tom infinity gives $\mathrm{Mod}_\infty^*(\Gamma) \le M$.

The other direction, $\mathrm{Mod}_\infty^*(\Gamma) \ge M$, follows from monotonicity (property (i) above).

(iv) Note that $\mathrm{Adm}(\Gamma_2) \subset \mathrm{Adm}(\Gamma_1)$, so the infimum is taken over a larger set. □

Definition 3.3.2. A curve family Γ is ∞-*exceptional*, if $\mathrm{Mod}_\infty^*(\Gamma) = 0$.

The next lemma is an expanded version of Lemma 5.7 in [13].

Lemma 3.3.3 ([6])**.** *Let Γ be a curve family. The following are equivalent:*

(a) Γ is ∞-exceptional

(b) There exists $\rho : X \to [0, \infty)$ such that $\|\rho\|_\infty < \infty$ and $\ell_\rho(\gamma) = \infty$ for all $\gamma \in \Gamma$.

(c) There exists $\rho : X \to [0, \infty)$ such that $\|\rho\|_\infty = 0$ and $\ell_\rho(\gamma) = \infty$ for all $\gamma \in \Gamma$.

(d) There is a Borel set F with $\mu(F) = 0$ such that $\Gamma \subset \Gamma_F^+$, i.e., in words: there is a set of measure zero such that every curve in the family spends positive time in that set.

Proof. (a) ⇒ (b): Assume $\mathrm{Mod}_\infty^*(\Gamma) = 0$. Then for $k = 1, 2, \dots$ there is $\rho_k \in \mathrm{Adm}(\Gamma)$ such that $\|\rho_k\|_\infty \le 2^{-k}$. Set $\rho := \sum_{k=1}^\infty \rho_k$. Then, $\|\rho\|_\infty \le 1$ and $\ell_\rho(\gamma) = \infty$ for all $\gamma \in \Gamma$.

(b) ⇒ (c): Assume $\|\rho\|_\infty \le 1$ and $\ell_\rho(\gamma) = \infty$ for all $\gamma \in \Gamma$. For $a \ge 0$, consider the level sets

$$S_a(\rho) = \{x : \rho(x) > a\}.$$

Since $\|\rho\|_\infty \leq 1$, we have $\mu(F) = 0$, where $F = S_1(\rho)$. Define $\tilde{\rho} := \rho\ \mathbb{1}_F$. Then, $\|\tilde{\rho}\|_\infty = 0$. Also

$$\int_\gamma \tilde{\rho}\, ds = \int_{\gamma\cap F} \rho\, ds \geq \int_{\gamma\cap F} \rho\, ds + \int_{\gamma\setminus F} (\rho - 1)\, ds \geq \int_\gamma \rho\, ds - \ell(\gamma) \geq \infty.$$

(c) $\Rightarrow$ (d): Assume $\|\rho\|_\infty = 0$ and $\ell_\rho(\gamma) = \infty$ for all $\gamma \in \Gamma$. Then $\mu(S_a(\rho)) = 0$ for every $a > 0$. Set $F = S_0(\rho)$. Then, by continuity of measures, $\mu(F) = 0$. However,

$$\infty = \int_\gamma \rho\, ds = \int_{\gamma\cap F} \rho\, ds$$

implies that $\int_{\gamma\cap F} ds > 0$, so $\Gamma \subset \Gamma_F^+$.

(d) $\Rightarrow$ (a): Assume $\mu(F) = 0$ and $\Gamma \subset \Gamma_F^+$. For $k = 1, 2, \ldots$, define

$$\Gamma_k = \left\{ \gamma \in \Gamma_F^+ : \int_{\gamma\cap F} ds \geq 1/k \right\}.$$

Let $\rho_k = k\mathbb{1}_F$. Then $\rho_k \in \mathrm{Adm}(\Gamma_k)$ and $\|\rho_k\|_\infty = 0$, so $\mathrm{Mod}_\infty^*(\Gamma_k) = 0$. Therefore, $\mathrm{Mod}_\infty^*(\Gamma_F^+) = 0$ by subadditivity of modulus and $\mathrm{Mod}_\infty^*(\Gamma) = 0$ by monotonicity of modulus.

$\square$

Remark 3.3.4. We say that a property holds for ∞-*a.e. curve*, if it fails only for an ∞-exceptional set of curves. For instance, Lemma 3.3.3 says that if $\mu(F) = 0$, then ∞-a.e. curve does not spend positive time in F.

We are now ready to define $\mathrm{Mod}_\infty(\Gamma)$ mentioned earlier.

Definition 3.3.5. We say that $\rho : X \to [0, \infty)$ is *weakly admissible* and write $\rho \in \text{w-Adm}(\Gamma)$, if

$$\int_\gamma \rho\, ds \geq 1 \qquad \text{for } \infty\text{-a.e. } \gamma \in \Gamma.$$

Then, ∞-*modulus* of a family Γ is

$$\mathrm{Mod}_\infty(\Gamma) := \inf_{\rho \in \text{w-Adm}(\Gamma)} \|\rho\|_\infty.$$

Since it is easier to be weakly admissible, $\mathrm{Mod}_\infty(\Gamma) \leq \mathrm{Mod}_\infty^*(\Gamma)$. In particular, an ∞-

exceptional family has ∞-modulus zero.

Lemma 3.3.6 ([6]). *We have*

$$\mathrm{Mod}_\infty(\Gamma) = \mathrm{Mod}_\infty^*(\Gamma).$$

Proof. In light of the above remark, it suffices to show that $\mathrm{Mod}_\infty(\Gamma) \geq \mathrm{Mod}_\infty^*(\Gamma)$. To this end, let ρ be weakly admissible for Γ. Then there is a family $\Gamma_0 \subset \Gamma$ with $\mathrm{Mod}_\infty^*(\Gamma_0) = 0$ such that whenever $\gamma \in \Gamma \setminus \Gamma_0$ we have $\int_\gamma \rho\, ds \geq 1$. By Lemma 3.3.3 there is a Borel function $\rho_0 : X \to [0, \infty]$ such that $\|\rho_0\|_\infty = 0$ and for each $\gamma \in \Gamma_0$ we have $\int_\gamma \rho_0\, ds = \infty$. Note then that $h := \rho + \rho_0$ belongs to $\mathrm{Adm}(\Gamma)$. Thus

$$\mathrm{Mod}_\infty^*(\Gamma) \leq \|h\|_\infty = \|\rho + \rho_0\|_\infty \leq \|\rho\|_\infty + \|\rho_0\|_\infty = \|\rho\|_\infty.$$

Taking the infimum over all such ρ that are weakly admissible for Γ yields that

$$\mathrm{Mod}_\infty^*(\Gamma) \leq \mathrm{Mod}_\infty(\Gamma)$$

as desired. $\square$

We will now show a result similar to (3.2.1) for ∞-Modulus.

Definition 3.3.7. Let Γ be a family of curves. For every $a \geq 0$, let

$$\Gamma(a) := \{\gamma \in \Gamma : \ell(\gamma) < a\}. \tag{3.3.5}$$

Define the *essential length* of Γ as

$$\mathrm{ess}\ell(\Gamma) := \sup\{a \geq 0 : \Gamma(a) \text{ is } \infty\text{-exceptional}\}$$

By definition, we always have

$$\ell(\Gamma) \leq \mathrm{ess}\ell(\Gamma).$$

Remark 3.3.8. Note that for all $a < \mathrm{ess}\ell(\Gamma)$, we have that $\Gamma(a)$ is ∞-exceptional. For simplicity of notation, write $a_0 := \mathrm{ess}\ell(\Gamma)$. By continuity from below of Mod_∞^*, this means that $\Gamma(a_0)$ is ∞-exceptional.

Theorem 3.3.9 ([6])**.** *Let (X, d, μ) be a metric measure space with μ Borel. Let Γ be a family of curves with $\mathrm{ess}\ell(\Gamma) > 0$. Then*

$$\mathrm{Mod}_\infty(\Gamma) = \frac{1}{\mathrm{ess}\ell(\Gamma)}.$$

Moreover, $\rho_0 \equiv \mathrm{ess}\ell(\Gamma)^{-1}$ is an extremal weakly admissible density.

Proof. Assume Γ is not ∞-exceptional.

Let $\rho \in \text{w-Adm}(\Gamma)$. Set $F = \{x : \rho(x) > \|\rho\|_\infty\}$. Then $\mu(F) = 0$, so Γ_F^+ is ∞-exceptional.

Also, by definition of weakly admissible, the family

$$\tilde{\Gamma} = \left\{\gamma \in \Gamma : \int_\gamma \rho \, ds < 1\right\}$$

is ∞-exceptional.

Finally, as seen in Remark 3.3.8, $\Gamma(a_0)$ is ∞-exceptional.

By subadditivity $\Gamma_F^+ \cup \tilde{\Gamma} \cup \Gamma(a_0)$ is ∞-exceptional, and since Γ is not ∞-exceptional, we can find at least one curve $\gamma \in \Gamma \setminus (\Gamma_F^+ \cup \tilde{\Gamma} \cup \Gamma(a_0))$. Then, for these curves, we have

$$1 \leq \int_\gamma \rho \, ds = \int_{\gamma \cap F} \rho \, ds + \int_{\gamma \setminus F} \rho \, ds = \int_{\gamma \setminus F} \rho \, ds \leq \|\rho\|_\infty \ell(\gamma).$$

Since for every $\varepsilon > 0$, $\Gamma(a_0 + \varepsilon)$ is not ∞-exceptional, it is in particular non-empty. So we find that

$$1 \leq \|\rho\|_\infty \mathrm{ess}\ell(\Gamma).$$

But since $\rho \in \text{w-Adm}(\Gamma)$ was arbitrary, we obtain that

$$\mathrm{Mod}_\infty(\Gamma) \geq \frac{1}{\mathrm{ess}\ell(\Gamma)}.$$

Conversely, let $\rho_0 \equiv \text{ess}\ell(\Gamma)^{-1}$. Then, for every $\gamma \in \Gamma \setminus \Gamma(a_0)$, we have $\ell(\gamma) \geq \text{ess}\ell(\Gamma)$. Thus

$$\int_\gamma \rho_0 \, ds = \frac{\ell(\gamma)}{\text{ess}\ell(\Gamma)} \geq 1.$$

And by Remark 3.3.8, $\Gamma(a_0)$ is ∞-exceptional. So ρ_0 is weakly admissible and

$$\text{Mod}_\infty(\Gamma) \leq \frac{1}{\text{ess}\ell(\Gamma)}.$$

It now remains to consider the case that Γ is ∞-exceptional. In this case, the zero function is weakly admissible for Γ and hence $\text{Mod}_\infty(\Gamma) = 0$. Furthermore, for each $a > 0$ we have that $\Gamma(a) \subset \Gamma$ and hence $\Gamma(a)$ is ∞-exceptional. Thus $\text{ess}\ell(\Gamma) = \infty$, and the claim follows from this.

Observe also that $\text{ess}\ell(\Gamma) = \infty$ if and only if Γ is ∞-exceptional (the collection of all locally rectifiable but unrectifiable curves have $\rho_\varepsilon = \varepsilon$ as an admissible function for all $\varepsilon > 0$). □

3.4 The Essential Metric

If we choose the "connecting" families $\Gamma(x, y)$ consisting of all curves connecting x to y, we can define

$$d_{\text{ess}}(x, y) := \text{Mod}_\infty(\Gamma(x, y))^{-1} = \text{ess}\ell(\Gamma(x, y)).$$

Note that $d_{\text{ess}}(x, y)$ could be infinite for some $x, y \in X$.

Theorem 3.4.1 ([2]). *$d_{\text{ess}}(x, y)$ is a pseudometric on X. Furthermore, if for each $x, y \in X$ with $x \neq y$ we have $\text{Mod}^*_\infty(\Gamma(x, y)) > 0$, then d_{ess} is a metric on X.*

Proof. Let $x, y \in X$ with $x \neq y$. Then as every curve γ connecting x to y satisfies $\ell(\gamma) \geq d(x, y)$, it follows that $d_{\text{ess}}(x, y) \geq d(x, y) > 0$. It is also clear that $d_{\text{ess}}(x, y) = d_{\text{ess}}(y, x)$.

Note that for each $a > 0$ the family $\Gamma(x, x)(a)$ (using the notation from (3.3.5)) contains the constant curve $\gamma(t) = x$, and hence $\text{Mod}_\infty(\Gamma(x, x)(a)) = \infty$. It follows that $\text{ess}\ell(\Gamma(x, x)) = 0$, that is, $d_{\text{ess}}(x, x) = 0$.

Next, we verify the triangle inequality. Fix three distinct points $a, b, c \in X$. For simplicity, let $\Gamma_1 = \Gamma(a, c)$, $\Gamma_2 = \Gamma(c, b)$ and $\Gamma_0 = \Gamma(a, b)$. Also assume that $d_0 = d_{\text{ess}}(a, b)$, $d_1 = d_{\text{ess}}(a, c)$, $d_2 = d_{\text{ess}}(c, b) > 0$. Let $\Gamma = \Gamma(a, b; c)$ be the family of all curves that start at a, end at b, and go through c. Then $\Gamma \subset \Gamma_0$, and so $d_0 = \text{ess}\ell(\Gamma_0) \leq \text{ess}\ell(\Gamma)$.

If $d_1 + d_2 = \infty$, then clearly $d_0 \leq d_1 + d_2$. So it suffices to verify the triangle inequality only in the case that both $d_1 < \infty$ and $d_2 < \infty$. Suppose that $\lambda = d_1 + d_2 + 2\varepsilon$ for some $\varepsilon > 0$. Then $\text{Mod}^*_\infty(\Gamma_j(d_j + \varepsilon)) > 0$ for $j = 1, 2$. Note that $h = \frac{1}{d(a,b)}$ is admissible for computing $\text{Mod}^*_\infty(\Gamma(\lambda))$, so $\text{Mod}^*_\infty(\Gamma(\lambda)) < \infty$. Let ρ be admissible for $\Gamma(\lambda)$. Then for $\gamma \in \Gamma_1(d_1 + \varepsilon)$ and $\beta \in \Gamma_2(d_2 + \varepsilon)$, we must have the concatenation $\gamma + \beta \in \Gamma(\lambda)$ and so $\int_{\gamma+\beta} \rho\, ds \geq 1$. For $j = 1, 2$ let

$$\ell_\rho(\Gamma_j(d_j + \varepsilon)) := \inf_{\gamma \in \Gamma_j(d_j+\varepsilon)} \int_\gamma \rho\, ds.$$

Then by the above,

$$\ell_\rho(\Gamma_1(d_1 + \varepsilon)) + \ell_\rho(\Gamma_2(d_2 + \varepsilon)) \geq 1. \tag{3.4.1}$$

Moreover $\rho \cdot \ell_\rho(\Gamma_j(d_j + \varepsilon))^{-1} \in \text{Adm}(\Gamma_j(d_j + \varepsilon))$ for $j = 1, 2$. So for $j = 1, 2$ we have

$$\ell_\rho(\Gamma_j(d_j + \varepsilon)) \leq \frac{\|\rho\|_\infty}{\text{Mod}^*_\infty(\Gamma_j(d_j + \varepsilon))}.$$

Now by (3.4.1),

$$1 \leq \|\rho\|_\infty \left[\frac{1}{\text{Mod}^*_\infty(\Gamma_1(d_1 + \varepsilon))} + \frac{1}{\text{Mod}^*_\infty(\Gamma_2(d_2 + \varepsilon))}\right],$$

and taking the infimum over all such ρ yields

$$\frac{1}{\text{Mod}^*_\infty(\Gamma(\lambda))} \leq \left[\frac{1}{\text{Mod}^*_\infty(\Gamma_1(d_1 + \varepsilon))} + \frac{1}{\text{Mod}^*_\infty(\Gamma_2(d_2 + \varepsilon))}\right] < \infty,$$

from which we conclude that $\text{Mod}^*_\infty(\Gamma(\lambda)) > 0$.

