A COMPARISON OF TWO ESTIMATORS OF THE VARIANCE IN THE TWO-FACTOR MULTIPLICATIVE INTERACTION MODEL

by

RONALD LEE WASSERSTEIN

B. A., Washburn University of Topeka, 1978

A MASTERS REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Statistics

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1984

Approved by:

Dallas E. Johnson

Major Professor

I. INTRODUCTION TO THE PROBLEM

Experiments involving two factors are common in statistical analysis, and the procedures for dealing with such experiments are well-known. Given certain assumptions about the data (normality, homogeneity of variance), such experiments can be analyzed by the analysis of variance technique. The goal in such experiments is to determine if differing levels of the factors result in different responses. In some situations, the effects of the various levels of one of the factors may depend on the level of the other factor. Such dependence is called interaction.

Suppose, for example, that the experimenter has two factors to study, and suppose that factor A has 4 levels (that is, 4 different possible values that are of interest) and factor B has 5 levels. A sample of size 20 is sufficient to give the experimenter one replication of the experiment, containing one observation for each of the 20 possible combinations of the levels of the two factors. Increasing the sample size by 20 gives the experimenter another look at (that is, another replication of) these combinations.

As long as there are at least two replications, the analysis of variance procedure is quite simple to perform. However, if the experiment contains only one replication, the experimenter is faced with a difficult problem. The analysis of variance technique compares the variation in the data caused by different combinations of the two factors to the amount of variation present in the data that is not caused by the variation of the levels of the factors. The latter amount is called the residual (or error) mean square. However, if interaction is present in the data, and if only one replication has been performed, the two types of variation cannot be separated. In such situations, the usual procedure for analyzing the effects of the factors is not valid.

A statistical model for this two-factor experimental situation is called a two-factor multiplicative interaction model. Let

(1.1) $$y_{ij} = \mu + \tau_i + \beta_j + \lambda\alpha_i\gamma_j + \varepsilon_{ij}, \quad \text{where } i = 1,2,\ldots,t \text{ and } j = 1,2,\ldots,b.$$

It is assumed, without loss of generality, that

$$\Sigma\tau_i = \Sigma\beta_j = \Sigma\alpha_i = \Sigma\gamma_j = 0 \text{ and } \Sigma\alpha_i^2 = \Sigma\gamma_j^2 = 1.$$

It is also assumed that $\varepsilon_{ij} \sim NID(0,\sigma^2)$, $i=1,\ldots,t$, $j=1,\ldots,b$. Implicit in the way this model is written is the presence of only one replication of the experiment.

If $\lambda = 0$, there is no interaction present, so one may proceed wth a standard additive two-way analysis of variance procedure, where the interaction mean square is used to measure experimental variation. However, if interaction is present, the interaction mean square is not an appropriate estimator of σ^2, and a different estimator is needed. Two such estimators of σ^2 have been proposed (Johnson and Graybill (1972), Carter and Srivastava (1980)). We have used Monte Carlo studies to examine properties of these estimators for different values of λ. The effect of b and t was also considered.

Let the t x b matrix $Z = \{z_{ij}\}$ be the matrix of residuals after fitting the additive model (that is, the above model with $\lambda = 0$) to the data. That is, let $z_{ij} = y_{ij} - \bar{y}_{i.} - \bar{y}_{.j} + \bar{y}_{..}$, $i = 1,2,\ldots,t$ and

$j = 1,2,\ldots,b$. Denote the nonzero characteristic roots of $Z'Z$ by $L_1 > L_2 > \ldots > L_p$, where $p = \min\{b-1, t-1\}$. Johnson and Graybill (1972) showed that the maximum likelihood estimator of σ^2 is

$$\hat{\sigma}^2 = (L_1 + L_2 + \ldots + L_p)/bt.$$

They also noted that the estimator

$$\tilde{\sigma}^2 = (L_2 + L_3 + \ldots + L_p)/(np - v_1),$$

where $n = \max\{b-1, t-1\}$ and $v_1 = E_{\lambda=0}(L_1/\sigma^2)$, is an unbiased estimator of σ^2 in model (1.1) when $\lambda = 0$ and not too badly biased when $\lambda \neq 0$. They suggested that $\tilde{\sigma}^2$ will be a satisfactory estimator of σ^2, bias notwithstanding.

Carter and Srivastava (1980) proposed an estimator of σ^2 which was shown to be asymptotically unbiased when $\lambda \neq 0$. The estimator they proposed is defined by

$$\hat{\hat{\sigma}}^2 = \frac{L_2 + L_3 + \ldots + L_p}{n(p-1)} \cdot \left(1 - \frac{L_2 + L_3 + \ldots + L_p}{n(p-1)L_1 - (L_2 + \ldots + L_p)}\right)$$

This estimator has a bias of the order $O(n^{-3/2})$ when $\lambda \neq 0$. Because the bias of $\tilde{\sigma}^2$ increases as λ increases, they recommend using $\hat{\hat{\sigma}}^2$ when $\lambda \neq 0$ in model (1.1).

II. THE MONTE CARLO STUDY

The Monte Carlo procedure used is similar to that used by Hegemann and Johnson (1976).

Both of the suggested estimators depend on the non-zero characteristic roots $L_1, L_2, \ldots, L_p$ of $Z'Z$. Johnson and Graybill showed that $L_1, L_2, \ldots, L_p$ are distributed as the roots of a $p \times p$ Wishart matrix. That is, they showed that the non-zero roots of $Z'Z$ are distributed the same as the non-zero characteristic roots of a matrix W, where $W \sim W_p(n, \sigma^2 I_p, M)$, and where $M = [\eta_{ij}]'[\eta_{ij}]$ is a noncentrality matrix with $\eta_{ij} = E(z_{ij})$. Furthermore, they noted that when the interaction term is of the form $\lambda\alpha_i\gamma_j$, as it is in model (1.1), M has rank one and, hence, has only one non-zero characteristic root, λ^2. The distribution of the characteristic roots of W depends only on the non-zero characteristic roots of M (Hegemann and Johnson (1976), James (1964)). Therefore, when generating a noncentral Wishart matrix W, one may take $M = \mathrm{diag}(\lambda^2, 0, \ldots, 0)$.

To generate W, the technique employed by Johnson and Hegemann (1974) was used. Without loss of generality, one may take $\sigma^2 = 1$, since the magnitude of $\tilde{\sigma}^2$ and $\hat{\sigma}^2$ are being compared relative to σ^2. Using $\sigma^2 = 1$ and $M = \mathrm{diag}(\lambda^2, 0, \ldots, 0)$, the procedure to generate $W \sim W_p(t-1, I_p, M)$ is as follows:

Let $\{N_{ij} : i < j = 1, 2, \ldots, p\}$ be a set of independent standard normal random deviates. Let $\{V_j : j = 1, 2, \ldots, p\}$ denote a sequence of independent chi-square random variates, where $V_1 \sim \chi'^2(n, \lambda^2/2)$ and $V_j \sim \chi^2(n-j+1)$, $j = 2, 3, \ldots, p$. Then the elements of W are given by

$$w_{11} = V_1$$

$$w_{jj} = V_j + \sum_{n=1}^{j-1} N_{ij}, \qquad j = 2,3,\ldots,p$$

$$w_{1j} = w_{j1} = N_{1j}V_1^{1/2}, \qquad j = 2,3,\ldots,p$$

$$w_{ij} = w_{ji} = N_{ij}V_i^{1/2} + \sum_{k=1}^{j-1} N_{ki}N_{kj}, \qquad i < j = 2,3,\ldots,p.$$

The FORTRAN code used to generate these non-central Wishart matrices appears in Appendix A. Marsaglia's Super-Duper random number generating package was used to generate the standard normal random variates. Subroutine GGCHS from the International Mathematical/Statistical Library (IMSL) was used to generate the chi-square random variates. Appendix B contains a test of the reliability of the subroutine. The noncentral X^2 deviate V_1 was generated as suggested in Johnson and Hegemann (1974). The variates $Z_1 \sim X^2(n-1)$ and $Z_2 \sim N(\lambda,1)$ were generated. Then $V_1 = Z_1 + Z_2^2$.