Hence, $d_0 \leq d_1 + d_2 + 2\varepsilon$, and by letting ε tend to 0, we see that $d_0 \leq d_1 + d_2$.

Finally, suppose that for each $x, y \in X$ with $x \neq y$ we have $\mathrm{Mod}_\infty^*(\Gamma(x,y)) > 0$. Then to show that d_{ess} is a metric on X it only remains to show that $d_{\mathrm{ess}}(x,y) < \infty$ for $x, y \in X$. Note that as $\mathrm{Mod}_\infty^*(\Gamma(x,y)) = \lim_{n\to\infty} \mathrm{Mod}_\infty^*(\Gamma(x,y)(n))$, by the hypothesis that $\mathrm{Mod}_\infty^*(\Gamma(x,y)) > 0$ we must have $\mathrm{Mod}_\infty^*(\Gamma(x,y)(n)) > 0$ for some $n \in \mathbb{N}$ (since the Mod_∞^*-measure of the collection of all non-rectifiable curves is zero). Thus we must have $d_{\mathrm{ess}}(x,y) \le n < \infty$, completing the proof. $\square$

The following definition is from [14], and is due to De Cecco and Palmieri.

Definition 3.4.2. Given a set $N \subset X$ with $\mu(N) = 0$, for $x, y \in X$ with $x \neq y$ we set

$$d_N(x,y) = \inf\{\ell(\gamma) \,:\, \gamma \text{ connects } x \text{ to } y, \text{ and } m_1(\gamma^{-1}(N)) = 0\}.$$

Then we define $\widehat{d} : X \times X \to \mathbb{R}$ by $\widehat{d}(x,x) = 0$ and for $x \neq y$,

$$\widehat{d}(x,y) = \sup\{d_N(x,y) \,:\, N \subset X \text{ with } \mu(N) = 0\}.$$

As with d_{ess}, it might well be that $\widehat{d}$ is not finite.

It was shown in that paper that if μ is doubling and X is complete, then $\widehat{d}$ is biLipschitz equivalent to the original metric d if and only if X supports an ∞-Poincaré inequality.

Proposition 3.4.3 ([6])**.** *We always have $\widehat{d} = d_{ess}$ on X.*

Proof. Fix $x, y \in X$ with $x \neq y$. We will first show that $\widehat{d}(x,y) \le d_{\mathrm{ess}}(x,y)$. To this end, note that if $d_{ess}(x,y) = \infty$, then the above inequality holds trivially. Therefore, let us assume that $d_{ess}(x,y) < \infty$. Recall that $\Gamma := \Gamma(x,y)$ denotes the collection of all rectifiable curves in X connecting x to y. Then for each $\varepsilon > 0$, if $a = [1+\varepsilon] d_{\mathrm{ess}}(x,y)$, then $\mathrm{Mod}_\infty(\Gamma(a)) > 0$. So, by Lemma 3.3.3(d), whenever $N \subset X$ with $\mu(N) = 0$ there must exist a curve γ in $\Gamma(a)$ that does not spend positive time in N. In particular, $m_1(\gamma^{-1}(N)) = 0$. Hence $d_N(x,y) \le [1+\varepsilon] d_{\mathrm{ess}}(x,y)$. Taking the supremum over all such nulls sets N gives

$$\widehat{d}(x,y) \le [1+\varepsilon] d_{ess}(x,y).$$

Letting $\varepsilon \to 0$ now gives the desired inequality.

We next show that $d_{\text{ess}}(x,y) \le \widehat{d}(x,y)$. First suppose that $d_{\text{ess}}(x,y) = \infty$. Then for each $a > 0$ we know that $\text{Mod}_\infty(\Gamma(a)) = 0$, and so there is a non-negative Borel function ρ on X with $\|\rho\|_{L^\infty(X)} = 0$ such that $\int_\gamma \rho\, ds = \infty$ for each $\gamma \in \Gamma(a)$. Let $N = \{x \in X : \rho(x) > 0\}$. Then $\mu(N) = 0$ and for every $\gamma \in \Gamma(a)$, $m_1(\gamma^{-1}(N)) > 0$. Therefore $\widehat{d}(x,y) \ge d_N(x,y) \ge a$. Since this holds true for each $a > 0$, we have that $\widehat{d}(x,y) = \infty$.

Now we consider the case that $d_{\text{ess}}(x,y) < \infty$. Then since $d_{\text{ess}}(x,y) \ge d(x,y) > 0$, we have that for $0 < \varepsilon < 1$, $\text{Mod}_\infty(\Gamma(a)) = 0$, with $a = [1-\varepsilon]d_{\text{ess}}(x,y)$. Therefore, there is some Borel function $\rho : X \to [0,\infty]$ such that $\|\rho\|_{L^\infty(X)} = 0$ and $\int_\gamma \rho\, ds = \infty$ for each $\gamma \in \Gamma(a)$. Thus, if $N = \{x \in X : \rho(x) > 0\}$, we see that $\mu(N) = 0$ and for each $\gamma \in \Gamma(a)$, $\mathcal{H}^1(\gamma^{-1}(N)) > 0$. It follows that

$$\widehat{d}(x,y) \ge d_N(x,y) \ge [1-\varepsilon]d_{\text{ess}}(x,y).$$

Letting $\varepsilon \to 0$ in the above gives $\widehat{d}(x,y) \ge d_{\text{ess}}(x,y)$ as desired. $\square$

Chapter 4

Duality of Modulus in $\mathbb{R}^N$

Here, we turn to the case when the metric space we are interested in is $\mathbb{R}^N$, $N \geq 2$, equipped with the Euclidean metric and N-dimensional Lebesgue measure. We will work towards a slightly modified version of the Corollary to Theorem 5 in [3], which is a generalization of the duality principle stated in Chapters one and two. See Theorem 4.2.6. The work in [3] is more general than what we need. Using the notation in [3], we will work in the setting where the matrix $\mathcal{A}(x)$ is the identity matrix for each x and the function $w(x) \equiv 1$. Throughout, p and q are assumed to be Hölder conjugates, that is $1/p + 1/q = 1$ and $p, q \in (1, \infty)$.

4.1 Definitions

4.1.1 Modulus of Measures

We begin by recalling Fuglede's definition of modulus of a family of measures [17]. Let $\mathcal{M}$ be a collection of (non-negative) Borel measures. We then define the *admissible set* and the *p-Energy* as follows:

$$\mathrm{Adm}(\mathcal{M}) := \left\{ \rho : \inf_{\mu \in \mathcal{M}} \int \rho \, d\mu \geq 1 \right\} \tag{4.1.1}$$

$$\mathcal{E}_p(\rho) := \int \rho^p \, dx, \tag{4.1.2}$$

where ρ is any Borel measurable function from an open bounded domain $\Omega \subset \mathbb{R}^N$ to $[0, \infty]$, and dx denotes the standard N-dimensional Lebesgue measure. We then define the p-Modulus as

$$\mathrm{Mod}_p(\mathcal{M}) := \inf_{\rho \in \mathrm{Adm}(\mathcal{M})} \mathcal{E}_p(\rho). \tag{4.1.3}$$

In the case that $\mathrm{Adm}(\mathcal{M}) = \emptyset$, we define $\mathrm{Mod}_p(\mathcal{M}) := \infty$.

Fuglede proved several properties of p-Modulus of a system of measures in [17]. We state those here as Lemma 4.1.1. Here, $|E|$ denotes the Lebesgue measure of E; also $\bar{\mu}$ denotes the completion of μ; and, as before, we say that a property holds for $p-$a.e. $\mu \in \mathcal{M}$, if it holds for $\mu \in \mathcal{M} \setminus \mathcal{M}_0$, where $\mathrm{Mod}_p(\mathcal{M}_0) = 0$.

Lemma 4.1.1 (Fuglede [17]). $\mathrm{Mod}_p(\mathcal{M})$ *has the following properties:*

(1) $\mathcal{M} \subset \mathcal{M}' \Rightarrow \mathrm{Mod}_p(\mathcal{M}) \leq \mathrm{Mod}_p(\mathcal{M}')$

(2) $\mathrm{Mod}_p(\cup_{j=1}^{\infty} \mathcal{M}_j) \leq \sum_{j=1}^{\infty} \mathrm{Mod}_p(\mathcal{M}_j)$

(3) $|E| = 0 \Rightarrow \bar{\mu}(E) = 0$ *for* $p-$*a.e.* $\mu \in \mathcal{M}$.

(4) If $\rho \in L^p(\mathbb{R}^N)$, *then* ρ *is* $\bar{\mu}$*-integrable for* $p-$*a.e.* $\mu \in \mathcal{M}$.

(5) If $\|\rho_i - \rho\|_p \to 0$, *then there is a subsequence* ρ_{i_k} *such that* $\int_{\mathbb{R}^N} |\rho_{i_k} - \rho| \, d\bar{\mu} \to 0$ *for* $p-$*a.e.* $\mu \in \mathcal{M}$.

(6) $\mathrm{Mod}_p(\mathcal{M}) = 0 \iff \exists \rho$ *such that* $\|\rho\|_p < \infty, \int_{R^N} \rho \, d\mu = \infty, \forall \mu \in \mathcal{M}$.

(7) There exists ρ^* *such that* $\int_{\mathbb{R}^N} \rho^* \, d\mu \geq 1$ *for* $p-$*a.e.* $\mu \in \mathcal{M}$ *and* $\mathrm{Mod}_p(\mathcal{M}) = \mathcal{E}_p(\rho^*)$.

4.1.2 Modulus of Vectors of Veasures

We will now use our definition of $\mathrm{Mod}_p(\mathcal{M})$ to help us define p-Modulus for vector measures. We say ν is a *vector measure* if it is a vector whose components are (finite) signed measures. That is, for any Borel set E,

$$\nu(E) = (\nu_1(E), \nu_2(E), \ldots, \nu_N(E)),$$

where ν_i is a finite signed measure. The *total variation* of a vector measure ν, denoted by $|\nu|$, is defined by

$$|\nu|(E) := \sup \sum_j \left(\sum_{i=1}^{N} \nu_i(E_j)^2 \right)^{1/2},$$

where the supremum is over all finite partitions $\{E_j\}$ of E into Borel sets. The total variation of a vector measure is a (non-negative) measure on $\mathbb{R}^N$. Given a function $f = (f_1, f_2, \ldots, f_N) : \mathbb{R}^N \to \mathbb{R}^N$ such that $\int |f_i| \, d|\nu_i| < \infty$ for each i, we define $\int f \cdot d\nu := \sum_{i=1}^{N} \int f_i \, d\nu_i$. In [22], it is shown that $|\int f \cdot d\nu| \leq \int |f| \, d|\nu|$.

Definition 4.1.2. Let $\mathcal{F}_0$ be a set of vector measures ν. Let $|\mathcal{F}_0| = \{|\nu| : \nu \in \mathcal{F}_0\}$. If $\mathrm{Mod}_p(|\mathcal{F}_0|) = 0$, we say $\mathcal{F}_0$ is p-exceptional. Also, if $\mathcal{F}_0 \subset \mathcal{F}$ and some property holds for all measures $\nu \in \mathcal{F} \backslash \mathcal{F}_0$, then we say that property holds p-a.e. in $\mathcal{F}$.

Because of a strange phenomenon that will be described below, it is necessary to use a notion of weak admissibility when defining p-Modulus for vector measures, unlike p-Modulus for curve families or systems of measures. First, for a Borel measurable function $f : \mathbb{R}^N \to \mathbb{R}^N$ and a set of vector measures $\mathcal{N}$, we define

$$\mathcal{N}_f := \left\{ \nu : \int f \cdot d\nu < 1 \right\}.$$

We define the *weakly admissible set* as

$$\text{w-Adm}(\mathcal{M}) := \{ f : \mathrm{Mod}_p(|\mathcal{N}_f|) = 0 \}. \tag{4.1.4}$$

Thus, we define *p-Modulus for a set of vector measures* as

$$\mathrm{Mod}_p(\mathcal{N}) := \inf \{ \mathcal{E}_p(f) : f \in \text{w-Adm}(\mathcal{N}) \}, \tag{4.1.5}$$

where $\mathcal{E}_p(f) := \int |f|^p \, dx$.

Now we return to our discussion of the necessity of using $\text{w-Adm}(\mathcal{N})$ instead of $\mathrm{Adm}(\mathcal{N})$.

Example 4.1.3. Consider a line segment S in $\mathbb{R}^3$ joining $(0, 0, 0)$ to $(1, 0, 0)$. Consider the family of vector measures $\mathcal{N} = \{(\mathcal{H}^1|_S, 0, 0), (-\mathcal{H}^1_S, 0, 0)\}$. Since

$$\int f \cdot d(\mathcal{H}^1, 0, 0) \geq 1 \Rightarrow \int f \cdot d(-\mathcal{H}^1, 0, 0) \leq -1,$$

we have $\{f : \inf_{\nu \in \mathcal{N}} \int f \cdot d\nu \geq 1\} = \varnothing$. In the case where we consider Modulus of families of measures, we would say the Modulus is infinite. However, $|\mathcal{N}| = \{\mathcal{H}^1|_S\}$ and we have $\mathrm{Mod}_p(|\mathcal{N}|) = 0$ since $\rho = (1, 0, 0)\chi_S$ is admissible, but $\|\rho\|_p = 0$ since $m_N(S) = 0$.

Now, although we need weak admissibility for general sets of vector measures, some of the specific examples we will work with will only have the empty set as a p-exceptional set. For these examples, we can disregard the need for weak admissibility.

Note that p-Modulus for a set of vectors of measures is increasing, as can be seen directly from the definition. We will make use of the following proposition, which is proved similarly to the case for p-Modulus of families of curves.

Proposition 4.1.4 ([3]). *Let $\{\mathcal{F}_j\}$ be a sequence of families of vectors of measures which satisfies $\mathcal{F}_j \subset \mathcal{F}_{j+1}$ and $\mathcal{F} = \cup_j \mathcal{F}_j$. Then*

$$\mathrm{Mod}_p(\mathcal{F}_j) \nearrow \mathrm{Mod}_p(\mathcal{F}).$$

Proof. Monotonicity gives us

$$\mathrm{Mod}_p(\mathcal{F}_j) \leq \mathrm{Mod}_p(\mathcal{F}_{j+1}) \leq \mathrm{Mod}_p(\mathcal{F})$$

and so we may assume $\mathrm{Mod}_p(\mathcal{F}_j) \to \alpha < \infty$. Thus, it is enough to show $\mathrm{Mod}_p(\mathcal{F}) \leq \alpha$. Let $\rho_j \in \mathrm{Adm}(\mathcal{F}_j)$ such that

$$\mathrm{Mod}_p(\mathcal{F}_j) \leq \|\rho_j\|_p^p \leq \mathrm{Mod}_p(\mathcal{F}_j) + \frac{1}{j} \tag{4.1.6}$$

for $j \geq 1$. Assume $j \leq i$ and consider $\frac{1}{2}(\rho_j + \rho_i)$. Then $\frac{1}{2}(\rho_j + \rho_i) \in \mathrm{Adm}(\mathcal{F}_j)$ since $\rho_i \in \mathrm{Adm}(\mathcal{F}_j)$. Thus $\mathrm{Mod}_p(\mathcal{F}_j) \leq \left\|\left(\frac{1}{2}(\rho_j + \rho_i)\right)\right\|_p^p$. If we assume $p \geq 2$, then Clarkson's inequality gives us

$$\left\|\frac{1}{2}(\rho_i+\rho_j)\right\|_p^p+\left\|\frac{1}{2}(\rho_i-\rho_j)\right\|_p^p\leq\frac{1}{2}\left\|\rho_i\right\|_p^p+\frac{1}{2}\left\|\rho_j\right\|_p^p. \tag{4.1.7}$$

Hence, using admissibility of $\frac{1}{2}(\rho_i+\rho_j)$ with respect to $\mathcal{F}_j$, we have

$$\begin{aligned}\left\|\frac{1}{2}(\rho_i-\rho_j)\right\|_p^p&\leq\frac{1}{2}\left(\|\rho_i\|_p^p+\|\rho_j\|_p^p\right)-\left\|\frac{1}{2}(\rho_i+\rho_j)\right\|_p^p\\&\leq\frac{1}{2}\left(\|\rho_i\|_p^p+\|\rho_j\|_p^p\right)-\mathrm{Mod}_p(\mathcal{F}_j)\end{aligned}$$

where the right side tends to zero by (4.1.6). So, we have $\limsup_{i,j\to\infty}\left\|\frac{1}{2}(\rho_i-\rho_j)\right\|_p\leq 0$. Let ρ' satisfy $\left\|\rho_i-\rho'\right\|_p\to 0$. Then $\|\rho'\|_p^p=\alpha$. Without loss of generality (by Lemma 4.1.1 (5)), we have $\int|\rho_i-\rho'|\,d|\mu|\to 0$ for p-a.e. $\mu\in\mathcal{F}$, which gives us

$$\int\rho' d\mu=\int\rho_i\,d\mu+\int(\rho'-\rho_i)\,d\mu\geq\liminf\int\rho_i d\mu\geq 1\qquad p-a.e.\ \mu\in\mathcal{F},$$

so $\mathrm{Mod}_p(\mathcal{F})\leq\alpha$.

For the case $1<p<2$, we use the other Clarkson inequality

$$\left\|\frac{1}{2}(\rho_i+\rho_j)\right\|_p^q+\left\|\frac{1}{2}(\rho_i-\rho_j)\right\|_p^q\leq\left(\frac{1}{2}(\|\rho_i\|_p^p+\|\rho_j\|_p^p\right)^{q/p}. \tag{4.1.8}$$

Then we have

$$\begin{aligned}\left\|\frac{1}{2}(\rho_i-\rho_j)\right\|_p^q&\leq\left(\frac{1}{2}(\|\rho_i\|_p^p+\|\rho_j\|_p^p\right)^{q/p}-\left\|\frac{1}{2}(\rho_i+\rho_j)\right\|_p^q\\&\leq\left(\frac{1}{2}(\|\rho_i\|_p^p+\|\rho_j\|_p^p)\right)^{q/p}-\mathrm{Mod}_p(\mathcal{F}_j)^{q/p},\end{aligned}$$

and we proceed exactly as in the case $p\geq 2$. $\square$

Now, the first family of vector measures we are interested in will be in terms of curve families. Let K_0 and K_1 be disjoint compact sets that satisfy $K_j\cap\overline{\Omega}\neq\varnothing$. Throughout, we will treat Ω as a fixed bounded and connected open subset of $\mathbb{R}^N$. In what follows, we will occasionally consider open subsets of Ω. The sets K_0 and K_1 can be assumed to satisfy $K_0 \cup K_1 \subset \overline{\Omega}$ by taking an intersection with $\overline{\Omega}$. Define

$$\Gamma = \Gamma(K_0, K_1, \Omega) = \{\text{rectifiable curves in } \Omega \text{ starting at } K_0 \cap \overline{\Omega} \text{ and ending at } K_1 \cap \overline{\Omega}\}.$$

Since $\gamma \in \Gamma$ is rectifiable, it can be parameterized by arc-length. Denote this parameterization by $\gamma_s(t) \in \mathbb{R}^N$. Since this parameterization is always 1-Lipschitz (See Proposition 3.1.7), it has derivatives at a.e. point in $[0, \ell(\gamma)]$. Then, we have a vector of measures defined by

$$\gamma(E) = \int_{[0,\ell(\gamma)]} \gamma_s'(t)\chi_E(\gamma_s(t))\, dt.$$

For all Borel measurable functions $f : \mathbb{R}^N \to \mathbb{R}^N$ define the *line integral of f along γ* as

$$\int f \cdot d\gamma = \int_{[0,\ell(\gamma)]} f(\gamma(t)) \cdot \gamma_s'(t)\, dt \tag{4.1.9}$$

provided that

$$\int |f|\, d|\gamma| = \int_{[0,\ell(\gamma)]} |f(\gamma_s(t))||\gamma_s'(t)|\, dt < \infty.$$

Here, we are abusing notation by refering to γ as a set, a function, and a vector of measures. Let $d\Gamma = \{d\gamma : \gamma \in \Gamma\}$ and $d|\Gamma| = \{d|\gamma| : \gamma \in \Gamma\}$. Note that using this notation, for a Borel function $\rho : \mathbb{R}^N \to [0, \infty]$, we have

$$\int_\gamma \rho\, ds = \int \rho\, d|\gamma|.$$

4.1.3 Gradients as Vectors of Measures

We now consider the gradients of certain classes of functions as vectors of measures. For functions that are 'nice enough,' the gradients will also be functions. Although the work in [3] is more general, we will work in the case where Ω is a bounded and connected open subset of $\mathbb{R}^N$.

Definition 4.1.5. Let u be a non-negative continuous function on Ω. Then we say u is *p-precise* if u is absolutely continuous on p-a.e. curve and $\nabla u \in L^p(\Omega)$.

Remark 4.1.6. Note that because a p-precise function u is absolutely continuous on p-a.e. curve, it has partial derivatives on m_{n-1} a.e. line segment parallel to a coordinate axis. For further details, see the discussion of Absolute Continuity on Lines on page 148 of [20].

Now, we wish to note the difference between p-precise functions and the Sobolev space $W^{1,p}(\Omega)$. We recall the definition of the Sobolev space $W^{1,p}(\Omega)$ and the following theorem relating p-precise functions which is Corollary 4.4.6 and Remark 4.4.1 of [22].

Definition 4.1.7. A function f is in the Sobolev space $W^{1,p}(\Omega)$ if f is in $L^p(\Omega)$ and the weak partial derivatives of f are in $L^p(\Omega)$.

Theorem 4.1.8 ([22])**.** *Every function f of the Sobolev space $W^{1,p}(\Omega)$ is equal a.e. to a p-precise function f_0 in Ω such that the distributional derivatives of f are equal to the derivatives of f_0. Further, every p-precise function in $L^p(\Omega)$ belongs to $W^{1,p}(\Omega)$.*

Definition 4.1.9. We define the following:

$$\mathcal{D}^*\left(K_0, K_1, \Omega\right) := \{u : u \text{ is } p\text{-precise on } \Omega,\ u = 0 \text{ on } K_0,\ u = 1 \text{ on } K_1\}$$

$$\mathcal{D}\left(K_0, K_1, \Omega\right) := \{u : u \text{ is } p\text{-precise on } \Omega,\ \lim_{x \xrightarrow{\gamma} K_0} u = 0,\ \lim_{x \xrightarrow{\gamma} K_1} u = 1, \text{ for } p\text{-a.e. } \gamma\}$$

$$\nabla\mathcal{D}^* := \nabla\mathcal{D}^*(K_0, K_1, \Omega) := \{\nabla u : u \in \mathcal{D}^*\}$$

$$\nabla\mathcal{D} = \nabla\mathcal{D}(K_0, K_1, \Omega) := \{\nabla u : u \in \mathcal{D}\}$$

Remark 4.1.10. For the families $\nabla\mathcal{D}^*$ and $\nabla\mathcal{D}$, we have if $\mathrm{Mod}_q(\mathcal{F}) = 0$ (that is, if $\mathcal{F}$ is q-exceptional), then $\mathcal{F} = \varnothing$. Indeed, suppose $\mathcal{F}$ is a q-exceptional subfamily of either of the above mentioned families. Then, by property 6 of Lemma 4.1.1, there is a ρ satisfying $\int \rho\,|\nabla u|\,dx = \infty$ for every $|\nabla u| \in \mathcal{F}$ and $\|\rho\|_q < \infty$. However, applying Hölder's inequality, we have

$$\int_\Omega \rho|\nabla u|\,dx \leq \|\nabla u\|_p\,\|\rho\|_q\,,$$

giving us a contradiction if $\mathcal{F} \neq \varnothing$, since the right hand side must be finite for all $|\nabla u|$. Consequently, for $\nabla\mathcal{D}^*$ and $\nabla\mathcal{D}$, weak admissibility is equivalent to requiring $\int f \cdot \nabla u \, dx \geq 1$ for every u.

Definition 4.1.11. Define the total variation norm for functions of bounded variation as

$$\|u\|_{BV} = \sup\left\{\int_\Omega u \operatorname{div} \xi dx : \xi \in C_0^\infty(\Omega, X), \|\xi\|_2 \leq 1\right\}$$

Definition 4.1.12. Define the perimeter of E relative to Ω as

$$P_\Omega(E) := \|\chi_{E\cap\Omega}\|_{BV}$$

Let $S = S(K_0, K_1, \Omega)$ be the family of functions $u \in BV(\Omega)$ such that $u = 0$ on some neighborhood of K_0, $u = 1$ on some neighborhood of K_1. Let $\Sigma = \Sigma(K_0, K_1, \Omega)$ be a family of characteristic functions, χ_E, where $E \subset \Omega$ with $P_\Omega(E) < \infty$ and $E = U \cap \Omega$, U open in $\mathbb{R}^N$, $K_0 \cap \overline{U} = \varnothing$, $K_1 \subset U$. Define ∇S, $|\nabla S|$, $\nabla\Sigma$, and $|\nabla\Sigma|$ as follows:

$$\nabla S = \nabla S(K_0, K_1, \Omega) = \{\nabla u : u \in S(K_0, K_1, \Omega)\}$$

$$|\nabla S| = |\nabla S(K_0, K_1, \Omega)| = \{|\nabla u| : u \in S(K_0, K_1, \Omega)\}$$

$$\nabla\Sigma = \nabla\Sigma(K_0, K_1, \Omega) = \{\nabla\chi_E : \chi_E \in \Sigma(K_0, K_1, \Omega)\}$$

$$|\nabla\Sigma| = |\nabla\Sigma(K_0, K_1, \Omega)| = \{|\nabla\chi_E| : \chi_E \in \Sigma(K_0, K_1, \Omega)\}$$

Definition 4.1.13. Define the following capacities:

$$\mathrm{C_p}(\mathrm{K_0}, \mathrm{K_1}, \Omega) := \inf_{u\in\mathcal{D}} \int_\Omega |\nabla u|^p dx$$

$$\mathrm{C_p^*}(\mathrm{K_0}, \mathrm{K_1}, \Omega) := \inf_{u\in\mathcal{D}^*} \int_\Omega |\nabla u|^p dx.$$

Note that since $\mathcal{D}^* \subset \mathcal{D}$, we have

$$\mathrm{C_p}(\mathrm{K_0}, \mathrm{K_1}, \Omega) \le \mathrm{C_p^*}(\mathrm{K_0}, \mathrm{K_1}, \Omega) \tag{4.1.10}$$

The other inequality is Theorem 5 of [3], see Theorem (4.2.9) below.

4.2 Aikawa-Ohtsuka duality

4.2.1 Approximation by C^∞

Typically when we are interested in approximating $f \in W^{1,p}(G)$ or $f \in BV(G)$ for some domain G, we use a convolution of f with a certain function ψ_ε that we call a mollifier. The standard mollifier is defined as

$$\psi(x) = \begin{cases} Ce^{1/(|x|^2-1)} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \ge 1, \end{cases} \tag{4.2.1}$$

where

$$C := \frac{1}{\int_{|x|<1} e^{1/(|x|^2-1)}\, dx},$$

so that $\int \psi\, dx = 1$ and $\psi \in C_0^\infty(\mathbb{R}^N)$. Let $\varepsilon > 0$, and define $\psi_\varepsilon(x) = \frac{1}{\varepsilon^N}\psi(\frac{x}{\varepsilon})$. Let $G_\varepsilon := \{x \in G : \mathrm{dist}(x, \partial G) > \varepsilon\}$. Then to approximate f, for $x \in G_\varepsilon$ we define

$$f^\varepsilon(x) := \int_U \psi_\varepsilon(x-y) f(y)\, dy. \tag{4.2.2}$$

Much is known about such approximations. For a more detailed discussion, see Chapters 4 and 5 of [15]. For our purposes, we desire a way of approximating p-precise functions on the whole domain G, rather than the subset G_ε. To that end, we come up with a homeomorphism $T_{y,r} : G \to G$ by using a "nice" function α.

Lemma 4.2.1 ([3]). *There exists $\alpha \in C^\infty(G)$ such that*

(1) $0 < \alpha \leq 1$,

(2) $|\nabla \alpha| \leq \frac{1}{2}$,

(3) $\alpha(x) \leq \frac{1}{2} \operatorname{dist}(x, \partial G)$.

The following lemma explains how to use α to define a homeomorphism that we can use to build an approximation to a p-precise function.

Lemma 4.2.2 ([3]). *Let $|y| \leq 1$ and $0 < r < 1$. Then the mapping $T_{y,r} : x \mapsto x + r\alpha(x)y$ gives a C^∞ homeomorphism of G onto itself.*

Now we define our approximation. Let $f \in L^1_{loc}(G)$. Define

$$(f)_r(x) := \int_{|y|<1} f(x + r\alpha(x)y)\psi(y)\, dy = \frac{1}{(r\alpha(x))^N} \int_{\mathbb{R}^N} f(y)\psi\left(\frac{y-x}{r\alpha(x)}\right) dy. \tag{4.2.3}$$

We observe that $(f)_r \in C^\infty$, $(f)_r \to f$ as $r \to 0$ and $|(f)_r(x)| \leq cMf(x)$ for a.e. $x \in G$. Applying the dominated convergence theorem yields the following proposition.