Values of t, b, and λ^2 were input at the start of each run. For each combination of t, b, and λ^2 used, 1000 random Wishart matrices were generated using the aforementioned procedure. For each matrix generated, the IMSL subroutine EIGRS was used to determine the characteristic roots, and the values of $\tilde{\sigma}^2$ and $\hat{\sigma}^2$ were calculated. The mean and mean square error of the 1000 values of $\tilde{\sigma}^2$ and $\hat{\sigma}^2$ were then computed.

In computing $\tilde{\sigma}^2$, the value $v_1 = E_{\lambda=0}(L_1/\sigma^2)$ is needed. Hegemann and Johnson (1976) and Mandel (1971) gave Monte Carlo values for v_1 for various values of t and b when $\sigma^2 = 1$. The Hegemann and Johnson values where used where possible, otherwise the Mandel values for v_1 were used.

Simulations were performed for this report for the following values of t, b, and λ^2:

t: 3-10, 12, 20, 32, 50, 100

b: 3-10

λ^2: 0, 1, 3, 6, 12, 24, 48, 96

III. RESULTS

Tables 1-8 show the resulting values of the mean and mean square error of the 1000 values of $\tilde{\sigma}^2$ and $\hat{\hat{\sigma}}^2$ obtained for the various values of t, b, and λ^2. The results of the Monte Carlo study cast doubt on the conclusion by Carter and Srivastava that $\hat{\hat{\sigma}}^2$ be used whenever $\lambda \neq 0$. The results also point to an instability in $\tilde{\sigma}^2$ when $\lambda \neq 0$ and both t and b are small. Such instability is predictable in an estimator of this type for small sample sizes.

Table 1 contains the Monte Carlo values for the case $\lambda^2 = 0$, that is, the case of no interaction. Note that $\tilde{\sigma}^2$ was approximately equal to 1 for all values of t and b, though it was closer to 1 for larger values of t and b. Note also that its mean square error for the case $t = b = 3$ was quite large (2.496) relative to other MSE values computed. Comparison of the MSE values for $t = b = 3$ for other values of λ^2 shows that MSE increased as λ^2 increased.

The estimator $\hat{\sigma}^2$ is not advised for $\lambda = 0$, and the Monte Carlo results make it easy to see why. The estimates of σ^2 are very poor even for the larger sample sizes. For example, mean($\hat{\sigma}^2$) = .525 for t = b = 6.

When $\lambda^2 = 1$, the mean values for $\tilde{\sigma}^2$ increased as expected, but still approached 1 as t and b increased. This improvement in $\tilde{\sigma}^2$ as t and b increased was consistent for all the values of λ^2 tested. This may suggest that $\tilde{\sigma}^2$ also has the property of asymptotic unbiasedness, though the tabled results show that mean($\tilde{\sigma}^2$) approached 1 at a slower rate than did mean($\hat{\sigma}^2$) as λ increased. When $\lambda^2 = 1$, mean($\tilde{\sigma}^2$) was closer to 1 than mean($\hat{\sigma}^2$) for all values of t and b used in this Monte Carlo study. As t and b increased, $\hat{\sigma}^2$ approached 1, but even for t=100 and b=10 it was still not as close as $\tilde{\sigma}^2$. More importantly, note that the former tended to underestimate the variance, while the latter overestimated, making $\tilde{\sigma}^2$ a more conservative estimate of $\hat{\sigma}^2$ than σ^2 when $\lambda^2 = 1$. Even for the large λ^2 values tested, this conservatism of $\tilde{\sigma}^2$ relative to σ^2 occurred consistently. Note also that for $\lambda^2 = 1$, MSE($\tilde{\sigma}^2$) > MSE($\hat{\sigma}^2$) for small t and b, but the opposite occurred for large t and b.

For $\lambda^2 = 3$, the tendencies mentioned above continued. Mean($\tilde{\sigma}^2$) was still closer to 1 than mean($\hat{\sigma}^2$) for all t and b. Note also the large value of MSE($\tilde{\sigma}^2$) (6.188) for t = b = 3.

When λ^2 = 6, we began to get parity between the two estimators for small values of t and b. Unfortunately, the parity was only in that both estimators were equally bad. For example, when t = 5 and b = 3, mean($\tilde{\sigma}^2$) = 1.503 and mean($\hat{\sigma}^2$) = 0.485. Again, however, $\tilde{\sigma}^2$ was the more conservative estimate of the two. As t and b increased, $\tilde{\sigma}^2$ remained better than $\hat{\sigma}^2$ in both mean and mean square error.

For the smaller values of t and b used, mean($\hat{\sigma}^2$) was closer to 1 than mean($\tilde{\sigma}^2$) when λ^2 = 12. However, larger values of t and b led to the opposite result. MSE($\tilde{\sigma}^2$) was extremely large for small t and b.

As λ^2 increased (to 24, 48, and 96, in this study), all these trends continued. That is, as λ^2 increased,

(1) mean($\hat{\sigma}^2$) was closer to 1 than mean($\tilde{\sigma}^2$) for the smaller values of t and b, but

(2) the threshold values of t and b, i.e., the values of t and b at which $\tilde{\sigma}^2$ became a closer estimate than $\hat{\sigma}^2$, increased. Both estimators approached 1 as t and b increased, but $\tilde{\sigma}^2$ tended to be the more conservative estimator.

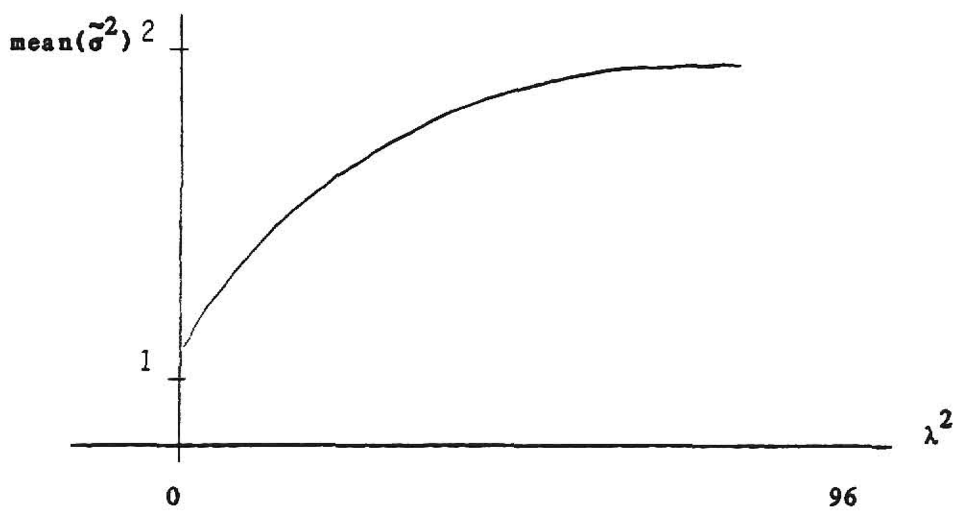
(3) Mean square errors for both estimators grew larger for small t and b.

IV. FURTHER STUDY SUGGESTED BY THE RESULTS

It appears that the choice between $\tilde{\sigma}^2$ and $\hat{\hat{\sigma}}^2$ is more difficult than the Carter and Srivastava suggestion to use the asymptotically unbiased estimate σ^2. The asymptotic property does not appear to help much in the range of likely sample sizes, and since $\hat{\hat{\sigma}}^2$ tends to underestimate σ^2, it may not be a desirable estimator.