Proposition 4.2.3 ([3]). *Let $f \in L^p(G)$. Then $\|(f)_r - f\|_p \to 0$ as $r \to 0$.*

Further, we obtain convergence of the derivatives by the following proposition.

Proposition 4.2.4 ([3],[22]). *Let $f \in L^p(G)$. Then $\|\nabla(f)_r - \nabla f\|_p \to 0$ as $r \to 0$ for all p-precise functions f on G.*

Suppose $u \in \mathcal{D}^*$. Then for sufficiently small $r > 0$, $(u)_r \in \mathcal{D}^*$. Letting $G = \Omega$ and applying Proposition 4.2.4 to $\mathrm{C}^*_\mathrm{p}(\mathrm{K}_0, \mathrm{K}_1, \Omega)$ and truncating, we obtain

Proposition 4.2.5 ([3]).

$$\mathrm{C}^*_\mathrm{p}(\mathrm{K}_0, \mathrm{K}_1, \Omega) = \inf_{\substack{u \in \mathcal{D}^* \cap C^\infty(\Omega) \\ 0 \leq u \leq 1}} \int_\Omega |\nabla u|^p\, dx.$$

4.2.2 Statements

As stated above, our goal is to first understand Theorem 5 and its Corollary in [3]. To that end, from [3], we state Theorem 1 as Theorem 4.2.10, Theorem 2 as Theorem as 4.2.7, Theorem 3 as Theorem 4.2.8, Theorem 4 as Theorem 4.2.11, Theorem 5 as Theorem 4.2.9, and its corollary as Theorem 4.2.6.

Our main interest is in the following duality result, which is reminiscent of Theorem 2.2.8. On one hand $|d\Gamma|$ can be thought as a family of curves connecting two compact sets, and on the other hand, $|\nabla\Sigma|$ is a family of separating surfaces.

Theorem 4.2.6 ([3]). *If* $\mathrm{Mod}_p(|d\Gamma|) = 0$*, then* $\mathrm{Mod}_q(|\nabla\Sigma|) = \infty$*; if* $\mathrm{Mod}_p(|d\Gamma|) > 0$*, then*

$$\mathrm{Mod}_p(|d\Gamma|)^{1/p}\,\mathrm{Mod}_q(|\nabla\Sigma|)^{1/q} = 1. \tag{4.2.4}$$

To prove this, the authors in [3] prove a chain of statements. First, they establish a duality statement for $\nabla\mathcal{D}^*$ and the capacity related to it.

Theorem 4.2.7 ([3]). $\mathrm{C}^*_{\mathrm{p}}(\mathrm{K}_0, \mathrm{K}_1, \Omega)$ *and* $\mathrm{Mod}_q(\nabla\mathcal{D}^*)$ *satisfy the following:*

(1) $\mathrm{C}^*_{\mathrm{p}}(\mathrm{K}_0, \mathrm{K}_1, \Omega) = 0 \Rightarrow \mathrm{Mod}_q(\nabla\mathcal{D}^*) = \infty$

(2) $\mathrm{C}^*_{\mathrm{p}}(\mathrm{K}_0, \mathrm{K}_1, \Omega) > 0 \Rightarrow \mathrm{C}^*_{\mathrm{p}}(\mathrm{K}_0, \mathrm{K}_1, \Omega)^{1/p}\,\mathrm{Mod}_q(\nabla\mathcal{D}^*)^{1/q} = 1.$

In particular, $\mathrm{Mod}_q(\nabla\mathcal{D}^*) > 0$.

Then, they relate $\mathrm{Mod}_q(\nabla\mathcal{D}^*)$ to $\mathrm{Mod}_q(|\nabla\Sigma|)$, by going through the family ∇S, which contains both of $\nabla\mathcal{D}^*$ and $\nabla\Sigma$.

Theorem 4.2.8 ([3]). *We have the chain of equalities*

$$\mathrm{Mod}_q(\nabla S) = \mathrm{Mod}_q(\nabla\Sigma) = \mathrm{Mod}_q(|\nabla S|) = \mathrm{Mod}_q(|\nabla\Sigma|) = \mathrm{Mod}_q(\nabla\mathcal{D}^*) > 0$$

This takes care of one term in the duality (4.2.4). To deal with $\mathrm{Mod}_p(|d\Gamma|)$, they first show that the inequality in (4.1.10) is an equality.

Theorem 4.2.9 ([3])**.** *We have the equality*

$$\mathrm{C_p}(\mathrm{K_0},\mathrm{K_1},\Omega) = \mathrm{C_p^*}(\mathrm{K_0},\mathrm{K_1},\Omega) \tag{4.2.5}$$

Moreover, suppose Ω *is bounded. If* $\mathrm{Mod}_P(d\Gamma) = 0$*, then* $\mathrm{Mod}_q(\nabla\Sigma) = \infty$*; if* $\mathrm{Mod}_q(d\Gamma) > 0$*, then*

$$\mathrm{Mod}_p(d\Gamma)^{1/p}\,\mathrm{Mod}_q(\nabla\Sigma)^{1/q} = 1.$$

Finally, they relate $\mathrm{Mod}_p(|d\Gamma|)$ to the capacity $\mathrm{C_p}(\mathrm{K_0},\mathrm{K_1},\Omega)$, which is a fairly standard fact in the theory of modulus.

Theorem 4.2.10 ([3])**.** *We have the equality*

$$\mathrm{C_p}(\mathrm{K_0},\mathrm{K_1},\Omega) = \mathrm{Mod}_p(|d\Gamma|) < \infty$$

In proving the equality in (4.2.5), the authors also need to establish the following technical convergence result.

Theorem 4.2.11 ([3])**.** *Let* K_0^j *and* K_1^j *be sequences of compact sets such that* $K_0^0 \cap K_1^0 = \emptyset$*,* $K_0^j \subset \operatorname{int} K_0^{j-1}$*,* $K_1^j \subset \operatorname{int} K_1^{j-1}$*,* $K_0 = \cap_{j=0}^{\infty} K_0^j$*,* $K_1 = \cap_{j=0}^{\infty} K_1^j$*. If* $\Gamma_j = \Gamma(K_0^j, K_1^j, \Omega)$*, then*

$$\mathrm{Mod}_p(|d\Gamma_j|) \downarrow \mathrm{Mod}_p(|d\Gamma|)$$

4.2.3 Proofs

We will proceed by repeating the proofs in the same order as in [3]. We begin by establishing Theorem 4.2.10, which is a common result in the theory of modulus, see for instance [19, Theorem 7.31].

Proof of Theorem 4.2.10. Clearly $\mathcal{D} \neq \emptyset$, so that $\mathrm{C_p}(\mathrm{K_0},\mathrm{K_1},\Omega) < \infty$. Let $u \in \mathcal{D}$. Then

$$1 = \int_\gamma \nabla u \, dx \leq \int_\gamma |\nabla u| \, ds.$$

So that $|\nabla u| \in \mathrm{Adm}(|d\Gamma|)$. Thus,

$$\mathrm{Mod}_p(|d\Gamma|) \leq \int |\nabla u|^p dx.$$

Taking the infimum over all $u \in \mathcal{D}$, we have $\mathrm{Mod}_p(|d\Gamma|) \leq \mathrm{C_p}(\mathrm{K}_0, \mathrm{K}_1, \Omega)$.

Conversely, fix $\rho \in L^p(\Omega) \cap \mathrm{Adm}(|d\Gamma|)$, for each $x \in \Omega$, let Γ_0^x be the family of curves starting in K_0 and ending at x, and define

$$u(x) = \inf_{\gamma \in \Gamma_0^x} \int_\gamma \rho \; ds$$

Claim:

(1) u is p-precise in Ω.

(2) for a.e. $x \in \Omega$, $|\nabla u(x)| \leq \rho(x)$.

(3) $u(x) \to 0$ as $x \to K_0$ along p-a.e. curve $\gamma \in \Gamma$.

(4) $\liminf u(x) \geq 1$ as $x \to K_1$ along p-a.e. curve $\gamma \in \Gamma$.

Since $\rho \in L^p(\Omega)$, we have $\int_\gamma \rho \; ds < \infty$ for p-a.e. curve $\gamma \in \Omega$, and so for p-a.e. curve γ

$$|u(b) - u(a)| \leq \int_{\tilde{ab}} \rho \; ds \text{ for all } a, b \in \gamma, \tag{4.2.6}$$

where $\tilde{ab}$ is the arc on γ connecting a to b. By Lemma 6.3.1 of [20], u is absolutely continuous for p-a.e. curve γ. Hence, u is ACL, and so has partial derivatives a.e. in Ω. Moreover, by (4.2.6), $|\partial u/\partial x_i| \leq \rho$ a.e. in Ω, and $\nabla u \in L^p(\Omega)$.

For (2), fix $a \in \Omega$ such that $\nabla u(a)$ exists, and let $\{\xi_j\}$ be a countable dense subset of $\{|v| = 1\}$ such that (4.2.6) holds, that is

$$|u(a + h\xi_j) - u(a)| \leq \int_0^h \rho(a + t\xi_j) dt.$$

Dividing by h and letting $h \to 0$, we obtain

$$|\xi_j \cdot \nabla u(a)| \leq \rho(a)$$

for a.e. $a \in \Omega$. Letting $\xi_j \to \nabla u(a)/|\nabla u(a)|$, we get $|\nabla u(a)| \leq \rho(a)$ and (2) is proved.

Since $u(a) = 0$ for all $a \in K_0$ and u is p-precise in Ω, (3) follows. Similarly, by definition, $u(b) \geq 1$ for all $b \in \Omega$, and since u is p-precise in Ω, (4) follows.

Letting $\tilde{u} = \min\{u, 1\}$, we have $\tilde{u} \in \mathcal{D}(K_0, K_1, \Omega)$. Hence,

$$\mathrm{C_p}(\mathrm{K_0}, \mathrm{K_1}, \Omega) \leq \int |\nabla \tilde{u}|^p dx \leq \int |\nabla u|^p dx \leq \int \rho^p dx.$$

Taking the infimum over $\rho \in \mathrm{Adm}(|d\Gamma|)$, we obtain the result. $\square$

Now we show the first duality result for $\mathrm{Mod}_q(\nabla\mathcal{D}^*)$ and the corresponding capacity.

Proof of Theorem 4.2.7. We repeat the statement of the Theorem here for convenience.

$\mathrm{C_p^*}(\mathrm{K_0}, \mathrm{K_1}, \Omega)$ and $\mathrm{Mod}_q(\nabla\mathcal{D}^*)$ satisfy the following:

(1) $\mathrm{C_p^*}(\mathrm{K_0}, \mathrm{K_1}, \Omega) = 0 \Rightarrow \mathrm{Mod}_q(\nabla\mathcal{D}^*) = \infty$

(2) $\mathrm{C_p^*}(\mathrm{K_0}, \mathrm{K_1}, \Omega) > 0 \Rightarrow \mathrm{C_p^*}(\mathrm{K_0}, \mathrm{K_1}, \Omega)^{1/p} \, \mathrm{Mod}_q(\nabla\mathcal{D}^*)^{1/q} = 1.$

In particular, $\mathrm{Mod}_q(\nabla\mathcal{D}^*) > 0$.

First, observe that $\mathcal{D}^*$ is a convex set. Indeed, if $u_1, u_2 \in \mathcal{D}^*(K_0, K_1, \Omega)$, then for $\lambda \in [0,1]$, we have $\lambda u_1(x) + (1-\lambda)u_2(x)$ is p-precise on Ω, $\lambda u_1(x) + (1-\lambda)u_2(x) = 0$ on K_0, and $\lambda u_1(x) + (1-\lambda)u_2(x) = \lambda + (1-\lambda) = 1$ on K_1. Second, observe that $B_q := \{\eta : \|\eta\|_q \leq 1\}$ is weakly compact and convex and

$$\Phi(u, \eta) := -\int_\Omega \nabla u \cdot \eta \, dx$$

is a bilinear functional on $\mathcal{D}^* \times B_q$ such that $\Phi(u, \cdot)$ is continuous with respect to the weak topology of B_q. Hence the minimax theorem [16] yields

$$\inf_{u\in\mathcal{D}^*}\sup_{\eta\in B_q}\Phi(u,\eta)=\sup_{\eta\in B_q}\inf_{u\in\mathcal{D}^*}\Phi(u,\eta). \tag{4.2.7}$$

Now, $\sup_{\eta\in B_q}\int_\Omega \nabla u\cdot\eta\,dx = \|\nabla u\|_p$ as can be seen by applying Hölder's inequality. For $\eta\in B_q$, we have

$$\int_\Omega \nabla u\cdot\eta\,dx \le \|\nabla u\|_p\,\|\eta\|_q \le \|\nabla u\|_p\,.$$

If $\|\nabla u\|_p\neq 0$, then letting η be defined by

$$\eta=\frac{|\nabla u|^{p-2}\nabla u}{\|\nabla u\|_p^{p-1}},$$

we have $\|\eta\|_q=1$ and

$$\int \nabla u\cdot\eta\,dx=\int\frac{|\nabla u|^p}{\|\nabla u\|_p^{p-1}}\,dx=\frac{\|\nabla u\|_p^p}{\|\nabla u\|_p^{p-1}}=\|\nabla u\|_p\,.$$

If $\|\nabla u\|_p=0$, choose $\eta\equiv 0\in\mathbb{R}^N$. Notice that the left hand side of (4.2.7) then becomes $\mathrm{C}^*_\mathrm{p}(\mathrm{K}_0,\mathrm{K}_1,\Omega)^{1/p}$. We will now work towards showing the right hand side is equal to $\mathrm{Mod}_q(\nabla\mathcal{D}^*)^{-1/q}$.

Suppose $\mathrm{C}^*_\mathrm{p}(\mathrm{K}_0,\mathrm{K}_1,\Omega)=0$. Then the right hand side of (4.2.7) is zero, and so $\inf_{u\in\mathcal{D}^*}\int_\Omega\nabla u\cdot\eta\,dx=0$ for every $\eta\in\mathbb{B}_q$. Since the only q-exceptional subfamily of $\mathcal{D}^*$ is the empty family (see Remark 4.1.10), there can be no admissible η satisfying $\|\eta\|_q<\infty$. Hence, $\mathrm{Mod}_q(\nabla\mathcal{D}^*)=\infty$.

Now, suppose $\mathrm{C}^*_\mathrm{p}(\mathrm{K}_0,\mathrm{K}_1,\Omega)>0$. We will establish that

$$\sup_{\eta\in B_q}\inf_{u\in\mathcal{D}^*}\int_\Omega\nabla u\cdot\eta\,dx=\sup\left\{\|\eta\|_q^{-1}:\inf_{u\in\mathcal{D}^*}\int_\Omega\nabla u\cdot\eta\,dx\ge 1\right\}. \tag{4.2.8}$$

Let $\alpha=\sup_{\eta\in B_q}\inf_{u\in\mathcal{D}^*}\int_\Omega\nabla u\cdot\eta\,dx$. Then $\|\eta\|_q\le 1$ implies $\inf_{u\in\mathcal{D}^*}\int\nabla u\cdot\eta\,dx\le\alpha$. Equivalently, if $\inf_{u\in\mathcal{D}^*}\int\nabla u\cdot\tilde{\eta}\,dx>1$, then $\|\tilde{\eta}\|_q>1/\alpha$. To see this, it is enough to consider the contrapositive and replace η with $\alpha\,\tilde{\eta}$. This is enough for us to conclude that

$$\sup\left\{\|\eta\|_q^{-1} : \inf_{u\in\mathcal{D}^*}\int_\Omega \nabla u\cdot\eta\,dx \geq 1\right\} \leq \alpha.$$

Conversely, let $\beta = \sup\left\{\|\eta\|_q^{-1} : \inf_{u\in\mathcal{D}^*}\int_\Omega \nabla u\cdot\eta\,dx \geq 1\right\}$. Then $\inf_{u\in\mathcal{D}^*}\int_\Omega \nabla u\cdot\eta\,dx \geq 1$ implies $\|\eta\|_q^{-1} \leq \beta$. Again, the contrapositive is $\|\eta\|_q^{-1} > \beta$ implies $\inf_{u\in\mathcal{D}^*}\int_\Omega \nabla u\cdot\eta\,dx < 1$. Hence, $\|\tilde{\eta}\|_q^{-1} > 1$ implies $\inf_{u\in\mathcal{D}^*}\int_\Omega \nabla u\cdot\tilde{\eta}\,dx < \beta$. Again, we see this via the substitution $\beta\,\eta = \tilde{\eta}$. From this, we conclude $\alpha \leq \beta$, and so we have established (4.2.8).

By definition of $\mathrm{Mod}_q(\nabla\mathcal{D}^*)$, we have

$$\beta = \left(\inf\left\{\|\eta\|_q : \inf_{u\in\mathcal{D}^*}\int_\Omega \nabla u\cdot\eta\,dx \geq 1\right\}\right)^{-1} = \mathrm{Mod}_q(\nabla\mathcal{D}^*)^{-1/q}.$$

From above, we have $\mathrm{C}_\mathrm{p}^*(\mathrm{K}_0,\mathrm{K}_1,\Omega)^{1/p} = \alpha$, and so we conclude

$$\mathrm{C}_\mathrm{p}(\mathrm{K}_0,\mathrm{K}_1,\Omega)^{1/p} = \mathrm{Mod}_q(\nabla\mathcal{D}^*)^{-1/q}. \tag{4.2.9}$$

□

Next, we will proceed to prove Theorem 4.2.8 by breaking it's contents into several lemmas. The first is a basic estimate that we will use for several of the families of vectors of measures we are interested in.

Lemma 4.2.12. *Let A be one of the families of functions $\mathcal{D},\mathcal{D}^*,S,$ or Σ. Then we have*

$$\mathrm{Mod}_q(\nabla A) \geq \mathrm{Mod}_q(|\nabla A|).$$

Proof. For every $f:\Omega\to\mathbb{R}^N$ satisfying $\int_\Omega f\cdot du \geq 1$, we have $\int_\Omega |f|\,d|u| \geq 1$. Thus for every $f\in$ w-Adm(∇A), we have $|f|\in$ w-Adm$(|\nabla A|)$, and $\|f\|_q = \||f|\|_q$. Hence, the set of numbers we take the infimum over for $\mathrm{Mod}_q(\nabla A)$ is a subset of the numbers we take the infimum over for $\mathrm{Mod}_q(|\nabla A|)$, and so we obtain $\mathrm{Mod}_q(\nabla A) \geq \mathrm{Mod}_q(|\nabla A|)$. Note, we have been careful to use du and $d|u|$. If $u\in\mathcal{D}$ or $u\in\mathcal{D}^*$, then $du = \nabla u dx$ and $d|u| = |\nabla u|dx$. If

$u \in S$ or $u \in \Sigma$, then du is a vector of measures and $d|u|$ is its total variation measure. $\square$

The following is Lemma 5.1 of [3]. It will establish the bulk of the inequalities in Theorem 4.2.8.

Lemma 4.2.13 ([3]).

$$\mathrm{Mod}_q(\nabla S) \geq \mathrm{Mod}_q(|\nabla S|) \geq \mathrm{Mod}_q(|\nabla \Sigma|) \tag{4.2.10}$$

$$\mathrm{Mod}_q(\nabla S) \geq \mathrm{Mod}_q(\nabla \Sigma) \geq \mathrm{Mod}_q(|\nabla \Sigma|) \tag{4.2.11}$$

Proof. The first inequality of (4.2.10) and second inequality of (4.2.11) are taken care of by Lemma 4.2.12. The other two inequalities follow from the fact that $\Sigma \subset S$. $\square$

The next Lemma we wish to discuss makes use of Sard's Theorem, which we state here for convenience. It also makes use of Proposition 5 of [3].

Theorem 4.2.14 ([24]). *Let $f : \mathbb{R}^M \to \mathbb{R}^N$ be an n times differentiable mapping. Let X be the set of critical points of f, that is the set where the Jacobian matrix of f has rank less than M. Then*

(1) If $M \leq N$ and $n \geq 1$, the image $f(X)$ is a set of M-dimensional Lebesgue measure 0.

(2) If $M > N$ and $n \geq M - N + 1$, the image $f(X)$ is a set of N-dimensional Lebesgue measure 0.

The next Lemma relates $|\nabla \Sigma|$ and $\nabla \mathcal{D}^*$. The proof makes use of the duality established by Theorem 4.2.7.

Lemma 4.2.15 ([3]).

$$\mathrm{Mod}_q(|\nabla \Sigma|) \geq \mathrm{Mod}_q(\nabla \mathcal{D}^*) > 0$$

Proof. Without loss of generality, we may assume $\mathrm{Mod}_q(|\nabla \Sigma|) < \infty$. By Theorem 4.2.7, it is equivalent to show $\mathrm{C}_\mathrm{p}^*(\mathrm{K}_0, \mathrm{K}_1, \Omega) > 0$ and

$$\mathrm{C}_\mathrm{p}^*(\mathrm{K}_0, \mathrm{K}_1, \Omega)^{1/p} \, \mathrm{Mod}_q(|\nabla \Sigma|)^{1/q} \geq 1. \tag{4.2.12}$$

Indeed, if $\mathrm{C}_\mathrm{p}^*(\mathrm{K}_0, \mathrm{K}_1, \Omega) > 0$, then $\mathrm{Mod}_q(\nabla\mathcal{D}^*) = 1/\, \mathrm{C}_\mathrm{p}^*(\mathrm{K}_0, \mathrm{K}_1, \Omega)^{q/p}$, and we have $\mathrm{Mod}_q(|\nabla\Sigma|) \geq \mathrm{Mod}_q(\nabla\mathcal{D}^*)$ if and only if $\mathrm{Mod}_q(|\nabla\Sigma|) \geq 1/\, \mathrm{C}_\mathrm{p}^*(\mathrm{K}_0, \mathrm{K}_1, \Omega)^{q/p}$. In light of Proposition 4.2.5, let $u \in \mathcal{D}^* \cap C^\infty(\Omega)$ and $0 \leq u \leq 1$. Let $N_t = \{x \in \Omega : u(x) > t\}$. We will now show $\chi_{N_t} \in \Sigma$ for a.e. t such that $0 < t < 1$. By part (2) of Sard's theorem, $\{x : u(x) = t\}$ is smooth for a.e. $t \in (0, 1)$. For these t, the divergence theorem gives us

$$\int_{N_t} \operatorname{div} \varphi \, dx = -\int_{u=t} \varphi \cdot \mathbf{n} \, dS \tag{4.2.13}$$

for $\varphi \in C_0^\infty(\Omega; \mathbb{R}^N)$, $|\varphi| \leq 1$, where $\mathbf{n}$ is the unit normal vector directed into N_t. Thus we have

$$\left| \int \chi_{N_t} \operatorname{div} \varphi \, dx \right| \leq \left| \int_{u=t} \varphi \cdot \mathbf{n} \, dS \right| \leq \mathcal{H}^{N-1}(\partial N_t). \tag{4.2.14}$$

Integrating and applying Proposition 2 of 3.4.4 of [15], we get

$$\int_0^1 \|\chi_{N_t}\|_{BV} \, dt \leq \int_0^1 \mathcal{H}^{N-1}(\partial N_t) \, dt = \int_\Omega |\nabla u| \, dx < \infty. \tag{4.2.15}$$

Hence, $\|\chi_{N_t}\|_{BV} < \infty$ for a.e. t.

Since $\mathrm{Mod}_q(|\nabla\Sigma|) < \infty$, there is a $\rho \in$ w-Adm$(|\nabla\Sigma|)$ with $\|\rho\|_p < \infty$. Thus, applying the Proposition 3 of 3.4.4 of [15], we have

$$1 \leq \int_0^1 \int_{u=t} \rho \, |\mathbf{n}| \, dS \, dt = \int_0^1 \int_{u=t} \frac{\rho \, |\nabla u|}{|\nabla u|} \, dS = \int_\Omega \rho \, |\nabla u| \, dx \leq \|\nabla u\|_p \, \|\rho\|_q \, . \tag{4.2.16}$$

Taking the infimum over ρ and u, we get $\mathrm{C}_\mathrm{p}^*(\mathrm{K}_0, \mathrm{K}_1, \Omega) > 0$ and (4.2.12) holds. □

All that remains of Theorem 4.2.8 is to show $\mathrm{Mod}_q(\nabla\mathcal{D}^*) \geq \mathrm{Mod}_q(\nabla S)$. This is the hardest part to prove, as $\nabla\mathcal{D}^* \subset \nabla S$. We will require four more lemmas in order to prove this. They are Lemmas 5.4, 5.5, 5.6, and 5.7 in [3]. Until now, we have abbreviated $\nabla\mathcal{D}^*(K_0, K_1, \Omega)$ as $\nabla\mathcal{D}^*$, and likewise for ∇S and $\nabla\Sigma$. However, now we will be comparing q-Modulus of these families with differing sets K_0, K_1 and Ω.

Lemma 4.2.16 ([3]). *Let Ω_1 be an open subset of Ω such that $K_0, K_1 \subset \overline{\Omega}$. Then*

$$\mathrm{Mod}_q(\nabla\mathcal{D}^*(K_0, K_1, \Omega_1)) \geq \mathrm{Mod}_q(\nabla\mathcal{D}^*(K_0, K_1, \Omega)). \tag{4.2.17}$$

Proof. Without loss of generality, assume $\mathrm{Mod}_q(\nabla\mathcal{D}^*(K_0, K_1, \Omega_1) < \infty$. Let $f \in$ w-Adm$(\nabla\mathcal{D}^*(K_0, K_1, \Omega_1))$ and let $\overline{f} = f\,\chi_{\Omega_1}$ be an extension of f to all of Ω. Then clearly $\overline{f} \in$ w-Adm$(\nabla\mathcal{D}^*(K_0, K_1, \Omega))$ since

$$\int_\Omega \overline{f} \cdot \nabla u\, dx = \int_{\Omega_1} f \cdot \nabla u|_{\Omega_1}\, dx \geq 1,$$

and $u \in \mathcal{D}^*(K_0, K_1, \Omega) \Rightarrow u|_{\Omega_1} \in \mathcal{D}^*(K_0, K_1, \Omega_1)$. Thus,

$$\|f\|_q^q = \left\|\overline{f}\right\|_q^q \geq \mathrm{Mod}_q(\nabla\mathcal{D}^*(K_0, K_1, \Omega_1)),$$

and so taking the infimum over all such f, we obtain the desired inequality. □

In light of the above monotonicity, we make the following definition.

Definition 4.2.17. Let $K_0, K_1 \subset \Omega$. We say that Ω *can be approximated from inside with respect to* $\mathrm{Mod}_q(\nabla\mathcal{D}^*(K_0, K_1, \Omega))$ if for each $\varepsilon > 0$, there exists an open set Ω' such that $K_0, K_1, \subset \Omega' \subset \overline{\Omega'} \subset \Omega$ and

$$\mathrm{Mod}_q(\nabla\mathcal{D}^*(K_0, K_1, \Omega)) \leq \mathrm{Mod}_q(\nabla\mathcal{D}^*(K_0, K_1, \Omega')) \leq \mathrm{Mod}_q(\nabla\mathcal{D}^*(K_0, K_1, \Omega)) + \varepsilon. \tag{4.2.18}$$

The next Lemma says that given an open set and a compactly contained open subset, we can always find an open subset between the two with the above approximation preoperty.

Lemma 4.2.18 ([3]). *Let $K_0, K_1 \subset \Omega_1 \subset \overline{\Omega_1} \subset \Omega$ as in definition 4.2.17. Suppose that* $\mathrm{Mod}_q(\nabla\mathcal{D}^*(K_0, K_1, \Omega_1)) < \infty$. *Then there exists Ω_2 satisfying $\overline{\Omega_1} \subset \Omega_2 \subset \Omega$ and Ω_2 can be approximated from inside with respect to* $\mathrm{Mod}_q(\nabla\mathcal{D}^*(K_0, K_1, \Omega_2))$.

Proof. For small $r > 0$, define

$$\Omega(r) := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > r\}.$$

Take note that $\Omega(r) \nearrow \Omega(0) = \Omega$ as $r \searrow 0$. Let $r_1 > 0$ satisfy $r_1 < \operatorname{dist}(\partial\Omega_1, \partial\Omega)$. Then $\overline{\Omega_1} \subset \Omega(r_1)$. Let $\varphi(r) := \operatorname{Mod}_q(\nabla\mathcal{D}^*(K_0, K_1, \Omega(r)))$. By Lemma 4.2.16, φ is nondecreasing, and we have $0 < \varphi(r) \leq \varphi(r_1) \leq \operatorname{Mod}_q(\nabla\mathcal{D}^*(K_0, K_1, \Omega_1)) < \infty$ for $0 < r < r_1$. Thus, φ is continuous from the right outside of a countable set. Hence, there exists r_2 such that $0 < r_2 < r_1$ and

$$\lim_{r \searrow r_2} \varphi(r) = \varphi(r_2).$$

In other words, as r decreases to r_2,

$$\operatorname{Mod}_q(\nabla\mathcal{D}^*(K_0, K_1, \Omega(r))) \searrow \nabla \operatorname{Mod}_q(\nabla\mathcal{D}^*(K_0, K_1, \Omega(r_2))).$$

Letting $\Omega_2 = \Omega(r_2)$, we see that $\overline{\Omega_1} \subset \Omega_2 \subset \Omega$ and Ω_2 can be approximated from inside with respect to $\operatorname{Mod}_q(\nabla\mathcal{D}^*(K_0, K_1, \Omega_2))$. $\square$

Next, we state a simple application of the continuity of p-Modulus in 4.1.4.