On the other hand, $\tilde{\sigma}^2$ provides a conservative estimate. Unfortunately, $\tilde{\sigma}^2$ is sensitive to large values of λ^2, which greatly decreases its desirability as an estimator of σ^2 when large interaction is present.

One can notice, however, a rather consistent relationship between the estimator $\tilde{\sigma}^2$ and λ^2 for the various values of t and b. In general, the mean of the $\tilde{\sigma}^2$ values generated, when plotted against λ^2, looks roughly like this:



This apparent relationship lead to the following question: Can the relationship between mean($\tilde{\sigma}^2$) and λ^2 be modeled? One possible model might be

$$\text{mean}(\tilde{\sigma}^2/\sigma^2) = \beta_0 + \beta_1\lambda^2\exp[-\beta_2\lambda^2 f(t,b)] + \varepsilon,$$

where f(t,b) is some function of t and b, the number of levels of each factor. Since when $\lambda^2 = 0$, mean($\tilde{\sigma}^2/\sigma^2$) is approximately equal to 1, the value of β_0 should be equal to 1. Such a model, if one could be found, would suggest a modification to the estimator $\tilde{\sigma}^2$:

$$\tilde{\sigma}^2_{new} = \tilde{\sigma}^2 - \beta_1\lambda^2\exp[-\beta_2\lambda^2 f(t,b)].$$

This new estimator would 'smooth' out the effect of large λ^2 on $\tilde{\sigma}^2$.

Using NLIN in SAS(c)(1982), the following model was fit for various combinations of t and b used in the Monte Carlo study:

$$\text{mean}(\tilde{\sigma}^2) = \beta_0 + \beta_1\lambda^2\exp[-\beta_2\lambda^2(t-b+1)] + \varepsilon.$$

The estimates for the coefficients β_0, β_1, and β_2 are given in table 9. These values were then used in a Monte Carlo study to produce 1000 simulations of

$$\tilde{\sigma}^2_{new} = \tilde{\sigma}^2 - \beta_1\lambda^2\exp[-\beta_2\lambda^2(t-b+1)]$$

as in the initial study. The largest characteristic root of Z'Z was used as the estimate of λ^2, since it is the maximum likelihood estimate (Johnson and Graybill (1972)).

Table 10 shows the results of this Monte Carlo study. The results indicate that $\tilde{\sigma}^2_{new}$ significantly improves upon $\tilde{\sigma}^2$ for larger values of λ^2.

while $\tilde{\sigma}^2$ remains somewhat better for small λ^2. The new estimator still provides reasonable estimates even for small λ^2, however.

V. CONCLUSIONS

The use of the modified $\tilde{\sigma}^2$ as an estimate for σ^2 in the two-factor multiplicative interaction model is suggested. It offers an intuitively appealing alternative, and the simulation results indicate that it also offers considerable improvement over the other estimators. The relatively poor results for $\tilde{\sigma}^2_{new}$ for small λ^2 may be the result of relatively poor fit (for small λ^2) of the model on which it is based. A better model, if one could be found, might well produce a still better estimator.

LITERATURE CITED

1. Johnson, D. E., and Graybill, F. A. 1972. An analysis of a two-way model with interaction and no replication. JASA 67:862-868.

2. Carter, E. M. and Srivastava, M. S. 1980. Asymptotic distribution of the latent roots of the noncentral Wishart distribution and the power of the likelihood ratio test for nonadditivity. Canadian Journal of Statistics 8:119-134.

3. Hegemann, V. and Johnson, D. E. 1976. On analyzing two-way AoV data with interaction. Technometrics 18:273-281.

4. James, A. T. 1964. Distributions of matrix variates and latent roots derived from normal samples. Annals of Mathematical Statistics 35:475-501.

5. Johnson, D. E. and Hegemann, V. 1974. Procedures to generate random matrices with noncentral distributions. Communications in Statistics 3:691-699.

6. Mandel, J. 1971. A new analysis of variance model for non-additive data. Technometrics 13:1-18.

7. Beyer, W. H., ed. 1968. <u>Handbook of Tables for Probability and Statistics</u>, second edition. CRC Press, Cleveland.

APPENDIX A

The following is the FORTRAN code used to generate the Monte Carlo values.

```
C  COMPARE THE TWC VARIANCE ESTIMATES FOR VARIOUS COMBINATIONS
C        OF N,P,LAMDA
      INTEGER P
      REAL*8 WW(55),W(10,10),R(55),V(10),NORM(10,10),DSEED
      REAL*8 LAMDA,NU1,LRT,VJG,VCS,SUMVJG,SUMVCS,SQDVJG,SQDVCS
      REAL*8 AVGVJG,AVGVCS,MSEVJG,MSEVCS
      REAL*8 WK(10),D(10),Z(10)
      REAL*8 TRACE,TR
C  THE FOLLOWING ARE THE MEANINGS OF THE VARIABLE NAMES:
C       N=NUMBER OF LEVELS OF FACTOR 1, MINUS 1
C      P=NUMBER OF LEVELS OF FACTOR2, MINUS 1
C      W=RANDOM WISHART MATRIX     R=WORK VECTOR FOR GGCHS
C      V=CHI-SQUARED DEVIATE VECTOR   NORM=NORMAL DEVIATE MATRIX
C      DSEED=SEED VALUE FOR GGCHS    WK=WORK VECTOR FOR EIGRS
C      WW=SYMMETRIC MODE VERSION OF W   D=EIGENVALUES
C       Z=WORK MATRIX FOR EIGRS
C      LAMDA=INTERACTION     NU1=EXPECTED L1 UNDER HO
C      LRT=LARGEST CHARACTERISTIC ROOT
C      VJG=JOHNSON ESTIMATE     VCS=CARTER ESTIMATE
C      SUMVJG=SUM OF VJG     SUMVCS=SUM OF VCS
C      AVGVJG=AVERAGE OF VJG     AVGVCS=AVERAGE OF VCS
      READ(5,100)DSEED
      WRITE(6,107)DSEED
      WRITE(6,103)
      WRITE(6,104)
  500 READ(5,101)N,P,LAMDA,NU1
      IF (N.EQ.0)GO TO 2100
      SUMVJG=0.
      SQDVJG=0.
      SUMVCS=0.
      SQDVCS=0.
      DO 2000 LCNT=1,1000
C ***GENERATE THE CHI-SQUARED DEVIATES***************************
      DO 1000 J=2,P
      K=N-J+1
      CHI2=0.
      CALL GGCHS(DSEED,K,R,CHI2)
 1000 V(J)=CHI2
      K=N-1
      CHI2=0.
      CALL GGCHS(DSEED,K,R,CHI2)
      Z2=RNOR(0)
      Z2=Z2+LAMDA**.5
      V(1)=CHI2+Z2*Z2
C ***FILL NORM AND W WITH ZEROS**********************************
      DO 1020 I=1,P
      DO 1010 J=1,P
      NORM(I,J)=0.
 1010 W(I,J)=0.
 1020 CONTINUE
C ***GENERATE THE STANDARD NORMAL DEVIATES***********************
      DO 1040 I=2,P
      K=I-1
      DO 1030 J=1,K
 1030 NORM(I,J)=RNOR(0)
 1040 CONTINUE
C ***CONSTRUCT THE WISHART MATRIX********************************
C ***DIAGONAL ELEMENTS**************
      W(1,1)=V(1)
```

```
      DO 1060 I=2,P
      K=I-1
      DO 1050 J=1,K
 1050 W(I,I)=W(I,I)+NORM(I,J)**2
 1060 W(I,I)=W(I,I)+V(I)
C ***ELEMENTS OF FIRST COLUMN*******
      SUM=0.
      DO 1070 I=2,P
 1070 W(I,1)=NORM(I,1)*V(1)**.5
C ***REMAINING LOWER DIAGONAL ELEMENTS*****
      DO 1100 I=3,P
      L=I-1
      DO 1090 J=2,L
      M=J-1
      DO 1080 K=1,M
 1080 SUM=SUM+NORM(I,K)*NORM(J,K)
      W(I,J)=SUM+NORM(I,J)*V(J)**.5
 1090 SUM=0.
 1100 CONTINUE
C ***STORE THE WISHART MATRIX IN SYMMETRIC MODE*****************
      K=1
      DO 1120 I=1,P
      DO 1110 J=1,I
      WW(K)=W(I,J)
 1110 K=K+1
 1120 CONTINUE
C ***FIND THE LARGEST CHARACTERISTIC ROOT***********************
      IP=10
      JOBN=0
      CALL EIGRS(WW,P,JOBN,D,Z,IP,WK,IER)
      IF(IER.EQ.0)GO TO 1130
      WRITE(6,102)IER
C ***FIND THE TRACE OF W****************************************
 1130 TRACE=0.
       DO 1135 I=1,P
 1135 TRACE=TRACE+W(I,I)
C ***CALCULATE JOHNSON & GRAYBILL'S ESTIMATE*******************
      LRT=D(P)
      TR=TRACE-LRT
      VJG=TR/(DFLOAT(N*P)-NU1)
C ***CALCULATE CARTER & SRIVASTAVA'S ESTIMATE******************
      VCS=(TR/(N*(P-1.)))*(1.-(TR/(N*((P-1.)*LRT-TR))))
      SUMVJG=SUMVJG+VJG
      SUMVCS=SUMVCS+VCS
      SQDVJG=SQDVJG+(VJG-1.)**2
      SQDVCS=SQDVCS+(VCS-1.)**2
 2000 CONTINUE
      AVGVJG=SUMVJG/1000.
      AVGVCS=SUMVCS/1000.
      MSEVJG=SQDVJG/1000.
      MSEVCS=SQDVCS/1000.
      WRITE(6,105)N,P,LAMDA,AVGVJG,MSEVJG,AVGVCS,MSEVCS
      GO TO 500
 2100 WRITE(6,108)DSEED
      STOP
  100 FORMAT(D12.6)
  101 FORMAT(2I2,F5.2,F6.2)
  102 FORMAT('0','IER=',I6)
  103 FORMAT('1',T15,'MEAN AND VARIANCE OF ESTIMATORS FOR 1000 SIMULATIO
     1NS')
```

```
104 FORMAT('-',T5,'N',T10,'P',T15,'LAMDA',T26,'AVG JG',T41,
   1'MSE JG',T60,'AVG CS',T77,'MSE CS')
105 FORMAT(' ',T4,I2,T9,I2,T16,F5.2,T24,F8.3,T39,F8.3,T58,F8.3,
   1T75,F8.3)
107 FORMAT('-','START SEED = ',D12.6)
108 FORMAT('0','END SEED= ',D12.6)
    END
```