Lemma 4.2.19 ([3])**.** *Let $K_0, K_1 \subset \Omega$ and let $\{K_0^j\}$ and $\{K_1^j\}$ be decreasing sequences of compact sets such that $K_i^j \subset \Omega$ for all j, $i = 0, 1$ and $\cap_{j=1}^{\infty} K_i^j = K_i$ for $i = 0, 1$. Then*

$$\operatorname{Mod}_q(\nabla S(K_0^j, K_1^j, \Omega)) \nearrow \operatorname{Mod}_q(\nabla S(K_0, K_1, \Omega)).$$

Proof. We claim that

$$\bigcup_{j=1}^{\infty} \nabla S(K_0^j, K_1^j, \Omega) = \nabla S(K_0, K_1, \Omega).$$

If $u \in S(K_0^j, K_1^j, \Omega)$ for some j, then we have $u \in BV(\Omega)$, $u = 0$ on some neighborhood U_0 of K_0^j, and $u = 1$ on some neighborhood U_1 of K_1^j. Since $K_i \subset K_i^j \subset U_i$ for $i = 0, 1$, we have $u \in S(K_0, K_1, \Omega)$.

If $u \in S(K_0, K_1, \Omega)$, then $u \in BV(\Omega)$, $u = 0$ on some neighborhood U_0 of K_0, and $u = 1$ on some neighborhood U_1 of K_1. Since $\{K_i^j\}$ is a decreasing sequence satisfying $\cap_{j=1}^{\infty} K_i^j = K_i$ for $i = 0, 1$, we must have some j_0, j_1 such that $K_i \subset K_i^{j_i} \subset U_i$ for $i = 0, 1$. Let $j^* = \max\{j_0, j_1\}$. Then we have that $u \in S(K_0^{j^*}, K_1^{j^*}, \Omega) \subset \cup_{j=1}^{\infty} S(K_0^j, K_1^j, \Omega)$. Applying Proposition 4.1.4, we obtain the result. □

Next, we show a special case of the desired inequality $\mathrm{Mod}_q(\nabla S) \leq \mathrm{Mod}_q(\nabla \mathcal{D}^*)$.

Lemma 4.2.20 ([3]). *Let $K_0, K_1 \subset \Omega$ and suppose Ω can be approximated from inside with respect to* $\mathrm{Mod}_q(\nabla \mathcal{D}^*(K_0, K_1, \Omega))$. *Then*

$$\mathrm{Mod}_q(\nabla S(K_0, K_1, \Omega)) \leq \mathrm{Mod}_q(\nabla \mathcal{D}^*(K_0, K_1, \Omega)).$$

Proof. Let $\varepsilon > 0$. By assumption, there exists Ω_1 such that $K_0, K_1, \subset \Omega_1 \subset \overline{\Omega_1} \subset \Omega$ such that

$$\mathrm{Mod}_q(\nabla \mathcal{D}^*(K_0, K_1, \Omega_1)) \leq \mathrm{Mod}_q(\nabla \mathcal{D}^*(K_0, K_1, \Omega)) + \varepsilon. \tag{4.2.19}$$

By Lemma 4.2.19, we can find compact sets K_0^1 and K_1^1 such that $K_i \subset \operatorname{int} K_i^1 \subset \Omega$ for $i = 0, 1$ and

$$\mathrm{Mod}_q(\nabla S(K_0, K_1, \Omega)) \leq \mathrm{Mod}_q(\nabla S(K_0^1, K_1^1, \Omega)) + \varepsilon. \tag{4.2.20}$$

We claim that

$$\mathrm{Mod}_q(\nabla S(K_0^1, K_1^1, \Omega)) \leq \mathrm{Mod}_q(\nabla \mathcal{D}^*(K_0, K_1, \Omega_1)). \tag{4.2.21}$$

Then combining equations (4.2.19), (4.2.20), and (4.2.21), we have

$$\begin{aligned} \mathrm{Mod}_q(\nabla S(K_0, K_1, \Omega)) &\leq \mathrm{Mod}_q(\nabla S(K_0^1, K_1^1, \Omega)) + \varepsilon \\ &\leq \mathrm{Mod}_q(\nabla \mathcal{D}^*(K_0, K_1, \Omega_1)) + \varepsilon \\ &\leq \mathrm{Mod}_q(\nabla \mathcal{D}^*(K_0, K_1, \Omega)) + 2\varepsilon, \end{aligned} \tag{4.2.22}$$

and letting $\varepsilon \to 0$, we obtain the result.

To prove (4.2.21), assume without loss of generality that $\mathrm{Mod}_q(\nabla \mathcal{D}^*(K_0, K_1, \Omega_1)) < \infty$, and let $f \in \mathrm{Adm}(\nabla \mathcal{D}^*(K_0, K_1, \Omega_1))$. Let $\overline{f}$ be an extension to Ω by defining $\overline{f}(x) = f(x)$ on Ω_1 and $\overline{f}(x) = 0$. Let $\psi(x)$ be the standard mollifier so that $\int \psi \, dx = 1$ and $\psi \in C_0^\infty(\mathbb{R}^N)$.

Let Ω_2 be defined by

$$\Omega_2 = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \frac{1}{2}\,\mathrm{dist}(\overline{\Omega_1}, \partial\Omega)\},$$

and

$$r_0 := \min\{\mathrm{dist}(K_0, \partial K_0^1), \mathrm{dist}(K_1, \partial K_1^1), \frac{1}{2}\,\mathrm{dist}(\overline{\Omega_1}, \partial\Omega)\} > 0.$$

Then for $0 < r < r_0$, $x \in \Omega_2$, $|y| < 1$, we have $x + ry \in \Omega$, and we can define

$$f_r(x) := \int_{|y|<1} \overline{f}(x + ry)\psi(y)\, dy.$$

We have $f_r \in C_0^\infty(\Omega_2)$. Further, $f_r \in \mathrm{Adm}(\nabla \mathcal{D}^*(K_0^1, K_1^1, \Omega))$ because for $u \in \mathcal{D}^*(K_0^1, K_1^1, \Omega)$

$$\begin{aligned}
\int_\Omega f_r \cdot \nabla u \, dx &= \int_{|y|<1} \psi(y) \int_\Omega \overline{f}(x + ry) \cdot \nabla u(x)\, dx\, dy \\
&= \int_{|y|<1} \psi(y) \int_{\Omega_1} f(x + ry) \cdot \nabla u(x)\, dx\, dy \\
&= \int_{|y|<1} \psi(y) \int_{\Omega_1} f(x) \cdot \nabla u(x - ry)\, dx\, dy \\
&\geq \int_{|y|<1} \psi(y)\, dy = 1,
\end{aligned} \tag{4.2.23}$$

where we have used the fact that $u(x - ry) \in \mathcal{D}^*(K_0^1, K_1^1, \Omega)$.

Now we will show that $f_r \in$ w-$\mathrm{Adm}(\nabla S(K_0^1, K_1^1, \Omega))$. We will use the approximation given by Theorems 2 and 3 of section 5.2.2 of [15]. Namely, if $u \in S(K_0^1, K_1^1, \Omega)$, then there exists $u_j \in C^\infty(\Omega)$ such that $u_j = i$ on a neigborhood of K_0^i for $i = 0, 1$, $u_j \to u$ in $L^1(\Omega)$, $\|u_j\|_{BV} \to \|u\|_{BV}$, and $\nabla u_j \to \nabla u$ weakly. Note that because $u \in S(K_0^1, K_1^1, \Omega)$, ∇u may not be absolutely continuous with respect to Lebesgue measure. Since $\|\nabla u_j\|_p$ may be infinite, we will truncate u_j with a function $g \in C_0^\infty(\Omega)$ such that $0 \leq g \leq 1$ on Ω and $g = 1$ on Ω_2.

Then we see that $gu_j \in C_0^\infty(\Omega)$, and so $gu_j \in \mathcal{D}^*(K_0^1, K_1^1, \Omega)$. Thus we have

$$1 \le \int_\Omega f_r \cdot \nabla(gu_j)\, dx = \int_\Omega f_r \cdot \nabla u_j \, dx \to \int_\Omega f_r \cdot \nabla u,$$

where the inequality follows from (4.2.23) and the equality follows since $f_r \in C_0^\infty(\Omega_2)$ and $g = 1$ on Ω_2. Hence $f_r \in$ w-Adm$(\nabla S(K_0^1, K_1^1, \Omega))$.

Now, since $f_r \in$ w-Adm$(\nabla S(K_0^1, K_1^1, \Omega))$, we have

$$\mathrm{Mod}_q(\nabla S(K_0^1, K_1^1, \Omega)) \le \int_\Omega |f_r|^q \, dx \to \int_\Omega |f|^q \, dx$$

as $r \to 0$, and thus, taking an infimum over $f \in \mathrm{Adm}(\nabla \mathcal{D}^*(K_0, K_1, \Omega_1))$, we obtain (4.2.21). □

We are now prepared to show the last part needed to prove Theorem 4.2.8. Note the Lemma below does not have the added constraints that Lemma 4.2.20 has. We will also be returning to our abbreviations $\nabla\mathcal{D}^*$ and ∇S.

Lemma 4.2.21 ([3])**.**

$$\mathrm{Mod}_q(\nabla \mathcal{D}^*) \ge \mathrm{Mod}_q(\nabla S).$$

Proof. Without loss of generality, assume $\mathrm{Mod}_q(\nabla\mathcal{D}^*) < \infty$. Recall, we assume $K_0, K_1 \subset \overline{\Omega}$. Let $\{K_0^j\}$ and $\{K_1^j\}$ be compact sets satisfying $K_i^{j+1} \subset \mathrm{int}\, K_i^j \subset \Omega$ for all j and $K_i = \cap_{j=1}^\infty K_i^j$ for $i = 0, 1$. Further, let $\Omega^j = \Omega \cup (\mathrm{int}\, K_0^j) \cup (\mathrm{int}\, K_1^j)$. We claim

$$\nabla S(K_0^j, K_1^j, \Omega^j) \nearrow \nabla S(K_0, K_1, \Omega). \tag{4.2.24}$$

Let $u \in S(K_0^j, K_1^j, \Omega^j)$. We will denote the distributional derivative of u over Ω^j by $\nabla_{\Omega^j} u$. Clearly, since $\nabla_{\Omega^j} u \equiv 0$ on open neighborhoods of K_0^j and K_1^j, we have $\nabla_{\Omega^j} u$ is supported in $\Omega^j \backslash (K_0^j \cup K_1^j) \subset \Omega$ and $\nabla_{\Omega^j} u = \nabla_{\Omega^{j+1}}(u|_{\Omega^{j+1}}) = \nabla_\Omega(u|_\Omega)$. Thus, $\nabla S(K_0^j, K_1^j, \Omega)$ is increasing and $\cup \nabla S(K_0^j, K_1^j, \Omega^j) \subset \nabla S(K_0, K_1, \Omega)$.

Now we show the other containment. Let $u \in S(K_0, K_1, \Omega)$. Then, for $i = 0, 1$, there are open sets U_i such that $u = i$ on U_i and $K_i \subset U_i$. Thus, there exist j such that $K_i^j \subset U_i$ for both $i = 0$ and $i = 1$ (taking the minimum of two j if necessary). Now we extend u to Ω^j by defining

$$\overline{u} := \begin{cases} u & \text{on } \Omega, \\ 0 & \text{on } \operatorname{int} K_0^j, \\ 1 & \text{on } \operatorname{int} K_1^j, \end{cases}$$

and hence, $\overline{u} \in S(K_0^j, K_1^j, \Omega^j)$. Further, $\nabla_{\Omega^j}\overline{u}$ is supported on $\Omega \backslash (K_0^j \cup K_1^j)$ and $\nabla_{\Omega^j}\overline{u} = \nabla_\Omega u$. Thus $\nabla u \in \nabla S(K_0^j, K_1^j, \Omega^j)$ and we have established (4.2.24).

Applying Proposition 4.1.4 to (4.2.24), we get

$$\operatorname{Mod}_q(\nabla S(K_0^j, K_1^j, \Omega^j) \nearrow \operatorname{Mod}_q(\nabla S). \tag{4.2.25}$$

Since $\Omega \subset \Omega^j$ we can apply Lemma 4.2.16 to get

$$\operatorname{Mod}_q(\nabla \mathcal{D}^*(K_0, K_1, \Omega^j)) \leq \operatorname{Mod}_q(\nabla \mathcal{D}^*(K_0, K_1, \Omega)) < \infty.$$

Using Lemma 4.2.18, we can modify K_0^j, K_1^j, and Ω^j so that Ω^j can be approximated from inside with respect to $\operatorname{Mod}_q(\nabla \mathcal{D}^*(K_0, K_1, \Omega^j))$. Further, we have $S(K_0^j, K_1^j, \Omega^j) \subset S(K_0, K_1, \Omega^j)$, and so $\operatorname{Mod}_q(\nabla S(K_0^j, K_1^j, \Omega^j)) \leq \operatorname{Mod}_q(\nabla S(K_0, K_1, \Omega^j))$. Applying (4.2.25), Lemma 4.2.19, and Lemma 4.2.20, we have

$$\begin{aligned} \operatorname{Mod}_q(\nabla S(K_0, K_1, \Omega)) &= \lim \operatorname{Mod}_q(\nabla S(K_0^j, K_1^j, \Omega^j)) \leq \operatorname{Mod}_q(\nabla S(K_0, K_1, \Omega^j)) \\ &\leq \lim \operatorname{Mod}_q(\nabla \mathcal{D}^*(K_0, K_1, \Omega^j)) \leq \operatorname{Mod}_q(\nabla \mathcal{D}^*(K_0, K_1, \Omega)). \end{aligned}$$

□

Combining Lemmas 4.2.13, 4.2.15, 4.2.21, we obtain Theorem 4.2.8.

We will now proceed to prove Theorem 4.2.11. We will require only one lemma to do so.

Lemma 4.2.22 ([22]). *Let $\rho \in L^p(\mathbb{R}^N)$ be a positive lower semicontinuous function which* *is continuous on $\Omega\backslash(K_0 \cup K_1)$. Let K_0^j and K_1^j be sequences of compact sets such that $K_i^{j+1} \subset \operatorname{int} K_i^j$ for all j and $K_i = \cap_{j=1}^{\infty} K_i^j$ for $i = 0, 1$. Then for each $\varepsilon > 0$ we can construct a function ρ' on Ω, $\rho' \geq \rho$ such that*

(1) $\int_\Omega \rho'^p \, dx \leq \int_\Omega \rho^p \, dx + \varepsilon$,

(2) Suppose for each j, there is $\gamma_j \in \Gamma_j := \Gamma(K_0^j, K_1^j, \Omega)$ such that $\int_{\gamma_j} \rho' \, ds \leq \alpha$. Then there exists $\tilde{\gamma} \in \Gamma(K_0, K_1, \Omega)$ such that $\int_{\tilde{\gamma}} \rho \, ds \leq \alpha + \varepsilon$.

Proof. Let $\varepsilon > 0$, $K^j := K_0^j \cup K_1^j$, $W^j := K^{j-1} \backslash \operatorname{int} K^j$, and $d_j := \operatorname{dist}(\partial K^{j-1}, \partial K^j) > 0$. Since $\rho > 0$ and continuous on $\Omega\backslash(K_0 \cup K_1)$, we have $\inf_{W^j \cap \Omega} \rho(x) \geq \min_{W^j} \rho(x) > 0$. Thus, we can find $\varepsilon_j \searrow 0$ such that

$$\sum_{j=1}^{\infty} (1 + \varepsilon_j^{-1})^p \varepsilon_j^{p+1} < \varepsilon, \tag{4.2.26}$$

$$\alpha \varepsilon_j < d_j \inf_{W^j \cap \Omega} \rho. \tag{4.2.27}$$

Since $\rho \in L^p(\mathbb{R}^N)$, we can find an increasing sequence of compact subsets $\Omega_j \subset \Omega$ such that $\cup \Omega_j = \Omega$ and

$$\int_{\Omega\backslash\Omega_j} \rho^p \, dx < \varepsilon_j^{p+1}.$$

Let $V^j = (\Omega\backslash\Omega_j) \cap W^j$ and

$$\rho'(x) = \begin{cases} (1 + \varepsilon_j^{-1})\rho(x) & \text{if } x \in V^j \\ \rho(x) & \text{if } x \in \Omega\backslash \cup V^j. \end{cases} \tag{4.2.28}$$

Computing, we have

$$\begin{aligned} \int_\Omega \rho'^p \, dx &= \sum_j \int_{V^j} [(1 + \varepsilon_j^{-1})\rho]^p \, dx + \int_{\Omega\backslash\cup V^j} \rho^p \, dx \\ &\leq \sum_j (1 + \varepsilon_j^{-1})^p \varepsilon_j^{p+1} + \int_\Omega \rho^p \, dx < \varepsilon + \int_\Omega \rho^p \, dx, \end{aligned} \tag{4.2.29}$$

and (1) holds.

Now we will prove (2). Fix $j \geq 1$. By definition, $\gamma_k \in \Gamma_j$ for $k \geq j$ since $\{K_i^j\}$ are decreasing for $i = 0, 1$. Consequently, γ_k has subarcs γ_k' and γ_k'' such that γ_k' connects ∂K_0^j and ∂K_0^{j-1}, γ_k'' connects ∂K_1^j and ∂K_1^{j-1}. We can see that γ_k' is not a subset of V^j, because otherwise

$$\alpha \geq \int_{\gamma_k} \rho' ds \geq \int_{\gamma_k'} \rho' ds \geq \varepsilon_j^{-1} \int_{\gamma_k'} \rho ds \geq \varepsilon_j^{-1} d_j \inf_{W^j \cap \Omega} \rho > \alpha,$$

a contradiction. Similarly, γ_k'' is not a subset of V^j. Thus, $\gamma_k \cap (\Omega_j \cap (K_i^{j-1} \backslash \operatorname{int} K_i^j) \neq \varnothing$ for $i = 0, 1$. Since Ω_j and $K_i^{j-1} \backslash \operatorname{int} K_i^j$ are both compact, so is $\Omega_j \cap (K_i^{j-1} \backslash \operatorname{int} K_i^j)$. Thus, since $j \geq 1$ is fixed, we can define a sequence of points $\{a_k\}_{k=j}^{\infty}$ by letting a_k be any point of γ_k. Then there must be a subsequence that converges to, say, $x_0^j \in \Omega_j \cap K_0^{j-1} \backslash K_0^j$. If we place a small closed ball $B_0^j \subset \Omega$ centered at x_0^j, then we may assume that the the corresponding subsequence of curves γ_k^j eventually intersects B_0^j. Since ρ is continuous at x_0^j, we can choose B_0^j small enough so that

$$\int_{\ell} \rho \, ds \leq \frac{\varepsilon}{2^{j+3}} \text{for any line segment } \ell \subset B_0^j. \tag{4.2.30}$$

Likewise, choose a small ball $B_1^j \subset \Omega$ with center $x_1^j \in \Omega_j \cap (K_1^{j-1} \backslash \operatorname{int} K_1^j)$ so that γ_k^j intersects B_1^j and choose subsequences $\{\gamma_k^j\}_k$ inductively such that γ_k^j intersects B_0^j and B_1^j.

Consider the diagonal γ_k^k. Then γ_k^k intersects B_0^j and B_1^j. To γ_k^k we add two suitable radii of B_i^j for each $i = 0, 1$ and for $1 \leq j \leq k$ so that we have a connected curve $\tilde{\gamma_k} \in \Gamma(K_0^k, K_1^K, \Omega)$ going through x_0^j and x_1^j. By (4.2.30), we have

$$\int_{\tilde{\gamma_k}} \rho \, ds \leq \int_{\gamma_k^k} \rho \, ds + 2 \sum_{j=1}^{k} \frac{\varepsilon}{2^{j+3}} \leq \alpha + \frac{\varepsilon}{4}.$$

Let Γ_0 be the family of all curves in $\Omega \backslash (K_0 \cup K_1)$ connecting x_0^1 and x_1^1. For $i = 0, 1$, let Γ_i^j be the family of all curves in $\Omega \backslash (K_0 \cup K_1)$ connecting x_i^j and x_i^{j+1}. Then

$$\inf_{\gamma \in \Gamma_0} \int_{\gamma} \rho \, ds + \sum_{j=1}^{k} \inf_{\gamma \in \Gamma_0^j} \int_{\gamma} \rho \, ds + \sum_{j=1}^{k} \inf_{\gamma \in \Gamma_1^j} \int_{\gamma} \rho \, ds \leq \int_{\tilde{\gamma_k}} \rho \, ds \leq \alpha + \frac{\varepsilon}{4}.$$

Thus, we can choose $C_0 \in \Gamma_0$ and $C_i^j \in \Gamma_i^j$ such that

$$\int_{C_0} \rho \, ds < \inf_{\gamma \in \Gamma_0} \int_\gamma \rho \, ds + \frac{\varepsilon}{2},$$

$$\int_{C_i^j} \rho \, ds < \inf \int_\gamma \rho \, ds + \frac{\varepsilon}{2^{j+3}}.$$

Let $\tilde{\gamma} = \cdots + C_0^1 + C_0 + C_1^1 + \ldots$ be the concatenation of the curves C_0 and C_i^j. Then $\tilde{\gamma} \in \Gamma(K_0, K_1, \Omega)$ and

$$\int_{\tilde{\gamma}} \rho \, ds \leq \alpha + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + 2 \sum_{j=1}^{\infty} \frac{\varepsilon}{2^{j+3}} = \alpha + \varepsilon.$$

□

Proof of 4.2.11. Here, we restate the Theorem for convenience.

Let K_0^j and K_1^j be sequences of compact sets such that $K_0^0 \cap K_1^0 = \emptyset$, $K_0^j \subset \operatorname{int} K_0^{j-1}$, $K_1^j \subset \operatorname{int} K_1^{j-1}$, $K_0 = \cap_{j=0}^{\infty} K_0^j$, $K_1 = \cap_{j=0}^{\infty} K_1^j$. If $\Gamma_j = \Gamma(K_0^j, K_1^j, \Omega)$, then

$$\operatorname{Mod}_p(|d\Gamma_j|) \downarrow \operatorname{Mod}_p(|d\Gamma|)$$

By the majorization principle (see (5.2.5) of [20]), it is clear that $\operatorname{Mod}_p(|d\Gamma|) \leq \lim_{j\to\infty} \operatorname{Mod}_p(|d\Gamma_j|)$. Let $M := \operatorname{Mod}_p(|d\Gamma|) < \infty$, $0 < \varepsilon < 1/2$, and $\tilde{\rho} \in \operatorname{Adm}(|d\Gamma|)$ such that $\|\tilde{\rho}\|_p^p < M + \varepsilon$. By the Vitali-Carathéodory Theorem (see p. 108 of [20]), we may assume $\tilde{\rho}$ is lower semicontinuous, and we can further assume $\tilde{\rho}$ is strictly positive (by adding a strictly positive lower semicontinuous function with small p-norm if necessary). Now, we wish to apply Lemma 4.2.22, so we need to come up with a continuous function that approximates $\tilde{\rho}$. Define

$$(\tilde{\rho})_r(x) = \int_{|y|<1} \tilde{\rho}(x + r\alpha(x)y)\psi(y) \, dy$$

as in 4.2.3 with $G = \Omega \backslash (K_0 \cup K_1)$. Note that $(\tilde{\rho})_r \in C^\infty(G)$ and $\|(\tilde{\rho})_r - \tilde{\rho}\|_p \to 0$ as $r \to 0$, and so if r is sufficiently small, then $\|(\tilde{\rho})_r\|_p^p < M + \varepsilon$.

Note that for fixed y with $|y| < 1$, we have that $T_{y,r}(x) = x + r\alpha(x)y$ is a C^∞ homeomorphism of G onto itself, and since $\alpha(x) \to 0$ as $x \to \partial G$, we can continuously extend $T_{y,r}(x)$ to the boundary by $T_{y,r}(x) = x$ for $x \in \partial G$. In view of this, if $\gamma \in \Gamma$, then so is $T_{y,r}(\gamma)$. Then we have

$$\begin{aligned} \int_\gamma (\tilde{\rho})_r \, ds &= \int_{|y|<1} \psi(y) \int_\gamma \tilde{\rho}(x + r\alpha(x)y) \, ds dy \\ &= \int_{|y|<1} \psi(y) \int_{T_{y,r}(\gamma)} \tilde{\rho}(z) \frac{1}{1 + r\nabla\alpha \cdot y} \, ds \, dy \\ &\geq \frac{1}{1 + r/2} \int_{|y|<1} \psi(y) \int_{T_{y,r}(\gamma)} \tilde{\rho}(z) \, ds \, dy \\ &\geq \frac{1}{1 + r/2}. \end{aligned} \tag{4.2.31}$$

Thus $(1 + r/2)(\tilde{\rho})_r \in \mathrm{Adm}(|d\Gamma|)$. Let $\rho = (1 + r/2)(\tilde{\rho})_r$. Then, we also have

$$\|\rho\|_p^p < (1 + r/2)^p (M + \varepsilon).$$

Let ρ' be as in Lemma 4.2.22. Then, for sufficiently large j, we have

$$\int_\gamma \rho' \, ds > 1 - 2\varepsilon,$$

because otherwise there would be a sequence $\{j_k\}$ and curves $\gamma_k \in \Gamma_{j_k}$ such that

$$\int_{\gamma_k} \rho' \, ds \leq 1 - 2\varepsilon.$$

Then by Lemma 4.2.22, there would exist $\tilde{\gamma} \in \Gamma$ such that

$$\int_{\tilde{\gamma}} \rho \, ds \leq 1 - 2\varepsilon + \varepsilon = 1 - \varepsilon,$$

contradicting the admissibility of ρ.

Then, for sufficiently large j, we have $\int_\gamma (1 - 2\varepsilon)^{-1} \rho' \, ds \geq 1$ for $\gamma \in \Gamma_j$. Thus, for sufficiently large j,

$$\begin{aligned}\operatorname{Mod}_p(|d\Gamma_j|) &\leq \left\|(1-2\varepsilon)^{-1}\rho'\right\|_p^p \leq (1-2\varepsilon)^{-p}(\|\rho\|_p^p+\varepsilon) \\ &< (1-2\varepsilon)^{-p}((1+r/2)^p(M+\varepsilon)+\varepsilon),\end{aligned} \tag{4.2.32}$$

and we obtain the result by letting $j \to \infty$, $r \to 0$, and $\varepsilon \to 0$.

□

We now turn our attention to 4.2.9. For its proof, we need two lemmas that can be found in Chapter 4 of [22]. In those two lemmas, we will make use of the following definition.

Definition 4.2.23. Given a set $X \subset \Omega$, we say X *is p-exceptional* if $\operatorname{Mod}_p(\Gamma_X) = 0$ where Γ_X is the family of curves in Ω which terminate at X. We say a property holds p-a.e. if the set of points where the property fails is p-exceptional.

Lemma 4.2.24 ([22])**.** *Let u be a p-precise function in Ω and let Γ be a family of curves in Ω whose end poins lie in Ω. Then u tends to the value of u at the end point along p-a.e. curve of Γ.*

Lemma 4.2.25 ([22])**.** *Let u be a p-precise function in Ω. Then any function equal to u p-a.e. in Ω is p-precise in Ω.*

Proof of 4.2.9. We state the Theorem again for convenience.

We have the equality

$$\mathrm{C_p}(\mathrm{K_0}, \mathrm{K_1}, \Omega) = \mathrm{C_p^*}(\mathrm{K_0}, \mathrm{K_1}, \Omega)$$

Moreover, suppose Ω is bounded. If $\operatorname{Mod}_P(d\Gamma) = 0$, then $\operatorname{Mod}_q(\nabla\Sigma) = \infty$; if $\operatorname{Mod}_q(d\Gamma) > 0$, then

$$\operatorname{Mod}_p(d\Gamma)^{1/p} \operatorname{Mod}_q(\nabla\Sigma)^{1/q} = 1.$$

Let K_0^j and K_1^j be as in Theorem 4.2.11. Let $u \in \mathcal{D}(K_0^j, K_1^j, \Omega)$ and set

$$\overline{u} = \begin{cases} 0 & \text{on } K_0^j \cap \Omega, \\ 1 & \text{on } K_1^j \cap \Omega, \\ u & \text{on } \Omega \backslash (K_0^j \cup K_1^j). \end{cases}$$

By definition of $\mathcal{D}(K_0^j, K_1^j, \Omega)$ and Lemma 4.2.24, we have $u = 0$ p-a.e. on $K_0^j \cap \Omega$ and $u = 1$ p-a.e. on $K_1^j \cap \Omega$, and so $u = \overline{u}$ p-a.e. in Ω. Thus, by Lemma 4.2.25, $\overline{u} \in \mathcal{D}^*(K_0, K_1, \Omega)$. Thus $\mathrm{C}_\mathrm{p}^*(\mathrm{K}_0, \mathrm{K}_1, \Omega) \leq \|\overline{u}\|_p^p = \|u\|_p^p$. Taking the infimum over u, we $\mathrm{C}_\mathrm{p}^*(\mathrm{K}_0, \mathrm{K}_1, \Omega) \leq \mathrm{C}_\mathrm{p}(\mathrm{K}_0^\mathrm{j}, \mathrm{K}_1^\mathrm{j}, \Omega)$. By Theorem 4.2.10, we have $\mathrm{C}_\mathrm{p}(\mathrm{K}_0^\mathrm{j}, \mathrm{K}_1^\mathrm{j}, \Omega) = \mathrm{Mod}_p(|d\Gamma(K_0^j, K_1^j, \Omega)|)$ and $\mathrm{C}_\mathrm{p}(\mathrm{K}_0, \mathrm{K}_1, \Omega) = \mathrm{Mod}_p(|d\Gamma|)$. By Theorem 4.2.11, we have $\mathrm{C}_\mathrm{p}(\mathrm{K}_0^\mathrm{j}, \mathrm{K}_1^\mathrm{j}, \Omega) \to \mathrm{C}_\mathrm{p}(\mathrm{K}_0, \mathrm{K}_1, \Omega)$ as $j \to \infty$, and thus $\mathrm{C}_\mathrm{p}^*(\mathrm{K}_0, \mathrm{K}_1, \Omega) \leq \mathrm{C}_\mathrm{p}(\mathrm{K}_0, \mathrm{K}_1, \Omega)$. As was stated earlier, since $\mathcal{D}^* \subset \mathcal{D}$, we have $\mathrm{C}_\mathrm{p}(\mathrm{K}_0, \mathrm{K}_1, \Omega) \leq \mathrm{C}_\mathrm{p}^*(\mathrm{K}_0, \mathrm{K}_1, \Omega)$, and hence

$$\mathrm{C}_\mathrm{p}(\mathrm{K}_0, \mathrm{K}_1, \Omega) = \mathrm{C}_\mathrm{p}^*(\mathrm{K}_0, \mathrm{K}_1, \Omega)\,. \tag{4.2.33}$$