APPENDIX B

To check the validity of the chi-square random variates generated by the IMSL subroutine GGCHS, 1000 random variates were generated for each degree of freedom from one to 30. These variates were compared with 95% confidence limits found in Beyer (1968). The table shows the proportion of the variates that fell within the 95% confidence range.

degrees of freedom	proportion of variates falling within 95% confidence range
1	.951
2	.956
3	.946
4	.950
5	.943
6	.947
7	.956
8	.942
9	.947
10	.950
11	.963
12	.945
13	.959
14	.948
15	.953
16	.944
17	.947
18	.945
19	.959
20	.950
21	.947
22	.948
23	.943
24	.956
25	.960
26	.958
27	.953
28	.943
29	.950
30	.940

APPENDIX C

The following table compares some of the Monte Carlo values for v_1 in Hegemann and Johnson (1976) and Mandel (1971) with the values the author of this paper obtained by simulation. In all cases, there do not appear to be any significant differences between the three values.

t	b	Hegemann	Mandel	Wasserstein
3	3	3.57*	3.55	3.57
7	7	17.22	17.17	17.32
8	5	14.96	15.03	14.87
8	6	16.91	16.97	17.14
10	3	12.66	12.77	12.83
10	10	28.21	27.99	28.01

*indicates exact values

The following table compares some of the Monte Carlo values for $E_{\lambda^2}(l_1)$ given by Hegemann and Johnson (1974) with those found by this author, using b = 5 and t = 8.

λ^2	Hegemann	Wasserstein
0	14.96	14.87
12	24.45	23.87
44	54.32	54.55

TABLE 1

MEAN AND MEAN SQUARE ERROR OF $\tilde{\sigma}^2$ AND $\hat{\sigma}^2$
FOR 1000 SIMULATIONS
WHEN $\lambda^2 = 0$

t	b	mean($\tilde{\sigma}^2$)	MSE($\tilde{\sigma}^2$)	mean($\hat{\sigma}^2$)	MSE($\hat{\sigma}^2$)
3	3	1.017	2.496	0.124	0.947
4	3	1.006	1.058	0.230	0.670
5	3	0.994	0.659	0.297	0.629
6	3	0.683	0.359	0.350	0.504
7	3	0.980	0.354	0.378	0.608
8	3	0.987	0.294	0.432	0.380
9	3	0.997	0.246	0.461	0.344
10	3	0.977	0.216	0.470	0.352
4	4	1.092	0.582	0.343	0.479
5	4	0.983	0.309	0.406	0.399
6	4	1.031	0.247	0.464	0.332
7	4	0.978	0.171	0.497	0.291
8	4	0.988	0.138	0.527	0.257
9	4	0.994	0.122	0.565	0.223
10	4	1.001	0.112	0.583	0.207
12	4	0.984	0.087	0.612	0.180
20	4	0.995	0.053	0.699	0.113
32	4	0.998	0.029	0.762	0.072
50	4	0.999	0.017	0.809	0.046
100	4	1.000	0.008	0.863	0.024
5	5	1.003	0.208	0.446	0.344
6	5	0.985	0.139	0.502	0.280
7	5	1.008	0.126	0.542	0.242
8	5	1.004	0.102	0.578	0.209
9	5	1.014	0.091	0.606	0.184
10	5	1.011	0.079	0.619	0.172
6	6	0.980	0.107	0.525	0.254
7	6	1.007	0.103	0.572	0.214
8	6	0.979	0.071	0.595	0.188
9	6	0.999	0.072	0.624	0.168
10	6	1.002	0.055	0.654	0.142
12	6	0.992	0.051	0.678	0.126
20	6	0.996	0.025	0.753	0.074
32	6	0.998	0.016	0.808	0.047
50	6	1.000	0.009	0.848	0.029
100	6	1.002	0.005	0.894	0.015
7	7	1.000	0.075	0.592	0.191
8	7	1.021	0.068	0.631	0.160
9	7	0.991	0.054	0.645	0.147
10	7	0.994	0.047	0.662	0.133

TABLE 1
Continued

MEAN AND MEAN SQUARE ERROR OF $\tilde{\sigma}^2$ AND $\hat{\sigma}^2$
FOR 1000 SIMULATIONS
WHEN λ^2 = 0

t	b	mean($\tilde{\sigma}^2$)	MSE($\tilde{\sigma}^2$)	mean($\hat{\sigma}^2$)	MSE($\hat{\sigma}^2$)
8	8	0.984	0.053	0.631	0.157
9	8	1.022	0.047	0.669	0.129
10	8	1.008	0.041	0.683	0.119
12	8	0.989	0.033	0.708	0.101
20	8	0.997	0.018	0.782	0.058
32	8	0.999	0.011	0.832	0.035
50	8	1.003	0.007	0.871	0.022
100	8	1.000	0.003	0.907	0.011
9	9	1.001	0.041	0.668	0.127
10	9	1.009	0.035	0.701	0.105
10	10	1.010	0.031	0.709	0.099
12	10	0.998	0.027	0.733	0.085
20	10	1.003	0.013	0.804	0.046
32	10	1.002	0.008	0.849	0.028
50	10	1.000	0.005	0.879	0.018
100	10	1.000	0.002	0.918	0.009

TABLE 2

MEAN AND MEAN SQUARE ERROR OF $\tilde{\sigma}^2$ AND $\hat{\sigma}^2$
FOR 1000 SIMULATIONS
WHEN $\lambda^2 = 1$

t	b	mean($\tilde{\sigma}^2$)	MSE($\tilde{\sigma}^2$)	mean($\hat{\sigma}^2$)	MSE($\hat{\sigma}^2$)
3	3	1.339	4.662	0.135	1.147
4	3	1.151	1.270	0.257	0.770
5	3	1.123	0.838	0.330	0.560
6	3	0.959	0.466	0.381	0.548
7	3	1.058	0.400	0.433	0.388
8	3	1.071	0.320	0.449	0.412
9	3	1.071	0.310	0.485	0.348
10	3	1.044	0.253	0.501	0.336
4	4	1.107	0.612	0.350	0.473
5	4	1.064	0.351	0.442	0.364
6	4	1.067	0.267	0.478	0.318
7	4	1.056	0.207	0.536	0.260
8	4	1.045	0.172	0.558	0.274
9	4	0.996	0.144	0.564	0.230
10	4	1.018	0.117	0.595	0.199
12	4	1.032	0.098	0.642	0.161
20	4	1.026	0.048	0.721	0.098
32	4	1.010	0.033	0.770	0.069
50	4	1.000	0.017	0.811	0.046
100	4	0.997	0.008	0.861	0.025
5	5	1.036	0.241	0.460	0.334
6	5	1.045	0.168	0.532	0.257
7	5	1.050	0.139	0.565	0.225
8	5	1.029	0.112	0.594	0.198
9	5	1.036	0.099	0.620	0.175
10	5	1.045	0.090	0.639	0.159
6	6	1.037	0.130	0.555	0.232
7	6	1.025	0.102	0.582	0.205
8	6	1.021	0.084	0.620	0.173
9	6	1.032	0.074	0.644	0.153
10	6	1.024	0.062	0.669	0.134
12	6	1.015	0.047	0.694	0.114
20	6	1.016	0.026	0.767	0.068
32	6	1.002	0.014	0.812	0.044
50	6	1.000	0.009	0.848	0.029
100	6	1.003	0.004	0.894	0.015
7	7	1.030	0.080	0.610	0.178
8	7	1.017	0.071	0.629	0.163
9	7	1.022	0.056	0.664	0.135
10	7	1.014	0.052	0.675	0.127

TABLE 2, cont.

MEAN AND MEAN SQUARE ERROR OF $\tilde{\sigma}^2$ AND $\hat{\sigma}^2$
FOR 1000 SIMULATIONS
WHEN $\lambda^2 = 1$

t	b	mean($\tilde{\sigma}^2$)	MSE($\tilde{\sigma}^2$)	mean($\hat{\sigma}^2$)	MSE($\hat{\sigma}^2$)
8	8	1.013	0.054	0.649	0.144
9	8	1.039	0.048	0.680	0.121
10	8	1.024	0.042	0.694	0.112
12	8	1.004	0.029	0.718	0.094
20	8	1.006	0.018	0.789	0.055
32	8	1.005	0.010	0.837	0.034
50	8	1.002	0.006	0.869	0.022
100	8	0.999	0.003	0.906	0.011
9	9	1.027	0.044	0.686	0.117
10	9	1.013	0.037	0.704	0.104
10	10	1.015	0.030	0.714	0.096
12	10	1.013	0.024	0.744	0.078
20	10	1.007	0.014	0.808	0.046
32	10	1.002	0.008	0.849	0.028
50	10	1.002	0.005	0.881	0.018
100	10	1.004	0.002	0.921	0.008

TABLE 3

MEAN AND MEAN SQUARE ERROR OF $\tilde{\sigma}^2$ AND $\hat{\sigma}^2$
FOR 1000 SIMULATIONS
WHEN $\lambda^2 = 3$

t	b	mean($\tilde{\sigma}^2$)	MSE($\tilde{\sigma}^2$)	mean($\hat{\sigma}^2$)	MSE($\hat{\sigma}^2$)
3	3	1.595	6.188	0.224	1.098
4	3	1.472	2.321	0.367	0.512
5	3	1.311	1.140	0.354	1.027
6	3	1.101	0.521	0.458	0.422
7	3	1.195	0.575	0.483	0.373
8	3	1.177	0.454	0.499	0.364
9	3	1.157	0.355	0.530	0.354
10	3	1.159	0.299	0.572	0.248
4	4	1.288	0.885	0.410	0.417
5	4	1.181	0.439	0.491	0.319
6	4	1.205	0.393	0.542	0.270
7	4	1.122	0.235	0.575	0.231
8	4	1.105	0.200	0.594	0.212
9	4	1.079	0.155	0.615	0.190
10	4	1.094	0.141	0.638	0.170
12	4	1.079	0.114	0.672	0.144
20	4	1.040	0.053	0.732	0.094
32	4	1.031	0.033	0.787	0.062
50	4	1.012	0.017	0.820	0.043
100	4	1.010	0.009	0.872	0.022
5	5	1.147	0.286	0.512	0.286
6	5	1.143	0.217	0.583	0.220
7	5	1.127	0.176	0.608	0.196
8	5	1.076	0.131	0.621	0.181
9	5	1.095	0.108	0.658	0.150
10	5	1.095	0.106	0.671	0.141
6	6	1.094	0.147	0.587	0.207
7	6	1.087	0.123	0.618	0.180
8	6	1.072	0.095	0.652	0.152
9	6	1.072	0.080	0.670	0.136
10	6	1.053	0.063	0.687	0.122
12	6	1.033	0.050	0.708	0.107
20	6	1.029	0.029	0.778	0.064
32	6	1.015	0.016	0.822	0.041
50	6	1.014	0.010	0.859	0.027
100	6	1.006	0.005	0.898	0.014
7	7	1.065	0.082	0.632	0.161
8	7	1.064	0.079	0.658	0.144
9	7	1.065	0.067	0.692	0.119
10	7	1.050	0.058	0.700	0.132

TABLE 3
Continued

MEAN AND MEAN SQUARE ERROR OF $\tilde{\sigma}^2$ AND $\hat{\sigma}^2$
FOR 1000 SIMULATIONS
WHEN λ^2 = 3

t	b	mean($\tilde{\sigma}^2$)	MSE($\tilde{\sigma}^2$)	mean($\hat{\sigma}^2$)	MSE($\hat{\sigma}^2$)
8	8	1.057	0.062	0.678	0.126
9	8	1.063	0.054	0.695	0.113
10	8	1.051	0.047	0.713	0.101
12	8	1.029	0.034	0.736	0.085
20	8	1.015	0.018	0.797	0.052
32	8	1.017	0.011	0.847	0.031
50	8	1.011	0.007	0.877	0.020
100	8	1.004	0.003	0.911	0.011
9	9	1.054	0.048	0.704	0.107
10	9	1.029	0.037	0.716	0.097
10	10	1.033	0.032	0.726	0.090
12	10	1.038	0.025	0.763	0.068
20	10	1.010	0.013	0.811	0.044
32	10	1.011	0.008	0.857	0.026
50	10	1.008	0.005	0.887	0.017
100	10	1.002	0.002	0.919	0.008