By Theorem 4.2.7, we have

(1) $\mathrm{C}_\mathrm{p}^*(\mathrm{K}_0, \mathrm{K}_1, \Omega) = 0 \Rightarrow \mathrm{Mod}_q(\nabla\mathcal{D}^*) = \infty$

(2) $\mathrm{C}_\mathrm{p}^*(\mathrm{K}_0, \mathrm{K}_1, \Omega) > 0 \Rightarrow \mathrm{C}_\mathrm{p}^*(\mathrm{K}_0, \mathrm{K}_1, \Omega)^{1/p}\,\mathrm{Mod}_q(\nabla\mathcal{D}^*)^{1/q} = 1.$

Applying Theorem 4.2.10 and (4.2.33), we have

(1) $\mathrm{Mod}_p(d\Gamma) = 0 \Rightarrow \mathrm{Mod}_q(\nabla\Sigma) = \infty$

(2) $\mathrm{Mod}_p(d\Gamma) > 0 \Rightarrow \mathrm{Mod}_p(d\Gamma)^{1/p}\,\mathrm{Mod}_q(\nabla\Sigma)^{1/q} = 1.$

□

Theorem 4.2.6 is proved by directly applying Theorems 4.2.10 and 4.2.8 to Theorem 4.2.9.

Chapter 5

Modulus Metrics in $\mathbb{R}^N$

In this chapter, we will use Aikawa-Ohtsuka duality to construct a metric like we did with Fulkerson Duality on graphs in Theorem 2.3.3 of chapter 2. The main Theorem we wish to establish is below.

Theorem 5.0.1. *Let Ω be a domain in $\mathbb{R}^2$. Given two points $x, y \in \Omega$, let $\Gamma = \Gamma(x,y)$ be the family of curves connecting x to y, and for $p > 2$, define $\delta_p(x,y)$ as follows:*

$$\delta_p(x,y) := \begin{cases} 0 & \text{if } x = y \\ \mathrm{Mod}_p(|d\Gamma|)^{-q/p} & \text{if } x \neq y. \end{cases}$$

Then δ_p is a pseudometric on Ω.

We also make the following conjecture.

Conjecture 5.0.2. *Let Ω be a domain in $\mathbb{R}^N$. Given two points $x, y \in \Omega$, let $\Gamma = \Gamma(x,y)$ be the family of curves connecting x to y, and for $p > N$, define $\delta_p(x,y)$ as follows:*

$$\delta_p(x,y) := \begin{cases} 0 & \text{if } x = y \\ \mathrm{Mod}_p(|d\Gamma|)^{-q/p} & \text{if } x \neq y. \end{cases}$$

Then δ_p is a pseudometric on Ω.

Before we go into the proof, we will establish a few lemmas to aid us. Many of these lemmas are common and can be found in, for instance, [20]. Our first lemma gives us a relation of the p-Modulus of a family of curves that is majorized by another. We say that Γ_1 *majorizes* Γ_2 if every curve $\gamma_1 \in \Gamma_1$ has a subcurve $\gamma_2 \in \Gamma_2$.

Lemma 5.0.3 ([20])**.** *If* Γ_1 *majorizes* Γ_2*, then* $\mathrm{Mod}_p(|d\Gamma_1|) \leq \mathrm{Mod}_p(|d\Gamma_2|)$.

Proof. Let $\rho \in \mathrm{Adm}(|d\Gamma_2|)$. Then $\rho \in \mathrm{Adm}(|d\Gamma_1|)$ since for each $\gamma_1 \in \Gamma_1$, there is a $\gamma_2 \in \Gamma_2$ such that γ_2 is a subcurve of γ_1. Thus, we have $\mathrm{Mod}_p(|d\Gamma_1|) \leq \int_\Omega \rho^p \, dx$. Taking the infimum over all ρ gives us the desired inequality. □

Our next lemma is a standard example, see for instance [20]. For any r such that $0 < r < \infty$, let $B_r(x)$ be the open ball centered at $x \in \Omega$ of radius r and $B_r = B_r(0)$. Further, for $N \geq 2$, let $\omega_{N-1} := \mathcal{H}^{N-1}(\partial B_1)$.

Lemma 5.0.4 ([20])**.** *Let* $0 < r < R < \infty$ *and* $\Gamma = \Gamma(\partial B_r, \partial B_R, \mathbb{R}^N)$*. Then*

$$\mathrm{Mod}_p(|d\Gamma|) = \begin{cases} \left|\frac{N-p}{p-1}\right|^{p-1} \omega_{N-1} \left| r^{(p-N)/(p-1)} - R^{(p-N)/(p-1)} \right|^{1-p}, & \textit{if } p \neq N, \\ \omega_{N-1} \left(\log \frac{R}{r}\right)^{1-N}, & \textit{if } p = N, \\ \omega_{N-1} r^{N-1}, & \textit{if } p = 1. \end{cases}$$

Proof. First we shall show the above are lower bounds. If $\rho \in \mathrm{Adm}(|d\Gamma|)$, then for the curves $\gamma = \{t\omega : t \in [r, R], \omega \in \partial B_1\}$, we have $\int \rho \, d|\gamma| \geq 1$. Multiplying and dividing by $t^{(N-1)/p}$ and applying Hölder's inequality, for $p \neq 1$ we have

$$1 \leq \left(\int_r^R \rho(t\omega)^p \, t^{N-1} \, dt \right)^{1/p} \left(\int_r^R t^{-(N-1)q/p} \, dt \right)^{1/q}.$$

Now, the second integral is of the form $\int_r^R t^\alpha \, dt$ which is $\frac{1}{\alpha+1}(R^{\alpha+1} - r^{\alpha+1})$ if $\alpha \neq -1$ and $\log(R/r)$ if $\alpha = -1$. With $\alpha = -(N-1)q/p$, $\alpha = -1$ if $p = N$. Further, $-(N-1)q/p =$

$-(N-1)/(p-1)$ so that $\alpha+1=(p-N)/(p-1)$. Hence, if $p \neq N$, we have

$$1 \leq \left(\int_r^R \rho(t\omega)^p\, t^{N-1}\, dt\right)^{1/p} \left(\frac{p-1}{|N-p|}\left|R^{(p-N)/(p-1)} - r^{(p-N)/(p-1)}\right|\right)^{1/q}.$$

Here, we have used absolute values to combine the two distinct cases $1 < p < N$ and $N < p$. If instead $p = N$, then we have

$$1 \leq \left(\int_r^R \rho(t\omega)^p\, t^{N-1}\, dt\right)^{1/p} (\log(R/r))^{1/q}.$$

Dividing both sides in the two above inequalities by the second term on the right, raising to the p-th power, then integrating over ∂B_1 (that is, with respect to ω), we get the desired lower bounds in the case $1 < p$ and $p \neq N$.

If $p = 1$, then Hölder's inequality instead yields

$$1 \leq \left(\int_r^R \rho(t\omega)\, t^{N-1}\, dt\right) \left\|t^{-(N-1)|}\right\|_\infty,$$

where $\left\|t^{-(N-1)|}\right\|_\infty = r^{1-N}$. Again, dividing, and then integrating over ∂B_1, we get the desired lower bound.

To see that these estimates are also upper bounds, for $p > 1$, let ρ be defined by

$$\rho(x) := \left(\int_r^R t^{-(N-1)q/p}\, dt\right)^{-q/p} |x|^{-(N-1)q/p} \chi_{B_R \setminus B_r}(x),$$

and for the case $p = 1$, let ρ_j be defined by

$$\rho_j(x) := j \chi_{B_{r+1/j} \setminus B_r}(x)$$

and let $j \to \infty$. These functions are defined so that $\int \rho\, d|\gamma| \geq 1$, and evaluating $\int_\Omega \rho^p\, dx$ gives the desired result. □

Our next lemma is obtained from combining the two previous lemmas.

Lemma 5.0.5 ([20]). *If Γ_x is the family of curves going through a point in $x \in \mathbb{R}^N$, then for $1 \leq p \leq N$, $\mathrm{Mod}_p(|d\Gamma|) = 0$.*

Proof. Fix $\varepsilon > 0$. Let $\Gamma_{r,R}$ be as in Lemma 5.0.4. Without loss of generality, let $x = 0$. We want to find r and R such that $0 < r < R < \varepsilon$ and $\mathrm{Mod}_p(|d\Gamma_{r,R}|) < \varepsilon$. If we can do so, then each curve that passes through 0 has a subcurve in $\Gamma_{r,R}$ and so the majorization principle gives us $\mathrm{Mod}_p(|d\Gamma_0|) \leq \mathrm{Mod}_p(|d\Gamma_{r,R}|) < \varepsilon$. The result then follows by taking $\varepsilon \to 0$.

If $1 \leq p < N$, let $r = R/2$. Then by Lemma 5.0.4, $\mathrm{Mod}_p(|d\Gamma_{r,R}|) = CR^{N-p}$, and so taking R sufficiently small, we are done. If $p = N$, let $r = R^2$. Then by Lemma 5.0.4, $\mathrm{Mod}_p(|d\Gamma_{r,R}|) = C\log(1/R)^{1-N}$, and again, taking R sufficiently small, we obtain the result. □

The next theorem is due to Fuglede and is very similar to Lemma 5.0.5, except instead of curves, we have k-dimensional surfaces, and instead of $1 \leq p \leq N$, we have $p \leq n/k$. We refer the reader to [17] for the proof.

Theorem 5.0.6 ([17]). *Fix $x \in \Omega$. Let L_x be the collection of surface measures of k-dimensional Lipschitz surfaces which pass through x. Then $\mathrm{Mod}_p(L_x) = 0$ if and only if $p \leq n/k$.*

We wish to now note that Lemma 5.0.3 holds when we consider p-Modulus of families of measure and is proved the same way. If $\mathcal{M}$ and $\mathcal{N}$ are two families of Borel measures such that every measure $\mu \in \mathcal{M}$ has a measure $\nu \in \mathcal{N}$ such that $\mu(B) = \nu(B)$ for all $B \subset \mathrm{supp}(\nu)$. Then whenever $\int \rho\, d\nu \geq 1$, we also have $\int \rho\, d\mu \geq 1$, and so $\mathrm{Mod}_p(\mathcal{M}) \leq \mathrm{Mod}_p(\mathcal{N})$. We use this to provide a corollary to the Theorem 5.0.6.

Corollary 5.0.7 ([17]). *For fixed $z \in \Omega$, let L_z be the collection of all $(n-1)$-dimensional surface measures of surfaces containing z that are Lipschitz in a neighborhood of z. Then $\mathrm{Mod}_q(L_z) = 0$ where q satisfies $1 \leq q \leq n/(n-1)$.*

Proof. Without loss of generality, assume $z = 0$. If we let L_0^r be the collection of surface measures in L_0 that are Lipschitz when restricted to B_r, then Theorem 5.0.6 and majorization tell us $\mathrm{Mod}_q(L_0^r) = 0$. Now, for every surface that is Lipschitz in a neighborhood of 0, there is n large enough such that for $r = 1/n$ the surface is Lipschitz on B_r. By majorization, Thus $L_0 \subset \cup_n L_0^{1/n}$, and by subadditivity, we have

$$\mathrm{Mod}_q(L_0) \leq \sum_{n=0}^{\infty} \mathrm{Mod}_q(L_0^{1/n}) = 0.$$

□

In general, finding a formula for δ_p, even in the case $\Omega = \mathbb{R}^N$ has proven difficult. In this section, we give two estimates, one upper and one lower. For the first estimate, we use the majorization principle.

Lemma 5.0.8. *Let $p > N$, $\Gamma = \Gamma(x, y, \Omega)$, $\Gamma_R = \Gamma(x, \partial B_R(x), \mathbb{R}^N)$ where $R = |x - y|$. Then $\delta_p(x, y) \geq \mathrm{Mod}_p(\Gamma_R)^{-q/p} = (\frac{p-1}{p-N})\omega_{N-1}^{1/(1-p)} R^{(p-N)/(p-1)}$*

Proof. We have that Γ majorizes Γ_R, so $\mathrm{Mod}_p(|d\Gamma|) \leq \mathrm{Mod}_p(|d\Gamma_R|)$. Raising both sides to the $-q/p$ power gives the result. □

We note that $\mathrm{Mod}_p(|d\Gamma_R|)$ is given by Lemma 5.0.4 and letting $r \to 0$. We are now ready to prove Theorem 5.0.1.

With this estimate, we are now ready to prove Theorem 5.0.1.

Proof of Theorem 5.0.1. Let $x, y, z \in \Omega$ be distinct. Then by Lemma 5.0.8, $\delta_p(x, y) > 0$. Symmetry is obvious since $\Gamma(x, y) = \Gamma(y, x)$. Let Σ_z be the collection of characteristic functions of sets of finite perimeter A such that $A = U \cap \Omega$ for some open set $U \subset \mathbb{R}^2$ where $x \notin \overline{U}$, $y \in U$, and $z \in \partial U$. The triangle inequality follows from Theorem 4.2.6. Computing, we have

$$\begin{aligned}
\delta_p(x, y) &= \mathrm{Mod}_p(|d\Gamma|)^{-q/p} \\
&= \mathrm{Mod}_q(|\nabla\Sigma(x, y, \Omega)|) \\
&\leq \mathrm{Mod}_q(|\nabla\Sigma(x, z, \Omega)|) + \mathrm{Mod}_q(|\nabla\Sigma(z, y, \Omega)|) + \mathrm{Mod}_q(|\nabla\Sigma_z|) \\
&= \delta_p(x, z) + \delta_p(z, y) + \mathrm{Mod}_q(|\nabla\Sigma_z|).
\end{aligned}$$

Now, since $N = 2$, we have that the sets A such that $\chi_A \in \Sigma$ have boundaries which are broken curves that pass through a point, so $\mathrm{Mod}_q(|\nabla \Sigma_z|) = 0$ by Lemma 5.0.5 since $1 < q < 2$. Thus the triangle inequality follows. $\square$

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