TABLE 4

MEAN AND MEAN SQUARE ERROR OF $\tilde{\sigma}^2$ AND $\hat{\sigma}^2$
FOR 1000 SIMULATIONS
WHEN $\lambda^2 = 6$

t	b	mean($\tilde{\sigma}^2$)	MSE($\tilde{\sigma}^2$)	mean($\hat{\sigma}^2$)	MSE($\hat{\sigma}^2$)
3	3	1.813	7.265	0.279	0.753
4	3	1.553	2.728	0.398	0.569
5	3	1.503	1.676	0.485	0.412
6	3	1.177	0.627	0.496	0.410
7	3	1.297	0.674	0.549	0.300
8	3	1.280	0.560	0.574	0.315
9	3	1.283	0.454	0.607	0.289
10	3	1.223	0.405	0.598	0.264
4	4	1.538	1.336	0.497	0.349
5	4	1.314	0.675	0.552	0.287
6	4	1.316	0.519	0.597	0.236
7	4	1.211	0.330	0.624	0.207
8	4	1.198	0.266	0.649	0.181
9	4	1.195	0.228	0.684	0.154
10	4	1.169	0.190	0.685	0.147
12	4	1.127	0.140	0.706	0.128
20	4	1.069	0.063	0.755	0.085
32	4	1.061	0.037	0.811	0.053
50	4	1.037	0.022	0.840	0.037
100	4	1.021	0.009	0.881	0.020
5	5	1.240	0.350	0.558	0.249
6	5	1.190	0.263	0.612	0.204
7	5	1.193	0.210	0.649	0.170
8	5	1.158	0.153	0.672	0.146
9	5	1.141	0.137	0.688	0.136
10	5	1.128	0.105	0.695	0.124
6	6	1.166	0.187	0.628	0.181
7	6	1.155	0.149	0.659	0.154
8	6	1.131	0.116	0.689	0.130
9	6	1.127	0.105	0.706	0.118
10	6	1.097	0.077	0.719	0.106
12	6	1.084	0.059	0.743	0.088
20	6	1.056	0.034	0.799	0.057
32	6	1.031	0.016	0.835	0.037
50	6	1.021	0.011	0.866	0.025
100	6	1.013	0.005	0.903	0.013
7	7	1.121	0.107	0.667	0.141
8	7	1.128	0.100	0.699	0.121
9	7	1.097	0.074	0.716	0.106
10	7	1.096	0.066	0.731	0.096

TABLE 4
Continued
MEAN AND MEAN SQUARE ERROR OF $\tilde{\sigma}^2$ AND $\hat{\sigma}^2$
FOR 1000 SIMULATIONS
WHEN $\lambda^2 = 6$

t	b	mean($\tilde{\sigma}^2$)	MSE($\tilde{\sigma}^2$)	mean($\hat{\sigma}^2$)	MSE($\hat{\sigma}^2$)
8	8	1.105	0.075	0.709	0.110
9	8	1.100	0.062	0.721	0.099
10	8	1.098	0.057	0.746	0.085
12	8	1.061	0.040	0.760	0.075
20	8	1.037	0.021	0.815	0.046
32	8	1.030	0.012	0.858	0.028
50	8	1.019	0.007	0.884	0.018
100	8	1.010	0.003	0.916	0.010
9	9	1.086	0.053	0.727	0.094
10	9	1.061	0.042	0.739	0.086
10	10	1.072	0.040	0.754	0.077
12	10	1.052	0.028	0.774	0.065
20	10	1.034	0.015	0.830	0.037
32	10	1.023	0.008	0.867	0.023
50	10	1.012	0.005	0.891	0.016
100	10	1.003	0.002	0.920	0.008

TABLE 5

MEAN AND MEAN SQUARE ERROR OF $\tilde{\sigma}^2$ AND $\hat{\sigma}^2$
FOR 1000 SIMULATIONS
WHEN $\lambda^2 = 12$

t	b	mean($\tilde{\sigma}^2$)	MSE($\tilde{\sigma}^2$)	mean($\hat{\sigma}^2$)	MSE($\hat{\sigma}^2$)
3	3	1.963	9.880	0.359	0.638
4	3	1.797	3.392	0.514	0.436
5	3	1.693	2.479	0.589	0.384
6	3	1.274	0.806	0.600	0.298
7	3	1.479	0.989	0.662	0.247
8	3	1.452	0.850	0.677	0.298
9	3	1.378	0.652	0.699	0.200
10	3	1.322	0.521	0.699	0.191
4	4	1.622	1.610	0.538	0.331
5	4	1.414	0.773	0.612	0.251
6	4	1.447	0.697	0.673	0.201
7	4	1.309	0.423	0.689	0.175
8	4	1.315	0.360	0.724	0.146
9	4	1.268	0.294	0.740	0.134
10	4	1.228	0.205	0.735	0.119
12	4	1.211	0.191	0.769	0.105
20	4	1.147	0.088	0.816	0.063
32	4	1.111	0.045	0.853	0.039
50	4	1.063	0.024	0.864	0.030
100	4	1.035	0.011	0.894	0.017
5	5	1.393	0.551	0.633	0.208
6	5	1.309	0.357	0.681	0.164
7	5	1.306	0.305	0.717	0.137
8	5	1.232	0.198	0.723	0.122
9	5	1.212	0.174	0.739	0.112
10	5	1.226	0.165	0.762	0.097
6	6	1.244	0.242	0.676	0.154
7	6	1.259	0.223	0.723	0.123
8	6	1.118	0.146	0.731	0.111
9	6	1.205	0.138	0.761	0.093
10	6	1.166	0.101	0.769	0.083
12	6	1.144	0.083	0.788	0.072
20	6	1.092	0.038	0.829	0.045
32	6	1.064	0.021	0.864	0.029
50	6	1.038	0.012	0.882	0.021
100	6	1.023	0.006	0.914	0.011
7	7	1.206	0.152	0.723	0.114
8	7	1.187	0.123	0.740	0.100
9	7	1.165	0.109	0.764	0.088
10	7	1.152	0.087	0.774	0.078

TABLE 5
Continued

MEAN AND MEAN SQUARE ERROR OF $\tilde{\sigma}^2$ AND $\hat{\sigma}^2$
FOR 1000 SIMULATIONS
WHEN λ^2 = 12

t	b	mean($\tilde{\sigma}^2$)	MSE($\tilde{\sigma}^2$)	mean($\hat{\sigma}^2$)	MSE($\hat{\sigma}^2$)
8	8	1.143	0.088	0.738	0.096
9	8	1.160	0.092	0.764	0.083
10	8	1.138	0.067	0.777	0.071
12	8	1.128	0.058	0.811	0.056
20	8	1.065	0.024	0.838	0.038
32	8	1.045	0.013	0.872	0.024
50	8	1.031	0.008	0.896	0.016
100	8	1.014	0.004	0.920	0.009
9	9	1.139	0.068	0.765	0.076
10	9	1.112	0.054	0.777	0.069
10	10	1.121	0.051	0.790	0.062
12	10	1.088	0.035	0.802	0.053
20	10	1.067	0.019	0.857	0.029
32	10	1.037	0.010	0.879	0.020
50	10	1.026	0.005	0.903	0.013
100	10	1.013	0.003	0.929	0.007

TABLE 6

MEAN AND MEAN SQUARE ERROR OF $\tilde{\sigma}^2$ AND $\hat{\sigma}^2$
FOR 1000 SIMULATIONS
WHEN $\lambda^2 = 24$

t	b	mean($\tilde{\sigma}^2$)	MSE($\tilde{\sigma}^2$)	mean($\hat{\sigma}^2$)	MSE($\hat{\sigma}^2$)
3	3	2.221	11.919	0.439	0.664
4	3	1.851	4.692	0.563	0.502
5	3	1.773	2.643	0.666	0.395
6	3	1.370	1.075	0.686	0.303
7	3	1.603	1.365	0.755	0.345
8	3	1.481	0.933	0.759	0.218
9	3	1.451	0.833	0.778	0.198
10	3	1.430	0.691	0.800	0.181
4	4	1.758	2.100	0.602	0.322
5	4	1.516	1.053	0.675	0.247
6	4	1.558	0.903	0.744	0.185
7	4	1.406	0.555	0.761	0.160
8	4	1.397	0.444	0.791	0.128
9	4	1.345	0.352	0.806	0.113
10	4	1.328	0.318	0.811	0.106
12	4	1.285	0.231	0.836	0.083
20	4	1.208	0.118	0.874	0.050
32	4	1.135	0.057	0.883	0.035
50	4	1.096	0.032	0.899	0.024
100	4	1.053	0.012	0.915	0.014
5	5	1.455	0.634	0.677	0.190
6	5	1.357	0.430	0.723	0.156
7	5	1.369	0.374	0.767	0.124
8	5	1.324	0.281	0.793	0.102
9	5	1.292	0.236	0.802	0.093
10	5	1.288	0.214	0.814	0.083
6	6	1.325	0.312	0.732	0.131
7	6	1.323	0.289	0.772	0.110
8	6	1.254	0.187	0.783	0.092
9	6	1.272	0.184	0.815	0.077
10	6	1.214	0.139	0.812	0.074
12	6	1.207	0.110	0.843	0.055
20	6	1.144	0.057	0.877	0.035
32	6	1.099	0.027	0.898	0.021
50	6	1.066	0.014	0.909	0.015
100	6	1.038	0.006	0.928	0.009
7	7	1.278	0.205	0.776	0.094
8	7	1.254	0.164	0.792	0.080
9	7	1.215	0.127	0.807	0.071
10	7	1.225	0.125	0.831	0.061

TABLE 6
Continued

MEAN AND MEAN SQUARE ERROR OF $\tilde{\sigma}^2$ AND $\hat{\sigma}^2$
FOR 1000 SIMULATIONS
WHEN $\lambda^2 = 24$

t	b	mean($\tilde{\sigma}^2$)	MSE($\tilde{\sigma}^2$)	mean($\hat{\sigma}^2$)	MSE($\hat{\sigma}^2$)
8	8	1.217	0.136	0.793	0.079
9	8	1.216	0.120	0.809	0.067
10	8	1.189	0.092	0.819	0.058
12	8	1.161	0.070	0.841	0.047
20	8	1.103	0.031	0.873	0.028
32	8	1.075	0.018	0.900	0.018
50	8	1.051	0.011	0.915	0.013
100	8	1.025	0.004	0.931	0.007
9	9	1.199	0.095	0.812	0.059
10	9	1.164	0.072	0.819	0.054
10	10	1.161	0.066	0.824	0.050
12	10	1.147	0.055	0.851	0.040
20	10	1.093	0.024	0.882	0.024
32	10	1.060	0.012	0.901	0.016
50	10	1.045	0.007	0.921	0.010
100	10	1.024	0.003	0.939	0.006

TABLE 7

MEAN AND MEAN SQUARE ERROR OF $\tilde{\sigma}^2$ AND $\hat{\sigma}^2$
FOR 1000 SIMULATIONS
WHEN λ^2 = 48

t	b	mean($\tilde{\sigma}^2$)	MSE($\tilde{\sigma}^2$)	mean($\hat{\sigma}^2$)	MSE($\hat{\sigma}^2$)
3	3	2.387	14.756	0.491	0.744
4	3	2.001	5.041	0.646	0.517
5	3	1.750	2.430	0.698	0.373
6	3	1.502	1.423	0.778	0.341
7	3	1.525	1.279	0.756	0.292
8	3	1.557	1.080	0.823	0.234
9	3	1.486	0.834	0.828	0.204
10	3	1.446	0.721	0.837	0.191
4	4	1.796	2.201	0.630	0.321
5	4	1.577	1.163	0.718	0.244
6	4	1.510	0.811	0.741	0.193
7	4	1.451	0.615	0.802	0.159
8	4	1.427	0.483	0.826	0.126
9	4	1.387	0.398	0.850	0.111
10	4	1.352	0.362	0.844	0.112
12	4	1.309	0.251	0.869	0.082
20	4	1.235	0.137	0.911	0.049
32	4	1.183	0.078	0.936	0.030
50	4	1.133	0.043	0.941	0.020
100	4	1.082	0.018	0.947	0.011
5	5	1.489	0.731	0.705	0.193
6	5	1.400	0.494	0.759	0.152
7	5	1.434	0.463	0.817	0.119
8	5	1.348	0.319	0.821	0.102
9	5	1.317	0.278	0.831	0.096
10	5	1.329	0.254	0.854	0.079
6	6	1.368	0.367	0.768	0.124
7	6	1.379	0.338	0.818	0.099
8	6	1.278	0.212	0.810	0.088
9	6	1.287	0.201	0.837	0.075
10	6	1.246	0.160	0.845	0.068
12	6	1.246	0.136	0.882	0.050
20	6	1.169	0.068	0.907	0.031
32	6	1.123	0.036	0.927	0.019
50	6	1.097	0.022	0.943	0.012
100	6	1.059	0.009	0.951	0.006
7	7	1.319	0.239	0.811	0.086
8	7	1.273	0.184	0.815	0.078
9	7	1.269	0.161	0.853	0.060
10	7	1.246	0.137	0.856	0.055

TABLE 7
Continued

MEAN AND MEAN SQUARE ERROR OF $\tilde{\sigma}^2$ AND $\hat{\sigma}^2$
FOR 1000 SIMULATIONS
WHEN $\lambda^2 = 48$

t	b	mean($\tilde{\sigma}^2$)	MSE($\tilde{\sigma}^2$)	mean($\hat{\sigma}^2$)	MSE($\hat{\sigma}^2$)
8	8	1.251	0.148	0.825	0.066
9	8	1.248	0.135	0.840	0.058
10	8	1.208	0.102	0.842	0.053
12	8	1.209	0.093	0.885	0.039
20	8	1.126	0.039	0.900	0.024
32	8	1.092	0.022	0.921	0.015
50	8	1.077	0.014	0.942	0.009
100	8	1.047	0.005	0.953	0.005
9	9	1.222	0.107	0.836	0.053
10	9	1.204	0.093	0.856	0.046
10	10	1.185	0.075	0.850	0.043
12	10	1.167	0.060	0.873	0.033
20	10	1.127	0.033	0.916	0.018
32	10	1.091	0.018	0.933	0.011
50	10	1.066	0.010	0.943	0.008
100	10	1.040	0.004	0.956	0.004

TABLE 8

MEAN AND MEAN SQUARE ERROR OF $\tilde{\sigma}^2$ AND $\hat{\sigma}^2$
FOR 1000 SIMULATIONS
WHEN $\lambda^2 = 96$

t	b	mean($\tilde{\sigma}^2$)	MSE($\tilde{\sigma}^2$)	mean($\hat{\sigma}^2$)	MSE($\hat{\sigma}^2$)
3	3	2.293	13.044	0.485	0.756
4	3	1.971	5.018	0.647	0.548
5	3	1.887	3.319	0.763	0.429
6	3	1.486	1.431	0.783	0.369
7	3	1.600	1.413	0.805	0.298
8	3	1.542	1.038	0.827	0.238
9	3	1.504	0.937	0.850	0.234
10	3	1.502	0.811	0.880	0.201
4	4	1.882	2.629	0.667	0.337
5	4	1.549	1.105	0.713	0.249
6	4	1.574	0.897	0.780	0.185
7	4	1.423	0.576	0.796	0.163
8	4	1.437	0.507	0.841	0.131
9	4	1.387	0.400	0.859	0.114
10	4	1.376	0.364	0.869	0.104
12	4	1.320	0.278	0.885	0.090
20	4	1.239	0.142	0.925	0.052
32	4	1.174	0.072	0.939	0.030
50	4	1.145	0.049	0.962	0.020
100	4	1.090	0.020	0.963	0.010
5	5	1.481	0.739	0.708	0.199
6	5	1.428	0.513	0.781	0.144
7	5	1.426	0.441	0.820	0.116
8	5	1.358	0.326	0.835	0.100
9	5	1.326	0.271	0.846	0.089
10	5	1.326	0.248	0.861	0.078
6	6	1.375	0.387	0.779	0.126
7	6	1.368	0.313	0.819	0.096
8	6	1.313	0.234	0.840	0.080
9	6	1.289	0.208	0.846	0.076
10	6	1.270	0.169	0.870	0.061
12	6	1.262	0.147	0.902	0.049
20	6	1.184	0.073	0.927	0.029
32	6	1.133	0.039	0.943	0.018
50	6	1.114	0.025	0.965	0.010
100	6	1.073	0.011	0.970	0.005
7	7	1.318	0.238	0.817	0.085
8	7	1.307	0.204	0.844	0.069
9	7	1.271	0.163	0.863	0.059
10	7	1.264	0.150	0.876	0.053

TABLE 8
Continued

MEAN AND MEAN SQUARE ERROR OF $\tilde{\sigma}^2$ AND $\hat{\sigma}^2$
FOR 1000 SIMULATIONS
WHEN λ^2 = 96

t	b	mean($\tilde{\sigma}^2$)	MSE($\tilde{\sigma}^2$)	mean($\hat{\sigma}^2$)	MSE($\hat{\sigma}^2$)
8	8	1.254	0.162	0.834	0.070
9	8	1.282	0.157	0.869	0.052
10	8	1.233	0.112	0.867	0.046
12	8	1.206	0.087	0.891	0.035
20	8	1.149	0.047	0.925	0.021
32	8	1.116	0.027	0.948	0.012
50	8	1.087	0.016	0.958	0.008
100	8	1.064	0.008	0.973	0.004
9	9	1.239	0.116	0.855	0.049
10	9	1.221	0.103	0.875	0.043
10	10	1.205	0.088	0.870	0.041
12	10	1.183	0.068	0.892	0.031
20	10	1.131	0.034	0.926	0.016
32	10	1.103	0.021	0.949	0.010
50	10	1.074	0.011	0.956	0.006
100	10	1.052	0.005	0.971	0.003

TABLE 9
ESTIMATED NONLINEAR REGRESSION COEFFICIENTS
FOR THE NEW ESTIMATOR

t	b	β_0	β_1	β_2
4	4	1.21475	0.0339516	0.0175206
5	4	1.10658	0.0277088	0.0097178
5	5	1.04996	0.0284706	0.0102013
6	4	1.13022	0.0263117	0.0065423
6	5	1.04926	0.0205445	0.0090859
6	6	1.02737	0.0192124	0.0181384
7	4	1.06740	0.0221559	0.0048124
7	5	1.04923	0.0216281	0.0061625
7	6	1.02800	0.0196412	0.0092484
7	7	1.02029	0.0162655	0.0178501
8	4	1.06098	0.0212346	0.0036800
8	5	1.02762	0.0184471	0.0045673
8	6	1.01621	0.0127188	0.0050639
8	7	1.03070	0.0131548	0.0082895
9	4	1.03142	0.0206524	0.0031174
9	5	1.03798	0.0157256	0.0035962
9	6	1.02508	0.0158532	0.0047872
9	7	1.01629	0.0127545	0.0056038
10	4	1.04555	0.0167255	0.0024495
10	5	1.03699	0.0161461	0.0030208
10	6	1.02077	0.0117183	0.0032660
10	7	1.01171	0.0126367	0.0042521
12	4	1.04038	0.0147547	0.0019486
12	6	1.00592	0.0119900	0.0023034
20	4	1.00888	0.0119889	0.0010164
20	6	1.00740	0.0077719	0.0010273
32	4	1.00672	0.0086647	0.0005869
32	6	0.99967	0.0056800	0.0005544

TABLE 10
MEAN AND MEAN SQUARE ERROR OF THE ESTIMATORS
(INCLUDING THE NEW ESTIMATOR)
FOR 1000 SIMULATIONS

λ^2	t	b	mean(JG)	MSE(JG)	mean(CS)	MSE(CS)	mean(NW)	MSE(NW)
3	4	4	1.343	0.956	0.428	0.399	1.085	0.800
	5	4	1.176	0.488	0.486	0.332	0.945	0.428
	6	4	1.211	0.372	0.546	0.264	0.968	0.302
	7	4	1.116	0.256	0.570	0.238	0.894	0.234
	7	7	1.072	0.093	0.633	0.163	0.857	0.100
	8	4	1.120	0.222	0.599	0.212	0.889	0.203
	8	7	1.066	0.077	0.659	0.142	0.879	0.080
	9	4	1.081	0.156	0.617	0.190	0.843	0.160
	9	7	1.059	0.064	0.690	0.120	0.866	0.074
	10	4	1.098	0.161	0.639	0.174	0.889	0.151
	10	7	1.063	0.060	0.708	0.109	0.864	0.069
	12	4	1.069	0.106	0.665	0.146	0.867	0.111
	20	4	1.046	0.058	0.737	0.094	0.835	0.078
	32	4	1.039	0.034	0.794	0.059	0.861	0.051
24	4	4	1.745	2.044	0.598	0.322	1.183	1.531
	5	4	1.495	0.971	0.666	0.243	1.035	0.726
	6	4	1.488	0.751	0.713	0.188	1.062	0.521
	7	4	1.413	0.559	0.764	0.159	1.046	0.387
	7	7	1.289	0.220	0.778	0.096	0.990	0.135
	8	4	1.371	0.416	0.777	0.131	1.002	0.276
	8	7	1.270	0.172	0.801	0.077	1.015	0.098
	9	4	1.316	0.334	0.789	0.120	0.958	0.235
	9	7	1.216	0.138	0.807	0.075	0.966	0.091
	10	4	1.320	0.340	0.806	0.117	1.008	0.236
	10	7	1.211	0.113	0.822	0.061	0.963	0.069
	12	4	1.273	0.221	0.829	0.086	0.993	0.145
	20	4	1.182	0.106	0.857	0.055	0.938	0.076
	32	4	1.135	0.059	0.883	0.035	0.951	0.043
96	4	4	1.875	2.361	0.665	0.309	1.298	1.696
	5	4	1.623	1.302	0.747	0.253	1.191	0.950
	6	4	1.572	0.942	0.779	0.196	1.213	0.663
	7	4	1.491	0.495	0.833	0.167	1.180	0.490
	7	7	1.320	0.231	0.815	0.083	1.064	0.136
	8	4	1.409	0.522	0.825	0.149	1.085	0.366
	8	7	1.305	0.210	0.842	0.073	1.070	0.124
	9	4	1.392	0.436	0.862	0.125	1.090	0.293
	9	7	1.273	0.164	0.864	0.059	1.051	0.093
	10	4	1.403	0.405	0.886	0.107	1.120	0.259
	10	7	1.260	0.146	0.874	0.053	1.049	0.082
	12	4	1.321	0.276	0.887	0.089	1.087	0.181
	20	4	1.247	0.150	0.931	0.053	1.062	0.094
	32	4	1.191	0.083	0.953	0.031	1.066	0.051

A COMPARISON OF TWO ESTIMATORS OF THE VARIANCE
IN THE TWO-FACTOR MULTIPLICATIVE INTERACTION MODEL

by

RONALD LEE WASSERSTEIN

B.A., Washburn University of Topeka, 1978

AN ABSTRACT OF A MASTERS REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Statistics

KANSAS STATE UNIVERSITY

Manhattan, Kansas

1984

ABSTRACT

Two estimators of the variance in the two-factor multiplicative interaction model have been proposed in the literature. The two estimators are compared by simulation, taking into account two aspects: (1) the number of levels of each of the two factors, and (2) the magnitude of the interaction. The choice between the two estimators is found to depend on both of these aspects. These simulations suggest a third estimator, which was found to be an improvement in many instances.